# *Hénon map*

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# NL3150 **Hénon Map**

By the mid-1970s examples such as the Lorenz equations had convinced researchers that strange attractors could arise in differential equations modelling physical systems. Unfortunately, the length of time needed to compute solutions coupled with strong contraction rates made it very difficult to observe fractal structures numerically in these examples with the computers then available. The Hénon map provided the first simple equation in which this fractal structure is easily observed.

Michel Hénon's approach was based on the idea of return maps. The dynamics of differential equations can be modelled by invertible maps, and so evidence of fractal structure in the attractor of an invertible map shows that these objects can exist in differential equations. The Hénon map is a very simple nonlinear difference equation

$$\begin{aligned} x_{n+1} &= y_n + 1 - ax_n^2 \\ y_{n+1} &= bx_n \end{aligned} \tag{1}$$

where the parameters $a$ and $b$ were chosen as $a = 1.4$ and $b = 0.3$ by Hénon (1976), although other values are also interesting. The attractor, together with some blow-ups of parts of the attractor, are shown in Figure 1. These pictures are very easy to generate: successive points on an orbit are obtained by evaluating the algebraic expressions on the right hand side of (1) as the following rough MATLAB programme shows

```
x(1)=0.3; y(1)=0.2;        %% Initial conditions
N=5000;                    %% N is the number of iterates
a=1.4;b=0.3;               %% Sets the parameters
for i=1:N                  %% Begin the iteration loop here
x(i+1)=y(i)+1-a*x(i)^ 2;
y(i+1)=b*x(i);
end                        %% That calculated the next point
plot(x,y,'k.')             %% Plot the iterates in the (x,y) plane
```

Despite the simplicity of this programme, Hénon used a mainframe (IBM 7040) to perform the 5 million iterates he needed to get a reasonable number of points in the equivalent of Figure 1(d). Figure 1 uses a slightly more sophisticated programme to generate the attractor and zoom in on the rectangular regions indicated so that 5000 points can be plotted in each of the blow-up regions. This involved 35724657 iterations of the map order to get 5000 points in the smallest blow-up region of Figure 1(d). Even this level of computation would have been almost unthinkable when Hénon wrote his paper.

The Hénon attractor pictured in Figure 1 is computationally cheap, requiring no more than the most simple algebraic operations. Also, the numerical evidence for fractal structure in the attractor is sufficiently convincing that most researchers have come to accept that it is a strange attractor, or at least, to suspend their disbelief. For this reason

**Figure 1.** (a) Numerically computed attractor of the Hénon map, (1) with $a = 1.4$ and $b = 0.3$; (b), (c), and (d) are blow-ups of the boxed regions of (a), (b), and (c) respectively, showing the fractal structure of the attractor. Each figure contains the first 5000 points to land in the displayed region.

it has become a canonical example of chaotic motion. Almost every new technique or relevant theoretical result is applied to the Hénon map as part of the evaluation of the method. Early papers on phase space reconstruction, dimension calculations, chaotic prediction, chaotic control and synchronization, periodic orbit expansions, invariant measure algorithms etc. have all used the Hénon map as an important test example. Given this general level of acceptance, it may come as a surprise to learn that it is still not known whether there really is a strange attractor for the Hénon map at the standard parameter values ($a = 1.4$, $b = 0.3$).

Hénon (1976) gave a number of reasons looking at orbits of (1):

> ... we try to find a model problem which is as simple as possible, yet exhibits the same essential properties as the Lorenz system. Our aim is (i) to make the numerical exploration faster and more accurate....; (ii) to provide a model which might lend itself more easily to mathematical analysis.

As we have seen, Hénon's aim of making the numerical exploration of apparently chaotic attractors more straightforward succeeded spectacularly. However, he could not have imagined how hard it would be to answer the theoretical questions posed by this deceptively simple map.

Hénon's intuitive explanation for his map in terms of folding, stretching, and contraction is much closer to the formation of horseshoes rather than to the Lorenz model, which has discontinuities in the natural return map. As such, the Hénon map has become a paradigm for the formation of horseshoes as parameters vary (see Devaney & Nitecki (1979)), and it is the more general question of how the attractors of the Hénon map change as the parameters vary which has occupied most theoretical approaches.

By defining a new $y$ variable $y_{\text{new}} = b^{-1}y_{\text{old}}$, equation (1) can be written in the form of a more general, Hénon-like map:

$$\begin{aligned} x_{n+1} &= -\epsilon y_n + f_a(x_n) \\ y_{n+1} &= x_n \end{aligned} \tag{2}$$

where $\epsilon = -b$ and $f_a(x) = 1 - ax^2$ gives the Hénon map in the new coordinates. This formulation emphasizes the relationship between Hénon-like maps and one-dimensional maps: if $\epsilon = 0$ then the $x$ equation decouples and $x$ evolves according to the one-dimensional difference equation $x_{n+1} = f_a(x_n)$, which in the original case, (1), is just the standard quadratic family. The Jacobian of the map is $\epsilon$, so positive $\epsilon$ corresponds to orientation-preserving maps, which is more natural in the context of return maps, although this means that $b < 0$ in Hénon's original formulation, (1). Early efforts towards proving that strange attractors exist in the Hénon map concentrated on extending results for one-dimensional maps to the two-dimensional case with $\epsilon > 0$ small. On the negative side, Holmes & Whitley (1984) showed that however small $\epsilon$ is, some periodic orbits of the Hénon map appear in a different order to the order in which they appear in the quadratic family. On the positive side, Gambaudo, van Strien & Tresser (1989) showed that for sufficiently small $\epsilon > 0$ the first complete period-doubling cascade is associated with the original period two orbit.

The major breakthrough on the existence of strange attractors was made by Benedicks & Carleson (1991). Using delicate mathematical analysis they were able to show that if $\epsilon > 0$ is small enough and $a$ is close to $a = -2$ (the equivalent of $\mu = 4$ for the standard formulation of the quadratic map, $\mu x(1-x)$) then there is a positive measure of parameter values for which the Hénon map has a strange attractor. This result was generalized by Mora & Viana (1993) who showed that Hénon-like maps arise naturally near homclinic bifurcations of maps. It had long been recognized that these bifurcations occur in the Hénon map (see, for example, Holmes & Whitley (1984)) so this made it possible to deduce the existence of strange attractors at values of $\epsilon$ which are not small. Indeed, this important paper provides a method of proving the existence of strange attractors for a set of parameter values with positive measure in a wide variety of model systems. Despite all these advances, these results only prove that there exist such parameter values, they do not give methods for proving that a strange attractor exists at a *given* parameter value.

Two other avenues of research suggested by the one-dimensional, $\epsilon = 0$, limit in (2) have led to interesting developments. We have already noted that the one-dimensional order of periodic orbits is not preserved if $\epsilon > 0$. However, a beautiful theory of partial orders based on period and knot type has emerged, which shows that the existence of some periodic orbits implies the existence of some others in two-dimensional maps. See Boyland (1994) for more details. The second adaptation of one-dimensional approaches is based on the idea of the symbolic dynamics (or *kneading theory*) of unimodal maps. The main idea, introduced by Cvitanovic, Gunaratne & Procaccia (1988) and developed by de Carvalho (1999), is to produce symbolic models of the dynamics of the Hénon map by relating the dynamics to modifications of the full horseshoe. This is done by *pruning* the horseshoe, i.e. identifying regions of the horseshoe with dynamics that is not present in the Hénon map under consideration and judiciously removing these regions together with their images and preimages, leaving a pruned horseshoe which can still be accurately described.

Much of the current interest in the Hénon map involves the existence and construction of invariant measures for the attractors. This work should lead to a good statistical description of properties of orbits and averages along orbits. So, even now, this simple two-dimensional map with a single nonlinear term is motivating important questions in dynamical systems.

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*See also* Attractors; Bifurcations; Chaotic dynamics; Difference equations; Horseshoes and hyperbolicity in dynamical systems; Markov partition; One-dimensional maps; Routes to chaos; Sinai–Ruelle–Bowen measures; Symbolic dynamics

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