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PLANCHEREL MEASURE FOR $\mathrm{GL}(n, F)$ AND $\mathrm{GL}(m, D)$: EXPLICIT FORMULAS AND BERNSTEIN DECOMPOSITION

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ABSTRACT. Let F be a nonarchimedean local field, let D be a division algebra over F, let $\mathrm{GL}(n) = \mathrm{GL}(n, F)$. Let ν denote Plancherel measure for $\mathrm{GL}(n)$. Let Ω be a component in the Bernstein variety $\Omega(\mathrm{GL}(n))$. Then Ω yields its fundamental invariants: the cardinality q of the residue field of F, the sizes $m_1, \ldots, m_t$, exponents $e_1, \ldots, e_t$, torsion numbers $r_1, \ldots, r_t$, formal degrees $d_1, \ldots, d_t$ and conductors $f_{11}, \ldots, f_{tt}$. We provide explicit formulas for the Bernstein component ν_Ω of Plancherel measure in terms of the fundamental invariants. We prove a transfer-of-measure formula for $\mathrm{GL}(n)$ and establish some new formal degree formulas. We derive, via the Jacquet-Langlands correspondence, the explicit Plancherel formula for $\mathrm{GL}(m, D)$.

Keywords: Plancherel measure, Bernstein decomposition, local harmonic analysis, division algebra.

AMS 2000 Mathematics subject classification: Primary 22E50, secondary 11F70, 11S40

1. INTRODUCTION

In this article we provide an explicit Plancherel formula for the p-adic group $\mathrm{GL}(n)$. Moreover, we determine explicitly the Bernstein decomposition of Plancherel measure, including all numerical constants.

Let F be a nonarchimedean local field with ring of integers $\mathfrak{o}_F$, let $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$. We will use the standard normalization of Haar measure on $\mathrm{GL}(n)$ for which the volume of $\mathrm{GL}(n, \mathfrak{o}_F)$ is 1. Plancherel measure ν is then uniquely determined by the equation

$$f(g) = \int \operatorname{trace} \pi(\lambda(g) f^\vee) d\nu(\pi)$$

for all $g \in G, f \in \mathcal{C}(G)$, where $f^\vee(g) = f(g^{-1})$.

The Harish-Chandra Plancherel Theorem expresses the Plancherel measure in the following form:

$$d\nu(\omega) \;=\; c(G|M)^{-2}\, \gamma(G|M)^{-1}\, \mu_{G|M}(\omega)\, d(\omega)\, d\omega$$

where M is a Levi subgroup of G, $\omega \in \mathrm{E}_2(M)$ the discrete series of M, $c(G|M)$ and $\gamma(G|M)$ are certain constants, $\mu_{G|M}$ is a certain rational function, $d(\omega)$ is the formal degree of ω, and $d\omega$ is the Harish-Chandra canonical measure.

In this article we determine explicitly

$$c(G|M)^{-2}\gamma(G|M)^{-1}\mu_{G|M}(\omega)\,d(\omega)\,d\omega$$

for $\mathrm{GL}(n)$.

The support of Plancherel measure ν admits a Bernstein decomposition [23] and therefore ν admits a canonical decomposition

$$\nu = \bigsqcup \nu_\Omega$$

where Ω is a component in the Bernstein variety $\Omega(G)$. We determine explicitly the Bernstein component ν_Ω for $\mathrm{GL}(n)$.

We can think of Ω as a vector of irreducible supercuspidal representations of smaller general linear groups. If the vector is

$$(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_t, \ldots, \sigma_t)$$

with σ_i repeated e_i times, $1 \le i \le t$, and $\sigma_1, \ldots, \sigma_t$ pairwise distinct (after unramified twist) then we say that Ω has *exponents* $e_1, \ldots, e_t$.

Each representation σ_i of $\mathrm{GL}(m_i)$ has a *torsion number*: the order of the cyclic group of all those unramified characters η for which $\sigma_i \otimes \eta \cong \sigma_i$. The torsion number of σ_i will be denoted r_i.

We may choose each representation σ_i of $\mathrm{GL}(m_i)$ to be unitary: in that case σ_i has a *formal degree* $d_i = d(\sigma_i)$. We have $0 < d_i < \infty$.

We will denote by $f_{ij} = f(\sigma_i^\vee \times \sigma_j)$ the conductor of the pair $\sigma_i^\vee \times \sigma_j$. An explicit conductor formula is obtained in the article by Bushnell, Henniart and Kutzko [9].

In this way, the Bernstein component Ω yields up the following *fundamental invariants*:

- the cardinality q of the residue field of F
- the sizes $m_1, m_2, \ldots, m_t$ of the smaller general linear groups
- the exponents $e_1, e_2, \ldots, e_t$
- the torsion numbers $r_1, r_2, \ldots, r_t$
- the formal degrees $d_1, d_2, \ldots, d_t$
- the conductors for pairs $f_{11}, f_{12}, \ldots, f_{tt}$.

Our Plancherel formulas are built from precisely these numerical invariants.

If Ω has the single exponent e, then the fundamental invariants yielded up by Ω are q, m, e, r, d, f. The component Ω determines a representation in the discrete series of $\mathrm{GL}(n)$, namely the generalized Steinberg representation $\mathrm{St}(\sigma, e)$. The formal degree of $\pi = \mathrm{St}(\sigma, e)$ is given by the following new formula, which is intricate, but depends only on the fundamental invariants of Ω, in line with our general philosophy:

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}e} \cdot q^{(e^2-e)(f(\sigma^\vee \times \sigma)+r-2m^2)/2} \cdot \frac{(q^r-1)^e}{q^{er}-1} \cdot \frac{|\mathrm{GL}(em,q)|}{|\mathrm{GL}(m,q)|^e}.$$

In section 2, we give a précis of the background material which we need, following the recent article of Waldspurger [34].

The Langlands-Shahidi formula gives the rational function $\mu_{G|M}$ as a ratio of certain L-factors and ϵ-factors [25]. In sections 3–4 we compute explicitly the expression

$$c(G|M)^{-2}\gamma(G|M)^{-1}\mu_{G|M}(\omega)d\omega$$

when M is a maximal parabolic. The resulting formula is stated in Theorem 4.4: in this formula we correct certain misprints in [26, p. 292 – 293].

In section 5, we compute the Plancherel density $\mu_{G|M}$ in the general case by using the Harish-Chandra product formula and we give the explicit Bernstein decomposition of Plancherel measure.

As a special case, we derive the explicit Plancherel formula for the (extended) affine Hecke algebra $\mathcal{H}(n,q)$.

We have, in effect, extended the classical formula of Macdonald [19], [20, Theorem 5.1.2] from the spherical component of GL(n) to the whole of the tempered dual.

The Plancherel formulas for GL(n,F) and GL(m,D) are dominated by *repeating patterns*, which we now attempt to explain. The repeating patterns are expressed by transfer-of-measure theorems, of which the first is as follows. With $j=1,2$, let F_j be a nonarchimedean local field and let Ω_j be a component in the Bernstein variety of GL(n_j,F_j). Let $\nu^{(j)}$ denote the Plancherel measure of GL(n_j,F_j). If Ω_1,Ω_2 share the same fundamental invariants, then

$$\nu^{(1)}_{\Omega_1} = \nu^{(2)}_{\Omega_2}.$$

The next transfer-of-measure theorem is more surprising. Let Ω be a component in the Bernstein variety of GL(n,F), and let ν be Plancherel measure. Let Ω have the fundamental invariants (q,m,e,r,d,f). Let K/F be an extension field with $q_K = q^r$. Let $G_0 := \mathrm{GL}(e,K)$, let Ω_0 be a component in the Bernstein variety of G_0, and let $\nu^{(0)}$ be Plancherel measure. If Ω_0 has fundamental invariants $(q^r,1,e,1,1,1)$ then ν_Ω and $\nu^{(0)}_{\Omega_0}$ are *proportional*, *i.e.*,

$$\nu_\Omega = \kappa \cdot \nu^{(0)}_{\Omega_0}$$

where $\kappa = \kappa(q, m, e, r, d, f)$. This phenomenon was first noted by Bushnell, Henniart, Kutzko [10, Theorem 4.1], working in the context of types and Hilbert algebras. We reconcile our result for $\mathrm{GL}(n)$ with (a special case of) their result by proving that

$$\kappa(q, m, e, r, d, f) = vol(J)^{-1} \cdot vol(I_0) \cdot \dim(\lambda)$$

where (J, λ) is an Ω-type, I_0 is an Iwahori subgroup of G_0: for this result, see Theorem 6.12. Theorem 5.7, which in essence is the Harish-Chandra product formula, then allows one to compute the Plancherel measure ν_Ω for any component Ω.

Using the explicit value for the formal degree of any representation in the discrete series of G previously obtained by Silberger and Zink, we show that the comparison formula between formal degrees, proved by Corwin, Moy, Sally in the tame case [14], is valid in general.

In the last section of the paper we consider the case of a group $\mathrm{GL}(n', D)$, where D is a central division algebra of index d over over F. We extend the transfer-of-measure result of Arthur and Clozel [1, pp. 88-90] to the case when F is of positive characteristic, by using results of Badulescu.

Let $G' = \mathrm{GL}(n', D)$, $G = \mathrm{GL}(n, F)$ with $n = dn'$. Let ν', ν denote the Plancherel measure for G', G, each with the standard normalization of Haar measure on G', G. Let $\mathrm{JL}\colon \mathrm{E}_2(G') \to \mathrm{E}_2(G)$ denote the Jacquet-Langlands correspondence. Then we have

$$d\nu'(\omega') = \lambda(D/F) \cdot d\nu(\mathrm{JL}(\omega'))$$

where

$$\lambda(D/F) = \prod (q^m - 1)^{-1}$$

the product taken over all m such that $1 \le m \le n-1, m \neq 0 \mod d$.

For example, let $G' = \mathrm{GL}(3, D), G = \mathrm{GL}(6, F)$ with D of index 2. Then we have

$$d\nu'(\omega') = (q-1)^{-1}(q^3-1)^{-1}(q^5-1)^{-1} \cdot d\nu(\mathrm{JL}(\omega'))$$

Our proof of this is in local harmonic analysis, cf [1, p. 88 - 90].

Historical Note. The Harish-Chandra Plancherel Theorem, and the Product Theorem for Plancherel Measure, were published posthumously in his collected papers in 1984, see [16]. The theorems were stated without proof (although Harish-Chandra had apparently written out the proofs). At this point, we quote from Silberger's article [29], published in 1996:

> In [16] Harish-Chandra has summarized the theory underlying the Plancherel formula for G and sketched a proof of the Plancherel theorem. To complete this sketch

> it seems to this writer that details need to be supplied justifying only one assertion of [16], namely Theorem 11. Every other assertion in this paper can be readily proved either by using prior published work of Harish-Chandra or the present author's notes on Harish-Chandra's lectures.

For Silberger's Notes, published in 1979, see [30]. Complete and detailed proofs were finally published by Waldspurger in 2003, see [34, V.2.1, VIII.1.1]. None of these sources contains any explicit computations for GL(n).

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2. The Plancherel Formula after Harish-Chandra

We shall follow very closely the notation and terminology in Waldspurger [34].

Let $\mathcal{K} = \mathrm{GL}(n, \mathfrak{o}_F)$. Let H be a closed subgroup of $G = \mathrm{GL}(n, F)$. We use the *standard* normalization of Haar measures, following [34, I.1, p.240]. Then Haar measure μ_H on H is chosen so that $\mu_H(H \cap \mathcal{K}) = 1$. If $Z = A_G$ is the centre of G then we have $\mu_Z(Z \cap \mathcal{K}) = 1$. If $H = G$ then Haar measure $\mu = \mu_G$ is normalized so that the volume of $\mathcal{K}$ is 1.

Denote by Θ the set of pairs $(\mathcal{O}, P = MU)$ where P is a semi-standard parabolic subgroup of G and $\mathcal{O} \subset \mathrm{E}_2(M)$ is an orbit under the action of $\mathrm{Im}\, X(M)$. (Here $\mathrm{E}_2(M)$ is the set of equivalence classes of the discrete series of the Levi subgroup M, and $\mathrm{Im}\, X(M)$ is the group of the unitary unramified characters of M.)

Two elements $(\mathcal{O}, P = MU)$ and $(\mathcal{O}', P' = M'U')$ are *associated* if there exists $s \in W^G$ such that $s \cdot M = M', s\mathcal{O} = \mathcal{O}'$. We fix a set $\Theta/assoc$ of representatives in Θ for the classes of association. For $(\mathcal{O}, P = MU) \in \Theta$, we set $W(G|M) = \{s \in W^G : s \cdot M = M\}/W^M$, and

$$\mathrm{Stab}(\mathcal{O}, M) = \{s \in W(G|M) : s\mathcal{O} = \mathcal{O}\}.$$

Let $\mathcal{C}(G)$ denote the Harish-Chandra Schwartz space of G and let $I_P^G\omega$ denote the normalized induced representation from ω. Let $f \in \mathcal{C}(G)$, $\omega \in \mathrm{E}_2(M)$. We will write

$$\pi = I_P^G\omega, \quad \pi(f) = \int f(g)\pi(g)dg, \quad \theta_\omega^G(f) = \mathrm{trace}\, \pi(f).$$

Theorem 2.1. The Plancherel Formula [34, VIII.1.1]. *For each $f \in \mathcal{C}(G)$ and each $g \in G$ we have*

$$f(g) = \sum c(G|M)^{-2}\gamma(G|M)^{-1}|\mathrm{Stab}(\mathcal{O}, M)|^{-1} \int_{\mathcal{O}} \mu_{G|M}(\omega)d(\omega)\theta_\omega^G(\lambda(g)f^\vee)d\omega$$

where the sum is over all the pairs $(\mathcal{O}, P = MU) \in \Theta/assoc$.

Note that

$$\mu_{G|M}(\omega) \cdot c(G|M)^{-2} \cdot \gamma(G|M)^{-1} = \gamma(G|M) \cdot j(\omega)^{-1}, \tag{1}$$

where j denotes the composition of intertwining operators defined in [34, IV.3 (2)].

The map

$$(\mathcal{O}, P = MU) \to \mathrm{Irr}^{\mathrm{t}}(G),\ \omega \mapsto I_P^G\omega$$

determines a *bijection*

$$\bigsqcup(\mathcal{O}, P = MU)/\mathrm{Stab}(\mathcal{O}, M) \longrightarrow \mathrm{Irr}^{\mathrm{t}}(G).$$

The tempered dual $\mathrm{Irr}^{\mathrm{t}}(G)$ acquires, by transport of structure, the structure of *disjoint union of countably many compact orbifolds.*

According to [34, V.2.1], the function $\mu_{G|M}$ is a rational function on $\mathcal{O}$. We have $\mu_{G|M}(\omega) \geq 0$ and $\mu_{G|M}(s\omega) = \mu(\omega)$ for each $s \in W^G, \omega \in \mathcal{O}$. This invariance property implies that μ *descends* to a function on the orbifold $\mathcal{O}/\mathrm{Stab}(\mathcal{O}, M)$. We can view μ either as an *invariant* function on the orbit $\mathcal{O}$ or as a function on the orbifold $\mathcal{O}/\mathrm{Stab}(\mathcal{O}, M)$.

We now define the *canonical measure* $\mathrm{d}\omega$. The map $\mathrm{Im}\, X(M) \to \mathcal{O}$ sends $\chi \mapsto \omega \otimes \chi$; the map $\mathrm{Im}\, X(M) \to \mathrm{Im}\, X(A_M)$ is determined by restriction. Let $(Y_i, \mathcal{B}_i, \mu_i)$ be finite measure spaces with $i = 1, 2$ and let $f : Y_1 \to Y_2$ be a measurable map. Then μ_1 is the *pull-back* of μ_2 if $\mu_1(f^{-1}E) = \mu_2(E)$ for all $E \in \mathcal{B}_2$. This surely is the meaning of *préserve localement les mesures* in [34, p.239, 302].

The compact group $\mathrm{Im}\, X(A_M)$ is assigned the Haar measure of total mass 1. Choose Haar measure on the compact orbit $\mathcal{O}$. Now $\mathrm{Im}\, X(M)$ admits two pull-back measures:

$$\mathrm{Im}\, X(A_M) \leftarrow \mathrm{Im}\, X(M) \to \mathcal{O}.$$

These must coincide: this fixes the Haar measure $\mathrm{d}\omega$ on $\mathcal{O}$, see [34, p. 239, 302].

Let E be a Borel set in $\mathcal{O}$ which is also a fundamental domain for the action of $\mathrm{Stab}(\mathcal{O}, M)$ on $\mathcal{O}$. Since $F(\omega) := \mu_{G|M}(\omega)\, d(\omega)\, \theta_\omega^G(\lambda(g)f^\vee)$ is $\mathrm{Stab}(\mathcal{O}, M)$-invariant, we have

$$|\mathrm{Stab}(\mathcal{O}, M)|^{-1} \cdot \int_{\mathcal{O}} F(\omega)\mathrm{d}\omega = \int_E F(\omega)\mathrm{d}\omega.$$

The *Plancherel density*, with respect to the canonical measure dω, is therefore

$$c(G|M)^2 \cdot \gamma(G|M)^{-1} \cdot \mu_{G|M}(\omega)\, d(\omega)$$

where $d(\omega)$ is the formal degree of ω. It is precisely this expression which we will compute explicitly for GL(n). To this end, we will use the following result.

Theorem 2.2. The Product Formula [34, V.2.1]. *With* $M = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k) \subset \mathrm{GL}(n)$ *and* $\omega = \omega_1 \otimes \cdots \otimes \omega_k$ *we have*

$$\mu_{G|M}(\omega) = \prod_{1 \le j < i \le k} \mu_{\mathrm{GL}(n_i+n_j)|\mathrm{GL}(n_i)\times\mathrm{GL}(n_j)}(\omega_i \otimes \omega_j).$$

The Plancherel measure ν is determined by the equation

$$f(g) = \int \mathrm{trace}\, \pi(\lambda(g) f^\vee) d\nu(\pi)$$

for all $f \in \mathcal{C}(G)$.

Theorem 2.3. The Bernstein Decomposition [23]. *The Plancherel measure* ν *admits a canonical Bernstein decomposition*

$$\nu = \bigsqcup \nu_\Omega$$

where Ω *is a component in the Bernstein variety* $\Omega(G)$. *The domain of each* ν_Ω *is a finite union of orbifolds of the form* $\mathcal{O}/\mathrm{Stab}(\mathcal{O}, M)$ *and is precisely a single extended quotient.*

We will use Theorem 2.3 to compute the Plancherel measure of the (extended) affine Hecke algebra $\mathcal{H}(n,q)$ (see Remark 5.6).

3. Calculation of the γ factors

Theorem 3.1. *We have*

$$\gamma(G|M) = q^{-2\sum_{1\le i<j\le k} n_i n_j} \frac{|\mathrm{GL}(n,q)|}{|\mathrm{GL}(n_1,q)| \times \cdots \times |\mathrm{GL}(n_k,q)|}.$$

Proof. By applying the formula given in [34, p.241, l.7] to the group $H = \mathrm{I}_n + \varpi \mathrm{M}(n, \mathfrak{o}_F)$, we obtain

$$\gamma(G|M) = q^{-2R} \frac{\mu(M \cap H)}{\mu(H)},$$

with $R = \Sigma(G)^+ - \Sigma(M)^+$, where $\Sigma(G)^+$ (resp. $\Sigma(M)^+$) denotes the set of positive roots in G (resp. M). We have

$$R = \sum_{1 \le i<j\le k} n_i n_j.$$

On the other hand, since the Haar measure on G is normalized so that the volume of $\mathcal{K}$ is 1, it follows from the exact sequence

$$1 \to H \to \mathcal{K} \to \mathrm{GL}(n,q),$$

that

$$\mu(H) = |\mathrm{GL}(n,q)|^{-1} \text{ and } \mu(H \cap M) = |\mathrm{GL}(n_1,q)|^{-1} \times \cdots \times |\mathrm{GL}(n_k,q)|^{-1}.$$

$\square$

Remark 3.2. Observe that $2\sum_{1\le i<j\le k} n_i n_j$ equals the length of the element $w = w_M w_{\mathrm{GL}(n)}$, where w_M (resp. $w_{\mathrm{GL}(n)}$) denotes the longest element in the Weyl group of M (resp. $\mathrm{GL}(n)$). Let $P_{S_n}(X)$ denote the Poincaré polynomial of the Coxeter group S_n. Then, using the fact that (see for instance [21, (2.6)])

$$P_{S_n}(q^{-1}) = \frac{|\mathrm{GL}(n,q)|}{q^{n^2-n}(q-1)^n}, \tag{2}$$

we obtain from Theorem 3.1

$$\gamma(G|M) = \frac{P_{S_n}(q^{-1})}{P_{S_{n_1}}(q^{-1}) \times \cdots \times P_{S_{n_k}}(q^{-1})}. \tag{3}$$

This gives the following expression for the c-function defined in [34, I.1]:

$$c(G|M) = \frac{\displaystyle\prod_{1\le i<j\le k} P_{S_{n_i+n_j}}(q^{-1})}{P_{S_n}(q^{-1}) \cdot \displaystyle\prod_{i=1}^{k} (P_{S_{n_i}}(q^{-1}))^{k-2}}. \tag{4}$$

4. The Langlands-Shahidi formula

Let ϖ denote a fixed uniformizer. We will choose a continuous additive character Ψ such that the conductor of Ψ is $\mathfrak{o}_F$. Note that Shahidi uses precisely this normalization in [27]. We shall need the L-factor $L(s, \pi_1 \times \pi_2)$ and the ϵ-factor $\epsilon(s, \pi_1 \times \pi_2, \Psi)$ for pairs, where s denotes a complex variable (see [18] and [25]). We define the conductor $f(\pi_1 \times \pi_2)$ (see [9]) and the γ-factor $\gamma(s, \pi_1 \times \pi_2, \Psi)$ (see [18, p. 374]) for pairs as

$$\epsilon(0, \pi_1 \times \pi_2, \Psi) = q^{f(\pi_1 \times \pi_2)} \cdot \epsilon(1, \pi_1 \times \pi_2, \Psi), \tag{5}$$

$$\gamma(s, \pi_1 \times \pi_2, \Psi) = \epsilon(s, \pi_1 \times \pi_2, \Psi) \cdot L(1-s, \pi_1^\vee \times \pi_2^\vee)/L(s, \pi_1 \times \pi_2). \tag{6}$$

We assume in this section that P is the upper block triangular maximal parabolic subgroup of G with Levi subgroup $M = \mathrm{GL}(n_1) \times$

GL(n_2). We have the Langlands-Shahidi formula for the Harish-Chandra μ-function, see [26, §7] or [27, §6]:

$$\mu_{G|M}(\omega_1 \otimes \omega_2) = \gamma(G|M)^2 \cdot \frac{\gamma(0, \omega_1^\vee \times \omega_2, \Psi)}{\gamma(1, \omega_1^\vee \times \omega_2, \Psi)}. \tag{7}$$

It is useful to note that

$$\frac{\gamma(0, \omega_1^\vee \times \omega_2, \Psi)}{\gamma(1, \omega_1^\vee \times \omega_2, \Psi)} = q^{f(\omega_1^\vee \times \omega_2)} \cdot L'' \tag{8}$$

where

$$L'' = \frac{L(1, \omega_1 \times \omega_2^\vee) L(1, \omega_1^\vee \times \omega_2)}{L(0, \omega_1 \times \omega_2^\vee) L(0, \omega_1^\vee \times \omega_2)}. \tag{9}$$

For any smooth representation π of G and any quasicharacter χ, we denote by $\chi\pi$ the twist of π by χ:

$$\chi\pi := (\chi \circ \det) \otimes \pi.$$

If σ_1 (resp. σ_2) is an irreducible supercuspidal representation of GL(m_1) (resp. GL(m_2)), then we have $L(s, \sigma_1 \times \sigma_2^\vee) = 1$ unless $\sigma_1 \cong \chi\sigma_2$ with χ an unramified quasicharacter of $F^\times$.

The next formula is from [26, p. 292] or [18, Prop. 8.1].

Lemma 4.1. *Let σ_2 have torsion number r and let $\sigma_1 \cong \chi\sigma_2$ with χ an unramified quasicharacter such that $\chi(\varpi) = \zeta$. Then we have*

$$L(s, \sigma_1 \times \sigma_2^\vee) = (1 - \zeta^{-r} q^{-rs})^{-1}.$$

Let χ_1, χ_2 be unramified (unitary) characters of $F^\times$. The group of unramified (unitary) characters $\operatorname{Im} X(M)$ of M has, via the map

$$(\chi_1 \circ \det) \otimes (\chi_2 \circ \det) \mapsto (\chi_1(\varpi), \chi_2(\varpi))$$

the structure of the compact torus $\mathbb{T}^2$.

Let π_i be in the discrete series of GL(n_i) with $i = 1, 2$, and let π_i have torsion number r. Consider now the *orbit* $\operatorname{Im}X(M) \cdot (\pi_1 \otimes \pi_2)$ in the Harish-Chandra parameter space $\Omega^{\mathrm{t}}(G)$. The action of $\operatorname{Im}X(M)$ creates a short exact sequence

$$1 \to \mathcal{G} \to \mathbb{T}^2 \to \mathbb{T}^2 \to 1$$

with

$$\mathbb{T}^2 \to \mathbb{T}^2,\ (\zeta_1, \zeta_2) \mapsto (\zeta_1^r, \zeta_2^r).$$

The finite group $\mathcal{G}$ is precisely the finite group in [5, Lemma 25] and is the product of cyclic groups:

$$\mathcal{G} = \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}.$$

We will write $z_1 = \zeta_1^r, z_2 = \zeta_2^r$ so that z_1, z_2 are precisely the coordinates of a point in the orbit.

Remark 4.2. We recall the following facts about the discrete series of $\mathrm{GL}(n)$. Let π_1 and π_2 be two discrete series representations of $\mathrm{GL}(n_1)$ and $\mathrm{GL}(n_2)$, respectively. By [35], there exist two pairs of integers (m_1, l_1) and (m_2, l_2) and two irreducible unitary supercuspidal representations σ_1 and σ_2 of $\mathrm{GL}(m_1)$ and $\mathrm{GL}(m_2)$ respectively such that, for $i = 1, 2$, we have $l_i m_i = n_i$ and the representation π_i is the unique irreducible quotient associated to the Zelevinsky segment

$$\{|\det|^{-g_i}\sigma_i, |\det|^{-g_i+1}\sigma_i, \ldots, |\det|^{g_i-1}\sigma_i, |\det|^{g_i}\sigma_i\},$$

where $2g_i + 1 = l_i$. We will follow the notation in Arthur-Clozel [1, p. 61] and write

$$\pi_i = \mathrm{St}(\sigma_i, l_i).$$

So π_i is a *generalized Steinberg representation*. We observe that

$$\chi\pi_i = \mathrm{St}(\chi\sigma_i, l_i).$$

It follows that the torsion numbers of π_i and σ_i are equal.

Theorem 4.3. *Let σ_1, σ_2 be irreducible unitary supercuspidal representations of $\mathrm{GL}(m_1), \mathrm{GL}(m_2)$. Let π_1, π_2 be discrete series representations of $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$ such that $\pi_i = \mathrm{St}(\sigma_i, l_i)$. Let χ_1, χ_2 be unramified characters. If $\sigma_1 \neq \chi\sigma_2$ for any unramified quasicharacter χ of $F^\times$ then, as a function on the compact torus $\mathbb{T}^2$, $\mu_{G|M}(\chi_1\pi_1 \otimes \chi_2\pi_2)$ is constant: we have*

$$\mu_{G|M}(\chi_1\pi_1 \otimes \chi_2\pi_2) = \gamma(G|M)^2 \cdot q^{l_1 l_2 f(\sigma_1^\vee \times \sigma_2)}$$

We also have

$$f(\pi_1^\vee \times \pi_2) = l_1 l_2 f(\sigma_1^\vee \times \sigma_2).$$

Proof. Let $\omega_i = \chi_i\pi_i$ and $\tau_i = \chi_i\sigma_i$ for $i = 1, 2$. We will use the multiplicative property of the γ-factors. From [18, Th. 3.1] or [17, p. 254], we have, with $b = g_1 + g_2$,

$$\gamma(s, \omega_1^\vee \times \omega_2, \Psi) = \prod_{i=0}^{l_1-1}\prod_{j=0}^{l_2-1} \gamma(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi).$$

On the other hand $\gamma(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi)$ equals

$$\epsilon(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi) \cdot \frac{L(1-s, |\ |^{-i-j+b}\tau_1 \times \tau_2^\vee)}{L(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2)}.$$

Since

$$\frac{\epsilon(0, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi)}{\epsilon(1, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi)} = q^{f(|\ |^{i+j-b}\tau_1^\vee \times \tau_2)} = q^{f(\tau_1^\vee \times \tau_2)} = q^{f(\sigma_1^\vee \times \sigma_2)},$$

it follows that

$$\frac{\gamma(0,\omega_1^\vee\times\omega_2,\Psi)}{\gamma(1,\omega_1^\vee\times\omega_2,\Psi)} = q^{l_1 l_2\cdot f(\sigma_1^\vee\times\sigma_2)}\cdot L', \tag{10}$$

with

$$L' = \prod_{i=0}^{l_1-1}\prod_{j=0}^{l_2-1}\frac{L(1,|\;|^{-i-j+b}\tau_1\times\tau_2^\vee)}{L(0,|\;|^{i+j-b}\tau_1^\vee\times\tau_2)}\cdot\frac{L(1,|\;|^{i+j-b}\tau_1^\vee\times\tau_2)}{L(0,|\;|^{-i-j+b}\tau_1\times\tau_2^\vee)}. \tag{11}$$

Since $\sigma_1\neq\chi\sigma_2$, then $\tau_1\neq\chi\tau_2$ for any unramified quasicharacter χ, and $L'=1$.

The multiplicative property of the L-factors [18, Theorem 8.2] implies that $L''=1$. Therefore, by (8) we have

$$\frac{\gamma(0,\omega_1^\vee\times\omega_2,\Psi)}{\gamma(1,\omega_1^\vee\times\omega_2,\Psi)} = q^{f(\omega_1^\vee\times\omega_2)} \tag{12}$$

Then the results follow from the Langlands-Shahidi formula (7), and from (10) and (12). □

Theorem 4.4. *Let σ be an irreducible unitary supercuspidal representations of* GL(m) *with torsion number r. Let π_1, π_2 be discrete series representations of* GL(n_1), GL(n_2), *with $n_i=l_i m$, such that $\pi_i=\mathrm{St}(\sigma,l_i)$. Let χ_1, χ_2 be unramified characters. Let $\chi_i(\varpi)=\zeta_i$, $z_i=\zeta_i^r$, $i=1,2$. Then, as a function on the compact torus $\mathbb{T}^2$ with co-ordinates (z_1,z_2), we have*

$$\mu_{G|M}(\chi_1\pi_1\otimes\chi_2\pi_2)=\gamma(G|M)^2\cdot q^{l_1l_2 f(\sigma^\vee\times\sigma)}\cdot\prod\left|\frac{1-z_2z_1^{-1}q^{gr}}{1-z_2z_1^{-1}q^{-(g+1)r}}\right|^2$$

where the product is over those g for which $|g_1-g_2|\leq g\leq g_1+g_2$. Note that g_1-g_2 and g_1+g_2 can both be half integers.

We also have

$$f(\pi_1^\vee\times\pi_2)=l_1l_2f(\sigma^\vee\times\sigma)+r(l_1l_2-\min(l_1,l_2)).$$

Proof. Let $\tau_i=\chi_i\sigma$. We have

$$L'=\prod_{i=0}^{l_1-1}\prod_{j=0}^{l_2-1}\frac{L(1-i-j+b,\tau_1\times\tau_2^\vee)}{L(i+j-b,\tau_1^\vee\times\tau_2)}\cdot\frac{L(i+j+1-b,\tau_1^\vee\times\tau_2)}{L(-i-j+b,\tau_1\times\tau_2^\vee)},$$

where L' is defined by (11).

Now we delve into the combinatorics. To this end, we make a change of variable, and a change of notation.

Let $\lambda(s)=L(s,\tau_1^\vee\times\tau_2)$, $\lambda^*(s)=L(s,\tau_1\times\tau_2^\vee)$. Note that, for all $s\in\mathbb{R}$, $\lambda^*(s)$ is the complex conjugate of $\lambda(s)$. Let now $k=i+j-b$.

We have

$$L' = \prod_{i=0}^{l_1-1} \prod_{j=0}^{l_2-1} \frac{\lambda^*(1-k)}{\lambda(k)} \cdot \frac{\lambda(1+k)}{\lambda^*(-k)}.$$

We now define the function

$$a : \{-b, -b+1, \ldots, b-1, b\} \longrightarrow \{1, 2, 3, \ldots, \min(l_1, l_2)\}$$

as follows:

$$a(k) = \sharp\{(i,j) : k = i + j - b, 0 \leq i \leq l_1 - 1, 0 \leq j \leq l_2 - 1\}.$$

Note that the function a is even: $a(-k) = a(k)$. It first increases, then is constant with its maximum value $\min(l_1, l_2)$, then decreases. Quite specifically, we have

- $a(-b) = 1$
- $-b \leq k < -|g_1 - g_2| \Rightarrow a(k+1) - a(k) = 1$
- $a(-|g_1 - g_2|) = \min(l_1, l_2)$
- $-|g_1 - g_2| \leq k < |g_1 - g_2| \Rightarrow a(k+1) = a(k)$
- $a(|g_1 - g_2|) = \min(l_1, l_2)$
- $|g_1 - g_2| \leq k < b \Rightarrow a(k+1) - a(k) = -1$
- $a(b) = 1.$

We have

$$\begin{aligned} L' &= \prod_{k=-b}^{b} \frac{\lambda^*(1-k)^{a(k)}}{\lambda(k)^{a(k)}} \cdot \frac{\lambda(1+k)^{a(k)}}{\lambda^*(-k)^{a(k)}} \\ &= \prod_{k=-b}^{b} \frac{\lambda^*(1+k)^{a(k)}}{\lambda(k)^{a(k)}} \cdot \frac{\lambda(1+k)^{a(k)}}{\lambda^*(k)^{a(k)}} \\ &= \prod_{k=-b}^{b} \left| \frac{\lambda(1+k)^{a(k)}}{\lambda(k)^{a(k)}} \right|^2 . \qquad (13) \end{aligned}$$

We also have, setting $a(1+b)=0$,

$$\begin{aligned}
\prod_{k=-b}^{b}\frac{\lambda(1+k)^{a(k)}}{\lambda(k)^{a(k)}} &= \frac{1}{\lambda(-b)}\cdot\prod_{k=-b}^{b}\frac{\lambda(1+k)^{a(k)}}{\lambda(1+k)^{a(1+k)}}\\
&= \frac{1}{\lambda(-b)}\prod_{k=-b}^{-|g_1-g_2|-1}\frac{1}{\lambda(k+1)}\cdot\prod_{k=|g_1-g_2|}^{b}\lambda(1+k)\\
&= \frac{\lambda(1+b)}{\lambda(-b)}\cdots\frac{\lambda(1+|g_1-g_2|)}{\lambda(-|g_1-g_2|)}\\
&= \prod_{g=|g_1-g_2|}^{g_1+g_2}\frac{\lambda(1+g)}{\lambda(-g)}. \qquad (14)
\end{aligned}$$

Note that $\tau_2=\chi\tau_1$ where $\chi(\varpi)=\zeta_2\zeta_1^{-1}$. Therefore $\chi(\varpi)^{-r}=z_1z_2^{-1}$. The first result now follows immediately from Lemma 4.1, since

$$\lambda(g)=L(g,\tau_1^\vee\times\tau_2)=L(g,\tau_2\times\tau_1^\vee)=(1-z_1z_2^{-1}q^{-gr})^{-1}.$$

Note also that $|1-z_2z_1^{-1}q^{-gr}|=|1-z_1z_2^{-1}q^{-gr}|$ since $z_2z_1^{-1},z_1z_2^{-1}$ are complex conjugates.

In addition we have

$$|1-z_2z_1^{-1}q^{gr}|^2=|q^{gr}-z_2z_1^{-1}|^2=q^{2gr}|1-z_2z_1^{-1}q^{-gr}|^2$$

and so we have

$$\left|\frac{\lambda(g)}{\lambda(-g)}\right|^2=q^{2gr}.$$

The multiplicative property of the L-factors [18, Theorem 8.2] leads to the equation

$$L''=\prod_{g=|g_1-g_2|}^{g_1+g_2}\left|\frac{\lambda(1+g)}{\lambda(g)}\right|^2$$

Therefore we have

$$\begin{aligned}
L'/L'' &= \prod_{g=|g_1-g_2|}^{g_1+g_2}\left|\frac{\lambda(g)}{\lambda(-g)}\right|^2\\
&= \prod_{g=|g_1-g_2|}^{g_1+g_2}q^{2rg}\\
&= q^{r(l_1l_2-\min(l_1,l_2))} \qquad (15)
\end{aligned}$$

thanks to the identity

$$2|g_1-g_2|+\cdots+2(g_1+g_2)=l_1l_2-\min(l_1,l_2)$$

which follows from the classic identity

$$2|g_1 - g_2| + 1 + \cdots + 2(g_1 + g_2) + 1 = l_1 l_2.$$

Since

$$\frac{\gamma(1, \omega_1^\vee \times \omega_2, \psi_F)}{\gamma(0, \omega_1^\vee \times \omega_2, \psi_F)} = q^{f(\omega_1^\vee \times \omega_2)} \cdot L'' = q^{l_1 l_2 f(\sigma^\vee \times \sigma)} \cdot L'$$

we have

$$q^{f(\omega_1^\vee \times \omega_2)} = q^{l_1 l_2 f(\sigma^\vee \times \sigma)} \cdot L'/L'' = q^{l_1 l_2 f(\sigma^\vee \times \sigma)} q^{r(l_1 l_2 - \min(l_1, l_2))}$$

and we conclude that

$$f(\pi_1^\vee \times \pi_2) = l_1 l_2 f(\sigma^\vee \times \sigma) + r(l_1 l_2 - \min(l_1, l_2)).$$

□

The above formulas are invariant under the map $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$ with λ a complex number of modulus 1, and under the map $(z_1, z_2) \mapsto (z_2, z_1)$. In section 6 of the paper we shall interpret q^r as the cardinality q_K of the residue field of a canonical extension field K/F.

For example, let $M = \mathrm{GL}(1) \times \mathrm{GL}(2) \subset \mathrm{GL}(3)$, $\omega_1 = 1$, $\omega_2 = \mathrm{St}(2) = \mathrm{St}(1, 2)$. We have $l_1 = 1$, $l_2 = 2$, $g_1 = 0$, $g_2 = 1/2$, $r = 1$. This gives the following (rational) function on the 2-torus:

$$\mu(\chi_1 \otimes \chi_2 \,\mathrm{St}(2)) = \gamma(\mathrm{GL}(3)|M)^2 \cdot q \cdot \left| \frac{1 - z_2 z_1^{-1} q^{-1/2}}{1 - z_2 z_1^{-1} q^{-3/2}} \right|^2.$$

Theorem 4.5. *Let $G = \mathrm{GL}(2m), M = \mathrm{GL}(m) \times \mathrm{GL}(m)$ and let σ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$ with torsion number r. Then we have*

$$\mu_{G|M}(\chi_1 \sigma \otimes \chi_2 \sigma) = \gamma(G|M)^2 \cdot q^{f(\sigma^\vee \times \sigma)} \cdot \left| \frac{1 - z_2 z_1^{-1}}{1 - z_2 z_1^{-1} q^{-r}} \right|^2$$

Proof. This follows from Theorem 4.4 by taking $l_1 = l_2 = 1$, so that $g_1 = g_2 = g = 0$. □

5. The Bernstein decomposition of Plancherel measure

5.1. **The one exponent case.** Let X be a space on which the finite group Γ acts. The extended quotient associated to this action is the quotient space $\tilde{X}/\Gamma$ where

$$\tilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}.$$

The group action on $\tilde{X}$ is $g.(\gamma, x) = (g\gamma g^{-1}, gx)$. Let $X^\gamma = \{x \in X : \gamma x = x\}$ and let $Z(\gamma)$ be the Γ-centralizer of γ. Then the extended quotient is given by:

$$\tilde{X}/\Gamma = \bigsqcup_\gamma X^\gamma/Z(\gamma)$$

where one γ is chosen in each Γ-conjugacy class. If $\gamma = 1$ then $X^\gamma/Z(\gamma) = X/\Gamma$ so *the extended quotient always contains the ordinary quotient*:

$$\tilde{X}/\Gamma = X/\Gamma \sqcup \ldots$$

We shall need only the special case in which X is the compact torus $\mathbb{T}^n$ of dimension n and Γ is the symmetric group S_n acting on $\mathbb{T}^n$ by permuting co-ordinates.

Let β be a partition of n, and let γ have cycle type β. Each cycle provides us with one circle, and cycles of equal length provide us with a symmetric product of circles. For example, the extended quotient $\widetilde{\mathbb{T}^5}/S_5$ is the following disjoint union of compact orbifolds (one for each partition of 5):

$$\mathbb{T} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}^2 \sqcup (\mathbb{T} \times \mathrm{Sym}^2\mathbb{T}) \sqcup (\mathbb{T} \times \mathrm{Sym}^2\mathbb{T}) \sqcup (\mathbb{T} \times \mathrm{Sym}^3\mathbb{T}) \sqcup \mathrm{Sym}^5\mathbb{T}$$

where $\mathrm{Sym}^n\mathbb{T}$ is the n-fold symmetric product of the circle $\mathbb{T}$. This extended quotient is a model of the arithmetically unramified tempered dual of GL(5).

Let $\Omega \subset \Omega(\mathrm{GL}(n))$ have one exponent e. Then we have $e|n$ and so $em = n$.

There exists an irreducible unitary supercuspidal representation σ of $\mathrm{GL}(m)$ such that the conjugacy class of the cuspidal pair $(\mathrm{GL}(m) \times \cdots \times \mathrm{GL}(m), \sigma \otimes \cdots \otimes \sigma)$ is an element in Ω. We have $\Omega \cong \mathrm{Sym}^e\,\mathbb{C}^\times$ as complex affine algebraic varieties. Consider now a partition $p = (l_1, \ldots, l_k)$ of e into k parts, and write $2g_1 + 1 = l_1$, …, $2g_k + 1 = l_k$. Let

$$\pi_i = \mathrm{St}(\sigma, l_i)$$

as in Remark 3.2. Then $\pi_1 \in \mathrm{E}_2(\mathrm{GL}(ml_1))$, …, $\pi_k \in \mathrm{E}_2(\mathrm{GL}(ml_k))$. Note that $ml_1 + \ldots + ml_k = n$ so that $\mathrm{GL}(ml_1) \times \ldots \times \mathrm{GL}(ml_k)$ is a standard Levi subgroup M of $\mathrm{GL}(n)$. Now consider

$$\pi = \chi_1\pi_1 \otimes \ldots \otimes \chi_k\pi_k$$

with $\chi_1, \ldots, \chi_k$ unramified (unitary) characters. Then $\pi \in \mathrm{E}_2(M)$. We have

$$\omega = \mathrm{I}_{MN}^{\mathrm{GL}(n)}(\pi \otimes 1) \in \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)$$

and each element $\omega \in \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)$ for which $inf.ch.\omega \in \Omega$ is accounted for on this way. As explained in detail in [23], we have

$$\widetilde{X}/\Gamma \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega \tag{16}$$

where $X = \mathbb{T}^e$, $\Gamma = S_e$, *i.e.*,

$$\bigsqcup_\gamma X^\gamma/Z(\gamma) \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega.$$

The partition $p = (l_1, \ldots, l_k)$ of e determines a permutation γ of the set $\{1, 2, \ldots, e\}$: γ is the product of the cycles $(1, \ldots, l_1) \cdots (1, \ldots, l_k)$. Then the fixed set X^γ is

$$\{(z_1, \ldots, z_1, \ldots, z_k, \ldots, z_k) \in \mathbb{T}^e : z_1, \ldots, z_k \in \mathbb{T}\}$$

and so $X^\gamma \cong \mathbb{T}^k$.

Explicitly, we have

$$X^\gamma \longrightarrow \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega$$

$$(z_1, \ldots, z_k) \mapsto \mathrm{I}_{MN}^{\mathrm{GL}(n)}(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k)$$

with $\chi_1(\varpi) = \zeta_1, \ldots, \chi_k(\varpi) = \zeta_k, z_1 = \zeta_1^r, \ldots, z_k = \zeta_k^r$ exactly as in Theorem 3.2. This map is constant on each $Z(\gamma)$-orbit and descends to an *injective* map

$$X^\gamma/Z(\gamma) \to \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega.$$

Taking one γ in each Γ-conjugacy class we have the bijective map

$$\bigsqcup_\gamma X^\gamma/Z(\gamma) \cong \mathrm{Irr}\,\mathrm{GL}(n)_\Omega.$$

This bijection, by transport of structure, equips $\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega$ with the structure of disjoint union of finitely many compact orbifolds.

We now describe the restriction μ_Ω of Plancherel density to the compact orbifold $X^\gamma/Z(\gamma)$.

Theorem 5.1. *Let σ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$ with torsion number r. For $i = 1, \ldots, k$, let*

$$\pi_i = \mathrm{St}(\sigma, l_i),$$

let χ_i be an unramified character with $\chi_i(\varpi) = \zeta_i$, and let $z_i = \zeta_i^r$.

Then, as a function on the compact torus $\mathbb{T}^k$ with co-ordinates $(z_1, \ldots, z_k)$ we have

$$\mu(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k) = const. \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2$$

where the product is taken over those i, j, g for which the following inequalities hold: $1 \le i < j \le k$, $|g_i - g_j| \le g \le g_i + g_j$, $2g_i + 1 = l_i$.

Proof. Apply Theorem 4.4 and the Harish-Chandra product formula, Theorem 2.2. Note that the function

$$(z_1, \ldots, z_k) \mapsto const. \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2$$

is a $Z(\gamma)$-invariant function on the γ-fixed set $X^\gamma = \mathbb{T}^k$, and descends to a non-negative function on the orbifold $X^\gamma/Z(\gamma)$:

$$X^\gamma/Z(\gamma) \longrightarrow \mathbb{R}_+.$$

□

In the above theorem, the co-ordinates $z_1, \ldots, z_k$ should be thought of as *generalized Satake parameters.* The k-tuple $t = (z_1, \ldots, z_k)$ is a point in the standard maximal torus T of the unitary group $U(k, \mathbb{C})$. In that case, the roots of the unitary group are given by

$$\alpha_{ij}(t) = z_i/z_j.$$

The μ-function may now be written in the more invariant form

$$\mu(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k) = const. \prod (1 - \alpha(t)q^{gr})(1 - \alpha(t)q^{-(g+1)r})^{-1}$$

where the product is taken over all roots $\alpha = \alpha_{ij}$ of $U(k, \mathbb{C})$ and all g for which the following inequalities hold: $1 \leq i \leq k, 1 \leq j \leq k, i \neq j$, $|g_i - g_j| \leq g \leq g_i + g_j$, $2g_i + 1 = l_i$.

Theorem 5.2. *We have the following numerical formula for const.*

$$const. = q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \gamma(G|M)^2 \cdot c(G|M)^2,$$

where $\ell(\gamma) = \sum_{1 \leq i < j \leq k} l_i l_j$.

Proof. The numerical constant is determined by Theorem 4.4 and Theorem 2.2. Explicitly, for $i, j \in \{1, \ldots, k\}$, setting

$$\gamma_{i,j} := \gamma(\mathrm{GL}(n_i + n_j)|\mathrm{GL}(n_i) \times \mathrm{GL}(n_j)),$$

for the γ-factor of the Levi subgroup $GL(n_i) \times \mathrm{GL}(n_j)$ of the maximal standard parabolic subgroup in $\mathrm{GL}(n_i + n_j)$,

$$\begin{aligned} const. &= q^{\sum_{1 \leq i < j \leq k} l_i l_j f(\sigma^\vee \times \sigma)} \cdot \prod_{1 \leq i < j \leq k} \gamma_{i,j}^2 \\ &= q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \gamma(G|M)^2 \cdot c(G|M)^2. \end{aligned}$$

□

Corollary 5.3. *We have*

$$j(\omega) = q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^{-(g+1)r}}{1 - z_j z_i^{-1} q^{gr}} \right|^2$$

Proof. This follows immediately from Theorems 5.1, 5.2 and the fact that

$$c(G|M)^{-2}\,\gamma(G|M)^{-1}\,\mu_{G|M}(\omega) \;= \gamma(G|M)\,j(\omega)^{-1}.$$

□

Given $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$ choose $e|n$ and let $m = n/e$. Let Ω be a Bernstein component in $\Omega(\mathrm{GL}(n))$ with one exponent e. The compact extended quotient attached to Ω has finitely many components, each component is a compact orbifold. We now have enough results to write down explicitly the component μ_Ω. Let $l_1 + \cdots + l_k = e$ be a partition of e, let $\gamma = (1, \ldots, l_1) \cdots (1, \ldots, l_k) \in S_e = \Gamma$, $g_1 = (l_1 - 1)/2$, $\ldots$, $g_k = (l_k - 1)/2$. Then we have the fixed set $X^\gamma = \mathbb{T}^k$. Let σ be an irreducible unitary supercuspidal representation of the group $\mathrm{GL}(m)$ and let the conjugacy class of the cuspidal pair $(\mathrm{GL}(m)^e, \sigma^{\otimes e})$ be a point in the Bernstein component Ω. Let r be the torsion number of σ and choose a field K such that $q_K = q_F^r$.

We have (16):

$$\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n, F)_\Omega \cong \widetilde{X}/\Gamma.$$

This compact Hausdorff space admits the Harish-Chandra *canonical measure* $d\omega$: on each connected component in the extended quotient $\widetilde{X}/\Gamma$, $d\omega$ restricts to the quotient by the centralizer $Z(\gamma)$ of the normalized Haar measure on the compact torus X^γ.

Let $d\nu$ denote Plancherel measure on the tempered dual of $\mathrm{GL}(n, F)$.

Theorem 5.4. *On the component $X^\gamma/Z(\gamma)$ of the extended quotient $\widetilde{X}/\Gamma$ we have:*

$$d\nu(\omega) = q^{\ell(\gamma)f(\sigma^\vee\times\sigma)} \cdot \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2 \cdot d\omega.$$

Proof. By (2.1), the Plancherel measure on $\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n, F)_\Omega$ is given by

$$d\nu(\omega) = c(G|M)^{-2}\,\gamma(G|M)^{-1}\,\mu(\omega)\,d(\omega)\,d\omega$$

Then, the result follows from Theorem 5.1 and Theorem 5.2. □

Let T be the diagonal subgroup of G and take for Ω the Bernstein component in $\Omega(G)$ which contains the cuspidal pair $(T, 1)$. Then Ω has the single exponent n and parametrizes those irreducible smooth representations of $\mathrm{GL}(n, F)$ which admit nonzero Iwahori fixed vectors.

Now let $l_1 + \cdots + l_k$ be a partition of n, and let

$$M = \mathrm{GL}(l_1, F) \times \cdots \times \mathrm{GL}(l_k, F) \;\subset\; \mathrm{GL}(n, F).$$

The formal degree of the Steinberg representation $\mathrm{St}(l_i)$ is given by

$$d(\mathrm{St}(l_i)) = \frac{q^{(l_i - l_i^2)/2}}{l_i} \cdot \frac{|\mathrm{GL}(l_i, q)|}{q^{l_i} - 1} = \frac{1}{l_i} \cdot \prod_{j=1}^{l_i - 1} (q^j - 1) \tag{17}$$

We also have the inner product identity in pre-Hilbert space:

$$\langle (\sigma_1 \otimes \cdots \otimes \sigma_k)(g) \xi_1 \otimes \cdots \xi_k, \xi_1 \otimes \cdots \otimes \xi_k \rangle \; = \; \prod \langle \sigma_j(g) \xi_j, \xi_j \rangle.$$

Let each $\xi_j \in V_j$ be a unit vector. With respect to the standard normalization of all Haar measures we then have (cf. [11, (7.7.9)])

$$1/d_{\sigma_1 \otimes \cdots \otimes \sigma_k} = \prod \int | < \sigma_j(g) \xi_j, \xi_j > |^2 d\dot{\mu}_j = \prod 1/d_{\sigma_j}$$

and so

$$d_{\sigma_1 \otimes \cdots \otimes \sigma_k} = \prod d_{\sigma_j}. \tag{18}$$

Using (18) and Theorem 3, we obtain the following result.

Corollary 5.5. *On the orbifold $X^\gamma / Z(\gamma)$ we have*

$$d\nu(\omega) = \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot d\omega$$

where

$$d(\omega) = \prod d(\mathrm{St}(l_i)).$$

So we have

$$\begin{aligned} d\nu(\omega) \quad &= \gamma(G|M) \cdot \prod_{i=1}^{k} \frac{1}{l_i} \prod_{j=1}^{l_i - 1} (q^j - 1) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot d\omega \\ &= \prod_{i=1}^{k} \frac{q^{\frac{l_i^2 - l_i}{2}} (q-1)^{l_i}}{l_i (q^{l_i} - 1)} \cdot P_{S_n}(q^{-1}) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot d\omega. \qquad (19) \end{aligned}$$

Remark 5.6. Using [10, Theorem 3.3], we obtain that the Plancherel measure of the (extended) affine Hecke algebra $\mathcal{H}(n, q)$ is given on $X^\gamma / Z(\gamma)$ by

$$\mu(I) \cdot \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot d\omega.$$

Concerning the volume $\mu(I)$: by [11, 5.4.3] we have

$$\mu(\mathrm{GL}(n, \mathfrak{o}_F)) = \sum_{w \in W_0} \mu(IwI) = \sum_{w \in W_0} \mu(I) \cdot q^{\ell(w)} = P_{S_n}(q) \cdot \mu(I).$$

The explicit formula is then (using (2)):

$$d\nu_{\mathcal{H}(n,q)}(\omega) = \prod_{i=1}^{k} \frac{q^{\frac{l_i^2 - l_i}{2}}(q-1)^{l_i}}{l_i(q^{l_i}-1)} \cdot q^{\frac{n-n^2}{2}} \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot d\omega,$$

where the second product is taken over those i, j, g for which the following inequalities hold: $1 \leq i < j \leq k$, $|g_i - g_j| \leq g \leq g_i + g_j$, $2g_i + 1 = l_i$. Note that Plancherel measure for Iwahori Hecke algebras has been already calculated by Opdam (see [22, 2.8.3]).

We will now consider a special case. The p-adic gamma function attached to the local field K (see [32, p. 51]) is the following meromorphic function of a single complex variable:

$$\Gamma_1(\zeta) = \frac{1 - q_K^{\zeta}/q_K}{1 - q_K^{-\zeta}}.$$

We will change the variable via $s = q_K^{\zeta}$ and write

$$\Gamma_K(s) = \frac{1 - s/q_K}{1 - s^{-1}},$$

a rational function of s. Let $s \in i\mathbb{R}$ so that s has modulus 1. Then we have

$$1/|\Gamma_K(s)|^2 = \left| \frac{1 - s}{1 - q_K^{-1} s} \right|^2.$$

Let T be the standard maximal torus in $\mathrm{GL}(n)$ and let $\widehat{T}$ denote the unitary dual of T. Then $\widehat{T}$ has the structure of a compact torus $\mathbb{T}^n$ (the space of Satake parameters) and the unramified unitary principal series of $\mathrm{GL}(n)$ is parametrized by the quotient $\mathbb{T}^n/S_n$. Let now $t = (z_1, \ldots, z_n) \in \mathbb{T}^n$. Applying the above formulas the Plancherel density $\mu_{G|T}$ is given by

$$\begin{aligned}
(20)\qquad \mu_{G|T} &= const \cdot \prod_{i<j} \left| \frac{1 - z_j z_i^{-1}}{1 - z_j z_i^{-1}/q} \right|^2 \\
(21)\qquad &= const \cdot \prod_{0<\alpha} \left| \frac{1 - \alpha(t)}{1 - \alpha(t)/q} \right|^2 \\
(22)\qquad &= const \cdot \prod_{\alpha} 1/\Gamma(\alpha(t))
\end{aligned}$$

where α is a root of the Langlands dual group $\mathrm{GL}(n, \mathbb{C})$ so that $\alpha_{ij}(t) = z_i/z_j$.

For GL(n), one connected component in the tempered dual is the compact orbifold $\mathbb{T}^n/S_n$, the symmetric product of n circles. On this component we have the Macdonald formula [19]:

$$d\mu(\omega_\lambda) = const. \cdot d\lambda / \prod_\alpha \Gamma(i\lambda(\alpha^\vee))$$

the product over all roots α where $\alpha^\vee$ is the coroot. This formula is a very special case of our formula for GL(n).

5.2. **General case.** We now pass to the general case of a component $\Omega \subset \Omega(\mathrm{GL}(n))$ with exponents $e_1, \ldots, e_t$. We first note that each component $\Omega \subset \Omega(\mathrm{GL}(n))$ yields up its fundamental invariants:

- the cardinality q of the residue field of F
- the sizes m_i of the small general linear groups
- the exponents e_i
- the torsion numbers r_i
- the formal degrees d_i
- the conductors $f_{ij} = f(\sigma_i^\vee \times \sigma_j)$

with $1 \le i \le t$.

We now construct the disjoint union

$$E = \Omega(\mathrm{GL}(\infty)) = \{\bigsqcup \Omega(\mathrm{GL}(n)) : n = 0, 1, 2, 3, \ldots\}$$

with the convention that $\Omega(\mathrm{GL}(0)) = \mathbb{C}$.

We will say that two components $\Omega_1, \Omega_2 \in E$ are *disjoint* if none of the irreducible supercuspidals which occur in Ω_1 is equivalent (after unramified twist) to any of the supercuspidals which occur in Ω_2. We now define a law of composition on *disjoint components* in E. With the cuspidal pair $(M_1, \sigma_1) \in \Omega_1$ and the cuspidal pair $(M_2, \sigma_2) \in \Omega_2$ we define $\Omega_1 \times \Omega_2$ as the unique component determined by

$$(M_1 \times M_2, \sigma_1 \otimes \sigma_2).$$

The set E admits a law of composition *not everywhere defined* such that E is unital, commutative and associative. Rather surprisingly, E admits prime elements: the prime elements are precisely the components with a single exponent. Each element in E admits a unique factorization into prime elements:

$$\Omega = \Omega_1 \times \cdots \times \Omega_t.$$

Plancherel measure respects the unique factorization into prime elements, modulo constants. Quite specifically, we have

Theorem 5.7. *Let Ω have the unique factorization*

$$\Omega = \Omega_1 \times \cdots \times \Omega_t$$

so that Ω has exponents $e_1, \ldots, e_t$ and $\Omega_1, \ldots, \Omega_t$ are pairwise disjoint prime elements with the individual exponents $e_1, \ldots, e_t$. Let

$$\nu = \bigsqcup \nu_\Omega$$

denote the Bernstein decomposition of Plancherel measure. Then we have

$$\nu_\Omega = const.\, \nu_{\Omega_1} \cdots \nu_{\Omega_t}$$

where $\nu_{\Omega_1}, \ldots, \nu_{\Omega_t}$ are given by Theorem 5.1 and the constant is given, in terms of the fundamental invariants, by Theorem 5.2.

Proof. In the Harish-Chandra product formula, all the cross-terms are constant, by Theorem 4.3. □

6. Transfer-of-measure, conductor, and the formal degree formulas

6.1. **Torsion number.** The theory of types of [11] produces a canonical extension K of F such that $q_K = q^r$. Indeed, let σ be an irreducible supercuspidal representation of $\mathrm{GL}(m)$, and let (J, λ) be a maximal simple type occurring in it. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order in $A = M(m, F)$ and let $E = F[\beta]$ be the field extension of F attached to the stratum (see [11, Definition 5.5.10 (iii)]). It is proved in [11, Lemma 6.2.5] that

$$r = \frac{m}{e(E|F)}, \tag{23}$$

where $e(E|F)$ denotes the ramification index of E with respect to F. Let B denote the centraliser of E in A. We set $\mathfrak{B} := \mathfrak{A} \cap B$. Then $\mathfrak{B}$ is a maximal hereditary order in B, see [11, Theorem 6.2.1]. Let K be an unramified extension of E which normalises $\mathfrak{B}$ and is maximal with respect to that property, as in [11, Proposition 5.5.14]. Then $[K : F] = m$, and (23) gives that r is equal to the residue index $f(K|F)$ of K with respect to F. Thus $Q = q^r$ is equal to the order q_K of the residue field of K.

Also the number Q is the one which occurs for the Hecke algebra $\mathcal{H}(\mathrm{GL}(m), \lambda)$ associated to (J, λ), see [11, Theorem 5.6.6]. Indeed, since the order of the residue field of E is equal to $q^{f(E|F)}$, that number is $(q^{f(E|F)})^f$, with

$$f = \frac{m}{[E : F]\, e(\mathfrak{B})},$$

where $e(\mathfrak{B})$ denotes the period of a lattice chain attached to $\mathfrak{B}$ as in [11, (1.1)]. Since σ is supercuspidal, $e(\mathfrak{B}) = 1$ (see [11, Corollary 6.2.3]). It follows that

$$f \cdot f(E|F) = \frac{m \cdot f(E|F)}{[E:F]} = \frac{m}{e(E|F)} = r. \tag{24}$$

6.2. **Normalization of measures.** We will relate our normalization of measures to the measures used in [11, (7.7)]. Bushnell and Kutzko work with a quotient measure $\dot{\mu}$, the quotient of μ_G by μ_Z.

Let Z denote the centre of GL(n). The second isomorphism theorem in group theory gives:

$$JZ/Z \cong J/J \cap Z.$$

We have

$$J \cap Z = \mathfrak{o}_F^\times.$$

One way to see this would be: J contains $\mathfrak{A}^\times \cap B$, where B is the centralizer in $M(n, F)$ of the extension E. Now certainly Z is contained in B. On the other hand, $\mathfrak{A}$ is an $\mathfrak{o}_F$-order so $\mathfrak{A}$ certainly contains $\mathfrak{o}_F$. Thanks to Shaun Stevens for this remark.

Then we have

$$JZ/Z \cong J/\mathfrak{o}_F^\times.$$

Now J is a principal $\mathfrak{o}_F^\times$-bundle over $J/\mathfrak{o}_F^\times$. Each fibre over the base $J/\mathfrak{o}_F^\times$ has volume 1. The quotient measure of the base space is then given by

$$\dot{\mu}(JZ/Z) = \mu(J). \tag{25}$$

Similar normalizations are done with $G_0 = \mathrm{GL}(e, K)$. We also need the corresponding quotient measure $\ddot{\mu}$ (see [11, (7.7.8)]). We have

$$\ddot{\mu}(IK^\times/K^\times) = \mu_{G_0}(I).$$

Let $M = \prod \mathrm{GL}(n_j)$. We have $Z_M = \prod Z_j$, $\mathcal{K} = \prod \mathcal{K}_j$, with $Z_j = Z_{\mathrm{GL}(n_j,F)}$ and $\mathcal{K}_j = \mathrm{GL}(n_j, \mathfrak{o}_F)$. With respect to the standard normalization of all Haar measures, we have $\mu_M = \prod \mu_j$ (where μ_j denotes $\mu_{\mathrm{GL}(n_i,F)}$) and $\mu_{Z_M} = \prod \mu_{Z_j}$. This then guarantees that

$$\dot{\mu}_M = \prod \dot{\mu}_j. \tag{26}$$

6.3. **Conductor formulas (the supercuspidal case).** We will first recall results from [9] in a suitable way for our purpose.

Let $(J^{\mathrm{s}}, \lambda^{\mathrm{s}})$ be a simple type in $\mathrm{GL}(2m)$ with associated maximal simple type (J, λ) (in the terminology of [11, (7.2.18) (iii)]). When (J, λ) is of positive level, we set $J_P = (J^{\mathrm{s}} \cap P) H^1(\beta, \mathfrak{A}) \subset J^{\mathrm{s}}$ (in notation [11, (3.1.4)]), where P denotes the upper-triangular parabolic subgroup of $\mathrm{GL}(2m)$ with Levi component $M = \mathrm{GL}(m) \times \mathrm{GL}(m)$, and unipotent radical denoted by N. Following [11, Theorem 7.2.17], we define λ_P as the natural representation of J_P on the space of $(J \cap N)$-fixed vectors in λ^{s}. The representation λ_P is irreducible and $\lambda_P \simeq \text{c–Ind}_{J_P}^{J^{\mathrm{s}}}(\lambda^{\mathrm{s}})$.

The pair $(J \times J, \lambda \otimes \lambda)$ is a type in M which occurs in $\sigma \otimes \sigma$, and, as shown in [13, prop. 1.4], (J_P, λ_P) is a $\mathrm{GL}(2m)$-cover of $(J \times J, \lambda \otimes \lambda)$.

Theorem 6.1. Conductor formulas, [9]. *Let $G_0 = \mathrm{GL}(2, K)$, let N_0 denote the unipotent radical of the standard Borel subgroup of G_0 and let I denote the standard Iwahori subgroup of G_0. We will denote by μ_0 the Haar measure on G_0 normalized as in subsection 6.2.*

Let $(J^{\mathrm{GL}(2m)}, \lambda^G)$ be any $\mathrm{GL}(2m)$-cover of $(J \times J, \lambda \otimes \lambda)$.

Then

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{-f(\sigma^\vee \times \sigma)} = \frac{j(\sigma \otimes \sigma)}{j_0(1)},$$

where j, j_0 denote the j-functions for the group G, G_0 respectively.

Proof. The first equality is [9, Theorem in §5.4], using the fact that $\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0) = q_K^{-1}$. The second equality is [9, Theorem in §5.4] (note that in *loc. cit.* the normalisations haven been taken so that $\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N}) = \mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)$). It also follows directly from our Corollary 5.3. $\square$

We will now extend the above Theorem to the case of $M = \mathrm{GL}(m)^{\times e}$, with e arbitrary.

Corollary 6.2. *Let $M = \mathrm{GL}(m)^{\times e}$ with $n = em$, et $G_0 = \mathrm{GL}(e, K)$, let N_0 denote the unipotent radical of the standard Borel subgroup of G_0 and let I denote the standard Iwahori subgroup of G_0.*

Let (J^G, λ^G) be a cover in $G = \mathrm{GL}(n)$ of $(J^{\times e}, \lambda^{\otimes e})$ (the existence of which is guaranteed by [13]).

Then

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{-\frac{e(e-1)}{2} f(\sigma^\vee \times \sigma)} = \frac{j(\sigma^{\otimes e})}{j_0(1)}.$$

Proof. Let M' be a Levi subgroup of a parabolic subgroup in G such that P is a maximal parabolic subgroup of M'. Then, $M'/M \simeq$

$\mathrm{GL}(2m)/\mathrm{GL}(m)\times\mathrm{GL}(m)$ and

$$\mu(J^G\cap M'\cap N)=\mu(J^{\mathrm{GL}(2m)}\cap\mathrm{GL}(2m)\cap N).$$

It follows from [12, Proposition 8.5 (ii)] that $(J^G\cap M',\lambda^G|J^G\cap M')$ is an M'-cover of $(J^{\times e},\lambda^{\otimes e})$.

Because of the unipotency of N, we have

$$\mu(J^G\cap N)=(\mu(J^{\mathrm{GL}(2m)}\cap\mathrm{GL}(2m)\cap N))^{\frac{e(e-1)}{2}}, \tag{27}$$

and similar equalities for the three others terms. Since $\mathrm{GL}(2m)\cap N$ is the unipotent radical of the parabolic subgroup of $\mathrm{GL}(2m)$ with Levi $\mathrm{GL}(m)\times\mathrm{GL}(m)$, the first equality in the Corollary follows from Theorem 6.1.

The second equality follows from our Corollary 5.3. It is also a direct consequence of Theorem 6.1, using the product formula for j and for j_0 from [34, IV.3. (5)]. □

6.4. **Formal degree formulas.** Using Corollary 6.2, we will deduce from [11, (7.7.11)] a formula relating the formal degree of any discrete series of $\mathrm{GL}(n)$ and the formal degree of a supercuspidal representation in its inertial support.

Given $G=\mathrm{GL}(n)=\mathrm{GL}(n,F)$ choose $e|n$ and let $m=n/e$. Let σ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$ and let (J,λ) be a maximal simple type occurring in it. Let $g=(e-1)/2$. We consider the standard Levi subgroup $M=\mathrm{GL}(m)^{\times e}$ of $\mathrm{GL}(n,F)$ and the supercuspidal representation

$$\sigma_M=|\det(\ )|^{-g}\sigma\otimes\cdots\otimes|\det(\ )|^{g}\sigma$$

of it. Then $(J_M,\lambda_M)=(J^{\times e},\lambda^{\otimes e})$ is a type in M occuring in σ_M.

Let $\pi=\mathrm{St}(\sigma,e)$ and let $(J^{\mathrm{s}},\lambda^{\mathrm{s}})$ be a simple type in $\mathrm{GL}(n)$ occuring in π (it has associated maximal simple type (J,λ)).

The following result is rather intricate, but note that only the *fundamental invariants* $m,e,r,d,f(\sigma^\vee\times\sigma)$ occur in it, in line with our general philosophy.

Theorem 6.3. *We have*

$$\frac{d(\pi)}{d(\sigma)^e}=\frac{m^{e-1}}{r^{e-1}e}\cdot q^{\frac{e^2-e}{2}(f(\sigma^\vee\times\sigma)+r-2m^2)}\cdot\frac{(q^r-1)^e}{q^{er}-1}\cdot\frac{|\mathrm{GL}(em,q)|}{|\mathrm{GL}(m,q)|^e}.$$

Remark 6.4. The right-hand side in the above equality can be rewritten, by using (17), as

$$r^{1-e}\cdot\frac{(q^{em}-1)(q^r-1)^e}{(q^m-1)^e(q^{er}-1)}\cdot q^{\frac{e^2-e}{2}(f(\sigma^\vee\times\sigma)+r-m^2)}\cdot\frac{\deg(\mathrm{St}(em))}{(\deg(\mathrm{St}(m)))^e}.$$

Proof. Let T denote the diagonal torus in $\mathrm{GL}(e,K)$ and let I denote the Iwahori subgroup of $G_0 = \mathrm{GL}(e,K)$ attached to the Bernstein component in $\Omega(\mathrm{GL}(e,K))$ which contains the cuspidal pair $(T,1)$. Note that $I \cap T = \mathrm{GL}(1,\mathfrak{o}_K)^{\times e}$. From [11, (7.7.11)], applied to the representations π and σ, we have

$$d(\pi) = \frac{\mu_0(I)}{\mu(J^{\mathrm{s}})} \cdot \frac{\dim(\lambda^{\mathrm{s}})}{e(E|F)} \cdot d(\pi)_0, \tag{28}$$

where $d(\pi)_0$ denotes the formal degree of $\pi \in \mathrm{E}_2(G_0)$, and

$$d(\sigma) = \frac{\mu(\mathrm{GL}(1,\mathfrak{o}_K))}{\mu(J)} \cdot \frac{\dim(\lambda)}{e(E|F)}. \tag{29}$$

Using (28), (29) and (24), we obtain

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}} \cdot \frac{\mu_0(I)}{\mu(J^{\mathrm{s}})} \cdot \frac{\mu(J^{\times e})}{\mu(\mathrm{GL}(1,\mathfrak{o}_K)^{\times e})} \cdot \frac{\dim(\lambda^{\mathrm{s}})}{(\dim(\lambda))^e} \cdot d(\pi)_0. \tag{30}$$

We set $J_P = (J^{\mathrm{s}} \cap P)H^1(\beta,\mathfrak{A}) \subset J^{\mathrm{s}}$, where P is the upper-triangular parabolic subgroup of G with Levi component M, and unipotent radical N. We define λ_P as the natural representation of J_P on the space of $(J \cap N)$-fixed vectors in λ^{s}. The representation λ_P is irreducible and $\lambda_P \simeq \mathrm{c\text{–}Ind}_{J_P}^{J^{\mathrm{s}}}(\lambda^{\mathrm{s}})$. Then (J_P,λ_P) is a G-cover of (J_M,λ_M). In the case where (J,λ) is of zero level, we denote by $(J^{\mathrm{s}},\lambda^{\mathrm{s}}) = (J_P,\lambda_P)$ an arbitrary G-cover of (J_M,λ_M).

Since $J^{\mathrm{s}} \cap M = J^{\times e} = J_M = J_P \cap M$, and

$$\dim(\lambda)^e = \dim(\lambda_M) = \dim(\lambda_P) = [J^{\mathrm{s}} : J_P]^{-1} \dim(\lambda^{\mathrm{s}}),$$

(30) gives

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}} \cdot \frac{\mu_0(I)}{\mu(J_P)} \cdot \frac{\mu(J_M)}{\mu_0(I \cap T)} \cdot d(\pi)_0.$$

On the other hand, by applying the formula [34, p.241, l.7] to the group J, we obtain

$$\gamma(G|M) = \frac{\mu(J_P \cap N) \cdot \mu(J_P \cap M) \cdot \mu(J_P \cap \overline{N})}{\mu(J_P)}. \tag{31}$$

Similarly we have

$$\gamma(G_0|T) = \frac{\mu_0(I \cap N_0) \cdot \mu_0(I \cap T) \cdot \mu_0(I \cap \overline{N}_0)}{\mu_0(I)}.$$

We then obtain

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{\gamma(G|M)}{\gamma(G_0|T)} \cdot \frac{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)}{\mu(J_P \cap N) \cdot \mu(J_P \cap \overline{N})} \cdot d(\pi)_0.$$

Applying Corollary 6.2, we get

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}} \cdot q^{\frac{e(e-1)}{2} f(\sigma^\vee \times \sigma)} \cdot \frac{\gamma(G|M)}{\gamma(G_0|T)} \cdot d(\pi)_0.$$

Since Haar measure on $\mathrm{GL}(e,K)$ has been normalised so that the volume of $\mathrm{GL}(e,\mathfrak{o}_K)$ is equal to one, the formal degree of the Steinberg representation of $\mathrm{GL}(e,K)$ is given as in (17) by

$$d(\pi)_0 = \frac{q_K^{(e-e^2)/2}}{e} \cdot \frac{|\mathrm{GL}(e,q_K)|}{q_K^e - 1}.$$

On the other hand, Theorem 3.1 gives

$$\gamma(G|M) = q^{mn-n^2} \cdot \frac{|\mathrm{GL}(n,q)|}{|\mathrm{GL}(m,q)|^e} \quad \text{and} \quad \gamma(G_0|T) = q^{e-e^2} \cdot \frac{|\mathrm{GL}(e,q_K)|}{(q_K-1)^e}.$$

The result follows. □

We will now recall the explicit formulas for $d(\pi)$ and $d(\sigma)$ from [31], using also [36]. We would like to thank Wilhelm Zink for explaining these works to us.

Let η be the Heisenberg representation of $J^1(\beta,\mathfrak{A})$ attached to a maximal simple type $(J(\beta,\mathfrak{A}),\lambda)$ occuring in the supercuspidal representation σ of $\mathrm{GL}(m)$ (see [11, (5.1.1), (5.5.10)]). Let $\mathfrak{P}$ denote the Jacobson radical of $\mathfrak{A}$ and let $U^i(\mathfrak{A}) = 1 + \mathfrak{P}^i$. Let π_β^1 be the compactly induced representation c–$\mathrm{Ind}_{J^1(\beta,\mathfrak{A})}^{U^1(\mathfrak{A})}(\eta)$. Then π_β^1 is irreducible, see [11, (5.2.3)]. More generally the restriction of η to $J^i(\beta,\mathfrak{A}) = J^1(\beta,\mathfrak{A}) \cap (1+\mathfrak{P}^i)$ is a multiple of an irreducible representation η^i which induces irreducibly to a representation π_β^i of $U^i(\mathfrak{A})$ (see [36, 2.2]). Let E_{-i} be any field such that

$$U^1(\mathfrak{A}) \cdot I_{\mathrm{GL}(m)}(\pi_\beta^{i+1}) \cdot U^1(\mathfrak{A}) \;=\; U^1(\mathfrak{A}) \cdot \mathrm{GL}(m/[E_{-i}:F], E_{-i}) \cdot U^1(\mathfrak{A}),$$

where $I_{\mathrm{GL}(m)}(\pi_\beta^{i+1})$ denotes the intertwining of π_β^{i+1} in $\mathrm{GL}(m,F)$. In particular, we have $E_0 = E$.

Theorem 6.5. Explicit formal degrees formulas, [31], [36]. *The formal degrees of σ and π are respectively given by*

$$d(\sigma) \;=\; r \cdot \frac{q^m - 1}{q^r - 1} \cdot q^{(r-m+\delta)/2} \cdot \deg(\mathrm{St}(m)),$$

$$d(\pi) \;=\; r \cdot \frac{q^{em} - 1}{q^{er} - 1} \cdot q^{(er-em+e^2\delta)/2} \cdot \deg(\mathrm{St}(em)),$$

where

$$\delta = rm \cdot \sum_{i\geq 0} (1 - [E_{-i}:F]^{-1}).$$

Proof. It follows directly from [31, Theorem 1.1] and [36, Corollary 6.7], using the fact that $r = f(K|F)$ and $m/e(E|F) = r$. $\square$

As immediate consequences, we obtain the following results.

Corollary 6.6.

$$\frac{d(\pi)}{d(\sigma)^{e^2}} = r^{1-e^2} \cdot \frac{(q^{em}-1)(q^r-1)^{e^2}}{(q^{er}-1)(q^m-1)^{e^2}} \cdot q^{(e^2-e)(m-r)/2} \cdot \frac{\deg(\mathrm{St}(em))}{(\deg(\mathrm{St}(m)))^{e^2}}.$$

Remark 6.7. We observe that the above formula extends to the general case the formula obtained in [14, Theorem 4.6] in the case where $(n,p)=1$ and F has characteristic zero. The existence of such a formula was expected in [14, Remark 4.7]. Our formula also extends [33, Theorem VII.3.2].

Corollary 6.8.

$$\frac{d(\pi)}{d(\sigma)^{e}} = r^{1-e} \cdot \frac{(q^{em}-1)(q^r-1)^{e}}{(q^{er}-1)(q^m-1)^{e}} \cdot q^{(e^2-e)\delta/2} \cdot \frac{\deg(\mathrm{St}(em))}{(\deg(\mathrm{St}(m)))^{e}}.$$

The comparison of Corollary 6.8 with Remark 6.4 gives the following expression for the conductor for pairs $f(\sigma^\vee \times \sigma)$.

Theorem 6.9. *We have*

$$f(\sigma^\vee \times \sigma) = \delta + m^2 - r.$$

Remark 6.10. In [10, §6.4] (see also [10, 6.13]) is introduced a certain discrimant function $C(\beta)$ and an integer $\mathfrak{c}(\beta)$ such that $C(\beta) = q^{\mathfrak{c}(\beta)}$. It follows from our Theorem 5.1 and [10, Theorem 6.5 (i)] that

$$\mathfrak{c}(\beta) = \frac{[E:F]^2}{m^2} \cdot \delta.$$

6.5. **Conductor formulas (the discrete series case).** Let σ be an irreducible supercuspidal representation of $\mathrm{GL}(m)$, and let (J,λ) be a maximal simple type occurring in it. Let $e|n$, and let $l_1 + \cdots + l_k = e$ be a partition of e. It determines the standard Levi subgroup

$$M = \mathrm{GL}(l_1 m) \times \cdots \times \mathrm{GL}(l_k m) \subset \mathrm{GL}(n,F). \tag{32}$$

Let $g_1 = (l_1-1)/2$, …, $g_k = (l_k-1)/2$, and let π_1, …, π_k be discrete series representations of $\mathrm{GL}(l_1 m)$, …, $\mathrm{GL}(l_k m)$ such that $\pi_i = \mathrm{St}(\sigma, l_i)$. Let $\pi = \pi_1 \otimes \cdots \otimes \pi_k$ be the corresponding discrete series representation of M. For each $i \in \{1,\ldots,k\}$, we fix a $\mathrm{GL}(l_i m)$-cover $(J^{\mathrm{GL}(l_i m)}, \lambda^{\mathrm{GL}(l_i m)})$ of $(J^{\times l_i}, \lambda^{\otimes l_i})$ (as in the proof of Theorem 6.3). Then

$$(J_M, \lambda_M) = (J^{\mathrm{GL}(l_1 m)} \times \cdots \times J^{\mathrm{GL}(l_k m)}, \lambda^{\mathrm{GL}(l_1 m)} \otimes \cdots \otimes \lambda^{\mathrm{GL}(l_k m)}) \tag{33}$$

is a M-cover of $(J^{\times e}, \lambda^{\otimes e})$. Then let (J^G, λ^G) denote a G-cover of (J_M, λ_M) (the existence of which is guaranteed by [13, Main Theorem (second version)]).

At the same time the partition $(l_1, \ldots, l_k)$ determines the standard Levi subgroup

$$M_0 = \mathrm{GL}(l_1) \times \cdots \times \mathrm{GL}(l_k) \subset \mathrm{GL}(e, K) = G_0. \tag{34}$$

Let P (resp. P_0) be the upper-triangular parabolic subgroup of G (resp. G_0) with Levi component M (resp. M_0), and unipotent radical denoted by N (resp. N_0). Let I denote the standard Iwahori subgroup of G_0.

Theorem 6.11. *We have*

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{-\ell(\gamma) f(\sigma^\vee \times \sigma)} = \frac{j(\sigma^{\otimes e})}{j_0(1)}.$$

Proof. The second equality follows from our Corollary 5.3.

We will prove the first equality. Let U denote the unipotent radical of the upper-triangular parabolic subgroup of G with Levi component $\mathrm{GL}(m)^{\times e}$, and, for $i = 1, \ldots, k$, let U_i denote the unipotent radical of the upper-triangular parabolic subgroup of $\mathrm{GL}(l_i m)$ with Levi component $\mathrm{GL}(m)^{\times l_i}$. We observe that

$$U = N \times (U \cap M) = N \times \prod_{i=1}^{k} U_i.$$

Similarly, let U_0 be the unipotent radical of the standard Borel subgroup of G_0, and, for $i = 1, \ldots, k$, let $U_{0,i}$ be the unipotent radical of the standard Borel subgroup of $\mathrm{GL}(l_i, K)$. We have

$$U_0 = N_0 \times (U_0 \cap M_0) = N_0 \times \prod_{i=1}^{k} U_{0,i}.$$

It follows from [12, Proposition 8.5 (i)] that (J^G, λ^G) is also a G-cover of $(J^{\times e}, \lambda^{\otimes e})$. Applying Theorem 6.2 to (J^G, U) and to $(J^{\mathrm{GL}(l_i m)}, U_i)$ for each $i \in \{1, \ldots, k\}$, we obtain

$$\frac{\mu(J^G \cap U) \cdot \mu(J^G \cap \overline{U})}{\mu_0(I \cap U_0) \cdot \mu_0(I \cap \overline{U}_0)} = q^{-\frac{e(e-1)}{2} f(\sigma^\vee \times \sigma)}$$

$$\frac{\mu(J^{\mathrm{GL}(l_i m)} \cap U_i) \cdot \mu(J^{\mathrm{GL}(l_i m)} \cap \overline{U}_i)}{\mu_0(I \cap U_{0,i}) \cdot \mu_0(I \cap \overline{U}_{0,i})} = q^{-\frac{l_i(l_i-1)}{2} f(\sigma^\vee \times \sigma)}.$$

Since $J^G \cap M = J_M$ (by definition of covers), it follows from (33) that $J^G \cap \mathrm{GL}(l_i m) = J^{\mathrm{GL}(l_i m)}$. Then using the fact that

$$\mu(J^G \cap N) = \mu(J^G \cap U) \times \prod_{i=1}^{k} \mu(J^{\mathrm{GL}(l_i m)} \cap U_i),$$

and the analogous equalities for the others terms, we obtain

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{(-\frac{e(e-1)}{2} + \sum_{i=1}^{k} \frac{l_i(l_i-1)}{2}) f(\sigma^\vee \times \sigma)} \tag{35}$$

$$= q^{-\ell(\gamma) f(\sigma^\vee \times \sigma)}. \tag{36}$$

□

6.6. **Transfer-of-measure.** The following result reduces the case of an arbitrary component Ω to the one (studied in Corollary 5.5) of a component (of a possibly different group G_0) which contains the cuspidal pair $(T, 1)$. We give a direct proof which is based on our previous calculations. It is worth noting that it is also a direct application of [10, Theorem 4.1].

Let $\Omega = \sigma^e$ be a Bernstein component in $\Omega(\mathrm{GL}(n))$ with single exponent e. Let T be the diagonal subgroup of $G_0 = \mathrm{GL}(e, K)$, and let Ω_0 be the Bernstein component in $\Omega(\mathrm{GL}(e, K))$ which contains the cuspidal pair $(T, 1)$. The components Ω, Ω_0 each have the single exponent e, and we have a homeomorphism of compact Hausdorff spaces

$$\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n, F)_\Omega \cong \mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(e, K)_{\Omega_0}. \tag{37}$$

This homeomorphism is determined by the map

$$\bigotimes_{i=1}^{k} \zeta_i^{\mathrm{val}_F \circ \det_F} \otimes \pi_i \mapsto \bigotimes_{i=1}^{k} (\zeta_i^r)^{\mathrm{val}_K \circ \det_K} \otimes \mathrm{St}(l_i).$$

This formula precisely allows for the fact that π_i has torsion number r and that $\mathrm{St}(l_i)$ has torsion number 1. Note that when ζ is replaced by $\omega\zeta$, where ω is an rth root of unity, each term remains unaltered.

The equation $r = f(K|F)$ and the standard formula

$$\mathrm{val}_K(y) = f(K|F)^{-1}\, \mathrm{val}_F(N_{K|F}(y))$$

lead to the more invariant formula:

$$\bigotimes_{i=1}^{k} (\chi_i \circ \det_F) \otimes \pi_i \mapsto \bigotimes_{i=1}^{k} (\chi_i \circ N_{K|F} \circ \det_K) \otimes \mathrm{St}(l_i)$$

where χ_i is an unramified character of $F^\times$.

Let (J^G, λ^G) be defined as in the previous subsection. It is a type in G attached to Ω. Recall that I denotes the standard Iwahori subgroup of G_0.

Theorem 6.12. *Let $d\nu$, $d\nu_0$ respectively denote Plancherel measure on* $\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n,F)_\Omega$, $\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(e,K)_{\Omega_0}$. *We have*

$$\frac{\mu(J^G)}{\dim(\lambda^G)} \cdot d\nu(\omega) = \mu_0(I) \cdot d\nu_0(\omega_0),$$

where

$$\omega = \chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k$$

and

$$\omega_0 = (\chi_1 \circ N_{K|F})\mathrm{St}(l_1) \otimes \cdots \otimes (\chi_k \circ N_{K|F})\mathrm{St}(l_k).$$

Proof. We first have to elucidate the canonical measures $d\omega, d\omega_0$. First, let $M = \mathrm{GL}(n)$, and let ω have torsion number r. Then the map $\mathrm{Im}\, X(M) \to \mathcal{O}$ is the r-fold covering map: $\mathbb{T} \to \mathbb{T}, z \mapsto z^r$. The map $\mathrm{Im}\, X(M) \to \mathrm{Im}\, X(A_M)$ sends the map $T \mapsto z^{\mathrm{val}(\det(T))}$ to the map $x \mapsto z^{val(\det(xI_n))} = (z^n)^{\mathrm{val}(\det(x))}$ and so induces the n-fold covering map $\mathbb{T} \to \mathbb{T}$. The canonical measure $\mathrm{d}\omega$ on the orbit $\mathcal{O}$ is the Haar measure of total mass n/r. If $M = \mathrm{GL}(l_1) \times \cdots \times \mathrm{GL}(l_k)$ and ω_j has torsion number r_j then the canonical measure $\mathrm{d}\omega$ on the orbit $\mathcal{O}$ of $\omega_1 \otimes \cdots \otimes \omega_k$ is the Haar measure of total mass $l_1 \cdots l_k/r_1 \cdots r_k$. For the canonical measures $d\omega, d\omega_0$ we therefore have

$$\mathrm{d}\omega = (ml_1 \cdots ml_k/r^k) \cdot \mathrm{d}\tau = l_1 \cdots l_k \cdot (m^k/r^k) \cdot \mathrm{d}\tau$$

$$\mathrm{d}\omega_0 = l_1 \cdots l_k \cdot \mathrm{d}\tau$$

where $\mathrm{d}\tau$ is the Haar measure on $\mathbb{T}^k$ of total mass 1. So, we have

$$\mathrm{d}\omega = (m^k/r^k) \cdot \mathrm{d}\omega_0. \tag{38}$$

By Theorem 5.4,

$$d\nu(\omega) = q^{\ell(\gamma)f(\sigma^\vee \times \sigma)} \cdot \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2 \cdot d\omega$$

and

$$d\nu_0(\omega_0) = \gamma(G_0|M_0) \cdot d(\omega_0) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2 \cdot d\omega_0.$$

Hence

$$\frac{d\nu(\omega)}{d\nu_0(\omega_0)} = q^{\ell(\gamma)f(\sigma^\vee \times \sigma)} \cdot \frac{\gamma(G|M)}{\gamma(G_0|M_0)} \cdot \frac{d(\omega)}{d(\omega_0)} \cdot \frac{d\omega}{d\omega_0}. \tag{39}$$

We keep the notation of section 6.5. It follows from (26), (25) that

$$\mu(J_M) = \mu(J^{\mathrm{GL}(l_1 m)}) \times \cdots \times \mu(J^{\mathrm{GL}(l_k m)}), \tag{40}$$

since $J_M = J^{\mathrm{GL}(l_1 m)} \times \cdots \times J^{\mathrm{GL}(l_k m)}$. In the same way, we have

$$\mu_0(I \cap M_0) = \mu_0(I \cap \mathrm{GL}(l_1 m)) \times \cdots \times \mu_0(I \cap \mathrm{GL}(l_k m)), \tag{41}$$

On the other hand, the formula [11, (7.7.11)] gives

$$\mu(J^{\mathrm{GL}(l_i m)}) \cdot d(\pi_i) = \mu_0(I \cap \mathrm{GL}(l_i, K)) \cdot \frac{\dim(\lambda^{\mathrm{GL}(l_i m)})}{e(E|F)} \cdot d(\mathrm{St}(l_i))$$

Then (40), (41), (33), and (18) imply

$$\mu(J_M) \cdot d(\omega) = \mu_0(I \cap M_0) \cdot \frac{\dim(\lambda_M)}{e(E|F)^k} \cdot d(\omega_0). \tag{42}$$

Applying (31) to both $\gamma(G|M)$ and $\gamma(G_0|M_0)$, we obtain

$$\frac{\gamma(G|M)}{\gamma(G_0|M_0)} = \frac{\mu(J^G \cap N)\mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0)\mu_0(I \cap \overline{N}_0)} \cdot \frac{\mu_0(I)}{\mu(J^G)} \cdot \frac{\mu(J_M)}{\mu_0(I \cap M_0)}. \tag{43}$$

It then follows from (39), (42) and (43) that

$$\frac{d\nu(\omega)}{d\nu_0(\omega_0)} = q^{\ell(\gamma)f(\sigma^\vee \times \sigma)} \cdot \frac{\mu(J^G \cap N)\mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0)\mu_0(I \cap \overline{N}_0)} \cdot \frac{\mu_0(I)}{\mu(J^G)} \cdot \frac{\dim(\lambda_M)}{e(E|F)^k} \frac{\mathrm{d}\omega}{\mathrm{d}\omega_0}.$$

Noting that $\dim(\lambda^G) = \dim(\lambda_M)$, and using equation (24) and Theorem 6.11, we have

$$\frac{\mathrm{d}\nu(\omega)}{\mathrm{d}\nu_0(\omega_0)} = \frac{\mu_0(I)}{\mu(J^G)} \cdot \dim(\lambda^G) \cdot \frac{r^k}{m^k} \cdot \frac{\mathrm{d}\omega}{\mathrm{d}\omega_0} = \frac{\mu_0(I)}{\mu(J^G)} \cdot \dim(\lambda^G),$$

using (38).

□

7. The central simple algebras case

Let D be a central division algebra of index d over F and ring of integers $\mathfrak{o}_D$, and let $A = A(n')$ denote the algebra of $n' \times n'$ matrices with coefficients in D. Then A is a central simple algebra with centre F of reduced degree $n = dn'$ and the group of units of A is the group $G' = \mathrm{GL}(n', D)$. In Theorem 7.2 we will prove a transfer of Plancherel measure formula for G': this will be deduced from properties of the Jacquet-Langlands correspondence. In order to do this, we will adapt the proof of [1, (2.5) p. 88] to the case when F is of positive characteristic by using results of A. Badulescu.

We use the *standard* normalization of Haar measures, in particular $\mu_{G'}$ is normalized so that the volume of $\mathcal{K}' = \mathrm{GL}(n', \mathfrak{o}_D)$ is 1.

7.1. **A transfer-of-measure formula.** The aim of this subsection is to prove the transfer-of-measure formula stated in Theorem 7.2.

An element x' in G' will be called *semisimple* (resp. *regular semisimple*) if its orbit $O_{G'}(x') = \{yx'y^{-1} \ : \ y \in G'\}$ is a closed subset of G' (resp. if its characteristic polynomial admits only simple roots in an algebraic closure of F). Let G'_{rs} denote the set of regular semisimple elements in G'.

Let $G'_{x'}$ denote the centralizer in G' of x'. Then the group $G'_{x'}$ is unimodular, and the choice of Haar measures on G' and $G'_{x'}$ induces an invariant measure dx on $G'/G'_{x'}$. The orbital integral of $f' \in C_c(G')$ at x' is defined as

$$\Phi(f', x') = \int_{G'/G'_{x'}} f'(y^{-1}x'y)dy. \tag{44}$$

Since the orbit $O_{G'}(x')$ is closed in G', the integral is absolutely convergent. Indeed, it is a finite sum, since the restriction of f' to $O_{G'}(x')$ is locally constant with compact support. Note that, if $x' \in G'_{\mathrm{rs}}$, then $G'_{x'}$ is a maximal torus in G'.

Orbital integrals have a local expansion, due to Shalika [28], which we will now recall. If O' is a unipotent orbit in G', let $\Lambda_{O'}$ denote the distribution given by integration over the orbit O'. There exist functions $\Gamma^{G'}_{O'} \colon G'_{\mathrm{rs}} \to \mathbb{R}$ (the *Shalika germs*) indexed by unipotent orbits of G' with the following property:

$$\Phi(f', x') = \sum_{O'} \Gamma^{G'}_{O'}(x') \ \cdot \ \Lambda_{O'}(f'), \tag{45}$$

for $x' \in G'_{\mathrm{rs}}$ sufficiently close to the identity. Observe that $\Lambda_1 = f'(1)$.

Harish-Chandra proved that the germ $\Gamma^{G'}_1$ associated to the trivial unipotent orbit is constant, and Rogawski [24] has determined its value assuming the characteristic of F to be zero:

$$\Gamma^{G'}_1 = \frac{(-1)^{n-n'}}{d(\mathrm{St}_{G'})}. \tag{46}$$

The equality (46) is still valid in the case when F is of positive characteristic. Indeed, let F be of positive characteristic and let E be a field of zero characteristic sufficiently close to F, that is, such that there exists a ring isomorphism from $\mathfrak{o}_F/\varpi^l\mathfrak{o}_F$ to $\mathfrak{o}_E/\varpi^l\mathfrak{o}_E$, for some sufficiently big integer $l \geq 1$. Let D_E be a central division algebra over E with the same index d. Then by [4, Lemma 3.8] the lifts f'_E of f' to $G'_E = \mathrm{GL}(m, D_E)$ (resp. f_E of f to $G_E = \mathrm{GL}(n, E)$) also satisfy $f_E \leftrightarrow (-1)^{n-n'}f'_E$. On the other hand, $f'_E(1) = f'(1)$, independently of m: since the way to lift f' to f'_E consists in cutting the group G' into compact open subsets on which f' is constant, in associating to these subsets compact open subsets in G'_E, and assigning to *these* subsets the same constants in order to define f'_E; but the compact open subset of G' containing 1 corresponds to the compact open subset in G'_E containing 1.

If π is a smooth representation of G or G' with finite length, we will denote by θ_π its character.

Theorem 7.1. The Jacquet-Langlands correspondence [15], [3]. *There exists a bijection*

$$\mathrm{JL}\colon \mathrm{E}_2(G') \to \mathrm{E}_2(G)$$

such that for each $\pi' \in \mathrm{E}_2(G')$*:*

$$\theta_{\pi'}(x') = (-1)^{n-n'}\,\theta_{\mathrm{JL}(\pi')}(x), \tag{47}$$

for any $(x, x') \in G \times G'$ *such that* $x \leftrightarrow x'$*.*

Recall that $A = A(n')$ denotes the algebra of $n' \times n'$ matrices with coefficients in D. Let $\mathrm{Nrd}_{A|F}\colon A \to F$ denote the reduced norm of A over F as defined in [8, § 12.3, p. 142]. We shall view the reduced norm $\mathrm{Nrd}_{A|F}$ as a homomorphism from G' to $F^\times$.

If η is a quasicharacter of $F^\times$ then we will write

$$\eta\pi' = (\eta \circ \mathrm{Nrd}_{A|F}) \otimes \pi'.$$

If η is an unramified quasicharacter then we will refer to $\eta\pi'$ as an *unramified twist* of π'.

Each representation π' of G' has a *torsion number*: the order of the cyclic group of all those unramified characters η of $F^\times$ for which

$$\eta\pi' \cong \pi'.$$

The Jacquet-Langlands correspondence has the property that

$$\eta(\mathrm{JL}(\pi')) = \mathrm{JL}(\eta\pi'), \tag{48}$$

for any square integrable representation π' of G' and any (unitary) character η of $F^\times$ (see [15, (4) p. 35]). It follows that the torsion number of π' is equal to that of $\mathrm{JL}(\pi')$.

For each Levi subgroup $M = \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F)$ of G such that d does not divide n_i for some $i \in \{1, \ldots, k\}$, we have

$$\theta^G_\omega(f) = 0, \quad \text{for any } \omega \in \mathrm{E}_2(M)$$

(see the beginning of [4, §3] and the proof of [4, Lem. 3.3]).

We consider now a Levi subgroup M of the form $M = \mathrm{GL}(dn'_1, F) \times \cdots \times \mathrm{GL}(dn'_k, F)$, and define $M' = \mathrm{GL}(n'_1, D) \times \cdots \times \mathrm{GL}(n'_k, D)$ (a

Levi subgroup of G'): M is the *transfer* of M'. The Jacquet-Langlands correspondence induces a bijection JL: $\mathrm{E}_2(M') \to \mathrm{E}_2(M)$, by setting

$$\mathrm{JL}(\omega_1' \otimes \cdots \otimes \omega_k') = \mathrm{JL}(\omega_1') \otimes \cdots \mathrm{JL}(\omega_k').$$

For any $\omega \in \mathrm{E}_2(M)$, there exists $\omega' \in \mathrm{E}_2(M')$ such that $\omega = \mathrm{JL}(\omega')$.

Let $\Omega^{\mathrm{t}}(G')$, $\Omega^{\mathrm{t}}(G)$ denote the Harish-Chandra parameter space of G', G. Each point in $\Omega^{\mathrm{t}}(G')$ is a G'-conjugacy class of discrete-series pairs (M', ω') with $\omega' \in E_2(M')$. The topology on $\Omega^{\mathrm{t}}(G')$ is determined by the unramified unitary twists: then $\Omega^{\mathrm{t}}(G')$ is a locally compact Hausdorff space. The map

$$(M', \omega') \mapsto (M, \mathrm{JL}(\omega')),$$

where M is the transfer of M', secures an *injective* map

$$\mathrm{JL}\colon \Omega^{\mathrm{t}}(G') \to \Omega^{\mathrm{t}}(G).$$

We will write $Y = \mathrm{JL}(\Omega^t(G'))$. Since the JL-map respects unramified unitary twists, we obtain a homeomorphism of $\Omega^{\mathrm{t}}(G')$ onto its image:

$$\mathrm{JL}\colon \Omega^{\mathrm{t}}(G') \cong Y \subset \Omega^{\mathrm{t}}(G).$$

Theorem 7.2. Transfer of Plancherel measure. *Let $G' = \mathrm{GL}(n', D), G = \mathrm{GL}(n, F)$ with $n = dn'$. Let ν', ν denote the Plancherel measure for G', G, each with the standard normalization of Haar measure on G', G. Then we have*

$$d\nu'(\omega') = \lambda(D/F) \cdot d\nu(\mathrm{JL}(\omega'))$$

where

$$\lambda(D/F) = \prod (q^m - 1)^{-1}$$

the product taken over all m such that $1 \le m \le n-1, m \ne 0 \mod d$.

Proof. If $x \in G$ and $x' \in G'$, we will write $x \leftrightarrow x'$ if x, x' are regular semisimple and have the same characteristic polynomial. If $x \in G$, we will say that x *can be transferred* if there exists $x' \in G'$ such that $x \leftrightarrow x'$.

Let $f' \in C_c(G')$. Then, by [4, Th. 3.2.], there exists $f \in C_c(G)$ such that

$$\Phi(f, x) = \begin{cases} (-1)^{n-n'} \cdot \Phi(f', x') & \text{for each } x' \in G' \text{ such that } x \leftrightarrow x', \\ 0 & \text{if } x \text{ cannot be transferred,} \end{cases}$$

for any $x \in G_{\mathrm{rs}}$.

It then follows from the germ expansion (45) that

$$f'(1) \cdot \Gamma_1^{G'} = (-1)^{n-n'} \cdot f(1) \cdot \Gamma_1^G,$$

that is, using (46),

$$\frac{f'(1)}{d(\mathrm{St}_{G'})} = \frac{f(1)}{d(\mathrm{St}_G)}. \tag{49}$$

We recall that $\theta^G_\omega(f) = 0$ on the complement of Y in $\Omega^t(G)$. Next, we use equation (49), and apply twice the Harish-Chandra Plancherel theorem, first for G', then for G. We obtain

$$\begin{aligned}
\int \theta^{G'}_{\omega'}(f')\, d\nu'(\omega') &= f'(1)\\
&= d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot f(1)\\
&= d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot \int \theta^G_\omega(f)\, d\nu(\omega)\\
&= d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot \int \theta^G_\omega(f)\, d\nu|_Y(\omega), \qquad (50)
\end{aligned}$$

for all $f' \in C_c(G')$.

We recall that the parameter space $\Omega^{\mathrm{t}}(G')$ is the *domain* of the Plancherel measure ν'.

By the refinement of the trace Paley-Wiener theorem due to Badulescu [4, Lemma 3.4] we have

$$\{\omega' \mapsto \theta^{G'}_{\omega'}(f'^{\vee}) : f' \in C_c(G'), \omega' \in \Omega^{\mathrm{t}}(G')\} = L(\Omega^{\mathrm{t}}(G')),$$

where $L(\Omega^{\mathrm{t}}(G'))$ is the space of compactly supported functions on $\Omega^{\mathrm{t}}(G')$ which, upon restriction to each connected component (a quotient of a compact torus $\mathbb{T}'^k$ by a product of symmetric groups), are Laurent polynomials in the co-ordinates $(z_1, z_2, \ldots, z_k)$.

Now $L(\Omega^{\mathrm{t}}(G'))$ is a dense subspace of $C_0(\Omega^{\mathrm{t}}(G'))$, the continuous complex-valued functions on $\Omega^{\mathrm{t}}(G')$ which vanish at infinity. On the other hand, it follows from [4, Prop. 3.6] that

$$\theta^{G'}_{\omega'}(f') = \theta^G_{\mathrm{JL}(\omega')}(f), \quad \text{for any } \omega' \in \mathrm{E}_2(M'). \tag{51}$$

Equation (50) therefore provides us with two Radon measures (continuous linear functionals) which agree on a *dense subspace* of $C_0(\Omega^{\mathrm{t}}(G'))$. Therefore the measures are equal:

$$d\nu'(\omega') = d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot d\nu|_Y(\omega) \tag{52}$$

At this point, we have to elucidate a normalization issue. Let $K' = \mathrm{GL}(n', \mathfrak{o}_D)$. The group $A_{G'}$ by definition is the F-split component of the centre of G' and can be identified with $F^\times$. As in section (6.2), we have $F^\times K'/F^\times = K'/K' \cap F^\times = K'/\mathfrak{o}_F^\times$. But the Haar measure on $A_{G'}$ has, as in [34, p.240], the standard normalization $\mathrm{mes}(K' \cap A_{G'}) = 1$, *i.e.*, $\mathrm{mes}(\mathfrak{o}_F^\times) = 1$. Since $\mathrm{mes}(K') = 1$, we have $\mathrm{mes}(F^\times K'/F^\times) = 1$. It follows (see for instance [31, 3.7]) that the formal degree of the Steinberg representation $\mathrm{St}_{G'}$ is given by

$$d(\mathrm{St}_{G'}) = \frac{1}{n}\prod_{j=1}^{n'-1}(q^{dj}-1)$$

We then have

$$d\nu'(\omega') = \lambda(D/F)\cdot d\nu(\omega) \tag{53}$$

where

$$\lambda(D/F) = (q^d-1)(q^{2d}-1)\cdots(q^{(n'-1)d}-1)(q-1)^{-1}(q^2-1)^{-1}\cdots(q^{n-1}-1)^{-1},$$

so that

$$\lambda(D/F) = \prod(q^m-1)^{-1} \tag{54}$$

the product taken over all m such that $1 \le m \le n-1$, $m \neq 0 \mod d$. □

This result may be expressed as follows

Theorem 7.3. *Let $(\Omega^{\mathrm{t}}G', \mathcal{B}', \nu')$ be the measure space determined by the Plancherel measure ν', let $(Y, \mathcal{B}, \lambda(D/F)\cdot\nu|_Y)$ be the measure space determined by the restriction of $\lambda(D/F)\cdot\nu$ to $Y = \mathrm{JL}(\Omega^{\mathrm{t}}(G') \subset \Omega^{\mathrm{t}}(G)$. Then these two measure spaces are isomorphic:*

$$(\Omega^{\mathrm{t}}G', \mathcal{B}', \nu') \cong (Y, \mathcal{B}, \lambda(D/F)\cdot\nu|_Y)$$

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