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Abstract

Explicit symmetry breaking occurs when a dynamical system having a certain symmetry group is perturbed in a way that the perturbation preserves only some symmetries of the original system. We give a geometric approach to study this phenomenon in the setting of Hamiltonian systems. We provide a method for determining the equilibria and relative equilibria that persist after a symmetry breaking perturbation. In particular a lower bound for the number of each is found, in terms of an equivariant Lyusternik-Schnirelmann category of the group orbit.

Keywords: Symmetry breaking, Hamiltonian systems, Lie group actions

1. Introduction

When we talk about symmetries, we either refer to the symmetry of a physical law (dynamical equations) or the symmetry of a physical state (solution of these equations). The *symmetry* or *symmetry group* of a physical law (or a physical state) is defined to be the group of transformations which leave these equations (or this solution) invariant. *Explicit symmetry breaking* is defined as a process of perturbing symmetric dynamical equations such that the resulting equations have a lower symmetry group. Any physical law observed in nature can be thought as a perturbation of a physical law having a bigger symmetry group. However the more symmetric a dynamical system is, the more simple its solutions are. In fact, complicated and interesting dynamical behaviours require low symmetry groups. For example, Lauterbach et al. [CL04, GL01, LR92] show that some periodic solutions of an unperturbed dynamical system persist under symmetry breaking perturbations and become heteroclinic cycles.

The lack of symmetries of a perturbed system can be due for example to the presence of terms whose origin is different from case to case. As explained in Brading and Castellani [BC03], such terms can be introduced artificially in order to match with theoretical or experimental observations. For example in quantum field theory, the Lagrangian for weak interactions is constructed so that the parity-symmetry and the charge-parity symmetry are violated, making the theory in the line with experimental observations. Besides, quantization processes might also be a cause for the appearance of such terms which are the so-called quantum anomalies. In this case, the terms are not artificially introduced but they appear after a renormalization procedure.

The dynamical systems we focus on are Hamiltonian systems. Some aspects of explicit symmetry breaking phenomena are studied by several authors including Ambrosetti et al. [ACZE87], Grabsi, Montaldi and Ortega [GMO04] and Gay-Balmaz and Tronci [GBT10]. Phase spaces of Hamiltonian systems are symplectic manifolds and the symmetries of such systems are encoded into Lie group actions on those manifolds. A symplectic manifold is a smooth manifold M equipped with a non-degenerate closed two-form ω. A (proper) action of a Lie group G on M is *canonical* if it is smooth and it preserves ω. A class of canonical group actions on symplectic manifolds are Hamiltonian. To those actions we can associate a Noether conserved quantity expressed in term of a momentum map $\Phi_G : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. This notion generalizes the notion of angular momentum in classical mechanics, when the phase space is $T^*\mathbb{R}^3$, acted on by the group of rotations $SO(3)$. By a *Hamiltonian (proper) G-manifold*, we mean a quadruple (M, ω, G, Φ_G) as described above, where $\Phi_G : M \to \mathfrak{g}^*$ is equivariant with respect to the coadjoint action $\mathrm{Ad}^* : (g, \mu) \in G \times \mathfrak{g}^* \mapsto \mathrm{Ad}^*_{g^{-1}}\mu \in \mathfrak{g}^*$.

The dynamics is governed by a Hamiltonian h which is a G-invariant real-valued function defined on M. The ring of such functions is denoted $C^\infty(M)^G$. The non-degeneracy of ω implies that, associated to any Hamiltonian $h \in C^\infty(M)^G$, there is a unique vector field X_h defined by $\iota_{X_h}\omega = -dh$. Since the action of G on M is canonical and h is G-invariant, the integral curve $\varphi_t(m)$ of X_h starting at $m \in M$ satisfies $\varphi_t(g \cdot m) = g \cdot \varphi_t(m)$ for all $g \in G$. The resulting Hamiltonian equations

$$\frac{d}{dt}\varphi_t(m) = X_h(\varphi_t(m)) \tag{1}$$

are thus G-equivariant and we say that G is the *symmetry group* of (1). We study the effect of a small Hamiltonian perturbation of these equations, which is invariant with respect to a subgroup of G.

Definition 1.1. Let $h \in C^\infty(M)^G$ and $H \subset G$ be a closed subgroup. An *H-pertubation* of h is a family of functions $h_\lambda \in C^\infty(M)^H$ such that the map $(m, \lambda) \in M \times \mathbb{R} \mapsto h_\lambda(m) \in \mathbb{R}$ is smooth, and $h_0 = h$.

We focus on specific solutions of (1), namely equilibria (fixed points under the dynamics) and relative equilibria (group orbits fixed under the dynamics). Under a specific non-degeneracy condition on a (relative) equilibrium of the unperturbed Hamiltonian h, there is a chance that this (relative) equilibrium persists under a small H-perturbation.

Section 3 is devoted to the question of persistence of equilibria. In this case, the required non-degeneracy condition on an equilibrium $m \in M$ of h is a particular case of Morse-Bott condition, when the critical manifold of h is the group orbit $G \cdot m$ (cf. Definition 3.1). We show that at least a certain number of H-orbits of equilibria persist under a small H-perturbation, in a tubular neighbourhood of $G \cdot m$ (cf. Theorem 3.1 and Corollary 3.2). This number is the positive integer $\mathrm{Cat}_H\,(G/G_m)$, which is the H-equivariant Lyusternik-Schnirelmann category of the group orbit. We present applications of our result, including the problem of an ellipse-shaped planar rigid body moving in a planar irrotational, incompressible fluid with zero vorticity and zero circulation around the body.

Extending Theorem 3.1 and Corollary 3.2 to the case of relative equilibria is more subtle because we must take into account the conservation of momentum. This question is treated in Section 4. Whereas equilibria are just critical points of the Hamiltonian function h, relative equilibria are critical points of the restriction of this same function to a level set $\Phi_G^{-1}(\mu)$ of the momentum map, the problem being that as the group changes, so do these level sets. Let $m \in M$ be one of those critical points. The element $\xi \in \mathfrak{g}$ playing the role of a Lagrange multiplier is called the velocity of m, which is in general not unique when the action is not free. For that reason, we refer to a relative equilibrium as a pair $(m, \xi) \in M \times \mathfrak{g}$. We denote the underlying Lagrange function associated to ξ by h^ξ.

A standard definition says that a relative equilibrium (m, ξ) of h is non-degenerate if the Hessian of h^ξ at m is a non-singular quadratic form when restricted to some symplectic subspace $N_1 \subset T_m M$, called the symplectic slice at m. If the perturbations h_λ are invariant with respect to the full symmetry group G, this notion of non-degeneracy is enough to guarantee the persistence of a relative equilibrium. This is no longer the case if h_λ has a smaller symmetry group than the one of h and we require a stronger non-degeneracy condition on the relative equilibrium (cf. Definition 4.1). In Grabsi, Montaldi and Ortega [GMO04] a step in that direction is taken, when the symmetry group is a torus that breaks into a subtorus. In addition, the group actions in consideration are assumed to be free. We extend their result to non-free actions and non-abelian symmetry groups.

A necessary condition for a relative equilibrium of h to persist under an H-perturbation is that the velocity ξ belongs to $\mathfrak{h}$, the Lie algebra of H. If the non-degeneracy condition on $(m, \xi) \in M \times \mathfrak{h}$ holds, and modulo some technicalities, the least number of H_μ-orbits of relative equilibria with velocity close to ξ, which persist under a small H-perturbation in some neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$, is the positive integer

$$\mathrm{Cat}_{H_\mu}\left(G_\mu / G_m\right).$$

This is the content of Theorem 4.3 and Corollary 4.4. We illustrate this result for the spherical pendulum on S^3, as a perturbation of the geodesic flow. This is an example of symmetry breaking from $SO(4)$ to $SO(3)$.

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2. Preliminaries

In this section we recall the equivariant version of the Lyusternik-Schnirelmann Theorem and the symplectic local model for a Hamiltonian proper G-manifold near a group orbit.

In their original paper [LS47], Lyusternik and Schnirelmann introduce a numerical homotopy invariant of a topological space M that they denote $\mathrm{Cat}(M)$. They define it to be the least number of open subsets of M, whose inclusion is nullhomotopic, that are required to cover M. They show that if M is a closed (i.e. compact without boundary) C^2-manifold, then any function $f \in C^1(M)$ has at least $\mathrm{Cat}(M)$ critical points. The infinite dimensional case has been studied by SCHWARTZ [Sch64] when M is a complete C^2-Hilbert manifold, and f satisfies a suitable compactness condition. The equivariant analogue has been proved by FADELL [Fad85] and MARZANTOWICZ [Mar89] in the case when G is a compact Lie group. The extension to proper Lie group actions can be found in AYALA, LASHERAS AND QUINTERO [ALQ01]. They define $\mathrm{Cat}_G(M)$ to be the least number of G-invariant open subsets of M, which admits a G-deformation retract onto a G-orbit, that are required to cover M.

If (M, G) is a proper G-manifold, there is a G-invariant Riemannian metric on M. Given a function $f \in C^\infty(M)^G$ the associated gradient vector field ∇f is G-equivariant. We say that $m \in M$ is a *critical point* of f if $\nabla f(m) = 0$. Denote by $\mathcal{C}_c(f)$ the set of critical points m such that $f(m) = c$. By G-invariance of f, if $m \in \mathcal{C}_c(f)$ then $G \cdot m \subset \mathcal{C}_c(f)$. The extension of the Lyusternik-Schnirelmann Theorem to non-compact manifolds requires some additional assumptions on M and f.

Definition 2.1 (Orbitwise Palais-Smale condition [ALQ01]). A proper G-manifold (M, G) and a function $f \in C^\infty(M)^G$ satisfy the *orbitwise Palais-Smale condition (OPS)* if the following holds:

(i) M is a complete Riemannian proper G-manifold, without boundary, modelled on a separable Hilbert G-space.

(ii) f is bounded below.

(iii) If $(x_n)_{n\in\mathbb{N}} \subset M$ is a sequence such that the associated sequence of images $f(x_n)$ is bounded and $\|\nabla f(x_n)\|$ converges to zero, then there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset G$ such that the sequence $(g_n \cdot x_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in M.

Condition (iii) is a compactness condition which implies that f restricted to its set of critical orbits modulo G is a proper map. In particular for any real number c, the subset $\mathcal{C}_c(f)$ is a closed bounded subset of M. Since M is complete, the Hopf-Rinow Theorem implies that $\mathcal{C}_c(f)$ is compact in M.

Note that if M is compact, or more generally if f is proper and bounded below then it satisfies (OPS).

THEOREM 2.1 (Equivariant Lyusternik-Schnirelmann Theorem [ALQ01, Bar93]). *If a proper G-manifold (M, G) and a function $f \in C^\infty(M)^G$ satisfy condition (OPS), then f has at least $Cat_G(M)$ group orbits of critical points.*

The Symplectic Tube Theorem

The Symplectic Tube Theorem is used to study the local dynamics and the local geometry of a Hamiltonian proper G-manifold (M, ω, G, Φ_G). It states essentially that every $m \in M$ admits a G-invariant neighbourhood, which is G-equivariantly symplectomorphic to a neighbourhood of the zero section of a symplectic associated bundle. This construction provides tractable semi-global coordinates for M near G-orbits. Those coordinates are sometimes referred as *slice coordinates*. This theorem was obtained by Guillemin and Sternberg [GS84] and by Marle [Mar85], for canonical Lie group actions with equivariant momentum maps. It has been extended independantly by Ortega and Ratiu [OR04] and Bates and Lerman [BL97], for general canonical Lie group actions. Schmah [Sch07] and Perlmutter et al. [PROSD08] studied the case when M is a cotangent bundle.

We briefly recall the construction underlying the Symplectic Tube Theorem. The reader is referred to Ortega and Ratiu (cf. [OR04] Chapter 7) or Cushman and Bates (cf. [OR04] Appendix B Section 3.2) for details. Let $m \in M$ with momentum $\mu = \Phi_G(m)$. Denote by G_m and G_μ the stabilizers of m and μ respectively and by $\mathfrak{g}_m$ and $\mathfrak{g}_\mu$ their respective Lie algebras. The stabilizer G_m is compact by properness of the action of G on M. We can thus split $\mathfrak{g}_\mu$ and $\mathfrak{g}$ into a direct sum of G_m-invariant subspaces

$$\mathfrak{g}_\mu = \mathfrak{g}_m \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{n}.$$

We denote by $\mathfrak{g} \cdot m$ the tangent space at m of $G \cdot m$. Elements of $\mathfrak{g} \cdot m$ are vectors of the form $x_M(m) := \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(tx) \cdot m$, where $x \in \mathfrak{g}$ and $\exp : \mathfrak{g} \to G$ is the group exponential. The tangent space T_mM can be decomposed into a direct sum of four G_m-invariant subspaces

$$T_mM = T_0 \oplus T_1 \oplus N_0 \oplus N_1 \tag{2}$$

defined as follow:

(i) $T_0 := \ker (D\Phi_G(m)) \cap \mathfrak{g} \cdot m = \mathfrak{g}_\mu \cdot m$.

(ii) $T_1 := \mathfrak{n} \cdot m$ which is a symplectic vector subspace of $(T_mM, \omega(m))$.

(iii) N_1 is a choice of G_m-invariant complement to T_0 in $\ker (D\Phi_G(m))$. It is a symplectic subspace of $(T_mM, \omega(m))$ and is called the *symplectic slice*. The linear action of G_m on N_1 is Hamiltonian with momentum map $\Phi_{N_1} : N_1 \to \mathfrak{g}_m^*$ given by $\langle \Phi_{N_1}(\nu), x\rangle = \frac{1}{2}\omega(x_{N_1}(\nu), \nu)$ for every $\nu \in N_1$ and $x \in \mathfrak{g}_m$.

(iv) N_0 is a G_m-invariant Lagrangian complement to T_0 in the symplectic orthogonal $(T_1 \oplus N_1)^{\omega(m)}$. There is an isomorphism $f : N_0 \to \mathfrak{m}^*$ given by $\langle f(w), y\rangle = \omega(m)\,(y_M(m), w)$ for every $w \in N_0$ and $y \in \mathfrak{m}$.

Since N_1 is a G_m-invariant subspace, there is a well-defined action of G_m on the product $G \times \mathfrak{m}^* \times N_1$ given by

$$k \cdot (g, \rho, \nu) = (gk^{-1}, \mathrm{Ad}^*_{k^{-1}}\rho, k \cdot \nu). \tag{3}$$

This action is free and proper by freeness and properness of the action on the G-factor. The orbit space Y is thus a smooth manifold whose points are equivalence classes of the form $[(g,\rho,\nu)]$. The group G acts smoothly and properly on Y, by left multiplication on the G-factor. Let $\mathfrak{m}_0^* \subset \mathfrak{m}^*$ and $(N_1)_0 \subset N_1$ be G_m-invariant neighbourhoods of zero in $\mathfrak{m}^*$ and N_1, respectively. Then

$$Y_0 := G \times_{G_m} (\mathfrak{m}_0^* \times (N_1)_0) \tag{4}$$

is a neighbourhood of the zero section in Y. It comes with a symplectic form ω_{Y_0} if it is chosen small enough (cf. [OR04] Proposition 7.2.2).

THEOREM 2.2 (Symplectic Tube Theorem). *Let (M,ω,G,Φ_G) be a Hamiltonian proper G-manifold. Let $m \in M$ with momentum $\mu = \Phi_G(m)$. If the neighbourhood Y_0 defined in (4) is sufficiently small, it admits a symplectic form ω_{Y_0}. In this case, there exists a G-invariant neighbourhood $U \subset M$ of m and a G-equivariant symplectomorphism $\varphi : (Y_0,\omega_{Y_0}) \to (U, \omega\big|_U)$ such that $\varphi([e,0,0]) = m$.*

We call the triplet (φ, Y_0, U) a *symplectic G-tube* at m and we also say that (Y_0,ω_{Y_0}) is a *symplectic local model* for $(U,\omega\big|_U)$. Besides the momentum map $\Phi_G : M \to \mathfrak{g}^*$ can be expressed in terms of the slice coordinates:

THEOREM 2.3 (Marle-Guillemin-Sternberg Normal Form). *Let (M,ω,G,Φ_G) be a Hamiltonian proper G-manifold and let (φ,Y_0,U) be a symplectic G-tube at $m \in M$. Then the G-action on Y_0 is Hamiltonian with associated momentum map $\widetilde{\Phi}_G : Y_0 \to \mathfrak{g}^*$ defined by*

$$\widetilde{\Phi}_G([g,\rho,\nu]) = Ad^*_{g^{-1}}(\Phi_G(m) + \rho + \Phi_{N_1}(\nu)). \tag{5}$$

If G is connected, (5) coincides with $\Phi_G\big|_U$ when pulled back along φ^{-1}.

3. Symmetry breaking for equilibria

The aim of this section is to give an estimate of the number of H-orbits of equilibria that persist under a small H-perturbation of some G-invariant Hamiltonian h. This is the content of Corollary 3.2. A point $m \in M$ is an *equilibrium* of $h \in C^\infty(M)^G$ if $dh(m) = 0$, or equivalently if $X_h(m) = 0$. Assume N is some vector space and $f : G \times N \to \mathbb{R}$ is a smooth function. We denote by $d_N f$ the partial derivative(s) of f with respect to the N-variable(s). By abuse of notations we write $D^2 f$ for the Hessian of f and $D^2_N f$ for the Hessian with respect to the N-variables.

Definition 3.1. A *G-nondegenerate equilibrium* of $h \in C^\infty(M)^G$ is a point $m \in M$ such that

(i) $dh(m) = 0$,

(ii) $D^2_N h(m)$ is non-singular where N is any subspace of $T_m M$ complementary to $\mathfrak{g} \cdot m$. In other words, the Hessian is non-singular in the directions normal to the group orbit.

If $m \in M$ is a G-nondegenerate equilibrium of h then so is any $p \in G \cdot m$, by G-invariance. For the same reason, the tangent space $T_p(G \cdot m)$ is contained in $\ker\left(D^2 h(p)\right)$ for any $p \in G \cdot m$. Definition 3.1 is a particular case of Morse-Bott non-degeneracy when $G \cdot m$ is the critical manifold of h (cf. [CB15] Appendix E.2). Note that Condition (ii) implies that the critical manifold $G \cdot m$ is isolated in the sense that there exists a tubular neighbourhood of $G \cdot m$ that does not contain any other critical points of h.

Persistence of equilibria

We say that a closed subgroup $H \subset G$ is *co-compact* (in G) if the left multiplication of H on G is co-compact i.e. the orbit space $H \setminus G$ under this action is compact (as a topological space). We can now state the main result of this section.

Theorem 3.1. *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold and let $H \subset G$ be a co-compact closed subgroup. Assume that $h_\lambda \in C^\infty(M)^H$ is an H-pertubation of some $h \in C^\infty(M)^G$ and that $m \in M$ is a G-nondegenerate equilibrium of h.*

Then there is a G-invariant neighbourhood $\widetilde{U} \subset M$ of m such that, if λ is sufficiently small, there exists a function $f_\lambda \in C^\infty(G/G_m)^H$ whose critical points are in one-to-one correspondence with those of h_λ in $\widetilde{U}$.

Proof — Let $m \in M$ be a G-nondegenerate equilibrium of h whose stabilizer is denoted by $K := G_m$. Following the notations of Section 2 the product space $N := \mathfrak{m}^* \times N_1$ is a K-vector space and it is isomorphic to some K-vector space complementary to $\mathfrak{g} \cdot m$ in $T_m M$. By the Symplectic Tube Theorem 2.2 and the G-nondegeneracy of m, we can choose a K-invariant neighbourhood $N_0 \subset N$ of zero, such that

(i) The associated bundle $G \times_K N_0$ is a symplectic local model of some G-invariant neighbourhood $U \subset M$ of m.

(ii) The only critical points of h in U are on $G \cdot m$.

In that model the point m reads $[(e, 0)]$ and the H-pertubation is identified with $h_\lambda : G \times_K N_0 \to \mathbb{R}$. Let $\rho : G \times N_0 \to G \times_K N_0$ be the orbit map. We define the lift of h_λ by

$$\widetilde{h}_\lambda := \rho^* h_\lambda : G \times N_0 \to \mathbb{R}$$

where ρ^* is the pullback map. The critical points of h_λ coincide with those of the lift $\widetilde{h}_\lambda$. Indeed since $\rho : G \times N_0 \to G \times_K N_0$ is a surjective submersion, we have

$$dh_\lambda\left([(g,\nu)]\right) = 0 \quad \Longleftrightarrow \quad \left(\rho^* dh_\lambda\right)\left((g,\nu)\right) = 0 \quad \Longleftrightarrow \quad d\widetilde{h}_\lambda\left((g,\nu)\right) = 0.$$

We may thus work with $\widetilde{h}_\lambda$ instead of h_λ.

We define a (left) action of the direct product $G \times K$ on $G \times N_0$ by

$$(h,k)\cdot(g,\nu) = (hgk^{-1}, k\cdot\nu).$$

By hypothesis, the lift $\widetilde{h}$ is $G\times K$-invariant whereas the perturbation $\widetilde{h}_\lambda$ is only $H \times K$-invariant. Since $(e,0) \in G \times N_0$ is a G-nondegenerate critical point of $\widetilde{h}$,

$$d\widetilde{h}(e,0) = 0 \quad \text{and} \quad D^2_N\widetilde{h}(e,0) \quad \text{is non-singular.} \tag{6}$$

In particular the map

$$d_N\widetilde{h} : G \times N_0 \to N_0^* \simeq N_0$$

satisfies $d_N\widetilde{h}(e,0) = 0$ and its derivative with respect to the N_0-variables, evaluated at $(e,0)$, is non-vanishing. The Implicit Function Theorem implies the existence of a neighbourhood $V_1 \times W_1$ of $(0,e)$ in $\mathbb{R} \times G$ such that, for any $(\lambda,g) \in V_1 \times W_1$, there is a unique $\phi^1_\lambda(g) \in N_0$ satisfying

$$d_N\widetilde{h}_\lambda(g, \phi^1_\lambda(g)) = 0. \tag{7}$$

By H-invariance of $d_N\widetilde{h}_\lambda$, we can choose W_1 to be H-invariant. Note that when we refer to H or K individually we are thinking of them as subgroups of $H \times K$. This procedure defines an H-invariant smooth function

$$\begin{aligned} \phi^1 : V_1 \times W_1 &\longrightarrow N_0 \\ (\lambda, g) &\longmapsto \phi^1_\lambda(g). \end{aligned}$$

By G-invariance of $\widetilde{h}$, (6) holds when replacing $(e,0)$ by any $(g,0) \in G \times N_0$. We apply the previous argument for every $(g,0)$ with $g \notin H$ and use the compacity of $H \setminus G$ to extract a finite collection of open subsets $\{V_i \times W_i\}_{i=2}^n$ with associated H-invariant smooth functions $\phi^i : V_i \times W_i \to N_0$ satisfying (7). Let $V \subset \cap_{i=1}^n V_i$ be an open interval containing $0 \in \mathbb{R}$. By uniqueness of each ϕ^i, we can glue them together to define an H-invariant smooth function

$$\begin{aligned} \phi : V \times G &\longrightarrow N_0 \\ (\lambda, g) &\longmapsto \phi_\lambda(g) \end{aligned}$$

such that

$$d_N\widetilde{h}_\lambda(g, \phi_\lambda(g)) = 0. \tag{8}$$

For every fixed parameter $\lambda \in V$,

$$\widetilde{h}_\lambda(hgk^{-1}, \phi_\lambda(hgk^{-1})) = \widetilde{h}_\lambda(g, \phi_\lambda(g)) \quad \text{for any} \quad (h,k) \in H \times K.$$

It thus descends to a function $f_\lambda \in C^\infty(G/K)^H$ given by

$$f_\lambda\left([g]_{\mathrm{K}}\right) := \widetilde{h}_\lambda\left([g]_{\mathrm{K}}, \phi_\lambda([g]_{\mathrm{K}})\right) \tag{9}$$

where $[g]_{\mathrm{K}}$ denotes a coset in G/K. For any pair $\beta = (\lambda, [g]_{\mathrm{K}}) \in V \times G/K$, we define the shift $\bar{h}_\beta : N_0 \to \mathbb{R}$ by

$$\bar{h}_\beta(\nu) := \widetilde{h}_\lambda([g]_{\mathrm{K}}, \nu + \phi_\lambda\left([g]_{\mathrm{K}}\right)) - f_\lambda([g]_{\mathrm{K}}). \tag{10}$$

This function has a non-degenerate critical point at $0 \in N_0$. Indeed by (8) and K-invariance of $\widetilde{h}_\lambda$,

$$d\bar{h}_\beta(0) = d_N\widetilde{h}_\lambda([g]_{\mathrm{K}}, \phi_\lambda([g]_{\mathrm{K}})) = 0.$$

Moreover, after possibly shrinking V, its Hessian $D^2\bar{h}_\beta(0)$ is non-singular, because non-degeneracy is a stable condition. By the Morse Lemma (cf. Lemma 2.2 in [Mil63]) there is a local coordinate system $\nu_\beta = (\nu^1, \ldots, \nu^\ell)$, defined in a neighbourhood $N_\beta \subset N_0$ of 0 with $\nu_\beta(0) = 0$, such that

$$\bar{h}_\beta(\nu) = \bar{h}_\beta(0) + \sum_{i=1}^{\ell} \varepsilon_i(\nu_i)^2 = \sum_{i=1}^{\ell} \varepsilon_i(\nu_i)^2 \quad \text{for all} \quad \nu \in N_\beta. \tag{11}$$

where $\varepsilon_i = \pm 1$ and $\nu_\beta(\nu) = (\nu_1, \ldots, \nu_\ell)$.

The Morse chart (N_β, ν_β) depends on $\beta = (\lambda, [g]_{\mathrm{K}})$. Since the functions defining (10) are H-invariant, the identity (11) holds on (N_β, ν_β) when replacing $\bar{h}_\beta$ by $\bar{h}_{\beta'}$ where $\beta' = (\lambda, [hg]_{\mathrm{K}})$ with $h \in H$. We repeat the previous argument for every $\beta = (\lambda, [g]_{\mathrm{H,K}}) \in V \times (H\backslash G/K)$, where $[g]_{\mathrm{H,K}}$ denotes the double coset of g. We thus obtain a collection of Morse charts (N_β, ν_β) indexed on $V \times (H\backslash G/K)$. The compacity of $H\backslash G$ is used next to extract a finite number of Morse charts $(N_{\beta_i}, \nu_{\beta_i})$ for $i = 1, \ldots, r$. Using a partition of unity, we construct a local coordinate system $\widetilde{\nu} = (\widetilde{\nu}^1, \ldots, \widetilde{\nu}^\ell)$, defined in a neighbourhood $\widetilde{N}_0 \subset \bigcap_{i=1}^{r} N_{\beta_i}$ of 0 with $\widetilde{\nu}(0) = 0$, such that

$$\bar{h}_\beta(\nu) = \sum_{i=1}^{\ell} \varepsilon_i(\widetilde{\nu}_i)^2 \quad \text{for every} \quad \beta \in V \times G/K, \quad \text{when} \quad \nu \in \widetilde{N}_0. \tag{12}$$

We may define a smooth map $\psi : V \times G/K \times \widetilde{N}_0 \to \widetilde{N}_0$ by

$$\psi(\lambda, [g]_{\mathrm{K}}, \nu) =: \psi_\lambda([g]_{\mathrm{K}}, \nu) = \nu + \phi_\lambda([g]_{\mathrm{K}}). \tag{13}$$

Replacing (13) in (10) yields

$$\widetilde{h}_\lambda([g]_{\mathrm{K}}, \psi_\lambda([g]_{\mathrm{K}}, \nu)) = \sum_i \varepsilon_i(\widetilde{\nu}_i)^2 + f_\lambda([g]_{\mathrm{K}}) \quad \text{whenever} \quad \nu \in \widetilde{N}_0 \tag{14}$$

where $\varepsilon_i = \pm 1$ and $\widetilde{\nu}(\nu) = (\widetilde{\nu}_1, \ldots, \widetilde{\nu}_\ell)$. Therefore $([g]_{\mathrm{K}}, \nu) \in G/K \times \widetilde{N}_0$ is a critical point of (14) if and only if

$$\left(\sum_{i=1}^{\ell} \varepsilon_i \widetilde{\nu}^i d\widetilde{\nu}^i\right)(\nu) = 0 \quad \text{and} \quad df_\lambda([g]_{\mathrm{K}}) = 0.$$

Let $\widetilde{U} \subset M$ be the G-invariant neighbourhood of m whose symplectic local model is $G \times_K \widetilde{N}_0$. In particular if $\lambda \in V$, the critical points of h_λ in $\widetilde{U}$ are in one-to-one correspondence with those of the function $f_\lambda \in C^\infty(G/K)^H$ defined in (9). ■

COROLLARY 3.2 (Persistence of Equilibria). *If the manifold G/G_m and the function $f_\lambda \in C^\infty(G/G_m)^H$ of Theorem 3.1 satisfy condition (OPS), then the number of H-orbits of equilibria that persist near $G \cdot m$ under a small H-perturbation is bounded below by $Cat_H(G/G_m)$.*

Proof — If λ is sufficiently small, Theorem 3.1 implies that the H-orbits of equilibria of h_λ in some neighbourhood of $G \cdot m$ are in one-to-one correspondence with those of the function $f_\lambda \in C^\infty(G/G_m)^H$ defined as in (9). By Theorem 2.1, the number of H-orbits of equilibria of h_λ is at least $\mathrm{Cat}_H(G/G_m)$. ■

Note that if G is compact, the (OPS) condition in Corollary 3.2 is automatically satisfied. Indeed, any compact manifold is automatically complete and the compactness condition on f_λ is fulfilled.

Example 1. Think of the cylinder $M = S^1 \times \mathbb{R}$ as embedded in $\mathbb{R}^3$ with coordinates (θ, z) and endow it with the standard symplectic form $\omega = d\theta \wedge dz$. The Lie group $G = O(2)$ acts on M by $R_\varphi \cdot (\theta, z) = (\theta + \varphi, z)$, if $R_\varphi \in O(2)$ is a rotation of angle φ; and by $r_\alpha \cdot (\theta, z) = (2\alpha - \theta, z)$, if $r_\alpha \in O(2)$ is a reflection about the line forming an angle α with the x-axis in $\mathbb{R}^3$. The action of G on M is Hamiltonian with momentum map $\Phi_G : (\theta, z) \in M \mapsto z \in \mathbb{R}$. Consider the 1-parameter family $h_\lambda : S^1 \times \mathbb{R} \to \mathbb{R}$ defined by

$$h_\lambda(\theta, z) = z^2 + \lambda \cos(n\theta).$$

Then $h = h_0$ is G-invariant and $m = (0, 0)$ is a G-nondegenerate equilibrium of h whose stabilizer is $G_m = \langle r_0 \rangle$. The perturbation h_λ is invariant under $H = D_n$, where D_n is the dihedral group of order $2n$. In fact, the full symmetry group is $D_n \times \mathbb{Z}_2$ since $\mathbb{Z}_2$ acts on the z-component by changing its sign. However such an action is not canonical. Since this discrete part does not contribute in the further application, we do not take it into account. The perturbed Hamiltonian h_λ has $2n$ critical points whose coordinates are $(\frac{\pi}{n}k, 0)$ for $k = 0, \ldots, 2n-1$, which form a regular $2n$-gone as shown in Figure 1 for the case $n = 3$. Since $G/G_m = O(2)/\langle r_0 \rangle$ is topologically a circle, we find $\mathrm{Cat}_H(G/G_m) = 2$ (cf. [Mar89] Corollary 1.17). Since G is compact, condition (OPS) of Corollary 3.2 is automatically satisfied. There are thus two H-orbits of equilibria of h which persist, each of them being a regular n-gone (cf. Figure 2).

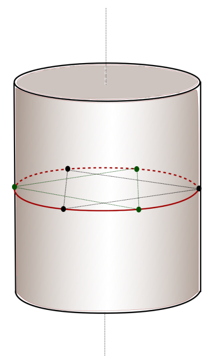

Figure 1: When $n = 3$, h has a G-orbit (red circle) consisting of G-nondegenerate equilibria, on which the six equilibria of h_λ lie.

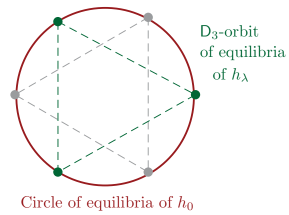

Figure 2: At the level of coordinate $z = 0$, the six equilibria of h_λ form two different D_3-orbits. One orbit is stable and one is unstable.

Dynamics of a 2D rigid body in a potential flow

We apply the result of Corollary 3.2 to the problem of a planar rigid body $\mathcal{B}$ of mass m moving in a planar irrotational, incompressible fluid with zero vorticity and zero circulation around the body. The motion is governed by Kirchhoff equations [Kir77]. Classical treatments of the problem can be found in Lamb [Lam93] and Milne-Thomson [MT60]. The configuration space of the body-fluid system is a submanifold Q of the product $SE(2) \times \mathrm{Emb}_{vol}\left(\mathcal{F}_0, \mathbb{R}^2\right)$, where $SE(2)$ is the special Euclidean group describing the motion of the body, and $\mathrm{Emb}_{vol}\left(\mathcal{F}_0, \mathbb{R}^2\right)$ is the space of volume-preserving embeddings of the fluid reference space $\mathcal{F}_0$ in $\mathbb{R}^2$. The symmetry group of this system is the direct product of $SE(2)$ (group of uniform body-fluid translations and rotations) and the particle relabeling symmetry group (volume-preserving diffeomorphisms of $\mathcal{F}_0$). Since these actions commute, the system can be reduced by the process of symplectic reduction by stages (cf. Marsden et al. [MMO+07]).

The Hamiltonian of the system is invariant under the particle relabeling symmetry group. Geometrically, eliminating the fluid variables amounts to carry out a symplectic reduction by this group. The particle relabeling symmetry group acts on T^*Q in a Hamiltonian fashion. The associated momentum map has two components corresponding to the vorticity and the circulation. The reduction at zero momentum corresponds to a fluid with zero circulation and zero vorticity. In this case, the symplectic reduced space is identified with $T^*SE(2)$, endowed with the canonical symplectic form and the $SE(2)$-invariant reduced Hamiltonian is the sum of the kinetic energy of the body-fluid system by the addition of the so-called “added masses”, and the kinetic energy of the body. Those added masses depend only on the body's shape and not on the mass distribution. The reader is refered to Kanso et al. [KMRMH05] and Vankerschaver et al. [VKM10] for details. Since $SE(2)$ acts symplectically on $T^*SE(2)$, the dynamics can be reduced a second time using Poisson reduction and thereby the reduced motion is governed by the Kirchhoff equations that are the Lie-Poisson equations on the dual

Lie algebra $\mathfrak{se}(2)^*$.

For the sake of simplicity we will assume that the body $\mathcal{B}$ is shaped as an ellipse with semi-axes of length $A > B > 0$. We will use the formulae and follow the notations of Fedorov et al. [FGNV13]. At the center of mass of $\mathcal{B}$ we attach a frame $\{E_1, E_2\}$ that is aligned with the symmetry axes of the body. Its position is related at any time to a fixed space frame $\{e_1, e_2\}$ by an element of $SE(2)$. An element of the Lie algebra $\xi \in \mathfrak{se}(2)$ is identified with a vector

$$(\dot{\theta}, v_1, v_2) \in \mathbb{R}^3 \tag{15}$$

where $\dot{\theta} \in \mathbb{R}$ is the angular velocity of $\mathcal{B}$ and $(v_1, v_2)^T \in \mathbb{R}^2$ is the linear velocity of its center of mass, expressed in the body's frame. In this setting the body has kinetic energy

$$T_{\mathcal{B}} = \frac{1}{2}\xi \cdot \mathbb{I}_{\mathcal{B}}\xi \tag{16}$$

with $\mathbb{I}_{\mathcal{B}} := \operatorname{diag}(I_{\mathcal{B}}, m, m)$, where $I_{\mathcal{B}}$ is the moment of inertia of the body about its center of mass. The kinetic energy of the fluid is given by

$$T_{\mathcal{F}} = \frac{1}{2}\xi \cdot \mathbb{I}_{\mathcal{F}}\xi \tag{17}$$

where $\mathbb{I}_{\mathcal{F}} = \frac{\rho\pi}{4}\operatorname{diag}((A^2 - B^2)^2, B^2, A^2)$ is the tensor of added masses, and ρ is the fluid density. In the absence of external forces, the Lagrangian of the body-fluid system $\mathcal{L} : TSE(2) \to \mathbb{R}$ is given by $\mathcal{L} = T_{\mathcal{B}} + T_{\mathcal{F}}$. It defines a Riemannian metric on $SE(2)$ with respect to which the motion of the body $\mathcal{B}$ is geodesic. Since $\mathcal{L}$ does not depend on the group variables, it is $SE(2)$-invariant and can thus be reduced to the function $\ell : \mathfrak{se}(2) \to \mathbb{R}$ given by

$$\ell(\xi) = \frac{1}{2}\xi \cdot (\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})\xi \tag{18}$$

with ξ as in (15). An element ν of the dual Lie algebra $\mathfrak{se}(2)^*$ is identified with a one by three matrix (x, α_1, α_2). The dual pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$ is thus given by

$$\langle \nu, \xi \rangle := (x, \alpha_1, \alpha_2)(\dot{\theta}, v_1, v_2)^T = x\dot{\theta} + \alpha_1 v_1 + \alpha_2 v_2. \tag{19}$$

We perform the Legendre transform $\mathbb{F}L : \xi \in \mathfrak{se}(2) \mapsto ((\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})\xi)^T \in \mathfrak{se}(2)^*$ to obtain the reduced Hamiltonian $h : \mathfrak{se}(2)^* \to \mathbb{R}$ defined by

$$h(\nu) = \frac{1}{2}\nu \cdot (\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})^{-1}\nu^T.$$

The Lie-Poisson equations on $\mathfrak{se}(2)^*$ that describe the motion of the body-fluid system are

$$\dot{\nu} = \operatorname{ad}^*_{\frac{\delta h}{\delta \nu}}\nu. \tag{20}$$

where $\operatorname{ad}^*_{\xi}\nu$ is identified with $(\alpha_1 v_2 - \alpha_2 v_1, \dot{\theta}\alpha_2, -\dot{\theta}\alpha_1)$. This problem turns out to exhibit symmetry breaking phenomena from different points of view:

(i) One point of view consists in looking at the body $\mathcal{B}$ without the fluid ($\rho = 0$). Adding the fluid amounts to seeing the fluid density ρ as a "parameter". The $O(2)$-symmetry of the kinetic reduced Hamiltonian breaks into a D_2-symmetry, where D_2 is the symmetry group of an ellipse.

(ii) On the other hand we can consider the original system as being a circular planar rigid body ($A = B$) in a fluid and the symmetry can be broken by deforming the body into an elliptical shaped body. This case exhibits the same pattern of symmetry breaking from $O(2)$ to the subgroup D_2.

These two approaches are the same from a group theoretical point of view. Contrary to Example 1, the Hamiltonian in consideration will not be perturbed by adding some potential energy. In this case, there is no potential energy involved, only the metric is perturbed giving rise to a modified kinetic energy. Let us now discuss the two cases mentioned above.

(i) The unperturbed system on the Poisson reduced space $\mathfrak{se}(2)^*$ is governed by the Hamiltonian

$$h(\nu) = \frac{1}{2}\nu \cdot \mathbb{I}\nu = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}}} + \frac{\alpha_1^2 + \alpha_2^2}{m}\right) \tag{21}$$

where $\nu := (x, \alpha_1, \alpha_2)$ and $\mathbb{I} := \mathbb{I}_{\mathcal{B}}^{-1}$. The Hamiltonian is invariant with respect to the group $G = O(2)$. In particular, for each $c \in \mathbb{R}$, the level sets $h(\nu) = c$ describe spheroids in $\mathbb{R}^3$. Note that the full symmetry group is in fact $O(2) \times \mathbb{Z}_2$ since $\mathbb{Z}_2$ acts on the x-component by swapping the sign. However this discrete part does not contribute to our analysis.

Adding a fluid to the system amounts to look at the variation of the dimensionless parameter

$$\lambda = d\rho \quad \text{where} \quad d := \frac{A^2 - B^2}{m} > 0 \quad \text{is fixed.}$$

This gives rise to the perturbed Hamiltonian $h_\lambda(\nu) = \frac{1}{2}\nu \cdot \mathbb{I}_\lambda \nu$ with

$$\mathbb{I}_\lambda = \operatorname{diag}\left(\frac{1}{I_{\mathcal{B}} + \lambda c_1}, \frac{1}{m + \lambda c_2}, \frac{1}{m + \lambda c_3}\right). \tag{22}$$

where $c_1 = \frac{m^2 d\pi}{4}$, $c_2 = \frac{\pi(A^2 - md)}{4d}$ and $c_3 = \frac{\pi(B^2 + md)}{4d}$ are fixed constants encoding the datas of the system. The perturbed Hamiltonian reads

$$h_\lambda(\nu) = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}} + \lambda c_1} + \frac{\alpha_1^2}{m + \lambda c_2} + \frac{\alpha_2^2}{m + \lambda c_3}\right) \tag{23}$$

and has symmetry $H = D_2$, the dihedral group of order four: recall that the group D_2 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and acts here by changing the signs of α_1 and α_2.

This perturbation coincides with h when $\lambda = 0$ and the function $(\lambda, \nu) \mapsto h_\lambda(\nu)$ is smooth. Therefore, h_λ is an H-pertubation of h. The symmetry is broken because the fluid influences the motion of the body if it is elliptical. If the body is circular ($A = B$), or if it moves in the vacuum, its center of mass would move at constant velocity and it would rotate at constant angular speed.

(ii) We carry out another kind of perturbation: rather than perturbing the rigid body motion by adding a fluid to the system, we start with a circular planar rigid body ($A = B$) in a fluid and break the symmetry by changing the body shape into an ellipse. The unperturbed Hamiltonian is given by

$$h(\nu) = \frac{1}{2}\nu \cdot \mathbb{I}\nu = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}}} + \frac{\alpha_1^2 + \alpha_2^2}{m + d_2}\right) \tag{24}$$

where $d_2 = \frac{\rho\pi B^2}{4}$, $\nu := (x, \alpha_1, \alpha_2)$, $\mathbb{I} := (\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})^{-1}$ and $A = B$ in the definition of $\mathbb{I}_{\mathcal{F}}$. The Hamiltonian is invariant with respect to $G = O(2)$. For each $c \in \mathbb{R}$, the level sets $h(\nu) = c$ also describe spheroids in $\mathbb{R}^3$.

We perturb the body shape by setting $\lambda = \frac{A^2 - B^2}{B^2}$ where $B > 0$ is fixed and $A \geq B > 0$ varies. This gives rise to the perturbed Hamiltonian $h_\lambda(\nu) = \nu \cdot \mathbb{I}_\lambda \nu$ with

$$\mathbb{I}_\lambda = \operatorname{diag}\left(\frac{1}{I_{\mathcal{B}} + \lambda^2 d_1}, \frac{1}{m + d_2}, \frac{1}{m + (\lambda + 1)d_2}\right) \tag{25}$$

where $d_1 = \frac{\rho\pi B^4}{4}$. The perturbed Hamiltonian is thus given by

$$h_\lambda(\nu) = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}} + \lambda^2 d_1} + \frac{\alpha_1^2}{m + d_2} + \frac{\alpha_2^2}{m + (\lambda + 1)d_2}\right) \tag{26}$$

and is again symmetric with respect to the action of $H = D_2$. In this case, if there was no fluid ($\rho = d_2 = 0$), no symmetries would have been broken.

Since the reduced motion is governed by the Lie-Poisson equations (20), it is constrained to the coadjoint orbits of $SE(2)$. As shown in [MR99] (Chapter 14.6), almost all of them are cylinders (the singular orbits consist of points on the vertical dashed line in Figure 3). In both cases, the level sets of h_λ are ellipsoids and those of $h = h_0$ are spheroids. Their intersections with a coadjoint orbit are shown in Figure 3. In particular, the circle of equilibria of h (in red in Figure 3) breaks into four fixed points of h_λ, two of which are connected by four heteroclinic cycles.

Let us go back to the first case we discussed above with h_λ as in (23). We will apply Corollary 3.2 to predict the existence of the four fixed points that persist (cf. Figure 3). The Fréchet derivative of h_λ is

$$\frac{\delta h_\lambda}{\delta \nu} = \left(\frac{x}{I_{\mathcal{B}} + \lambda c_1}, \frac{\alpha_1}{m + \lambda c_2}, \frac{\alpha_2}{m + \lambda c_3}\right). \tag{27}$$

Therefore, the Lie-Poisson equations (20) reduce to

$$\begin{cases} \dot{x} &= \frac{\lambda(c_2 - c_3)}{(m+\lambda c_3)(m+\lambda c_2)}\alpha_1\alpha_2 \\ \dot{\alpha_1} &= \frac{x\alpha_2}{m+\lambda c_1} \\ \dot{\alpha_2} &= -\frac{x\alpha_1}{m+\lambda c_1} \end{cases} \tag{28}$$

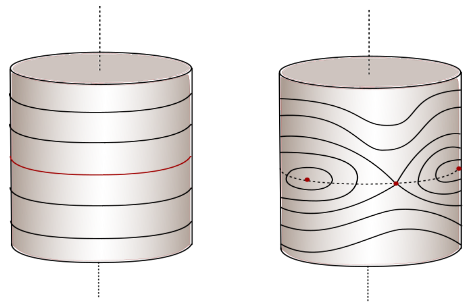

Figure 3: The flow lines are given by intersecting the level sets of h_λ (the ellipsoids) and the coadjoint orbits. On the left hand side, we see the flow lines of h on a coadjoint orbit. On the right hand side, the flow has been perturbed.

Setting $\lambda = 0$ in (28), we see that the fixed points of $h = h_0$ are either of the form $(0, \alpha_1, \alpha_2)$ with $(\alpha_1, \alpha_2) \in (\mathbb{R}^2)^*$, or of the form $(x, 0, 0)$ which correspond to points on the singular coadjoint orbit.

Let $\mu := (0, \alpha_1, \alpha_2)$ with $\alpha_1^2 + \alpha_2^2 = 1$ be a fixed point of the unperturbed hamiltonian h. The isotropy subgroup of μ is $G_\mu = \langle r_\vartheta \rangle$ where r_ϑ is a reflection in the plane. The quotient $G/G_\mu = O(2)/\langle r_\vartheta \rangle$ is topologically a circle yielding $\mathrm{Cat}_{D_2}(S^1) = 2$. The four fixed points appearing in Figure 3 are the two H-orbits that persist.

4. Symmetry breaking for relative equilibria

In this section, we extend Theorem 3.1 and Corollary 3.2 to the case of relative equilibria which is more subtle for two reasons: firstly we must take into account the conservation of momentum, and secondly for a non-zero velocity the so-called augmented Hamiltonian no longer has symmetry G.

We start by briefly recalling some standard facts about relative equilibria, the reader is invited to consult the book of Marsden [Mar92] (Chapter 4) for a more detailed exposition. Given a Hamiltonian proper G-manifold (M, ω, G, Φ_G), a *relative equilibrium* of a Hamiltonian $h \in C^\infty(M)^G$ is a pair $(m, \xi) \in M \times \mathfrak{g}$ such that $X_h(m) = \xi_M(m)$. Equivalently, if (m, ξ) is a relative equilibrium of h, then m is a critical point of the *augmented Hamiltonian*

$$h^\xi := h - \phi_G^\xi \in C^\infty(M)^{G_\xi}$$

where $\phi_G^\xi(m) := \langle \Phi_G(m), \xi \rangle$, which is a G_ξ-invariant function which depends linearly on ξ. A standard fact about relative equilibria is that the velocity ξ and the momentum $\mu = \Phi_G(m)$ commute i.e. $\xi \in \mathfrak{g}_\mu$. Note that, if the isotropy group G_m is non trivial and (m, ξ) is a relative equilibrium of h, then $(m, \xi + \eta)$ is also a relative equilibrium of h, for any $\eta \in \mathfrak{g}_m$. Moreover if (m, ξ) is a relative equilibrium of h then so is $(g \cdot m, \mathrm{Ad}_g \xi)$ for every $g \in G$. In general a relative equilibrium is said to be non-degenerate if the Hessian $D^2 h^\xi(m)$ is a non-singular quadratic form, when restricted to the symplectic slice N_1 at m relative to the G-action. However, this definition of non-degeneracy is not enough to guarantee that a relative equilibrium of some $h \in C^\infty(M)^G$ persists under an H-perturbation. For that reason, we need a stronger version of non-degeneracy.

Induced momentum map

Let H be a closed subgroup of G. The dual of the inclusion of Lie algebras $i_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the projection $\mathfrak{i}_{\mathfrak{h}}^* : \mathfrak{g}^* \to \mathfrak{h}^*$ and is given by $i_{\mathfrak{h}}^*(\mu) = \mu\big|_{\mathfrak{h}}$, which is the restriction of the linear form μ to the Lie subalgebra $\mathfrak{h}$. The action of H on M is still both canonical and Hamiltonian. A momentum map for this action is given by $\Phi_H = i_{\mathfrak{h}}^* \circ \Phi_G : M \to \mathfrak{h}^*$ and is called the *induced momentum map* for the H-action.

PROPOSITION 4.1. *Assume that G is connected and consider the decomposition of T_mM as in (2), and define the subspace $\mathcal{M} := \{z_M(m) + w \in T_1 \oplus N_0 \mid -ad_z^*\mu + f(w) \in \mathfrak{h}^\circ\}$ where f denotes the isomorphism between N_0 and $\mathfrak{m}^*$, and $\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Then* $\ker(D\Phi_H(m)) = \ker(D\Phi_G(m)) \oplus \mathcal{M}$.

Proof — It is clear from the definitions that there is an inclusion of subspaces

$$\ker(D\Phi_G(m)) \subset \ker(D\Phi_H(m)). \tag{29}$$

Let $(\varphi, G \times_{G_m} (\mathfrak{m}_0^* \times (N_1)_0), U)$ be a symplectic G-tube at m as in Theorem 2.2. Linearising φ^{-1} at m yields a linear symplectomorphism

$$T_m\varphi^{-1} : T_0 \oplus T_1 \oplus N_0 \oplus N_1 \to T_{\varphi^{-1}(m)}(G \times_{G_m} (\mathfrak{m}^* \times N_1)).$$

For $x + y \in \mathfrak{g}_m \oplus \mathfrak{m}$ and $z \in \mathfrak{n}$ we have

$$T_m\varphi^{-1} \cdot ((x+y)_M(m) + z_M(m) + w + \nu) = T_{(e,0,0)}\rho \cdot (x + y + z, f(w), \nu)$$

where $\rho : G \times \mathfrak{m}^* \times N_1 \to G \times_{G_m} (\mathfrak{m}^* \times N_1)$ is the orbit map. By definition, the subspace $\ker(D\Phi_H(m))$ consists of the elements

$$((x+y)_M(m) + z_M(m) + w + \nu) \in T_0 \oplus T_1 \oplus N_0 \oplus N_1$$

satisfying $D(\Phi_H \circ \varphi \circ \rho)(m) \cdot (x + y + z, f(w), \nu) = 0$. Equivalently

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Phi_H \circ \varphi([(\exp(t(x+y+z)), tf(w), t\nu)]) \\ &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} i_{\mathfrak{h}}^* \left(\mathrm{Ad}^*_{\exp(-t(x+y+z))}(\mu + tf(w) + \Phi_{N_1}(t\nu))\right) \\ &= i_{\mathfrak{h}}^*(-\mathrm{ad}_z^*\mu + f(w)) \end{aligned}$$

where the normal form for the momentum map is given by Theorem 2.3. As required $-\mathrm{ad}_z^*\mu + f(w) \in \mathfrak{h}^\circ$ since the kernel of $i_{\mathfrak{h}}^*$ is equal to $\mathfrak{h}^\circ$. ■

Non-degeneracy condition and regularity condition

We now state a stronger version of non-degeneracy of a relative equilibrium.

Definition 4.1. Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold with G connected, $H \subset G$ be a closed subgroup, and $\Phi_H : M \to \mathfrak{h}^*$ be the induced momentum map. Setting $\alpha := \Phi_H(m)$, a relative equilibrium $(m, \xi) \in M \times \mathfrak{g}$ of $h \in C^\infty(M)^G$ is said to be α-*nondenegerate* if $D^2 h^\xi(m)$ is a non-singular quadratic form on $N_1 \oplus \mathcal{M}$ with $\mathcal{M}$ as in Proposition 4.1.

Definition 4.1 only depends on α and not on the underlying Witt-Artin decomposition of $T_m M$. If G is non-abelian, the space $\mathcal{M}$ might have an non-trivial intersection with $\mathfrak{g} \cdot m$. This intersection is the subspace $\mathfrak{q} \cdot m \subset \mathfrak{g} \cdot m$ where $\mathfrak{q}$ is an H_m-invariant complement to $\mathfrak{g}_\mu$ in the "symplectic orthogonal"

$$\mathfrak{h}^{\perp_\mu} := \left\{ x \in \mathfrak{g} \mid x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)} \right\}.$$

The non-singularity of $D^2 h^\xi(m)$ along $\mathfrak{g} \cdot m$ depends only on that of $D^2 \phi^\xi_G(m)$ which has symmetry group G_ξ. The condition is a consequence of the following lemma which is proved in Section 5.

Lemma 4.2. *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold. Let $m \in M$ with momentum $\mu = \Phi_G(m)$ and an element $\xi \in \mathfrak{g}_\mu$. If $\mathfrak{g}$ is semi-simple then the Hessian $D^2 \phi^\xi_G(m)$ restricted to $\mathfrak{g} \cdot m$ is singular only along $(\mathfrak{g}_\xi + \mathfrak{g}_\mu) \cdot m$.*

Therefore if an equilibrium $(m, \xi) \in M \times \mathfrak{g}$ of some $h \in C^\infty(M)^G$ with momentum $\mu = \Phi_G(m)$ is α-nondegenerate in the sense of Definition 4.1, then $\mathfrak{g}_\xi$ has trivial intersection with $\mathfrak{q}$. In Theorem 4.3 we show that a number of orbits of relative equilibria of h persist under H-perturbations. Such relative equilibria must have their velocity ξ in $\mathfrak{h}_\mu$. We assume an additional regularity assumption

$$\mathfrak{g}_\mu \subset \mathfrak{g}_\xi \tag{R}$$

This says essentially that μ needs to be more regular (cf. [Fon18] Definition 6.2.2). However if $\xi \in \mathfrak{h}_\mu$, this assumption depends on the embedding of $\mathfrak{h} \hookrightarrow \mathfrak{g}$ as shown in the example below. This is not a problem for us because isomorphic Lie algebras have different underlying Lie groups.

Example 2. In this example we show when condition (R) holds for $\mathfrak{g} = \mathfrak{so}(4)$ and a subalgebra isomorphic to $\mathfrak{h} = \mathfrak{so}(3)$. The Lie algebra $\mathfrak{g}$ is identified with the set of pairs $(x, a) \in \mathbb{R}^3 \times \mathbb{R}^3$ with Lie bracket

$$[(x, a), (y, b)] = (x \times y + a \times b, x \times b + a \times y). \tag{30}$$

The dual Lie algebra $\mathfrak{g}^*$ consists of pairs $(\chi, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3$ which satisfy

$$\langle(\chi, \rho), (x, a)\rangle = \chi \cdot x + \rho \cdot a.$$

The linearized coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by

$$\mathrm{ad}^*_{(x,a)}(\chi, \rho) = (\chi \times x + \rho \times a, \chi \times a + \rho \times x). \tag{31}$$

Lie subalgebras isomorphic to $\mathfrak{so}(3)$. Elements of $\mathfrak{h} = \mathfrak{so}(3)$ are identified with vectors $x \in \mathbb{R}^3$. We consider two Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$, namely

(i) The Lie algebra of rotations in $\mathbb{R}^3$ denoted $\mathfrak{so}(3)_r = \left\{(x, 0) \in \mathbb{R}^6 \mid x \in \mathbb{R}^3\right\}$ with Lie bracket $[(x, 0), (y, 0)] = (x \times y, 0)$.

(ii) The diagonal elements denoted $\mathfrak{so}(3)_d = \left\{\left(\frac{x}{2}, \frac{x}{2}\right) \in \mathbb{R}^6 \mid x \in \mathbb{R}^3\right\}$ with Lie bracket $[(\frac{x}{2}, \frac{x}{2}), (\frac{y}{2}, \frac{y}{2})] = (\frac{x\times y}{2}, \frac{x\times y}{2})$.

Regularity condition. Given a fixed momentum $\mu := (\chi, \rho) \in \mathfrak{g}^*$, the stabilizer Lie subalgebra is

$$\mathfrak{g}_\mu = \{(x, a) \in \mathfrak{g} \mid \chi \times x + \rho \times a = 0 \quad \text{and} \quad \chi \times a + \rho \times x = 0\}$$

by (31). We show below whether condition (R) is satisfied for our two different choices of Lie subalgebras isomorphic to $\mathfrak{h}$.

(i) Let $\mathfrak{h} = \mathfrak{so}(3)_r$ with inclusion map $i_\mathfrak{h} : x \in \mathfrak{h} \mapsto (x, 0) \in \mathfrak{g}$. To compute the dual of this inclusion $i^*_\mathfrak{h} : \mathfrak{g}^* \to \mathfrak{h}^*$, we take $(\chi, \rho) \in \mathfrak{g}^*$ and $x \in \mathfrak{h}$ and we compute

$$\langle i^*_\mathfrak{h}(\chi, \rho), x\rangle = \langle(\chi, \rho), i_\mathfrak{h}(x)\rangle = \langle(\chi, \rho), (x, 0)\rangle = \chi \cdot x.$$

Then

$$i^*_\mathfrak{h}((\chi, \rho)) = \chi \in \mathfrak{h}^*.$$

The symplectic orthogonal is $\mathfrak{h}^{\perp_\mu} = \{(x, a) \in \mathfrak{g} \mid \chi \times x + \rho \times a = 0\}$. Since the velocity $\xi \in \mathfrak{h}$ must commute with μ, it has to belong to the subspace $\mathfrak{h}_\mu = \mathfrak{g}_\mu \cap \mathfrak{h}$. Using equation (31),

$$\mathfrak{h}_\mu = \{(x, 0) \in \mathfrak{so}(3)_r \mid \chi \times x = 0 \text{ and } \rho \times x = 0\}.$$

There are three cases to consider:

(a) If $\chi = \rho = 0$ then $\mathfrak{g}_\mu = \mathfrak{g}$ and $\mathfrak{h}_\mu = \mathfrak{h}$. We choose $\xi = (y, 0) \in \mathfrak{h}$ where $y \in \mathbb{R}^3$ is arbitrary. Using (30) we get

$$\mathfrak{g}_\xi = \{(\lambda_1 y, \lambda_2 y) \in \mathfrak{g} \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$$

and clearly (R) does not hold.

(b) If χ and ρ are not collinear, $\mathfrak{h}_\mu = \{(0,0)\}$. In this case, the only available velocity is $\xi = 0$ and thus $\mathfrak{g}_\xi = \mathfrak{g}$. In particular (R) holds.

(c) If $\mu = (\chi, \rho)$ is such that $\chi = s\rho$ for some $s \in \mathbb{R}$, we choose ξ of the form

$$\xi := (\lambda\chi, 0) \in \mathfrak{h}_\mu \quad \text{for some} \quad \lambda \in \mathbb{R}$$

and thus $\mathfrak{g}_\xi = \{(x,a) \in \mathfrak{g} \mid x \times \chi = 0 \text{ and } a \times \chi = 0\}$. Note that in particular, $\mathfrak{g}_\xi \subset \mathfrak{g}_\mu$. To see whether $\mathfrak{g}_\mu \subset \mathfrak{g}_\xi$, pick an element $(x,a) \in \mathfrak{g}_\mu$. By definition, it satisfies

$$x \times \chi = \rho \times a \quad \text{and} \quad \chi \times a = x \times \rho. \tag{32}$$

Using (32) and the fact that $\chi = s\rho$ we get,

$$x \times \chi = s(x \times \rho) = s(\chi \times a) = s^2(\rho \times a) = s^2(x \times \chi).$$

Similarly

$$a \times \chi = s(a \times \rho) = s(\chi \times x) = s^2(\rho \times x) = s^2(a \times \chi).$$

Therefore, $(x,a) \in \mathfrak{g}_\xi$ as long as $s^2 \neq 1$ i.e. (R) holds as long as $\mu \neq (\chi, \pm\chi)$, as shown in Figure 4.

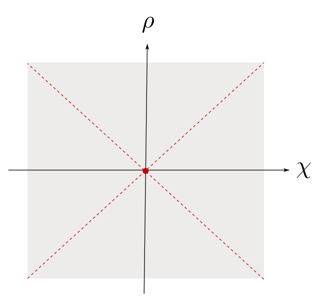

Figure 4: Condition (R) holds as long as μ is away form the red dashed lines which represent subspaces of codimension three in $\mathbb{R}^6$.

(ii) Let $\mathfrak{h} = \mathfrak{so}(3)_d$ with inclusion map

$$i_\mathfrak{h} : x \in \mathfrak{h} \mapsto \left(\frac{x}{2}, \frac{x}{2}\right) \in \mathfrak{g}.$$

To compute the dual of this inclusion $i_\mathfrak{h}^* : \mathfrak{g}^* \to \mathfrak{h}^*$, we take $(\chi, \rho) \in \mathfrak{g}^*$ and $x \in \mathfrak{h}$ and we compute

$$\langle i_\mathfrak{h}^*(\chi,\rho), x\rangle = \langle(\chi,\rho), i_\mathfrak{h}(x)\rangle = \langle(\chi,\rho), \left(\frac{x}{2}, \frac{x}{2}\right)\rangle = \frac{\chi+\rho}{2} \cdot x,$$

Then

$$i_{\mathfrak{h}}^*((\chi,\rho)) = \frac{\chi+\rho}{2} \in \mathfrak{h}^*.$$

Set $\mu := (\chi,\rho) \in \mathfrak{g}^*$ and $\alpha := i_{\mathfrak{h}}^*(\mu) = \frac{\chi+\rho}{2} \in \mathfrak{h}^*$. Using Equation (31) we get

$$\mathfrak{h}_\mu = \left\{ \left(\frac{x}{2}, \frac{x}{2}\right) \in \mathfrak{so}(3)_d \mid \alpha \times x = 0 \right\}.$$

We thus choose a velocity of the form

$$\xi := (\lambda\alpha, \lambda\alpha) \in \mathfrak{h}_\mu$$

for some $\lambda \in \mathbb{R}$. By (30) the stabilizer Lie algebra of ξ is

$$\mathfrak{g}_\xi = \{(x,a) \in \mathfrak{g} \mid x \times \alpha + a \times \alpha = 0\}. \tag{33}$$

In particular, $\mathfrak{g}_\mu \subset \mathfrak{g}_\xi$ and (R) is automatic for any choice of μ.

Persistence of relative equilibria

We are now ready to state the corresponding version of Theorem 3.1 for relative equilibria. The proof follows the same steps as Theorem 3.1. For that reason some details have been skipped.

Theorem 4.3. *Let (M,ω,G,Φ_G) be a Hamiltonian proper G-manifold with G connected. Assume $h \in C^\infty(M)^G$ has a relative equilibrium $(m,\xi) \in M \times \mathfrak{h}$ with momentum $\mu = \Phi_G(m)$. Let α be the restriction of μ to $\mathfrak{h}$. We assume that*

(i) $\Phi_H^{-1}(\alpha)$ is a smooth manifold,

(ii) (m,ξ) is α-nondegenerate and (R) is satisfied,

(iii) $G_\mu \subset H_\alpha$,

(iv) $H_\mu \subset G_\mu$ is co-compact.

Then there is a G_μ-invariant neighbourhood $\widetilde{U} \subset \Phi_H^{-1}(\alpha)$ of m and a neighbourhood $V \subset \mathbb{R} \times \mathfrak{h}$ of $(0,\xi)$ such that, for each $(\lambda,\eta) \in V$, there is a function $f_\lambda^\eta \in C^\infty(G_\mu/G_m)^{H_\mu}$ whose critical points are in one-to-one correspondence with those of h_λ^η in $\widetilde{U}$.

Proof — Let $(m,\xi) \in M \times \mathfrak{h}$ be an α-nondegenerate relative equilibrium h, where α is the restriction of the momentum $\mu = \Phi_G(m)$ to $\mathfrak{h}$. By assumption $\Phi_H^{-1}(\alpha)$ is a smooth manifold on which $G_\mu \subset H_\alpha$ acts canonically and properly.

Let $K = G_m$ and consider the K-vector space $N := N_1 \oplus \mathcal{M}$, where N_1 is a symplectic slice at m relative to the G-action, and $\mathcal{M}$ is as in Proposition 4.1. By construction

N is isomorphic to some K-vector space complementary to $\mathfrak{g}_\mu \cdot m$ in $T_m\left(\Phi_H^{-1}(\alpha)\right)$. By the Tube Theorem (cf. [OR04] Theorem 2.3.28), there is a K-invariant neighbourhood $N_0 \subset N$ of zero, such that

(i) The associated bundle $G_\mu \times_K N_0$ is a local model for some G_μ-invariant neighbourhood $U \subset \Phi_H^{-1}(\alpha)$ of m.

(ii) The only critical points of h^ξ in U are on $G_\mu \cdot m$.

In that model, the point m corresponds to $[(e,0)]$ and the augmented Hamiltonian of an H-perturbation $h_\lambda \in C^\infty(M)^H$ of h is identified with $h_\lambda^\xi : G_\mu \times_K N_0 \to \mathbb{R}$. According to the proof of Theorem 3.1, the critical points of h_λ^ξ are in bijective correspondence with those of the lift

$$\widetilde{h}_\lambda^\xi := \rho^* h_\lambda^\xi : G_\mu \times N_0 \to \mathbb{R}$$

where $\rho : G_\mu \times N_0 \to G_\mu \times_K N_0$ is the orbit map. We may thus work with $\widetilde{h}_\lambda^\xi$ instead of h_λ^ξ.

We define a (left) action of the direct product $G_\mu \times K$ on $G_\mu \times N_0$ by

$$(h,k)\cdot(g,\nu) = (hgk^{-1}, k\cdot\nu).$$

As $G_\mu \subset G_\xi$ by the (R) assumption, the lift $\widetilde{h}^\xi$ is $G_\mu \times K$-invariant whereas $\widetilde{h}_\lambda^\xi$ is only $H_\mu \times K$-invariant. By α-nondegeneracy of (m,ξ),

$$d\widetilde{h}^\xi(e,0) = 0 \quad \text{and} \quad D_N^2 \widetilde{h}^\xi(e,0) \quad \text{is non-singular.}$$

As in the proof of Theorem 3.1, we can use the Implicit Function Theorem and the compacity of $H_\mu \backslash G_\mu$ to get an H_μ-invariant smooth function $\phi_\lambda^\eta : G_\mu \to N_0$, depending on parameters (λ,η) taken in a neighbourhood $V \subset \mathbb{R} \times \mathfrak{h}$ of $(0,\xi)$, satisfying

$$d_N \widetilde{h}_\lambda^\eta(g, \phi_\lambda^\eta(g)) = 0 \quad \text{for every} \quad g \in G_\mu.$$

For every fixed parameters $(\lambda,\eta) \in V$, the $H_\mu \times K$-invariance of $\widetilde{h}_\lambda^\eta$ allows us to define a function $f_\lambda^\eta \in C^\infty(G_\mu/K)^{H_\mu}$ by

$$f_\lambda^\eta([g]_{\mathrm{K}}) := \widetilde{h}_\lambda^\eta([g]_{\mathrm{K}}, \phi_\lambda^\eta([g]_{\mathrm{K}}))$$

where $[g]_{\mathrm{K}}$ is a coset in G_μ/K. An application of the Morse Lemma with parameters gives us a Morse chart $(\widetilde{N}_0, \widetilde{\nu})$ centered at $0 \in N_0$, where $\widetilde{N}_0 \subset N_0$ and $\widetilde{\nu} = (\widetilde{\nu}^1, \ldots, \widetilde{\nu}^n)$. Then there is a smooth map $\psi_\lambda^\eta : G_\mu/K \times \widetilde{N}_0 \to \widetilde{N}_0$, depending on $(\lambda,\eta) \in V$ such that

$$\widetilde{h}_\lambda^\eta([g]_{\mathrm{K}}, \psi_\lambda^\eta([g]_{\mathrm{K}}, \nu)) = \sum_{i=1}^n \varepsilon_i \widetilde{\nu}_i^2 + f_\lambda^\eta([g]_{\mathrm{K}}) \tag{34}$$

where $\varepsilon_i = \pm 1$ and $\widetilde{\nu}(\nu) = (\widetilde{\nu}_1, \ldots, \widetilde{\nu}_n)$. Therefore $([g]_{\mathrm{K}}, \nu) \in G_\mu/K \times \widetilde{N}_0$ is a critical point of (34) if and only if

$$\left(\sum_{i=1}^n \varepsilon_i \widetilde{\nu}^i d\widetilde{\nu}^i\right)(\nu) = 0 \quad \text{and} \quad df_\lambda^\eta([g]_{\mathrm{K}}) = 0.$$

Let $\widetilde{U} \subset \Phi_H^{-1}(\alpha)$ be the G_μ-invariant neighbourhood of m whose local model is $G_\mu \times_K \widetilde{N}_0$. In particular for $(\lambda, \eta) \in V$, the critical points of h_λ^η in $\widetilde{U}$ are in one-to-one correspondence with those of the function f_λ^η. ■

Corollary 4.4 (Persistence of relative equilibria). *If the manifold G_μ/G_m and the function $f_\lambda^\eta \in C^\infty(G_\mu/G_m)^{H_\mu}$ of Theorem 4.3 satisfy condition (OPS) then the number of H_μ-orbits of relative equilibria of h with velocity close to ξ, that persist under a small H-perturbation in a neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$, is bounded below by $Cat_{H_\mu}(G_\mu/G_m)$.*

Proof — We apply Theorem 2.1 to $f_\lambda^\eta \in C^\infty(G_\mu/G_m)^{H_\mu}$ and we obtain that the number of H_μ-orbits of critical points of f_λ^η is bounded below by $\mathrm{Cat}_{H_\mu}(G_\mu/G_m)$. In other words, as long as $(\lambda, \eta) \in V$, the number of H_μ-orbits of relative equilibria of h_λ with velocity η in a neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$ is at least $\mathrm{Cat}_{H_\mu}(G_\mu/G_m)$. ■

Example 3 (Torus action). As a first application, we recover the result of Grabsi, Montaldi and Ortega [GMO04] for compact abelian groups and free actions. Let $(M, \omega, \mathrm{T}^n, \Phi_{\mathrm{T}^n})$ be a Hamiltonian T^n-manifold where T^n is a n-dimensional torus acting freely on M and let T^r be a subtorus of T^n. Assume $h \in C^\infty(M)^{\mathrm{T}^n}$ has an α-nondegenerate relative equilibrium $(m, \xi) \in M \times \mathfrak{t}^r$ with momentum $\mu = \Phi_{\mathrm{T}^n}(m)$ and where $\alpha = \mu\big|_{\mathfrak{t}^r}$. As T^n and T^r are abelian, condition (R) always hold. By compactness of T^n, condition (OPS) is automatic and then any T^r-perturbation h_λ with λ small enough has at least $\mathrm{Cat}_{\mathrm{T}^r}(\mathrm{T}^n)$ T^r-orbit of relative equilibria with velocity closed to ξ in a neighbourhood of $\mathrm{T}^n \cdot m$ in $\Phi_{\mathrm{T}^r}^{-1}(\alpha)$. Since T^n acts freely on T^r by left multiplication,

$$\mathrm{Cat}_{\mathrm{T}^r}(\mathrm{T}^n) = \mathrm{Cat}(\mathrm{T}^n/\mathrm{T}^r) = \mathrm{Cat}(\mathrm{T}^{n-r}).$$

Hence $\mathrm{Cat}(\mathrm{T}^{n-r}) = (n-r) + 1$.

Example 4 (Spherical pendulum on S^3). As an application of Corollary 4.4, we consider the case of a spherical pendulum on S^3, whose Hamiltonian is viewed as a perturbation of the free Hamiltonian on S^3. Endow $\mathbb{R}^4$ with the standard inner product $\langle \cdot, \cdot \rangle$ and let e_1, e_2, e_3, e_4 be the standard basis. The phase space is $(T^*S^3, \omega, G, \Phi_G)$ where $G = SO(4)$ acts on

$$T^*S^3 = \left\{(x, y) \in S^3 \times \mathbb{R}^4 \mid \langle x, y \rangle = 0\right\}$$

by matrix multiplication $A \cdot (x, y) = (Ax, Ay)$. The associated momentum map $\Phi_G : T^*S^3 \to \bigwedge^2(\mathbb{R}^4)$ is given by $\Phi_G(x, y) = y \wedge x$.

Let $H = SO(3) \subset SO(4)$ be the rotations about the e_4-axis with Lie algebra $\mathfrak{h} = \mathfrak{so}(3)_r$ as defined in Example 2. The Hamiltonian of the spherical pendulum

$$h_\lambda(x, y) = \frac{1}{2}\|y\|^2 + \lambda \langle x, e_4 \rangle \tag{35}$$

is an H-perturbation of the free Hamiltonian $h(x,y) = \frac{1}{2}\|y\|^2$. By definition, the relative equilibria of (35) are pairs $((x,y),\xi) \in T^*S^3 \times \mathfrak{h}$ such that

$$dh_\lambda(x,y) = d\phi_H^\xi(x,y) \tag{36}$$

where

$$\phi_H^\xi(x,y) := -\frac{1}{2}\mathrm{Tr}((y \wedge x)\xi) = \langle \xi x, y\rangle.$$

Solving (36) is a straightforward calculation. The result is summarized as follows:

Lemma 4.5. *Denote by* $\xi = \begin{pmatrix} \hat{w} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}$ *and* $p_V(x)$ *the projection of* x *on* $V =$ *span*(e_1, e_2, e_3). *The relative equilibria* $((x,y),\xi) \in T^*S^3 \times \mathfrak{h}$ *of* (35) *satisfy the equations*

(i) $\langle x, e_4\rangle = -\lambda\|w\|^{-2}$

(ii) $\|p_V(x)\|^2 = 1 - \langle x, e_4\rangle^2$ *and* $w \cdot p_V(x) = 0$ *(dot product in* $\mathbb{R}^3$*)*

(iii) $y = \xi x$ *is of the form* $p_V(y) = w \times p_V(x)$ *and* $\langle y, e_4\rangle = 0$.

When $\lambda = 0$, (35) is just the free Hamiltonian $h(x,y) = \frac{1}{2}\|y\|^2$ on T^*S^3. The integral curves of the corresponding Hamiltonian vector field project to the great circles on S^3. Given $\xi \in \mathfrak{h}$ with $w = (0,0,1)^T$, the relative equilibria $((x,y),\xi)$ of h are such that $\langle x, e_4\rangle = 0$ and $\mathrm{p}_V(x)$ lies on the unit sphere in the hyperplane orthogonal to the line $[w]$, and y is tangent to this sphere. We can thus take $m = (x,y) = (e_1, e_2)$ as a relative equilibrium of h with velocity ξ. The momentum is then $\mu = \Phi_G(e_1 \wedge e_2) = \xi$ and its projection on $\mathfrak{h}^*$ is $\alpha = w^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. The stabilizer G_μ is thus a copy of $SO(3)$ in $SO(4)$ and the orbit $G_\mu \cdot m$ is the unit sphere $S^2 \subset S^3$ lying on the hyperplane of equation $\langle x, e_4\rangle = 0$.

We want to find the relative equilibria $((\tilde{x},\tilde{y}),\eta)$ of the perturbed Hamiltonian (35) which lie on $\Phi_H^{-1}(\alpha)$ where

$$\Phi_H(\tilde{x},\tilde{y}) = (\mathrm{p}_V(\tilde{x}) \times \mathrm{p}_V(\tilde{y}))^T$$

is the induced momentum map. Writing $\eta = \begin{pmatrix} \hat{u} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}$, those relative equilibria satisfy the equation

$$\|\mathrm{p}_V(\tilde{x})\|^2 u - (\mathrm{p}_V(\tilde{x}) \cdot u)\mathrm{p}_V(\tilde{x}) = w \tag{37}$$

with $w = (0,0,1)^T$, as fixed earlier. In addition they satisfy the equations of Lemma 4.5, which require in particular that $\mathrm{p}_V(\tilde{x}) \cdot u = 0$. Replacing in (37) we obtain $u = \|\mathrm{p}_V(\tilde{x})\|^{-2} w$. From Lemma 4.5 we get $\langle x, e_4\rangle = -\lambda\|\mathrm{p}_V(\tilde{x})\|^4$ and

$$\|\mathrm{p}_V(\tilde{x})\|^2 + \lambda^2\|\mathrm{p}_V(\tilde{x})\|^8 = 1 \tag{38}$$

Setting $t = \|\mathrm{p}_V(\widetilde{x})\|^2$ in (38), we obtain the equation of an algebraic curve

$$\lambda^2 t^4 + t - 1 = 0 \qquad t > 0.$$

For a fixed λ there is exaclty one solution representing the square of the radius $r(\lambda)$ of the sphere on which $\mathrm{p}_V(\widetilde{x})$ lies. This sphere is an H_μ-orbit of relative equilibria of (35). Since $r(0) = 1$, it lies in a neighbourhood of the orbit $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$. Furthermore η is such that $u = r(\lambda)^{-2} w$ and thus η is close to ξ in $\mathfrak{h}$. We also see from (38) that λ must be chosen small enough such that

$$\lambda < \|u\|^2 < r(\lambda)^{-4} < c.$$

where c is some constant coming from the fact that $r(\lambda)$ is bounded below.

We conclude that for λ small enough, h_λ has exactly one H_μ-orbit of relative equilibria in a neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$ with velocity close to ξ. For this example, we verify the assumptions of Theorem 4.3. We have $G_\mu = H = SO(3)$ and the stabilizer G_m is an $SO(2)$, as it is the subgroup of rotations in $SO(4)$ which preserve both axis e_1 and e_2. The quotient G_μ/G_m is thus a unit sphere S^2 and $H_\mu = G_\mu \cap H = SO(3)$. Furthermore, as $\mu = \xi$, the assumption (R) is satisfied, as well as the other assumptions of Theorem 4.3, since we work with compact groups. As expected, we have

$$\mathrm{Cat}_{H_\mu}(G_\mu/G_m) = \mathrm{Cat}_{SO(3)}(S^2) = 1.$$

5. Proof of Lemma 4.2

This section is devoted to the proof of Lemma 4.2 where we assume that $\mathfrak{g}$ is semi-simple. Further details are available in [Fon18]. For each $\xi \in \mathfrak{g}$ a momentum map $\Phi_G : M \to \mathfrak{g}^*$ defines a smooth function $\phi_G^\xi : M \to \mathbb{R}$ depending linearly on ξ

$$\phi_G^\xi(m) := \langle \Phi_G(m), \xi \rangle.$$

Assume that $(m, \xi) \in M \times \mathfrak{g}$ is a relative equilibrium of some Hamiltonian $h \in C^\infty(M)^G$ with momentum $\mu = \Phi_G(m)$. By definition of a relative equilibrium, ξ and μ commute i.e. $\mathrm{ad}_\xi^* \mu = 0$. We would like to describe the space of degeneracy of the Hessian $D^2\phi_G^\xi(m)$ along the orbit $\mathfrak{g} \cdot m$. A straightforward calculation yields

$$D^2\phi_G^\xi(m)\,(y_M(m), x_M(m)) = \langle \Phi_G(m), [x, [y, \xi]] \rangle. \tag{39}$$

Set $\mu = \Phi_G(m)$ and note that the Jacobi identity of the Lie bracket and the fact that $\mathrm{ad}_\xi^* \mu = 0$ imply that $\langle \mu, [x, [y, \xi]] \rangle = \langle \mu, [y, [x, \xi]] \rangle$, reflecting the symmetric property of the Hessian. The non-degeneracy space of $D^2\phi_G^\xi(m)$ along $\mathfrak{g} \cdot m$ consists of the elements $y \in \mathfrak{g}$ such that

$$\langle \mu, [y, [x, \xi]] \rangle = 0 \quad \text{for all} \quad x \in \mathfrak{g}. \tag{40}$$

Since μ and ξ commute, we can fix a maximal commutative Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ such that $\xi \in \mathfrak{t}$ and $\mu \in \mathfrak{t}^*$. We complexify both of them

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g} \quad \text{and} \quad \mathfrak{t}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{t}$$

with extended Lie bracket $[\cdot,\cdot]_{\mathbb{C}}$. After this step the velocity and momentum read $\xi = 1\otimes_{\mathbb{R}}\xi$ and $\mu = 1\otimes_{\mathbb{R}}\mu$ and there respective stabilizer subalgebras are

$$\mathfrak{g}_\xi := \{x\in\mathfrak{g}_{\mathbb{C}} \mid [x,\xi]_{\mathbb{C}} = 0\} \quad \text{and} \quad \mathfrak{g}_\mu := \{x\in\mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}^*_x\mu = 0\}\,.$$

Consider the Cartan Lie subalgebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$. Since $\xi\in\mathfrak{h}$ and $\mu\in\mathfrak{h}^*$, it is clear that $\mathfrak{h}$ is a subspace of both $\mathfrak{g}_\xi$ and $\mathfrak{g}_\mu$. We thus write

$$\mathfrak{g}_\xi = \mathfrak{h}\oplus\left(\bigoplus_{\beta\in S_f}\mathfrak{g}_\beta\right) \quad \text{and} \quad \mathfrak{g}_\mu = \mathfrak{h}\oplus\left(\bigoplus_{\alpha\in D_f}\mathfrak{g}_\alpha\right) \tag{41}$$

for some finite subsets S_f and D_f of the root space $\mathcal{R}$ with the property:

$$\alpha\in S_f \text{ (resp. } D_f) \quad\Longrightarrow\quad -\alpha\in S_f \text{ (resp. } D_f).$$

Definition 5.1. ξ (resp. μ) is *regular* if $S_f=\varnothing$ (resp. $D_f=\varnothing$).

Since $\mathfrak{g}_{\mathbb{C}}$ is semi-simple, the Killing form κ induces an isomorphism $\kappa^\sharp:\mathfrak{h}^*\to\mathfrak{h}$. Let $t_\mu\in\mathfrak{h}$ be the image of μ by this isomorphism and let $\mathcal{O}_{t_\mu}$ be the adjoint orbit of t_μ. There is an identification

$$T_{t_\mu}\mathcal{O}_{t_\mu} = \sum_{\alpha\in\mathcal{R}\setminus D_f}\mathfrak{g}_\alpha.$$

The problem stated in (40), after complexification of the Lie algebra $\mathfrak{g}$, reduces to find all the $y\in\mathfrak{g}_{\mathbb{C}}$ satisfying

$$\kappa\left([y^*,t_\mu]_{\mathbb{C}},[x,\xi]_{\mathbb{C}}\right) = 0 \quad \text{for all} \quad x\in\mathfrak{g}_{\mathbb{C}}. \tag{42}$$

Let $\{H_1,\dots,H_k\}\cup\{X_\alpha\mid\alpha\in\mathcal{R}\}$ be a Weyl-Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$, where the H_i's form a basis of $\mathfrak{h}$. Let $y\in\mathfrak{g}_{\mathbb{C}}$ be an arbitrary element and let $y^*=-\bar{y}$. With respect to the Weyl-Chevalley basis, this element is expressed as

$$y^* = \sum_{i=1}^k a_iH_i + \sum_{\alpha\in\mathcal{R}}\mu_\alpha X_\alpha \quad \text{for some unique} \quad a_i,\mu_\alpha\in\mathbb{C}. \tag{43}$$

Hence

$$\begin{aligned}
[y^*,t_\mu]_{\mathbb{C}} &= [\sum_{i=1}^k a_iH_i + \sum_{\alpha\in\mathcal{R}}\mu_\alpha X_\alpha, t_\mu]_{\mathbb{C}} \\
&= \sum_{\alpha\in\mathcal{R}}\mu_\alpha[X_\alpha,t_\mu]_{\mathbb{C}} \quad \text{as} \quad t_\mu\in\mathfrak{h} \\
&= -\sum_{\alpha\in\mathcal{R}}\mu_\alpha\alpha(t_\mu)X_\alpha \\
&= -\sum_{\alpha\in\mathcal{R}\setminus D_f}\mu_\alpha\alpha(t_\mu)X_\alpha
\end{aligned}$$

where the last equality follows because

$$[y^*, t_\mu]_{\mathbb{C}} \in T_{t_\mu}\mathcal{O}_{t_\mu}.$$

Similarly (41) allows us to write an element $[x,\xi]_{\mathbb{C}} \in T_\xi \mathcal{O}_\xi$ as

$$[x,\xi]_{\mathbb{C}} = \sum_{\beta \in \mathcal{R}\setminus S_f} \lambda_\beta X_\beta \quad \text{with} \quad \lambda_\beta \in \mathbb{C}.$$

Solving (42) is equivalent to solve

$$\sum_{\alpha\in\mathcal{R}\setminus D_f} \sum_{\beta\in\mathcal{R}\setminus S_f} \mu_\alpha \lambda_\beta \alpha(t_\mu) \kappa\left(X_\alpha, X_\beta\right) = 0 \quad \text{for any} \quad \lambda_\beta \in \mathbb{C}.$$

Using the fact that the $\mathfrak{g}_\alpha$'s appearing in the root decomposition are mutually orthogonal with respect to κ (except for those corresponding to the same root with opposite sign), we get

$$\begin{aligned}
0 &= \sum_{\alpha\in\mathcal{R}\setminus D_f} \sum_{\beta\in\mathcal{R}\setminus S_f} \mu_\alpha \lambda_\beta \alpha(t_\mu) \kappa\left(X_\alpha, X_\beta\right) \\
&= \sum_{\alpha,\beta\in\mathcal{R}\setminus (D_f\cup S_f)} \mu_\alpha \lambda_\beta \alpha(t_\mu) \kappa\left(X_\alpha, X_\beta\right) \\
&= \sum_{\alpha\in\mathcal{R}\setminus (D_f\cup S_f)} \mu_\alpha \lambda_\alpha \alpha(t_\mu) \kappa\left(X_\alpha, X_\alpha\right) \\
&\quad + \sum_{\alpha\in\mathcal{R}\setminus (D_f\cup S_f)} \mu_\alpha \lambda_{-\alpha} \alpha(t_\mu) \kappa\left(X_\alpha, X_{-\alpha}\right) \\
&= \sum_{\alpha\in\mathcal{R}\setminus (D_f\cup S_f)} \mu_\alpha \alpha(t_\mu) \left(\lambda_\alpha \kappa\left(X_\alpha, X_\alpha\right) + \lambda_{-\alpha} \kappa\left(X_\alpha, X_{-\alpha}\right)\right).
\end{aligned}$$

This is true for any $\lambda_\alpha \in \mathbb{C}$ if and only if $\mu_\alpha = 0$ for all $\alpha \in \mathcal{R}\setminus(D_f \cup S_f)$ as such roots satisfy $\alpha(t_\mu) \neq 0$ and both $\kappa(X_\alpha, X_\alpha)$ and $\kappa(X_\alpha, X_{-\alpha})$ do not vanish. We conclude that $y \in \mathfrak{g}_{\mathbb{C}}$ fulfils (42) for all $x \in \mathfrak{g}_{\mathbb{C}}$ if and only if y^* decomposes as

$$y^* = \sum_{i=1}^{k} a_i H_i + \sum_{\alpha\in D_f\cup S_f} \mu_\alpha X_\alpha. \tag{44}$$

Therefore,

$$y^* \in \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in D_f\cup S_f} \mathfrak{g}_\alpha\right) = \mathfrak{g}_\xi + \mathfrak{g}_\mu.$$

In particular this shows that the degeneracy set of the Hessian $D^2\Phi_G(m)$ along $\mathfrak{g}\cdot m$ belongs to $\mathfrak{g}_\xi + \mathfrak{g}_\mu$, by considering only the elements $y \in \mathfrak{g}_{\mathbb{C}}$ which are real. This proves the lemma because the other inclusion is clear.

References

[ACZE87] A. Ambrosetti, V. Coti Zelati, and I. Ekeland. Symmetry breaking in Hamiltonian systems. *J. Differential Equations*, 67(2):165–184, 1987.

[ALQ01] R. Ayala, F. F. Lasheras, and A. Quintero. The equivariant category of proper G-spaces. *Rocky Mountain J. Math.*, 31(4):1111–1132, 2001.

[Bar93] T. Bartsch. *Topological methods for variational problems with symmetries*, volume 1560 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1993.

[BL97] L. Bates and E. Lerman. Proper group actions and symplectic stratified spaces. *Pacific J. Math.*, 181(2):201–229, 1997.

[BC03] K. Brading and E. Castellani. *Symmetries in physics: philosophical reflections*. Cambridge Univ. Press, Cambridge, 2003.

[CL04] D. Chillingworth and R. Lauterbach. Dynamics and geometry in forced symmetry breaking: a tetrahedral example. *Math. Proc. Cambridge Philos. Soc.*, 137(2):411–432, 2004.

[CB15] R. Cushman and L. Bates. *Global aspects of classical integrable systems*. Birkhäuser/Springer, Basel, second edition, 2015.

[Fad85] E. Fadell. The equivariant Ljusternik-Schnirelmann method for invariant functionals and relative cohomological index theories. In *Topological methods in nonlinear analysis*, volume 95 of *Sém. Math. Sup.*, pages 41–70. Presses Univ. Montréal, Montreal, QC, 1985.

[FGNV13] Y. N. Fedorov, L. C. García-Naranjo, and J. Vankerschaver. The motion of the 2D hydrodynamic Chaplygin sleigh in the presence of circulation. *Discrete Contin. Dyn. Syst.*, 33(9):4017–4040, 2013.

[Fon18] M. Fontaine. *Explicit Symmetry Breaking and Hamiltonian Systems*. PhD thesis, The University of Manchester, 2018.

[GBT10] F. Gay-Balmaz and C. Tronci. Reduction theory for symmetry breaking with applications to nematic systems. *Phys. D*, 239(20-22):1929–1947, 2010.

[GMO04] F. Grabsi, J. Montaldi, and J.-P. Ortega. Bifurcation and forced symmetry breaking in Hamiltonian systems. *C. R. Math. Acad. Sci. Paris*, 338(7):565–570, 2004.

[GS84] V. Guillemin and S. Sternberg. A normal form for the moment map. In *Differential geometric methods in mathematical physics (Jerusalem, 1982)*, volume 6 of *Math. Phys. Stud.*, pages 161–175. Reidel, Dordrecht, 1984.

[GL01] F. Guyard and R. Lauterbach. Forced symmetry breaking and relative periodic orbits. In *Ergodic theory, analysis, and efficient simulation of dynamical systems*, pages 453–468. Springer, Berlin, 2001.

[KMRMH05] E. Kanso, J. E. Marsden, C. W. Rowley, and J. B. Melli-Huber. Locomotion of articulated bodies in a perfect fluid. *J. Nonlinear Sci.*, 15(4):255–289, 2005.

[Kir77] G. R. Kirchhoff. *Vorlesunger Über Mathematische Physik, Band I, Mechanik*. Teubner, Leipzig, 1877.

[Lam93] H. Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, sixth edition, 1993. With a foreword by R. A. Caflisch [Russel E. Caflisch].

[LR92] R. Lauterbach and M. Roberts. Heteroclinic cycles in dynamical systems with broken spherical symmetry. *J. Differential Equations*, 100(1):22–48, 1992.

[LS47] L. Lyusternik and L. Schnirelmann. Topological methods in variational problems and their application to the differential geometry of surfaces. *Uspehi Matem. Nauk (N.S.)*, 2(1(17)):166–217, 1947.

[Mar85] C.-M. Marle. Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique. *Rend. Sem. Mat. Univ. Politec. Torino*, 43(2):227–251 (1986), 1985.

[Mar92] J. E. Marsden. *Lectures on mechanics*, volume 174 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1992.

[MMO$^+$07] J. E. Marsden, G. Misiołek, J.-P. Ortega, M. Perlmutter, and T. S. Ratiu. *Hamiltonian reduction by stages*, volume 1913 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[MR99] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1999. A basic exposition of classical mechanical systems.

[Mar89] W. Marzantowicz. A G-Lusternik-Schnirelman category of space with an action of a compact Lie group. *Topology*, 28(4):403–412, 1989.

[MT60] L. M. Milne-Thomson. *Theoretical hydrodynamics*. 4th ed. The Macmillan Co., New York, 1960.

[Mil63] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[OR04] J.-P. Ortega and T. S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004.

[PROSD08] M. Perlmutter, M. Rodríguez-Olmos, and M. Sousa-Dias. The symplectic normal space of a cotangent-lifted action. *Differential Geom. Appl.*, 26(3):277–297, 2008.

[Sch07] T. Schmah. A cotangent bundle slice theorem. *Differential Geom. Appl.*, 25(1):101–124, 2007.

[Sch64] J. T. Schwartz. Generalizing the Lusternik-Schnirelman theory of critical points. *Comm. Pure Appl. Math.*, 17:307–315, 1964.

[VKM10] J. Vankerschaver, E. Kanso, and J. E. Marsden. The dynamics of a rigid body in potential flow with circulation. *Regul. Chaotic Dyn.*, 15(4-5):606–629, 2010.

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