Generalized point vortex dynamics on CP^2

Montaldi, James and Shaddad, Amna

2018

MIMS EPrint: **2018.26**

ISSN 1749-9097

Generalized point vortex dynamics on $\mathbb{CP}^2$

James Montaldi & Amna Shaddad
University of Manchester

Dedicated to Darryl Holm on the occasion of his 70th birthday

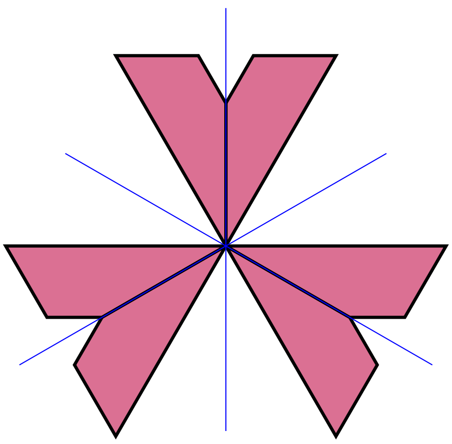
Abstract

This is the second of two companion papers. We describe a generalization of the point vortex system on surfaces to a Hamiltonian dynamical system consisting of two or three points on complex projective space $\mathbb{CP}^2$ interacting via a simple Hamiltonian function. The system has symmetry group SU(3). The first paper describes all possible momentum polytopes for this system, and here we apply methods of symplectic reduction and geometric mechanics to analyze the possible relative equilibria of interacting generalized vortices.

The different types of polytope depend on the values of the 'vortex strengths', which are manifested as coefficients of the symplectic forms on the copies of $\mathbb{CP}^2$. We show that the reduced spaces for this Hamiltonian action for 3 vortices is generically a 2-sphere, and proceed to describe the reduced dynamics under simple hypotheses on the type of Hamiltonian interaction. For 2 vortices, the reduced spaces are just points, and the motion is governed by a collective Hamiltonian.

MSC 2010: 37J15, 53D20

Keywords: Hamiltonian systems, momentum map, symplectic geometry, eigenvalues



Contents

1 Introduction

A point vortex is a point of isolated vorticity traditionally in a background of irrotational fluid, although in some cases one now allows a background of constant vorticity (necessary for arbitrary point vortices on a compact surface). It was discovered by Helmoltz in 1858 that point vortices interact and evolve in a way that depends only on their strengths and mutual positions. Some years later (1876) Kirchhoff noticed that the equations of motion could be written in Hamiltonian form—perhaps the first example of a Hamiltonian system not governed by kinetic and potential energy. It was through the work of Novikov in the 1970s that the Hamiltonian formulation of point vortex dynamics came to the fore, and this approach has produced many outstanding advances. Among these are the complete integrability of the motion of three vortices on a sphere proved by Kidambi & Newton [7], the existence of quasiperiodic orbits on invariant tori for lattice vortex systems demonstrated by Lim [11], and several other related results. In more recent years, the subject has been extended to the study of point vortices on the sphere [21, 10, 12], on the hyperbolic plane [16], and other less symmetric surfaces [3, 22]. The study of point vortices has been described as a mathematics playground by H. Aref [1], by which he meant that many different areas of (classical) mathematics can be brought to bear to study these point vortex systems.

The general Hamiltonian set-up is as follows. Let (S, ω_0) be a symplectic surface (smooth manifold of dimension 2, often endowed with a Riemannian metric). A configuration is an ordered set of n distinct points in S, and hence an element of

$$M = S \times S \times \cdots \times S \ \setminus \ \Delta.$$

Here Δ is the 'large diagonal', the subset consisting of all possible collisions: it is customary to rule collisions out of the model. Each of the n points $x_j \in S$ has a fixed non-zero real number Γ_j associated to it, called the *vortex strength*. The symplectic form Ω on M is defined to be

$$\Omega = \Gamma_1\omega_0 \oplus \Gamma_2\omega_0 \oplus \cdots \oplus \Gamma_n\omega_0.$$

More formally, if $\pi_j : M \to S$ is the Cartesian projection to the j^{th} component of M, then

$$\Omega = \Gamma_1 \pi_1^*(\omega_0) + \Gamma_2 \pi_2^*(\omega_0) + \cdots + \Gamma_n \pi_n^*(\omega_0).$$

For this to be a symplectic form one requires that all $\Gamma_j \neq 0$.

The dynamics is defined by a pairwise interaction: let $h_0 : S \times S \setminus \Delta \to \mathbb{R}$ be a given smooth function (usually taken to be the negative of the Green's function of the Laplacian relative to the Riemannian metric). The Hamiltonian function $H : M \to \mathbb{R}$ is then[1]

$$H(x_1, \dots, x_n) = \sum_{i<j} \Gamma_i \Gamma_j h_0(x_i, x_j),$$

The evolution of the system is given by,

$$\dot{x} = X_H(x)$$

where the vector field X_H is defined by Hamilton's equation $i_{X_H}\Omega = -dH$.

If in addition there is a group G acting on the surface S preserving the symplectic form, and preserving the function h_0 (that is $h_0(g \cdot x, g \cdot y) = h_0(x, y)$ for all $g \in G,\ x, y \in S$), then the point vortex system has symmetry G and the vector field is equivariant. Moreover, if the action on (S, ω_0) is *Hamiltonian*, meaning that there is a momentum map $J_0 : S \to \mathfrak{g}^*$, then so is the action on (M, Ω) with momentum map $J : M \to \mathfrak{g}^*$ given by

$$J(x_1, \dots, x_n) = \sum_j \Gamma_j J_0(x_j). \tag{1.1}$$

This approach highlights the application of Noether's theorem, which states that the components of the momentum map are preserved by the dynamics.

If S is the plane, the sphere or the hyperbolic plane then the respective actions of the groups $SE(2)$, $SO(3)$ and $SL(2)$ are indeed Hamiltonian (see above references and [15]). However, if the point vortices lie on a torus or cylinder then the action, which is symplectic, fails to be Hamiltonian [17]. See also the recent paper about vortices on the round torus [23].

In this paper, we extend the playground to higher dimensional symplectic manifolds (S, ω_0) and in particular to $\mathbb{CP}^2$, and we apply systematic geometric methods to the analysis of relative equilibria for such a system. We do not assume a particular form for the pairwise interaction, although it would be reasonable to choose the Green's function as in 2-dimensional point vortices; the only assumption we make is that $h_0(x, y)$ depends only on the distance between x and y. The set-up is otherwise identical to the description above. It should be pointed out that there is no supposition that this model is related to vortex dynamics for 4-dimensional fluids (indeed, it is not clear that isolated points of vorticity can exist in 4 dimensions).

The paper is organized as follows. In order to perform symplectic reduction for this system of generalized point vortices, it is important to know the possible values of the momentum map and its singularities. This was carried out in the companion paper [MS-I]. In Section 2 we

[1] More generally [5] there may be a 'self-interaction' term $\sum_j \Gamma_j^2 R(x_j)$ for some function R, encoding the interaction of the point vortex with the geometry or asymmetry of the space, we ignore this as $\mathbb{CP}^2$ is highly symmetric

discuss the reduced spaces; for 2 vortices these are just single points, while for 3 they are usually diffeomorphic to a sphere, sometimes to a pinched sphere and for some points on the boundary of the polytope they reduce to a single point. Finally, in Section 3 we consider the resulting reduced dynamics and in particular the reduced and relative equilibria and their stability. For two generalized vortices, every configuration is a relative equilibrium, but for three this is not the case in general, but is of course when the reduced space is a point.

This work forms part of the PhD thesis of the second author [25], where further details and alternatives for some of the calculations may be found.

2 Symplectic reduction

When studying the dynamics of symmetric Hamiltonian systems, a first approach is to study the reduced systems. Recall that the symplectic reduction at $\mu \in \mathfrak{g}^*$ is the space

$$M_\mu := J^{-1}(\mu)/G_\mu \simeq J^{-1}(\mathcal{O}_\mu)/G,$$

where these two versions of the reduced space are known as point reduction and orbit reduction. A proof of the fact that they are equivalent for compact groups can be found in [20, Theorems 6.4.1 & 8.4.4].

One advantage of orbit reduction is that it allows for studying the variation of reduced spaces as the momentum values varies. Indeed, since the momentum map is equivariant, it descends to a map between orbit spaces we call the *orbit momentum map* and denote $\mathcal{J}$, according to the following diagram,

$$\begin{array}{ccc} M & \xrightarrow{J} & \mathfrak{su}(3)^* \\ \downarrow & & \downarrow \\ M/G & \xrightarrow{\mathcal{J}} & \mathfrak{g}^*/G \end{array} \tag{2.1}$$

where the vertical maps are the quotient maps. The fibres of $\mathcal{J}$ are the reduced spaces, using orbit reduction.

The momentum map on the phase space of interest, products of $\mathbb{CP}^2$, is determined by (1.1), where $J_0 : \mathbb{CP}^2 \to \mathfrak{su}(3)^*$ is given by $J_0(Z) = Z \otimes \overline{Z} - \frac{1}{3} I_3$ (see [MS-I]). Explicitly,

$$J_0([x:y:z]) = \begin{pmatrix} |x|^2 - \frac{1}{3} & x\bar{y} & x\bar{z} \\ \bar{x}y & |y|^2 - \frac{1}{3} & y\bar{z} \\ \bar{x}z & \bar{y}z & |z|^2 - \frac{1}{3} \end{pmatrix}.$$

Here we consider $\mathbb{CP}^2$ as $S^5/U(1)$, so that $|x|^2 + |y|^2 + |z|^2 = 1$, and the matrix has trace 0 as required.

The distance between two points in $\mathbb{CP}^2$ can be given by the dihedral angle between the complex lines represented by the points. A formula is,

$$d(Z_1, Z_2) = \arccos \left| \widehat{Z}_1^\dagger \, \widehat{Z}_2 \right|. \tag{2.2}$$

Here $\widehat{Z}$ is any representative of Z in S^5, and $W^\dagger = \overline{W}^T$ is the conjugate transpose, so the expression within the modulus is the complex inner product. Note that this expression is well-defined: $|W_1^\dagger W_2| = \left|(e^{i\theta}W_1)^\dagger (e^{i\phi}W_2)\right|$.

The minimal distance between two points on $\mathbb{CP}^2$ is of course 0, and the maximum is $\pi/2$, when the points are orthogonal[2]. Given any two points separated by a distance θ, there is an element of $SU(3)$ that transforms them into e_1 and $\cos\theta e_1 + \sin\theta e_2$ (where addition is understood in $\mathbb{C}^3$, before taking the quotient to $\mathbb{CP}^2$). Here and throughout we denote,

$$e_1 = [1:0:0], \quad e_2 = [0:1:0], \quad e_3 = [0:0:1].$$

There are, up to conjugacy, three special subgroups of $SU(3)$ that we will refer to frequently:

$$\begin{aligned} U(2) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \mid A \in U(2) \right\}, \\ \mathbb{T}^2 &= \left\{ \mathrm{diag}[e^{i\theta}, e^{i\phi}, e^{-i(\theta+\phi)}] \mid \theta, \phi \in [0, 2\pi] \right\}, \\ U(1) &= \left\{ \mathrm{diag}[e^{i\theta}, e^{i\theta}, e^{-2i\theta}] \mid \theta \in [0, 2\pi] \right\}. \end{aligned} \tag{2.3}$$

The stabilizer of any point in $\mathbb{CP}^2$ is conjugate to $U(2)$, and it is not hard to see that any stabilizer subgroup for points in $\mathbb{CP}^2 \times \mathbb{CP}^2$ is conjugate to one of these 3 subgroups. As given, it is straightforward to see that $U(2)$ is the normalizer of the subgroup $U(1)$, that $U(1)$ is the centre of $U(2)$ and that $\mathbb{T}^2$ is a maximal torus of both $SU(3)$ and $U(2)$. The fixed point spaces in $\mathbb{CP}^2$ of these subgroups are,

$$\begin{aligned} \mathrm{Fix}(U(2), \mathbb{CP}^2) &= \{e_3\}, \\ \mathrm{Fix}(\mathbb{T}^2, \mathbb{CP}^2) &= \{e_1, e_2, e_3\}, \\ \mathrm{Fix}(U(1), \mathbb{CP}^2) &= \{[x:y:0] \mid [x:y] \in \mathbb{CP}^1\} \cup \{e_3\}. \end{aligned} \tag{2.4}$$

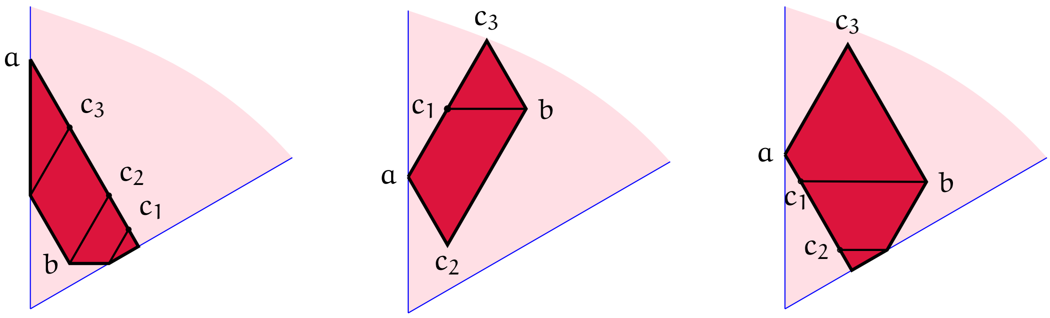
2.1 Reduction for 2 copies of $\mathbb{CP}^2$

Let $M = \mathbb{CP}^2 \times \mathbb{CP}^2$. Recall from [MS-I, Sec. 3] that the momentum polytope for the $SU(3)$ action on M is a line segment. One endpoint of the segment arises as momentum values for points on the diagonal in M (distance 0), while the other endpoint corresponds to orthogonal pairs of points (distance $\pi/2$).

Theorem 2.1 *If $\mu \in \Delta(M)$ then the reduced space M_μ is a single point.*

Proof. Since the orbit space $M/SU(3)$ is 1-dimensional (parametrized by the distance θ as pointed out above), the orbit momentum map is a map between two 1-dimensional manifolds with boundary, and is injective as the only singular points are the endpoints. The fibres, which coincide with the reduced spaces, therefore consist of a single point. □

[2] note that under the correspondence of $\mathbb{CP}^1$ with S^2, points are orthogonal in $\mathbb{CP}^1$ iff the corresponding points in S^2 are antipodal



(i) Polytope A (ii) Polytope E (iii) Polytope H

Figure 2.1: Three of the generic momentum polytopes from [MS-I]. They illustrate in particular how a and b are always vertices, but the c_j may or may not be vertices of the polytope.

2.2 Reduction for 3 copies of $\mathbb{CP}^2$

Now let $M = \mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2$. By a dimension count, if μ is a regular value of the momentum map, then at points of the fibre $J^{-1}(\mu)$ the group action is free, and the reduced spaces are smooth of dimension

$$\dim M_\mu = \dim M - \dim G - \mathrm{rk} G = 2.$$

We claim that for generic Γ_j there are 3 types of reduced space for the $SU(3)$ action on M. Generically, the reduced space is diffeomorphic to a smooth 2-sphere S^2. For exceptional values of momentum, it is either diffeomorphic to a ***pinched sphere***, which is a topological S^2 with a single conical singular point, or equal to a single point.

We distinguish a few special configurations in M, which have stabilizers given in Figure 2.2i:

Definition 2.2 A ***triple point*** in M is where all three elements of $\mathbb{CP}^2$ coincide, a ***double point*** is where two coincide and the third is different, a ***double+orthogonal point*** is a double point where the third point is orthogonal to the two coincident ones, a ***coplanar point*** is where the three points lie in a common copy of $\mathbb{CP}^1$, a ***semi-orthogonal point*** is where one point is orthogonal to the other two, which are distinct, and a ***totally orthogonal point*** is one where all three points of $\mathbb{CP}^2$ are mutually orthogonal.

Recall from [MS-I] that the possible stabilizer subgroups for 3 points on $\mathbb{CP}^2$ is as in Figure 2.2i. In particular, we distinguish the points in $\Delta(M)$ arising from points with stabilizer equal to a (maximal) torus:

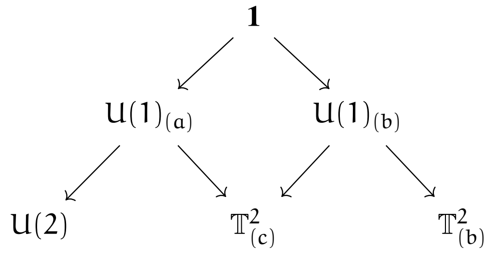
$$a = J(e_j, e_j, e_j), \quad b = J(e_i, e_j, e_k),$$

$$c_1 = J(e_i, e_j, e_j), \quad c_2 = J(e_j, e_i, e_j), \quad c_3 = J(e_j, e_j, e_i),$$

where i, j, k are distinct, and are chosen so that the corresponding point lies in $\mathfrak{t}^*_+$. Thus, a is the image of a triple point, b of a totally orthogonal point, while the c_j are images of 'dou-

geometry	stabilizer
triple point	$U(2)$
double+orthogonal	$\mathbb{T}^2$ — (c)
totally orthogonal	$\mathbb{T}^2$ — (b)
coplanar	$U(1)$ — (a)
semi-orthogonal	$U(1)$ — (b)
general position	**1**

(i) Stabilizers for 3 points in $\mathbb{CP}^2$.



(ii) Adjacencies of the different orbit type strata: $X \to Y$ means $Y \subset \overline{X}$.

Figure 2.2: These two figures show the orbit type stratification of M; the table in (i) shows the geometry corresponding to the different stabilizers, while in (ii) we see the adjacencies of the strata. The (a) and (b) refer in each case to two different geometry types for the same stabilizer, and hence different components of the corresponding fixed point space. (Note that strata marked with (a) contain the vertex a in their image, while strata marked with (b) contain the vertex b in their image, and the image of $\mathbb{T}^2_{(c)}$ consists of the points c_j.)

ble+orthogonal' points. Moreover, with reference to Figure 2.2, the points a, c_1, c_2, c_3 are images of points in $\mathrm{Fix}(U(1), M)_{(a)}$ (they are all coplanar configurations), while b, c_1, c_2, c_3 are in the image of points in $\mathrm{Fix}(U(1), M)_{(b)}$.

Before progressing to the statement of the classification of reduced spaces, let us examine this fixed point space of $U(1)$ and its momentum map. This subgroup consists of matrices given in (2.3), and the set of fixed points in $\mathbb{CP}^2$ has two components (2.4). It follows that $\mathrm{Fix}(U(1), M)$ has $2^3 = 8$ connected components. In this discussion, we identify $\mathbb{CP}^1$ with the subspace of $\mathbb{CP}^2$ orthogonal to e_3.

The two components of interest of this fixed point space are,

$$M_{(a)} = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1, \quad \text{and} \quad M_{(b)} = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \{e_3\}. \tag{2.5}$$

(We only consider these components of the fixed point spaces as the others are either permutations of the components of $M_{(b)}$, or are better studied as points in the fixed point space of a conjugate copy of $U(1)$.) The symplectic submanifold $M_{(a)}$ consists of coplanar configurations, while $M_{(b)}$ consists of semi-orthogonal ones. The restriction of the momentum map to each is as follows. With $\mathbb{CP}^1 = \{[x : y : 0] \in \mathbb{CP}^2\}$,

$$J_{(a)}(m) = \sum_{j=1}^{3} \Gamma_j \begin{pmatrix} |x_j|^2 - \frac{1}{3} & x_j\overline{y_j} & 0 \\ \overline{x_j}\, y_j & |y_j|^2 - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \tag{2.6}$$

while for $M_{(b)}$, the restriction of the momentum map is

$$J_{(b)}(m) = \sum_{j=1}^{2} \Gamma_j \begin{pmatrix} |x_j|^2 - \frac{1}{3} & x_j\overline{y_j} & 0 \\ \overline{x_j}\,y_j & |y_j|^2 - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} + \Gamma_3 \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}, \tag{2.7}$$

Remark 2.3 In both cases, $M_{(a)}$ and $M_{(b)}$ are invariant under the subgroup $U(2)$ (equal to the normalizer of $U(1)$ in $SU(3)$), and the momentum maps $J_{(a)}$ and $J_{(b)}$ can be identified with the momentum maps for this $U(2)$ action. Moreover, $U(1) \subset U(2)$ acts trivially, giving an effective action of $U(2)/U(1) \simeq SO(3)$ on $M_{(a)}$ and $M_{(b)}$. The image of these momentum maps is then contained in an affine copy of $\mathfrak{so}(3)^*$ in $\mathfrak{su}(3)^*$. After identifying $\mathbb{CP}^1$ as the Riemann sphere, the two momentum maps can be identified with those for the $SO(3)$ actions on $M_{(a)} \simeq S^2 \times S^2 \times S^2$ and $M_{(b)} \simeq S^2 \times S^2$. For $M_{(a)}$ the momentum map can be rewritten as,

$$J_{(a)}(m) = \mu_0 + \sum_{j=1}^{3} \Gamma_j \begin{pmatrix} s_j & x_j\overline{y_j} & 0 \\ \overline{x_j}\,y_j & -s_j & 0 \\ 0 & & 0 \end{pmatrix}, \tag{2.8}$$

where $\mu_0 = \frac{1}{6}\left(\sum_j \Gamma_j\right) \mathrm{diag}[1, 1, -2]$ and $s_j = \frac{1}{2}(|x_j|^2 - |y_j|^2)$. The first term is a constant while the image of the second lies in $\mathfrak{su}(2)^* \simeq \mathfrak{so}(3)^*$.

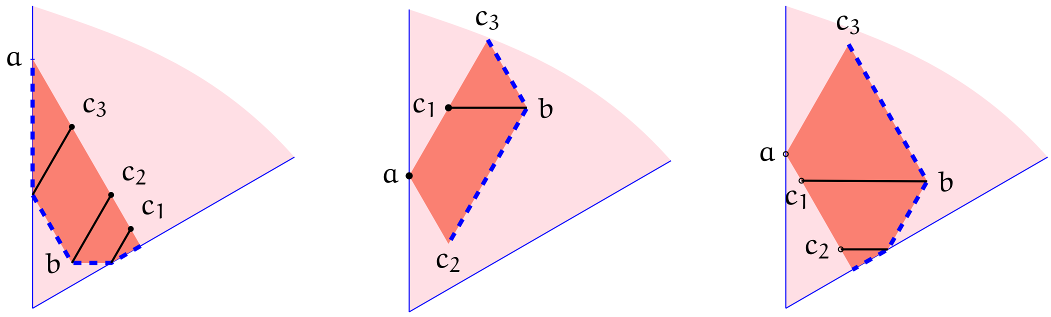
Theorem 2.4 *Suppose the Γ_j are generic, in the sense that $\Delta(M)$ is one of the 8 types A, B, …, H described in [MS-I]. If $\mu \in \Delta(M)$ then the reduced space M_μ is diffeomorphic to a point, a smooth sphere or a singular space of dimension 2, as follows,*

μ is an interior point of $\Delta(M)$	*regular value of J*	*sphere*
	singular value of J	*dimension 2*
μ in an edge of $\Delta(M)$	$\mu \in \partial(\mathfrak{t}^*_+)$	*point*
	on edge adjacent to b	*point*
	$\mu \in \mathrm{Int}(\mathfrak{t}^*_+)$, $\mu \neq c_j$	*sphere*
	$\mu \in \mathrm{Int}(\mathfrak{t}^*_+)$, $\mu = c_j$	*dimension 2*
μ is a vertex of $\Delta(M)$	$\mu = a$	*dimension 0 or 2*
	$\mu = b$	*point*
	$\mu = c_j$	*point*
	$\mu = x$	*point*

Here x is a vertex in the wall of the Weyl chamber, but not the image of a point fixed by $\mathbb{T}^2$. The space M_a is of dimension 0 if and only if the weights Γ_j are such that the polytope is of type A, B, *or* D.

There are two approaches to proving this. One is by direct calculation, as carried out in [25], and the other, which we give here, is to use the following particular consequence of a theorem of Kirwan [8, Chapter 5]:

Lemma 2.5 *Suppose the compact Lie group G acts in a Hamiltonian fashion on the symplectic manifold P, and suppose $H^1(P, \mathbb{R}) = 0$. If $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map then $H^1(P_\mu, \mathbb{R}) = 0$, where P_μ is the reduced space at μ.*



(i) Polytope A (ii) Polytope E (iii) Polytope H

Figure 2.3: Three of the generic momentum polytopes from [MS-I], showing the type of reduced space. The salmon coloured regions (including some boundary points) are where the reduced space is a smooth 2-sphere, the black lines or points where it is a 2-dimensional singular space (conjecturally a pinched 2-sphere), and the thick dashed lines represent where the reduced space is a point.

(Kirwan's theorem is about more general cohomology, but this particular case suffices for our needs.)

Outline of proof. We proceed case by case; the numbering refers to the rows in the table.

(1) In the first case, for regular values of the momentum map, this either follows from an explicit local calculation (see [25], where this is carried out), or it follows from Kirwan's lemma above. In this case M is a product of copies of $\mathbb{CP}^2$ which is simply connected, and it follows that the reduced spaces also have vanishing rational cohomology in degree 1. The only smooth compact orientable surface with this property is the sphere (orientable because symplectic).

(2) When μ is in the interior of the polytope but is not regular (i.e., on one of the black lines in the diagrams) then the reduced space is still homeomorphic to a sphere but is now singular, and has a single conical point. These singular points arise from points in M with non-trivial stabilizer, and since they are not vertices, they have stabilizer $U(1)$—see the table in Figure 2.2i.

(3) $\mu \in \partial(\mathfrak{t}^*_+)$ (edge, not vertex): the action of $SU(3)$ here is locally free (in fact free), so the fibre of the momentum map is a submanifold of dimension $12-8=4$. Now, $G_\mu = U(2)$ is also of dimension 4 and is acting (locally) freely, whence the reduced space is of dimension 0.

For much of the analysis below, note that if μ is in the interior $\mathrm{Int}(\mathfrak{t}^*_+)$ of the Weyl chamber and lies on an edge of $\Delta(M)$, then every point in $J^{-1}(\mu)$ has stabilizer at least $U(1)$ (otherwise the image under J of a neighbourhood of that point would contain an entire neighbourhood of μ in $\mathfrak{t}^*_+$), and the same argument applies to the point a. The set of points with stabilizer containing $U(1)$ are of two types described in Figure 2.2. We can therefore restrict attention for these edges to the two fixed point sets described above in (2.5). Note that the image of $M_{(a)}$ consists of edges containing the points a and c_j, while the image of $M_{(b)}$ contains the point b and the c_j (see Fig. 2.2ii).

First consider $M_{(b)}$ and the momentum map $J_{(b)} : M_{(b)} \to \mathfrak{so}(3)^*$.

(4,8,9) $\mu \in \mathrm{Int}(\mathfrak{t}^*_+)$ lies in any edge adjacent to vertex b (including the vertex b): this implies $J^{-1}(\mu) \subset M_{(b)}$. Since $M_{(b)} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, the same argument as used in the proof of Theorem 2.1 shows the orbit momentum map $\mathcal{J}_{(b)}$ is a bijection, and the reduced spaces for this stratum are therefore points.

Now consider $M_{(a)}$. As pointed out in Remark 2.3, the momentum map $J_{(a)} : M_{(a)} \to (\mu_0 + \mathfrak{so}(3)^*) \subset \mathfrak{su}(3)^*$ can be identified with the one for the $SO(3)$ action on $S^2 \times S^2 \times S^2$.

(5) $\mu \in \mathrm{Int}(\mathfrak{t}^*_+)$ lies in an edge not containing b: Then $J^{-1}(\mu) \subset M_{(a)}$. Now, in $M_{(a)}$ the three points are 'coplanar', that is they lie in a common $\mathbb{CP}^1$ as discussed above, and the momentum map is $J_{(a)}$. Now $M_{(a)}/SU(2)$ is of dimension 3 (parametrized by the 3 distances), while the image of the orbit momentum map is a segment (the edge in question). The regular fibres are therefore of dimension 2, and it follows from the lemma above that the reduced spaces are smooth spheres. The singular values are the points c_j and a, treated below.

(6) $\mu = c_j$ when not a vertex: refer to the expressions for the symplectic slice momentum map J_{N_1} for c_1, c_2 and c_3 in [MS-I, Eq. §4]. For example, for c_2 it is shown that $J_{N_1}(u_1, v_2, w_3) = R\alpha_1 + S\alpha_3$ where $R = -(\Gamma_3/\Gamma_1)(\Gamma_1 + \Gamma_3)|w_3|^2$, and

$$S = \left(\frac{\Gamma_2}{\Gamma_3}(\Gamma_2 - \Gamma_3)|v_2|^2 + \frac{\Gamma_1\Gamma_2}{\Gamma_3}(u_1 v_2 + \overline{u_1 v_2}) + \frac{\Gamma_1}{\Gamma_3}(\Gamma_1 + \Gamma_3)|u_1|^2\right).$$

Restricting to $M_{(a)}$ imposes $w_3 = 0$, which implies $R = 0$. The point c_2 is a vertex if and only if the coefficient S of α_3 is a definite quadratic form (which is the case if $\Gamma_1\Gamma_2\Gamma_3(\Gamma_2 - \Gamma_1 - \Gamma_3) > 0$). If it is indefinite, which we are assuming here, then the set of solutions to $S = 0$ is a 3-dimensional conical subspace of $\mathbb{C}^2$ (in fact a cone over a 2-torus since S has index 2). After factoring out by the remaining $U(1)$-action (here $U(1) = SO(3)_\mu$ for non-zero $\mu \in \mathfrak{so}(3)^*$), one concludes that the reduced space is 2-dimensional, with a single conical point.

(7) $\mu = a$: as in the discussion above, every point of $J^{-1}(a)$ has stabilizer at least $U(1)$ (and one point has stabilizer $U(2)$, namely the triple point), and in particular $J^{-1}(a)$ lies in $M_{(a)}$. It therefore suffices to consider $J_{(a)} : M_{(a)} \to \mathfrak{su}(2)^*$ (see also Remark 2.3). Now, in a neighbourhood of the triple point, (e_1, e_1, e_1) the full momentum map is determined by the action on the symplectic slice (see [MS-I, §4])

$$J_{N_1} = \begin{pmatrix} -\sum_j \Gamma_j(|v_j|^2 + |w_j|^2) & 0 & 0 \\ 0 & \sum_j \Gamma_j |v_j|^2 & \sum_j \Gamma_j \overline{v_j} w_j \\ 0 & \sum_j \Gamma_j v_j \overline{w_j} & \sum_j \Gamma_j |w_j|^2 \end{pmatrix}.$$

(see [MS-I, §4]), subject to $\sum_j \Gamma_j v_j = \sum_j \Gamma_j w_j = 0$. Restricting to $M_{(a)}$ imposes $w_j = 0$, and hence, locally in $M_{(a)}$,

$$J_{N_1,(a)}(v_1, v_2, v_3) = \big(\sum_j \Gamma_j |v_j|^2\big)\,\mathrm{diag}\big[-1,\ 1,\ 0\big]$$

again, subject to $\sum_j \Gamma_j v_j = 0$. This quadratic form on $\mathbb{C}^2$ (after putting $\sum_j \Gamma_j v_j = 0$) is definite if and only if $\Gamma_1\Gamma_2\Gamma_3(\Gamma_1 + \Gamma_2 + \Gamma_3) > 0$. If instead this expression is negative, then the zero-set of $J_{N_1,(a)}$ is a cone over a 2-torus, and the quotient by $U(1)$ gives an ordinary conical point in a 2-dimensional surface.

(10) $\mu = x$, which is the intersection of an edge of $\Delta(M)$ with a wall of the Weyl chamber. There are two cases. Firstly, suppose the edge in question is adjacent to vertex b. Then the reduced space is a point as follows from (4,8,9) above. If on the other hand, the edge is adjacent to vertex a, the preimage $J^{-1}(x)$ is contained in $M_{(a)}$ (see Figure 2.1). This preimage is of dimension $6-3=3$, with an effective action of $G_x = SU(2)$, and hence the reduced space is a point. □

We conjecture that for generic Γ_j the singular reduced spaces M_a and M_{c_j} when of dimension 2 are pinched spheres.

Transition polytopes. Some of the arguments above extend to the cases where Γ_j are 'transition' cases, that is, in the boundaries between the regions A, B, ..., H. Here we discuss some of theses.

$\Gamma_1 = \Gamma_2 \neq \Gamma_3$: the vertex b is contained in a wall of the Weyl chamber, and $c_1 = c_2$ (or similar). In that case, at $m = (e_1, e_2, e_3)$, $G_m = \mathbb{T}^2$ and $G_b = U(2)$ the Witt-Artin decomposition satisfies $\dim T_0 = 2 = \dim N_0$ and hence the symplectic slice is only of dimension 4 instead of 6. The momentum map on the symplectic slice (which we identify with $\mathbb{C}^2$) is

$$J_{N_1}(w_1, v_3) = \frac{\Gamma_3}{\Gamma_2}(\Gamma_2 - \Gamma_3)|v_3|^2\alpha_1 \ + \ \frac{\Gamma_1}{\Gamma_3}(\Gamma_3 - \Gamma_1)|w_1|^2\alpha_2.$$

(cf. [MS-I, Eq, (4.2)], with $\Gamma_1 = \Gamma_2$). It is clear that $J_{N_1}^{-1}(0)$ is just the origin, and this (or its quotient by G_m) provides a local model for the reduced space over μ, which is therefore just a point. The reduced space over $c_1 = c_2$ will be again of dimension 2, but with two singular points: we conjecture it is a twice-pinched sphere.

$\Gamma_1 = \Gamma_2 = \Gamma_3$: In this case $b = 0$ and $c_1 = c_2 = c_3$ (see [MS-I, Fig. 4.9b]). In this case the Witt-Artin decomposition has $\dim T_0 = \dim N_0 = 6$, and hence $N_1 = 0$. Then the MGS normal form is

$$J_Y([g, \sigma]) = g\sigma g^{-1}$$

with $\sigma \in N_0 \simeq \mathfrak{t}^\circ \subset \mathfrak{su}(3)^*$. It follows that $J^{-1}(0)$ is just a single orbit, and again the reduced space M_b is a single point. On the other hand, we conjecture that the reduced space over $c_1 = c_2 = c_3$ is a thrice-pinched sphere. The other reduced spaces will be as described in the theorem.

Other transition cases can be treated similarly.

3 Dynamics and relative equilibria

We now consider aspects of the dynamics for the generalized point vortex system, described in the introduction, with 2 or 3 point vortices, and in particular possible relative equilibria. We assume the pairwise interaction is governed by an $SU(3)$ invariant Hamiltonian,

$$h_0 : \mathbb{CP}^2 \times \mathbb{CP}^2 \setminus \Delta \longrightarrow \mathbb{R},$$

as described in the introduction (here Δ is the diagonal). One may also consider interactions allowing collisions, where h_0 extends to a smooth function on $\mathbb{CP}^2 \times \mathbb{CP}^2$. We write M° for the open subset of $M = \mathbb{CP}^2 \times \cdots \times \mathbb{CP}^2$ obtained by removing the large diagonal (or collision set). Such an invariant function on M or M° will be a smooth function of the distance defined in (2.2).

Given the symplectic form $\Omega = \sum_j \Gamma_j \pi_j^* \omega_0$, as described in the introduction, the dynamics is given by Hamilton's equation $\dot{x} = X_H(x)$ where X_H is the vector field satisfying $dH = \Omega(-, X_H)$, and $H : M \to \mathbb{R}$ is given by

$$h(x_1, \ldots, x_N) = \sum_{i<j} \Gamma_i \Gamma_j h_0(x_i, x_j). \tag{3.1}$$

3.1 Relative equilibria and allowed velocity vectors

We consider for the moment the general setting of a G-invariant Hamiltonian system on a symplectic manifold $\mathcal{P}$. See for example [13] or [15] for definitions. For $\xi \in \mathfrak{g}$, the associated vector field on $\mathcal{P}$ is denoted by $\xi_{\mathcal{P}}$ and, given a Hamiltonian function $H : P \to \mathbb{R}$, the associated vector field is denoted by X_H.

A relative equilibrium is a trajectory that lies in a group orbit or, almost equivalently, an invariant group orbit. The fact that the trajectory lies in the group orbit means that the Hamiltonian vector field is always tangent to this orbit, so that x lies on a relative equilibrium if and only if there is a $\xi \in \mathfrak{g}$ for which $X_H(x) = \xi_{\mathcal{P}}(x)$. Such a value of ξ is an ***angular velocity*** of the relative equilibrium in question. Using the symplectic form this becomes $dH_x = \xi \cdot DJ_x$ and so is equivalent to requiring x to be a critical point of $H_\xi = H - \xi \cdot J$. If the level set $J^{-1}(\mu)$ is non-singular, then it follows that $x \in J^{-1}(\mu)$ lies on a relative equilibrium if and only if x is a critical point of the restriction of H to $J^{-1}(\mu)$. Thus the relative equilibria are given by constrained critical points of H in much the same way that equilibria are given by ordinary critical points.

If the point x has a particular symmetry, then so must the angular velocity ξ, as the following result shows.

Proposition 3.1 *Let $x \in P$ be a relative equilibrium for a G-invariant Hamiltonian system H. Then*

$$X_H(x) \in R_0 := (\mathfrak{g}_\mu \cdot x)^{G_x} \subset T_x M. \tag{3.2}$$

We call this subspace R_0 the space of ***allowed velocity vectors***. We emphasize this is only a restriction on the velocity if we know that x is a relative equilibrium. Since the kernel of the map $\mathfrak{g} \to T_x\mathcal{P}$ given by $\xi \mapsto \xi_{\mathcal{P}}(x)$ is precisely $\mathfrak{g}_x$, it follows that

$$R_0 \simeq (\mathfrak{g}_\mu / \mathfrak{g}_x)^{G_x}. \tag{3.3}$$

Note that if $R_0 = 0$ then a relative equilibrium is necessarily a (group orbit of) equilibria.

Proof. This is a combination of conservation of symmetry (which holds for any symmetric dynamical system) with conservation of momentum. Since x is a relative equilibrium, there is a

$\xi \in \mathfrak{g}$ for which $X_H(x) = \xi_{\mathcal{P}}(x)$. By conservation of momentum, $DJ_x(X_H(x)) = 0$ and hence

$$\xi_{\mathcal{P}}(x) \in \mathfrak{g} \cdot x \cap \ker DJ_x = \mathfrak{g}_\mu \cdot x.$$

Now, for any symmetric dynamical system $\dot{x} = f(x)$, the vector field is tangent to the fixed point spaces: $f(x) \in T_x(\mathrm{Fix}(G_x, \mathcal{P}))$. This latter subspace of $T_x\mathcal{P}$ is equal to $\mathrm{Fix}(G_x, T_x\mathcal{P})$. Combining these shows that indeed for a relative equilibrium,

$$X_H(x) \in \mathfrak{g}_\mu \cdot x \cap \mathrm{Fix}(G_x, T_x\mathcal{P}) = (\mathfrak{g}_\mu \cdot x)^{G_x}$$

as reqired. □

3.2 Dynamics for 2 generalized point vortices

Since the reduced spaces in this instance are single points (Theorem 2.1), every trajectory is a relative equilibrium, or in another language, this is an instance of collective motion. The dynamics on M is therefore integrable, and every motion takes place on a torus of dimension at most 2 (the rank of $SU(3)$).

Moreover, since the Hamiltonian is a function of the distance, the distance for 2 point vortices is a conserved quantity and the generalized vortices cannot collide under the dynamics. In other words, the dynamics on M° is complete.

A final observation is that, the set of orthogonal points (those at a distance of $\pi/2$) is an extremum of the Hamiltonian h, and consequently they necessarily form a group orbit of *equilibria*. If the Hamiltonian is a strictly monotonic function of the distance, these will be the only equilibria in M°. Of course, if the Hamiltonian extends to M then it will have critical points at the diagonal, which will therefore also consist of equilibria. In both cases, since the group orbits of equilibria are extremal for the Hamiltonian, they will be $SU(3)$-Lyapunov stable [14], and even $SU(3)_\mu$-stable ($SU(3)_\mu$ is equal to $SU(3)$, or $U(2)$ or $\mathbb{T}^2$ according to the configuration and the values of the Γ_j).

Theorem 3.2 *Consider the $SU(3)$ action on two vortices on $\mathbb{CP}^2$, where every motion is a relative equilibrium. For any non-zero values of Γ_1, Γ_2, the space of allowed velocities is of dimension at most 1 according to the configuration type, as follows,*

$$\dim R_0 = \begin{cases} 0 & \text{if equal or orthogonal} \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. Table 3.1 lists the space R_0 for every configuration, generic and otherwise, on $\mathbb{CP}^2 \times \mathbb{CP}^2$. □

3.3 Dynamics for 3 generalized point vortices

Now let $M = \mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2$, with 'symplectic weights' Γ_j, and let M° be M with the large diagonal removed (that is, omitting all collisions).

$\Gamma_1 = \Gamma_2$	equal: $G_x = U(2) = G_\mu$	$R_0 \simeq (\mathfrak{u}(2)/\mathfrak{u}(2))^{U(2)} = \{0\}$
	orthogonal: $G_x = \mathbb{T}^2$, G_μ =U(2)	$R_0 \simeq (\mathfrak{u}(2)/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
	generic: $G_x = U(1)$, $G_\mu = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{u}(1))^{U(1)} = \mathbb{R}$
$\Gamma_1 = -\Gamma_2$	equal: $G_x = U(2)$, $G_\mu = SU(3)$	$R_0 \simeq (\mathfrak{su}(3)/\mathfrak{u}(2))^{U(2)} = \{0\}$
	orthogonal: $G_x = G_\mu = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
	generic: $G_x = U(1)$, $G_\mu = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{u}(1))^{U(1)} = \mathbb{R}$
Otherwise	equal: $G_x = G_\mu = U(2)$	$R_0 = \{0\}$
	orthogonal: $G_x = G_\mu = \mathbb{T}^2$	$R_0 = \{0\}$
	generic: $G_x = U(1)$, $G_\mu = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{u}(1))^{U(1)} = \mathbb{R}$

Table 3.1: The allowed velocity spaces for relative equilibria for 2 vortices on $\mathbb{CP}^2$ (allowing for collisions).

As discussed above (Theorem 2.4), each regular reduced space M_μ is diffeomorphic to a 2-sphere. The reduced dynamics thereon will be Hamiltonian, with reduced Hamiltonian function H_μ.

There are some obvious conclusions to make: if M_μ is a sphere then there are at least 2 relative equilibria with that value μ of J. If on the other hand, M_μ is a pinched sphere, there are also at least 2 relative equilibria, one of which must lie at the singular point. Moreover, if there are just the two critical points on M_μ, both relative equilibria are extremal and hence G_μ-Lyapunov stable [14]. If there are more than this minimum number of critical points, then some will be saddle points and hence unstable.

Finally, if M_μ is a point, then it is a relative equilibrium, and trivially extremal, and hence G_μ-Lyapunov stable.

More interesting is the allowed velocities of the relative equilibria. In Table 3.2 we assume the Γ_j are generic, so the polytope is one of the 8 forms A, B,..., G, H described in [MS-I, Section 4].

Special configurations. Consider the subgroup $\mathbb{T}^2 \subset SU(3)$ of diagonal matrices. The fixed points of $\mathbb{T}^2$ in $\mathbb{CP}^2$ are e_1, e_2 and e_3, and therefore in M the fixed points are the 27 points $(e_i, e_j, e_k) \in M$ for any $i, j, k \in \{1, 2, 3\}$. Since these are isolated, $\mathrm{Fix}(\mathbb{T}, T_m M) = 0$ and they are all necessarily equilibria for any $SU(3)$-invariant Hamiltonian. If the Hamiltonian does not extend to allow collisions then the only ones of these allowed are the 6 totally orthogonal configurations (which map to the vertex b under the orbit momentum map).

Consider now the semi-orthogonal configurations, those where one of the points is orthogonal to the other two. These points have stabilizer conjugate to $U(1)$, and in particular are all equivalent under the group action to points in $M_{(a)}$, and are therefore necessarily relative equilibria.

Finally, consider the coplanar configurations, corresponding to points in $M_{(b)}$. Now $M_{(b)} \simeq$

$\mu \in$ Wall: $G_\mu = U(2)$

triple point	$G_x = U(2)$	$R_0 \simeq (\mathfrak{u}(2)/\mathfrak{u}(2))^{U(2)} = \{0\}$
other vertices	$G_x = U(1)$	$R_0 \simeq (\mathfrak{u}(2)/\mathfrak{u}(1))^{U(1)} = \mathbb{R}^3$
generic	$G_x = \mathbf{1}$	$R_0 \simeq \mathfrak{u}(2) = \mathbb{R}^4$

$\mu \notin$ Wall: $G_\mu = \mathbb{T}^2$

double point	$G_x = U(1)$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{u}(1))^{U(1)} = \mathbb{R}$
double+orthogonal	$G_x = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
distinct coplanar	$G_x = U(1)$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{u}(1))^{U(1)} = \mathbb{R}$
totally orthogonal	$G_x = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
semi-orthogonal	$G_x = U(1)$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{u}(1))^{U(1)} = \mathbb{R}$
generic	$G_x = \mathbf{1}$	$R_0 \simeq \mathfrak{t}^2 = \mathbb{R}^2$

Table 3.2: The allowed velocity spaces for relative equilibria for 3 vortices on $\mathbb{CP}^2$, and for generic Γ_j (allowing for collisions). See text for explanations.

$S^2 \times S^2 \times S^2$, and the system reduces to that of three point vortices on the sphere. See for example [21] for discussions of this system.

Identical vortices. In this special case where $\Gamma_1 = \Gamma_2 = \Gamma_3$, there is a further symmetry of the system given by permutations of the point vortices. Thus the full symmetry group becomes $G = SU(3) \times S_3$.

Now let $A \in SU(3)$ be any element of order 3; that is, one satisfying $A^3 = I$, $A \neq I$. In $SO(3)$ any element of order 3, if not the identity, is a rotation by $2\pi/3$ about some axis, and all such elements are conjugate. However, in $SU(3)$ there are different (non-conjugate) elements of interest:

$$\begin{aligned} A_1 &= \mathrm{diag}[1,\, e^{2i\pi/3},\, e^{-2i\pi/3}], \\ A_2 &= \mathrm{diag}[e^{2i\pi/9},\, e^{2i\pi/9},\, e^{-4i\pi/9}]. \end{aligned}$$

(In both cases A_j^3 is a scalar matrix which therefore acts trivially on $\mathbb{CP}^2$.) Let $\sigma = (1\ 2\ 3) \in S_3$, and let Σ_j be the subgroup of order 3 of G generated by (A_j, σ). Now $m = (m_1, m_2, m_3) \in \mathrm{Fix}(\Sigma_j, M)$ if and only if

$$m_2 = A_j m_1, \quad \text{and} \quad m_3 = A_j^2 m_1.$$

It follows that $\mathrm{Fix}(\Sigma_j, M) \simeq \mathbb{CP}^2$, parametrized by say $m_1 \in \mathbb{CP}^2$ (for each of $j = 1, 2$). The normalizer of Σ_1 is $\mathbb{T} \times \mathbb{Z}_3$, while that of Σ_2 is $U(2) \times \mathbb{Z}_3$ (here $\mathbb{Z}_3$ is the cyclic subgroup of S_3 generated by σ). There are therefore actions of $\mathbb{T}$ and $U(2)$ on $\mathrm{Fix}(\Sigma_1, M)$ and $\mathrm{Fix}(\Sigma_2, M)$

respectively, and the momentum maps for these actions are

$$J_{\Sigma_2}([x:y:z]) = \Gamma_1 \begin{pmatrix} 3|x|^2-1 & 3x\overline{y} & 0 \\ 3\overline{x}y & 3|y|^2-1 & 0 \\ 0 & 0 & 3|z|^2-1 \end{pmatrix} \in \mathfrak{u}(2)^*$$

and $J_{\Sigma_1}([x:y:z]) = \mathrm{diag}[3|x|^2-1,\, 3|y|^2-1,\, 3|z|^2-1] \in \mathfrak{t}^*$, where $m_1 = [x:y:z] \in \mathbb{CP}^2$. A standard argument (see for example [12, §3.2]) then shows that for identical vortices, every configuration in $\mathrm{Fix}(\Sigma_j, M)$ is a relative equilibrium (for $j = 1, 2$).

Remark 3.3 In this section we have only used the $SU(3)$-invariance of the Hamiltonian, and not the 'pairwise interaction' form of (3.1). It would be interesting to know (in general, not just in this context) what differences there are between Hamiltonian dynamics based on pairwise interactions, and more general (symmetric) Hamiltonian systems. In the study of molecular dynamics, for example, the interactions between the atoms is not assumed to be a pairwise interaction, and the potential may depend on more general shape information.

References

[1] H. Aref, Point vortex dynamics: a classical mathematics playground. *J. Math. Phys.* **48** (2007), 065401.

[2] H. Aref, P.K. Newton, M.A. Stremler, T. Tokieda, & D.L. Vainchtein, Vortex crystals. *Adv. in Appl. Mech.* **39** (2003), 1–79.

[3] S. Boatto & J. Koiller, Vortices on closed surfaces. *Geometry, Mechanics and Dynamics: The Legacy of Jerry Marsden*. Springer, 2015.

[4] P.-L. Buono, F. Laurent-Polz and J. Montaldi, *Symmetric Hamiltonian Bifurcations*, Geometric Mechanics and Symmetry: The Peyresq Lectures, London Mathematical Society Lecture Note Series: 306, P. 357-402, Cambridge University Press, 2005.

[5] D.G. Dritschel and S. Boatto, The motion of point vortices on closed surfaces. *Proc. R. Soc.* A 471 (2015), 20140890. [DOI:10.1098/rspa.2014.0890]

[6] V. Guillemin & S. Sternberg, *Symplectic Techniques in Physics*. CUP, 1984.

[7] R. Kidambi & P. Newton, Motion of three point vortices on a sphere. *Physica D* **116** (1998), 143–175.

[8] F.C. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Math. Notes 31 (Princeton University Press, 1984).

[9] F.C. Kirwan, Momentum maps and reduction in algebraic geometry. *Diff. Geom. and Appl.* **9** (1998), 135–171.

[10] F. Laurent-Polz, J. Montaldi & M. Roberts, Point vortices on the sphere: stability of symmetric relative equilibria, *J. Geom. Mech.* **3** (2011), 439–486.

[11] C.C. Lim, Existence of Kolmogorov–Arnold–Moser tori in the phase-space of lattice vortex systems. *Z. Angew. Math. Phys.* **41** (1990), 227–244.

[12] C. Lim, J. Montaldi & M. Roberts, Relative equilibria of point vortices on the sphere. *Phys. D.* **148** (2001), 97–135.

[13] J. Marsden and T. Ratiu *Introduction to Mechanics and Symmetry*, Springer Verlag, New York, Berlin, Heidelberg, 1994.

[14] J. Montaldi, Persistence and stability of relative equilibria. *Nonlinearity* **10** (1997), 449-466.

[15] J. Montaldi, Relative equilibria and conserved quantities in symmetric Hamiltonian systems, in *Peyresq lectures on nonlinear phenomena.* World Sci. Publ., (2000), pp. 239–280.

[16] J. Montaldi & C. Nava-Gaxiola, Point vortices on the hyperbolic plane. *J. Math. Phys.* **55** (2014), 102702. doi: 10.1063/1.4897210

[17] J. Montaldi, A. Soulière & T. Tokieda, Vortex dynamics on cylinders. *SIAM J. on Appl. Dyn. Sys.*, **2** (2003), 417–430.

[MS-I] J. Montaldi & A. Shaddad, Non-Abelian momentum polytopes for products of $\mathbb{CP}^2$. *Preprint* (2018)

[18] J. Montaldi, & T. Tokieda, Openness of momentum maps and persistence of extremal relative equilibria. *Topology* **42** (2003), 833–844.

[19] P. Newton, *The N-Vortex Problem*, Springer Verlag, New York, Berlin, Heidelberg, 2001.

[20] J.-P. Ortega and T.S. Ratiu, *Momentum Maps and Hamiltonian Reduction*, Springer, 2004.

[21] S. Pekarsky & J.E. Marsden, Point vortices on a sphere: stability of relative equilibria. *J. Math. Phys.*, **39** (1998), 5894–5907.

[22] A.R. Rodrigues, C. Castilho & J. Koiller, Vortex dynamics on a triaxial ellipsoid and Kimura's conjecture. *J. Geom. Mech.* (2018, to appear).

[23] T. Sakajo & Y. Shimizu, Point vortex interactions on a toroidal surface. *Proc. R. Soc. A 472 (2016): 20160271.* http://dx.doi.org/10.1098/rspa.2016.0271

[24] I. Schur, Über eine Klasse von Mittelbildungen mit Anwendungen auf der Determinantentheorie (On a class of averaging with application to the theory of determinants), Sitzunsberichte der Berliner Mathematischen Gesellschaft, **22** (1923), 9-20.

[25] A. Shaddad, *The classification and dynamics of the momentum polytopes of the SU(3) action on points in the complex projective plane with an application to point vortices.* Ph.D. thesis, University of Manchester, 2018.

[26] R. Sjamaar, Convexity properties of the moment mapping re-examined. *Advances in Math.* **138** (1998) 46–91.

JM: j.montaldi@manchester.ac.uk
AS: amna.shaddad@manchester.ac.uk

School of Mathematics
University of Manchester
Manchester M13 9PL, UK