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Cuspidal Characters of Sporadic Simple Groups

Peter Rowley and David Ward

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Abstract

In this paper the notion of an irreducible cuspidal character for finite groups of Lie type is generalized to any finite group. All the irreducible cuspidal characters for the finite sporadic simple groups are then determined.

1 Introduction

For a finite group $G$, its complex irreducible character table encodes a diverse range of details relating to the structure and properties of $G$. For example the structure constants may be extracted from the character table (see [7, Theorem 4.2.12]). While there are results connecting character values with the orders of various subgroups of $G$ (see [7, Theorems 4.2.8 and 4.2.11]). Since the birth and rapid development of character theory by Frobenius, Schur and Burnside ([5]), there has been a fruitful interplay between characters and finite groups. A particular jewel in the crown being Frobenius’s theorem ([7, Theorem 5.1]), for which no proof without characters is known. This has resulted in extensive efforts to calculate character tables of “interesting” finite groups. Tables for all the sporadic simple groups are to be found in the ubiquitous ATLAS ([4]). While the case when $G$ is a Lie type group is the subject of the mammoth text by Carter ([3]), and continues to be a very active area of research.

When $G$ is a group of Lie type, a particular type of irreducible character, called a cuspidal character, plays an important role. This is because of the fact that every irreducible character of $G$ is a constituent of the induced character $\phi^G_P$ for some cuspidal character $\phi$ of some $P_J$, where $P_J$ is a proper parabolic subgroup of $G$ (see [3, Chapter 9] for more details). The aim of the present paper is to generalize the notion of a cuspidal character to an arbitrary finite group, and then to determine all irreducible characters for the sporadic simple groups. Since the complex irreducible character tables are known for all the sporadic simple groups, this begs the question as to whether this is a worthwhile enterprise. Unlike the situation of groups of Lie type where the cuspidal characters are being used to determine further irreducible characters, our motivation here is to better understand the sporadic simple groups in a wider context. We shall return to this issue shortly.

The generalization of cuspidal characters is set against the following backdrop.
Definition 1.1. Suppose that $G$ is a finite group, $X$ a subgroup of $G$ and $I$ an index set with $|I| = n$. Then an $X$-parabolic system of rank $n$ is a set of pairs of subgroups of $G$, $(P_J, Q_J)$, indexed by subsets $J$ of $I$ such that

(i) for each $J \subseteq I$, $X \leq P_J$, $Q_J \leq P_J$;

(ii) for $K \subseteq J \subseteq I$, $Q_J \leq Q_K$;

(iii) $P_I = G$ and $Q_I = 1$; and

(iv) $X = P_\emptyset$.

We shall write $\mathcal{X} = \{(P_J, Q_J)\}_{J \subseteq I}$ and note that by part (iii) of Definition 1.1, all $Q_J$ are subgroups of $\emptyset$ and that $Q_\emptyset \leq P_\emptyset = X$. We allow the possibility that $(P_J, Q_J) = (P_K, Q_K)$ with $J \neq K$, but this will not arise in most of the cases that follow. If our index set is $I = \{1, 2, \ldots, n\}$ for some $n \geq 1$, then given a subset $\{i_1, i_2, \ldots, i_r\} \subseteq I$ with $i_j < i_{j+1}$ for all $j$, we will often denote the subgroups $P_{\{i_1, i_2, \ldots, i_r\}}$ and $Q_{\{i_1, i_2, \ldots, i_r\}}$ by $P_{i_1 \ldots i_r}$ and $Q_{i_1 \ldots i_r}$, respectively. Given an $X$-parabolic system $\mathcal{X} = \{(P_J, Q_J)\}_{J \subseteq I}$ of a finite group $G$ and $J \subseteq I$, we set $\mathcal{P}_J := P_J/Q_J$. Furthermore, for any subgroup $Q_J \leq Y \leq P_J$, we use the standard bar notation $\bar{Y} := Y/Q_J$. We may use $\mathcal{X}$ to form an $X$-parabolic system, $\mathcal{X}_J$, of rank $|J|$ for $\mathcal{P}_J$ given by

$$\mathcal{X}_J = \{(\mathcal{P}_J \cap \mathcal{P}_K, \mathcal{Q}_K)| (P_K, Q_K) \in \mathcal{X}, K \subseteq J\}.$$ 

We now describe a particular type of $X$-parabolic system of interest here. Suppose that $G$ is a finite group, $p$ a prime and $S \in \text{Syl}_p(G)$. Set $B = N_G(S)$. A subgroup $P$ of $G$ is called $p$-minimal (with respect to $B$) if $B$ is a proper subgroup of $P$ and $B$ is contained in a unique maximal subgroup of $P$. We recall that for $H$ a finite group, $O_p(H)$ is the largest normal $p$-subgroup of $H$, and we shall refer to $O_p(H)$ as the $p$-core of $H$.

Defining

$$\mathcal{M}(G, B) = \{P|P \text{ is a } p\text{-minimal subgroup of } G \text{ (with respect to } B)\},$$

then a set

$$\mathcal{M}_0 = \{P_i| P_i \in \mathcal{M}(G, B), i \in I\}$$

is called a minimal parabolic system of characteristic $p$ for $G$ or a $p$-minimal parabolic system of $G$, if $G = \{P_i|i \in I\}$ and $G \neq \{P_j|j \in I \setminus \{i\}\}$ for any $i \in I$. The rank of $\mathcal{M}_0$ is $|I|$. We call $\mathcal{M}_0$ a geometric $p$-minimal parabolic system if for all $J, K \subseteq I$ we have $P_{J \cap K} = P_J \cap P_K$. Otherwise $\mathcal{M}_0$ is called non-geometric.

Provided $B \neq G$, we always have $G = \langle \mathcal{M}(G, B) \rangle$ and so there is always at least one minimal parabolic system of characteristic $p$ for $G$. 

Now suppose that $O_p(G) = 1$. For a minimal parabolic system $\mathcal{M}_0 = \{P_i|P_i \in \mathcal{M}(G, B), i \in I\}$ of $G$ we define a $B$-parabolic system $\mathcal{X} = \{(P_J, Q_J)\}_{J \subseteq I}$ by

$$P_J = \begin{cases} \{P_j| j \in J\} & \text{if } \emptyset \neq J \subseteq I; \\ B & \text{if } J = \emptyset. \end{cases}$$
and \( Q_J = O_p(P_J) \) for all \( J \subseteq I \). Should \( G \) be a simple group of Lie type of characteristic \( p \), then \( B \) would be the Borel subgroup of \( G \) and \( \{ P_J, J \subseteq I \} \) the parabolic subgroups of \( G \) (containing \( B \)). Further, \( Q_J \) would be the unipotent radical of \( P_J \) for \( J \subseteq I \). Turning to the sporadic simple groups, the \( p \)-minimal parabolic systems were catalogued by Ronan and Stroth [14] for groups whose Sylow \( p \)-subgroups are non-cyclic. (We note that in [14] they require their \( p \)-minimal subgroups to have a non-trivial \( p \)-core, which we do not need to assume here.)

For \( G \) a finite group, \( \text{Irr}(G) \) will denote the set of complex irreducible characters of \( G \). We now give the promised generalization of cuspidal characters.

**Definition 1.2.** Let \( X \) be an \( X \)-parabolic system of \( G \) where \( X \subseteq G \), and let \( \chi \in \text{Irr}(G) \). Then \( \chi \) is called \( X \)-cuspidal if for all \( (P_J, Q_J) \in X \) with \( Q_J \neq 1 \) we have

\[
\sum_{g \in Q_J} \chi(g) = 0. \tag{1}
\]

The condition (1) will be known as the cuspidal condition on \( Q_J \) and is equivalent to \( (\chi_{Q_J}, 1_{Q_J}) = 0 \). In an abuse of terminology, we will also sometimes refer to the cuspidal relation holding for \( P_J \) when (1) occurs. Clearly, when the index set \( I = 0 \), we have \( G = P_0 = X \) and \( Q_0 = 1 \), and hence every irreducible character is vacuously \( X \)-cuspidal. When \( X \) is a \( B \)-parabolic system associated to a \( p \)-minimal parabolic system of \( G \), then any \( X \)-cuspidal character will also be called a \( p \)-cuspidal character of \( G \). Consulting [3] Proposition 9.1.1 we see that this generalizes the situation when \( G \) is a simple Lie type group of characteristic \( p \).

Our main theorem is as follows – in its statement the names for the sporadic simple groups and their irreducible characters are as they appear in [4].

**Theorem 1.3.** Suppose that \( G \) is a finite sporadic simple group with \( X \) an \( X \)-parabolic system of \( G \) given by one of the \( p \)-minimal parabolic systems of \( G \). Let \( \chi \in \text{Irr}(G) \). Then \( \chi \) is \( X \)-cuspidal for the following pairs \((G, \{\chi_J\})\).

(i) \( p = 2 \). \((M_{11}, \{\chi_3, \chi_4\}), (M_{22}, \{\chi_3, \chi_4\}), (M_{23}, \{\chi_3, \chi_4\}), (M_{24}, \{\chi_3, \chi_4\}), (Co_2, \{\chi_3, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{31}, \chi_{32}\}), (Co_1, \{\chi_2, \chi_8, \chi_{11}\}), (Ru, \{\chi_2, \chi_3\}), (Th, \chi_2), (J_4, \{\chi_2, \chi_3\})\).

(ii) \( p = 3 \). \((M_{11}, \{\chi_6, \chi_7\}), (M_{12}, \{\chi_4, \chi_5\}), (Fi_{22}, \{\chi_2\}), (Th, \{\chi_2\})\).

(iii) \( p \geq 5 \). \((J_2, \{\chi_6\}), (HN, \{\chi_4\}), (Th, \{\chi_2\}), (Ly, \{\chi_2, \chi_3\}) (p = 5); (He, \{\chi_2, \chi_3\}) (p = 7); (M_{11}, \{\chi_2, \chi_3, \chi_4\}) (p = 11); (M_{23}, \{\chi_2\}) (p = 23)\).

We direct the reader to Tables [1] [2] and [3] for which \( X \)-parabolic systems arise in Theorem 1.3 as well as the degrees of the various irreducible \( X \)-cuspidal characters. Note that in these tables we have used the same notation for the \( p \)-minimal subgroups as in [14] when the Sylow \( p \)-subgroup of \( G \) is non-cyclic.

We now return to the question of motivation for this study. As we note shortly, the existence of a number of the \( X \)-cuspidal irreducible characters for the sporadic simple groups coincides with interesting and exceptional behaviour.
For example, when \( p = 3 \) and \( G \cong \text{Th} \), we have that \( \chi_2 \) is an irreducible 3-cuspidal character of degree 248. This representation was instrumental in the original construction of \( \text{Th} \) in which \( \text{Th} \) was shown to be a subgroup of \( E_6(3) \), but not a subgroup of \( E_6(q) \) for any other prime \( q \neq 3 \) (see [10], [17]). In [11], Margolin looked at a geometry for the Mathieu group \( M_{24} \). Margolin’s interest stemmed from the two 1333-dimensional irreducible \( GF(2)J_4 \)-representations. Since \( 2^{11} : M_{24} \) is a maximal subgroup of \( J_4 \), Margolin considered the restriction of these representations to \( 2^{11} : M_{24} \), namely as a faithful 1288-dimensional representation and a 45-dimensional representation having kernel \( 2^{11} \). Hence Margolin sought to find a simple explanation for this 45-dimensional representation and this resulted in the construction of a geometry. We note that both 1333-dimensional \( J_4 \)-characters are 2-cuspidal, as are both of the resulting irreducible \( M_{24} \)-characters of degree 45, along with their irreducible restrictions to \( M_{22} \) and \( M_{23} \). So the work presented here, may further highlight certain characters and/or \( \mathcal{X} \)-parabolic systems (and associated geometries) where one might prospect for interesting nuggets.

The sporadic simple groups are something of an unruly bunch, so we can not expect the light cuspidal characters shines on them to reveal all their secrets. Indeed, looking at Theorem 1.3(i) we see that though \( M^{o}L \) (and to a lesser extent \( Fi_{24}^{+} \)) has more minimal parabolic systems than you can shake a stick at, it (and \( Fi_{23} \)) fail to have any cuspidal characters. Moreover, the minimal parabolic systems for \( M_{24} \) and \( He \) (with \( p = 2 \)) have the same diagram (see [14]) yet \( M_{24} \) has 2-cuspidal characters, but \( He \) does not. For the tuples \( (G, p) \) given by \( (M_{11}, 11) \) and \( (M_{23}, 23) \), it is unlikely that the \( p \)-cuspidal characters of \( G \) will give rise to any interesting geometries. Indeed, in both cases a Sylow \( p \)-subgroup of \( G \) is cyclic of order \( p \), and the resulting \( p \)-cuspidal characters have degree \( p-1 \). Similarly, when \( (G, p) \) is \( (M_{11}, 3) \) or \( (Th, 5) \), there is a unique class of elements of order \( p \), and a Sylow \( p \)-subgroup has exponent \( p \). Hence it is improbable that the resulting \( p \)-cuspidal characters lead to interesting geometries. As an aside, we mention that representation theory and minimal parabolic systems have interacted in the modular case (see [13]). Finally we note that \( Co_1 \) and \( Co_2 \) are the proud owners of many irreducible cuspidal characters for \( p = 2 \), and these are definitely worthy of further scrutiny.

This paper is organized as follows. Section 2 consists of some general results on cuspidal characters. Theorem 2.3 and Proposition 2.5 are the exact analogues of [3] 9.1.3 and [3] 9.1.2 respectively. But other results do not generalize from the Lie type group case (see Example 2.4). The last two results of this section are useful for our later calculations. Sections 3, 4, 5 and 6 determine the \( p \)-cuspidal characters for the sporadic groups in the cases where, respectively, \( p = 2 \), \( p = 3 \), \( p = 5 \) and \( p > 5 \).
<table>
<thead>
<tr>
<th>Family</th>
<th>Group</th>
<th>2-Minimal Parabolic System</th>
<th>Rank</th>
<th>Geometric/Non-Geometric</th>
<th>2-Cuspidal Characters (character degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathieu Groups</td>
<td>$M_{11}$</td>
<td>${ P_1 \sim 2^{1+2}, \text{Sym}(3), P_2 \sim 3^2, S D_{16} }$</td>
<td>2</td>
<td>2</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$M_{12}$</td>
<td>${ P_1 \sim 2^{1+2}, \text{Sym}(3), P_3 \sim 3^2, \text{Alt}(6) }$</td>
<td>2</td>
<td>2</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$M_{22}$</td>
<td>${ P_1 \sim 2^{1+2}, \text{Sym}(3), P_2 \sim 3^2, \text{Alt}(6) }$</td>
<td>2</td>
<td>2</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$M_{23}$</td>
<td>${ P_1 \sim 2^{1+2}, \text{Sym}(3), P_2 \sim 3^2, \text{Sym}(5) }$</td>
<td>2</td>
<td>2</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$M_{24}$</td>
<td>${ P_1 \sim 2^{1+2}, \text{Sym}(3), P_2 \sim 3^2, \text{Sym}(5) }$</td>
<td>2</td>
<td>2</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$M^* L$</td>
<td>${ P_2 \sim 2^{1+2}, \text{Sym}(3), P_3 \sim 2^4, \text{Sym}(5) }$</td>
<td>3</td>
<td>3</td>
<td>Non-Geometric</td>
</tr>
<tr>
<td></td>
<td>$C_{01}$</td>
<td>${ P_1 \sim [2^2]^{m}, \text{Sym}(3), P_2 \sim [2^2]^{m}, \text{Sym}(3), P_3 \sim [2^2]^{m}, \text{Sym}(3) }$</td>
<td>4</td>
<td>4</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$C_{02}$</td>
<td>${ P_1 \sim [2^{15}], \text{Sym}(5), P_2 \sim [2^{17}], \text{Sym}(3), P_3 \sim [2^{17}], \text{Sym}(3) }$</td>
<td>3</td>
<td>3</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$C_{03}$</td>
<td>${ P_1 \sim 2^{1+3}, \text{Sym}(3), P_2 \sim 2^{1+3}, \text{Sym}(3), P_3 \sim 2^{1+3}, \text{Sym}(3) }$</td>
<td>3</td>
<td>3</td>
<td>Geometric</td>
</tr>
<tr>
<td></td>
<td>$C_{04}$</td>
<td>${ P_1 \sim 2^{1+3}, \text{Sym}(5), P_2 \sim 2^{1+3}, \text{Sym}(5), P_3 \sim 2^{1+3}, \text{Sym}(5) }$</td>
<td>3</td>
<td>3</td>
<td>Geometric</td>
</tr>
<tr>
<td>Leech Lattice</td>
<td>$S U_{2}$</td>
<td>${ P_1 \sim 2^{1+3} L_2(4), P_2 \sim 2^{1+3}, (3 \times \text{Sym}(3)), P_3 \sim 2^{1+3} (3 \times \text{Sym}(3)) }$</td>
<td>3</td>
<td>3</td>
<td>Geometric</td>
</tr>
<tr>
<td>and Conway</td>
<td>Groups</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Mathieu, Leech Lattice and Conway sporadic simple groups.
<table>
<thead>
<tr>
<th>Family</th>
<th>Group</th>
<th>2-Minimal Parabolic System</th>
<th>Rank</th>
<th>Geometric/Non-Geometric</th>
<th>2-Cuspidal Characters (character degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>He</strong></td>
<td></td>
<td>{P_1 \sim 2^{6+3}.\text{Sym}(3), P_2 \sim 2^{6+3}.\text{Sym}(3), P_3 \sim 2^{6+3}.\text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td></td>
<td>{P_1 \sim 2^{6+3}.\text{Sym}(3), P_1 \sim 2^{6+3}.\text{Sym}(3), P_1 \sim 2^{6+3}.\text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>HN</strong></td>
<td></td>
<td>{P_1 \sim 2^{1+5}.\text{Alt}(5) \times \mathbb{Z}_2, P_2 \sim 2^{1+5+2+2},\text{Sym}(3)}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>Th</strong></td>
<td></td>
<td>{P_1 \sim 2^{1+5}, \text{Alt}(9), P_2 \sim 2^{1+5+2+1}.\text{Sym}(3)}</td>
<td>2</td>
<td>Geometric</td>
<td>\chi_2 (248)</td>
</tr>
<tr>
<td><strong>Fi_{22}</strong></td>
<td></td>
<td>{P_1 \sim [2^{15}], \text{Sym}(3), P_2 \sim [2^{10}], \text{Sym}(3), P_3 \sim [2^{4}],\text{Sym}(5)}</td>
<td>3</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>Fi_{23}</strong></td>
<td></td>
<td>{P_1 \sim [2^{17}], \text{Sym}(3), P_1 \sim [2^{17}], \text{Sym}(3), P_2 \sim [2^{17}], \text{Sym}(3), P_3 \sim [2^{15}],\text{Sym}(5)}</td>
<td>4</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td></td>
<td>{P_1 \sim [2^{17}], \text{Sym}(3), P_2 \sim [2^{17}], \text{Sym}(3), P_3 \sim [2^{17}], \text{Sym}(3), P_4 \sim [2^{17}],\text{Sym}(5)}</td>
<td>4</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td></td>
<td>{P_1 \sim [2^{17}], \text{Sym}(3), P_2 \sim [2^{17}], \text{Sym}(3), P_3 \sim [2^{17}], \text{Sym}(3), P_4 \sim [2^{17}],\text{Sym}(5)}</td>
<td>4</td>
<td>Non-Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>Fi_{24}</strong></td>
<td></td>
<td>{P_1 \sim [2^{20}], L_2(2), P_2 \sim [2^{6}], L_2(2), P_3 \sim [2^{6}], L_2(2), P_4 \sim [2^{6}], L_2(2)}</td>
<td>4</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>J_4</strong></td>
<td></td>
<td>{P_1 \sim [2^{20}], P_2 \sim [2^{8}], P_2 \sim [2^{8}], L_2(2), P_3 \sim [2^{8}], L_2(2)}</td>
<td>5</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td></td>
<td>{J_1}</td>
<td>1</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td></td>
<td>{J_1}</td>
<td>1</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>O’N</strong></td>
<td></td>
<td>{P_1 \sim 4^{*}, 2^{4}, \text{Sym}(3), P_2 \sim 4.L_3(4),2}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>J_3</strong></td>
<td></td>
<td>{P_1 \sim 2^{4+1}.(3 \times \text{Sym}(3)), P_2 \sim 2^{4+1}, L_2(4)}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td><strong>R_8</strong></td>
<td></td>
<td>{P_1 \sim 2^{8+4}, \text{Sym}(5), P_2 \sim 2^{8+4},\text{Sym}(3)}</td>
<td>2</td>
<td>Geometric</td>
<td>\chi_2 (378), \chi_3 (378)</td>
</tr>
<tr>
<td><strong>J_4</strong></td>
<td></td>
<td>{P_1 \sim [2^{4}], \text{Sym}(3), P_2 \sim [2^{4}], \text{Sym}(3), P_3 \sim [2^{4}],\text{Sym}(5)}</td>
<td>3</td>
<td>Geometric</td>
<td>\chi_2 (1333), \chi_3 (1333)</td>
</tr>
<tr>
<td><strong>L_2y</strong></td>
<td></td>
<td>{P_1 \sim [2^{4}], \text{Sym}(3), P_2 \sim [2^{4}], \text{Sym}(5)}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
</tr>
</tbody>
</table>

Table 2: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Monster group, its subquotients and the pariah sporadic simple groups.
<table>
<thead>
<tr>
<th>Family</th>
<th>Group</th>
<th>Prime, $p$</th>
<th>$p$-Minimal Parabolic System</th>
<th>Rank</th>
<th>Geometric/Non-Geometric</th>
<th>$p$-Cuspidal Characters (character degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathieu Groups</td>
<td>$M_{11}$</td>
<td>3</td>
<td>{ $M_{11}$ }</td>
<td>1</td>
<td>Geometric</td>
<td>$\chi_6$ (16), $\chi_7$ (16)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>{ $M_{11}$ }</td>
<td>1</td>
<td>Geometric</td>
<td>$\chi_8$ (10), $\chi_9$ (10), $\chi_4$ (10)</td>
</tr>
<tr>
<td></td>
<td>$M_{12}$</td>
<td>3</td>
<td>{ $P_1 \sim 3^2.GL_2(3), P_2 \sim 3^2.GL_2(3)$ }</td>
<td>2</td>
<td>Geometric</td>
<td>$\chi_4$ (16), $\chi_5$ (16)</td>
</tr>
<tr>
<td></td>
<td>$M_{23}$</td>
<td>23</td>
<td>{ $M_{23}$ }</td>
<td>1</td>
<td>Geometric</td>
<td>$\chi_2$ (22)</td>
</tr>
<tr>
<td>Leech Lattice and Conway Groups</td>
<td>$J_2$</td>
<td>5</td>
<td>{ $J_2$ }</td>
<td>1</td>
<td>Geometric</td>
<td>$\chi_6$ (36)</td>
</tr>
<tr>
<td>Monster Group and Subquotients</td>
<td>$He$</td>
<td>7</td>
<td>{ $He$ }</td>
<td>1</td>
<td>Geometric</td>
<td>$\chi_2$ (51), $\chi_3$ (51)</td>
</tr>
<tr>
<td></td>
<td>$HN$</td>
<td>5</td>
<td>{ $P_1 \sim 5^{1+4} + 4.A(5), P_2 \sim 5^{1+4} + 4.A(5)$ }</td>
<td>2</td>
<td>Geometric</td>
<td>$\chi_4$ (760)</td>
</tr>
<tr>
<td></td>
<td>$Th$</td>
<td>3</td>
<td>{ $P_1 \sim 3^{1+2} + 4 + 3^{1+2} + 4.GL_2(3), P_2 \sim 3^{1+2} + 4 + 3^{1+2} + 4.GL_2(3)$ }</td>
<td>2</td>
<td>Geometric</td>
<td>$\chi_2$ (248)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>{ $Th$ }</td>
<td>1</td>
<td>Geometric</td>
<td>$\chi_2$ (248)</td>
</tr>
<tr>
<td></td>
<td>$Fi_{22}$</td>
<td>3</td>
<td>{ $P_1 \sim 3^3 + 2.GL_2(3), P_2 \sim 3^3 + 2.GL_2(3), P_3 \sim 3^3 + 2.GL_2(3)$ }</td>
<td>3</td>
<td>Geometric</td>
<td>$\chi_2$ (78)</td>
</tr>
<tr>
<td></td>
<td>$Ly$</td>
<td>5</td>
<td>{ $P_1 \sim 5^{1+4} + 4.A.PGL_2(5), P_2 \sim 5^{1+4} + 4.A.PGL_2(5), P_3 \sim 5^{1+4} + 4.A.PGL_2(5)$ }</td>
<td>3</td>
<td>Geometric</td>
<td>$\chi_2$ (2480), $\chi_3$ (2480)</td>
</tr>
</tbody>
</table>

Table 3: The $p$-cuspidal characters and associated $p$-minimal parabolic systems of the sporadic simple groups in the case that $p > 2$. 
2 Elementary Properties of $\mathfrak{X}$-Cuspidal Characters

We recall the notion of the intertwining number of two modules.

**Definition 2.1.** [10] Let $F$ be a field of characteristic 0, with algebraic closure $F^c$, let $G$ be a finite group and let $V$ and $W$ be $FG$-modules. The intertwining number, denoted $i(V,W)$, is defined by

$$i(V,W) := \dim_F \text{Hom}_{FG}(V,W)$$

The intertwining number of modules will be of importance due to its connection with the inner product of the associated characters.

**Theorem 2.2.** [10, Chapter 3, Theorem 1.1] Let $F$ be an arbitrary field of characteristic 0 and let $\lambda$ and $\mu$ be arbitrary characters of $G$ afforded by $FG$-modules $V$ and $W$ respectively. Then

$$(\lambda,\mu) = i(V,W).$$

We may use Theorem 2.2 to prove an analogue of Proposition 9.1.3 of [3], the proof of which is almost identical to that used in Carter’s Proposition.

**Theorem 2.3.** Let $\mathfrak{X}$ be an $X$-parabolic system of $G$ and $\chi \in \text{Irr}(G)$. Then there exists $(P_J,Q_J) \in \mathfrak{X}$ and an $\mathfrak{X}_J$-cuspidal character $\psi$ of $P_J = P_J/Q_J$ such that $$(\chi,\psi) \neq 0.$$ We may use Theorem 2.2 to prove an analogue of Proposition 9.1.3 of [3], the proof of which is almost identical to that used in Carter’s Proposition.

**Proof.** Let $S = \{ J \subseteq I | (\chi_{Q_J},1_{Q_J}) \neq 0 \}$. Note that $S \neq \emptyset$ as $Q_I = 1$. Let $J$ be a minimal element of $S$ and let $V$ be an irreducible $C_G$-module that affords $\chi$.

Define

$$V' = \{ v \in V | v \cdot u = v \text{ for all } u \in Q_J \}.$$

By Theorem 2.2 as $(\chi_{Q_J},1_{Q_J}) \neq 0$, there exists a non-zero $C_{Q_J}$-homomorphism from the trivial $C_{Q_J}$-module to $V$, and hence $V'$ is non-empty.

Clearly $V'$ is a linear subspace of $V$, and given $g \in P_J$ and $u \in Q_J$ we have that

$$(vg)u = vg^{-1}g = vg,$$

as $Q_J \leq P_J$. Hence $V'$ is a $CP_J$-module.

Consider $V'$ as a $CP_J$-module, having associated character $\phi = \sum_i \phi_i$ (with the $\phi_i$ irreducible $C_{P_J}$-characters). So $V'$ affords $\phi_{P_J} = \sum_i (\phi_i)_{P_J}$ and $V$ affords the character $\chi_{P_J}$. Now $V'$ is a $CP_J$-submodule of $V$, hence each $(\phi_i)_{P_J}$ is a component of $\chi_{P_J}$. Consequently

$$(\phi_{P_J},\chi) = ((\phi_i)_{P_J},\chi_{P_J}) \neq 0,$$

and $\chi$ is a component of $(\phi_i)^G$. Thus it remains to prove that $\phi_i$ is a cuspidal character.

If $\phi_i$ is not cuspidal, then $((\phi_i)_{Q_K},1_{Q_K}) \neq 0$ for some $K \subsetneq J$. It follows that $\dim_C \text{Hom}_{Q_K}(1,V) \neq 0$ and hence $(\chi_{Q_K},1_{Q_K}) \neq 0$. Hence $K \in S$, contradicting the minimality of $J$. Thus the result holds true. □
We illustrate this behaviour with an example.

Example 2.4. $G = \text{Alt}(7)$ has a 2-minimal parabolic system of rank 2 given by $\{P_1, P_2\} \subseteq \mathcal{P}(G, B)$ with $B \cong \text{Dih}(8)$, $P_1 \cong \text{Dih}(3)$ and $P_2 \cong \text{Sym}(4)$. Take $B = \langle (1, 2), (3, 4), (1, 3)(5, 6) \rangle$, $P_1 = \langle (5, 6, 7) \rangle$ and $P_2 = \langle (1, 2, 5)(3, 4, 6) \rangle$. Let $X$ be the $B$-parabolic system given by $\{\langle P_J, Q_J \rangle | J \subseteq \{1, 2\}\}$ where $P_0 = B$. So $Q_0 = B$, $Q_1 = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$, $Q_2 = \langle (1, 3)(5, 6), (2, 4)(5, 6) \rangle$ and $Q_{\{1, 2\}} = 1$. It follows using $[\mathfrak{B}]$ that for $i = 1, 2$ we have

$$\sum_{g \in Q_i} \chi(g) = \chi(1) + 3\chi(2A) \neq 0$$

for any $\chi \in \text{Irr}(G)$. Thus $G$ has no $X$-cuspidal characters.

For $\mathcal{P}_i = P_i/Q_i \cong \text{Sym}(3)$ we have that $\mathcal{Q}_0 \cong C_2$. It follows that there is one $\mathcal{X}_i$-cuspidal character, namely the sign character. We denote this character by $\phi_1$ and also think of it as a $P_i$-character. Using the notation from $[\mathfrak{B}]$, calculations show that the constituent characters of $\phi_1^G$ are $\chi_3, \chi_4, \chi_7, \chi_9$. Meanwhile the constituent characters of $\phi_2^G$ are $\chi_3, \chi_4, \chi_5, \chi_7, \chi_8, \chi_9$.

For $\mathcal{P}_0 = 1$ we note that trivial character will be $\mathcal{X}_0$-cuspidal, and it lifts to the trivial character $1_B$. We have that the constituent characters of $1_B^G$ are $\chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9$. Thus we observe - not very surprisingly - that Proposition 9.1.5 of $[\mathfrak{B}]$ does not extend to our more general situation, since $(\phi_1^G, \phi_2^G) = 5$, whilst $\phi_1^G \neq \phi_2^G$.

Although by definition, to determine whether or not a character is cuspidal we must check the cuspidal condition for every subgroup in the $X$-parabolic system, we shall shortly see that this is not actually necessary. First we give the analogue of $[\mathfrak{B}]$ Proposition 9.1.2.

Proposition 2.5. Let $G$ be a group, $X \leq G$ and $\chi \in \text{Irr}(G)$. If $X$ is an $X$-parabolic system of $G$ of rank $n$ having underlying indexing set $I$, then the following are equivalent:

(i) $\chi$ is a $X$-cuspidal character of $G$.

(ii) $(\chi_{Q_J}, 1_{Q_J}) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(iii) $(\chi, 1^G_J) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(iv) $\sum_{x \in Q_J} \chi(xg) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$ and all $g \in G$.

(v) $\sum_{x \in Q_J} \chi(xg) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$ and all $g \in G$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\chi$ is a $X$-cuspidal character of $G$. Thus for each pair of subgroups $(P_J, Q_J)$ for $J \subseteq I$ we have that either $Q_J = 1$ or

$$\sum_{x \in Q_J} \chi(x) = 0.$$

In particular

$$\sum_{x \in Q_J} \chi(x)1_{Q_J}(x) = 0$$

and hence \((\chi_{Q_J}, 1_{Q_J}) = 0\) for all \(J \subseteq I\) such that \(Q_J \neq 1\).

(ii) \(\Rightarrow\) (iv) Let \(Q_J\) be such that \(Q_J \neq 1\) (if no such \(Q_J\) exists, the result is vacuously true). Let \(\rho\) be an irreducible representation corresponding to \(\chi\), let \(\rho'\) be an irreducible constituent of \(\rho|_{Q_J}\) and let \(d\) denote the degree of \(\rho'\). The module corresponding to \(\rho'\) has basis \(\{e_1, \ldots, e_d\}\) and hence we may define coefficient functions \(\rho'_{ij}\) for \(i, j = 1, \ldots, d\) by

\[
e_i g = \sum_{j=1}^d \rho'_{ij}(g)e_j.
\]

By the orthogonality relations for the coefficient functions (as given in [3, Section 6.1]), it follows that

\[
(\rho'_{ij}, (1_{Q_J}, 1_{Q_J})) = 0
\]

for all \(i, j = 1, \ldots, d\), as \(1_{Q_J}\) is not an irreducible constituent of \(\chi_{Q_J}\). Thus

\[
\sum_{x \in Q_J} \rho'_{ij}(x) = 0
\]

for all \(i, j\) and hence

\[
\sum_{x \in Q_J} \rho'(x) = 0
\]

for all \(i, j\). Since this holds for all irreducible components \(\rho'\) of \(\chi_{Q_J}\) we deduce that

\[
\sum_{x \in Q_J} \rho(x) = 0.
\]

(2)

Now let \(g \in G\) be given. Multiplying (2) on the right by \(\rho(g)\) gives

\[
\sum_{x \in Q_J} \rho(xg) = \left(\sum_{x \in Q_J} \rho(x)\right) \rho(g) = 0.
\]

Consequently, taking traces we obtain

\[
\sum_{x \in Q_J} \chi(xg) = 0.
\]

(iv) \(\Rightarrow\) (i) Taking \(g = 1\) we see that \(\chi\) is a \(\mathfrak{X}\)-cuspidal character of \(G\).

(ii) \(\Rightarrow\) (v) \(\Rightarrow\) (i) This follows analogously by multiplying on the left by \(\rho(g)\) in (2).

(ii) \(\Leftrightarrow\) (iii) This follows by Frobenius reciprocity.

Proposition 2.5 infers that we only have to check that the cuspidal condition holds for certain “maximal” subgroups of a parabolic system to ascertain whether a character is cuspidal.
Corollary 2.6. Let $G$ be a group, $X \leq G$, $I = \{1, \ldots, n\}$ and let $\mathcal{X} = \{(P_J, Q_J)|J \subseteq I\}$ be an $X$-parabolic system of $G$. Define
$$\mathfrak{Y} := \{J \subseteq I|Q_J \neq 1 \text{ and if } J \subseteq K \subseteq I, \text{ then } Q_K = 1\}.$$ Then $\chi \in \text{Irr}(G)$ is cuspidal precisely when
$$\sum_{x \in Q_J} \chi(x) = 0$$
for all $Q_J$ such that $J \in \mathfrak{Y}$.

Proof. The condition is clearly necessary. To see that it is sufficient, let $J' \subseteq I$ be such that $Q_{J'} \neq 1$. We shall show that
$$\sum_{x \in Q_{J'}} \chi(x) = 0. $$
Since $Q_{J'} \neq 1$, we see that $J' \subseteq I$ and there exists some $J \in \mathfrak{Y}$ such that $J' \subseteq J$. Consequently $Q_J \leq Q_{J'}$. By assumption
$$\sum_{x \in Q_J} \chi(x) = 0,$$
and so $(\chi|_{Q_J}, 1_{Q_J}) = 0$. The proof of Proposition 2.5 asserts that
$$\sum_{x \in Q_J} \chi(xg) = 0$$
for all $g \in G$.

Let $T$ denote a right transversal of $Q_J$ in $Q_{J'}$. Then
$$\sum_{x \in Q_{J'}} \chi(x) = \sum_{t \in T} \left( \sum_{x \in Q_J} \chi(xt) \right) = 0$$
as required. \hfill \qed

The final result that we will use in classifying the $p$-cuspidal characters of the sporadic simple groups concerns irreducible characters of odd degree.

Lemma 2.7. Let $G$ be a finite group and $p$ an odd prime such that $|G| = p^a m$ for some $a \geq 1$ with $(p, m) = 1$. Assume that $G$ has a $p$-minimal parabolic system containing a parabolic subgroup with non-trivial $p$-core. If for each $G$-conjugacy class, $C$, of elements of order $p^b$ for $b \leq a$ and all $g \in C$ we have that
$$\langle g \rangle \cap \{g \in G \mid |g| = p^b\} \subseteq C,$$
then every $p$-cuspidal character of $G$ has even degree.

Proof. Assume that condition (3) holds for all non-trivial powers of $p$. Then a non-trivial $p$-core, $Q$, of a parabolic subgroup will intersect every conjugacy class of $p$-elements in a set of even order. Thus if the degree of $\chi \in \text{Irr}(G)$ is odd, then the same is true of
$$\sum_{g \in Q} \chi(g),$$
and hence $\chi$ is not a $p$-cuspidal character of $G$. \hfill \qed
We now work systematically through the sporadic simple groups, determining for each group \( G \) and each 2-minimal parabolic system of \( G \), which characters \( \chi \in \text{Irr}(G) \) are 2-cuspidal. A summary of our results is given in Tables 1 and 2. Throughout, the notation \( \chi_i \in \text{Irr}(G) \) is the same as that used in [4]. We shall also use the standard notation from [4] for the conjugacy classes of \( G \).

### 3.1 The Mathieu Groups

**\( M_{11} \)**

There are three 2-minimal parabolic subgroups of \( M_{11} \), namely

\[
P_1 \sim 2^{1+2} \cdot \text{Sym}(3), \quad P_2 \sim 3^2 \cdot \text{SD}_{16}, \quad \text{and} \quad P_3 \sim \text{Alt}(6) : 2,
\]

(where \( \text{SD}_{16} \) is the semidihedral group of order 16) and these give rise to three 2-minimal parabolic systems, each of rank 2. Since \( O_2(P_2) = O_2(P_3) = 1 \), we must consider a Sylow 2-subgroup of \( M_{11} \). Such a subgroup will intersect the \( M_{11} \)-conjugacy classes 1, 2, 4, 8 and 8 in 1, 5, 6, 2 and 2 elements respectively. It follows that the cuspidal relation for a Sylow 2-subgroup holds for \( \chi_3, \chi_4 \in \text{Irr}(M_{11}) \) (both of degree 10). Consequently, \( \chi_3 \) and \( \chi_4 \) are 2-cuspidal characters of the minimal parabolic system \( \{P_2, P_3\} \). Finally, as \( O_2(P_1) \) contains 1, 1 and 6 elements from the classes 1, 2 and 4 respectively and \( \chi_i(1A) + 12 \cdot \chi_i(2A) + 7 \cdot \chi_i(4A) + 6 \cdot \chi_i(4B) = 64 \), for \( i = 3, 4 \), we see that the minimal parabolic systems containing \( P_1 \) admit no 2-cuspidal characters.

**\( M_{12} \)**

There are no 2-cuspidal characters for the unique 2-minimal parabolic system of \( M_{12} \) given by

\[
\{ P_1 \sim 4^2 \cdot \text{Sym}(3), P_2 \sim 2^{1+4} \cdot \text{Sym}(3) \}.
\]

To see this, we observe that \( O_2(P_1) \) intersects the \( M_{12} \)-conjugacy classes 1, 2, 2, 4 and 4 in 1, 4, 15, 6 and 6 elements respectively, whilst \( O_2(P_2) \) intersects these classes in 1, 12, 7, 6 and 6 elements respectively. Consequently the only character satisfying the cuspidal relation for \( O_2(P_1) \) is \( \chi_{13} \) (of degree 120). However

\[
\chi_{13}(1A) + 12 \cdot \chi_{13}(2A) + 7 \cdot \chi_{13}(2B) + 6 \cdot \chi_{13}(4A) + 6 \cdot \chi_{13}(4B) = 64.
\]

**\( M_{22} \)**

There is a unique 2-minimal parabolic system for \( M_{22} \), namely

\[
\{ P_1 \sim 2^{4+2} \cdot \text{Sym}(3), P_2 \sim 2^4 \cdot \text{Sym}(5) \}.
\]
The 2-minimal parabolic systems are given by those subgroups which feature in 2-minimal parabolic systems of $M_{24}$. For each sub-maximal parabolic, the only irreducible characters of degree 45, $\chi$, are the two characters of degree 45, $\chi_3$ and $\chi_4$.

$M_{23}$

The group $M_{23}$ has seven conjugacy classes of 2-minimal parabolic subgroups, six of which feature in 2-minimal parabolic systems of $M_{23}$. Using the notation of [14], these subgroups are

$$P_1 \sim 2^{4+2}, \text{Sym}(3), \quad P_5 \sim 2^{4+2}, \text{Sym}(3), \quad P_3 \sim 2^{4+2}, \text{Sym}(3)$$

The 2-minimal parabolic systems are given by $\{P_1, P_3, P_7\}, \{P_3, P_4, P_7\}, \{P_2, P_3, P_7\}, \{P_1, P_6, P_7\}, \{P_2, P_6, P_7\}, \{P_3, P_6, P_7\}$ and $\{P_4, P_6, P_7\}$.

Considering the maximal 2-parabolic subgroups of these systems, we see that the maximal 2-parabolic subgroups involving $P_6$ and $P_7$ have trivial 2-cores. Thus we need to check sub-maximal parabolics in order to apply Corollary 2.6.

The 2-cores $O_2(P_i)$ and $O_2(P_7)$ intersect the $M_{22}$-conjugacy classes 1A, 2A, 4A and 4B in 1, 27, 12, 24 and 1, 15, 0 and 0 elements respectively. The only elements $\chi \in \text{Irr}(M_{22})$ satisfying

$$\chi(1A) + 27 \cdot \chi(2A) + 12 \cdot \chi(4A) + 24 \cdot \chi(4B) = \chi(1A) + 15 \cdot \chi(2A) = 0$$

are the two characters of degree 45, $\chi_3$ and $\chi_4$.

$M_{24}$

The Mathieu group $M_{24}$ has a unique 2-minimal parabolic system given by

$$\{P_1 \sim 2^{6+3}, \text{Sym}(3), P_2 \sim 2^{6+3}, \text{Sym}(3), P_3 \sim 2^{6+3}, \text{Sym}(3)\}.$$

The maximal parabolic subgroups $P_{12}, P_{13}$ and $P_{23}$ all have non-trivial 2-cores, and their intersections with the $M_{24}$-conjugacy classes are summarised in Table 4. It follows that the characters $\chi_3$, $\chi_4$, $\chi_{12}$, $\chi_{13}$, $\chi_{15}$ and $\chi_{16}$ satisfy the cuspidal relation for $P_{12}$, as do $\chi_3$, $\chi_4$, $\chi_5$, $\chi_6$, $\chi_{12}$, $\chi_{13}$, $\chi_{15}$ and $\chi_{16}$ for $P_{13}$ and $\chi_3$, $\chi_4$, $\chi_5$, $\chi_6$ and $\chi_8$ for $P_{23}$. We conclude that $\chi_3$ and $\chi_4$ - both of degree 45 - are the only 2-cuspidal characters of $M_{24}$. 

<table>
<thead>
<tr>
<th>Parabolic Subgroup</th>
<th>2-core</th>
<th>Order of intersection with $M_{24}$-class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{12}$</td>
<td>$2^9$</td>
<td>1 45 18 0 0</td>
</tr>
<tr>
<td>$P_{13}$</td>
<td>$2^{6+2}$</td>
<td>1 57 54 72 72</td>
</tr>
<tr>
<td>$P_{23}$</td>
<td>$2^{4+3}$</td>
<td>1 29 42 56 0</td>
</tr>
</tbody>
</table>

Table 4: The 2-cores of maximal parabolic subgroups of $M_{24}$. 
The intersections of the 2-cores of the maximal 2-parabolic subgroups of $Co_1$ with the $Co_1$-conjugacy classes.

### Table 5: The intersections of the 2-cores of the maximal 2-parabolic subgroups of $Co_1$ with the $Co_1$-conjugacy classes.

<table>
<thead>
<tr>
<th>Conjugacy Class</th>
<th>1A</th>
<th>2A</th>
<th>2B</th>
<th>2C</th>
<th>4A</th>
<th>4B</th>
<th>4C</th>
<th>4D</th>
<th>4E</th>
<th>4F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{123}$</td>
<td>1</td>
<td>1095</td>
<td>1344</td>
<td>4984</td>
<td>336</td>
<td>22512</td>
<td>18816</td>
<td>38976</td>
<td>0</td>
<td>43008</td>
</tr>
<tr>
<td>$P_{124}$</td>
<td>1</td>
<td>1095</td>
<td>576</td>
<td>6264</td>
<td>720</td>
<td>13680</td>
<td>17280</td>
<td>25920</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_{134}$</td>
<td>1</td>
<td>759</td>
<td>0</td>
<td>1288</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_{234}$</td>
<td>1</td>
<td>551</td>
<td>0</td>
<td>2520</td>
<td>240</td>
<td>15120</td>
<td>896</td>
<td>13440</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

#### 3.2 The Leech Lattice and Conway Groups

**HS**

The Higman-Sims group has a unique 2-minimal parabolic system of the form

$$\{ P_1 \sim 4.2^4.\text{Sym}(5), P_2 \sim 4^3.2^2.\text{Sym}(3) \}.$$

Considering $O_2(P_1)$, we see that it intersects the $HS$-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$ and $4C$ in $1$, $31$, $0$, $2$, $30$ and $0$ elements respectively. Consequently, there are no 2-cuspidal characters of $HS$, as the cuspidal relation does not hold for $P_1$.

**J2**

There is a unique 2-minimal parabolic system of $J_2$ given by

$$\{ P_1 \sim 2^{2+4}.3.\text{Sym}(3), P_2 \sim 2^{1+4}.L_2(4) \}.$$

The intersections of $O_2(P_1)$ with the $J_2$-classes $1A$, $2A$, $2B$ and $4A$ have orders $1$, $3$, $24$ and $36$ respectively. Consequently, the cuspidal relation on $P_1$ holds for the irreducible characters $\chi_4$, $\chi_5$, $\chi_{14}$ and $\chi_{15}$. Meanwhile, the 2-core $O_2(P_2)$ intersects the given $J_2$-classes in $1$, $11$, $0$ and $20$ elements respectively, meaning that the cuspidal relation holds on $P_2$ for the characters $\chi_8$, $\chi_9$ and $\chi_{18}$. We conclude that $J_2$ admits no 2-cuspidal characters.

**Co1**

The largest Conway group, $Co_1$, admits a unique 2-minimal parabolic system, having rank 4. Its minimal parabolic subgroups are given by $P_i \sim [2^{20_i}].\text{Sym}(3)$ for $i = 1, \ldots, 4$ and the corresponding 2-maximal parabolic subgroups have the form $P_{123} \sim 2^{2+12+3}.(\text{Sym}(3) \times L_3(2))$, $P_{124} \sim 2^{4+12}.(\text{Sym}(3) \times 3.\text{Sp}_4(2))$, $P_{134} \sim 2^{11}.M_{24}$ and $P_{234} \sim 2^{1+8+6}.L_4(2)$. The orders of the intersections of the 2-cores of the maximal parabolic subgroups with the $Co_1$-conjugacy classes are given in Table 5. A summary of the elements of $\text{Irr}(Co_1)$ which satisfy the cuspidal relation for each of the maximal parabolics is given in Table 6. We conclude that $Co_1$ admits three 2-cuspidal characters, namely $\chi_2$, $\chi_8$ and $\chi_{11}$.

**Co2**

The group $Co_2$ has a 2-minimal parabolic system of the form $\{ P_1, P_2, P_3 \}$, where $P_1 \sim [2^5].\text{Sym}(5)$ and $P_i \sim [2^{17_i}].\text{Sym}(3)$ for $i = 2, 3$. This system has maximal
Parabolic Subgroup | Characters satisfying the cuspidal relation (character degrees)
---|---
$P_{23}$ | $\chi_2 (276), \chi_4 (1771), \chi_5 (8855), \chi_8 (37674), \chi_{11} (94875), \chi_{13} (345345), \chi_{15} (483000), \chi_{21} (1434510), \chi_{23} (1771000), \chi_{27} (2464749), \chi_{29} (2464749)$
$P_{24}$ | $\chi_2 (276), \chi_8 (37674), \chi_{11} (94875), \chi_{21} (1434510), \chi_{27} (2464749), \chi_{29} (2464749)$
$P_{34}$ | $\chi_2 (276), \chi_4 (1771), \chi_5 (8855), \chi_8 (37674), \chi_{11} (94875), \chi_{13} (345345), \chi_{15} (483000), \chi_{21} (1434510), \chi_{23} (1771000), \chi_{27} (2464749), \chi_{29} (2464749)$
$P_{234}$ | $\chi_2 (276), \chi_4 (1771), \chi_8 (37674), \chi_9 (44275), \chi_{11} (94875), \chi_{13} (345345)$

Table 6: The elements of $\text{Irr}(Co_1)$ satisfying the cuspidal relation for each maximal 2-parabolic subgroup of $Co_1$.

<table>
<thead>
<tr>
<th>2-core</th>
<th>Order of intersection with $Co_2$-conjugacy class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_2(P_{12})$</td>
<td>1 125 490 2328 240 1440 2400 1680 1920 5760 0 0</td>
</tr>
<tr>
<td>$O_2(P_{13})$</td>
<td>1 77 330 616 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>$O_2(P_{23})$</td>
<td>1 141 634 1848 240 3808 2464 336 8064 8064 0</td>
</tr>
</tbody>
</table>

Table 7: The 2-cores of the maximal parabolic subgroups of $Co_2$.

parabolic subgroups $P_{12} \sim 2^{4+10} \cdot \text{Sym}(3) \times \text{Sym}(5)$, $P_{13} \sim 2^{10} \cdot M_{22}, 2$ and $P_{23} \sim 2^{1+8+6} \cdot L_3(2)$. The orders of the intersections of the 2-cores of these maximal parabolics with the relevant $Co_2$-conjugacy classes are given in Table 7.

It follows that the cuspidal relation holds on $P_{12}$ for the characters $\chi_2, \chi_3, \chi_10, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{21}, \chi_{22}, \chi_{23}, \chi_{31}, \chi_{32}$ and $\chi_{37}$, it holds on $P_{13}$ for $\chi_3, \chi_5, \chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{25}, \chi_{31}$ and $\chi_{32}$, and the cuspidal relation holds on $P_{23}$ for the irreducible characters $\chi_2, \chi_3, \chi_5, \chi_7, \chi_8, \chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{20}, \chi_{21}, \chi_{22}, \chi_{23}, \chi_{25}, \chi_{31}, \chi_{32}, \chi_{36}, \chi_{37}$ and $\chi_{40}$. Consequently, $Co_2$ admits eight 2-cuspidal characters namely $\chi_3$ (of degree 253), $\chi_{10}$ (9625), $\chi_{11}$ (9625), $\chi_{12}$ (10395), $\chi_{13}$ (10395), $\chi_{16}$ (31625), $\chi_{31}$ (239085) and $\chi_{32}$ (239085).

$Co_3$

There is a unique 2-minimal parabolic system of $Co_3$ given by

$$\{ P_1 \sim 2^{4+4+1} \cdot \text{Sym}(3), P_2 \sim 2^{4+4+1} \cdot \text{Sym}(3), P_3 \sim 2^{4+4+1} \cdot \text{Sym}(3) \},$$

which has maximal parabolic subgroups $P_{12} \sim 2^{2+6} \cdot (\text{Sym}(3) \times \text{Sym}(3))$, $P_{13} \sim 4^3.2.L_3(2)$ and $P_{23} \sim 4^3.2.\text{Sp}_4(2)$. We have that $O_2(P_{23})$ intersects the $Co_3$-conjugacy classes $1A, 2A, 2B, 4A$ and $4B$ in 1, 31, 0, 2 and 30 elements respectively, and hence the cuspidal condition does not hold for $P_{23}$. Thus $Co_3$ admits no 2-cuspidal characters.

$M^\sigma L$

The McLaughlin group has a multitude of 2-minimal parabolic systems comprising of the 2-minimal parabolic subgroups $P_i$ and $P_i^\sigma$ for $i = 1, \ldots, 5$, where $\sigma$ is a non-trivial outer automorphism of $M^\sigma L$ of order 2. Here $P_3^\sigma = P_3, P_4^\sigma = P_4,$
rise to the minimal parabolic systems

\[ \{ P_i, P_5, P_5^5 \}, \{ P_5^2, P_5, P_5^7 \}, \{ P_5^2, P_5, P_5^7 \}, \{ P_5^2, P_5, P_5^7 \}. \]

There is a unique 2-minimal parabolic system of \( HN \) in \([8, \text{Table IV}]\), and we adopt the notation given in \([8]\) for the 2-minimal parabolic subgroups satisfying

\[ \{ P_1, P_5, P_5^7 \}, \{ P_1^2, P_2, P_5 \}, \{ P_1, P_3, P_5^2 \}, \{ P_1, P_3, P_5^5 \}, \{ P_1, P_3, P_5^{25} \}, \{ P_1, P_1^2, P_2 \}, \{ P_1, P_1^2, P_2^3 \}, \{ P_1, P_1^2, P_3 \}. \]

Since the 2-cores of \( P_5 \) and \( P_5^5 \) are elementary abelian of rank 4, we see that any minimal parabolic system containing either of these minimal parabolics will not admit a 2-cuspidal character. Conversely, any 2-minimal parabolic system not containing these subgroups will contain the parabolic subgroup \( P_{11^r} := (P_1, P_1^r) \). Since \( O_2(P_{11^r}) \) intersects the \( M^e L \)-conjugacy classes 1A, 2A and 4A in 1, 19 and 12 elements respectively, we see that the cuspidal relation does not hold for \( P_{11^r} \) and hence none of the 2-minimal parabolic systems of \( M^e L \) admit a 2-cuspidal character.

\( Suz \)

The group \( Suz \) has a unique 2-minimal parabolic system, which has rank 3. Its minimal parabolic subgroups satisfy \( P_1 \sim 2^{4+6+1}.L_2(4) \), \( P_2 \sim 2^{4+6+2}.(3 \times L_2(2)) \) and \( P_3 \sim 2^{6+4+2}.(3 \times L_2(2)) \). The maximal parabolic subgroups are given by \( P_{12} \sim 2^{1+6}.U(2) \), \( P_{13} \sim 2^{1+8}.(\text{Sym}(3) \times L_2(4)) \) and \( P_{23} \sim 2^{1+6}.3.3p_4(2)' \). The 2-core \( O_2(P_{12}) \) intersects the \( Suz \)-conjugacy classes 1A, 2A and 4A in 1, 55 and 72 elements respectively. We deduce that there are no 2-cuspidal characters of \( Suz \).

### 3.3 The Monster Group and its Subquotients

\( He \)

There are four 2-minimal parabolic subgroups of \( He \) given by \( P_1 \cong P_4 \sim 2^{6+3}.\text{Sym}(3) \) and \( P_2 \cong P_5 \sim 2^{6+3}.\text{Sym}(3) \). These give rise to the 2-minimal parabolic systems \( \{ P_1, P_2, P_4 \} \) and \( \{ P_1, P_3, P_4 \} \). Considering the maximal parabolic subgroups \( P_{14} \) and \( P_{13} \cong P_{24} \) we see that \( O_2(P_{14}) \) intersects the \( He \)-conjugacy classes 1A, 2A, 2B, 4A, 4B and 4C in 1, 18, 45, 0, 0 and 0 elements respectively, whilst \( O_2(P_{13}) \) intersects the respective conjugacy classes in 1, 42, 29, 0, 56 and 0 elements. It follows that the cuspidal relation for \( P_{14} \) holds for the characters \( \chi_7, \chi_8 \in \text{Irr}(He) \), whilst for \( P_{13} \cong P_{24} \) the cuspidal relation holds for \( \chi_4, \chi_5 \in \text{Irr}(He) \). Since each 2-minimal parabolic system contains \( P_{14} \) and either \( P_{13} \) or \( P_{24} \), we conclude that there are no 2-cuspidal characters of \( He \).

\( HN \)

There is a unique 2-minimal parabolic system of \( HN \) given by

\[ \{ P_1 \sim 2^{1+8}.\text{Alt}(5) \wr \mathbb{Z}_2, P_2 \sim 2^{2+3+6+2}.3.\text{Sym}(3) \}. \]

We consider the minimal parabolic subgroup \( P_1 \), whose character table is given in \([8, \text{Table IV}]\), and we adopt the notation given in \([8]\) for the \( P_1 \)-conjugacy
classes. We have that each $P_i$-class is either contained in, or is disjoint from $O_2(P_i)$. It follows that

$$O_2(P_i) = 1_1 \cup 2_1 \cup 2_2 \cup 2_3 \cup 4_1.$$  

Considering the centralizer orders of $1_1$, $2_1$, $2_2$, $2_3$ and $4_1$ in $P_i$ and the orders of the centralizers of 2-elements in $HN$, we see that the $P_i$-classes $2_1$ and $2_2$ are contained in the $HN$-class $2B$, the $P_i$-class $2_3$ lies in either $HN$-class $2A$ or $2B$, and the $P_i$-class $4_1$ is contained in the $HN$-class $4A$. It follows that $|O_2(P_i) \cap 2A| = 0$ or 120, $|O_2(P_i) \cap 2B| = 151$ or 271 and $|O_2(P_i) \cap 4A| = 240$. It follows that the cuspidal relation on $P_i$ does not hold for any $\chi \in \text{Irr}(HN)$, and hence there are no 2-cuspidal characters of $HN$.

$Th$

The 2-minimal parabolic system

$$\{P_1 \sim 2^{1+8}, \text{Alt}(9), P_2 \sim 2^{5+6+2+1}.\text{Sym}(3)\}$$

of $Th$ is unique. Considering fusion within the maximal subgroup $2^5\cdot L_5(2) > P_2$ we find that $O_2(P_2)$ intersects the $Th$-conjugacy classes 1A, 2A, 4A, 4B, 8A and 8B in 1, 687, 656, 7104, 4864 and 3072 elements respectively. It follows that the cuspidal relation holds on $P_2$ for $\chi_2, \chi_6 \in \text{Irr}(Th)$. Considering the normal subgroups of a Sylow 2-subgroup of $2^5\cdot L_5(2)$ having order $2^9$ and exponent 4, we see that for each such subgroup the only element of $\text{Irr}(Th)$ for which the cuspidal relation holds is $\chi_2$ of degree 248. Thus $\chi_2$ is the unique 2-cuspidal character of $Th$.

$Fi_{22}$

There is a unique 2-minimal parabolic system of $Fi_{22}$ given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{16}].\text{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{14}].\text{Sym}(5)$. This system has maximal parabolic subgroups $P_{12} \sim 2^{9+4+2}.(\text{Sym}(3) \times \text{Sym}(3))$, $P_{13} \sim 2^{4+8}.U_4(2).2$ and $P_{23} \sim 2^{10}.M_{22}$. The 2-core $O_2(P_{13})$ intersects the $Fi_{22}$-conjugacy classes 1A, 2A, 2B, 2C, 4A, 4B, 4C, 4D and $4E$ in 1, 2, 271, 270, 480, 0, 0, 0 and 0 elements respectively. We see that the cuspidal relation does not hold on $P_{13}$ and hence $Fi_{22}$ admits no 2-cuspidal characters.

$Fi_{23}$

The group $Fi_{23}$ has eight 2-minimal parabolic subgroups, seven of which feature in 2-minimal parabolic systems.Using the notation of [14] these have the form $P_i \sim [2^{17}].\text{Sym}(3)$ for $i = 1, \ldots, 5$ and $P_i \sim [2^{15}].\text{Sym}(5)$ for $i = 7, 8$. These give rise to the geometric 2-minimal parabolic systems $\{P_1, P_3, P_5, P_8\}$ and $\{P_1, P_4, P_5, P_8\}$ and the non-geometric systems $\{P_1, P_2, P_3, P_8\}$ and $\{P_1, P_7, P_8\}$.

The maximal parabolic subgroups of these systems are

$$P_{325} \sim 2^{10+4}.(\text{Sym}(3) \times \text{Alt}(7)), \quad P_{128} = P_{129} = P_{148} \sim 2^2 \times 2^{1+8}.(3 \times U_4(2)).2,$$
$$P_{135} \sim [2^{14}].(\text{Sym}(3) \times L_3(2)), \quad P_{345} \sim [2^{14}].(\text{Sym}(3) \times L_3(2)),$$
$$P_{158} \sim 2.Fi_{22}, \quad P_{17} \sim [2^{14}].(\text{Sym}(3) \times \text{Sym}(5)),$$
$$P_{18} \sim [2^{11}].U_4(2).2, \quad P_{258} = P_{368} = P_{258} \sim 2^{11}.M_{23},$$
$$P_{28} \sim 2^{11}.M_{21}.2.$$
It is easy to check that the cuspidal relation does not hold for $P_{18}$. and hence the three 2-minimal parabolic systems of rank 4 do not admit any 2-cuspidal characters.

Finally, we consider the maximal parabolic subgroup $P_{18} < P_{158}$. We see that the 2-core $O_2(P_{18})$ intersects the $Fi_{24}$-conjugacy classes $1A, 2A, 2B, 2C, 4A, 4B, 4C$ and $4D$ in 1, 3, 273, 811, 0, 960, 0 and 0 elements respectively. Consequently, the cuspidal relation does not hold for $P_{18}$, and hence there are no 2-cuspidal characters of $Fi_{24}$.

$Fi_4$

There is a unique 2-minimal parabolic system of $Fi_4$, which has rank 4. The maximal parabolic subgroups are $P_a \sim 2^{1+12}.(3.U_4(3).2)$, $P_b \sim 2^{3+12+2}.(\text{Sym}(3) \times Sp_4(2^4))$, $P_c \sim 2^{8+6+3}.(L_3(2) \times \text{Sym}(3))$ and $P_d \sim 2^{11}.M_{24}$. Since $O_2(P_d)$ is elementary abelian, we consider the minimum value that each $\chi \in \text{Irr}(Fi_4)$ takes on elements of order 2. We immediately deduce that the only possible 2-cuspidal character of $Fi_4$ is $\chi_2$ of degree 8671. For $\chi_2$ to be 2-cuspidal, we would require an integer solution to

$$\chi_2(1A) + j \cdot \chi_2(2A) + (2^{11} - j - 1) \cdot \chi_2(2B) = 0.$$ 

Since no such solution exists, we conclude that there are no 2-cuspidal characters of $Fi_4$.

$B$

The baby monster has five conjugacy classes of 2-minimal parabolic subgroups having representatives $P_i \sim [2^{4i}], \text{Sym}(3)$ for $i = 1, \ldots, 4$ and $P_5 \sim [2^{38}], \text{Sym}(5)$. These give rise to a unique 2-minimal parabolic system $\{P_1, P_2, P_3, P_4\}$. The maximal parabolic subgroups of this system are given by $P_{123} \sim 2^{9+16+6+4}.L_4(2)$, $P_{125} \sim 2^{3+32}.(L_3(2) \times \text{Sym}(5))$, $P_{135} \sim 2^{2+10+20}.(\text{Sym}(3) \times M_{24})$ and $P_{235} \sim 2^{1+22}.C_{O_2}$. All of these maximal parabolic subgroups are 2-radical. Indeed, from [20] we observe that all 2-parabolic subgroups generated by $P_1, \ldots, P_5$ are 2-radical with the exception of $P_3$, $P_4$ and $P_{34}$. The reader can find further information regarding the structure of the 2-radical parabolic subgroups in [20].

The fusion of elements within the 2-cores of $P_{125}$, $P_{135}$ and $P_{235}$ is given in [15]. We see that there are no characters of $B$ satisfying the cuspidal condition for the 2-core $O_2(P_{335})$ and hence the baby monster admits no 2-cuspidal characters.

$M$

The monster group has a unique 2-minimal parabolic system, $\{P_1, P_2, P_3, P_4, P_5\}$, where $P_i \sim [2^{4i}].L_2(2)$ for $i = 1, \ldots, 5$. The maximal parabolic subgroups are given by $P_{1234} \sim 2^{3+5+16+10}.L_5(2)$, $P_{1235} \sim 2^{4+1+2+8+8+12+4}.(L_4(2) \times \text{Sym}(3))$, $P_{1245} \sim 2^{3+36}.(L_3(2) \times 3.Sp_4(2))$, $P_{1345} \sim 2^{2+11+22}.(\text{Sym}(3) \times M_{24})$ and $P_{2345} \sim 2^{1+24}.C_{O_1}$.

We observe that there is no $\chi \in \text{Irr}(M)$ that satisfies the cuspidal relation for $O_2(P_{2345})$. Indeed, let $z$ be an involution of $M$ in class $2B$ and let $\Lambda$ be the Leech lattice as defined in [1]. Moreover, let $\Lambda_i$ be the set of all vectors in $\Lambda$ of
type $i$ defined as

$$\Lambda_i := \{v \in \Lambda | (v, v)/16 = i \}.$$ 

Then calculations in \cite{1} show that

- $|\Lambda_2| = 196,560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$,
- $|\Lambda_3| = 2^{12}(2^{12} - 1) = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, and
- $|\Lambda_4| = 398,034,000 = 2^4 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13$.

Let $G := G/\langle \varepsilon_X \rangle$ (where $\varepsilon_X$ is the scalar map defined on $\Lambda$ by $-1$). Thus $\widetilde{G}$ is equal to $CO_1$. Since $\varepsilon_X$ acts trivially on $\Lambda := \Lambda/2\Lambda$, and $G$ acts transitively on the set $\Lambda$ (Lemma 22.12(1)) and on each of the sets $\Lambda_2, \Lambda_3, \Lambda_4$ (Lemma 22.14(1)), it follows that $\widetilde{G}$ acts transitively on the sets $\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$ (where $\Lambda_i$ is the image of $\Lambda_i$ in $\Lambda$).

Without loss, we may assume that $z$ is the central involution of the extraspecial group $O_2(2345)$ and we define $P_{2345} := P/\langle z \rangle$. By considering the action of $G$ on $\Lambda$ we have that

$$\begin{align*}
|\text{Stab}_G(\lambda_2)| & = 2^{17} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_G(\lambda_3)| & = 2^{9} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_G(\lambda_4)| & = 2^{17} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23
\end{align*}$$

(4)

(where $\lambda_i \in \Lambda_i$). It follows that

$$\begin{align*}
|\text{Stab}_{P_{2345}}(\lambda_2)| & = 2^{41} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_{P_{2345}}(\lambda_3)| & = 2^{43} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_{P_{2345}}(\lambda_4)| & = 2^{41} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.
\end{align*}$$

The question remains, how does the element $\lambda_i$ lift to the extraspecial group $2^{1+24} = O_2(2345)$? Once this is established, we may then use the centralizer orders

$$\begin{align*}
|C_M(2A)| & = 2^{42} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47, \\
|C_M(2B)| & = 2^{46} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23, \\
|C_M(4A)| & = 2^{34} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|C_M(4B)| & = 2^{27} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17, \\
|C_M(4C)| & = 2^{24} \cdot 3^4 \cdot 5^7, \text{ and} \\
|C_M(4D)| & = 2^{27} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13
\end{align*}$$

(5)

to determine the fusion within $O_2(2345)$. Indeed, we have that $C_{P_{2345}}(\lambda_2) \sim 2^{24}.CO_2$, $C_{P_{2345}}(\lambda_3) \sim 2^{24}.CO_3$ and $C_{P_{2345}}(\lambda_4) \sim 2^{24}.(2^{11} : M_{24})$.

There are two possible ways in which a $\lambda_i$ can lift into $2^{1+24}$, namely to an abelian subgroup of order 4 of the form $(\lambda_i, z)$ having exponent 2 or 4. The former case occurs when $|\Lambda_i| = 2$, whilst the latter case occurs when $|\Lambda_i| = 4$. 

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and hence $\lambda_3^2 = z$. Since $(4A)^3 = 4A$, $(4B)^3 = 4B$, $(4C)^3 = 4C$ and $(4D)^3 = 4D$, we may use the centralizer and stabilizer orders from [5] and [4] to see that the only possible elements of order 4 in $P_{2345}$ must lie in the $M$-conjugacy class 4A. Since the exponent of $2^{1+24}$ is 4, we conclude that the elements of the orbit $\lambda_3^G$ lift to cyclic groups of order 4 containing 1, $z$ and two elements from the $M$-class 4A. This means that

$$|O_2(P_{2345}) \cap 4A| = 2 \cdot |\lambda_3| = 2^{12}(2^{12} - 1) = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 16,773,120.$$ 

Next we consider the lifts of $\lambda_2$ and $\lambda_4$. We see that these must lift to the elementary abelian subgroups $\langle \lambda_2, z \rangle$ and $\langle \lambda_4, z \rangle$ respectively. Since there are only two $M$-classes of involutions, we have that $(\lambda_2 z)^{g_2} = \lambda_2$ and $(\lambda_4 z)^{g_4} = \lambda_4$ for some $g_2, g_4 \in M$. To determine which element lifts to class 2A and which element lifts to 2B, we note that by [12, Lemma 4.4] for $x \neq z$ an involution of $P_{2345}$, either $C_{P_{2345}}(x) \approx 2^{1+23}.Co_2$ (if $x$ is not 2-central) or $C_{P_{2345}}(x) \approx 2^{1+23}.(2^{11} : M_{24})$ if $x$ is 2-central. Here

$$|2^{1+23}.Co_2| = 2^{42} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$$

$$|2^{1+23}.(2^{11} : M_{24})| = 2^{45} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$ 

Combining [6] with [4] and the fact that the 2-central elements of $P_{2345}$ lie in $2B$, we have that $\lambda_2$ lifts to an elementary abelian subgroup generated by $z$ and an involution of the $M$-class 2A, whilst $\lambda_4$ lifts to a subgroup generated by $z$ and an element of $2B$. Since $z$ also lies in $2B$, we conclude that

$$|O_2(P_{2345}) \cap 2A| = 2 \cdot |\lambda_2| = |A_2| = 196,560,$$

$$|O_2(P_{2345}) \cap 2B| = 2 \cdot |\lambda_4| + 1 = |A_4|/24 + 1 = 16,584,751 \quad \text{and}$$

$$|O_2(P_{2345}) \cap 4A| = 2 \cdot |\lambda_3| = |A_3| = 16,773,120.$$ 

As there is no $\chi \in \text{Irr}(M)$ satisfying

$$\chi(1A) + 196560 \cdot \chi(2A) + 16584751 \cdot \chi(2B) + 16773120 \cdot \chi(4A) = 0,$$

it follows that there are no 2-cuspidal characters of $M$.

### 3.4 The Pariahs

$J_1$

The normalizer of a Sylow 2-subgroup of $J_1$ is maximal. Thus as the cuspidal relation does not hold for such a Sylow subgroup, $J_1$ has no 2-cuspidal characters.

$O'N$

The group $O'N$ admits a unique 2-minimal parabolic system of the form

$$\{P_1 \sim 4^3.2^2.\text{Sym}(3), P_2 \sim 4.L_3(4).2\}.$$ 

The generators of $O_2(P_2)$ are elements of the $O'N$-conjugacy class 4A. Thus we see that the cuspidal relation does not hold on $P_2$ for any $\chi \in \text{Irr}(O'N)$. Hence $O'N$ admits no 2-cuspidal characters.
Table 8: The intersections of the 2-cores of the maximal parabolic subgroups of $J_4$ with the $J_4$-conjugacy classes.

<table>
<thead>
<tr>
<th>$J_4$-conjugacy class, $C'$</th>
<th>$1A$</th>
<th>$2A$</th>
<th>$2B$</th>
<th>$4A$</th>
<th>$4B$</th>
<th>$4C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_2(P_{12}) \cap C$</td>
<td>1</td>
<td>3579</td>
<td>4868</td>
<td>22848</td>
<td>100800</td>
<td>72704</td>
</tr>
<tr>
<td>$O_2(P_{13}) \cap C$</td>
<td>1</td>
<td>3067</td>
<td>5892</td>
<td>21312</td>
<td>54720</td>
<td>46080</td>
</tr>
<tr>
<td>$O_2(P_{23}) \cap C$</td>
<td>1</td>
<td>1387</td>
<td>2772</td>
<td>4032</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$J_3$

There is a unique 2-minimal parabolic system of $J_3$, given by

$$\{P_1 \sim 2^{2+4}, (3 \times \text{Sym}(3)), P_2 \sim 2^{1+4}, L_2(4)\}.$$ 

Considering the 2-core $O_2(P_2)$, it contains 1, 11 and 20 elements from the $J_3$-conjugacy classes $1A, 2A$ and $4A$ respectively. It follows that $J_3$ does not admit any 2-cuspidal characters.

$Ru$

There are three 2-minimal parabolic subgroups of $Ru$ given by $P_1 \sim 2^{5+6}, \text{Sym}(5)$ and $P_i \sim 2^{5+6+2}, \text{Sym}(3)$ for $i = 2, 3$. Since $P_3 \leq P_1$, we obtain a unique 2-minimal parabolic system, namely $\{P_1, P_2\}$. Considering the 2-cores of $P_1$ and $P_2$, we see that $O_2(P_i)$ intersects the $Ru$-conjugacy classes $1A, 2A, 2B, 4A, 4B, 4C, 4D, 8A, 8B$ and $8C$ in 1, 271, 512, 64, 240, 960, 0, 0 and 0 respectively, whilst $O_2(P_2)$ intersects the given $Ru$-classes in respectively 1, 367, 192, 608, 448, 1296, 1440, 1536, 768 and 1536 elements. It follows that the cuspidal relation holds on $P_1$ for $\chi_2, \chi_3 \in \text{Irr}(Ru)$ and on $P_2$ for $\chi_2, \chi_3, \chi_4 \in \text{Irr}(Ru)$. We conclude that the two characters $\chi_2$ and $\chi_3$ of degree 378 are the only 2-cuspidal characters of $Ru$.

$J_4$

There is a unique 2-minimal parabolic system of $J_4$ given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{20}], \text{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{18}], \text{Sym}(5)$. The maximal 2-parabolic subgroups of this system are given by $P_{12} \sim [2^{18}], L_3(2)$, $P_{13} \sim [2^{17}], (\text{Sym}(3) \times \text{Sym}(5))$ and $P_{23} \sim 2^{1+12}.3.M_{22}.2$. By considering centralizer orders, powering up classes and conjugation of representatives of certain $P_{12}$, $P_{13}$- and $P_{23}$-conjugacy classes by random elements in $J_4$, we may determine the fusion of $O_2(P_{12})$-, $O_2(P_{13})$- and $O_2(P_{23})$-classes in $J_4$. We detail the orders of the intersections of the 2-cores of the maximal parabolic subgroups with the $J_4$-conjugacy classes in Table 8. Consequently, the cuspidal relation holds on $P_{12}$ for the characters $\chi_2, \chi_3, \chi_4$ and $\chi_5$, it holds on $P_{13}$ for $\chi_2$, $\chi_3$, $\chi_4$, $\chi_5$, $\chi_6$, $\chi_7$, $\chi_9$, $\chi_{10}$, $\chi_{12}$ and $\chi_{13}$, whilst the cuspidal relation holds on $P_{23}$ for the
irreducible characters $\chi_2$ and $\chi_3$. We conclude that $J_4$ admits two 2-cuspidal characters, $\chi_2$ and $\chi_3$, both of degree 1333.

$L_y$

There are six 2-minimal parabolic subgroups of $L_y$, three of which feature in the two 2-minimal parabolic systems of $L_y$. These are $P_1 \sim [2^7].\Sym(3)$, $P_2 \sim [2^5].\Sym(5)$, $P_3 \sim 2.\Sym(9)$ and they give rise to the systems $\{P_1, P_2\}$ and $\{P_1, P_3\}$. Since $|O_2(P_1)| = 2^7$, $|O_2(P_2)| = 2^5$ and $|O_2(P_3)| = 2$, it is easy to see that the cuspidal relation does not hold for any of the 2-minimal parabolic subgroups of $L_y$, and hence there are no 2-cuspidal characters of $L_y$.

4 3-Cuspidal Characters

We now describe the 3-cuspidal characters for each of the sporadic groups.

4.1 The Mathieu Groups

$M_{11}$

The normalizer of a Sylow 3-subgroup of $M_{11}$ is the maximal subgroup $M_9 : 2$ of $M_{11}$. Consequently, we see that $M_{11}$ admits two 3-cuspidal characters, $\chi_6$ and $\chi_7$, both of degree 16.

$M_{12}$

The group $M_{12}$ has a unique 3-minimal parabolic system

$$\{ P_1 \sim 3^2.GL_2(3), P_2 \sim 3^2.GL_2(3) \}.$$  

The non-trivial elements of the 3-cores $O_3(P_1)$ and $O_3(P_2)$ lie in the $M_{12}$-conjugacy class $3A$. It follows that the 3-cuspidal characters of $M_{12}$ are $\chi_4$ and $\chi_5$, both of which have degree 16.

$M_{22}$

There is a unique 3-minimal parabolic system of $M_{22}$, namely

$$\{ P_1 \cong M_{11}, P_2 \cong L_3(4) \}.$$  

Since $O_3(P_i) = 1$ for $i = 1, 2$, an element $\chi \in \Irr(M_{22})$ will be 3-cuspidal if and only if the cuspidal relation holds for a Sylow 3-subgroup. Since this is never the case, $M_{22}$ has no 3-cuspidal characters.

$M_{23}$

The 3-cores of the two 3-minimal parabolic subgroups comprising the unique 3-minimal parabolic system

$$\{ P_1 \cong M_{11}, P_2 \sim L_3(4) : 2 \}$$

of $M_{23}$ are both trivial. Since the cuspidal relation does not hold for a Sylow 3-subgroup, we conclude that $M_{23}$ admits no 3-cuspidal characters.
There are three 3-minimal parabolic subgroups of $M_{24}$ given by $P_1 \sim 3. \text{Sym}(6)$, $P_2 \sim 2^6.3^{1+2}.2_1^{1+2}$ and $P_3 \sim M_{12} : 2$. These give rise to two 3-minimal parabolic systems of $M_{24}$, namely $\{P_1, P_3\}$ and $\{P_2, P_3\}$. It is easy to observe that there is no $\chi \in \text{Irr}(M_{24})$ satisfying the cuspidal relation for a Sylow 3-subgroup of $M_{24}$. It follows that neither 3-minimal parabolic system of $M_{24}$ admits a 3-cuspidal character.

### 4.2 The Leech Lattice and Conway Groups

#### HS

As a Sylow 3-subgroup of $HS$, it is easy to see that the cuspidal relation will not hold for such a subgroup, and hence $HS$ admits no 3-cuspidal characters.

#### J$_2$

There is a unique 3-minimal parabolic system of $J_2$ given by $\{P_1 \sim 3. \text{Alt}(6), P_2 \cong U_3(3)\}$. The non-trivial elements of $O_3(P_1)$ are contained in the $J_2$-conjugacy class $3A$. We immediately see that $J_2$ has no 3-cuspidal characters.

#### Co$_1$

There is a solitary 3-minimal parabolic system of $Co_1$, which has rank 3. Its maximal parabolic subgroups are $P_{12} \sim 3^{1+4}.Sp_4(3).2$, $P_{13} \sim 3^{1+4}.GL_2(3)^2$ and $P_{23} \sim 3^6.2.M_{12}$.

The $Co_1$-fusion of the 3-core of $P_{23}$ is described in [6]. We see that $O_3(P_{23})$ intersects the $Co_1$-conjugacy classes $1A, 3A, 3B, 3C$ and $3D$ in 1, 2, 240 and 0 elements respectively. It follows that the cuspidal relation does not hold for $P_{23}$ and hence $Co_1$ admits no 3-cuspidal characters.

#### Co$_2$

The unique 3-minimal parabolic system of $Co_2$ has the form $\{P_1 \sim 3^{1+4}.2^{1+4}.\text{Sym}(5), P_2 \sim 3^4.L_2(9).\text{Dih}(8)\}$. Since $O_3(P_3)$ intersects the respective $Co_2$-conjugacy classes $1A, 3A$ and $3B$ in 1, 20 and 60 elements, we see that there are no 3-cuspidal characters of $Co_2$.

#### Co$_3$

There are two 3-minimal parabolic subgroups of $Co_3$ and they form the unique 3-minimal parabolic system $\{P_1 \sim 3^{1+4}.4.\text{Sym}(6), P_2 \sim 3^5.(M_{11} \times 2)\}$. The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the $Co_3$-classes $1A, 3A, 3B$ and $3C$ in 1, 2, 240 and 0 and 1, 110, 132 and 0 elements respectively. It follows that the cuspidal relation holds on $P_1$ for $\chi_6, \chi_7 \in \text{Irr}(Co_3)$ and on $P_2$ for $\chi_{10}, \chi_{11} \in \text{Irr}(Co_3)$. Consequently, there are no 3-cuspidal characters of $Co_3$. 

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\(M^cL\)

The McLaughlin group has a unique 3-minimal parabolic system given by
\[ \{ P_1 \sim 3^4.M_{10}, P_2 \sim 3^{1+4} \cdot 2\cdot \Sym(5) \} . \]

The 3-core \(O_3(P_1)\) intersects the \(M^cL\)-conjugacy classes 1A, 3A and 3B in 1, 20 and 60 elements respectively. It follows that no \(\chi \in \Irr(M^cL)\) satisfies the cuspidal relation for \(P_1\) and hence \(M^cL\) has no 3-cuspidal characters.

\(Suz\)

The unique 3-minimal parabolic system
\[ \{ P_1 \sim 3^2.M_{11}, P_2 \sim 3^{2+4} \cdot (\Alt(4) \times 2^2) \cdot 2 \} \]

of \(Suz\) does not admit any 3-cuspidal characters. To see this, we note that the 3-core \(O_3(P_1)\) intersects the \(Suz\)-classes 1A, 3A, 3B and 3C in 1, 22, 220 and 0 elements respectively, meaning that the cuspidal relation does not hold on \(P_1\) for any \(\chi \in \Irr(Suz)\).

4.3 The Monster Group and its Subquotients

\(He\)

Since a Sylow 3-subgroup of \(He\) has order 27 and exponent 3, we easily observe that there are no 3-cuspidal characters of \(He\).

\(HN\)

The unique 3-minimal parabolic system of \(HN\) is
\[ \{ P_1 \sim 3^{1+4} \cdot 2\cdot \Sym(5), P_2 \sim 3^{4+2} \cdot (\Alt(4) \times \Alt(4)).4 \} . \]

The character table of the subgroup \(M = 3^{1+4} \cdot \SL_2(5) \leq HN\) is given as \([8\, \text{TABLE II}]\). Since \(M \leq P_1\), it follows that \(O_3(M) = O_3(P_1)\). Moreover, every \(M\)-conjugacy class is either contained in, or disjoint from \(O_3(M)\). Using the notation from \([8]\), we have that
\[ O_3(M) = 1 \cup 3_1 \cup 3_2^\uparrow \cup 3_2 \cup 3_3. \]

Considering centralizer orders in \(M\) and \(HN\), we see that \(3_1\) and \(3_2^\uparrow\) are contained in the \(HN\)-class 3B, whilst \(3_2\) and \(3_3\) lie in either 3A or 3B. It follows that \(|O_3(P_1) \cap 3A| = |O_3(M) \cap 3A| = 0, 120\) or 240 and \(|O_3(P_1) \cap 3B| = |O_3(M) \cap 3B| = 2, 122\) or 242. For each of these possibilities, we see that the cuspidal relation would not hold for \(P_1\), and hence there are no 3-cuspidal characters of \(HN\).

\(Th\)

The Thompson group has a single 3-minimal parabolic system. It is of rank 2 and has the form
\[ \{ P_1 \sim 3^{(1+2)+4+2} \cdot GL_2(3), P_2 \sim 3^{(2+3)+4} \cdot GL_2(3) \} . \]
The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the respective $Th$-conjugacy classes $1A$, $3A$, $3B$, $3C$, $9A$, $9B$ and $9C$ in 1, 270, 2186, 4104, 2106, 5184 and 5832 elements respectively. It follows that for both $P_1$ and $P_2$ there is a unique element of $\text{Irr}(Th)$ satisfying the cuspidal relation, namely $\chi_2$ of degree 248. We conclude that $\chi_2$ is the unique 3-cuspidal character of $Th$.

$Fi_{22}$

There is a unique 3-minimal parabolic system of $Fi_{22}$, which has rank 3 and minimal parabolic subgroups of the form $P_i \sim [3^8].2.PGL_2(3)$ for $i = 1, 2, 3$. The maximal parabolic subgroups of this system are

$P_{12} \cong P_{13} \sim 3^{4+2}.L_3(3)$ and $P_{23} \sim 3^{1+6}.2^2.SL_2(3).\text{Sym}(4)$.

The maximal parabolics $P_{12}$ and $P_{13}$ are submaximal subgroups of $Fi_{22}$, being contained in the maximal subgroups isomorphic to $O_7(3)$, and hence their 3-cores can be easily computed. Meanwhile, by [2, (39.6)], the 3-core $O_3(P_{23})$ is isomorphic to the Fitting subgroup of the normalizer in $Fi_{22}$ of an element of the $Fi_{22}$-class $3B$. The orders of the intersections of the 3-cores of these maximal parabolics with the $Fi_{22}$-conjugacy classes is summarized in Table 9.

We see that the cuspidal relation holds on $P_{12}$ and $P_{13}$ for the characters $\chi_2, \chi_5 \in \text{Irr}(Fi_{22})$, and it holds on $P_{23}$ for $\chi_2$. Consequently, $\chi_2$ (of degree 78) is the unique 3-cuspidal character of $Fi_{22}$.

$Fi_{23}$

The group $Fi_{23}$ has a unique 3-minimal parabolic system given by

\[ \{ P_1 \sim [3^{12}].2^2.PGL_2(3), P_2 \sim [3^{12}].2^2.PGL_2(3), P_3 \sim [3^9].2.L_2(3)^3.2.\text{Sym}(3) \}. \]

The corresponding 3-maximal parabolic subgroups are

$P_{12} \sim 3^{3+7}.GL_3(3)$, $P_{13} \sim 3^{1+8}.2^{1+6} : 3^{1+2} : 2.\text{Sym}(4)$ and $P_{23} \sim D_4(3).\text{Sym}(3)$.

Using the information on $Fi_{23}$-fusion within $O_3(P_{13})$ given in [13, Table 2], and the fact that the non-trivial elements of $Z(3^{1+8})$ lie in the $Fi_{23}$-class $3B$, we see that $O_3(P_{13})$ intersects the $Fi_{23}$-classes $3A$, $3B$, $3C$ and $3D$ in 864, 1538, 3456 and 13824 elements respectively. (We note that the above fusion can also be calculated within $P_{23}$. Indeed, there are nine $P_{23}$-classes of elements of order 3, say $3a, \ldots, 3i$, having centralizer orders in $P_{23}$ of 408146688, 37791360, 37791360, 12737088, 2834352, 944784, 314928, 78732 and 17496 respectively.)
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Element, } m & \text{Stab}_{U(2)}(m) & \text{Stab}_{\mathbb{A}}(m) & m_{\mathbb{A}}(2) \\
\hline
m_1 & 3^3, \text{Alt}(5) & 4860 & 2816 \\
m_2 & 3, \text{Alt}(6) & 1080 & 12672 \\
m_3 & 2^{1+3}.3^{1+2}.3 & 10368 & 1320 \\
m_4 & 3^3, \text{Alt}(4) & 324 & 42240 \\
\hline
\end{array}
\]

Table 10: The non-trivial orbits of the unique irreducible 10-dimensional \(GF(3)U_5(2)\)-module.

It follows that the \(P_{23}\)-classes satisfy the following inclusions: 3c, 3d \(\subset\) 3A, 3a, 3e \(\subset\) 3B, 3b, 3g \(\subset\) 3C and 3f, 3h, 3i \(\subset\) 3D. Consequently, the cuspidal relation does not hold on \(P_{13}\) for any \(\chi \in \text{Irr}(Fi_{23})\), and \(Fi_{23}\) admits no 3-cuspidal characters.

\(Fi_{24}\)

As with \(Fi_{23}\) we see that there is a unique 3-minimal parabolic system of \(Fi_{24}\), namely

\[
\{P_1 \sim [3^{15}].2^2.PGL_2(3), P_2 \sim [3^{15}].2^2.PGL_2(3), P_3 \sim [3^{15}].2.\text{Sym}(5)\},
\]

having maximal parabolics \(P_{12} \sim 3^{3+7+3}.L_3(3).2, P_{13} \sim 3^{2+4+8}.(SL_2(3) \times \text{Alt}(5))).2\) and \(P_{23} \sim 3^{1+10}.U_5(2).2\).

Consider the extra-special 3-core, \(O_3(P_{23}) \cong 3_+^{1+10}\). Since there is a unique irreducible 10-dimensional \(GF(3)U_5(2)\)-module, \(M\), we can explicitly determine the sizes of the orbits of elements of this module. These are summarized in Table 10. We see that there are at most four classes of non-central elements of \(3_+^{1+10}\).

Let \(z \in Z(3_+^{1+10}) \setminus \{1\}\) and let \(x \in 3_+^{1+10} \setminus Z(3_+^{1+10})\). Thus \(x\) represents a non-zero vector in \(M\). Then \(x, x^2, zx, z^2x, z^2x^2\) and \(z^2x^2\) are all \(Fi_{24}\)-conjugate. Indeed, from the ATLAS we have that for any 3-element \(w \in Fi_{24}\), the elements \(w\) and \(w^2\) are \(Fi_{24}\)-conjugate. Suppose that \(g \in Fi_{24}\) is such that \(x^9 = x^2\). Then \((zx)^9 = zx^2\) as \(z\) is central. Thus \(zx\) and \(zx^2\) lie in a common \(Fi_{24}\)-class, and are joined by \((zx)^2 = z^2x^2\) and \((zx)^2 = z^2x\). Finally, as \(zx\) and \(z^2x\) are \(Fi_{24}\)-conjugate, there exists \(h \in Fi_{24}\) satisfying \(z^2x = (zx)^h = zx^h\) and hence \(zx = zx^h\). We conclude that the orbits of \(m_1, m_2, m_3\) and \(m_4\) give rise to orbits of \(3_+^{1+10} \setminus Z(3_+^{1+10})\) of respective sizes 8448, 38016, 3960 and 126720.

Label the orbit of \(3_+^{1+10}\) arising from \(m_i\) by \(M_i\) for \(i = 1, \ldots, 4\). Considering the orders of stabilizers given in Table 10 together with the centralizer orders of elements of order 3 in \(Fi_{24}\) given in the ATLAS, we deduce that \(M_1, M_2, M_3 \subset 3A \cup 3B \cup 3C\), whilst elements in \(M_4\) could form a subset of \(3A, 3B, 3C, 3D\) or \(3E\). Since \(N(3B) \cong P_{23}\), it follows that the non-trivial central elements of \(3_+^{1+10}\) lie in the \(Fi_{24}\)-class \(3B\).

The menagerie of information obtained above results in 135 different possibilities for the fusion of 3-elements within \(O_3(P_{23}) \cong 3_+^{1+10}\). Feeding each possibility into MAGMA and allowing it to roam over all 108 complex characters of \(Fi_{24}\), we see that the cuspidal relation never holds for \(P_{23}\), and hence \(Fi_{24}\) admits no 3-cuspidal characters.
Aside 4.1. We note that the $Fi'_{24}$-fusion within the 3-core $O_3(P_{23})$ has previously been studied by Wilson. Indeed, in [18, Section 2.2] Wilson calculates that $O_3(P_{23})$ contains $3960, 8450, 38016$ and $126720$ elements from the respective $Fi'_{24}$-classes $3A, 3B, 3C$ and $3D$. However, these calculations are based heavily on an unpublished preprint, and we have been unable to verify them.

$B$

There is a unique 3-minimal parabolic system of $B$, which has rank 3. The maximal parabolic subgroups of this system are $P_{12} \sim 3^{3+7}GL_3(3)$, $P_{13} \sim 3^{2+3+6}GL_2(3)\times M_{11}$ and $P_{23} \sim 3^{1+8}PSp_4(3)\times S_5$. Considering the minimum value that each $\chi \in \text{Irr}(B)$ takes on 3-elements, we see that the only possible 3-cuspidal character of $B$ is $\chi_2$ of degree 4371. However, as $B$ satisfies the condition of Lemma 2.7, we see that $\chi_2$ cannot be 3-cuspidal, and hence $B$ admits no 3-cuspidal characters.

$M$

The monster group, $M$, has a unique 3-minimal parabolic system, which has rank 3. Its maximal parabolic subgroups are given by

$P_{12} \sim 3^{3+8+6}L_3(3)$, $P_{13} \sim 3^{2+5+10}(GL_2(3) \times M_{11})$ and $P_{23} \sim 3^{1+12}.2.Suz_2$.

By considering the 3-core $O_3(P_{23})$, we see that it has exponent 3. Moreover, appealing to [9, Lemma 1.5] we see that

$$|O_3(P_{23}) \cap 3A| = 196,560 \quad \text{and} \quad |O_3(P_{23}) \cap 3B| = 1,397,762.$$ 

It immediately follows that $M$ admits no 3-cuspidal characters.

4.4 The Pariahs

$J_1$

The cuspidal relation does not hold for a Sylow 3-subgroup of $J_1$. Hence there are no 3-cuspidal characters for the unique 3-minimal parabolic system, $\{J_1\}$, of $J_1$.

$O'N$

A Sylow 3-subgroup of $O'N$ has order 81. Moreover, since there is a unique $O'N$-conjugacy class of non-trivial 3-elements, we see that the cuspidal relation does not hold for a Sylow 3-subgroup of $O'N$. Consequently, $O'N$ admits no 3-cuspidal characters.

$J_3$

The normalizer of a Sylow 3-subgroup of $J_3$ is maximal. Moreover, such a Sylow subgroup intersects the $J_3$-conjugacy classes $1A, 3A, 3B, 9A, 9B$ and $9C$ in $1, 18, 8, 72, 72$ and 72 elements respectively. Checking the cuspidal relation for each $\chi \in \text{Irr}(J_3)$ for a Sylow 3-subgroup, we see that there are no 3-cuspidal characters of $J_3$. 

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Since a Sylow 3-subgroup of $Ru$ has order 27 and there is a unique $Ru$-conjugacy class of non-trivial 3-elements, it is easy to see that $Ru$ admits no 3-cuspidal characters.

The non-trivial elements of a Sylow 3-subgroup of $J_4$ lie in the $J_4$-class 3A. Since such a subgroup has order 27, it is easy to see that the cuspidal relation does not hold for a Sylow 3-subgroup of $J_4$. Hence, $J_4$ has no 3-cuspidal characters.

$Ly$

There is a unique 3-minimal parabolic system of $Ly$ given by

$$\{ P_1 \sim 3^{2+4}.\Sym(5), P_2 \sim 3^5.(M_{11} \times 2) \}.$$

Considering the minimum value that each $\chi \in \Irr(Ly)$ takes on elements of order 3, we see that the only possible candidates for 3-cuspidal characters are $\chi_7$ and $\chi_8$ of degree 120064. However, these characters take strictly negative values on all 3-elements, and hence the cuspidal relation cannot hold for them for both $O_3(P_2)$ and a Sylow 3-subgroup of $Ly$. We conclude that there are no 3-cuspidal characters of $Ly$.

5 5-Cuspidal Characters

The groups $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, $Suz$, $He$, $Fi_{22}$, $Fi_{23}$, $Fi'_{24}$, $J_1$, $O'N$, $J_3$, $Ru$ and $J_4$ have Sylow 5-subgroups of exponent 5 and of order at most $5^3$. Thus, considering the minimal values that an irreducible character takes on elements of order 5 together with the character degree, we see that none of these groups admit a 5-cuspidal character. We now consider the remaining eleven sporadic groups in turn.

$HS$

A Sylow 5-subgroup of $HS$ intersects the $HS$-conjugacy classes 1A, 5A, 5B and 5C in 1, 4, 40 and 80 elements respectively. It follows immediately that $HS$ has no 5-cuspidal characters.

$J_2$

The normalizer in $J_2$ of a Sylow 5-subgroup, $S$, is maximal and $S$ intersects each of the $J_2$-conjugacy classes 5A, 5B, 5C and 5D in 6 elements. From this we deduce that $J_2$ has a unique 5-cuspidal character given by $\chi_6$ of degree 36.

$Co_1$

There is a unique 5-minimal parabolic system of $Co_1$ given by

$$\{ P_1 \sim 5^3.(4 \times \Alt(5)).2, P_2 \sim 5^{1+2}.GL_2(5) \}.$$
Considering the minimum value that each $\chi \in \text{Irr}(Co_1)$ takes on elements of order 5, and the order of the 5-cores of the minimal parabolic subgroups, we see that there are no 5-cuspidal characters of $Co_1$.

$Co_2, Co_3, M^cL$

If $G \in \{Co_2, Co_3, M^cL\}$, then a Sylow 5-subgroup of $G$ intersects the $G$-conjugacy classes $1A$, $5A$ and $5B$ in $1$, $4$ and $120$ elements respectively. It follows that the cuspidal relation does not hold for a Sylow 5-subgroup for any $\chi \in \text{Irr}(G)$ and hence $G$ has no 5-cuspidal characters.

$HN$

There is a unique 5-minimal parabolic system of $HN$ given by
\[
\{ P_1 \sim 5^{1+4} \cdot (2^{1+4} \cdot 5 \cdot 4), P_2 \sim 5^{2+1+2+4} \cdot \text{Alt}(5) \}.
\]
The character table of $P_1$ is given as [8, Table III], whilst a partial character table of $P_2$ - featuring the conjugacy classes of elements of order 2 and classes contained in $O_5(P_2)$ - is given in Table 11. By considering the restriction of $\chi_2 \in \text{Irr}(HN)$ to $P_1$ and $P_2$, we may calculate the $HN$-fusion within $O_5(P_1)$ and $O_5(P_2)$. We see that $O_5(P_1)$ intersects the $HN$-classes $1A$, $5A$, $5B$, $5C$, $5D$, and $5E$ in $1$, $400$, $324$, $800$, $800$ and $800$ elements respectively, whilst $O_5(P_2)$ intersects the given classes in $1$, $0$, $624$, $650$, $650$, and $1200$ elements respectively. Consequently, the cuspidal relation holds on $P_1$ for $\chi_4 \in \text{Irr}(HN)$ and on $P_2$ for $\chi_4, \chi_5 \in \text{Irr}(HN)$.

$Th$

The normalizer of a Sylow 5-subgroup of $Th$ is a maximal subgroup. Thus a character $\chi \in \text{Irr}(Th)$ will be 5-cuspidal precisely when the cuspidal condition holds for the Sylow subgroup. Since $Th$ has a unique conjugacy class of elements of order 5, we observe that there is a unique 5-cuspidal character of $Th$, namely $\chi_2$ of degree 248.

$B$

There is a unique 5-minimal parabolic system of $B$ given by
\[
\{ P_1 \sim 5^{1+4} \cdot 2^{1+4} \cdot \text{Sym}(5).2, P_2 \sim 5^{2+1+2}.\text{GL}_2(5) \}.
\]
Considering the minimum value that each $\chi \in \text{Irr}(B)$ takes on $B$-classes of 5-elements, we see that the only possible 5-cuspidal character of $B$ is $\chi_2$. However, $\deg(\chi_2) = 4371$, and $B$ satisfies the conditions of Lemma 2.7. Thus there are no 5-cuspidal characters of $B$.

$M$

The monster has a unique 5-minimal parabolic system, namely
\[
\{ P_1 \sim 5^{1+6} \cdot (J_2 \times 2).2, P_2 \sim 5^{2+2+4}.\text{Sym}(3).\text{GL}_2(5) \}.
\]
Table 11: A partial character table of the 5-minimal parabolic subgroup $P_2 \sim 5^{2r+1+2}.A$. Alt(5) of $HN$ (where $w = \exp(2\pi i/5)$).

<table>
<thead>
<tr>
<th>Class</th>
<th>$1_1$</th>
<th>$2_1$</th>
<th>$2_2$</th>
<th>$5_1$</th>
<th>$5_2$</th>
<th>$5_3$</th>
<th>$5_4$</th>
<th>$5_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
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<td>2</td>
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Table 11: A partial character table of the 5-minimal parabolic subgroup $P_2 \sim 5^{2r+1+2}.A$. Alt(5) of $HN$ (where $w = \exp(2\pi i/5)$).
Using a similar approach to that used for the baby monster, with the 5-core $O_5(P_1)$, we deduce that there are no 5-cuspidal characters of $M$.

$Ly$

From [14] we see that there is a unique 5-minimal parabolic system of $Ly$, having rank 3. Its minimal parabolic subgroups are $P_1 \sim 5^{1+2}_+ A.PGL_2(5)$, $P_2 \sim 5^{1+2}_+ A.PGL_2(5)$ and $P_3 \sim 5^{1+4}_+ A.PGL_2(5)$. The maximal 5-parabolic subgroups are given by

$$P_{12} \sim 5^3.SL_3(5), \quad P_{13} \sim 5^{1+4}.2.Alt(6).4 \quad \text{and} \quad P_{23} \cong G_2(5).$$

By considering the elementary abelian subgroup $O_5(P_{12}) \cong 5^3$, we see that the only possible 5-cuspidal characters of $Ly$ are $\chi_2$ and $\chi_3$ (both of degree 2480), and that for these characters to be 5-cuspidal, we must have that the non-trivial elements of $O_5(P_{12})$ are contained in the $Ly$-conjugacy class 5A. Defining $S$ to be our given Sylow 5-subgroup of $Ly$, we see that $S$ has a unique normal elementary abelian 5-subgroup of order $5^3$. Considering this within the maximal subgroup $G_2(5) \leq Ly$, we see that the non-trivial elements of $O_5(P_{12})$ are indeed contained in the $Ly$-class $5A$. It remains to check the cuspidal relation for $\chi_2$ and $\chi_3$ for the extra-special 5-core $O_5(P_{13})$ of order $5^5$.

By constructing $O_5(P_{13})$ within both $P_{13}$ and $P_{23}$ we may deduce that it intersects the $Ly$-conjugacy classes $1A$, $5A$ and $5B$ in 1, 724 and 2400 elements respectively. It follows that the cuspidal relation holds on $O_5(P_{13})$ for both $\chi_2$ and $\chi_3$, and hence we see that $\chi_2$ and $\chi_3$ are 5-cuspidal characters of $Ly$.

6 $p$-Cuspidal Characters ($p > 5$)

In the case that $p > 5$, most sporadic groups with order divisible by $p$ have a cyclic Sylow $p$-subgroup of order $p$. The exceptions are $(Co_1, p = 7)$, $(He, p = 7)$, $(Th, p = 7)$, $(Fi_{24}', p = 7)$, $(B, p = 7)$, $(M, p = 7, 11, 13)$, $(O'N, p = 7)$ and $(J_4, p = 11)$.

The normalizer of a Sylow 7-subgroup, $S$, of $He$ is maximal in $He$, and hence there is a unique 7-minimal parabolic system given by $\{He\}$. Consequently, an element $\chi \in \text{Irr}(He)$ will be 7-cuspidal precisely when the cuspidal condition holds for $S$. We have that $S$ contains 1, 42, 42, 132, 63 and 63 elements from the $He$-classes $1A$, $7A$, $7B$, $7C$, $7D$ and $7E$ respectively. It follows that the 7-cuspidal characters of $He$ are $\chi_2$ and $\chi_3$ of degree 51.

In all other cases, since the sporadic group in question contains no elements of order $p^a$ for $a > 1$, we may consider the minimum value that each irreducible character takes on elements of order $p$, to conclude that there are no $p$-cuspidal characters.

We conclude by considering the $p$-cuspidal characters arising from sporadic groups having a cyclic Sylow $p$-subgroup of order $p$. Since such a subgroup will necessarily be the $p$-core of its normalizer, it is easy to see that the cuspidal relation must hold for the Sylow subgroup. Moreover, a character will be cuspidal precisely when this is true. This gives an additional four cuspidal characters for the sporadic groups; the 10-dimensional characters $\chi_2, \chi_3, \chi_4 \in \text{Irr}(M_{11})$ are 11-cuspidal and the 22-dimensional character $\chi_2 \in \text{Irr}(M_{23})$ is 23-cuspidal.
References


