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DETERMINANTS OF NORMALIZED BOHEMIAN UPPER HESSENBERG MATRICES*

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Abstract. A matrix is Bohemian if its elements are taken from a finite set of integers. We enumerate all possible determinants of Bohemian upper Hessenberg matrices with subdiagonal fixed to one, and consider the special case of families of matrices with only zeros on the main diagonal, whose determinants proved to be related to a generalization of Fibonacci numbers. Several conjectures recently stated by Corless and Thornton follow from our results.

Key words. Bohemian matrix, upper Hessenberg matrix, determinant, Fibonacci number.

MSC2010. 11C20, 15A15, 15B36.

1. Introduction. Matrices whose entries are drawn from a finite discrete set $\mathcal{D}$ are called Bohemian. The name is a partial acronym for "BOunded HEight Matrix of Integers", and is due to the fact that $\mathcal{D}$, being finite, is bounded and is typically a subset of the integers. Even though this terminology is of quite recent introduction [5], Bohemian matrices have been a subject of interest for at least a century and a half, with early work of Sylvester [14] and Hadamard [9] that dates back to the second half of the 19th century. These early efforts, in particular, represent a first attempt to shed light on what is now known as the *Hadamard conjecture*, which was formulated in its current form by Paley [12], but is in fact much older [10] and remains an active research topic still today [6], [11].

The investigation of families of matrices with integer elements flourished in the sixties, when both theoretical [15] and practical [8] applications started to be considered. It is around this period that many others conjectures on these objects started to appear; a good example is the expository survey of "computational problems involving integral matrices" collected by Taussky [16], where these combinatorial problems are tackled, for small dimensions, by means of a brute force, computer-aided approach.

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More recently, Chan et al. considered families of Bohemian upper Hessenberg [3] and upper Hessenberg Toeplitz matrices [2]. Despite the fact that their work focused on the localization of the eigenvalues and the height of the characteristic polynomials of these matrices, they collected many of their observations on their determinants in the form of conjectures in the Characteristic Polynomial Database [17].

Some of these conjectures have been either proved or disproved since, but many remain still open. Here we focus on those that state properties of the determinants of families of normalized upper Hessenberg matrices, that is, upper Hessenberg matrices whose subdiagonal elements are fixed to one and those in the upper triangular part drawn from the sets $\{0, 1\}$, $\{0, -1\}$, $\{-1, 1\}$, $\{-1, 0, 1\}$, and $\{0, 1, 2\}$. In most cases, we were able to work with more general domains, thus proving the aforementioned conjectures as special cases. For the convenience of the reader, we state here the conjectures we considered, in the order they are addressed in following sections. They all reference sequences in The On-Line Encyclopedia of Integer Sequences (OEIS) [13].

Conjecture 1 ([17, Conj. 18])**.** *The number of distinct determinants of an $n \times n$ normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 0\}$ is given by a suitable shift of the OEIS sequence A000051.*

Conjecture 2 ([17, Conj. 19])**.** *The number of distinct determinants of an $n \times n$ normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 0, 1\}$ is given by a suitable shift of the OEIS sequence A000051.*

Conjecture 3 ([17, Conj. 20])**.** *The number of distinct determinants of an $n \times n$ normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 1\}$ is given by a suitable shift of the OEIS sequence A000051.*

Conjecture 4 ([17, Conj. 13])**.** *The maximum absolute determinant of an $n \times n$ zero-diagonal normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 0\}$ is given by the nth element of the OEIS sequence A000045.*

Conjecture 5 ([17, Conj. 15])**.** *The maximum absolute determinant of an $n \times n$ zero-diagonal normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 0, 1\}$ is given by the nth element of the OEIS sequence A000045.*

Conjecture 6 ([17, Conj. 17])**.** *The maximum absolute determinant of an $n \times n$ zero-diagonal normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 1\}$ is given by the nth element of the OEIS sequence A000045.*

Conjecture 7 ([17, Conj. 12])**.** *The number of distinct determinants of an $n \times n$ zero-diagonal normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 0\}$ is given by the nth element of the OEIS sequence A001611.*

Conjecture 8 ([17, Conj. 14])**.** *The number of distinct determinants of an $n \times n$ zero-diagonal normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 0, 1\}$ is given by the nth element of the OEIS sequence A001588.*

Conjecture 9 ([17, Conj. 16])**.** *The number of distinct determinants of an $n \times n$ zero-diagonal normalized Bohemian upper Hessemberg matrix with entries from the set $\{-1, 1\}$ is given by the nth element of the OEIS sequence A001611.*

The following section introduces the notation we use in the rest of the paper, and recalls background definitions and a result that will be used in later sections. In sections 3 and 4 we discuss the determinants of families of normalized and normalized zero-diagonal Bohemian upper Hessenberg matrices, respectively, and prove a generalization of most of the open conjectures in the Characteristic Polynomial Database. Finally, in section 5 we provide a proof of [17, Conj. 8], the only conjecture we were not able to prove in its generalized form, and state our generalized conjecture. This will be the subject of future investigation.

2. Background and notation.

We denote by $\mathbb{N}_d$ the set of integers between 0 and d, by $-\mathbb{N}_d$ the set integers between $-d$ and 0, by $\mathbb{Z}_d$ the set of integers between $-d$ and d, and by $\mathbb{Z}_d^0 = \mathbb{Z}_d \setminus \{0\}$ the set of non-zero integers between $-d$ and d. We say that the matrix $H \in \mathbb{C}^{n\times n}$ is upper Hessenberg if $h_{ij} = 0$ for $i > j+1$. An upper Hessenberg matrix $H \in \mathbb{C}^{n\times n}$ is normalized if $h_{i+1,i} = 1$, for $i = 1, \dots, n-1$, and is zero-diagonal if $h_{ii} = 0$, for $i = 1, \dots, n$. For any $\mathcal{D} \subset \mathbb{C}$, we denote by $\mathcal{H}^n(\mathcal{D})$ the family of normalized upper Hessenberg matrices with elements from $\mathcal{D}$ in the upper triangular part, and by $\widehat{\mathcal{H}}^n(\mathcal{D})$ the family of zero-diagonal normalized upper Hessenberg matrix with elements from $\mathcal{D}$ in the strictly upper triangular part.

We will often make use of the following result, which allows us to write the determinant of a normalized upper Hessenberg matrix in terms of those of its leading principal minors.

Lemma 1 ([7, sect. 7.11])**.** *Let $H \in \mathcal{H}^n(\mathcal{D})$, and let $H^{(i)} \in \mathcal{H}^i(\mathcal{D})$, for $i = 1, \dots, n-1$, be the ith leading principal minor of H. Then*

$$\det H = (-1)^{n+1}\left(h_{1n} + \sum_{i=2}^{n}(-1)^{i-1}h_{in}\det H^{(i-1)}\right).$$

Proof. The identity can be verified by expanding along the last column of H. ∎

3. Normalized Bohemian upper Hessenberg matrices.

In this section we analyze the case of normalized Bohemian upper Hessenberg matrices with elements in the upper triangular part taken from $\mathbb{Z}_d$, $-\mathbb{N}_d$, and $\mathbb{Z}_d^0$. We will start by finding a closed expression for the maximum absolute value of the determinant of matrices in these classes. Then we will show how to build matrices with given determinants.

Proposition 1. *For all $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$, one has that*

$$\operatorname*{argmax}_{H\in\mathcal{H}^n(\mathbb{Z}_d)} |\det H| = \operatorname*{argmax}_{H\in\mathcal{H}^n(-\mathbb{N}_d)} |\det H| = \operatorname*{argmax}_{H\in\mathcal{H}^n(\mathbb{Z}_d^0)} |\det H| = d(d+1)^{n-1},$$

where the maximum is attained by the matrix $K^{(d,n)}$, defined by

$$k_{ij}^{(d,n)} = \begin{cases} 0, & i > j+1, \\ 1, & i = j+1, \\ -d, & i \le j, \end{cases} \tag{1}$$

for which

$$\det K^{(d,n)} = (-1)^n d(d+1)^{n-1}. \tag{2}$$

Proof. We begin by showing (2). It is easy to verify that the identity holds for $n = 1$. Let $K^{(d,n)}$ be

$$K^{(d,n)} = \begin{bmatrix} -d & -d & -d & \dots & -d \\ 1 & -d & -d & \dots & -d \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & -d & -d \\ & & & 1 & -d \end{bmatrix}.$$

For the inductive step, from Lemma 1 we have that

$$\begin{aligned} \det K^{(d,n)} &= (-1)^{n+1}\left(k_{1n}^{(d,n)} + \sum_{i=2}^{n}(-1)^{i-1}k_{in}^{(d,n)} \det K^{(d,i-1)}\right) \\ &= (-1)^n\left(d + \sum_{i=2}^{n} d^2(d+1)^{i-2}\right) \\ &= (-1)^n\left(d + d^2\frac{1-(d+1)^{n-3}}{1-d+1}\right) \\ &= (-1)^n d(d+1)^{n-1}. \end{aligned} \tag{3}$$

On the other hand, using the notation in Lemma 1 gives, for any $H \in \mathcal{H}^n(\mathbb{Z}_d)$,

$$\begin{aligned} |\det H| &= \left| h_{1n} + \sum_{i=2}^{n}(-1)^{i-1}h_{in} \det H^{(i)} \right| \\ &\leq d + \sum_{i=2}^{n} d^2(d+1)^{i-2} \\ &= |\det K^{(d,n)}|, \end{aligned}$$

where the last equality follows from (3). Noting that $K^{(d,n)}$ also belongs to both $\mathcal{H}^n(-\mathbb{N}_d)$ and $\mathcal{H}^n(\mathbb{Z}_d^0)$, and that $\widehat{\mathcal{H}}^n(-\mathbb{N}_d)$ and $\mathcal{H}^n(\mathbb{Z}_d^0)$ are both subsets of $\mathcal{H}^n(\mathbb{Z}_d)$ concludes the proof. $\square$

This Proposition generalizes [17, Conj. 4]: for $d = 1$, it shows that the sequence of maximal absolute determinants of normalized upper Hessenberg matrices with entries from the set $\mathbb{Z}_1$ is given by the OEIS sequence A03433.

Next we prove a few technical lemmas we will need later on in order to show how to construct a Bohemian matrix with a given determinant.

Lemma 2. *Let $d \in \mathbb{N}^+$ and let $\gamma \in \mathbb{N}_{d(d+1)^{n-1}}$. Then there exist $\beta, \alpha_0, \dots, \alpha_{n-2} \in \mathbb{N}_d$ such that*

$$\gamma = \beta + d\sum_{i=0}^{n-2} \alpha_i(d+1)^i. \tag{4}$$

Proof. If γ is strictly smaller than $d(d+1)^{n-1}$, then there exists a unique pair of integers $\mu \in \mathbb{N}_{d-1}$ and $\nu \in \mathbb{N}_{(d+1)^{n-1}-1}$ such that $\gamma = \mu + d\nu$. Since $\nu \leq (d+1)^{n-1} - 1$, we can write

$$\nu = \sum_{i=0}^{n-2} \alpha_i(d+1)^i,$$

Algorithm 1: Conversion of representation coefficients from $\mathbb{N}_d$ to $\mathbb{Z}_d^0$.

Input: $\beta', \alpha'_0, \ldots, \alpha'_{n-2} \in \mathbb{N}_d$ that satisfy (6).
Output: $\beta, \alpha_0, \ldots, \alpha_{n-2} \in \mathbb{Z}_d^0$ that satisfy (5).
$(\beta, \alpha_0, \ldots, \alpha_{n-2}) \leftarrow (\beta', \alpha'_0, \ldots, \alpha'_{n-2})$;
for $i = n-2$ ***down to 1*** **do**
  if $\alpha_i = 0$ **then**
    $(\alpha_{i-1}, \alpha_i) \leftarrow (\alpha_{i-1} - d - 1, 1)$;
if $\alpha_0 = 0$ **then**
  if $\beta = d$ **then**
    $(\beta, \alpha_0) \leftarrow (-d, 2)$;
  else
    $(\beta, \alpha_0) \leftarrow (\beta - d, 1)$;

and the representation (4) is obtained by setting β to μ and $\alpha_0, \ldots, \alpha_{n-2} \in \mathbb{N}_d$ to the $n-2$ digits in the representation of ν in radix $d+1$. If $\gamma = d(d+1)^{n-1}$, on the other hand, setting $\beta, \alpha_0, \ldots, \alpha_{n-2} = d$ gives

$$d + d\sum_{i=0}^{n-2} d(d+1)^i = d + d((d+1)^{n-1} - 1) = d(d+1)^{n-1},$$

which concludes the proof. □

Corollary 1. *Let $d \in \mathbb{N}^+$ and let $\gamma \in \mathbb{Z}_{d(d+1)^{n-1}}$. Then there exist $\beta, \alpha_0, \ldots, \alpha_{n-2} \in \mathbb{Z}_d$ such that*

$$\gamma = \beta + d\sum_{i=0}^{n-2} \alpha_i (d+1)^i.$$

Proof. If γ is nonnegative, then the result follows directly from Lemma 2. Otherwise, $\beta, \alpha_0, \ldots, \alpha_{n-2}$ can be obtained by changing the sign of the corresponding coefficients in the representation (4) for $-\gamma \in \mathbb{N}_d$. □

Lemma 3. *Let $d \in \mathbb{N}^+ \setminus \{1\}$ and let $\gamma \in \mathbb{Z}_{d(d+1)^{n-1}}$. Then there exist $\beta, \alpha_0, \ldots, \alpha_{n-2} \in \mathbb{Z}_d^0$ such that*

$$\gamma = \beta + d\sum_{i=0}^{n-2} \alpha_i (d+1)^i. \tag{5}$$

Proof. If $\gamma \geq 0$, then by Lemma 2 there exist $\beta', \alpha'_0, \ldots, \alpha'_{n-2} \in \mathbb{N}_d$ such that

$$\gamma = \beta' + d\sum_{i=0}^{n-2} \alpha'_i (d+1)^i. \tag{6}$$

Algorithm 1 shows how the coefficients of the representation (5) can be computed from those of (6). To prove the correctness of the algorithm, note that the two sets of coefficients represent the same number, that all zero coefficients in (6) are changed to either 1 or 2, and that no new zero coefficients are introduced.

If $\gamma < 0$, it suffices to find the representation (6) for $-\gamma$ in $\mathbb{N}_d$ and change the sign of its coefficients. $\square$

Lemma 2 and Corollary 1 essentially say that any integer number in $\mathbb{N}_{d(d+1)^{n-1}}$ or $\mathbb{Z}_{d(d+1)^{n-1}}$ has at least one representation in the form (4) and (5), respectively. Lemma 3 tells us that, for $d > 1$, we can achieve the same result even without using any zero coefficients. On the other hand, this is no longer true for $d = 1$: for example, we cannot build $n = 3$, because we would need to find β, a_0 and a_1 such that $\beta + a_0 + 2a_1 = 3$, but no such triple exists, since the left-hand side must be even while the right-hand side must be odd. Note that, in Algorithm 1, we must require $d > 2$ because of the assignment $\alpha_0 \leftarrow 2$ on line 1.

The next two propositions will keep this difference between the cases $d \neq 1$ and $d = 1$.

Proposition 2. *Let $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+ \setminus \{1\}$. The set of possible determinants for matrices in the family $\mathcal{H}^n(\mathbb{Z}_d^0)$ is $\mathbb{Z}_{d(d+1)^{n-1}}$.*

Proof. In this case we show how to construct a matrix $H \in \mathcal{H}^n(\mathbb{Z}_d^0)$ such that $\det H = k$ for any $k \in \mathbb{Z}_{d(d+1)^{n-1}}$. Here we take the matrix

$$H = \left[\begin{array}{ccc|c} & & & b \\ & & & a_0 \\ & K^{(d,n-1)} & & \vdots \\ & & & a_{n-3} \\ \hline 0 & \dots & 1 & a_{n-2}, \end{array}\right], \tag{7}$$

where $K^{(d,n-1)}$ is defined in (21). By Lemma 1 and Proposition 1, we have that

$$\begin{aligned} \det H &= (-1)^{n+1} b + \sum_{i=0}^{n-2} (-1)^{n+i} a_i \det K^{(d,i+1)} \\ &= \widetilde{b} + d \sum_{i=0}^{n-2} \widetilde{a}_i (d+1)^i, \end{aligned}$$

where $\widetilde{b} = (-1)^{n+1} b$ and $\widetilde{a}_i = (-1)^{n+1} a_i$, for $i = 0, \dots, n-2$. Since $b, a_0, \dots, a_{n-2} \in \mathbb{Z}_d^0$, by Lemma 3 we can conclude that by choosing the last column of H in a suitable way, one can obtain a matrix H such that $\det H = k$ for any $k \in \mathbb{Z}_{d(d+1)^{n-1}}$. $\square$

Proposition 3. *Let $n \in \mathbb{N}^+$. The set of possible determinants for matrices in the family $\mathcal{H}^n(\mathbb{Z}_1^0)$ is $\{2k \mid k \in \mathbb{Z}_{n-1}\}$.*

Proof. As noted by Ching [4], there are only 2^{n-1} possibly nonzero terms in the determinant expansion of an $n \times n$ Hessenberg matrix. If the matrix is in $\mathcal{H}^n(\mathbb{Z}_1^0)$, then each of these 2^{n-1} monomials evaluates to either $+1$ or -1, which implies that the determinant of any such matrices must be even and cannot be larger than 2^{n-1} in absolute value. Now we explain how to construct a matrix $H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_1^0)$ such that $\det H = 2k$ for any $k \in \mathbb{N}_{n-1}$. Matrices with negative determinants can be obtained by changing the sign of any row or columns of the matrices thus obtained.

Let us consider the matrix

$$H = \left[\begin{array}{ccc|c} & & & (-1)^{n+1} \\ & K^{(1,n-1)} & & a_0 \\ & & & \vdots \\ & & & a_{n-3} \\ \hline 0 & \dots & 1 & (-1)^{n+1}, \end{array}\right],$$

where $K^{(1,n-1)}$ is defined in (21). Using Lemma 1 followed by Proposition 1, we obtain that

$$\begin{aligned} \det H &= 1 + \sum_{i=0}^{n-3} (-1)^{n+1} a_i \det K^{(1,i+1)} + (-1)^{n+1} K^{(1,n-1)} \\ &= 1 + (-1)^{n+1} \sum_{i=0}^{n-3} a_i 2^i + 2^{n-2} \end{aligned}$$

Let $k' := 2^{n-2} - k$ and let $b_0, \dots, b_{n-3} \in \{0, 1\}$ be such that

$$k' = \sum_{i=0}^{n-3} b_i 2^i = \sum_{i \in I} b_i 2^i,$$

where $I := \{i \in \mathbb{N}_{n-3} : b_i \neq 0\}$. If we set $a_i = (-1)^{n+1+b_i}$, we are taking the coefficients that would create the matrix $K^{(d,n)}$ with the last column multiplied by $(-1)^{n+1}$ and changing the signs in the corresponding nonzero coefficients of the binary representation of k'. It follows that

$$\begin{aligned} \det H &= 1 + (-1)^{b_0} + \dots + (-1)^{b_{n-3}} 2^{n-3} + 2^{n-2} \\ &= 1 + \sum_{i=0}^{n-2} 2^i - 2 \sum_{i \in I} b_i 2^i \\ &= 2^{n-1} - 2k' = 2k, \end{aligned}$$

which concludes the proof. $\square$

Proposition 4. *Let $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$. The set of possible determinants for matrices in the family $\mathcal{H}^n(\mathbb{Z}_d)$ is $\mathbb{Z}_{d(d+1)^{n-1}}$.*

Proof. The proof is analogous to that of Proposition 2, if Lemma 3 is replaced by Corollary 1. $\square$

Proposition 5. *Let $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$. The set of possible determinants for matrices in the family $\mathcal{H}^n(-\mathbb{N}_d)$ is $(-1)^n \cdot \mathbb{N}_{d(d+1)^{n-1}}$.*

Proof. This proof retraces the steps of that of Proposition 2. Note that $\mathcal{H}^n(-\mathbb{N}_d) \subset \mathcal{H}^n(\mathbb{Z}_d)$, and that since $K^{(d,n-1)} \in \mathcal{H}^n(-\mathbb{N}_d)$, the matrix H in (7) is in $\mathcal{H}^n(-\mathbb{N}_d)$ if we require that $b, a_0, \dots, a_{n-2} \in -\mathbb{N}_d$. Combining Lemma 1 and Proposition 1 in this case gives

$$\begin{aligned}\det H &= (-1)^{n+1} b + \sum_{i=0}^{n-2} (-1)^{n+i} a_i \det K^{(d,i+1)} \\ &= (-1)^n \left(\widetilde{b} + d \sum_{i=0}^{n-2} \widetilde{a}_i (d+1)^i \right),\end{aligned}$$

where where $\widetilde{b} = -b$ and $\widetilde{a}_i = -a_i$, for $i = 0, \dots, n-2$. Since $\widetilde{b}, \widetilde{a}_0, \widetilde{a}_{n-2} \in \mathbb{N}_d$, by Lemma 2 we can choose the last column of H so that $\det H = (-1)^n k$ for all $k \in \mathbb{N}_{d(d+1)^{n-1}}$. $\square$

Note that Propositions 2, 3, 4, and 5 prove a more general version of [17, Conj. 18], [17, Conj. 19], and [17, Conj. 20], as they show that the number of distinct determinants of normalized Bohemian upper Hessenberg matrices with entries from the sets $-\mathbb{N}_1$, $\mathbb{Z}_1$, and $\mathbb{Z}_1^0$ is given by a suitable shift of the OEIS sequence A000051.

4. Zero-diagonal normalized Bohemian upper Hessenberg matrices. In the previous section we explained how to build a Bohemian upper Hessenberg matrix for every possible determinant, and by doing so we generalized some of the conjectures in [17]. Now, we follow the same steps and prove the corresponding conjectures for the normalized zero-diagonal Bohemian upper Hessenberg matrices in the family $\widehat{\mathcal{H}}^n(\mathcal{D})$ for the same finite domains in the previous section.We begin with a definition.

Definition 1 (Generalized Fibonacci number)**.** *Let $d \in \mathbb{N}^+$. We call the numbers generated by the two-term recurrence*

$$\begin{cases} f_1^d = 0, \\ f_2^d = d, \\ f_n^d = f_{n-1}^d + d\, f_{n-2}^d, \qquad n \in \mathbb{N} \setminus \{0,1,2\}, \end{cases} \tag{8}$$

generalized Fibonacci numbers.

Note that for $d = 1$ one obtains the sequence of Fibonacci numbers, in which case we drop the superscript and write $f_n := f_n^1$ to indicate the term $n+1$ in the OEIS sequence A000045.

The next lemma shows one possible way of constructing a matrix with maximum absolute determinant for all families of interest in this section.

Proposition 6. *For all $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$, one has that*

$$\operatorname*{argmax}_{H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_d)} |\det H| = \operatorname*{argmax}_{H \in \widehat{\mathcal{H}}^n(-\mathbb{N}_d)} |\det H| = \operatorname*{argmax}_{H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_d^0)} |\det H| = f_n^d, \tag{9}$$

where the maximum is attained by the matrix $\widehat{K}^{(d,n)}$, defined by

$$k_{ij}^{(d,n)} = \begin{cases} 0, & i > j+1 \text{ or } i = j, \\ 1, & i = j+1, \\ -d, & i < j, \end{cases} \tag{10}$$

for which

$$\det \widehat{K}^{(d,n)} = (-1)^n f_n^d. \tag{11}$$

Proof. First we prove (11) by induction. For the base cases, a direct computation shows that $\det \widehat{K}^{(d,1)} = 0$ and $\det \widehat{K}^{(d,2)} = d$. For the inductive step, by subtracting the ith row from the row below, for i from $n-1$ down to 1, we obtain

$$\det \widehat{K}^{(d,n)} = \begin{vmatrix} 0 & -d & -d & \dots & -d \\ 1 & 0 & -d & \dots & -d \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & 0 & -d \\ & & & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -d & & & \\ 1 & -1 & -d & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & -1 & -d \\ & & & 1 & 0 \end{vmatrix}.$$

By using the Laplace expansion by minors along the first column of the tridiagonal matrix thus obtained, we get

$$\det \widehat{K}^{(d,n)} = -\det \widehat{K}^{(d,n-1)} - \begin{vmatrix} -d & & & & \\ 1 & -1 & -d & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & -1 & -d \\ & & & 1 & 0 \end{vmatrix},$$

and by expanding along the first row, we can conclude that

$$\det \widehat{K}^{(d,n)} = -\det \widehat{K}^{(d,n-1)} + d \det \widehat{K}^{(d,n-2)} = (-1)^n f_{n-1}^d + (-1)^{n-2} d\, f_{n-2}^d = (-1)^n f_n^d.$$

In order to prove the optimality result, we define the family of Bohemian Hessenberg matrices $\mathcal{B}^n = \{H + x\, e_n e_n^T : H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_d), x \in \mathbb{Z}_d\}$, and show by induction that for all $H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_d)$ and all $B \in \mathcal{B}^n$, we have that $|\det H| \le f_n^d$ and $|\det B| \le f_{n+1}^d$.

The only matrix in $\widehat{\mathcal{H}}^1(\mathbb{Z}_d)$ has determinant 0, and it is easy to check by exhaustion that the absolute value of the determinant of matrices in $\widehat{\mathcal{H}}^2(\mathbb{Z}_d)$, $\mathcal{B}^1$, and $\mathcal{B}^2$ is at most d. For the inductive step, let us consider the matrices

$$H = \begin{bmatrix} 0 & \times & \times & \dots & \times \\ 1 & 0 & \times & \dots & \times \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & 0 & \times \\ & & & 1 & 0 \end{bmatrix} \in \widehat{\mathcal{H}}^n(\mathbb{Z}_d), \qquad B = \begin{bmatrix} 0 & \times & \times & \dots & \times \\ 1 & 0 & \times & \dots & \times \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & 0 & \times \\ & & & 1 & b_{nn} \end{bmatrix} \in \mathcal{B}^n.$$

For the first matrix, we have that $|\det H| = |-\det B'|$ for some $B' \in \mathcal{B}^{n-1}$, and by the inductive hypothesis we can conclude that $|\det H| \le f_n^d$. For the matrix B, by observing that $\det B = b_{nn} \det H' - \det B'$, for some $H' \in \widehat{\mathcal{H}}^{n-1}(\mathbb{Z}_d)$ and $B' \in \mathcal{B}^{n-1}$, we obtain that

$$|\det B| = |b_{nn} \det H' - \det B'| \le |b_{nn}||\det H'| + |\det B'| \le d\, f_{n-1}^d + f_n^d = f_{n+1}^d.$$

Therefore, the absolute value of the determinant of matrices in the family $\widehat{\mathcal{H}}^n(\mathbb{Z}_d)$ is bounded by f_n^d, and observing that $\widehat{K}^{(d,n)}$ belongs to $\widehat{\mathcal{H}}^n(\mathbb{Z}_d)$ shows that the bound is attained by a matrix in that family. The optimality result in (9) follows from the fact that $\widehat{\mathcal{H}}^n(-\mathbb{N}_d)$ and $\widehat{\mathcal{H}}^n(\mathbb{Z}_d^0)$ are subfamilies of $\widehat{\mathcal{H}}^n(\mathbb{Z}_d)$ and that $\widehat{K}^{(d,n)}$ belongs to $\widehat{\mathcal{H}}^n(-\mathbb{N}_d) \cap \widehat{\mathcal{H}}^n(\mathbb{Z}_d^0)$. $\square$

For $d = 1$, the three equalities in (9) prove [17, Conj. 13], [17, Conj. 15], and [17, Conj. 17]. We stress that $\widehat{K}^{(d,n)}$ is not the only matrix $H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_d)$ such that $|\det H| = f_n^d$.

Now we can use these results to build matrices with specific determinants, but first we need a few technical results corresponding to those in Lemma 2, Corollary 1 and Lemma 3 for our generalized Fibonacci numbers.

Lemma 4. *Let $d \in \mathbb{N}^+$ and let $\gamma \in \mathbb{N}_{f_n^d}$. Then there exist $\beta, \alpha_2, \ldots, \alpha_{n-3} \in \mathbb{N}_d$ such that*

$$\gamma = \beta + \sum_{i=2}^{n-3} \alpha_i f_i^d,$$

where the sequence $(f_i^d)_{i \in \mathbb{N}^+}$ is defined in (8).

Proof. The proof is by induction on n. The base case is trivially satisfied, since $f_1^d = 0$, and 0 is obtained for $\beta = 0$.

For the inductive step, if $\gamma \leq f_n^d$, then the statement is true by the inductive hypothesis. Otherwise we have that $f_n^d < \gamma \leq f_{n+1}^d$, which implies that $\gamma - d f_{n-1}^d \leq f_n^d$. Therefore, there exist $\alpha_1', \ldots, \alpha_{n-3}', \beta' \in \mathbb{N}_d$ such that

$$\gamma - d f_{n-1}^d = \beta' + \sum_{i=2}^{n-3} \alpha_i' f_i^d. \tag{12}$$

It follows that

$$\gamma = \beta' + \sum_{i=2}^{n-3} \alpha_i' f_i^d + d f_{n-1}^d = \beta + \sum_{i=2}^{n-1} \alpha_i f_i^d,$$

where $\beta = \beta'$, $\alpha_i = \alpha_i'$ for $i = 1, \ldots, n-3$, and $\alpha_{n-2} = d$. $\square$

Corollary 2. *Let $\gamma \in \mathbb{Z}_{f_n^d}$. Then there exist $\beta, \alpha_2, \ldots, \alpha_{n-3} \in \mathbb{Z}_d$ such that*

$$\gamma = \beta + \sum_{i=2}^{n-3} \alpha_i f_i^d, \tag{13}$$

where the sequence $(f_i^d)_{i \in \mathbb{N}^+}$ is defined in (8).

Proof. If $\gamma \geq 0$, then (13) follows from Lemma 4. For $\gamma < 0$, the coefficients in (13) can be obtained by changing the sign of those in the representation of $-\gamma$ in (12). $\square$

Lemma 5. *Let $d \in \mathbb{N}^+ \setminus \{1\}$ and let $\gamma \in \mathbb{Z}_{f_n^d}$. Then there exist $\beta, \alpha_2, \ldots, \alpha_{n-3} \in \mathbb{Z}_d^0$ such that*

$$\gamma = \beta + \sum_{i=2}^{n-3} \alpha_i f_i^d, \tag{14}$$

where the sequence $(f_i^d)_{i \in \mathbb{N}^+}$ is defined in (8).

Algorithm 2: Conversion of representation coefficients from $\mathbb{N}_d$ to $\mathbb{Z}_d^0$.

Input: $\beta', \alpha'_0, \ldots, \alpha'_{n-3} \in \mathbb{N}_d$ that satisfy (15).

Output: $\beta, \alpha_0, \ldots, \alpha_{n-3} \in \mathbb{Z}_d^0$ that satisfy (14).

```
(β, α_0, ..., α_{n-3}) ← (β', α'_0, ..., α'_{n-3});
i ← n − 3;
while i ≥ 3 do
    if α_i = 0 then
        if α_{i−1} ≠ 1 and α_{i−1} ≠ −d then
            (α_{i−2}, α_{i−1}, α_i) ← (α_{i−2} − d, α_{i−1} − 1, 1)
        else
            if α_{i−2} = 0 then
                (α_{i−2}, α_{i−1}, α_i) ← (d, 2, −1);
                i ← i − 2;
            else
                (α_{i−2}, α_{i−1}, α_i) ← (α_{i−2} − d, 0, 1);
    i ← i − 1;
if α_2 = 0 then
    if α_1 ≠ 1 and α_1 ≠ −d then
        (α_1, α_2) ← (α_1 − 1, α_2 + 1);
    else
        (α_1, α_2) ← (α_1 + 1, α_2 − 1);
if β = 0 then
    if α_1 ≠ 1 and α_1 ≠ −d then
        (β, α_1) ← (d, α_1 − 1);
    else
        (β, α_1) ← (−d, α_1 + 1);
```

Proof. From Lemma 4, if $\gamma \geq 0$ then there exist $\beta', \alpha'_0, \ldots, \alpha'_{n-3} \in \mathbb{N}_d$ such that

$$\gamma = \beta' + \sum_{i=0}^{n-3} \alpha'_i f_i^d. \tag{15}$$

The pseudocode in Algorithm 2 shows that the coefficients in the representation (14) can be computed from those of (15). The case $\gamma < 0$ is analogous, it suffices to consider the representation in (15) for $-\gamma$ and change the sign of the coefficients thus obtained. ∎

Lemma 4 can be related to a famous and interesting result for standard Fibonacci numbers, known as *Zeckendorf's theorem* [1], which we now state.

Theorem 1 (Zeckendorf's theorem)**.** *Every nonzero natural number $n \in \mathbb{N}$ can be written as the sum of one or more distinct Fibonacci numbers in such a way that the sum does* *not contain two consecutive Fibonacci numbers, i.e.,*

$$n = \sum_{i}^{k} f_{c_i}, \tag{16}$$

where $c_i \geq 3$, $c_{i+1} > c_i + 1$. Moreover, this representation is unique.

We need one last technical lemma. In Proposition 3 we used a result by Ching [4] to count the number of nonzero terms in the determinant expression of a Hessenberg matrix. Lemma 6 achieves the same goal for matrices in $\widehat{\mathcal{H}}^n(\mathcal{D})$.

Lemma 6. *Let $H \in \widehat{\mathcal{H}}^n(\mathcal{D})$. Let $p_n(h_{12}, \dots, h_{(n-1)n})$ be the determinant of H seen as a polynomial in the $n(n-1)/2$ variables $h_{12}, \dots, h_{(n-1)n}$. Denote with $\mathbf{a}_j$ a monomial of p_n. Then*

$$p_n(h_{12}, \dots, h_{(n-1)n}) = (-1)^n \sum_{j=1}^{f_n} (-1)^{\deg \mathbf{a}_j} \mathbf{a}_j, \tag{17}$$

where f_n is the n–th Fibonacci number as defined in (8) and $\deg \mathbf{a}_j$ is the degree of the monomial $\mathbf{a}_j$

Proof. We will prove this claim by induction for the matrix

$$H = \begin{bmatrix} 0 & h_{12} & h_{13} & \dots & h_{1n} \\ 1 & 0 & h_{23} & & h_{2n} \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & 0 & h_{(n-1)n} \\ & & & 1 & 0 \end{bmatrix}.$$

The claim is true for the basic cases $n = 2$ and $n = 3$, where

$$p_2(h_{12}) = -h_{12} = (-1)^2 \cdot (-1) \cdot h_{12},$$

and

$$p_3(h_{12}, h_{13}, h_{23}) = h_{13} = (-1)^3 \cdot (-1) \cdot h_{13}.$$

Then, by using the Laplace expansion twice, we get

$$\det H_n = -h_{12} \begin{vmatrix} 0 & h_{34} & h_{35} & h_{36} & \dots & h_{3n} \\ 1 & 0 & h_{45} & h_{46} & \dots & h_{4n} \\ & 1 & 0 & h_{56} & \dots & h_{5n} \\ & & 1 & \ddots & \ddots & \vdots \\ & & & \ddots & 0 & h_{(n-1)n} \\ & & & & 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & h_{13} & h_{14} & h_{15} & \dots & h_{1n} \\ 1 & 0 & 0 & 0 & \dots & 0 \\ & 1 & 0 & h_{45} & \dots & h_{4n} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & h_{(n-1)n} \\ & & & & 1 & 0 \end{vmatrix},$$

which reads, using the inductive hypothesis,

$$\begin{aligned}p_n(h_{12},\dots,h_{(n-1)n}) &= -h_{12}p_{n-2}(h_{34},\dots,h_{(n-1)n}) - p_{n-1}(h_{13},\dots,h_{(n-1)n})\\ &= -h_{12}(-1)^{n-2}\sum_{k=1}^{f_{n-2}}(-1)^{\deg\widehat{\mathbf{a}}_k}\widehat{\mathbf{a}}_k - (-1)^{n-1}\sum_{j=1}^{f_{n-1}}(-1)^{\deg\mathbf{a}_j}\mathbf{a}_j\\ &= (-1)^n\sum_{k=1}^{f_{n-2}}(-1)^{\deg\mathbf{a}_k}\mathbf{a}_k + (-1)^n\sum_{j=1}^{f_{n-1}}(-1)^{\deg\mathbf{a}_j}\mathbf{a}_j\end{aligned} \tag{18}$$

This proves the first part of the claim about the sign of the monomials. Next, we know that the total number of monomials in p_n cannot exceed $f_{n-1}+f_{n-2}=f_n$, but could be smaller due to possible cancellations. However, these do not occur because every monomial in $p_{n-2}(h_{34},\dots,h_{(n-1)n})$ contains one coefficient among $h_{34},h_{35},\dots h_{3n}$ that does not appear in $p_{n-1}(h_{13},\dots,h_{(n-1)n})$. This concludes the proof. □

Proposition 7. *Let $n\in\mathbb{N}^+$ and $d\in\mathbb{N}^+\setminus\{1\}$. The set of possible determinants for matrices in the family $\widehat{\mathcal{H}}^n(\mathbb{Z}_d^0)$ is $\mathbb{Z}_{f_n^d}$, where f_n^d is the nth generalized Fibonacci number as defined in (8).*

Proof. In this case we show how to construct a matrix $H\in\widehat{\mathcal{H}}^n(\mathbb{Z}_d^0)$ such that $\det H = k$ for any $k\in\mathbb{Z}_{f_n^d}$. Let

$$H = \left[\begin{array}{ccc|c} & & & b\\ & \widehat{K}^{(d,n-1)} & & a_0\\ & & & \vdots\\ & & & a_{n-3}\\ \hline 0 & \dots & 1 & 0,\end{array}\right],$$

where $\widehat{K}^{(d,n-1)}$ is defined in (10). By Lemma 1 and Proposition 6, we have that

$$\begin{aligned}\det H &= (-1)^{n+1}b + \sum_{i=0}^{n-3}(-1)^{n+i}a_i\det\widehat{K}^{(d,i+1)}\\ &= \widetilde{b} + \sum_{i=0}^{n-3}\widetilde{a}_i f_i^d,\end{aligned}$$

where $\widetilde{b}=(-1)^{n+1}b$ and $\widetilde{a}_i=(-1)^{n+i}a_i$, for $i=0,\dots,n-3$. Since $b,a_0,\dots,a_{n-3}\in\mathbb{Z}_d^0$, by Lemma 5 we can conclude that the last column of H can be chosen so that $\det H=k$ for any $k\in\mathbb{Z}_{f_n^d}$. □

Proposition 8. *Let $n\in\mathbb{N}^+$. The set of possible determinants for matrices in the family $\widehat{\mathcal{H}}^n(\mathbb{Z}_1^0)$ is*

$$\{-f_n,-f_n+2,-f_n+4,\dots,f_n-4,f_n-2,f_n\},$$

where f_n is the nth Fibonacci number as defined in (8).

Proof. By Lemma 6, there are only f_n possibly nonzero terms in the determinant expression of a $n\times n$ Hessenberg matrix with a zero diagonal. If a matrix is in $\widehat{\mathcal{H}}^n(\mathbb{Z}_1^0)$, then each of these f_n monomials evaluates to either $+1$ or -1, which implies that the determinant of any such matrices must have the same parity of f_n. Now we explain how to build a matrix $H \in \widehat{\mathcal{H}}^n(\mathbb{Z}_1^0)$ such that $\det H = f_n - 2k$ for $k \in \mathbb{N}_{f_n^d}$. Let

$$H = \left[\begin{array}{ccc|c} & & & b \\ & \widehat{K}^{(1,n-1)} & & a_0 \\ & & & \vdots \\ & & & a_{n-3} \\ \hline 0 & \dots & 1 & 0, \end{array}\right],$$

where $\widehat{K}^{(1,n-1)}$ is defined in (10). By Lemma 1 and Proposition 6, we have that

$$\begin{aligned}\det H &= (-1)^{n+1} b + \sum_{i=0}^{n-3} (-1)^{n+i} a_i \det \widehat{K}^{(d,i+1)} \\ &= \widetilde{b} + \sum_{i=0}^{n-3} \widetilde{a}_i f_i^d,\end{aligned}$$

where $\widetilde{b} = (-1)^{n+1} b$ and $\widetilde{a}_i = (-1)^{n+i} a_i$, for $i = 0, \dots, n-3$. By Lemma 4 we can choose $\widetilde{b}$ and $\widetilde{a}_i$ in $\mathbb{N}_1$ so that $\widetilde{b} + \sum_{i=0}^{n-3} \widetilde{a}_i f_i^d = k$ for $k \in \mathbb{N}_{f_n^d}$. If we then substitute the coefficients that are zeros with -1, then the last column is such that

$$\det H = \widetilde{b} + \sum_{i=0}^{n-3} \widetilde{a}_i f_i^d = f_n^d - 2k,$$

which concludes the proof. □

Proposition 9. *Let $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$. The set of possible determinants for matrices in the family $\widehat{\mathcal{H}}^n(\mathbb{Z}_d)$ is $\mathbb{Z}_{f_n^d}$, where f_n is the nth Fibonacci number as defined in (8).*

Proof. The proof is analogous to that of Proposition 7: it suffices to use Corollary 2 in lieu of Lemma 5. □

Proposition 10. *Let $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$. The set of possible determinants for matrices in the family $\widehat{\mathcal{H}}^n(-\mathbb{N}_d)$ is $(-1)^n \cdot \mathbb{N}_{f_n^d}$, where f_n is the nth Fibonacci number as defined in (8).*

Proof. This proof retraces the lines of Propositions 9. Note that $\widehat{\mathcal{H}}^n(-\mathbb{N}_d) \subset \widehat{\mathcal{H}}^n(\mathbb{Z}_d)$, and that since $\widehat{K}^{(d,n-1)} \in \widehat{\mathcal{H}}^n(-\mathbb{N}_d)$, the matrix H in (7) is in $\widehat{\mathcal{H}}^n(-\mathbb{N}_d)$ if we require that $b, a_0, \dots, a_{n-3} \in -\mathbb{N}_d$. Combining Lemma 1 and Proposition 6, in this case gives

$$\begin{aligned}\det H &= (-1)^{n+1} b + \sum_{i=0}^{n-3} (-1)^{n+i} a_i \det \widehat{K}^{(d,i+1)} \\ &= (-1)^n \left(\widetilde{b} + \sum_{i=0}^{n-2} \widetilde{a}_i f_i^d \right),\end{aligned}$$

where $\widetilde{b} = -b$ and $\widetilde{a}_i = -a_i$, for $i = 0, \dots, n-3$. Since $\widetilde{b}, \widetilde{a}_0, \widetilde{a}_{n-2} \in \mathbb{N}_d$, by Lemma 4 we can choose the last column of H so that $\det H = (-1)^n k$ for all $k \in \mathbb{N}_{f_n^d}$. ∎

Propositions 7, 8, 9, and 10 generalize [17, Conj. 12], [17, Conj. 14], and [17, Conj. 16]. For $d = 1$, these results show that the number of distinct determinants of normalized zero-diagonal Bohemian upper Hessenberg matrices of size n with entries from the sets $-\mathbb{N}_1$ and $\mathbb{Z}_1^0$ is given by the nth element of the OEIS sequence A001611. For the family of matrices with elements drawn from $\mathbb{Z}_1$, an analogous result holds for the OEIS sequence A001588.

5. Future developments. In the previous sections, we have generalized all conjectures in [17] regarding upper Hessenberg matrices, except [17, Conj. 8] and [17, Conj. 9]. In this section, we show that the former is true and state its generalization to more general domains we were unable to prove. On the other hand, the latter has so far resisted our efforts and will be the subject of future investigation. We will make use of a generalization of Fibonacci numbers which differs from that in Definition 1 and grows significantly faster, namely the sequence

$$\begin{cases} g_1^d &= d, \\ g_2^d &= d^2, \\ g_n^d &= d g_{n-1}^d + g_{n-2}^d, \qquad n \in \mathbb{N} \setminus \{0,1,2\}. \end{cases} \tag{19}$$

Conjecture 10. *For all $n \in \mathbb{N}^+$ and $d \in \mathbb{N}^+$, one has that*

$$\operatorname*{argmax}_{H \in \mathcal{H}^n(\mathbb{N}_d)} |\det H| = \det J^{(d,n)} = g_n^d, \tag{20}$$

where g_n is the nth generalized Fibonacci number as defined in (19), and

$$j_{ij}^{(d,n)} = \begin{cases} 0, & i > j+1 \text{ or } i < j \text{ and } i+j \text{ is odd}, \\ 1, & i = j+1, \\ -d, & i \le j \text{ and } i+j \text{ is even}. \end{cases} \tag{21}$$

The proof for $d = 1$ is a special case of [4, Thm. 1]. In the following we supply a proof for the case $d = 2$.

Proof of Conjecture 10 for $d = 2$. The second equality in (20) can be proven easily by induction, by expanding along the last row of $J^{(2,n)}$ and noting that $j_{nn}^{(2,n)} = d$. The first equality can also be proven by induction. It is straightforward to check that the result holds for $n = 1$, 2, and 3. For $n > 4$, let $H \in \mathcal{H}^n(\mathbb{N}_2)$. By expanding along the first column of H we have

$$\det H = h_{11} \begin{vmatrix} h_{22} & h_{23} & \dots & h_{2n} \\ 1 & h_{33} & \dots & h_{3n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix} - \begin{vmatrix} h_{12} & h_{13} & \dots & h_{1n} \\ 1 & h_{33} & \dots & h_{3n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix}.$$

If $h_{11} \neq 2$, then $|\det H| \leq 2g_{n-1}^2 < g_n^2$, and the proof is completed. Otherwise we can expand along the first column of the two resulting matrices, to obtain

$$\det H = (2h_{22} - h_{12}) \begin{vmatrix} h_{33} & h_{34} & \dots & h_{3n} \\ 1 & h_{44} & \dots & h_{4n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix} - 2 \begin{vmatrix} h_{23} & h_{33} & \dots & h_{2n} \\ 1 & h_{44} & \dots & h_{4n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix} + \begin{vmatrix} h_{13} & h_{14} & \dots & h_{1n} \\ 1 & h_{44} & \dots & h_{4n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix}. \quad (22)$$

If $h_{23} = 0$, then $|\det H| \leq 4g_{n-2}^2 < 2g_{n-1}^2$, which as above would conclude the proof. Otherwise, the last two determinants can be rewritten, by expanding along the last row and collecting like terms, as

$$(h_{13} - 2h_{23}) \begin{vmatrix} h_{44} & h_{45} & \dots & h_{4n} \\ 1 & h_{55} & \dots & h_{5n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix} + 2 \begin{vmatrix} h_{24} & h_{25} & \dots & h_{2n} \\ 1 & h_{55} & \dots & h_{5n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix} - \begin{vmatrix} h_{14} & h_{15} & \dots & h_{1n} \\ 1 & h_{55} & \dots & h_{5n} \\ & \ddots & \ddots & \vdots \\ & & 1 & h_{nn} \end{vmatrix}, \quad (23)$$

and substituting (23) into (20) yields

$$|\det H| \leq |2h_{22} - h_{12}|\, g_{n-2}^d + |h_{13} + 2h_{23}|\, g_{n-3}^d + g_{n-3}^d < 2g_{n-1}^d < g_n^d. \qquad \square$$

This proof does not generalize to $d > 2$, but provides a proof of [17, Conj. 8], as it shows that the maximum determinant in absolute value of matrices in $\widehat{\mathcal{H}}^n(\mathbb{N}_2)$ is the nth element of the OEIS sequence A052542.

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