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Carr, Emily and Rowley, Peter

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Emily L. Carr, Peter J. Rowley

Abstract

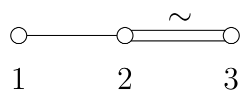
This paper investigates the structure of the chamber graph associated with the minimal parabolic geometries of rank 3 for the groups M_{24}, the Mathieu group of degree 24, He, Held's simple group, and $3^{7\cdot}Sp_6(2)$, a non-split extension of an elementary abelian 3-group with $Sp_6(2)$. These three minmal parabolic geometries all have the same diagram. Executable files containing data describing the disc structure of a fixed chamber of these graphs accompany this paper. Those chambers at maximal distance from a given chamber are studied, as are the maximal opposite sets. Also the geodesic closure of opposite chambers are described for each of these three geometries.

1 Introduction

The finite non-abelian simple groups, excluding the twenty six sporadic simple groups, display a number of unifying features. Firstly, and also historically, some of them may be viewed as, essentially, matrix groups defined over finite fields which preserve some form. A very different viewpoint emerged in the middle of the twentieth century, as well as some further then unknown families of simple groups, with the work of Chevalley, Steinberg, Ree and others (see for example Carter [6], Steinberg [19] and Wilson [23]). This perspective revealed these groups as automorphism groups of Lie algebras or fixed points of automorphisms of algebraic groups. Hence their description as groups of Lie type. Then, at the hands of Tits [20], there emerged a geometric description in the guise of buildings. These combinatorial objects have had a considerable impact on the study of finite simple groups. Subsequently Tits [21] recast the theory of buildings, placing the emphasis on chamber graphs. All the essential features of a building are encapsulated in the chamber graph of a building.

To this day, the sporadic simple groups stubbornly refuse to fit into any discernable overall pattern. This is despite a considerable amount of effort being put into the study of this ramshackle, yet fascinating, collection of groups. One approach to trying to produce a conceptual framework which will also include the sporadic simple groups has been to weaken the concept of a building. Such combinatorial structures are loosely referred to as geometries or group geometries. There is now a vast array of different sorts of geometries, a selection of which may be seen in [2], [3] and [4]. One species of geometry attempts to parallel the theory of parabolic subgroups in groups of Lie type, resulting in the so-called minimal parabolic geometries. Such geometries for the sporadic simple groups were catalogued by Ronan and Stroth in [14]. Closely related to these are the maximal parabolic geometries as introduced in [13] by Ronan and Smith. Geometries give rise to chamber systems and then to chamber graphs. The study of chamber graphs related to the minimal parabolic and maximal parabolic geometries for the sporadic groups has been limited so far. This is undoubtedly due to the fact that we have little understanding of these structures and the number of chambers for these geometries is very large. Some preliminary investigations into the chamber graphs of a number of small geometries are recorded in [18], see also [16]. While a very detailed analysis of the chamber graph of the maximal parabolic geometry for the Mathieu group M_{24} is presented in [17].

Associated to minimal and maximal parabolic geometries is the notion, modeled on that of a Dynkin diagram, of a diagram. Here we shall be examining the chamber graphs of minimal parabolic geometries whose diagram is

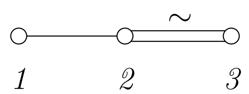


Parabolic systems which give rise to such geometries were first studied in Rowley[15] and then later in Heiss [10]. Together these papers showed that, essentially, there are three such systems, to be observed in the sporadic simple groups M_{24} (Mathieu group of degree 24) and He (Held's group) together with $3^{7\cdot}Sp_6(2)$ (a non-split extension of an elementary abelian 3-group of order 3^7 and $Sp_6(2)$). These are the three examples we shall be scrutinizing in this paper. A more detailed discussion of these will be given Section 2 along with notation relating to geometries and their chamber graphs. For $\gamma \in \mathcal{C}$, $\mathcal{C}$ a chamber graph, we recall that the i^{th} disc of γ, where $i \in \mathbb{N} \cup \{0\}$, is the set

$$\Delta_i(\gamma) = \{\gamma' \in \mathcal{C} \mid d(\gamma, \gamma') = i\}.$$

Here $d(\ ,\ )$ is the usual graph theoretic distance on $\mathcal{C}$. Our first result concerns the disc structure of the chamber graphs around a fixed chamber.

Theorem 1.1 *Let G denote one of the three groups M_{24}, He and $3^{7\cdot}Sp_6(2)$ and let Γ be the rank 3 geometry of G with diagram*



with $\mathcal{C}$ being the chamber graph of Γ. Let γ_0 be a fixed chamber of $\mathcal{C}$, and put $B = Stab_G(\gamma_0)$.

(i) If $G = M_{24}$, then the disc structure of $\mathcal{C}$ is

Disc	1	2	3	4	5	6	7	8	9
Size	6	20	56	144	368	848	1800	3810	8040
Number of B-orbits	3	5	7	9	13	18	24	31	39

10	11	12	13	14	15	16	17
16920	32832	55200	62336	47616	6656	2048	384
53	71	93	78	47	10	6	2

.

(ii) If $G = He$, then the disc structure of $\mathcal{C}$ is

Disc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Size	6	20	56	144	368	848	1800	3810	8040	16920	32832	62496	118048	222048	401688	614768	869376	965376	562432	54784	64
Number of B-orbits	3	5	7	9	13	18	24	31	39	53	71	107	189	321	528	761	994	1015	569	72	1

.

(iii) If $G = 3^7{}^{\cdot}Sp_6(2)$, then the disc structure of $\mathcal{C}$ is

Disc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Size	6	20	56	144	368	848	1800	3810	8040	16920	32832	62496	118048	222048	404248	632744	951564	1243568	1145744	769696	399856	150016	33128	2016	128
Number of B-orbits	3	5	7	9	13	18	24	31	43	68	112	191	340	594	1011	1516	2146	2713	2549	1868	1131	550	185	20	2

.

So, the chamber graph for the groups M_{24}, He and $3^7{}^{\cdot}Sp_6(2)$ has diameter equal to, respectively, $17, 21$ and 25. The information presented in Theorem 1.1 is just the tip of the iceberg – much more detail is obtained. See Sections 3 and 4.

In order to compare and contrast the data given in Theorem 1.1 we amplify our earlier remarks on the chamber graphs of buildings associated with finite groups of Lie type. Suppose that G is a finite group of Lie type defined over $GF(q)$ where $q = p^a$, p a prime, and let W be its Coxeter group. We denote the chamber graph of the building of G by $\mathcal{C}$. Let γ_0 be a chamber of $\mathcal{C}$. The Coxeter group of G has a heavy influence on the disc structure of $\mathcal{C}$. For example,

$$|\Delta_i(\gamma_0)| = q^i \times (\text{size of the } i^{th}\text{disc of the chamber graph of } W),$$

and the diameter of $\mathcal{C}$, d, is the Coxeter number of W. Moreover $|\Delta_d(\gamma_0)| = |U|$ where U is the unipotent radical of $B = Stab_G(\gamma_0)$. Recall that $U \in Syl_p(G)$, B is the Borel subgroup of G and $N_G(U) = B$. Also we have that B acts transitively on $\Delta_d(\gamma_0)$. Apartments play a central role in buildings and may be realized, via their chambers, as the geodesic closure of γ_0 and γ, where $\gamma \in \Delta_d(\gamma_0)$. See Section 4 for the definition of geodesic closure, and consult Ronan[12] for the results on buildings mentioned above, and much more.

We now return to the three chamber graphs featuring in Theorem1.1. Sharing the same diagram means all three geometries are "locally" similar. It is surprising (to the authors) how much this similarity persists in the chamber graphs. We have that the sizes of $\Delta_i(\gamma_0)$ for $i = 1, \ldots, 11$ are the same in all three cases. For $G \cong M_{24}$ and $G \cong He$, the number of B-orbits in each of these discs are the same. While for $G \cong He$ and $G \sim 3^{7\cdot}Sp_6(2)$ the phenomena of the same size of discs $\Delta_i(\gamma_0)$ even persists up to $i = 14$. The last disc of $\mathcal{C}$ (that is $\Delta_i(\gamma_0), i = 17(M_{24}), 21(He), 25(3^{7\cdot}Sp_6(2))$ is either the union of two B-orbits or is a B-orbit (where $B = Stab_G(\gamma_0)$). This is very reminiscent of the situation observed in the chamber graph of a building. Chambers at maximal possible distance distance in our three geometries appear worthy of further study. Borrowing from the lexicon of buildings we call two chambers of $\mathcal{C}$ opposite chambers if their distance apart is the diameter of $\mathcal{C}$. So, in Section 4, we examine geodesic closures of opposite chambers. Our group theoretic notation is standard as given in [1] and [9].

A related quest is the study of maximal opposite sets of chambers. By a maximal opposite set of chambers we mean a set of chambers of maximal size subject to having the property that any two chambers are opposite to each other. In Section 4 we discover that for all three chamber graphs in Theorem 1.1, the size of a maximal opposite set is always 3. When $G \cong He$ or $3^{7\cdot}Sp_6(2)$ there is only one G-orbit of maximal opposite sets (see Theorem 4.3) while for $G \cong M_{24}$ things are much more complicated. In this case, in Theorem 4.2 we show that there are 14 G-orbits on maximal opposite sets of $\mathcal{C}$. We note that for the M_{24} maximal 2-local geometry maximal opposite sets have size 5 – see [16] for a detailed analysis of such sets.

It is intended that this paper is a first step in analysing these chamber graphs in extensive detail. Accordingly we have made available various files and programs with this paper. More details of these files may be found in Section 3. One final (comforting) point is that the disc sizes for M_{24} here agree with those in [18], where a much more primitive program was used.

2 The Minimal Parabolic Geometries

We begin this section recapping the definitions of geometries and their chamber graphs while also introducing relevant notation. Then we take a detailed look at the three minimal parabolic geometries we shall be interested in.

A geometry over a set I is a triple $(\Gamma, \tau, \star)$ where Γ is a non-empty set, τ is an onto map from Γ to I and $\star$ is a symmetric relation on Γ with the property that for $x, y \in \Gamma$, $x \star y$ implies $\tau(x) \neq \tau(y)$. The relation $\star$ is called the incidence relation of Γ and for $x \in \Gamma$ with $\tau(x) = i$ we say x has type i (or is an object of type i). We shall use Γ_i, $i \in I$, to denote the set of objects of type i. So $\Gamma = \bigcup_{i \in I} \Gamma_i$. Rather than referring to the triple we just say Γ is a geometry. For $x \in \Gamma$ the residue of x, Γ_x, is defined to be

$$\Gamma_x = \{y \in \Gamma \mid x \star y\},$$

and this is also a geometry. Some of the most interesting geometries are intimately connected with groups. We give details of this only in the context of minimal parabolic systems as this is our principal interest here. First we recall the definition of a minimal parabolic system. Suppose that G is a finite group and p is a prime. Let $S \in Syl_p G$, and set $B = N_G(S)$. A subgroup P of G containing B is called a minimal parabolic subgroup of G (with respect to B) if B is contained in a unique maximal subgroup of P. Let $\{P_1, \ldots, P_n\}$ be a set of minimal parabolic subgroups of G with respect to B, and set $I = \{1, \ldots, n\}$. Then $\{P_1, \ldots, P_n\}$ is a minimal parabolic system for G

if $G = \langle P_i \mid i \in I \rangle \neq \langle P_i \mid i \in J \rangle$ for each proper subset J of I. We refer to n as the rank of this system and p as its characteristic. For J a subset of I let P_J denote $\langle P_i \mid i \in J \rangle$. Then taking the right cosets of $P_{(I \setminus \{i\})}$ to be the type i elements and the incidence relation between two such cosets to be given by having a non-empty intersection in G, yields a geometry over I.

Suppose that Γ is a geometry over I. A set F of pairwise incidence elements of Γ is called a flag and its type, $\tau(F)$, is $\{\tau(x) \mid x \in F\}$. A maximal flag is a flag F such that $\tau(F) = I$, and such flags are sometimes also called chambers of Γ. Let $\mathcal{C}(\Gamma)$ denote the set of chambers of Γ. Two chambers F and F' are said to be i-adjacent if either $F = F'$ or $\tau(F \cap F') = I \setminus \{i\}$. The chamber graph of Γ has as its vertex set $\mathcal{C}(\Gamma)$ with chambers F and F' adjacent if $F \neq F'$ with F and F' being i-adjacent for some $i \in I$. Returning to the situation when our geometry Γ has been obtained from a minimal parabolic system $\{P_1, \ldots, P_n\}$ of a group G, as above. In the geometries we consider here we shall have $B = \bigcap_{i \in I} P_i$, and we shall assume that to be the case now. As a consequence the chambers of Γ may be identified with the set of right cosets of B in G with two chambers Bg and Bh being i-adjacent whenever $gh^{-1} \in P_i$. Thus the chamber graph may be analysed within the group G, which is the strategy we employ here.

The three geometries whose chamber graphs we are investigating have rank 3 (so $I = \{1, 2, 3\}$) and arise from a minimal parabolic system. So let G be one of M_{24}, He and $3^7{}^{\cdot}Sp_6(2)$, with $\{P_1, P_2, P_3\}$ the minimal parabolic system, $B = P_1 \cap P_2 \cap P_3$, Γ the associated geometry and $\mathcal{C}$ its chamber graph. We have that $P_i/O_2(P_i) \cong SL_2(2) \cong Sym(3), i = 1, 2, 3$. For $i, j \in I, i \neq j$ set $P_{ij} = \langle P_i, P_j \rangle$ and $Q_{ij} = O_2(P_{ij})$. Then $P_{12}/Q_{12} \cong L_3(2), P_{13}/Q_{13} \cong Sym(3) \times Sym(3)$ and $P_{23}/Q_{23} \cong 3{}^{\cdot}Sym(6)$ (the triple cover $3{}^{\cdot}Alt(6)$ extended by an involution inverting the central element of order 3). From a geometric point of view, Γ_x is the generalized $Sp_4(2)$ quadrangle when x is of type 1 (see [15] for more details on this geometry), it is a 3-gon for x of type 2 and the projective plane over $GF(2)$ when x is of type 3.

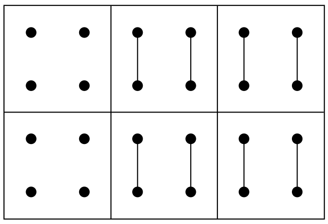
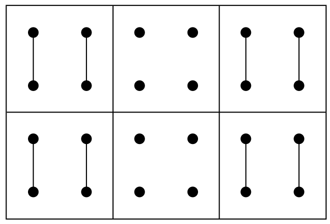
2.1 $G \cong M_{24}$

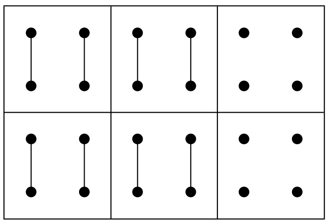
So $|G| = 2^{10}.3^3.5.7.11.23$, $[G : B] = |\mathcal{C}| = 239,085$ and the permutation rank of G on $\mathcal{C}$ is 510. The shapes of P_{12}, P_{13}, P_{23} are as follows: $P_{12} \sim 2^{1+3+3}L_3(2)$, $P_{13} \sim 2^{2+(2+4)}(Sym(3) \times Sym(3))$ and $P_{23} \sim 2^6 : 3{}^{\cdot}Sym(6)$. Employing Curtis's MOG [8] on the 24 element set Ω we may give an explicit description of Γ. For the objects of type 1 we take all the sextets of Ω, the objects of type 2 we take all fours groups of G whose non-trivial elements y_1, y_2, y_3 are such that $Fix_\Omega(y_1), Fix_\Omega(y_2), Fix_\Omega(y_3)$ are the octads of a trio while type 3 objects are the involutions of G which fix an octad point-wise. (For the definitions of sextets, trios and octads and much more consult [8].) An object of type 1, a sextet S is incident with an object of type 2, a fours group $F = \{1, y_1, y_2, y_3\}$ if the y_i stabilize S and the trio $\{Fix_\Omega(y_1), Fix_\Omega(y_2), Fix_\Omega(y_3)\}$ may be obtained from S by pairing the tetrads of S. An object of type 3, an involution y, is incident with a fours group F if $y \in F$ and is incident with a sextet S if it stabilizes S and $Fix_\Omega(y)$ is an octad which is the union of two tetrads of S.

We label the elements of Ω as in Curtis [8]. Thus we think of Ω as

$$\Omega = \begin{array}{|cc|cc|cc|}\hline \infty & 14 & 17 & 11 & 22 & 19 \\ 0 & 8 & 4 & 13 & 1 & 9 \\ 3 & 20 & 16 & 7 & 12 & 5 \\ 15 & 18 & 10 & 2 & 21 & 6 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline & & \\ O_1 & O_2 & O_3 \\ & & \\ \hline \end{array},$$

where O_1, O_2 and O_3 are the heavy bricks. In our calculations we shall take B (our initial chamber) to be the one for whom

$y = y_1 =$  , $y_2 =$  ,

$y_3 =$  ,

and S is the standard sextet. So

$$S := \begin{array}{|cc|cc|cc|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array},$$

the positions labelled i where $i = 1, \ldots, 6$ indicating the tetrads of S. (Also note that, for example,

$$\begin{array}{|cc|cc|cc|}\hline 2 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 2 & 5 & 5 & 5 & 5 \\ 1 & 2 & 6 & 6 & 6 & 6 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|cc|cc|cc|}\hline 5 & 4 & 1 & 1 & 1 & 1 \\ 4 & 5 & 2 & 2 & 2 & 2 \\ 4 & 5 & 3 & 3 & 3 & 3 \\ 4 & 5 & 6 & 6 & 6 & 6 \\ \hline \end{array}$$

describe the same sextet.) We may describe chambers of $\mathcal{C}$ in the following concise manner, similar to that used in [17]. We illustrate this notation with an example by describing B as follows

$$\begin{array}{|cc|cc|cc|}\hline 1^* & 2^* & 3^* & 4^* & 5^* & 6^* \\ 1^* & 2^* & 3^* & 4^* & 5^* & 6^* \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}$$
$$\underline{12}|34|56$$

The positioning of $1, 2, 3, 4, 5, 6$ indicates the sextet, and the partition underneath gives the trio (which here consists of the three octads O_1, O_2, O_3). The $*$ gives a partition of each tetrad and tells us which pairs are interchanged by the involutions y_i and, finally, the underlining of 1 and 2 identify $y_1 = y$ by giving the octad of fixed points of y, namely O_1 (the union of tetrads 1 and 2).

2.2 $G \cong He$

Here we have $|G| = 2^{10}.3^3.5^2.7^3.17$, $[G : B] = |\mathcal{C}| = 3,935,925$ with the permutation rank of G on $\mathcal{C}$ being $4,831$. In this case the shapes of P_{12}, P_{13} and P_{23} are the same as in (2.1). The smallest non-trivial permutation representation of G has degree $2,058$ (with point stabilizer isomorphic to $Sp_4(4) : 2$). Let Ω be a $2,058$ set upon which G acts (in our calculations we employ the representation available from [22]). We may give an alternative description of Γ, using certain subsets of Ω which we now define.

$$\Gamma_1 = \{Fix(Q_{23})^g \mid g \in G\}$$

$$\Gamma_2 = \{Fix(Z(Q_{13}))^g \mid g \in G\}$$

$$\Gamma_3 = \{Fix(Z(P_{12}))^g \mid g \in G\}$$

Subsets in Γ_1 have size 6, those in Γ_2 size 18 and those in Γ_3 size 42. The symmetric incidence relation is just (symmetrized) containment. Thus a chamber γ may be identified with a triple of subsets of Ω which we denote by $\{F_1(\gamma), F_2(\gamma), F_3(\gamma)\}$ where $F_1(\gamma) \subseteq F_2(\gamma) \subseteq F_3(\gamma)$, $F_i(\gamma)$ the appropriate fixed point set with $|F_1(\gamma)| = 6, |F_2(\gamma)| = 18$ and $|F_3(\gamma)| = 42$. For two chambers γ, γ' of $\mathcal{C}$ we may define the intersection matrix $I(\gamma, \gamma')$ to be the 3×3 matrix whose $(i,j)^{th}$ – entry is $|F_i(\gamma) \cap F_j(\gamma')|$.

2.3 $G \sim 3^{7} Sp_6(2)$

In this case we have $|G| = 2^9.3^{11}.5.7$ with $[G : B] = |\mathcal{C}| = 6,200,145$. The permutation rank of G on $\mathcal{C}$ is $15,150$. The shape of P_{12}, P_{13} and P_{23} are as follows: $P_{12} \sim 2^{3+3}L_3(2), P_{13} \sim 2^{2+1+4}(Sym(3) \times Sym(3))$ and $P_{23} \sim 2^{1+4}3Sym(6)$. For our calculations we obtain a copy of G by first availing ourselves of the construction given in Section 3 of [10]. Specifically, we take the presentation given in Lemma 12 of [10] for, in the notation there, G_0. We have that $G \cong G_0$ and by selecting $H = \langle u_1, \ldots, u_9, w_2, w_3 >$ (notation as in [10]) we get $[G_0 : H] = 3^8.7 = 45,927$ with $core_{G_0}(H) = 1$. Hence we have a $45,927$ degree permutation representation of G. Further in G (or more accurately in G_0) we have another core-free subgroup, of shape $3^6U_4(2)2$, which then yields a permutation representation for G of degree 84, and this is the representation we employ in our calculations.

3 Structure of the Chamber Graphs

As indicated earlier we will carry out our calculations in each of the three target groups whose minimal parabolic geometry were detailed in Section 2. Let G denote any one of those three groups with $\{P_1, P_2, P_3\}$ the rank 3 minimal parabolic system for G with diagram as in Theorem 1.1. We use all the associated notation introduced in Section 2. In particular, Γ will denote the minimal parabolic geometry over $I = \{1,2,3\}$ and $\mathcal{C}$ its chamber graph. We recall that $B = N_G(S) = S \in Syl_2(G)$. Since the right cosets of B in G are in one-to one correspondence with the chambers of Γ, we identify the vertices (chambers) of $\mathcal{C}$ with this set of right cosets. Employing the Magma command DoubleCosetRepresentation(G,B,B) delivers these as an ordered sequence, which we denote by DB. In order to save on storage space we now further identify a vertex (chamber) of $\mathcal{C}$ with $i \in \{1, \ldots, \ell\}$ where ℓ is the length of DB. So we have chambers Bg identified by i where $g = DB[i]$. And we take B (which corresponds to $i = 1$) as our fixed vertex (chamber) of $\mathcal{C}$ whose discs we determine. First we itemize the neighbours of B in $\mathcal{C}$ as an ordered sequence thus (recall for $\mathcal{C}$ we have that the valency is 6):

$$[Tr1[2], Tr1[3], Tr2[2], Tr2[3], Tr3[2], Tr3[3]].$$

Here Tri for $i = 1, 2, 3$ denotes the ordered sequence $Transversal(P_i, B)$. Since Tri[1] is the identity element, this gives the chambers incident with B in $\mathcal{C}$. We call this sequence $NeighboursofB$, and have put this data in a sequence so as to keep track of i-adjacency ($i \in I$). The first two entries are 1-adjacent to B, the next two are 2-adjacent to B and the final two are 3-adjacent to B. The sequence $Neighbours$ captures the essential structure of $\mathcal{C}$. So the i^{th} entry concerns the neighbours of Bg where $g = DB[i]$ and is a sequence of length 6, $[j_1, j_2, j_3, j_4, j_5, j_6]$ ($j_1, j_2, j_3, j_4, j_5, j_6 \in \{1, \dots, \ell\}$) and tells us, where $g_i = DB[j_i]$, that the chambers Bg_1 and Bg_2 are both 1-adjacent to Bg, Bg_3 and Bg_4 are 2-adjacent to Bg while Bg_5 and Bg_6 are 3-adjacent to Bg. Along the way we also assemble $BorbitsDiscs$, a sequence whose i^{th} entry is a set $\{j_1, \dots, j_r\}$ which indicates that the i^{th} disc of B is the union of the B-orbits of Bg_{j_k} ($1 \leq k \leq r$) where $g_{j_k} = DB[j_k]$. From this information the size of the i^{th} disc of B is easily obtained and the length of the sequence $BorbitsDiscs$ is the diameter of $\mathcal{C}$.

The files accompanying this paper contain the following information and programs. First there are three files, one for each of M_{24}, He and $3^{7\cdot}Sp_6(2)$, and called, respectively, $ChamberGraphM_{24}$, $ChamberGraphHe$ and $ChamberGraph3^7Sp(6,2)$. Each contains permutations which generate G, along with generators for P_1, P_2, P_3 and B (of degree 24, 2058 and 84 respectively). These permutation representations are the ones used in our calculations. Next we give DB, the sequence of double coset representatives for B in G. This is followed by the ordered sequences $TrP1$, $TrP2$ and $TrP3$ which are then used to define the ordered sequence $NeighboursofB$. The last two pieces of data are the output from running the the programs, namely $Neighbours$ and $BorbitsDiscs$. Then there is one further file, $ProgramsDiscStructureandGeodesicClosure$, in which is to be found the programs for determining the disc structure and the geodesic closure of maximal opposite chambers. There are four programs for calculating geodesic closures. The first is a generic version which may give duplication of some of the chambers. The other three programs (taking the first program output as their input) removes these possible duplicates for each of the three geometries, and describes the chambers in a more combinatorial fashion. For $G \cong M_{24}$, chambers are displayed using the MOG, for $G \cong He$ we use the fixed-point sets mentioned in (2.2) and for $G \sim 3^{7\cdot}Sp_6(2)$ the orbits of B^g (for the coset Bg) on Ω, where $|\Omega| = 84$. We do not record the output from the geodesic closure programs as this is easily (and quickly) obtained by running the programs.

4 The Last Disc and Geodesic Closures

We first look at the structure of the last disc of $\mathcal{C}$, that is the disc of maximal distance from our initial chamber γ_0 (which corresponds to B).

Theorem 4.1 *Let G be one of M_{24}, He and $3^{7\cdot}Sp_6(2)$, and $\mathcal{C}$ the chamber graph of Γ.*

1. *If $G = M_{24}$, then the last disc of $\mathcal{C}$ is the union of two B-orbits $\Delta^1_{17}(\gamma_0)$ and $\Delta^2_{17}(\gamma_0)$ with $|\Delta^1_{17}(\gamma_0)| = 128$ and $|\Delta^2_{17}(\gamma_0)| = 256$. Further every chamber in $\Delta^1_{17}(\gamma_0)$ is 1-adjacent to two chambers in $\Delta^2_{17}(\gamma_0)$*

2. *If $G = He$, then the last disc of $\mathcal{C}$, $\Delta_{21}(\gamma_0)$, is a B-orbit. Moreover, $\Delta_{21}(\gamma_0)$ is a co-clique.*

3. *If $G = 3^{7\cdot}Sp_6(2)$, then the last disc of $\mathcal{C}$ is the union of two B-orbits $\Delta^1_{25}(\gamma_0)$ and $\Delta^2_{25}(\gamma_0)$ with $|\Delta^1_{25}(\gamma_0)| = 64 = |\Delta^2_{25}(\gamma_0)|$. Further, $\Delta_{25}(\gamma_0)$ is a co-clique.*

As preparation for Theorem 4.2 where we look at $G \cong M_{24}$, we introduce some chambers of $\mathcal{C}$. Let γ_0 be the chamber corresponding to B. Thus

$$\gamma_0 := \begin{array}{|cc|cc|cc|} \hline 1^* & 2^* & 3^* & 4^* & 5^* & 6^* \\ 1^* & 2^* & 3^* & 4^* & 5^* & 6^* \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}.$$

$\underline{12}|34|56$

We have that $\Delta_{17}(\gamma_0) = \Delta^1_{17}(\gamma_0) \cup \Delta^2_{17}(\gamma_0)$ and we take $\gamma_1 \in \Delta^1_{17}(\gamma_0)$ and $\gamma_2 \in \Delta^2_{17}(\gamma_o)$ where

$$\gamma_1 := \begin{array}{|cc|cc|cc|} \hline 5 & 6 & 2^* & 1^* & 6^* & 5 \\ 4^* & 4 & 4 & 3^* & 1^* & 1 \\ 3^* & 3 & 4^* & 3 & 2^* & 2 \\ 6^* & 5^* & 1 & 2 & 6 & 5^* \\ \hline \end{array} \quad \text{and} \quad \gamma_2 := \begin{array}{|cc|cc|cc|} \hline 6^* & 6 & 1^* & 1 & 5^* & 6^* \\ 3^* & 4^* & 4^* & 3 & 1 & 2 \\ 3 & 4 & 3^* & 4 & 1^* & 2^* \\ 5^* & 5 & 2 & 2^* & 6 & 5 \\ \hline \end{array}$$

$\underline{12}|34|56$ $\qquad$ $\underline{12}|34|56$

We shall also encounter the following batch of chambers.

$$\gamma_3 := \begin{array}{|cc|cc|cc|} \hline 2^* & 6 & 6 & 2 & 5^* & 4 \\ 1^* & 1 & 4^* & 4 & 1^* & 3 \\ 3^* & 3 & 5 & 5^* & 1 & 3^* \\ 6^* & 2 & 6^* & 2^* & 4^* & 5 \\ \hline \end{array} \qquad \gamma_4 := \begin{array}{|cc|cc|cc|} \hline 2^* & 5 & 3 & 1^* & 4^* & 4 \\ 1^* & 1 & 4^* & 6 & 2 & 5 \\ 3^* & 3 & 6^* & 4 & 2^* & 5^* \\ 5^* & 2 & 3^* & 1 & 6 & 6^* \\ \hline \end{array}$$

$13|26|\underline{45}$ $\qquad$ $13|25|\underline{46}$

$$\gamma_5 := \begin{array}{|cc|cc|cc|} \hline 5 & 2^* & 1^* & 3 & 6^* & 6 \\ 1^* & 1 & 4^* & 6 & 2^* & 5^* \\ 3^* & 3 & 6^* & 4 & 2 & 5 \\ 2 & 5^* & 1 & 3^* & 4 & 4^* \\ \hline \end{array} \qquad \gamma_6 := \begin{array}{|cc|cc|cc|} \hline 5^* & 2 & 1 & 3^* & 6 & 6^* \\ 1^* & 1 & 4^* & 6 & 2^* & 5^* \\ 3^* & 3 & 6^* & 4 & 2 & 5 \\ 2^* & 5 & 1^* & 3 & 4^* & 4 \\ \hline \end{array}$$

$13|25|\underline{46}$ $\qquad$ $13|26|\underline{45}$

$$\gamma_7 := \begin{array}{|cc|cc|cc|} \hline 2^* & 2 & 2 & 6 & 4 & 4^* \\ 1^* & 5^* & 3^* & 4^* & 1^* & 5 \\ 1 & 5 & 3 & 4 & 5^* & 1 \\ 6^* & 6 & 6^* & 2^* & 3^* & 3 \\ \hline \end{array} \qquad \gamma_8 := \begin{array}{|cc|cc|cc|} \hline 5^* & 2^* & 1^* & 1 & 4 & 6^* \\ 1^* & 3 & 4^* & 6^* & 2^* & 2 \\ 3^* & 1 & 4 & 6 & 5 & 5^* \\ 5 & 2 & 3 & 3^* & 6 & 4^* \\ \hline \end{array}$$

$15|26|\underline{34}$ $\qquad$ $13|25|\underline{46}$

$\gamma_9 :=$

6^*	6	3^*	1	4^*	5
1^*	3^*	4	4^*	2^*	6^*
1	3	5	5^*	6	2
2^*	2	1^*	3	4	5^*

$13|26|\underline{45}$

$\gamma_{10} :=$

2^*	5	3^*	3	6^*	4^*
1^*	3	4^*	6	2^*	2
3^*	1	4	6^*	5^*	5
2	5^*	1	1^*	4	6

$13|25|\underline{46}$

$\gamma_{11} :=$

2^*	2	1^*	5^*	4	3^*
1	5^*	3^*	3	2^*	6
1^*	5	4^*	4	6^*	2
6	6^*	5	1	4^*	3

$15|26|\underline{34}$

$\gamma_{12} :=$

5^*	3^*	5^*	5	6	4
1^*	2	4^*	6	2^*	2
2^*	1	6^*	4	1	1^*
5	3	3	3^*	6^*	4^*

$12|35|\underline{46}$

$\gamma_{13} :=$

5^*	2	1	1^*	4^*	6
1^*	3	4^*	6^*	2^*	2
3^*	1	4	6	5^*	5
5	2^*	3^*	3	6^*	4

$13|25|\underline{46}$

$\gamma_{14} :=$

5	2^*	1^*	1	4	6^*
1^*	3^*	4^*	6^*	2^*	2
3	1	4	6	5^*	5
5^*	2	3^*	3	6	4^*

$13|25|\underline{46}$

$\gamma_{15} :=$

2^*	5	3^*	3	6^*	4
1^*	3^*	4^*	6^*	2	2^*
3	1	4	6	5	5^*
2	5^*	1^*	1	4^*	6

$13|25|\underline{46}$

$\gamma_{16} :=$

2	5	3^*	3	6	4^*
1^*	3	4^*	6^*	2	2^*
3^*	1	4	6	5^*	5
2^*	5^*	1	1^*	4	6^*

$13|25|\underline{46}$

Theorem 4.2 *Suppose $G \cong M_{24}$ and $\mathcal{C}$ is the chamber graph of Γ. Then a maximal opposite set of chambers of $\mathcal{C}$ consists of three chambers. There are* 14 *G-orbits on the set of maximal opposite sets of $\mathcal{C}$ with representatives $\{\gamma_0, \gamma_1, \gamma_i\}(3 \leq i \leq 11)$ and $\{\gamma_0, \gamma_2, \gamma_i\}(12 \leq i \leq 16)$.*

Proof For these calculations, as we wish to express our chambers in terms of the MOG, we identify γ_0 (which corresponds to B) with the ordered triple

$$[\langle y\rangle, \langle y, y_2\rangle, O_2(Stab_G(S))].$$

Recall that S is the standard sextet of the MOG. For an arbitrary chamber Bg we identify it with

$$[\langle y\rangle^g, \langle y, y_2\rangle^g, O_2(Stab_G(S))^g].$$

We easily recover the combinatorial description of Bg from this, by examining y^g, y_2^g and the orbits of $O_2(Stab_G(S))^g$ on Ω. Using the representatives $\gamma_1 \in \Delta^1_{17}(\gamma_0)$ and $\gamma_2 \in \Delta^2_{17}(\gamma_0)$, applying B we calculate $\Delta^1_{17}(\gamma_0)$ and $\Delta^2_{17}(\gamma_0)$, whence we obtain $\Delta_{17}(\gamma_0) = \Delta^1_{17}(\gamma_0) \cup \Delta^2_{17}(\gamma_0)$. Applying $DB[388]$ (number 388 of the double coset representatives) to $\Delta_{17}(\gamma_0)$ yields $\Delta_{17}(\gamma_1)$, and we then see that $|\Delta_{17}(\gamma_0) \cap \Delta_{17}(\gamma_1)| = 24$. First we examine those maximal opposite sets of chambers where two of its chambers γ, γ' are such that $\gamma' \in \Delta^1_{17}(\gamma)$. Up to G-action we may assume $\gamma = \gamma_0$ and $\gamma' = \gamma_1$.

Now $|\Delta_{17}(\gamma_0) \cap \Delta^1_{17}(\gamma_1)| = 8$ and $|\Delta_{17}(\gamma_0) \cap \Delta^2_{17}(\gamma_1)| = 16$. We have $G_{\gamma_0\gamma_1} \cong Dih(8)$, and we consider the action of $G_{\gamma_0\gamma_1}$ on $\Delta_{17}(\gamma_0) \cap \Delta^1_{17}(\gamma_1)$. There are four $G_{\gamma_0\gamma_1}$-orbits on $\Delta_{17}(\gamma_0) \cap \Delta^1_{17}(\gamma_1)$, of sizes $4, 2, 1$ and 1 with representatives being, respectively, $\gamma_3, \gamma_4, \gamma_5$, and γ_6. Let γ'' denote any of these four representatives. Calculation reveals that

$$\Delta_{17}(\gamma_0) \cap \Delta_{17}(\gamma_1) \cap \Delta_{17}(\gamma'') = \emptyset$$

and so $\{\gamma_0, \gamma_1, \gamma_j\}$ for $j = 3, 4, 5, 6$ are maximal opposite sets of chambers. Since $G_{\gamma_0\gamma_1\gamma_5} = G_{\gamma_0\gamma_1\gamma_6} = G_{\gamma_0\gamma_1} \cong Dih(8)$, $|G_{\gamma_0\gamma_1\gamma_4}| = 4$ and $|G_{\gamma_0\gamma_1\gamma_3}| = 2$, only $\{\gamma_0, \gamma_1, \gamma_5\}$ and $\{\gamma_0, \gamma_1, \gamma_6\}$ could possibly be in the same G-orbit. These two sets are in the same G-orbit if and only if they are in the same $N_G(D)$-orbit where $D = G_{\gamma_0\gamma_1}$. Since $|N_G(D)| = 2^6.3$, we quickly check that they are not in the same $N_G(D)$-orbit. Thus $\{\gamma_0, \gamma_{1,j}\}$ for $j = 3, 4, 5, 6$ are representatives from different G-orbits.

We repeat the above analysis looking first at the $G_{\gamma_0\gamma_1}$-orbits on $\Delta_{17}(\gamma_0) \cap \Delta^2_{17}(\gamma_1)$, and calculate that there are five $G_{\gamma_0\gamma_1}$-orbits. Their sizes are $8, 2, 2, 2, 2$ with representatives, respectively, $\gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}$. Since

$$\Delta_{17}(\gamma_0) \cap \Delta_{17}(\gamma_1) \cap \Delta_{17}(\gamma') = \emptyset$$

for γ' any one of these five representatives, we obtain further maximal opposite sets $\{\gamma_0, \gamma_1, \gamma_j\}, j = 7, 8, 9, 10, 11$. We claim that no two of $\{\gamma_0, \gamma_1, \gamma_j\}$ and $\{\gamma_0, \gamma_1, \gamma_k\}$ $(j, k \in \{7, 8, 9, 10, 11\}, j \neq k)$ are in the same G-orbit. For, if there exists $g \in G$ such that $\{\gamma_0, \gamma_1, \gamma_j\}^g = \{\gamma_0, \gamma_1, \gamma_k\}$, then, as $\gamma_1 \in \Delta^1_{17}(\gamma_0), \gamma_1 \in \Delta^2_{17}(\gamma_j) \cap \Delta^2_{17}(\gamma_k), \gamma_0 \in \Delta^2_{17}(\gamma_j) \cap \Delta^2_{17}(\gamma_k)$ (and these are symmetric relations), we must have either $\gamma_0^g = \gamma_0, \gamma_1^g = \gamma_1$ or $\gamma_0^g = \gamma_1, \gamma_1^g = \gamma_0$. Hence $g \in N_G(D)$ where $D = G_{\gamma_0\gamma_1}$. Checking in $N_G(D)$ we see that no such g exists, so establishing the claim. So $\{\gamma_0, \gamma_1, \gamma_j\}, j = 7, 8, 9, 10, 11$ are in different G-orbits.

It remains to look at those maximal opposite sets where for any two different chambers γ, γ' in the set, $\gamma' \in \Delta^2_{17}(\gamma)$. Up to G-action, we may assume such sets contain γ_0 and γ_2. We have that $G_{\gamma_0\gamma_2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For further chambers to add to γ_0 and γ_2 we must look in $\Delta^2_{17}(\gamma_0) \cap \Delta^2_{17}(\gamma_2)$. Now $|\Delta^2_{17}(\gamma_0) \cap \Delta^2_{17}(\gamma_2)| = 8$ and this set has five $G_{\gamma_0\gamma_2}$-orbits of sizes $4, 1, 1, 1, 1$ with representatives, respectively, $\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{16}$. Since

$$\Delta^2_{17}(\gamma_0) \cap \Delta^2_{17}(\gamma_2) \cap \Delta^2_{17}(\gamma_j) = \emptyset$$

for $j = 12, 13, 14, 15, 16$, we get maximal opposite sets $\{\gamma_0, \gamma_2, \gamma_j\}, j = 12, 13, 14, 15, 16$. Clearly any two of $\{\gamma_0, \gamma_2, \gamma_j\}, j = 13, 14, 15, 16$ are in the same G-orbit if and only if they are in the same $N_G(E)$-orbit where $E = G_{\gamma_0\gamma_2}$. We have $|N_G(E)| = 2^9.3^2.5$ and we check that no two of these maximal opposite sets are in the same $N_G(E)$-orbit. Thus we obtain five further G-orbit representatives of maximal opposite sets and this completes the census of G-orbit representatives and the proof of Theorem 4.2.

□

Theorem 4.3 *Suppose $G \cong He$ or $G \sim 3^7 Sp_6(2)$ and $\mathcal{C}$ is the chamber graph of Γ. Then a maximal opposite set of $\mathcal{C}$ consists of three chambers and the set of maximal opposite sets of $\mathcal{C}$ form one G-orbit.*

Proof Suppose first that $G \cong He$. Since G_{γ_0} is transitive on $\Delta_{21}(\gamma_0)$, we may assume our maximal opposite set contains $\{\gamma_0, \gamma_1\}$ where $\gamma_1 \in \Delta_{21}(\gamma_0)$ is the chamber corresponding to $B \star DB[4556]$ (the right coset of B containing $DB[4556]$). For these calculations we identify a chamber γ with $\{F_1(\gamma), F_2(\gamma), F_3(\gamma)\}$, as detailed in (2.2). Using the action of B, we determine $\Delta_{21}(\gamma_0)$ (as a set), and by applying $DB[4556]$ to this set we then obtain $\Delta_{21}(\gamma_1)$. As a consequence we see that $|\Delta_{21}(\gamma_0) \cap \Delta_{21}(\gamma_1)| = 4$. Put $D = G_{\gamma_0\gamma_1}$. Now $|D| = 2^4$ (and D has nilpotency class 2) with D acting transitively on $\Delta_{21}(\gamma_0) \cap \Delta_{21}(\gamma_1)$, Selecting $\gamma_2 \in \Delta_{21}(\gamma_0) \cap \Delta_{21}(\gamma_1)$, we calculate that

$$\Delta_{21}(\gamma_0) \cap \Delta_{21}(\gamma_1) \cap \Delta_{21}(\gamma_2) = \emptyset.$$

Hence $\{\gamma_0, \gamma_1, \gamma_2\}$ is a maximal opposite set and there is just one G-orbit on the set of maximal opposite chambers of $\mathcal{C}$.

In the case when $G \sim 3^{7\cdot}Sp_6(2)$, we choose $\gamma' \in \Delta^1_{25}(\gamma_0)$, corresponding to the chamber $B \star DB[38]$. Calculation shows that $|\Delta_{25}(\gamma_0) \cap \Delta_{25}(\gamma')| = 1$ with $\Delta_{25}(\gamma_0) \cap \Delta_{25}(\gamma') \subseteq \Delta^2_{25}(\gamma_0)$. From this we deduce that maximal opposite sets have size 3 and they form a G-orbit.

□

If $\gamma, \gamma' \in \mathcal{C}$, then a geodesic between γ and γ' is a path in $\mathcal{C}$ between γ and γ' whose length is $d(\gamma, \gamma')$. For X a set of chambers of $\mathcal{C}$, we define the *geodesic closure* of X, $\bar{X}$, to be the set of chambers which are in a geodesic between γ and γ' where γ and γ' are any chambers in X.

Our final result concerns geodesic closures of pairs of chambers at maximal distance from each other.

Theorem 4.4 *(i) Suppose that $G = M_{24}$. Set $n_{1,j} = |\overline{\{\gamma_0, \gamma_1\}} \cap \Delta_j(\gamma_0)|$ and $n_{2,j} = |\overline{\{\gamma_0, \gamma_2\}} \cap \Delta_j(\gamma_0)|$. Then*

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n_{1,j}$	1	4	8	24	24	16	16	16	24	24	16	16	16	24	24	8

j	16	17
$n_{1,j}$	4	1

and

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n_{2,j}$	1	4	12	28	36	44	42	50	50	50	50	42	44	36	28	12

j	16	17
$n_{2,j}$	4	1

Hence $|\overline{\{\gamma_0, \gamma_1\}}| = 266$ and $|\overline{\{\gamma_0, \gamma_2\}}| = 534$.

(ii) Suppose that $G = He$, and let $\gamma' \in \Delta_{21}(\gamma_0)$. Set $n_j = |\overline{\{\gamma_0, \gamma'\}} \cap \Delta_j(\gamma_0)|$. Then

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n_j	1	6	12	24	24	32	40	42	32	42	52	52	42	32	42	40

j	16	17	18	19	20	21
n_j	32	24	24	12	6	1

So $|\overline{\{\gamma_0, \gamma'\}}| = 614$.

(iii) Suppose that $G = 3^7{}^{\cdot}Sp_6(2)$*, and let* $\gamma' \in \Delta^1_{25}(\gamma_0)$*. Set* $n_j = |\overline{\{\gamma_0, \gamma'\}} \cap \Delta_j(\gamma_0)|$*. Then*

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
n_j	1	6	18	25	36	64	88	128	160	168	180	176	184	192

j	14	15	16	17	18	19	20	21	22	23	24	25
n_j	176	184	164	152	116	68	56	40	25	20	6	1

.

So $|\overline{\{\gamma_0, \gamma'\}}| = 2,434$*. The corresponding table for* $\gamma' \in \Delta^2_{25}(\gamma_0)$ *is obtained by reading the lower row in reverse order.*

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