Integrability and dynamics of the n-dimensional symmetric Veselova top

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Integrability and dynamics of the n-dimensional symmetric Veselova top[*]

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Abstract

We consider the the n-dimensional generalisation of the nonholonomic Veselova problem. We derive the reduced equations of motion in terms of the mass tensor of the body and determine some general properties of the dynamics. In particular we give a closed formula for the invariant measure, we indicate the existence of steady rotation solutions, and obtain some results on their stability.

We then focus our attention on bodies whose mass tensor has a specific type of symmetry. We show that the phase space is foliated by invariant tori that carry quasi-periodic dynamics in the natural time variable. Our results enlarge the known cases of integrability of the multi-dimensional Veselova top. Moreover, they show that in some previously known instances of integrability, the flow is quasi-periodic without the need of a time reparametrisation.

Contents

1   Introduction

The *n*-dimensional generalisation of some classical nonholonomic systems considered recently by Fedorov and Jovanović [12, 13, 21, 20] provides a remarkable family of examples of nonholonomic systems whose dynamics, after a time reparametrisation, is quasi-periodic in large dimensional invariant tori. A crucial feature of these examples is that they possess an invariant measure and admit a Hamiltonization by Chaplygin's multiplier method [4]. After a time reparametrisation, the reduced equations of motion of these examples become Hamiltonian and, moreover, turn out to be related to classical integrable Hamiltonian systems. In this manner the authors establish the integrability of the systems and prove that the flow on the invariant, large dimensional tori is quasiperiodic in the new time. The Hamiltonization of these systems is hence central to their approach and is truly remarkable considering that Chaplygin's method is only guaranteed to work for systems with 2 degrees of freedom. This Hamiltonization relies on the very particular type of inertia operators considered by the authors that, as we explain below, are generally unphysical.

In this paper we analyse the dynamics of the *n*-dimensional Veselova top (described below) and treated before by Fedorov and Jovanović [12, 13]. A fundamental difference in our study of the problem with respect to these references is that we assume that the inertia tensor is *physical* (see below).

Our main contribution is to show that under certain symmetry assumptions on the mass distribution of the body, the dynamics takes place on invariant tori that carry quasi-periodic flow *without the need of a time reparametrisation*. We prove this by performing a detailed symmetry analysis of the problem, and in particular by performing a reduction to a system that is manifestly integrable and in fact periodic. The quasi-periodicity of the flow of the unreduced system (in the natural time variable) then follows from a theorem of Field [14]. This type of analysis to establish integrability of nonholonomic systems had been previously followed in e.g. [18, 8, 10].

Our approach allows us to recover all cases of integrability of the multi-dimensional Veselova top determined in [12, 13] that correspond to physical inertia tensors and we show that these correspond to *axisymmetric* bodies. We also determine new cases of integrability that we term *cylindrical* bodies. For this type of body, the system does not seem to allow a Chaplygin Hamiltonization (Remark 3.8), and the generic motion takes place in 4-dimensional invariant tori.

This paper also considers some properties of the motion of the general *n*-dimensional Veselova top (without any symmetry assumptions on the body). We prove the existence of steady rotation solutions that are periodic solutions with constant angular velocity, and determine some of their properties. In particular, we prove that some of these solutions are stable. The existence of these solutions appears not to have been observed before, not even in the 3D case. We also give the explicit form of the reduced equations of motion in terms of the mass tensor of the body, and a closed formula for the known invariant measure.

The *n*-dimensional Veselova top.   The 3D Veselova top, introduced by Veselova in her thesis [26], is a rigid body rotating under its own inertia (like Euler's rigid body) and subject to a nonholonomic constraint which forces the projection of the angular velocity to a distinguished axis fixed in space, to vanish at all time (see also [25]).

This paper deals with a multi-dimensional generalization of this model that was introduced by Fedorov and Kozlov [11]. They consider the motion of an *n*-dimensional rigid body subject to the nonholonomic constraint that requires the angular velocity matrix $\Omega \in \mathfrak{so}(n)$ to have rank two and to define a rotation in a plane containing a fixed axis in space.

The Veselova top and its *n*-dimensional generalisation are examples of *LR systems* which form a remarkable class of nonholonomic systems possessing an invariant measure [25].

The inertia tensor.   Throughout this paper, we work with the commonly accepted *n*-dimensional generalisation of rigid body dynamics (see e.g. [24]), where the inertia tensor is a linear operator $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ of the form

$$\mathbb{I}(\Omega) = \mathbb{J}\Omega + \Omega\mathbb{J}, \tag{1.1}$$

where $\mathbb{J}$ is the **mass tensor**. $\mathbb{J}$ is a constant $n \times n$ matrix that depends on the mass distribution of the body which, by an appropriate choice of a body frame, may be assumed to be diagonal with positive entries (see Section 2.1 for details). We say that a linear operator $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is a **physical inertia tensor** if and only if it may be written as above for a certain mass tensor $\mathbb{J}$.

Summary of results

The configuration space of the *n*-dimensional Veselova top problem is the orthogonal group[1] $\mathrm{O}(n)$. The phase space for the *n*-dimensional Veselova top is a subbundle $D \subset T\mathrm{O}(n)$ of rank $n-1$ determined by the nonholonomic constraints.

Our first contribution is to show that the system admits steady rotation solutions on D. These are periodic solutions where the body steadily rotates in a principal plane of the body. Because of the constraint, the orientation of the body along these motions is such that the distinguished axis is contained in the plane of rotation. Along these solutions, the nonholonomic constraint forces vanish.

Next we perform the reduction of the system by the symmetry group $G_L = \mathrm{O}(n-1)$ that corresponds to invariance of the system under rotations and reflections of the space frame that fix the distinguished axis (the subscript L indicates that this action is by left multiplication in the configuration space $\mathrm{O}(n)$). As shown in [12, 13], this is a generalised Chaplygin reduction and the **first reduced space** D/G_L is isomorphic to the cotangent bundle $T^*\mathrm{S}^{n-1}$. We give the reduced equations of motion and the explicit expression for the invariant measure in terms of the mass tensor $\mathbb{J}$. Moreover, with the reduced system at hand, we are able to show that the steady rotation solutions correspond to the singularities of an energy-momentum map and we show stability of those occurring on *extremal* planes. In particular this proves that in 3D, the steady rotations about the smallest and largest axes of inertia of the body are stable.

We then focus on the study of symmetric *n*-dimensional rigid bodies whose mass tensor has just 2 distinct eigenvalues:

$$\mathbb{J} = \mathrm{diag}[J_1, \ldots, J_1, J_2, \ldots, J_2],$$

with $J_1 \neq J_2$. We assume that the multiplicities of J_1 and J_2 are r and $r' = n - r$, with $1 \leq r \leq r'$. Under this hypothesis, the mass tensor $\mathbb{J}$ has symmetry $G_R := \mathrm{O}(r) \times \mathrm{O}(r') \subset \mathrm{O}(n)$: if

$$h = \begin{bmatrix} h_{11} & 0 \\ 0 & h_{22} \end{bmatrix},$$

with $h_{11} \in \mathrm{O}(r)$ and $h_{22} \in \mathrm{O}(r')$, then $h\mathbb{J}h^T = \mathbb{J}$. The subindex R indicates that G_R acts by right multiplication on the configuration space $\mathrm{O}(n)$. Since this action commutes with the left multiplication action of $G_L = \mathrm{O}(n-1)$, it passes down to a proper action on $T^*\mathrm{S}^{n-1}$ that turns out not to be free.

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[1]Readers may be more familiar with using the connected component $\mathrm{SO}(n)$ as the configuration space, however allowing both components simplifies our exposition in Section 5; moreover Arnold [1, p.133] suggests that $\mathrm{O}(n)$ is the 'correct' configuration space.

The **second reduced space** $\mathcal{R} = T^*\mathsf{S}^{n-1}/G_R$ is not a smooth manifold, but a stratified space where, as we shall show, the dynamics is periodic. We apply a theorem of Field [14] to conclude that the reconstructed motion, both on D and on $T^*\mathsf{S}^{n-1}$, is quasi-periodic on invariant tori. We emphasise that this proves quasi-periodicity of the flow without a time reparametrisation.

In order to estimate the dimension of the invariant tori on D and on $T^*\mathsf{S}^{n-1}$ it is necessary to determine the details of the stratification. For this, we distinguish two cases depending on the multiplicity r of the eingenvalue J_1 as follows.

$r = 1$. The mass tensor is

$$\mathbb{J} = \mathrm{diag}[J_1, J_2, \ldots, J_2],$$

that corresponds to an **axisymmetric body**. In this case the orbit space $\mathcal{R}$ is isomorphic to a singular semi-algebraic subspace in $\mathbb{R}^3$. We show that the dynamics on D is essentially that of the 3D axisymmetric Veselova top (see Theorem 4.3 for a precise statement), and is generically quasi-periodic on tori of dimension 2 both in D and on $T^*\mathsf{S}^{n-1}$. We also give a physical description of the dynamics.

$r > 1$. A body with this type of symmetry is called a **cylindrical body** and can only exist in dimension 4 or higher. For these bodies the orbit space $\mathcal{R}$ is isomorphic to a singular semi-algebraic subspace in $\mathbb{R}^4$. For $n > 4$ we prove that the system evolves as a 4D cylindrical Veselova top (a precise statement is given in Theorem 5.7) and we prove that the generic motion is quasi-periodic on tori of dimension 4 on D and of dimension 3 on $T^*\mathsf{S}^{n-1}$. We also give an argument to prove that the generic motions do not take place in lower dimensional invariant tori.

Previous work. Fedorov and Jovanović [12, 13] show that the G_L-reduction of the n-dimensional Veselova top to $T^*\mathsf{S}^{n-1}$ is Hamiltonizable and integrable under the assumption that the action of the inertia operator on rank 2 matrices[2] $a \wedge b \in \mathfrak{so}(n)$ satisfies

$$\mathbb{I}(a \wedge b) = (Aa) \wedge (Ab), \tag{1.2}$$

for a diagonal matrix $A = \mathrm{diag}[A_1, \ldots, A_n]$. In particular, this implies that the space of rank 2 matrices in $\mathfrak{so}(n)$ is invariant under $\mathbb{I}$. This condition is vacuous in dimension 3 since every non-zero matrix in $\mathfrak{so}(3)$ has rank 2. However, for $n \geq 4$ it is a very restrictive assumption, as we discuss in Appendix B. It turns out that an inertia tensor satisfying (1.2) is physical as defined by Equation (1.1) if and only if the body is axisymmetric. In this case $A = \mathrm{diag}[A_1, A_2, \ldots, A_2]$, and the mass tensor $\mathbb{J}$ is determined by the relations $A_2^2 = 2J_2$ and $A_1 A_2 = J_1 + J_2$.

As mentioned above, their approach only allows them to conclude quasi-periodicity of the flow after a time reparametrisation.

Structure of the paper

In Section 2 we begin by recalling the kinematics of the rigid body in arbitrary dimensions, governed by a mass tensor $\mathbb{J}$. We then recall from [11] the n-dimensional generalisation of the Veselova top and we discuss its symmetries, which depend on $\mathbb{J}$. We also describe the steady rotations which play an important role later. In Section 3 we perform the reduction of the system by $G_L = \mathrm{O}(n-1)$. We obtain the reduced equations of motion in terms of $\mathbb{J}$ and determine general properties of the motion.

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[2]For $a, b \in \mathbb{R}^n$ we denote $a \wedge b = ab^T - ba^T \in \mathfrak{so}(n)$.

Our analysis of axisymmetric and cylindrical bodies is respectively contained in Sections 4 and 5. In Section 6 we present some conclusions and open questions.

The paper ends with two appendices. The first recalls a well-known result about group actions, the reconstruction theorem of Field which we use in the proofs of quasiperiodicity, as well as some details on isotropy subgroups. The second appendix considers in detail the physical implications of the inertia tensor hypothesis (1.2) that Fedorov and Jovanović make in [12, 13].

2 The Veselova system

2.1 The n-dimensional rigid body

We begin by recalling the model of an n-dimensional rigid body $\mathcal{B}$ that moves in $\mathbb{R}^n$ about a fixed point O; further details can be found in [24, 11] and the more recent work [19]. The configuration of such a body is given by an element $g \in \mathrm{O}(n)$ called the attitude matrix that relates an inertial frame in $\mathbb{R}^n$ with a body frame rigidly attached to $\mathcal{B}$, with both frames having their origin in O. The velocity is given by the derivative $\dot{g} \in T_g \mathrm{O}(n)$. Given a motion of the body, a material point $\mathbf{x} \in \mathcal{B}$ moves along a curve $t \mapsto g(t)\mathbf{x} \in \mathbb{R}^n$, whose velocity at time t is $\dot{g}(t)\mathbf{x}$. If the point has mass m then its kinetic energy is $\frac{1}{2}m\|\dot{g}\mathbf{x}\|^2$. We make use of the left-trivialization of the tangent bundle of $\mathrm{O}(n)$:

$$
\begin{array}{rcl}
T\mathrm{O}(n) & \longrightarrow & \mathrm{O}(n) \times \mathfrak{so}(n) \\
(g, \dot{g}) & \longmapsto & (g, \Omega)
\end{array}
\tag{2.1}
$$

where $\Omega = g^{-1}\dot{g}$. The kinetic energy of the motion of the point $\mathbf{x}$ above becomes

$$
\frac{1}{2}m\|\dot{g}\mathbf{x}\|^2 = \frac{1}{2}m\|\Omega\mathbf{x}\|^2.
$$

If the body $\mathcal{B}$ is a finite collection of particles, then the total kinetic energy is a sum over the constituent particles, while if it is a continuum this becomes an integral; in what follows we write it as an integral. This is conveniently done by introducing the ***mass tensor*** of the body

$$
\mathbb{J} = \int_{\mathcal{B}} \mathbf{x}\mathbf{x}^T \, dm(\mathbf{x}),
$$

which is a symmetric $n \times n$ matrix. We will assume the body's geometry to be such that $\mathbb{J}$ is positive definite. The kinetic energy, and hence the Lagrangian, $L : T\mathrm{O}(n) \simeq \mathrm{O}(n) \times \mathfrak{so}(n) \to \mathbb{R}$, may be written in terms of the mass tensor as

$$
L(g, \Omega) = \tfrac{1}{2}\operatorname{tr}(\Omega \mathbb{J} \Omega^T).
$$

Equivalently, we may write

$$
L(g, \Omega) = \tfrac{1}{2}\langle M, \Omega \rangle,
\tag{2.2}
$$

where the ***angular momentum*** in the body frame $M \in \mathfrak{so}(n)$ is given by

$$
M = \mathbb{I}(\Omega) = \mathbb{J}\Omega + \Omega\mathbb{J},
\tag{2.3}
$$

and $\langle \cdot, \cdot \rangle$ denotes the usual invariant pairing on $\mathfrak{so}(n)$

$$
\langle \Lambda, \Omega \rangle = \tfrac{1}{2}\operatorname{tr}(\Lambda^T \Omega),
\tag{2.4}
$$

that we use to identify $\mathfrak{so}(n)^*$ with $\mathfrak{so}(n)$. The symmetric, positive definite, linear map $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ defined by (2.3) is the *inertia tensor* (the Legendre transform). We will sometimes call $\mathbb{I}$ a *physical inertia tensor* when we wish to emphasise that it is not a generic symmetric and positive definite, linear operator on $\mathfrak{so}(n)$, but rather one that arises from the existence of a mass tensor $\mathbb{J}$ as in (2.3). As explained in the introduction, several works in nonholonomic mechanics [12, 13, 21, 20, 22, 16] depend on assumptions on $\mathbb{I}$ that are generally incompatible with it being physical.

A basis $\{f_1, \ldots, f_n\}$ of the body frame that diagonalizes the symmetric mass tensor $\mathbb{J}$ is known as a *principal basis*, and for such a basis we write

$$\mathbb{J} = \text{diag}[J_1, J_2, \ldots, J_n].$$

The quantities $J_i + J_j$ are called the *principal moments of inertia* of the body. Suppose a, b are linearly independent vectors of $\mathbb{R}^n$: it is not hard to show that $a \wedge b$ is an eigenvector of the inertia operator $\mathbb{I}$ if and only if the plane Π spanned by a and b is invariant under $\mathbb{J}$. Such planes are called the *principal planes* of the body, and it follows from $\mathbb{J}$ being symmetric that such a plane must contain two eigenvectors of $\mathbb{J}$, and if J_i and J_j are the corresponding eigenvalues, then $\mathbb{I}(a \wedge b) = (J_i + J_j)(a \wedge b)$. Given such a principal plane Π we say the *moment of inertia* of that plane is $\text{in}(\Pi) = J_i + J_j$. If the eigenvalues of $\mathbb{J}$ are simple then there are $\frac{1}{2}n(n-1)$ such principal planes, but if they are not simple then there are infinitely many.

2.2   The Veselova constraint

An n-dimensional version of the Veselova constraint was introduced by Fedorov and Kozlov [11] and proceeds as follows. Choose a fixed axis in $\mathbb{R}^n$ (space), and the (generalized) Veselova constraint allows only rotations in planes containing that axis Note that [12] considers more general constraints. In the notation of that reference, we only consider the case $r = 1$. We call this axis the *distinguished axis*.

In terms of the attitude matrix, let e_1 be a unit vector parallel to the distinguished axis, and complete to an orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$ (space). Fix a principal basis for the body frame $\{f_1, \ldots, f_n\}$, and as before denote by $g \in \text{O}(n)$ the attitude matrix of the body and by $\Omega = g^{-1}\dot{g} \in \mathfrak{so}(n)$ its angular velocity expressed in these body coordinates. The matrix $\Omega_s := g\Omega g^{-1}$ describes the angular velocity in the space frame, and the n-dimensional Veselova constraint is that Ω_s is of the form $e_1 \wedge w$ for some $w \in \mathbb{R}^n$, or equivalently

$$\Omega = q \wedge v \qquad \text{for some} \quad v \in \mathbb{R}^n, \quad \text{where} \quad q := g^{-1}e_1. \qquad (2.5)$$

Let us write

$$\mathfrak{d} = e_1 \wedge \mathbb{R}^n = \text{span}\{e_1 \wedge e_2, \ldots, e_1 \wedge e_n\}.$$

The nonholonomic constraint distribution $D \subset T\text{O}(n)$ is then defined by $\Omega_s \in \mathfrak{d}$, or $\dot{g} \in D_g = \mathfrak{d}g \subset T_g\text{O}(n)$. By construction, D is right invariant and is non-integrable since $\mathfrak{d}$ is not a subalgebra of $\mathfrak{so}(n)$. A vector $\dot{g} = g\Omega \in D_g$ if and only if $\text{Ad}_g \Omega \in \mathfrak{d}$ which is equivalent to having

$$\langle E_r \wedge E_s, \Omega \rangle = 0, \qquad 2 \le r < s \le n, \qquad (2.6)$$

where $E_r = g^{-1}e_r$ so that $\text{Ad}_{g^{-1}}(e_r \wedge e_s) = E_r \wedge E_s$.

The left invariance of the Lagrangian (2.2), together with the right invariance of the constraint distribution, signifies that this is an example of an *LR system* as introduced by Veselova [26]. Following

this reference, we write the equations of motion for the system in Euler-Poincaré-Arnold form:

$$\frac{d}{dt}\left(\mathbb{I}(\Omega)\right) \;=\; [\mathbb{I}(\Omega),\Omega] \;+\; \sum_{2\leq r<s\leq n}\lambda_{rs}E_r\wedge E_s,$$
$$\frac{d}{dt}\left(E_r\wedge E_s\right) \;=\; [E_r\wedge E_s,\Omega]. \tag{2.7}$$

Here, $[\cdot,\cdot]$ denotes the matrix commutator in $\mathfrak{so}(n)$, and the multipliers λ_{rs} are the unique solution to the linear system of equations

$$\sum_{2\leq r<s\leq n}\langle \mathbb{I}^{-1}(E_\rho\wedge E_\sigma),E_r\wedge E_s\rangle\lambda_{rs} = -\langle\mathbb{I}^{-1}(E_\rho\wedge E_\sigma),[\mathbb{I}(\Omega),\Omega]\rangle,\qquad 2\leq\rho<\sigma\leq n. \tag{2.8}$$

This choice of λ_{rs} guarantees that $\langle E_r\wedge E_s,\Omega\rangle$ are first integrals of (2.7) for all $2\leq r<s\leq n$. The dynamics of the Veselova system on D is determined by considering the system (2.7) restricted to the invariant manifold defined by (2.6), together with the kinematical equation $\dot{g}=g\Omega$.

Both the restriction of (2.7) to (2.6), and the system on D, preserve the energy $H=\frac{1}{2}\langle\mathbb{I}(\Omega),\Omega\rangle$, and possess an invariant measure [26].

Symmetries Let $G_L=\mathrm{O}(n-1)$ be the subgroup of $\mathrm{O}(n)$ that fixes the distinguished axis, and let $G_R\subset\mathrm{O}(n)$ be the group of symmetries of $\mathbb{J}$, that is

$$g\in G_R\qquad\text{if and only if}\qquad g\mathbb{J}g^{-1}=\mathbb{J}. \tag{2.9}$$

The natural action of $G=G_L\times G_R$ on the configuration space is, for $(A,B)\in G_L\times G_R$, $(A,B)\cdot g = AgB^{-1}$. The tangent lift of this to the tangent bundle is

$$(A,B)\cdot(g,\dot{g})=(AgB^{-1},\,A\dot{g}B^{-1}).$$

Equivalently, using the left trivialization (2.1),

$$(A,B)\cdot(g,\Omega)=(AgB^{-1},\,B\Omega B^{-1}). \tag{2.10}$$

It is clear that the Lagrangian (2.2) is invariant under this action, and furthermore, because G_L fixes the distinguished axis, the constraint distribution D is also invariant under this action. Since the only ingredients that determine the dynamics of the Veselova top are the Lagrangian and the distribution, it follows that the dynamics of the system is equivariant with respect to the action of $G=G_L\times G_R$.

2.3 Steady rotations

We now show the Veselova system admits ***steady rotation solutions***, which we define to be motions with constant angular velocity. We show such motions are necessarily periodic solutions in which the body steadily rotates in a principal plane of inertia of the body and the constraint forces vanish. In view of the nonholonomic constraint, the orientation of the body along these solutions is such that the principal plane of rotation contains the distinguished axis.

We recall that the n-dimensional free rigid body also allows steady rotation solutions, and that these are in general quasi-periodic if $n>3$. The periodicity of the steady rotations of the n-dimensional Veselova system is a consequence of the nonholonomic constraint that forces the angular velocity matrix to have rank 2.

Proposition 2.1. *The n-dimensional Veselova system admits steady rotation solutions. These solutions are all steady rotations in principal planes, and are characterised by the condition that the angular velocity matrix Ω satisfies $[\mathbb{I}(\Omega),\Omega] = 0$. The steady rotations in the principal plane Π are periodic with period $2\pi/\|\Omega\|$, energy $H = \frac{1}{2}\operatorname{in}(\Pi)\|\Omega\|^2$ and momentum M satisfying $\|M\|^2 = (\operatorname{in}(\Pi))^2\|\Omega\|^2$. Here $\|\cdot\|$ denotes the norm in $\mathfrak{so}(n)$ induced by the pairing (2.4).*

It may seem the definition of steady rotation is ambiguous since it was not specified whether it is the angular velocity in the body or in space that should be constant. In fact the two are equivalent: write Ω and Ω_s for the two angular velocities. If Ω is constant, then the motion is given by $g(t) = g_0\exp(t\Omega)$ for some $g_0 \in \mathrm{O}(n)$. Then $\Omega_s = \dot{g}g^{-1} = g_0\Omega g_0^{-1}$, which is constant. The argument is easily reversed to show the equivalence.

Proof. First we show the existence of such solutions. If Ω is a constant rank two matrix that defines a rotation on a principal plane then $[\mathbb{I}(\Omega),\Omega] = 0$ and the multipliers λ_{rs} defined by (2.8) vanish, so the first equation in (2.7) holds since Ω is constant. The motion on $\mathrm{O}(n)$ is then $g(t) = g_0\exp(\Omega t)$ for a constant $g_0 \in \mathrm{O}(n)$, and is periodic with the stated period since Ω has rank 2. The formulas for H and $\|M\|^2$ follow immediately from $\mathbb{I}(\Omega) = \operatorname{in}(\Pi)\Omega$.

To complete the proof that $(g(t),\Omega)$ is a solution we should check that the nonholonomic constraints are satisfied along the motion. We have $\mathrm{Ad}_{g(t)}\Omega = \mathrm{Ad}_{g_0}\Omega$ so the constraints are indeed satisfied provided that the initial orientation of the body g_0 satisfies $\mathrm{Ad}_{g_0}\Omega \in \mathfrak{d}$. This is accomplished in the following way. If $\Omega = a\wedge b$ then $g_0 \in \mathrm{O}(n)$ should be such that $g_0^{-1}e_1$ lies on the plane spanned by a and b; that is, the distinguished axis should lie on the plane of rotation.

Next, if $(g(t),\Omega(t))$ is a solution for which $[\mathbb{I}(\Omega(t)),\Omega(t)] = 0$, then the multipliers λ_{rs} defined by (2.8) vanish, and the first equation in (2.7) implies that Ω is constant.

Suppose now that $(g(t),\Omega)$ is a solution with constant $\Omega \neq 0$. It remains to show that Ω defines a rotation on a principal plane (the condition that $[\mathbb{I}(\Omega),\Omega] = 0$ follows immediately from this). From (2.5), $\Omega = q\wedge v$ for some $v \in \mathbb{R}^n$ which can be chosen to be orthogonal to q. Recall that $q = g^{-1}e_1$. Then differentiating shows that $\dot{q} = -g^{-1}(g\Omega)q = -(q\wedge v)q = v$. On the other hand, given that Ω is constant we have

$$0 = \dot{\Omega} = \dot{q}\wedge v + q\wedge\dot{v} = q\wedge\dot{v},$$

which shows that $\dot{v}$ is parallel to q. These observations, together with the boundedness of q, imply that the vectors $q(t)$ and $v(t)$ describe simple harmonic motion on the plane that they span, which remains constant throughout the motion. We will now prove that this plane is invariant under $\mathbb{J}$ which is equivalent to Ω defining a rotation on a principal plane.

Using the definition of $\mathbb{I}$ given in (2.3), one shows that $[\mathbb{I}(\Omega),\Omega] = [\mathbb{J},\Omega^2]$, and with $\Omega = q\wedge v$ this becomes

$$[\mathbb{I}(\Omega),\Omega] = -\|v\|^2(\mathbb{J}qq^T - qq^T\mathbb{J}) - (\mathbb{J}vv^T - vv^T\mathbb{J}). \tag{2.11}$$

Since Ω is constant, then so is $\mathbb{I}(\Omega)$, and the first of the equations of motion (2.7) becomes

$$0 = [\mathbb{I}(\Omega),\Omega] \mod q^\perp\wedge q^\perp,$$

where $q^\perp$ is the subspace orthogonal to q. Applying both sides of the above relation to q and using (2.11) gives

$$0 = -\|v\|^2\mathbb{J}q - \|v\|^2(q\cdot\mathbb{J}q)q + (q\cdot\mathbb{J}v)v, \tag{2.12}$$

where $\cdot$ is the Euclidean scalar product in $\mathbb{R}^n$. This shows that $\mathbb{J}q$ is in the span of q and v for all time. The harmonic dynamics of $q(t)$ and $v(t)$ described above imply that the same is true about $\mathbb{J}v$. $\qquad\square$

3 First reduction of the general Veselova top

We now perform the reduction of the system by the group $G_L = \mathrm{O}(n-1)$ introduced above. This corresponds to invariance of the system with respect to the rotations and reflections of the space frame that fix the distinguished axis. This reduction was first considered in [12, Section 5] in a more general framework. Here we specialise some of their results and give their expressions in terms of the mass tensor $\mathbb{J}$ of the body, which is necessary for our purposes. The geometry is clarified by the decomposition introduced in [12], $\mathfrak{so}(n) = \mathfrak{d} \oplus \mathfrak{h}$ where $\mathfrak{d}$ is defined above and $\mathfrak{h}$ is given by

$$\mathfrak{h} = \mathrm{span}\{e_r \wedge e_s : 2 \leq r < s \leq n\},$$

where e_1 is a unit vector along the distinguished axis. This is the orthogonal complement of $\mathfrak{d}$ with respect to the Killing metric given above. Note moreover that $\mathfrak{h}$ is the Lie algebra of the subgroup $G_L = \mathrm{O}(n-1)$.

The splitting $\mathfrak{so}(n) = \mathfrak{d} \oplus \mathfrak{h}$ implies that we have a generalized Chaplygin system [23, 2]. The reduced space $D/\mathrm{O}(n-1)$ is isomorphic to the tangent bundle TS^{n-1}. An isomorphism may be constructed by viewing D as the horizontal space of a connection on the principal $\mathrm{O}(n-1)$-bundle $\mathrm{O}(n) \to S^{n-1}$, as described in [23]. Explicitly, this map $D \to TS^{n-1}$ is given by

$$(g, \Omega) \longmapsto (q, v)$$

where $q = g^{-1} e_1$ and v is the unique vector orthogonal to q satisfying

$$\Omega = q \wedge v. \tag{3.1}$$

Note that with this definition, (q, v) satisfies $\|q\| = 1$ and $q \cdot v = 0$ showing that (q, v) is indeed in TS^{n-1}.

The Legendre transform provides a further isomorphism of TS^{n-1} with T^*S^{n-1} and in our case is given as follows.

Proposition 3.1. *For the Lagrangian given in* (2.2)*, the Legendre transform* $TS^{n-1} \to T^*S^{n-1}$ *is given by* $(q, v) \longmapsto (q, p)$*, where*

$$p = -\mathbb{I}(\Omega)q. \tag{3.2}$$

and Ω *is given in* (3.1)*.*

Note that since $\mathbb{I}(\Omega)$ is skew symmetric, it follows that $p \cdot q = 0$, which means T^*S^{n-1} is realized as the submanifold of $\mathbb{R}^{2n}$ defined by the conditions

$$\|q\| = 1, \qquad p \cdot q = 0. \tag{3.3}$$

Proof. The fibre derivative of the Lagrangian

$$L(q, v) = \frac{1}{2} \langle \mathbb{I}(q \wedge v), (q \wedge v) \rangle$$

in the direction w is

$$\frac{1}{2} \langle \mathbb{I}(q \wedge v), (q \wedge w) \rangle + \frac{1}{2} \langle \mathbb{I}(q \wedge w), (q \wedge v) \rangle.$$

By the symmetry of $\mathbb{I}$ this is equal to $\langle \mathbb{I}(q \wedge v), (q \wedge w) \rangle$. Writing $\Omega = q \wedge v$ this is

$$
\begin{aligned}
-\frac{1}{2} \operatorname{tr} \left(\mathbb{I}(\Omega)(qw^T - wq^T) \right) &= -\frac{1}{2} \operatorname{tr} \left(\mathbb{I}(\Omega)(qw^T) \right) + \frac{1}{2} \operatorname{tr} \left(\mathbb{I}(\Omega)(wq^T) \right) \\
&= -\frac{1}{2} \operatorname{tr} \left(w^T \mathbb{I}(\Omega)q \right) + \frac{1}{2} \operatorname{tr} \left(q^T \mathbb{I}(\Omega)w \right) \\
&= -\mathbb{I}(\Omega)q \cdot w.
\end{aligned}
$$

This shows that $p = -\mathbb{I}(\Omega)q$ as required.   □

Composing the two isomorphisms gives

$$
D/\mathrm{O}(n-1) \longrightarrow T^*\mathrm{S}^{n-1}, \quad (g, \Omega) \longmapsto (q, p),
$$

where $q = g^{-1}e_1$ and $p = -\mathbb{I}(\Omega)q$. This map is easily shown to be well-defined.

Following the action of $G = G_L \times G_R$ on D (see Section 2.2) through the reduction process above it is easy to see that,

Corollary 3.2. *The symmetry $G = G_L \times G_R$ of the Veselova top descends to the natural action of G_R on the spaces $T\mathrm{S}^{n-1}$ and $T^*\mathrm{S}^{n-1}$; that is $g \in G_R$ acts by $g \cdot (q, v) = (gq, gv)$ and $g \cdot (q, p) = (gq, gp)$.*

The following lemma is a particular instance of [12, Theorem 5.4]:

Lemma 3.3. *The vectors q and p are constant in the space frame, that is, they satisfy*

$$
\dot{q} = -\Omega q, \qquad \dot{p} = -\Omega p. \tag{3.4}
$$

Proof. Since $q = g^{-1}e_1$, in space it becomes e_1 which is fixed by definition (alternatively, differentiating $q = g^{-1}e_1$ shows $\dot{q} = -\Omega q$). Next, starting with the relation $p = -\mathbb{I}(\Omega)q$ and using (2.7) shows that

$$
\dot{p} = \left(-\frac{d}{dt}(\mathbb{I}(\Omega)) + \mathbb{I}(\Omega)\Omega \right) q = \Omega \mathbb{I}(\Omega)q = -\Omega p, \tag{3.5}
$$

where we have used $(E_r \wedge E_s)q = 0$ for all $2 \leq r < s \leq n$.   □

Combining the first equation in (3.4) with Equation (3.1) shows that along the motion

$$
\Omega = q \wedge \dot{q}. \tag{3.6}
$$

We will use this repeatedly in what follows.

Let C be the $n \times n$ matrix defined by,

$$
C = C(q) = (\mathbb{J} + (q \cdot \mathbb{J}q)\mathrm{Id}_n)^{-1}, \tag{3.7}
$$

where Id_n denotes the identity matrix. (It is easy to see the matrix in brackets is diagonal and positive definite and hence invertible.) The following proposition gives the explicit form of the reduced equations for a general physical inertia tensor (i.e., one derived from a mass tensor $\mathbb{J}$).

Proposition 3.4. *The reduced equations of motion on $T^*\mathrm{S}^{n-1}$ are*

$$
\begin{cases}
\dot{q} = C \left[p - \left(\dfrac{p \cdot Cq}{q \cdot Cq} \right) q \right], \\
\dot{p} = -2H(q, p)q.
\end{cases} \tag{3.8}
$$

Here,

$$H(q,p) = \frac{1}{2}\left(p \cdot Cp - \frac{(p \cdot Cq)^2}{q \cdot Cq}\right), \tag{3.9}$$

is the energy integral. Furthermore, this system is invariant under the action of G_R described in Corollary 3.2.

Proof. In view of (3.6) we have

$$p = -\mathbb{I}(\Omega)q = -(\mathbb{J}(q \wedge \dot{q}) + (q \wedge \dot{q})\mathbb{J})q.$$

We would like to solve the above equation for $\dot{q}$ in terms of q and p. This is not possible for general vectors in $\mathbb{R}^n$ since the right hand side is a linear expression in $\dot{q}$ that vanishes whenever $\dot{q}$ is parallel to q. However, on T^*S^{n-1} the equations (3.3) hold and, since $\dot{q} \cdot q = 0$, we may uniquely write $\dot{q}$ as in the first equation in (3.8).

Using that $p \cdot q = 0$ along T^*S^{n-1}, the equation $\dot{p} = -\Omega p$ (see lemma above) simplifies to $\dot{p} = -(\dot{q} \cdot p)q = -2H(q,p)q$ which is the second equation in (3.8).

It remains to show that $H(q,p)$ given by (3.9) coincides with the energy $\frac{1}{2}\langle\mathbb{I}(\Omega),\Omega\rangle$. Using once again (3.6) we obtain

$$\tfrac{1}{2}\langle\mathbb{I}(\Omega),\Omega\rangle = -\tfrac{1}{2}\operatorname{tr}(\mathbb{J}(q \wedge \dot{q})^2) = \tfrac{1}{2}C^{-1}\dot{q} \cdot \dot{q} = \tfrac{1}{2}p \cdot \dot{q} = H(q,p).$$

Finally, to see the invariance, note that (2.9) implies that $C(gq) = C(q)$ for all $g \in G_R$. $\qquad\square$

Lemma 3.3 above allows us to conclude the following proposition (which is an instance of a general result on kinetic LR systems given in [12, Proposition 2.2]).

Proposition 3.5. *The square $\|p\|^2$ of the momentum is conserved under the dynamics on T^*S^{n-1}.*

Proof. This is clear from (3.8), since $p \cdot \dot{p} = -2Hp \cdot q = 0$. $\qquad\square$

Another application of Lemma 3.3 is the following reconstruction formula, valid for the 3-dimensional system, and found in a slightly different form in [25] (not using p).

Proposition 3.6. *In dimension 3, if $(q(t), p(t))$ is a solution of the reduced equations of motion on T^*S^2, then for any $g_0 \in G_L = O(2)$ the curve*

$$g(t) = g_0 \begin{pmatrix} q(t)^T \\ \frac{1}{\|p(t)\|}p(t)^T \\ \frac{1}{\|p(t)\|}(q(t) \times p(t))^T \end{pmatrix} \tag{3.10}$$

is a solution to the full Veselova system, and all solutions with $p \neq 0$ can be written in this way.

Proof. We know q, p are constant in space, whence $q \times p$ is also constant in space. Thus each row satisfies $\dot{r} = r\Omega$ and hence $\dot{g} = g\Omega$. Furthermore, for such g_0, the velocity $\dot{g}$ lies in the distribution D. This is because the angular velocity in space Ω_s satisfies

$$\Omega_s = \dot{g}(t)g^{-1}(t) = g_0 \begin{pmatrix} 0 & 2H/\|p\| & (*) \\ -2H/\|p\| & 0 & 0 \\ (*) & 0 & 0 \end{pmatrix} g_0^{-1}$$

and since the lower left 2×2 block vanishes, and this is preserved by g_0, Ω_s satisfies the constraint. $\qquad\square$

3.1   Reduced dynamics of the general Veselova top

With the equations of motion above, we are able to discuss some aspects of the reduced dynamics of the general (physical) n-dimensional Veslova top.

Note that since the system on D is an LR system it follows from a theorem of Fedorov and Jovanović [12, Theorem 3.3] that Equations (3.8) possess a smooth invariant measure on $T^*\mathrm{S}^{n-1}$; namely the push forward of the invariant measure on D mentioned already. A formula for this volume form, valid in a more general framework than the one we consider, is given in [12, Theorem 5.5]. The following proposition gives a non-trivial simplification of this formula to our setting.

Proposition 3.7. *The reduced equations of motion* (3.8) *on* $T^*\mathrm{S}^{n-1}$ *have invariant measure*

$$\sqrt{\frac{\det(C)}{q\cdot Cq}}\,\sigma,$$

where σ *is the Liouville volume form on* $T^*\mathrm{S}^{n-1}$.

Proof. The Liouville form σ on $T^*\mathrm{S}^{n-1}$ coincides with the volume form on $T^*\mathrm{S}^{n-1}$ inherited from the Euclidean measure $dq\wedge dp$ in the ambient space $\mathbb{R}^{2n}$. To prove the result we compute the divergence of the vector field on $\mathbb{R}^{2n}$ defined by the equations (3.8) with respect to $\mu(q)\,dq\wedge dp$, where $\mu(q)=\sqrt{\frac{\det(C)}{q\cdot Cq}}$, and show that its restriction to $T^*\mathrm{S}^{n-1}$ vanishes. Given that μ is independent of p, this is equivalent to showing that

$$y(q,p):=\mu\sum_{i=1}^{n}\left(\frac{\partial\dot{q}^i}{\partial q^i}+\frac{\partial\dot{p}_i}{\partial p_i}\right)+\frac{\partial\mu}{\partial q}\cdot\dot{q}$$

is zero along $T^*\mathrm{S}^{n-1}$. In the calculations below we will repeatedly use that the matrices $\mathbb{J}$ and C are symmetric and commute (which is obvious since they are diagonal).

First notice that (3.8) implies

$$\sum_{i=1}^{n}\frac{\partial\dot{p}_i}{\partial p_i}=-2\frac{\partial H}{\partial p}\cdot q=-2\dot{q}\cdot q,$$

which vanishes along $T^*\mathrm{S}^{n-1}$. Next, by direct differentiation of μ and $\dot{q}^i$ defined by (3.8) one obtains

$$\frac{\partial\mu}{\partial q}=\mu\left(-X(q)\mathbb{J}q-\frac{Cq}{q\cdot Cq}\right),$$

$$\sum_{i=1}^{n}\frac{\partial\dot{q}^i}{\partial q^i}=\left(C\left(-2\mathbb{J}+\frac{2(q\cdot\mathbb{J}Cq)-1}{q\cdot Cq}\mathrm{Id}_n\right)q\right)\cdot\dot{q}-X(q)\frac{q\cdot Cp}{q\cdot Cq},$$

where $X(q):=\mathrm{tr}(C)-\frac{q\cdot C^2 q}{q\cdot Cq}$. Therefore, along $T^*\mathrm{S}^{n-1}$ we may write

$$y(q,p)=-2\mu\left(C\left(\mathbb{J}+\frac{1-(q\cdot\mathbb{J}Cq)}{q\cdot Cq}\mathrm{Id}_n\right)q\right)\cdot\dot{q}-\mu X(q)\left(\mathbb{J}q\cdot\dot{q}+\frac{q\cdot Cp}{q\cdot Cq}\right).\qquad(3.11)$$

We now make use of the equality

$$\mathbb{J}Ck\cdot q+(Ck\cdot q)(\mathbb{J}q\cdot q)=k\cdot q,$$

which holds for all $k \in \mathbb{R}^n$ and is easily established. By respectively applying it to $k = q$ and $k = p$ shows that along $T^* \mathrm{S}^{n-1}$ we have

$$q \cdot \mathbb{J}Cq = 1 - (q \cdot Cq)(q \cdot \mathbb{J}q), \qquad \text{and} \qquad q \cdot \mathbb{J}Cp = -(q \cdot Cp)(q \cdot \mathbb{J}q).$$

Using these identities and the expression for $\dot{q}$ given in (3.8) one deduces the formulae

$$\mathbb{J} + \frac{1 - (q \cdot \mathbb{J}Cq)}{q \cdot Cq} \mathrm{Id}_n = C^{-1}, \qquad \mathbb{J}q \cdot \dot{q} = -\frac{q \cdot Cp}{q \cdot Cq},$$

that are valid along $T^* \mathrm{S}^{n-1}$. Therefore, (3.11) implies

$$y(q,p)|_{T^* \mathrm{S}^{n-1}} = -2\mu q \cdot \dot{q} = 0.$$

<div align="right">□</div>

Remark 3.8. It is unknown whether the reduced system (3.8) can be Hamiltonized by Chaplygin's reducing multiplier method. This would amount to the introduction of a time reparametrization $dt = \nu(q) \, d\tau$, and a rescaling of the momenta $p \to \nu(q)p$, that would transform the equations into a Hamiltonian system. The factor $\nu(q)$, if it exists, is the so-called Chaplygin multiplier. The specific form of the invariant measure implies that, up to a constant factor, the multiplier $\nu(q)$ would be given by

$$\nu(q) = \left(\frac{\det C}{q \cdot Cq} \right)^{-\frac{1}{2(n-2)}}.$$

The details about the method may be found in e.g. [12, 6]. For $n = 3$ the Hamiltonization is possible and follows from Chaplygin's last multiplier theorem [4]. For $n \geq 4$, using the above expression for $\nu(q)$, we were able to verify the Hamiltonization of the system for axisymmetric tops (treated in Section 4). This in fact follows from the work of Fedorov and Jovanović [12, 13]. On the other hand, some preliminary investigations of us suggest that the system is *not* Hamiltonizable for $n \geq 4$ and more general mass tensor $\mathbb{J}$.

3.2 Steady rotations

Proposition 3.9. *The (non-trivial) steady rotation solutions on D described in Proposition 2.1 project to periodic solutions of the reduced system (3.8) on $T^* \mathrm{S}^{n-1}$, where q and p rotate uniformly in a principal plane of the body. The projected motion has the same period and the same energy as the motion on D.*

We prove the stability of some of these steady rotations in Theorem 3.12 below.

Proof. Let $\mathbb{J} = \mathrm{diag}[J_1, \ldots, J_n]$ be the mass tensor and consider a steady rotation solution with angular velocity Ω. In view of Proposition 2.1, by rotating the body frame if necessary, we may assume that $\Omega = \omega f_i \wedge f_j$, where $\omega = \|\Omega\| \in \mathbb{R}$. Then $p = -\mathbb{I}(\Omega)q = -(J_i + J_j)\Omega q = (J_i + J_j)\dot{q}$. It follows from the second equation in (3.8) that the motion of q, p, is simple harmonic with period $2\pi \sqrt{(J_i + J_j)/2H} = 2\pi/|\omega|$. Moreover, q and p are contained in the plane spanned by f_i and f_j in view of (3.6). That the energies are equal is obvious, by the definition of H. □
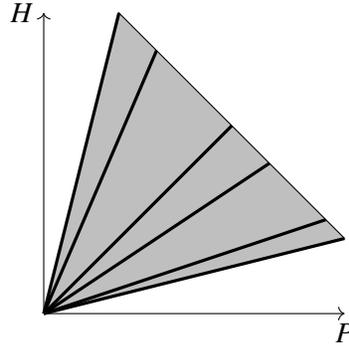
Figure 3.1: The image and critical rays L_{ij} (shown as dark lines, see Proposition 3.10) of the energy-momentum map for the Veselova top. The slopes of the critical rays as shown are equal to $[2(J_i + J_j)]^{-1}$.

Consider the integral of motion $P = \|p\|^2$ and the resulting energy-momentum map

$$(P,H) : T^*\mathsf{S}^{n-1} \to \mathbb{R}^2.$$

The following proposition refers to the singularities of this map. Since H and P are both non-negative, the image lies in the (closed) positive quadrant $\mathbb{R}^2_+ \subset \mathbb{R}^2$. If $p = 0$ then $(P,H)(q,0) = (0,0)$ and such points are all critical points of (P,H). As we see in the following proposition, the other critical points occur at the steady rotations.

Proposition 3.10. *Let* $\mathbb{J} = \mathrm{diag}[J_1, \ldots, J_n]$ *with* $J_j > 0$ *for all* j.

(i) *The critical points of the energy-momentum map* (P,H) *with* $p \neq 0$ *occur at the steady rotations described above.*

(ii) *The critical values of the energy-momentum map* (P,H) *are the rays* L_{ij} *given by* $P = 2(J_i + J_j)H$, *for* $i < j$.

The image and critical values of the energy-momentum map are illustrated in Figure 3.1. Let us emphasize that, in the case of eigenvalues with multiplicity, this statement does not assume $J_i \neq J_j$, only that $i \neq j$. Moreover, recall from Section 2.1 that a rotation of the body in the principal plane $\Pi_{i,j}$ has angular velocity Ω equal to a multiple of $f_i \wedge f_j$, and this is an eigenvector of the inertia operator with eigenvalue $\mathrm{in}(\Pi_{i,j}) = J_i + J_j$.

Proof. (i) Let $(q_0, p_0) \in T^*\mathsf{S}^{n-1}$ with $p_0 \neq 0$. Writing $H_0 = H(q_0, p_0)$ and $P_0 = \|p_0\|^2$, this implies $H_0 > 0$ and $P_0 > 0$. At such points, neither $dH = 0$ nor $dP = 0$. Thus, considering H and P as functions on the ambient space $\mathbb{R}^n \times \mathbb{R}^n$, the condition for a critical point at (q_0, p_0) is that the gradients of H and P are linearly dependent (on $T^*\mathsf{S}^{n-1}$), and using Lagrange multipliers for the constraints defining this cotangent bundle gives

$$\nabla H = \lambda_1 \nabla P + \lambda_2 \nabla (p \cdot q) + \lambda_3 \nabla \|q\|^2, \tag{3.12}$$

and (q_0, p_0) are such that

$$q_0 \cdot p_0 = 0, \qquad \|q_0\|^2 = 1, \qquad H(q_0, p_0) = H_0, \qquad \|p_0\|^2 = P_0. \tag{3.13}$$

At such points, (3.12) is equivalent to

$$\nabla H(q_0, p_0) = 2\lambda_1(0, p_0) + \lambda_2(p_0, q_0) + 2\lambda_3(q_0, 0). \tag{3.14}$$

Denote by

$$v(q, p) := \dot{q} = Cp - \frac{p \cdot Cq}{q \cdot Cq} Cq.$$

One calculates

$$\frac{\partial H}{\partial p} = v, \qquad \frac{\partial H}{\partial q} = -\|v\|^2 \mathbb{J}q - \frac{(p \cdot Cq)}{(q \cdot Cq)} v.$$

Then (3.14) can be rewritten as 2 equations,

$$v_0 = 2\lambda_1 p_0 + \lambda_2 q_0, \tag{3.15a}$$

$$-\left(\frac{p_0 \cdot Cq_0}{q_0 \cdot Cq_0}\right) v_0 - \|v_0\|^2 \mathbb{J}q_0 = \lambda_2 p_0 + 2\lambda_3 q_0, \tag{3.15b}$$

where $v_0 = v(q_0, p_0)$. Taking the inner product on both sides of equation (3.15a) with q_0 and p_0, and using (3.13), leads to the relations:

$$\lambda_2 = 0, \qquad \lambda_1 = \frac{H_0}{P_0}. \tag{3.16}$$

Substituting this into (3.15b) and taking the inner product with q_0 gives $\lambda_3 = -\frac{1}{2}\|v_0\|^2 (q_0 \cdot \mathbb{J}q_0)$. Putting first $\lambda_2 = 0$ in (3.15a) shows that $v_0 = 2(H_0/P_0)p_0$, and secondly taking the inner product of (3.15b) with p_0 shows

$$\lambda_3 = -2\frac{H_0^2}{P_0}(q_0 \cdot \mathbb{J}q_0), \quad \text{and} \quad \frac{p_0 \cdot Cq_0}{q_0 \cdot Cq_0} = -\frac{2H_0}{P_0}(p_0 \cdot \mathbb{J}q_0).$$

Now consider $\mathbb{J}q_0$ and $\mathbb{J}p_0$. First from (3.15a), $v_0 = (2H_0/P_0)p_0$ and then from (3.15b) we deduce that $\mathbb{J}q_0$ is a linear combination of q_0 and p_0. Again from (3.15a), substituting for the definition of v_0, and using the definition of the matrix C one finds

$$\mathbb{J}p_0 = \left(\frac{P_0}{2H_0} - (q_0 \cdot \mathbb{J}q_0)\right)p_0 + \frac{P_0}{2H_0}\left(\frac{p_0 \cdot Cq_0}{q_0 \cdot Cq_0}\right)q_0,$$

which is also a linear combination of p_0 and q_0. In this way we have shown that the plane spanned by q_0, p_0 is invariant under $\mathbb{J}$, which is precisely the condition for it to be a principal plane.

Finally, if p_0, q_0 lie in a principal plane, and $p_0 \neq 0$, they define a steady rotation with $\Omega = \frac{2H_0}{P_0}q_0 \wedge p_0$ as is seen from Equations (3.6), (3.15a) and (3.16).

(ii) Without loss of generality, we may assume that the body frame $\{f_1, \ldots, f_n\}$ is chosen such that $\mathbb{J} = \text{diag}[J_1, \ldots J_n]$ and $\text{span}\{q_0, p_0\} = \text{span}\{f_i, f_j\}$ $(i \neq j)$. We may then write

$$q_0 = \cos\alpha f_i + \sin\alpha f_j, \qquad p_0 = \sqrt{P_0}\left(-\sin\alpha f_i + \cos\alpha f_j\right),$$

for a certain $\alpha \in [0, 2\pi)$. The relation $P_0 = 2H_0(J_i + J_j) = 2H_0 \text{in}(\Pi_{i,j})$ follows from Proposition 2.1, since $P = \|M\|^2$. $\qquad\square$

Stability of steady rotations.   Recall that for a dynamical system, a compact subset S of the phase space is Lyapunov stable if for each neighbourhood U of S there is a neighbourhood V such that any trajectory that enters V lies entirely in U. For the free n-dimensional rigid body, Izosimov [19] has produced a fairly complete analysis of the stability of the relative equilibria. In particular, in Example 2.3 he shows that if $\mathbb{J}$ has simple eigenvalues, with $J_1 < J_2 < \cdots < J_n$ then the 2-dimensional rotations in the principal plane $\Pi_{i,j}$ are (Lyapunov) stable if and only if $|i - j| = 1$ (that is, if J_i and J_j are adjacent in the ordering). It is not known whether the same result holds with the Veselova constraint. However, we do have the following result which follows from the geometry of the energy-momentum map. We do not require the eigenvalues of $\mathbb{J}$ to be simple (motivated by the symmetric bodies considered in later sections).

Recall that if Π is a principal plane of inertia then there is a corresponding moment of inertia, which we denote in(Π); if $\Pi = \Pi_{i,j}$ for some principal basis then $\text{in}(\Pi_{i,j}) = J_i + J_j$.

Definition 3.11.   A principal plane Π is ***extremal*** if either in(Π) $\leq$ in(Π') for all principal planes Π' (in which case it is minimal) or in(Π) $\geq$ in(Π') for all principal planes Π' (in which case it is maximal).

If the eigenvalues of $\mathbb{J}$ are all simple and ordered as usual, then the only minimal principal plane is $\Pi_{1,2}$ and the only maximal one is $\Pi_{n-1,n}$. If some of the eigenvalues of $\mathbb{J}$ are not simple, then the dynamics has a further symmetry G_R described by Equation (2.9), and one does not in that case expect a steady rotation to be Lyapunov stable. Instead the natural notion is G_R-Lyapunov stable, which is defined by saying that a G_R-invariant subset S of phase space is G_R-Lyapunov stable if for every G_R-invariant neighbourhood U of S there is a G_R-invariant neighbourhood V of S such that any trajectory that intersects V is entirely contained in U.

Theorem 3.12.   *Steady rotations in extremal principal planes are G_R-Lyapunov stable.*

If as above, the eigenvalues are simple then the statement reduces to saying the steady rotations in extremal principal planes are Lyapunov stable. Before the proof of the theorem, we state the following immediate consequence.

Corollary 3.13.   *The steady rotations of the 3D Veselova top about the largest and smallest principal axes of inertia are Lyapunov stable (O(2)-Lyapunov stable for axisymmetric tops).*

Proof of Theorem 3.12.   We shall prove this for minimal principal planes of inertia, and leave the maximal case to the reader. We define a function on the phase space which, under the hypotheses of the theorem, is a Lyapunov function. Let $(q_0, p_0) \in \Pi$ be a point of steady rotation, and let $f : T^* \mathrm{S}^{n-1} \to \mathbb{R}$ be defined by

$$f(q,p) = H - \lambda P - \tfrac{1}{2}(P - c)^2,$$

where $\lambda = (2\,\text{in}(\Pi))^{-1}$ and $c > 0$ is a constant to be fixed. This function is clearly invariant under the dynamics, and is moreover invariant under the action of G_R. Let S be the G_R-orbit of the trajectory through (q_0, p_0). We show that (i) f has a critical point at every point of S, and (ii) under the condition that Π is minimal, the Hessian matrix of f transverse to S is negative definite. It then follows that for $\varepsilon > 0$ the set

$$V_\varepsilon := \{(q,p) \in T^* \mathrm{S}^{n-1} \mid f(q,p) > f(q_0, p_0) - \varepsilon\}$$

is a G_R-invariant neighbourhood of S, and that any G_R-invariant neighbourhood U of S contains V_ε for some $\varepsilon > 0$, and we can deduce the G_R-Lyapunov stability of S.

For (i), since f is G_R-invariant, it suffices to prove it has a critical point at any point of the trajectory. Now, writing T_0 for the tangent space $T_0 = T_{(q_0, p_0)}(T^* \mathrm{S}^{n-1})$, we have

$$df = (dH - \lambda dP)_{|T_0} - (P - c) dP_{|T_0}.$$

The first term vanishes by the proof of Proposition 3.10 above, and in particular the value of λ is given by λ_1 in that proof—see (3.16). The second term vanishes when $P = c$, so from now on we assume $c = \|p_0\|^2$.

We now wish to compute the Hessian of f at (q_0, p_0) in directions transverse to the submanifold S. Let us choose a principal basis so that $\Pi = \mathrm{span}\{f_1, f_2\}$, in which case

$$q_0 = \cos\alpha f_1 + \sin\alpha f_2, \quad p_0 = \|p_0\|(-\sin\alpha f_1 + \cos\alpha f_2).$$

We can further simplify the calculations by noting that since f is invariant under the dynamics, the signature of the Hessian will also be invariant (by Sylvester's law of inertia), so we can suppose $\alpha = 0$, and we take

$$q_0 = f_1, \quad p_0 = \|p_0\| f_2. \tag{3.17}$$

Now we consider a slice V (of dimension $2n - 3$) to the trajectory in the tangent space T_0. With (q_0, p_0) as above, and putting $\hat{p} = (0, f_2)$ (the radial direction in p) we can choose the following basis for V:

$$\{\hat{p},\ (af_3, 0),\ (0, b_3 f_3),\ \ldots,\ (af_n, 0),\ (0, b_n f_n)\}$$

(in $\mathbb{R}^n \oplus \mathbb{R}^n$), where $b_k = \sqrt{(J_1 + J_2)(J_1 + J_k)}$ and $a = \omega^{-1}$, where ω is the frequency of the steady rotation. With respect to this basis, one finds the Hessian matrix of f at (q_0, p_0) to be in block form,

$$\begin{bmatrix} -8\omega^2(J_1 + J_2)^2 & 0 & 0 & 0 & \ldots & 0 \\ \hline 0 & A_3 & 0 & 0 & \ldots & 0 \\ 0 & 0 & A_4 & 0 & \ldots & 0 \\ 0 & 0 & 0 & A_5 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & \ldots & A_n \end{bmatrix},$$

where the 2×2 matrix A_k is given simply by

$$A_k = \begin{pmatrix} J_1 - J_k & 0 \\ 0 & J_2 - J_k \end{pmatrix}. \tag{3.18}$$

The remainder of the proof is simply combining the signs of these eigenvalues with the subspace of V that suffices for a transversal to S.

Firstly, note that if all the eigenvalues J_k are distinct, then the Hessian matrix is negative definite if and only if $J_k > \max\{J_1, J_2\}$ for all $k \geq 3$, which is precisely the condition for a minimal principal plane. Indeed this holds more generally, provided $\max\{J_1, J_2\} < \min\{J_k \mid k \geq 3\}$ (whether or not $J_1 = J_2$ or some of the J_k for $k \geq 3$ are multiple).

Now assume $J_1 \leq J_2 = J_3$ (which covers all remaining cases).

First consider the case where $J_1 = J_2$ and suppose this has multiplicity $m \geq 3$. Let E_1 be the m-dimensional eigenspace of $\mathbb{J}$ with eigenvalue J_1. The tangent space to the $O(m)$-group orbit through $(q_0, p_0) = (f_1, \|p_0\| f_2)$ is

$$\mathfrak{g} \cdot (q_0, p_0) = \{(\xi f_1, \|p_0\|\xi f_2) \mid \xi \in \mathfrak{so}(m)\}.$$

Since $m \geq 3$ it follows that

$$\mathfrak{g} \cdot (q_0, p_0) = \mathbb{R}\{(f_2, -\|p_0\| f_1)\} \oplus (F_1 \times \{0\}) \oplus (\{0\} \times F_1),$$

where F_1 is the span of $f_3, \ldots, f_m$ in E_1. The first component here is the tangent space to the trajectory, so is not required in V, and then

$$T_0 = \mathbb{R}(\dot{q}_0, \dot{p}_0) \oplus V = \mathfrak{g} \cdot (q_0, p_0) \oplus V_1$$

where V_1 has basis

$$\{\hat{p}, (af_{r+1}, 0), (0, b_{r+1}f_{r+1}), \ldots, (af_n, 0), (0, b_nf_n)\},$$

using the notation introduced above. It is clear that the restriction to this space of the Hessian matrix given above is again negative definite.

Finally suppose $J_1 < J_2 = J_3$, and let m_2 be the multiplicity of J_2, and E_2 the eigenspace. Since J_1 is simple there is only a finite group acting on that space, so that

$$\mathfrak{g} \cdot (q_0, p_0) = \{0\} \times E_2$$

where E_2 is the span of $\{f_3, \ldots, f_{m+1}\} \subset E_2$. Again writing,

$$T_0 = \mathbb{R}(\dot{q}_0, \dot{p}_0) \oplus V = \mathbb{R}(\dot{q}_0, \dot{p}_0) \oplus \mathfrak{g} \cdot (q_0, p_0) \oplus V_2$$

we have V_2 is spanned by $\hat{p}$ and $(f_r, 0)$ ($r = 3, \ldots, m+1$) and by $(f_k, 0), (0, f_k)$ for $k > m+1$. The eigenvalues of the Hessian above that are relevant are just the first (for the space spanned by $\hat{p}$) and the first element of each A_k (for $k = 3, \ldots, m+1$, which are all negative), and the remaining 2×2 blocks, which are all negative definite. $\qquad\square$

4   The axisymmetric Veselova top

This section considers ***axisymmetric tops*** for which the mass tensor relative to a frame $\{f_1, \ldots f_n\}$ takes the form

$$\mathbb{J} = \mathrm{diag}[J_1, J_2, \ldots, J_2]. \tag{4.1}$$

From its definition in (2.9), it follows that the symmetry group $G_R = \mathrm{O}(n-1)$, which corresponds to rotations and reflections of the body frame that fix the symmetry axis.

In order to analyse the ***second reduced space*** $\mathcal{R} := D/G = T^*\mathrm{S}^{n-1}/G_R$, we require the invariants for the action of $G_R = \mathrm{O}(n-1)$ on $T^*\mathrm{S}^{n-1}$. Now, it is well-known that the ring of invariants for $\mathrm{O}(n-1)$ acting on $(q_2, p_2) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is generated by three independent quadratic expressions:

$$\|q_2\|^2, \quad \|p_2\|^2, \quad \text{and} \quad q_2 \cdot p_2.$$

By the Cauchy-Schwarz inequality, these satisfy $|p_2 \cdot q_2|^2 \leq \|q_2\|^2\|p_2\|^2$.

It follows that the ring of invariants for $\mathrm{O}(n-1)$ acting on $(q_1, q_2, p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}$ is generated by the 5 invariants

$$q_1, \quad p_1, \quad \|q_2\|^2, \quad \|p_2\|^2, \quad q_2 \cdot p_2.$$

We now restrict to $T^*\mathrm{S}^{n-1}$ by requiring $\|q\|^2 = q_1^2 + \|q_2\|^2 = 1$ and $q \cdot p = q_1 p_1 + q_2 \cdot p_2 = 0$. In place of 5 invariants, we now have 3:

$$q_1, \quad p_1, \quad \text{and} \quad P := \|p\|^2 = p_1^2 + \|p_2\|^2.$$

Proposition 4.1. *The orbit space* $\mathcal{R} = T^*\mathrm{S}^{n-1}/\mathrm{O}(n-1) = D/G$ *is a stratified semi-algebraic subspace of* $\mathbb{R}^3$ *given by*

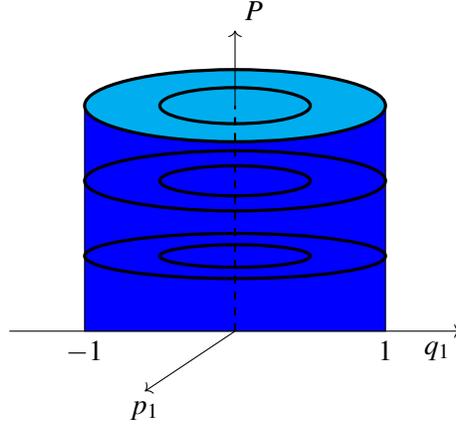$$\mathcal{R} = \{(q_1, p_1, P) \in \mathbb{R}^3 : p_1^2 \leq (1 - q_1^2)P, \ -1 \leq q_1 \leq 1\}.$$

Figure 4.1: The reduced space $\mathcal{R}$ with boundary the 'canoe surface'

The boundary of this space is often called the *canoe surface*. The reduced space $\mathcal{R}$ and the dynamics on it (to be discussed below) are illustrated in Figure 4.1. Basic notions of the stratification of orbits spaces are given in Appendix A. Further details may be found for example in [5].

Proof. It was discussed above that the quantities q_1, p_1 and P generate the invariants. The Cauchy-Schwarz inequality $(q_2 \cdot p_2)^2 \leq \|q_2\|^2 \|p_2\|^2$ leads to the inequality $p_1^2 \leq P(1 - q_1^2)$. □

Th orbit space $\mathcal{R}$ has a natural stratification arising from the isotropy subgroups of the $O(n-1)$ action on T^*S^{n-1}. Accordingly, there are four strata of $\mathcal{R}$ as follows (recall that strata are by definition connected).

S_0, S_0': If $q_2 = p_2 = 0$, the point (q, p) is fixed by $O(n-1)$; this determines two zero-dimensional strata consisting of the points $(q_1, p_1, P) = (\pm 1, 0, 0)$. This corresponds to the top being placed with its axis aligned with the distinguished axis, but stationary since $P = 0$; from the dynamical point of view it is no more special than any other stationary configuration of the top.

S_2: This represents elements in T^*S^{n-1} for which q_2 and p_2 are parallel and do not both vanish. The isotropy group of such (q, p) is conjugate to $O(n-2)$. The image in $\mathcal{R}$ of this set is the canoe surface $p_1^2 = (1 - q_1^2)P$, with the points $(q_1, p_1, P) = (\pm 1, 0, 0)$ removed.

S_3: This 3-dimensional stratum consists of the points (q_1, p_1, P) corresponding to points of T^*S^{n-1} for which $q_2 \wedge p_2 \neq 0$ and their isotropy group is conjugate to $O(n-3)$ (where, for $n = 4$, $O(1) = \mathbb{Z}_2$, and for $n = 3$, $O(0)$ is the trivial group). This stratum corresponds to the interior of the orbit space, where $p_1^2 < (1 - q_1^2)P$.

4.1 Reduced dynamics

The reduced equations on $\mathcal{R}$ can easily be obtained from (3.8). One finds,

$$\dot{q}_1 = \frac{p_1}{J_1 + J_2}, \qquad \dot{p}_1 = -2Hq_1, \qquad \dot{P} = 0, \qquad (4.2)$$

where, after some calculations, H is given by

$$H = \frac{1}{2(J_1 + J_2)} \left(\frac{(J_2 - J_1)p_1^2 + (J_1 + J_2)P}{(J_1 - J_2)q_1^2 + 2J_2} \right), \qquad (4.3)$$

and is a first integral. As expected, the flow of these equations leaves the strata of $\mathcal{R}$ invariant.

Dynamics: Since H is a first integral, the equations of motion above are linear, with solutions

$$q_1(t) = A\cos(\omega t) + B\sin(\omega t), \qquad p_1(t) = (J_1 + J_2)\omega(-A\sin(\omega t) + B\cos(\omega t)),$$

where $\omega^2 = 2h/(J_1 + J_2)$ and A, B are arbitrary constants of integration and h is the constant value of H along the solution. There are two classes of equilibria: along the q_1-axis (where $p_1 = P = 0$ and hence $H = 0$) and along the P-axis (where $q_1 = p_1 = 0$). In Figure 4.1, the ellipses illustrate the trajectories; they lie in horizontal planes $P = \text{const}$ and are centred on the P-axis.

Steady rotations Following Section 2.3, we see that for an axisymmetric top there are two kinds of principal plane of inertia:

- Let $a \in \mathbb{R}^n$ be orthogonal to f_1. Then $\{f_1, a\}$ spans a principal plane containing the axis of symmetry, an ***axial*** principal plane, with associated moment of inertia $\text{in}(\Pi) = J_1 + J_2$, and the corresponding steady rotation is simply a steady rotation in this plane.

- Now let Π be any plane orthogonal to the axis of symmetry. This is also a principal plane of inertia, called an ***equatorial*** plane, and with $\text{in}(\Pi) = 2J_2$. The resulting motion generalizes the 3D steady rotations where the body rotates about its symmetry axis.

Proposition 4.2. *(i) The steady rotations in an axial principal plane are contained in S_2, and conversely all points in S_2 with $P > 0$ correspond to such steady rotations.*

(ii) Steady rotations in an equatorial plane of inertia are contained in S_3 and have $q_1 = p_1 = 0$, and conversely, all points with $q_1 = p_1 = 0$ correspond to such steady rotations.

(iii) Both the axial and equatorial steady rotations are Lyapunov stable relative to $O(n-1)$.

Proof. First notice that if $\Omega = a \wedge b$ defines a steady rotation solution then, by Equation (3.6), $a \wedge b = q \wedge \dot{q}$ and it follows that $q(t)$ and $\dot{q}(t)$ are contained in the plane spanned by a and b at all time. We will make use of this in the following.

(i) For a steady rotation in an axial plane we can write $\Omega = f_1 \wedge a$ for a constant vector a such that $f_1 \cdot a = 0$. Since q lies on the plane spanned by f_1 and a, we have $q = q_1 f_1 \pm \frac{\sqrt{1-q_1^2}}{\|a\|} a$. Hence,

$$p = -\mathbb{I}(\Omega)q = -(J_1 + J_2)\Omega q = (J_1 + J_2)\left(\mp\sqrt{1-q_1^2}\,\|a\|f_1 + q_1 a\right).$$

The above formulae imply that q_2 and p_2 are parallel so the solution is indeed contained in S_2.

For the converse, recall that along S_2 the vectors q_2 and p_2 are parallel. Therefore, in view of (3.8) we conclude that $\dot{q}_2$ is also parallel to q_2. Writing $\dot{q}_2 = \lambda q_2$ we obtain

$$\Omega = q \wedge \dot{q} = (q_1 f_1 + (0, q_2)) \wedge (\dot{q}_1 f_1 + \lambda(0, q_2)) = f_1 \wedge a,$$

where $a = (\lambda q_1 - \dot{q}_1)(0, q_2)$. Due to our symmetry assumptions on $\mathbb{I}$, this implies that $\mathbb{I}(\Omega) = (J_1 + J_2)\Omega$ so $[\mathbb{I}\Omega, \Omega] = 0$ which, by Proposition 2.1, implies that the solution is a steady rotation. Therefore a is constant and the steady rotation is of axial type. (Note that the condition $P > 0$ implies $\Omega \neq 0$).

(ii) If $\Omega = a \wedge b$ with a, b, f_1, mutually perpendicular, then, since q is contained in the plane spanned by a and b, $q_1 = q \cdot f_1 = 0$. But also $p_1 = p \cdot f_1 = 0$ since $p = -\mathbb{I}(\Omega)q = -2J_2(a \wedge b)q$ is also contained in the plane spanned by a and b.

For the converse, note that these points are equilibria on $\mathcal{R}$ satisfying $q_1 = \dot{q}_1 = 0$. Considering that $\Omega = q \wedge \dot{q}$, and our symmetry assumptions on the inertia tensor, we have $\mathbb{I}(\Omega) = 2J_2\Omega$. Therefore $[\mathbb{I}(\Omega), \Omega] = 0$, and Proposition 2.1 implies that the motion is a steady rotation which takes place on an equatorial plane since f_1 is orthogonal to q and $\dot{q}$.

(iii) This follows immediately from Theorem 3.12. $\qquad\square$

4.2 Reconstruction

We now give the global details of the reconstruction of the dynamics both to T^*S^{n-1} and to D. The first conclusion is that the dynamics is essentially that of the axisymmetric 3D Veselova top, in a sense that is made precise in the following theorem.

Theorem 4.3. *Consider the n-dimensional axisymmetric Veselova top, with $n > 3$ and mass tensor $\mathbb{J}$ given in (4.1). Consider any solution to the reduced equation on $\mathcal{R}$, and a choice of initial lift to both T^*S^{n-1} and D. Then there exist choices of basis of $\mathbb{R}^n$ in the body and in space, the first containing f_1 and the second e_1, such that*

$$q(t) = \begin{pmatrix} \bar{q}(t) \\ 0 \end{pmatrix}, \quad p(t) = \begin{pmatrix} \bar{p}(t) \\ 0 \end{pmatrix}, \quad g(t) = \begin{pmatrix} \bar{g}(t) & 0 \\ 0 & \mathrm{Id}_{n-3} \end{pmatrix}, \quad \Omega(t) = \begin{pmatrix} \bar{\Omega}(t) & 0 \\ 0 & 0_{n-3} \end{pmatrix} \qquad (4.4)$$

where $(\bar{q}(t), \bar{p}(t)) \in T^*S^2$ and $(\bar{g}(t), \bar{\Omega}(t)) \in D(3) \subset TO(3)$ satisfy the equations of motion of the axisymmetric 3D Veselova top (for $\bar{g}, \bar{\Omega}$) and its reduced equations (for $(\bar{q}, \bar{p})$), with mass tensor $\bar{\mathbb{J}} = \mathrm{diag}[J_1, J_2, J_2]$.

Proof. Suppose that the given solution in $\mathcal{R}$ is in the open stratum S_3 (the other cases follow similarly), and let q_0, p_0, g_0, Ω_0 be corresponding initial values for the lifted dynamics. We describe the necessary changes of basis.

To begin, let V_0 be the subspace of $\mathbb{R}^n$ spanned by q_0, p_0, f_1 (which are necessarily linearly independent by virtue of being in the stratum S_3), and choose an orthonormal basis of V_0 consisting of f_1 and two further vectors. Finally complete this to an orthonormal basis $\{f_1, \dots, f_n\}$ of $\mathbb{R}^n$; being orthogonal to f_1, the remaining vectors will all be principal directions. The change of basis matrix $P_b \in G_R = O(n-1)$ maps V_0 to $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^n$. Then q_0, p_0 has the form given in (4.4).

For the matrices, define now a basis in space as follows: let $V_0' = gV_0$. Clearly $e_1 \in V_0'$ since $q_0 \in V_0$. Extend $\{e_1\}$ to an orthonormal basis of V_0' and then further extend to $\mathbb{R}^n$ by putting $e_j = gf_j$ for $j > 3$. Let $P_s \in G_L = O(n-1)$ be the change of basis matrix. This maps V_0' to $\mathbb{R}^3 \times \{0\}$.

With this choice of basis, q_0, p_0, g_0 are of the form given in (4.4). It remains to show the same is true of Ω and then that the space is fixed by the dynamics. Now, from Equation (3.6), $\Omega = q \wedge \dot{q}$ and under the choice of basis transforms to $P_b \Omega P_b^T = (P_b q) \wedge (P_b \dot{q})$. Since from the equations of motion (3.8), $\dot{q} \in V_0$ it follows that $P_b \Omega P_b^T$ is of the required form since both $P_b q$ and $P_b \dot{q}$ belong to $\mathbb{R}^3 \times \{0\}$.

We need now to show that the space of such matrices and vectors is invariant under the dynamics. There are two ways to do this, either a symmetry argument or an inspection of the equations of motion. For the symmetry argument, first consider the $G_R = O(n-1)$ action on T^*S^{n-1} (Corollary 3.2) and note that V_0 is the fixed point set of $O(n-3) \subset O(n-1)$. It follows that $(q(t), p(t))$ is fixed by $O(n-3)$ for all t, and hence that it evolves on $(V_0 \times V_0) \cap T^*S^{n-1} = T^*S^2$. Secondly, consider the action of $G = G_L \times G_R = O(n-1) \times O(n-1)$ on the pair of matrices $(g, \Omega) \in D$ (Equation (2.10)).

Similar to the previous argument, the set matrix pairs (g, Ω) given in (4.4) is the fixed point set of the diagonal subgroup

$$O(n-3)_\Delta = \{(k,k) \in G \mid k \in O(n-3)\},$$

and hence that too is invariant under the dynamics.

From the equations of motion (2.7) and (3.8) it is easy to see that the resulting equations of motion on these fixed point sets are precisely those arising from the 3D axisymmetric Veselova top with the given mass tensor. $\square$

It follows from this theorem that in order to study the dynamics of the axisymmetric n-dimensional Veselova top, it suffices to reconstruct the dynamics for the 3D one. And because of the reconstruction formula (3.10) given in Proposition 3.6, each motion on T^*S^2 lifts to a similar motion on D. Recall that equatorial and axial planes are defined just prior to Proposition 4.2.

Theorem 4.4. *Consider a solution on $\mathcal{R}$ of the reduced 3-dimensional axisymmetric Veselova top, with mass tensor $\mathbb{J} = \mathrm{diag}[J_1, J_2, J_2]$. The possible reconstructions to both T^*S^2 and D are given in the following table,*

stratum	reconstruction
S_3, with $(q_1, p_1) \neq 0$	quasiperiodic motion on 2-tori
$(0, 0, P) \in S_3$	steady rotations in equatorial plane
S_2 with $P > 0$	steady rotations in axial plane
all remaining points	equilibrium

Proof. The proofs of the statements in the second and third rows of the table were given in Proposition 4.2.

For the first row, the solutions through these points in the second reduced space $\mathcal{R}$ are periodic. Moreover, the action of $G_R = O(2)$ on T^*S^2 restricts to a free action on S_3. Since $O(2)$ has rank 1, Field's theorem (Theorem A.1) implies that the motion is quasi-periodic of tori of dimension two on T^*S^2.

In the remaining case $P = 0$ in which case $\Omega = 0$ and the kinetic energy vanishes. Since there is no potential, there is no motion. $\square$

Remark 4.5. We note the application of Field's theorem to the reconstruction of orbits on S_3 to D proves that the motion is quasi-periodic on invariant 3-tori since the group $G = G_L \times G_R = O(2) \times O(2)$ has rank 2. Theorem 4.4 instead shows that the dynamics takes place in a finer foliation by invariant 2-dimensional sub-tori. A physical interpretation of this phenomenon is given in Section 4.3 below. This scenario, where the invariant tori on T^*S^{n-1} and on D have the same dimension, also holds for general non-physical inertia tensors satisfying (1.2) (see [12, Section 7]).

The energy-momentum map   The global description of the dynamics of the n-dimensional axisymmetric Veselova top may now be conveniently illustrated by looking at the image of the energy-momentum map (P, H). We assume that $J_1 < J_2$ (the opposite inequality leads to a similar description). Using that $p_1^2 \leq P(1 - q_1^2)$, it is straightforward to show that $P/4J_2 \leq H(q,p) \leq P/2(J_1 + J_2)$. Therefore, the image of the energy-momentum map (P, H) is the wedge shown in Figure 4.2, see also Proposition 3.10.

Moreover, for $(P, H) \neq (0, 0)$, the equalities $H = P/2(J_1 + J_2)$ and $H = P/4J_2$ are respectively attained on S_2 and along the points of S_3 where $q_1 = p_1 = 0$ and $P > 0$. Therefore, combining the conclusion of Theorem 4.3 with the different items of Theorem 4.4, we obtain:
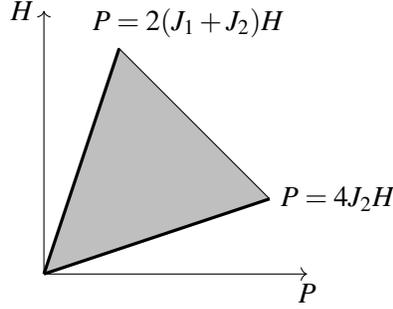
Figure 4.2: Image of the energy-momentum map for the axisymmetric Veselova top, assuming $J_1 < J_2$; see Figure 3.1

(i) The pre-image of the points in the interior of the wedge (the regular values of the map) is foliated by invariant tori of dimension 2 on both T^*S^{n-1} and D that carry quasiperiodic flow.

(ii) The pre-image of the lower boundary line $H = P/4J_2$, $P > 0$, consists of steady rotations in equatorial principal planes of inertia.

(iii) The pre-image of the upper boundary line $H = P/2(J_1 + J_2)$, $P > 0$, consists of steady rotations in axial principal planes of inertia.

(iv) The pre-image of the vertex of the wedge consists of all the equilibria both on D and on T^*S^{n-1}.

4.3   Physical description of the dynamics

The dynamics of the n-dimensional axisymmetric Veselova top is now easily described. In view of Theorem 4.3, we restrict our attention to the 3D case. Consider a solution of the system with $H, P \neq 0$. Without loss of generality, we may choose the second axis e_2 of the space frame to be aligned with the space representation of the momentum vector $p(t)$. The attitude matrix $g(t)$ is then given by Equation (3.10) with $g_0 = \mathrm{Id}_3$. It follows that the axis of symmetry of the body, as seen in space, traces the following path along the motion:

$$g(t)f_1 = \left(q_1(t), \frac{p_1(t)}{\sqrt{P}}, \pm\sqrt{L(t)} \right)^T,$$

where $L(t) = 1 - q_1(t)^2 - \frac{p_1(t)^2}{\sqrt{P}}$. The above is a closed curve in space since all entries of the above vector are periodic with the same frequency.

Let us focus on the case where the solution is quasi-periodic on a 2-torus (see Theorem 4.4). Then $0 < L(t) < 1$ and, as follows from the conservation of energy (4.3), the axis of the body rotates about the e_3 axis of the space frame along the curve given as the intersection of the cylinder

$$\left\{ (x_1, x_2, x_3) \,\middle|\, x_1^2 + \frac{P}{2H(J_1 + J_2)} x_2^2 = \frac{4HJ_2 - P}{2H(J_2 - J_1)} \right\}$$

with the unit sphere. Under our assumptions, the connected components of this curve are not contained on a plane with constant x_3. Therefore, apart from the periodic precession of the body about the e_3-axis, the body undergoes a periodic nutation with half its period (see Figure 4.3). Apart from this
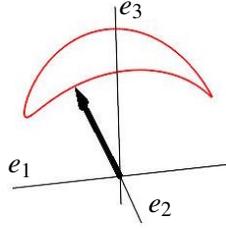
Figure 4.3: Periodic motion of the axis of symmetry of the top. The distinguished axis that defines the nonholonomic constraint is e_1. The axis e_2 is aligned with the space representation of the momentum vector $p(t)$.

periodic translational motion, the body rotates about its symmetry axis at an angular speed that is compatible with the nonholonomic constraint.

Note that even though the system has the same symmetry group as the classical Lagrange top, whose motions are generically quasi-periodic with three frequencies (see e.g. [1]), the motion of the axisymmetric Veselova top only has two independent frequencies since the nutation and precession motions of the body are commensurate. This gives a physical interpretation of Remark 4.5.

5   The cylindrical Veselova top

We now consider the more general case of **cylindrical tops**: that is, we assume

$$\mathbb{J} = \mathrm{diag}[J_1, \ldots, J_1, J_2, \ldots, J_2], \tag{5.1}$$

with $J_1 \neq J_2$. Let r be the multiplicity of eigenvalue J_1, and $r' = n - r$ the multiplicity of J_2, and we may assume $2 \leq r \leq r'$ (the case $r = 1$ was treated in the previous section). In particular the dimension n of the body must be at least 4.

The symmetry group G_R defined by Equation (2.9) is $G_R := \mathrm{O}(r) \times \mathrm{O}(r') \subset \mathrm{O}(n)$. Indeed, if

$$h = \begin{bmatrix} h_{11} & 0 \\ 0 & h_{22} \end{bmatrix},$$

with $h_{11} \in \mathrm{O}(r)$ and $h_{22} \in \mathrm{O}(r')$, then $h\mathbb{J}h^T = \mathbb{J}$. Corresponding to the splitting $\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{r'}$, write $q = (q_1, q_2)$ and $p = (p_1, p_2)$. Then, the action of $h \in G_R$ on $(q, p) \in T^*\mathrm{S}^{n-1}$ (see Corollary 3.2) is

$$hq = (h_{11}q_1, h_{22}q_2)$$

and similarly for p.

5.1   Second reduction

As for the axisymmetric tops, denote the **second reduced space** $T^*\mathrm{S}^{n-1}/G_R$ by $\mathcal{R}$. To determine $\mathcal{R}$ for a cylindrical body, we find the invariant functions for the G_R action on $T^*\mathrm{S}^{n-1}$. As mentioned before, the ring of invariants for $\mathrm{O}(k)$ acting on $(q, p) \in \mathbb{R}^k \times \mathbb{R}^k$ is generated by three independent quadratic expressions:

$$\|q\|^2, \quad \|p\|^2, \quad \text{and} \quad q \cdot p.$$

By the Cauchy-Schwarz inequality, these satisfy $|q \cdot p|^2 \leq \|q\|^2 \|p\|^2$.

It follows that the ring of invariants for $G_R = O(r) \times O(r')$ acting on $(q_1, q_2, p_1, p_2) \in \mathbb{R}^r \times \mathbb{R}^{r'} \times \mathbb{R}^r \times \mathbb{R}^{r'}$ is generated by the 6 invariants

$$\|q_1\|^2, \quad \|p_1\|^2, \quad q_1 \cdot p_1, \quad \|q_2\|^2, \quad \|p_2\|^2, \quad q_2 \cdot p_2.$$

We now restrict to $T^* S^{n-1}$ by requiring $\|q\|^2 = \|q_1\|^2 + \|q_2\|^2 = 1$ and $q \cdot p = q_1 \cdot p_1 + q_2 \cdot p_2 = 0$. In place of 6 invariants, we now have 4:

$$A = \|q_1\|^2, \quad B = \|p_1\|^2, \quad P = \|p_1\|^2 + \|p_2\|^2, \quad \text{and} \quad D = p_1 \cdot q_1. \tag{5.2}$$

The remaining invariants are given simply by $\|q_2\|^2 = 1 - A$, $\|p_2\|^2 = P - B$, and $q_2 \cdot p_2 = -D$. The reason for using P rather than $\|p_2\|^2$ is that P is conserved by the dynamics. (Note that D refers both to this variable as well as the distribution in $TO(n)$—we hope no confusion will be caused.)

Proposition 5.1. *The second reduced space $\mathcal{R}$ is the semialgebraic variety given by the set consisting of those $(A, B, P, D) \in \mathbb{R}^4$ satisfying*

$$0 \le A \le 1, \quad 0 \le B \le P, \quad D^2 \le AB, \quad D^2 \le (1 - A)(P - B).$$

This reduced space $\mathcal{R}$ has a natural stratification arising from the isotropy groups for the action of G_R on $T^* S^{n-1}$. These subgroups and the associated strata, both in $T^* S^{n-1}$ and their images in $\mathcal{R}$, are given in Table 5.1.

A section of $\mathcal{R}$ with constant $P > 0$ and the adjacencies between the strata are shown in Figure 5.1.

stratum	the below are zero	ISG	equations of stratum
		$T^* S^{n-1}$	$\mathcal{R}$
S_0	q_1, p_1, p_2	$O(r) \times O(r' - 1)$	$A = B = P = D = 0$
S_0'	q_2, p_1, p_2	$O(r - 1) \times O(r')$	$A = 1, B = P = D = 0$
S_1	q_1, p_1	$O(r) \times O(r' - 2)$	$A = B = D = 0$
S_1'	q_2, p_2	$O(r - 2) \times O(r')$	$A = 1, P - B = D = 0$
S_2	$\begin{cases} q_1 \wedge p_1 \\ q_2 \wedge p_2 \end{cases}$	$O(r - 1) \times O(r' - 1)$	$\begin{cases} D^2 = AB \\ \quad = (1 - A)(P - B) \end{cases}$
S_3	$q_1 \wedge p_1$	$O(r - 1) \times O(r' - 2)$	$D^2 = AB$
S_3'	$q_2 \wedge p_2$	$O(r - 2) \times O(r' - 1)$	$D^2 = (1 - A)(P - B)$
S_4	–	$O(r - 2) \times O(r' - 2)$	the rest

Table 5.1: Strata for the action of $G_R = O(r) \times O(r')$ on $T^* S^{n-1}$, valid for all $r, r' \ge 2$ with $r + r' = n$. Here $O(1) = \mathbb{Z}_2$ and $O(0)$ is the trivial group. See Proposition 5.1. The notation is such that $\dim S_k = \dim S_k' = k$ in $\mathcal{R}$.

Proof. The inequalities arise from the definition of A and the Cauchy-Schwarz inequality.

For the stratification and isotropy subgroups, consider first the natural action of $O(k)$ on $x = (a, b) \in \mathbb{R}^k \times \mathbb{R}^k$, with $k \ge 2$. Clearly, the stabilizer of the point $x = (0, 0)$ is $O(k)$, and that is the only

point fixed by the whole group. Next, if a and b are parallel ($a \wedge b = 0$) and not both zero, then the stabilizer is (conjugate to) $O(k-1)$ (if $k = 2$ then $O(1) = \mathbb{Z}_2$ is generated by the reflection in the line containing a, b). Finally, if a, b are linearly independent then the stabilizer is $O(k-2)$, which is the group of all orthogonal transformations in the subspace orthogonal to the space spanned by a, b. (For the case of $k = 2$, the group $O(0)$ is the trivial group.)

These observations can now be applied to the action of $G_R = O(r) \times O(r')$ on $\mathbb{R}^r \times \mathbb{R}^{r'} \times \mathbb{R}^r \times \mathbb{R}^{r'}$ and its restriction to $T^* S^{n-1}$ to obtain the strata in Table 5.1. □
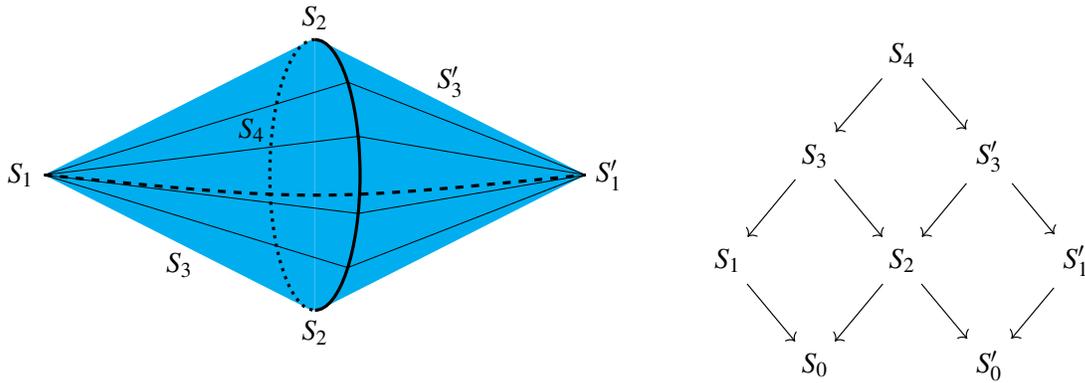

Figure 5.1: On the left, a section at constant $P > 0$ of the reduced space $\mathcal{R}$ for the cylindrical Veselova top; the S_j label the strata; it is a double solid cone, meeting at the 'equator' which is the stratum S_2. The dashed curve shows the (relative) equilibria in the open stratum S_4 (the interior of the cone). As P decreases, the cone becomes 'thinner' and in the limit when $P = 0$ it becomes a line segment (the A-axis in $\mathcal{R}$). On the right is shown the adjacencies between the strata, where $X \longrightarrow Y$ means $Y \subset \overline{X}$.

Description of strata Here we give a brief description of the strata, in particular in the constraint distribution D; the statements about dynamics are proved in Proposition 5.6 and Theorem 5.8.

S_0, S_0': These points in $\mathcal{R}$ correspond to points in $T^* S^{n-1}$ where $p = 0$, and either $q_1 = 0$ or $q_2 = 0$ respectively. This means the top is oriented in space in such a way that the distinguished axis is contained in one of the eigenspaces of $\mathbb{J}$, and the top is stationary. These strata do not appear in the cone in Figure 5.1 since the cone represents a section of $\mathcal{R}$ with $P > 0$.

S_1, S_1': Here the top is such that again the distinguished axis lies in one of the eigenspaces of $\mathbb{J}$; we will see below that the corresponding motion is necessarily a steady rotation (of *pure type*; see below). These strata correspond to the vertices of the cones in Figure 5.1.

S_2: The configuration is arbitrary, but we show below the motion is a steady rotation, here of *mixed type* (see below). In Figure 5.1 the stratum S_2 is the circle where the two cones meet.

S_3, S_3': In Figure 5.1 these strata are the boundaries of the two solid cones. We show below that in fact the motion on these strata is 'isomorphic' to the motion of an axisymmetric top.

S_4: In the figure, this corresponds to the interior of the solid cone. This stratum contains the 'genuine' general motion of the cylindrical top, not reducible to other cases. For generic points, the

solutions are quasiperiodic on tori of dimension 4; there are also some relative equilibria where the tori are of dimension 3, represented by the dashed curve in the figure.

Proposition 5.2. *The reduced equations of motion for the cylindrical Veselova top on $\mathcal{R}$ are,*

$$\begin{cases} \dot{A} = \dfrac{2D}{J_1 + J_2}, \\[2mm] \dot{B} = -4HD, \\[2mm] \dot{P} = 0, \\[2mm] \dot{D} = -4HA + \dfrac{P - 4J_2 H}{J_1 - J_2}. \end{cases} \tag{5.3}$$

Here

$$H = \frac{1}{2}\left((\beta_1 - \beta_2)B + \beta_2 P - \frac{(\beta_1 - \beta_2)^2 D^2}{\beta_1 A + \beta_2(1 - A)} \right), \tag{5.4}$$

which is an integral of motion, and

$$\beta_1 = ((J_1 - J_2)A + J_1 + J_2)^{-1}, \qquad \beta_2 = ((J_1 - J_2)A + 2J_2)^{-1}. \tag{5.5}$$

Proof. We derive these from the equations of motion given in Proposition 3.4. Under our assumption on $\mathbb{J}$ we have

$$q \cdot \mathbb{J}q = J_1 \|q_1\|^2 + J_2 \|q_2\|^2 = J_1 A + J_2(1 - A).$$

It follows that the matrix C of (3.7) satisfies

$$C^{-1} = \begin{pmatrix} (J_1(1+A) + J_2(1-A))\,\mathrm{Id}_r & 0 \\ 0 & (J_1 A + J_2(2-A))\,\mathrm{Id}_{r'} \end{pmatrix}$$

Then $C = \begin{pmatrix} \beta_1\,\mathrm{Id}_r & 0 \\ 0 & \beta_2\,\mathrm{Id}_{r'} \end{pmatrix}$ where β_1, β_2 are given in (5.5). Using this notation, we find

$$\begin{aligned} q \cdot Cq &= \beta_1 A + \beta_2(1 - A) \\ p \cdot Cp &= \beta_1 B + \beta_2 C \\ q \cdot Cp &= (\beta_1 - \beta_2)D, \end{aligned}$$

and the Hamiltonian (3.9) is indeed given by (5.4).

Now,

$$\dot{A} = 2q_1 \cdot \dot{q}_1 = 2q_1 \cdot \left(\beta_1 p_1 - \frac{(\beta_1 - \beta_2)D}{\beta_1 A + \beta_2(1 - A)} \beta_1 q_1 \right) = 2\beta_1 D \left(1 - \frac{(\beta_1 - \beta_2)A}{\beta_1 A + \beta_2(1 - A)} \right).$$

and after some simple calculations one finds the expression for $\dot{A}$ given above.

More simple is,

$$\dot{B} = 2p_1 \cdot \dot{p}_1 = -4HD, \quad \dot{P} = 0,$$

and

$$\dot{D} = -2HA + \beta_1 B - \frac{\beta_1(\beta_1 - \beta_2)D^2}{\beta_1 A + \beta_2(1 - A)},$$

which can be shown to be equal to the required expression. □

Remark 5.3. The invariant measure for the general (first reduced) n-dimensional Veslova top given in (3.7) pushes forward to an invariant measure on $\mathcal{R}$ given simply by

$$\beta_1 \beta_2 \, \mathrm{d}A \, \mathrm{d}B \, \mathrm{d}P \, \mathrm{d}D.$$

Reduced dynamics   We can now describe the reduced dynamics on $\mathcal{R}$. Since P, H are constant, the equations of motion (5.3) are linear, and easy to integrate. First consider the equations for (A, D): the general solution is

$$A(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + A_*$$
$$D(t) = \frac{1}{2}(J_1 + J_2)\omega\left(-C_1 \sin(\omega t) + C_2 \cos(\omega t)\right), \tag{5.6}$$

where C_1, C_2 and A_* are arbitrary constants (if the constants are chosen so that the initial value lies within $\mathcal{R}$, then the entire solution must lie within $\mathcal{R}$). Here

$$\omega^2 = \frac{8h}{J_1 + J_2}, \tag{5.7}$$

where h is the constant value of H on the solution. Substituting into the differential equation for B shows,

$$B(t) = -2h(J_1 + J_2)\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right) + B_*. \tag{5.8}$$

The quantities A_* and B_* (depending on the initial conditions) are the respective mean values of A and B over one period. The mean value for D is $D_* = 0$.

Equilibria   From $\dot{A} = 0$ we require $D = 0$ which then implies $\dot{B} = 0$. There remains to consider the equation $\dot{D} = 0$.

- As already noted above, along the A-axis, the points are equilibria (these correspond to $p = 0$ so are the equilibria of the full system, and are the points where $H = 0$). The end points of this axis are the points on S_0 and S_0', while the other points belong to S_2.

- The strata S_0, S_0', S_1, S_1' consist entirely of equilibrium points (for example, on S_1, if $A = B = D = 0$ then $P = 4J_2 H$ showing $\dot{D} = 0$).

- More generally, equilibria occur with $H = h > 0$, if $A = A_0$ and $B = B_0$, where

$$A_0 = \frac{P - 4J_2 h}{4(J_1 - J_2)h}, \quad \text{and} \quad B_0 = \frac{(P - 4J_2 h)(P + 4J_1 h)}{8(J_1 - J_2)h}. \tag{5.9}$$

  If $P = 4J_2 h$ then $A_0 = B_0 = 0$ and the point lies in S_1; if $P = 4J_1 h$ then $A = 1, B = P$ and the point lies in S_1'. All other equilibrium points lie in S_4, and are illustrated by the dashed curve in $\mathcal{R}$ in Figure 5.1.

Theorem 5.4. *The motion on the reduced space $\mathcal{R}$ has the following properties.*

  (i) *It is integrable, with the 3 first integrals P, H and*

$$F := 2(J_1 + J_2)AH + B.$$

  (ii) *Each point with $H = 0$ is an equilibrium.*

  (iii) *For each (h, P) with $h > 0$ in the image of the energy-momentum map (see Figure 5.2) there is a unique equilibrium with $H = h$; this occurs at the point $(A, B, P, D) = (A_0, B_0, P, 0)$, where A_0 and B_0 are the expressions given in (5.9).*

(iv) All other solutions are periodic with period $\pi\sqrt{\frac{J_1+J_2}{2h}}$.

In particular the period of the periodic orbits given in (iv) depends only on the value of the energy; this is reminiscent of the period-energy relation in Hamiltonian systems, even though this system is not known to be Hamiltonian (not even Poisson-Hamilton systems in general satisfy this property).

Proof. (i) For P this is clear, for H we know this already, while for F it is a simple calculation:

$$\begin{aligned} \dot{F} &= 2(J_1+J_2)\dot{A}H+\dot{B} \\ &= 2H\left((J_1+J_2)\dot{A}-2D\right) = 0. \end{aligned}$$

(ii) The set of points with $H=0$ is the A-axis in $\mathcal{R}$, and we have already seen above that these are equilibria.

(iii) Solving the equation $\dot{D}=0$ with $H=h$ for A shows immediately that $A=A_0$. Putting $A=A_0$ the equation $H=h$ can easily be solved for B to find $B=B_0$.

(iv) If a point is not an equilibrium then it lies on a periodic orbit, as shown in (5.6) and (5.8), with period $2\pi/\omega$ with ω given in (5.7). $\qquad\square$

Conclusion: In $\mathcal{R}$ each invariant set given by fixing values of H and P is a surface bounded by the intersection with a circle in the boundary of the cone in Figure 5.1, with a unique equilibrium point surrounded by periodic orbits all of the same period.

Remark 5.5. Since the dynamics on $\mathcal{R}$ is entirely periodic, one may find Poisson structures of rank 2 for which the system is Poisson-Hamiltonian, with Hamiltonian H, and with Casimir functions P and F [7]. It would be interesting to know if any of these Poisson structures arises as the reduction of an almost Poisson structure on D as described in [17]. This possibility to express the dynamics on $\mathcal{R}$ in Hamiltonian form contrasts with the dynamics on T^*S^{n-1}, c.f. Remark 3.8.

Steady rotations It follows from Proposition 3.10 that there are three critical rays in the image of the momentum map; namely where $P=4J_1H$, $P=2(J_1+J_2)H$ and $P=4J_2H$. The image of the energy-momentum map is therefore as shown in Figure 5.2.
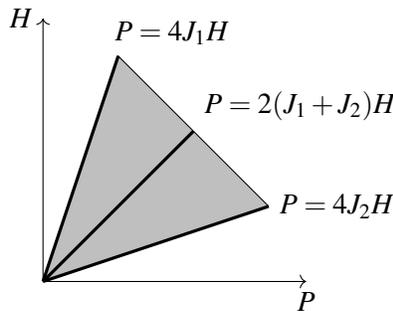

Figure 5.2: Image and critical rays (see Proposition 3.10) of the energy-momentum map for the cylindrical Veselova top, assuming $J_1 < J_2$

Each critical ray in the image of the momentum map corresponds to a type of steady rotation of the cylindrical Veselova top:

- Any plane Π contained in the r-dimensional J_1-eigenspace of $\mathbb{J}$ is a principal plane of inertia. Such a plane Π satisfies $\text{in}(\Pi) = 2J_1$.

- Any plane Π contained in the r'-dimensional J_2-eigenspace of $\mathbb{J}$ is a principal plane of inertia. Such a plane Π satisfies $\text{in}(\Pi) = 2J_2$.

- Let $\mathbb{J}a = J_1 a$ and $\mathbb{J}b = J_2 b$, with a, b non-zero. The plane $\Pi_{a,b} = \mathbb{R}\{a,b\}$ is a principal plane in the body with moment of inertia $\text{in}(\Pi) = J_1 + J_2$.

The inertial planes of the first two types we call ***pure inertial planes***, while those of the third type we call ***mixed inertial planes***. The steady rotations in pure inertial planes are relative equilibria, while the ones in mixed inertial planes are not (see Theorem 5.8).

Analogous to Proposition 4.2, we have:

Proposition 5.6. *(i) Steady rotations of mixed type are contained in S_2, and conversely, all points in S_2 with $P > 0$ correspond to such steady rotations.*

 (ii) The steady rotations in a pure inertia plane associated to J_1 are contained in S'_1, and conversely all points in S'_1 correspond to such steady rotations. The same statement holds replacing J_1 by J_2 and S'_1 by S_1.

 (iii) The pure steady rotations are Lyapunov stable relative to the action of G_R.

Proof. The proof is a straightforward adaptation of the proof of Proposition 4.2. For example, let us show that all points in S_2 with $P > 0$ correspond to steady rotations of mixed type. Recall from Table 5.1 that along this stratum q_j is parallel to p_j, $j = 1, 2$. In view of the cylindrical symmetry of $\mathbb{J}$, Equations (3.8) then imply that $\dot{q}_j$ is parallel to q_j. Writing $\dot{q}_j = \lambda_j q_j$ we obtain

$$\Omega = q \wedge \dot{q} = ((q_1, 0) + (0, q_2)) \wedge (\lambda_1(q_1, 0) + \lambda_2(0, q_2)) = (\lambda_2 - \lambda_1)(q_1, 0) \wedge (0, q_2).$$

The cylindrical symmetry of $\mathbb{J}$ now implies that $\mathbb{I}(\Omega) = (J_1 + J_2)\Omega$ so $[\mathbb{I}\Omega, \Omega] = 0$ which, by Proposition 2.1, implies that the solution is a steady rotation, that is clearly of mixed type. (Note that the condition $P > 0$ implies $\Omega \neq 0$.)

The details of the proof of the other statements are left to the reader. The final statement follows from Theorem 3.12.   □

5.2   Reconstruction to T^*S^{n-1} and D

Recall that a Veselova top with cylindrical symmetry is necessarily of dimension at least 4. The analogous reduction to 3D for the axisymmetric top in this case is a reduction to 4D.

Theorem 5.7. *Consider the n-dimensional cylindrical Veselova top, with $n > 4$ and mass tensor $\mathbb{J}$ given in (5.1). Consider any solution to the reduced equation on $\mathcal{R}$, and a choice of initial lift to both T^*S^{n-1} and D. Then there exists choices of basis of $\mathbb{R}^n$ in the body and in space, such that*

$$q(t) = \begin{pmatrix} \bar{q}(t) \\ 0 \end{pmatrix}, \quad p(t) = \begin{pmatrix} \bar{p}(t) \\ 0 \end{pmatrix}, \quad g(t) = \begin{pmatrix} \bar{g}(t) & 0 \\ 0 & \text{Id}_{n-4} \end{pmatrix}, \quad \Omega(t) = \begin{pmatrix} \bar{\Omega}(t) & 0 \\ 0 & 0_{n-4} \end{pmatrix}, \quad (5.10)$$

where $(\bar{q}(t), \bar{p}(t)) \in T^*S^3$ and $(\bar{g}(t), \bar{\Omega}(t)) \in D(4) \subset TO(4)$ satisfy the equations of motion of the axisymmetric 4D Veselova top (for $(\bar{g}, \bar{\Omega})$) and its reduced equations (for $(\bar{q}, \bar{p})$), with mass tensor $\bar{\mathbb{J}} = \text{diag}[J_1, J_1, J_2, J_2]$.

Proof. This is similar to the proof of Theorem 4.3, and we highlight the differences. Here we consider a point in the stratum S_4. Let $V = V_1 \oplus V_2$ be the 4-dimensional subspace of $\mathbb{R}^n$, where

$$V_1 = \mathbb{R} \left\{ \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1 \\ 0 \end{pmatrix} \right\}, \quad \text{and} \quad V_2 = \mathbb{R} \left\{ \begin{pmatrix} 0 \\ q_2 \end{pmatrix}, \begin{pmatrix} 0 \\ p_2 \end{pmatrix} \right\}.$$

The assumption that the initial point is in S_4 implies these 4 vectors are linearly independent. We now construct an orthonormal body basis $\{f_1, \dots, f_n\}$ as follows. First choose $\{f_1, f_2\}$ and $\{f_3, f_4\}$ as orthonormal bases of V_1 and V_2 respectively. Next choose $\{f_5, \dots, f_{r+2}\}$ to be an orthonormal basis of the complement of V_1 in $\mathbb{R}^r \times \{0\}$ and the remaining vectors $\{f_{r+3}, \dots, f_n\}$ to be a basis of the complement of V_2 in $\{0\} \times \mathbb{R}^{r'}$.

With this choice of body frame, q_0 and p_0 are of the form given in (5.10). However, contrary to the situation encountered in the proof of Theorem 4.3, the corresponding change of basis matrix $P_b \notin G_R$. With the new choice of body frame, the mass tensor $\mathbb{J}$ transforms to

$$\mathbb{J}' = \mathrm{diag}[J_1, J_1, J_2, J_2, \underbrace{J_1, \dots, J_1}_{r-2}, \underbrace{J_2, \dots, J_2}_{r'-2}].$$

Now, for the space frame, $e_1 = g_0 q$ so is contained in the image $g_0(V)$, and complete $\{e_1\}$ to an orthonormal basis of $g_0(V)$. Finally let $e_j = g_0 f_j$ for $j > 4$. The change of basis matrix $P_s \in G_L$ and the matrices g_0 and Ω_0 are in the forms given in (5.10).

To conclude invariance of the set (5.10) under the dynamics we apply a symmetry argument as in the proof of Theorem 4.3. This time we consider the group H with elements:

$$h = \begin{bmatrix} \mathrm{Id}_4 & 0 & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & h_{22} \end{bmatrix}, \qquad h_{11} \in \mathrm{O}(r-2), \quad h_{22} \in \mathrm{O}(r'-2).$$

Note that for $h \in H$ we clearly have $h^T \mathbb{J}' h = \mathbb{J}'$ so $H \subset G_R$. Similar to the argument given in the proof of Theorem 4.3, all vectors (q, p) in (5.10) are fixed by the H action on $T^* S^{n-1}$ and the matrices (g, Ω) are fixed by the action of the diagonal subgroup H_Δ on D where

$$H_\Delta = \{(h, h) \in G = G_L \times G_R \, | \, h \in H\}.$$

It can readily be seen that the equations of motion on these fixed point subspaces is the same as those of the 4D cylindrical Veselova top. $\square$

We now describe the dynamics of the cylindrical Veselova top in D, as well as its first reduction in $T^* S^{n-1}$. Since in $\mathcal{R}$ the motion is entirely periodic, the lifted motions are known as *relative periodic orbits*. We take advantage of the previous theorem and assume, without loss of generality, that $n = 4$.

Theorem 5.8. *The reconstruction to $T^* S^3$ and to D of the solutions of the reduced equations are as follows.*

(i) *In S_0, S_0' the solutions are equilibria.*

(ii) *In S_1, S_1', the relative equilibria are steady rotations in pure inertial planes (for S_1 they rotate in a principal plane Π with $\mathrm{in}(\Pi) = 2J_2$ while in S_1' it has $\mathrm{in}(\Pi) = 2J_1$).*

(iii) *In S_2 we have two following possibilities. If $P = 0$ the top is in equilibrium, whereas if $P > 0$, the solutions are steady rotations in mixed inertial planes, and they are not relative equilibria.*

(iv) In S_3, S'_3 the motion lies on tori of dimension at most 2 in both T^*S^3 and D and behaves in the same way as the 3D axisymmetric Veselova top.

(v) In S_4 we have two possibilities. In the (relative) equilibria in S_4 the motion lies on tori of dimension at most 2 in T^*S^3 and at most 3 in D. These relative equilibria are G_R-Lyapunov stable. Finally, over the periodic orbits in S_4 the motion lies on tori of dimension at most 3 in T^*S^3 and at most 4 in D.

Proof. (i) S_0 and S'_0 have $p = 0$ and hence are equilibria.

(ii) and (iii): the proofs are entirely analogous to the proof of Proposition 4.2 (the case where $P = 0$ in (iii) is not, but is immediate).

(iv) Suppose $(q, p) \in S_3$ (S'_3 is similar). Then their projections q_1 and p_1 are parallel, say to a vector u, while q_2 and p_2 are linearly independent. The line $\mathbb{R}u$ is the fixed point set of the isotropy group $O(1) = \mathbb{Z}_2$, and hence the dynamics will take place on the 3-dimensional subspace spanned by u, q_2 and p_2. The resulting equations of motion are precisely those of the 3-dimensional axisymmetric top with $\mathbb{J} = \text{diag}[J_1, J_2, J_2]$, and the conclusions about the invariant tori are to be found in Theorem 4.4.

(v) Denote by $\widehat{S}_4$ the stratum in T^*S^3 corresponding to the stratum S_4 in $\mathcal{R}$. It follows from Table 5.1 that the $G_R = O(2) \times O(2)$ action on $\widehat{S}_4$ is free. The conclusion about the dynamics on T^*S^3 then follows from Field's theorem (Theorem A.1) since $O(2) \times O(2)$ has rank 2. Similarly, if $\widehat{\widehat{S}}_4$ denotes the corresponding stratum on D, then the action of $G = G_L \times G_R = O(3) \times O(2) \times O(2)$ on $\widehat{\widehat{S}}_4$ is free and the conclusion follows from the same theorem (since G has rank 3). The stability statement follows from the Lyapunov stability of the equilibria on $\mathcal{R}$. $\qquad\square$

Note that the above theorem only gives upper bounds for the dimension of the invariant tori corresponding to the reconstruction of the solutions on the stratum S_4. We will now present some evidence to show that these bounds are generically attained and the dynamics does not take place in lower dimensional sub-tori (compare with Remark 4.5). We begin with the following proposition that considers the reconstruction of the non-trivial relative equilibria in S_4.

Proposition 5.9. *Let (h, P) with $h > 0$ be a point in the image of the energy-momentum map (see Figure 5.2) and consider the corresponding relative equilibrium described by item (iii) of Theorem 5.4. A reconstruction of this solution to T^*S^3 and to D is respectively determined by:*

$$q(t) = \left(\sqrt{A_0} \cos \omega_1 t, -\sqrt{A_0} \sin \omega_1 t, \sqrt{1 - A_0} \cos \omega_2 t, -\sqrt{1 - A_0} \sin \omega_2 t \right)^T,$$

$$p(t) = \left(-\sqrt{B_0} \sin \omega_1 t, -\sqrt{B_0} \cos \omega_1 t, -\sqrt{P - B_0} \sin \omega_2 t, -\sqrt{P - B_0} \cos \omega_2 t \right)^T,$$

and

$$g(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \omega_3 t & -\sin \omega_3 t \\ 0 & 0 & \sin \omega_3 t & \cos \omega_3 t \end{bmatrix} g_0 \begin{bmatrix} \cos \omega_1 t & -\sin \omega_1 t & 0 & 0 \\ \sin \omega_1 t & \cos \omega_1 t & 0 & 0 \\ 0 & 0 & \cos \omega_2 t & -\sin \omega_2 t \\ 0 & 0 & \sin \omega_2 t & \cos \omega_2 t \end{bmatrix},$$

where

$$g_0 = \begin{bmatrix} \sqrt{A_0} & 0 & \sqrt{1 - A_0} & 0 \\ 0 & \sqrt{B_0/P} & 0 & \sqrt{(P - B_0)/P} \\ \sqrt{1 - A_0} & 0 & -\sqrt{A_0} & 0 \\ 0 & \sqrt{(P - B_0)/P} & 0 & -\sqrt{B_0/P} \end{bmatrix}.$$

The frequencies in the above formulae are given by

$$\omega_1 = \frac{2\sqrt{2}h}{\sqrt{4J_1 h + P}}, \qquad \omega_2 = \frac{2\sqrt{2}h}{\sqrt{4J_2 h + P}}, \qquad \omega_3 = \frac{-4h\sqrt{P}}{\sqrt{4J_1 h + P}\sqrt{4J_2 h + P}}. \tag{5.11}$$

The proof for $T^*\mathrm{S}^3$ is a direct verification that $(q(t), p(t))$ is a solution of Equations (3.4). For D, one may compute $\Omega = g^{-1}\dot{g}$ and verify by a direct calculation that the nonholonomic constraints and relations such as (3.2) hold. For generic (h, P) the reconstruction frequencies (5.11) are non-resonant so these relative equilibria indeed reconstruct to motions on invariant 3-tori on D and on an invariant 2-tori on $T^*\mathrm{S}^3$.

Now consider the motion along a relative periodic orbit on $\mathcal{R}$ that is close to the relative equilibrium treated above. Both the reconstructed solutions on $T^*\mathrm{S}^3$ and on D are quasi-periodic. They will posses the natural frequency given by Equation (5.7) (that only depends on h) and reconstruction frequencies that are close to those given in (5.11). The joint set of these frequencies will generically be non-resonant and the motion will take place on a 4-torus on D and on a 3-torus on $T^*\mathrm{S}^3$.

6 Conclusions and open question

Our study shows that the dynamics of the axisymmetric and cylindrical Veselova tops is quasiperiodic in the natural time variable. More precisely, at the level of the first reduced space $D/G_L = T^*\mathrm{S}^{n-1}$ the flow possesses an invariant measure and:

(i) for the axisymmetric case (we may take $n = 3$), the generic motion is quasi-periodic on invariant 2-tori on $T^*\mathrm{S}^2$ and there are two integrals of motion H and P;

(ii) for the cylindrical case (we may take $n = 4$), the generic motion is quasi-periodic on invariant 3-tori on $T^*\mathrm{S}^3$ and there are three integrals of motion H, P and F.

These scenarios are remarkably similar to the situation found in Liouville-Arnold integrable Hamiltonian systems. We leave it as an open problem to find a Hamiltonian structure for these systems on these cotangent bundles (one that does not involve a time reparametrisation) or to identify an obstruction to its existence. The interest in this question arises in view of the great quantity of research that in recent years has been devoted to identifying the differences between nonholonomic and Hamiltonian systems.

A Field's theorem on reconstruction

We briefly recall the language and main properties of (compact) group actions and equivariant vector fields, needed for the proofs of quasiperiodicity.

Suppose a (compact Lie) group G acts smoothly on a manifold M. If two points lie in the same orbit, their isotropy subgroups are conjugate. This motivates the partition or stratification of M into "orbit types," where the orbit type stratum $M_{(H)}$ consists of those points with isotropy subgroup (ISG) conjugate to H, or more precisely a stratum is a connected component of such points. Moreover, if M_H is (a connected component of) the set of points with ISG equal to H, then, up to connected components, $M_{(H)} = G.M_H$ (the image of M_H under the group action). The orbit space M/G is similarly stratified by the orbit type; one denotes its corresponding stratum by $(M/G)_H$. All of these spaces $M_{(H)}, M_H$ and $(M/G)_H$ are manifolds.

If in addition there is an equivariant dynamical system on M, then it descends to a dynamical system on M/G which respects each stratum. The question of reconstruction is, if one knows some properties of the dynamics on M/G what does this imply about the dynamics on M. We use the following simplified version of an important theorem due to Field [14] (see also [15, Chapter 8] or [3]).

Theorem A.1 (Field [14])**.** *Consider a smooth equivariant dynamical system, with symmetry group G acting freely on the phase space M. The dynamics passes down to the smooth orbit space M/G.*

   (i) Let $x \in M/G$ be an equilibrium point of the reduced equations. Then the inverse image of x in M is a group orbit which is foliated by invariant tori of dimension at most the rank of G.

   (ii) Let γ be a periodic orbit of the reduced dynamics in M/G. Then the inverse image of this curve is also foliated by invariant tori, but now of dimension at most $\mathrm{rk}(G) + 1$.

In both cases the dynamics in the invariant tori is quasiperiodic.

In case (i) the dynamics on M is called a *relative equilibrium*, while in (ii) it is a a *relative periodic orbit*. If the dynamics on M/G include an invariant quasiperiodic torus, then it is unknown except in special cases, see [9], what the corresponding reconstructed dynamics may be. The more general version of Field's theorem does not require the action to be free, but this suffices for our purposes.

B   Physical inertia tensors satisfying the hypothesis of Fedorov-Jovanović

Recall that Fedorov and Jovanović [12, 13] work under the assumption that there is an $n \times n$ diagonal matrix A such that the inertia tensor $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ satisfies

$$\mathbb{I}(a \wedge b) = (Aa) \wedge (Ab), \quad (\forall a, b \in \mathbb{R}^n). \tag{B.1}$$

In this appendix we examine the feasibility of this condition within the family of physical inertia tensors. The following proposition, that was already suggested in [13, 21], shows that for $n \geq 4$ a physical inertia tensor that satisfies (B.1) necessarily corresponds to an axisymmetric rigid body.

Proposition B.1. *Let $\mathbb{I}$ be a physical inertia tensor defined by $\mathbb{I}(\Omega) = \mathbb{J}\Omega + \Omega\mathbb{J}$ (see (1.1)).*

   (i) If $n = 3$ then the identity (B.1) holds for arbitrary $a, b \in \mathbb{R}^3$ with

$$A_i = \sqrt{\frac{(J_i + J_j)(J_i + J_k)}{J_j + J_k}},$$

   for $\{i, j, k\} = \{1, 2, 3\}$, where $\mathbb{J} = \mathrm{diag}[J_1, J_2, J_3]$ (with $J_i + J_j > 0$ for $i \neq j$).

   (ii) If $n \geq 4$, there exists a diagonal matrix A satisfying (B.1) if and only if $\mathbb{I}$ is the inertia tensor of an axisymmetric body. In this case, the body frame can be chosen in such way that $\mathbb{J}$ is given by

$$\mathbb{J} = diag\left[A_1 A_2 - \tfrac{1}{2}A_2^2, \tfrac{1}{2}A_2^2, \ldots, \tfrac{1}{2}A_2^2\right] \tag{B.2}$$

   and $A = \mathrm{diag}[A_1, A_2, \ldots, A_2]$, where $A_1, A_2 \in \mathbb{R}$ satisfy $A_1 \geq A_2/2 > 0$.

Proof. (i) is a calculation.

(ii) "Only if": by selecting the body frame with f_1 parallel to the symmetry axis of the body we have $\mathbb{J} = (J_1, J_2, \ldots, J_2)$ with $J_1 \geq 0$ and $J_2 > 0$. The statement follows by putting $A_1 = (J_1 + J_2)/\sqrt{2J_2}$ and $A_2 = \sqrt{2J_2}$

"If": suppose that (2.3) holds and $\mathbb{I}$ is a physical inertia tensor with corresponding diagonal matrix $\mathbb{J}$. Then there are real numbers $A_1, \ldots, A_n$ such that for each $i \neq j$,

$$J_i + J_j = A_i A_j. \tag{E_{ij}}$$

We claim that if the J_i satisfy these conditions, then at least $n - 1$ of the A_i coincide. In that case suppose (w.l.o.g.) $A_2 = \cdots = A_n$. Then comparing the equations E_{1i} shows that all $J_1 + J_i$ coincide for $i > 1$, and hence $J_2 = \cdots = J_n$ as required.

To prove the claim, note that for any collection $\{i, j, k, \ell\}$ of 4 distinct indices,

$$E_{ij} + E_{ik} - E_{jk} \text{ becomes } 2J_i = A_i A_j + A_i A_k - A_j A_k,$$

and similarly

$$E_{ij} + E_{i\ell} - E_{j\ell} \text{ becomes } 2J_i = A_i A_j + A_i A_\ell - A_j A_\ell.$$

Subtracting these shows

$$(A_i - A_j)(A_k - A_\ell) = 0.$$

Suppose $J_1 \neq J_2$ and apply this to the indices $1, 2, i, j$ with $i, j > 2$. It follows that $A_3 = A_4 = \cdots = A_n$. Finally, if $A_1 = A_3$ we are done, while if $A_1 \neq A_3$ apply the previous reasoning to the indices $1, 3, 2, 4$ and one can conclude that $A_2 = A_4$, and hence indeed $A_2 = \cdots = A_n$. $\square$

The usefulness of a condition of type (B.1) for the developments in [12, 13] (also [21, 20]) seems to arise from the fact that an inertia tensor satisfying (B.1) maps the set of rank 2 matrices in $\mathfrak{so}(n)$ into itself. For interest, although we make no use of this, we show that for $n \geq 4$, the only physical inertia tensors with this property are axisymmetric.

Proposition B.2. *Let $\mathbb{I}$ be a physical inertia tensor defined by* (1.1). *If $n \geq 4$, then $\mathbb{I}$ maps the set of rank two matrices in $\mathfrak{so}(n)$ into itself if and only if the body is axisymmetric.*

Proof. Let us consider the case $n = 4$. Suppose $\mathbb{I}$ maps the space of rank two matrices in $\mathfrak{so}(4)$ into itself and that the body is not axisymmetric. Choose the body frame so that $\mathbb{J} = \text{diag}[J_1, \ldots, J_4]$ with $J_1 \neq J_3$, $J_2 \neq J_4$. A direct calculation shows that

$$\det(\mathbb{I}((f_1 + f_3) \wedge (f_2 + f_4))) = (J_1 - J_3)^2 (J_2 - J_4)^2 \neq 0,$$

which shows $\mathbb{I}((f_1 + f_3) \wedge (f_2 + f_4))$ has rank 4, reaching a contradiction. A similar argument shows this implication for $n > 4$.

The converse statement follows from part (ii) of the Proposition B.1. $\square$

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