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# REDUCED $C^*$ -ALGEBRA OF THE $p$ -ADIC GROUP $GL(n)$ II

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ABSTRACT. The reduced  $C^*$ -algebra of the  $p$ -adic group  $GL(n)$  admits a Bernstein decomposition. We give a minimal refinement of this decomposition, and provide structure theorems for the reduced Iwahori-Hecke  $C^*$ -algebra and the reduced spherical  $C^*$ -algebra. This leads to a very explicit description of the tempered dual of  $GL(n)$  in terms of Bernstein parameters and extended quotients. We also prove that Plancherel measure (on the tempered dual of a reductive  $p$ -adic group) is rotation-invariant.

*Key Words:* Reduced  $C^*$ -algebra, tempered dual, Bernstein parameters, Plancherel measure.

## 1. INTRODUCTION

Harish-Chandra, in his search for the Plancherel measure on a reductive  $p$ -adic group, had a clear conception of the *support* of the Plancherel measure [14]. The support is a  $C^\infty$ -manifold with countably many connected components. Each component is a compact torus  $T$  and Plancherel measure is absolutely continuous with respect to Haar measure on  $T$ . The Plancherel density is a real analytic function which is invariant under a certain finite group which acts on  $T$ , see [14, p. 353 – 367].

Let  $F$  be a nonarchimedean local field, let  $G = GL(n) = GL(n, F)$ , and let  $C_r^*(G)$  denote the reduced  $C^*$ -algebra of  $G$ . In [17] [18] we proved that  $C_r^*(G)$  is strongly Morita equivalent to a commutative  $C^*$ -algebra:

$$C_r^*(G) \sim C_0(\Omega^t(G))$$

where  $\Omega^t(G)$  is the Harish-Chandra parameter space. In [19], we constructed an equivalence bimodule which effects this strong Morita equivalence. For equivalence bimodules and strong Morita equivalence, see [12, II.Appendix A, p.152].

Let  $\mathcal{H}(G)$  be the Hecke algebra of  $G$ , and let  $\Omega$  be a component in the Bernstein variety  $\Omega(G)$ , see [5, 6]. The Bernstein decomposition

$$\mathcal{H}(G) = \bigoplus_1 \mathcal{H}(\Omega)$$

is a canonical decomposition of  $\mathcal{H}(G)$  into 2-sided ideals  $\mathcal{H}(\Omega)$ . This determines the Bernstein decomposition of the  $C^*$ -algebra  $A$  (where  $A = C_r^*(G)$ ):

$$A = \bigoplus A(\Omega).$$

This is a canonical decomposition of  $A$  into a  $C^*$ -direct sum of  $C^*$ -ideals  $A(\Omega)$ .

Let  $Irr^t(G)$  denote the tempered dual of  $G$ . The Bernstein decomposition of  $A$  determines the Bernstein decomposition of  $Irr^t(G)$ :

$$Irr^t(G) = \bigsqcup Irr^t(G)_\Omega.$$

It is sometimes convenient to assimilate the component  $\Omega$  to a point  $s$ , in which case the Bernstein decomposition is

$$A = \bigoplus A(s)$$

where  $s$  is a point in the Bernstein spectrum  $\mathcal{B}(G)$ .

In this article we refine the Bernstein decomposition of  $A$ . Each  $C^*$ -summand  $A(s)$  is itself a finite direct sum :

$$A(s) = A(t_1) \oplus \dots \oplus A(t_j)$$

where  $j$  depends on  $s$ . This refinement is minimal: the dual of each  $A(t)$  is a compact *connected* orbifold.

In [19] we construct an equivalence bimodule  $\mathcal{E}$  which effects a strong Morita equivalence between  $A$  and a commutative  $C^*$ -algebra  $D$ . For each  $s$ , a direct summand  $\mathcal{E}(s)$  of  $\mathcal{E}$  effects a strong Morita equivalence between  $A(s)$  and a commutative sub- $C^*$ -algebra  $D(s)$  of  $D$ . These commutative  $C^*$ -algebras admit a very simple, uniform description as follows.

Let  $s$  be a point in the Bernstein spectrum  $\mathcal{B}(G)$ . We can think of  $s$  as a vector  $(\tau_1, \dots, \tau_k)$  of irreducible supercuspidal representations of smaller general linear groups: the entries of this vector are determined up to tensoring with unramified quasicharacters and permutation. If the vector is  $(\sigma_1, \dots, \sigma_1, \dots, \sigma_r, \dots, \sigma_r)$  with  $\sigma_j$  repeated  $e_j$  times,  $1 \leq j \leq r$ , and  $\sigma_1, \dots, \sigma_r$  pairwise distinct (after unramified twist) then we say that  $s$  has *exponents*  $e_1(s), \dots, e_r(s)$ . Let

$$d(s) = e_1(s) + \dots + e_r(s)$$

and let

$$W(s) = S_{e_1(s)} \times \dots \times S_{e_r(s)}$$

a product of symmetric groups. Let  $\mathbb{T}^d$  denote the standard compact torus of dimension  $d$ . We prove the following strong Morita equivalence:

$$A(s) \sim C(\widehat{\mathbb{T}^{d(s)}}/W(s))$$

the *extended* quotient of the torus  $\mathbb{T}^{d(s)}$  by the finite group  $W(s)$ .

Note that this strong Morita equivalence *depends only on the exponents of  $s$* . If  $\Omega_1 \cong \Omega_2$  as complex algebraic varieties then we will write  $s_1 \sim s_2$ . Note that  $\Omega_1 \cong \Omega_2$  if and only if  $s_1 \sim s_2$  if and only if  $s_1, s_2$  have the same exponents. Then  $s_1 \sim s_2$  if and only if  $A(s_1)$  and  $A(s_2)$  are strongly Morita equivalent:

$$s_1 \sim s_2 \iff A(s_1) \sim A(s_2).$$

The support of Plancherel measure is therefore given by

$$Irr^t(G) = \bigsqcup_s \widehat{\mathbb{T}^{d(s)}}/W(s).$$

We do not have a numerical formula for Plancherel density on  $Irr^t(G)$ , nor does [10], but we hope that the above structure theorems will be a useful step on the way towards the final numerical formula. Meanwhile (in joint work with Paul Baum) we prove that *Plancherel measure is rotation-invariant*. This implies that Plancherel density is a constant on each circle in the discrete series of  $GL(n)$ .

There is one summand in the Bernstein decomposition of  $A$  which is conspicuous, namely the component determined by the cuspidal pair  $(M, 1)$  where  $M$  is the diagonal subgroup of  $G$  and  $1$  is the trivial representation of  $M$ . This component is itself strongly Morita equivalent to the reduced Iwahori-Hecke  $C^*$ -algebra  $C_r^*(G//I)$ . We prove, in great detail, a structure theorem for this unital  $C^*$ -algebra, using standard techniques of noncommutative Fourier analysis.

The dual of the reduced Iwahori-Hecke  $C^*$ -algebra  $C_r^*(G//I)$  is a compact Hausdorff space which is best viewed as a space of Deligne-Langlands parameters, as done in [15].

Although it is not strictly necessary, the geometry of the tempered dual  $Irr^t(G)$  is best viewed in terms of Langlands parameters, see [9].

In some ways, the Schwartz algebra  $\mathcal{C}(G)$  is more appropriate to Plancherel measure than the reduced  $C^*$ -algebra. This was certainly the point of view of Harish-Chandra [14], and also the one adopted in [4, 5]. We should also mention the Harish-Chandra Product Formula for Plancherel measure, see [14, p. 92 – 93], [14, Theorem 5, p.359].

This article is an updated version of [20, 21], which have lain dormant for several years. The present article owes much to noncommutative

geometry, specifically the periodic cyclic homology computations which appear in [4, 5, 8, 9].

## 2. BERNSTEIN DECOMPOSITION

Let  $\Omega$  be a component in the Bernstein variety  $\Omega(G)$ . The Hecke algebra  $\mathcal{H}(G)$  can be decomposed into a direct sum of 2-sided ideals:

$$\mathcal{H}(G) = \bigoplus \mathcal{H}(\Omega).$$

This is the Bernstein decomposition of the Hecke algebra  $\mathcal{H}(G)$ , see [5, 6].

Choose left-invariant Haar measure on  $G$ . The left regular representation  $\lambda$  of  $L^1(G)$  on  $L^2(G)$  is defined as follows:

$$(\lambda(f))(h) = f * h,$$

where  $f \in L^1(G)$ ,  $h \in L^2(G)$  and  $*$  denotes convolution. The reduced  $C^*$ -algebra is the closure (in the norm topology) of the image of  $\lambda$ :

$$A = C_r^*(G) = \overline{\lambda(L^1(G))} \subset \mathcal{L}(L^2(G)).$$

The dual of  $A$  is homeomorphic to the tempered dual of  $G$ , see [13].

We have  $\mathcal{H}(G) \subset A$ . Let  $A(\Omega)$  be the norm closure in  $A$  of  $\mathcal{H}(G)$ . Then  $A(\Omega)$  is a  $C^*$ -ideal of  $A$ . Let  $\Omega_1, \Omega_2$  be distinct components of the Bernstein variety  $\Omega(G)$ , and let  $I = A(\Omega_1), J = A(\Omega_2)$ . Now  $\widehat{(I \cap J)} = \hat{I} \cap \hat{J}$  by [13, 3.2.3]. So  $I \cap J \neq 0$  if and only if there exists  $\omega \in \hat{I} \cap \hat{J}$  by [13, 3.2.3]. But points in the dual of  $A(\Omega)$  correspond to tempered representations  $\omega$  such that

$$\text{inf.ch. } \omega \in \Omega.$$

Since the infinitesimal character map is finite-to-one [6], we must have  $I \cap J = 0$ .

As a consequence, the Bernstein decomposition of  $\mathcal{H}(G)$  determines a  $C^*$ -direct sum of  $C^*$ -ideals

$$A = \bigoplus A(\Omega).$$

This is the Bernstein decomposition of  $A$ . It corresponds to the following disjoint union:

$$\text{Irr}^t(G) = \bigsqcup \text{Irr}^t(G)_\Omega.$$

This is a partition of the tempered dual of  $G$ , the points in  $\text{Irr}^t(G)_\Omega$  corresponding to tempered representations  $\omega$  such that  $\text{inf.ch. } \omega \in \Omega$ .

Let us consider the following special case. Let  $G = GL(n)$ , let  $M_0$  be the diagonal subgroup of  $G$ , and let  $1$  be the trivial representation

of  $M_0$ . Then  $(M_0, 1)$  is a cuspidal pair. Let  $\Omega_0$  be the component containing  $(M_0, 1)$ . We shall call this component the *Borel* component. As an algebraic variety, the Borel component is the symmetric product of  $n$  copies of  $\mathbb{C}^\times$ . This is a nonsingular complex affine algebraic variety.

### 3. REDUCED IWAHORI-HECKE $C^*$ -ALGEBRA FOR $GL(n)$

We begin with an elementary account of the *extended* quotient, see [1]. Let  $X$  be a space on which the finite group  $\Gamma$  acts. The extended quotient associated to this action is the quotient space  $\hat{X}/\Gamma$  where

$$\hat{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}.$$

The group action on  $\hat{X}$  is  $g \cdot (\gamma, x) = (g\gamma g^{-1}, gx)$ . Let  $X^\gamma = \{x \in X : \gamma x = x\}$  and let  $Z(\gamma)$  be the  $\Gamma$ -centralizer of  $\gamma$ . Then the extended quotient is given by:

$$\hat{X}/\Gamma = \bigsqcup_{\gamma} X^\gamma/Z(\gamma)$$

where one  $\gamma$  is chosen in each  $\Gamma$ -conjugacy class. If  $\gamma = 1$  then  $X^\gamma/Z(\gamma) = X/\Gamma$  so *the extended quotient always contains the ordinary quotient*:

$$\hat{X}/\Gamma = X/\Gamma \sqcup \dots$$

We shall need only the special case in which  $X$  is the compact torus  $\mathbb{T}^n$  of dimension  $n$  and  $\Gamma$  is the symmetric group  $S_n$  acting on  $\mathbb{T}^n$  by permuting co-ordinates.

Let  $\alpha$  be a partition of  $n$ , and let  $\gamma$  have cycle type  $\alpha$ . Each cycle provides us with one circle, and cycles of equal length provide us with a symmetric product of circles. For example, the extended quotient  $\widehat{\mathbb{T}^5}/S_5$  is the following disjoint union of compact orbifolds (one for each partition of 5):

$$\mathbb{T} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}^2 \sqcup (\mathbb{T} \times \text{Sym}^2 \mathbb{T}) \sqcup (\mathbb{T} \times \text{Sym}^2 \mathbb{T}) \sqcup (\mathbb{T} \times \text{Sym}^3 \mathbb{T}) \sqcup \mathbb{T}^5$$

where  $\text{Sym}^n \mathbb{T}$  is the  $n$ -fold symmetric product of the circle  $\mathbb{T}$ . We shall see that this extended quotient is a model of the arithmetically unramified tempered dual of  $GL(5)$ .

**3.1 THEOREM** Let  $I$  be the Iwahori subgroup of  $GL(n)$ . Let  $n = n_1 + \dots + n_k$  be a partition of  $n$ , let  $St(n_j)$  be the Steinberg representation of  $GL(n_j)$  and let  $\chi_1, \dots, \chi_k$  be unramified characters of  $GL(1)$ . Then the representation

$$(\chi_1 \circ \det)St(n_1) \times \dots \times (\chi_k \circ \det)St(n_k)$$

is unitary, irreducible, tempered and admits  $I$ -fixed vectors. Moreover, all such representations are accounted for in this way.

*Proof.* We use the Langlands classification for  $GL(n)$ , see [16]. Each tempered representation of  $GL(n)$  is of the form  $Q(\Delta_1) \times \dots \times Q(\Delta_k)$  where the Langlands quotient  $Q(\Delta_i)$  is square-integrable for each  $i = 1, \dots, k$ . We now use transitivity of parabolic induction [2] and Borel's theorem [7] to infer that (up to unramified unitary twist) we must have

$$\Delta_i = \{ | \cdot |_F^{(1-n_i)/2}, \dots, | \cdot |_F^{(n_i-1)/2} \}$$

with  $i = 1, \dots, k$ . But then  $Q(\Delta_i)$  is the Steinberg representation  $St(n_i)$  of  $GL(n_i)$ . Note that  $Q(\chi \otimes \Delta_i) = (\chi \circ \det) \otimes Q(\Delta_i)$  and that  $Q(\Delta_1) \times \dots \times Q(\Delta_k)$  is irreducible.  $\square$

**3.2 Theorem.** The parameter space for the tempered representations of  $GL(n)$  which admit  $I$ -fixed vectors is the extended quotient  $\widehat{\mathbb{T}^n}/S_n$ .

*Proof.* Suppose that there are  $r_j$  blocks of size  $n_j$  with  $1 \leq j \leq l$ . Then the Weyl group of the Levi factor  $M = GL(n_1) \times \dots \times GL(n_l)$  is

$$W(M) = S_{r_1} \times \dots \times S_{r_l}.$$

This Weyl group permutes blocks of the same size. By standard Bruhat theory, the Weyl group controls equivalences of parabolically induced representations. It follows that the parameter space for the tempered representations which admit  $I$ -fixed vectors is

$$X = \bigsqcup \mathbb{T}^{r_1}/S_{r_1} \times \dots \times \mathbb{T}^{r_l}/S_{r_l}.$$

The disjoint union is over all partitions

$$n_1 + \dots + n_1 + \dots + n_l + \dots + n_l = r_1 n_1 + \dots + r_l n_l = n.$$

Let now  $\gamma$  be an element in  $S_n$  whose cycle type is the above partition. Then the centralizer  $Z(\gamma)$  is the product of wreath products:

$$Z(\gamma) = (\mathbb{Z}/n_1 \wr S_{r_1}) \times \dots \times (\mathbb{Z}/n_l \wr S_{r_l}).$$

But the cyclic groups  $\mathbb{Z}/n_1, \dots, \mathbb{Z}/n_l$  act trivially on the fixed-point set  $(\mathbb{T}^n)^\gamma$ . We then have

$$\begin{aligned} \widehat{\mathbb{T}^n}/S_n &= \bigsqcup (\mathbb{T}^n)^\gamma / Z(\gamma) \\ &= \bigsqcup \{(a, \dots, a, \dots, b, \dots, b, \dots, c, \dots, c, \dots, d, \dots, d)\} / W(M) \\ &= \bigsqcup Sym^{r_1} \mathbb{T} \times \dots \times Sym^{r_l} \mathbb{T} \\ &= X. \end{aligned}$$

$\square$

Left-invariant Haar measure on  $G$  is now chosen so that the Iwahori subgroup has volume 1. Let  $e : G \rightarrow \mathbb{R}$  be defined as follows:  $e(x) = 1$  if  $x \in I$ ,  $e(x) = 0$  if  $x \notin I$ . Then  $e$  is a projection in  $A$ . It generates the non-unital  $C^*$ -ideal  $AeA$ . Now consider the corner  $eAe$ . This is a unital algebra with unit  $e$ .

**3.3 LEMMA** The equivalence bimodule  $eA$  effects a strong Morita equivalence between  $eAe$  and  $AeA$ .

*Proof.* Let  $B = eAe$ ,  $C = AeA$ ,  $\mathcal{E} = eA$ . We have to check 3 points.

(1)  $\mathcal{E}$  admits a  $C$ -valued inner product given by

$$(x, y)_{\mathcal{E}} = x^*y.$$

Then  $\|x\|_{\mathcal{E}} = \|x^*x\|_C^{1/2} = \|x\|$ . Since  $C = AeA$  is a  $C^*$ -ideal,  $C$  is  $\|\cdot\|$ -complete and  $\mathcal{E}$  is a  $C^*$ -module.

The set  $\{(x, y)_{\mathcal{E}} : x, y \in \mathcal{E}\}$  is

$$\{x^*y : x, y \in \mathcal{E}\} = \{a^*eb : a, b \in A\} = AeA = C$$

hence  $\mathcal{E}$  is a full  $C^*$ -module.

(2)  $\mathcal{E}$  is a right  $C$ -module given by  $\mathcal{E} \times C \rightarrow \mathcal{E}$ ,  $(ea, c) \mapsto eac$ .

(3) The standard rank 1 operators are given by  $\theta_{x,y}(z) = x(y, z) = xy^*z$  with  $x, y, z \in \mathcal{E}$ . Then  $\theta_{ex,e}(z) = (exe)z$  and so the linear span of the  $\theta_{x,y}$  is isomorphic to  $eAe$ . Since  $eAe$  is complete, the closure of the linear span of the  $\theta_{x,y}$  is isomorphic to  $eAe$ . So we have

$$eAe \cong \text{End}^0(\mathcal{E})$$

the compact endomorphisms of  $\mathcal{E}$ . Hence  $\mathcal{E}$  is an equivalence bimodule.  $\square$

**3.4 THEOREM** The dual of  $eAe$  is homeomorphic to the extended quotient  $\widehat{\mathbb{T}^n/S_n}$ .

*Proof.* The equivalence bimodule  $\mathcal{E}$  determines a homeomorphism of dual spaces:

$$\widehat{eAe} \cong \widehat{AeA}.$$

Also we have

$$\pi(e) = \int e(x)\pi(x) dx = \int_I \pi(x) dx$$

which is the projection onto the subspace of  $I$ -fixed vectors occurring in  $\pi$ . So  $\pi(e) \neq 0$  if and only if  $\pi$  admits nonzero  $I$ -fixed vectors. Since  $\pi(exe) = \pi(e)\pi(x)\pi(e)$ , the dual of  $eAe$  is precisely that part of  $\text{Irr}^t(G)$  which admits nonzero  $I$ -fixed vectors. Therefore we have  $AeA = A(\Omega)$  where  $\Omega$  is the Borel component in the Bernstein variety  $\Omega(G)$ . We now apply Theorem 3.2.  $\square$

Let  $\mathcal{H}(G//I)$  be the Iwahori-Hecke algebra. This comprises all complex-valued functions  $\phi$  on  $G$  which are compactly supported and bi-invariant with respect to  $I$ :

$$\phi(i_1 x i_2) = \phi(x)$$

for all  $i_1, i_2 \in I, x \in G$ . The product in  $\mathcal{H}(G//I)$  is the convolution product.

Since  $\phi = e * \phi * e$  it is immediate that  $\mathcal{H}(G//I) \subset eAe$ . In fact  $\mathcal{H}(G//I)$  is dense in  $eAe$  in the reduced  $C^*$ -algebra norm, and we refer to  $eAe$  as the reduced Iwahori-Hecke  $C^*$ -algebra. The notation is

$$C_r^*(G//I) = eAe.$$

A hermitian vector bundle  $S$  now presents itself. The base space  $X$  is the dual of  $eAe$ . By Theorem 3.4, the base space is the extended quotient  $\widehat{\mathbb{T}^n}/S_n$ . The fibre  $S_\pi$  comprises all  $I$ -fixed vectors in the representation  $\pi$ . This bundle is a trivial vector bundle.

The bundle  $S$  admits an endomorphism bundle

$$End(S) = S \otimes S^*.$$

Since  $S$  is hermitian, the continuous sections of  $End(S)$  form a unital  $C^*$ -algebra whose dual is homeomorphic to  $X$ .

*Definition.* Let  $\phi \in \mathcal{H}(G//I)$ . The Fourier Transform  $\hat{\phi}$  is defined as

$$\hat{\phi}(\pi) = \pi(\phi) = \int \phi(g)\pi(g)dg.$$

Since  $\phi = e\phi e$  it follows that

$$\hat{\phi}(\pi) = \pi(e\phi e) = \pi(e)\pi(\phi)\pi(e)$$

where  $\pi(e)$  projects onto the  $I$ -fixed subspace of  $\pi$ . Therefore we have  $\hat{\phi}(\pi) \in End(S_\pi)$ . Define

$$\alpha : \mathcal{H}(G//I) \longrightarrow C(End S)$$

$$\phi \mapsto \hat{\phi}.$$

So we have

$$(\alpha(\phi))(\pi) = \hat{\phi}(\pi) \in End(S_\pi).$$

**3.5 THEOREM** The Fourier Transform extends uniquely to an isomorphism of unital  $C^*$ -algebras:

$$C_r^*(G//I) \cong C(End S).$$

This is a finite direct sum of homogeneous  $C^*$ -algebras.

*Proof.* We have already shown that the Fourier Transform determines a map

$$\mathcal{H}(G//I) \longrightarrow C(\text{End } S).$$

*Injectivity.* Let  $y \in eAe$ . Then there exists  $\pi$  in  $\widehat{eAe}$  such that  $\|y\| = \sup\|\pi(y)\|$  by [13, 3.3.6]. Therefore  $y \neq 0$  implies there exists  $\pi$  such that  $\pi(y) \neq 0$ , and  $y_1 \neq y_2$  implies there exists  $\pi$  such that  $\pi(y_1) \neq \pi(y_2)$  so that

$$\alpha : \mathcal{H}(G//I) \longrightarrow C(\text{End } S)$$

is injective.

*Surjectivity.* The image  $\alpha(eAe)$  is a sub- $C^*$ -algebra of  $C(\text{End } S)$ . Let  $B = \alpha(eAe)$ . Note that  $C(\text{End } S)$  is a liminal  $C^*$ -algebra with compact Hausdorff dual  $X$ . Let

$$C = C(\text{End } S).$$

We shall now apply [13, 11.1.4], which is a preliminary version of the Stone-Weierstrass theorem for  $C^*$ -algebras.

Let  $\pi \in \hat{C}$ . Then  $\pi \in X$ . Consider the restriction  $\pi|_B$ . This is irreducible because

$$\pi : \mathcal{H}(G//I) \longrightarrow \text{End}(S_\pi)$$

is a simple  $\mathcal{H}(G//I)$ -module,  $\mathcal{H}(G//I)$  is dense in  $eAe$ , and  $B$  is isomorphic to  $eAe$ . The simplicity of the  $\mathcal{H}(G//I)$ -module of  $I$ -fixed vectors in  $\pi$  is a classical result of Borel [7].

Let  $\pi, \psi \in \hat{C}$  and suppose  $\pi \neq \psi$ . Then the simple  $\mathcal{H}(G//I)$ -modules  $S_\pi$  and  $S_\psi$  are distinct owing to the bijection between irreducible representations of  $G$  which admit nonzero  $I$ -fixed vectors and simple  $\mathcal{H}(G//I)$ -modules. Once again  $\mathcal{H}(G//I)$  is dense in  $eAe$  which is isomorphic to  $B$ . So we conclude that  $\pi|_B$  and  $\psi|_B$  are distinct irreducible representations of  $B$ . By [13, 11.1.4], we have  $B = C$ .  $\square$

There are in fact two equivalence bimodules at work here. We recall that the Borel component  $\Omega$ , as a complex algebraic variety, is the symmetric product of  $n$  copies of  $\mathbb{C}^\times$ . The points in the dual of  $A(\Omega)$  correspond to tempered representations of  $GL(n)$  whose infinitesimal characters lie in  $\Omega$ .

The  $C^*$ -ideal  $A(\Omega)$  is given by  $A(\Omega) = AeA$ . The first strong Morita equivalence is

$$AeA \sim eAe$$

with equivalence bimodule  $\mathcal{E} = eA$ . The unital corner  $eAe$  is the reduced Iwahori-Hecke  $C^*$ -algebra:  $eAe = C_r^*(G//I)$ . The second strong Morita equivalence is

$$eAe \sim C(X)$$

where  $X$  is the extended quotient  $\widehat{\mathbb{T}^n}/S_n$ . The equivalence bimodule comprises all continuous sections of the complex Hermitian bundle  $S$  of all Iwahori fixed vectors.

So we have

$$A(\Omega) \sim eAe \sim C(X).$$

The extended quotient  $X$  is a disjoint union: one connected component is the ordinary quotient  $\mathbb{T}^n/S_n$ . The corresponding direct summand of  $A(\Omega)$  is discussed in the next section.

#### 4. REDUCED SPHERICAL $C^*$ -ALGEBRA FOR $GL(n)$

Let  $K$  be a maximal compact subgroup of  $GL(n)$ . We may take  $K = GL(n, \mathcal{O})$ . We have

$$C_r^*(G//K) \subset C_r^*(G//I).$$

In the strong Morita equivalence

$$C_r^*(G//I) \sim C(X)$$

the sub- $C^*$ -algebra  $C_r^*(G//K)$  determines a strong Morita equivalence

$$C_r^*(G//K) \sim C(\mathbb{T}^n/S_n)$$

with equivalence bimodule  $C(L)$ , where  $L$  is the line bundle of all  $K$ -fixed vectors. This of course means that

$$C_r^*(G//K) \cong C(\mathbb{T}^n/S_n).$$

This is the  $C^*$ -algebra version of the Satake isomorphism [11, p. 147]:

$$\mathcal{H}(G//K) \cong \mathbb{C}[\Lambda]^W$$

where the lattice  $\Lambda = \mathbb{Z}^n$ . The group algebra  $\mathbb{C}[\Lambda]$  will Fourier Transform to a dense subalgebra of  $C(\mathbb{T}^n)$  and the  $W$ -invariant part  $\mathbb{C}[\Lambda]^W$  will Fourier Transform to a dense subalgebra of  $C(\mathbb{T}^n/W)$ .

The  $C^*$ -algebra  $C_r^*(G//K)$  is the reduced *spherical*  $C^*$ -algebra for the group  $GL(n)$ , cf. [11, 4.4]. We conclude with the following

**4.1 THEOREM** Let  $C_r^*(G//K)$  be the reduced spherical  $C^*$ -algebra for  $GL(n)$ . Then we have

$$C_r^*(G//K) \cong C(\mathbb{T}^n/S_n).$$

5. STRUCTURE OF THE  $C^*$ -SUMMAND  $A(s)$ 

Let  $s$  be a point in the Bernstein spectrum of  $GL(n)$  and let  $A(s)$  be the corresponding  $C^*$ -summand in the Bernstein decomposition of  $A$ .

**5.1 THEOREM** The  $C^*$ -algebra  $A(s)$  is strongly Morita equivalent to a commutative  $C^*$ -algebra:

$$A(s) \sim C(\widehat{\mathbb{T}^{d(s)}}/W(s)).$$

*Proof.* We begin by reviewing the Bernstein variety  $\Omega(G)$ , see [6]. The Bernstein variety is a disjoint union of ordinary quotients:

$$\Omega(G) = \bigsqcup D/W(M, D)$$

where  $D$  is a complex torus, and  $W(M, D)$  is a certain finite group which acts on  $D$ . We now replace, as in [9], the ordinary quotient by the extended quotient to create a new variety  $\Omega^+(G)$ . So we have

$$\Omega^+(G) = \bigsqcup \hat{D}/W(M, D).$$

We call  $\Omega^+(G)$  the *extended* Bernstein variety. Each component in the extended Bernstein variety is the quotient of a complex torus by a finite group, and so is itself a complex algebraic variety. So  $\Omega^+(G)$  is a complex algebraic variety with countably many irreducible components.

In [9] we construct a bijection:

$$Irr(GL(n)) \longrightarrow \Omega^+(GL(n))$$

This bijection is constructed in terms of Langlands parameters, and also depends on a short but intricate piece of combinatorics [9, p. 217]. Then, by transport of structure, the smooth dual acquires the structure of complex algebraic variety with countably many irreducible components.

We will write

$$\Omega = D/W(M, D)$$

for a component in  $\Omega(G)$ . If we now assimilate  $\Omega$  to a point  $s$  in the Bernstein spectrum, then we can write

$$\hat{D}/W(M, D) = \widehat{T_c^{d(s)}}/W(s)$$

where  $T_c^n$  denotes the complex torus of dimension  $n$ .

Now the complex commutative Lie group  $T_c^n$  maps by a deformation retraction onto its maximal compact subgroup  $\mathbb{T}^n$ :

$$(z_1, \dots, z_n) \mapsto (|z_1|^{-1}z_1, \dots, |z_n|^{-1}z_n).$$

There is also a deformation retraction of  $Irr(G)$  onto the tempered dual  $Irr^t(G)$ , see [9, Theorem 2]. These deformation retractions are compatible in the sense that the following diagram is commutative:

$$\begin{array}{ccc} Irr(G) & \longrightarrow & \bigsqcup \widehat{T_c^{d(s)}}/W(s) \\ \downarrow & & \downarrow \\ Irr^t(G) & \longrightarrow & \bigsqcup \widehat{\mathbb{T}^{d(s)}}/W(s) \end{array}$$

In this diagram, the horizontal maps are bijective.

We now return to  $C^*$ -algebras. The main result of [18] is the following isomorphism of  $C^*$ -algebras:

$$C_r^*(G) \cong C_0(Irr^t(G), \mathcal{K}(H))$$

where  $\mathcal{K}(H)$  is the  $C^*$ -algebra of compact operators on the standard Hilbert space  $H$ . For each point  $s$  in the Bernstein spectrum we then have

$$A(s) \cong C(Irr_\Omega^t(G), \mathcal{K}(H)).$$

The equivalence bimodule  $\mathcal{E} = C(Irr_\Omega^t(G), H)$  then effects the following strong Morita equivalence:

$$A(s) \sim C(\widehat{\mathbb{T}^{d(s)}}/W(s)).$$

□

Let  $\mathcal{C}(G)$  denote the Schwartz algebra of  $G$ . Let  $S$  be a complete set of standard tori in  $G$  no two of which are conjugate in  $G$ . The following decomposition is due to Harish-Chandra [14, p. 367]:

$$\mathcal{C}(G) = \bigoplus_{A \in S} \mathcal{C}_A(G).$$

There is a refinement of this decomposition:

$$\mathcal{C}(G) = \bigoplus \mathcal{C}(\mathfrak{o})$$

where  $\mathfrak{o}$  is an  $\mathfrak{a}^*$ -orbit in  $E_2(M)$ , see [14, p. 359]. This "wave-packet" decomposition is not stated explicitly by Harish-Chandra, but is implied by his formula for the component  $f_{\mathfrak{o}}$ , see [14, p. 360].

## 6. ROTATION INVARIANCE OF PLANCHEREL MEASURE

This section is joint work with Paul Baum. Let  $G$  be a reductive  $p$ -adic group and let  $\Psi(G)$  denote the group of unramified quasicharacters of  $G$ . It acts naturally on  $Irr(G)$  by  $\psi : \pi \mapsto \psi \otimes \pi$ . As in [3], define an algebraic action of  $\Psi(G)$  on  $\Omega(G)$  by  $\psi : (M, \rho) \mapsto (M, \psi|_M \cdot \rho)$ . Then *inf. ch* is a  $\Psi(G)$ -equivariant map.

If  $M$  is a standard Levi subgroup, then  $\Psi(M)$  is a complex commutative Lie group, and its maximal compact subgroup is denoted  $\Psi^t(M)$ , the group of unramified (unitary) characters of  $M$ .

We recall the definition of the Harish-Chandra parameter space  $\Omega^t(G)$ . We call a *discrete-series pair* a pair  $(M, \sigma)$  where  $M$  is a standard Levi subgroup,  $\sigma$  is an irreducible unitary representation in the discrete series of  $M$ . We denote by  $\Omega^t(G)$  the set of all discrete-series pairs up to conjugation by  $G$ . For any discrete-series pair  $(M, \sigma)$  the image of the map  $\Psi^t(M) \rightarrow \Omega^t(G)$ , given by  $\chi \mapsto (M, \chi\sigma)$ , is called a *connected component* of  $\Omega^t(G)$ . Each connected component is a compact connected orbifold.

The group  $\Psi^t(G)$  acts naturally on  $Irr^t(G)$  by  $\chi : \pi \mapsto \chi \otimes \pi$ . We now define a smooth action of  $\Psi^t(G)$  on  $\Omega^t(G)$  by  $\chi : (M, \sigma) \mapsto (M, \chi|_M \cdot \sigma)$ .

We emphasize that

$$\text{inf. ch} : Irr^t(G) \longrightarrow \Omega^t(G)$$

is a homeomorphism of the tempered dual in its standard topology onto the Harish-Chandra parameter space in its natural topology. Also, *inf. ch* is a  $\Psi^t(G)$ -equivariant map.

Now  $\Omega^t(G)$  is the support of Plancherel measure, and  $\Psi^t(G)$  is a compact torus. We shall say that  $\Psi^t(G)$  *rotates*  $\Omega^t(G)$ , and that it *rotates* Plancherel measure  $\nu$ . If  $\chi \in \Psi^t(G)$  then the rotated measure is given by

$$E \mapsto \nu(\chi^{-1}(E))$$

where  $E$  is a Borel set in the tempered dual of  $G$ .

If  $G = GL(n)$ , then  $\Psi^t(G) = \mathbb{T}$ . The representations in the discrete series of  $G$  arrange themselves into circles.

**6.1 THEOREM.** Let  $G$  be a reductive  $p$ -adic group. Then Plancherel measure is rotation-invariant. In the case of  $GL(n)$ , Plancherel measure induces Haar measure on each circle in the discrete series.

*Proof.* We begin with the Plancherel formula

$$f(1) = \int \text{tr } \hat{f}(\omega) d\nu(\omega)$$

where  $d\nu$  is Plancherel measure. We have

$$\begin{aligned}
\hat{f}(\chi.\omega) &= \int f(g)(\chi.\omega)(g) dg \\
&= \int f(g)\chi(g)\omega(g) dg \\
(1) \qquad &= \widehat{f.\chi}(\omega).
\end{aligned}$$

Let  $f \in \mathcal{C}(G)$ . Making a change of variable, and applying (1) we get

$$\begin{aligned}
\int tr \hat{f}(\omega) d\nu(\chi^{-1}.\omega) &= \int tr \hat{f}(\chi.\omega) d\nu(\omega) \\
&= \int tr \widehat{f.\chi}(\omega) d\nu(\omega) \\
&= (f.\chi)(1) \\
&= f(1).
\end{aligned}$$

We now apply the trace-Paley-Wiener theorem [3] and properties of the  $q$ -projection (twisted projection) in [9]. As  $f$  varies in the Hecke algebra  $\mathcal{H}(G)$  the integrand  $tr \hat{f}(\omega)$  varies over all finite Fourier series on the component  $\Omega^t(G)$ . So the Radon measures  $d\nu(\omega)$  and  $d\nu(\chi^{-1}.\omega)$  determine the same continuous linear functional on a dense subset of  $C_0(\Omega^t(G))$ . Therefore these two Radon measures are equal, i.e.  $\nu$  is rotation-invariant.

In the case of  $GL(n)$ , each circle in the discrete series is rotated by the group  $\Psi^t(G)$ , hence Plancherel measure induces Haar measure on each circle. □

Let  $\Omega^t$  be a component in  $\Omega^t(G)$ . Then  $\Omega^t = T/W(M, T)$ . Let  $d$  be the depth of  $M$ , i.e. the dimension of  $\Psi(M)$ . Then  $T = \mathbb{T}^d$ . When  $G = GL(n)$  we have  $\Psi^t(G) \cong \mathbb{T}$  and the action of  $\Psi^t(G)$  on  $\Omega^t$  is induced by the *diagonal* action of  $\mathbb{T}$  on  $\mathbb{T}^d$ . So Plancherel density is invariant under this diagonal action.

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