Explicit symmetry breaking and Hamiltonian systems

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EXPLICIT SYMMETRY BREAKING AND HAMILTONIAN SYSTEMS

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Contents

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The central topic of this thesis is the study of persistence of stationnary motion under explicit symmetry breaking perturbations in Hamiltonian systems. Explicit symmetry breaking occurs when a dynamical system having a certain symmetry group is perturbed in a way that the perturbation preserves only some symmetries of the original system. We give a geometric approach to study this phenomenon in the setting of equivariant Hamiltonian systems. A lower bound for the number of orbits of equilibria and orbits of relative equilibria which persist after a small perturbation is given. This bound is given in terms of the equivariant Lyusternik-Schnirelmann category of the group orbit. We also find a localization formula for this category in terms of the closed orbit-type strata. We show that this formula holds for topological spaces admitting a particular cover, made of tubular neighbourhoods of their minimal orbit-type strata. Finally we propose a construction of symplectic slices for subgroup actions.

Declaration

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INTRODUCTION

When we talk about symmetries, we either refer to the symmetry of a physical law (dynamical equations) or the symmetry of a physical state (solution of these equations). The *symmetry* or *symmetry group* of a physical law (or a physical state) is defined to be the group of transformations which leave these equations (or this solution) invariant. Although the symmetry of a system of dynamical equations is reflected into the range of solutions of these equations in the sense that two solutions are related to one another by an element of this group of transformations, there might be solutions which do not exhibit this symmetry. Such a solution has a lower symmetry group than the one of the dynamical equations from which it is originated. In a theoretical approach we say that the symmetry group of this physical state is a subgroup of the symmetry group of the physical law. This phenomenon is called *spontaneous symmetry breaking* and is widely studied in mathematical physics.

This thesis focuses on another notion of symmetry breaking called *explicit symmetry breaking*. We define it as a process of perturbing symmetric dynamical equations such that the resulting equations have a lower symmetry group. In fact any physical law observed in nature can be thought as a perturbation of a physical law having a bigger symmetry group. However the more symmetric a dynamical system is, the more simple its solutions are. In fact, complicated and interesting dynamical behaviours require low symmetry group. For example, Lauterbach et al. [11, 27, 36] show that some periodic solutions of an unperturbed dynamical system persist under symmetry breaking perturbations and become heteroclinic cycles.

The lack of symmetries of a perturbed system can be due for example to the presence of terms whose origin is different from case to case. As explained in

Brading and Castellani [8], such terms can be introduced artificially in order to match with theoretical or experimental observations. For example in quantum field theory, the Lagrangian for weak interactions is constructed so that the parity-symmetry and the charge-parity symmetry are violated, making the theory in the line with experimental observations. Besides, quantization processes might also be a cause for the appearance of such terms which are the so-called quantum anomalies. In this case, the terms are not artificially introduced but they appear after a renormalization procedure.

The dynamical systems studied in this thesis are Hamiltonian systems. While spontaneous symmetry breaking phenomena are discussed in many papers related to bifurcations theory, fewer papers treat the case of explicit symmetry breaking phenomena and many results holding for general dynamical systems need to be adapted to the Hamiltonian case. Some aspects are studied by several authors including Ambrosetti et al. [1], Grabsi, Montaldi and Ortega [23] and Gay-Balmaz and Tronci [22]. For what we are concerned, we are interested in the number of solutions of such systems which persist under explicit symmetry breaking perturbations. In the Hamiltonian formulation of classical mechanics, explicit symmetry breaking perturbations can arise as described above but they can also arise as metric perturbations. Morally, a Hamiltonian dynamical system is partially determined by a scalar function called the *Hamiltonian.* This function is regarded as the total energy of the system, which is generally the sum of the kinetic energy, defined by mean of a metric, and the potential energy. If the Hamiltonian is written in term of the kinetic energy only, an example of explicit symmetry breaking perturbation would be a metric perturbation so that the perturbed metric has not the same invariance properties as the original metric.

Phase spaces (space of positions and momenta) of Hamiltonian systems are symplectic manifolds and the symmetries of such systems are encoded into Lie group actions on those manifolds. A symplectic manifold is a smooth manifold M equipped with a non-degenerate closed two-form ω. A (proper) action of a (connected) Lie group G on M is canonical if it is smooth and it preserves ω. A class of canonical Lie group actions on symplectic manifolds are Hamiltonian. To those actions we can associate a Noether conserved quantity expressed in term of a momentum map $\Phi_G : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G.

This notion generalizes the notion of angular momentum in classical mechanics, when the phase space is $T^*\mathbb{R}^3$, acted on by the group of rotations $SO(3)$. By a Hamiltonian (proper) G-manifold, we mean a quadruple (M, ω, G, Φ_G) as described above, with G connected.

The dynamics is governed by a Hamiltonian h which is a G-invariant real-valued function defined on M. We refer to G as the symmetry group of the system. The non-degeneracy and the G-invariance of the symplectic form allow us to associate to h a G-equivariant vector field X_h, whose flow lines $\varphi_t(m)$ are the evolution maps. Together with M, they define a dynamical system and are solutions of the equations of motion $\dot{x} = X_h(x)$. We focus on specific solutions of these equations of motion, namely equilibria (fixed points under the dynamics) and relative equilibria (group orbits fixed under the dynamics). For example, for a good choice of Riemannian metric on the two-sphere S^2, the norm square of an element in a fiber of T^*S^2 defines an $SO(3)$-invariant Hamiltonian function, understood as the kinetic energy. The relative equilibria of the associated Hamiltonian system project to the great circles on S^2.

To study explicit symmetry breaking phenomena, we consider pertubations h_λ of h that are smooth in the real parameter λ, and invariant with respect to a closed subgroup $H \subset G$. Such perturbations are called H-perturbations. Under a specific non-degeneracy condition on a (relative) equilibrium of the unperturbed Hamiltonian h, there is a chance that this (relative) equilibrium persists under a small H-perturbation.

Section 4.1 is devoted to the question of persistence of equilibria. In this case, the required non-degeneracy condition on an equilibrium $m \in M$ of h is a particular case of Morse-Bott condition, when the critical manifold of h is the group orbit $G \cdot m$ (cf. Definition 4.1.1). We show that at least a certain number of H-orbits of equilibria persist under a small H-perturbation, in a tubular neighbourhood of $G \cdot m$ (cf. Theorem 4.1.2 and Corollary 4.1.3). This number is the positive integer $\mathrm{Cat}_H(G/G_m)$, which is the H-equivariant Lyusternik-Schnirelmann category of the group orbit. For technical reasons, our result requires the additional assumption that the set $H\backslash G$ of right cosets is compact, as a topological space.

The proof uses the well-known Symplectic Tube Theorem (cf. Theorem 2.4.1 [26, 40]), which states that there is a tubular neighbourhood of a group orbit which can be identified (by mean of a G-equivariant symplectomorphism) with a neighbourhood of the zero section of the normal bundle of this group orbit. This setting provides good (semi-global) coordinates on M, with variables along the group orbit and variables along the normal directions to the group orbit. An application of the Implicit Function Theorem and a Morse Lemma with parameters allows us to forget about the normal coordinates, reducing the proof to an application of the equivariant Lyusternik-Schnirelmann Theorem 3.4.5 on the group orbit. If G is compact, we do not require any assumption on the perturbation h_λ. However, for non-compact Lie groups, a compactness assumption on h_λ must be fulfilled in order to apply Theorem 3.4.5, as explained in Section 3.4.4.

We present applications of our result, including the problem of an ellipse-shaped planar rigid body moving in a planar irrotational, incompressible fluid with zero vorticity and zero circulation around the body (cf. Section 4.1.2). The reduced motion at zero vorticity and zero circulation is governed by Kirchhoff equations. Classical treatments of the problem can be found for example in [48, 35, 30, 68]. This problem turns out to exhibit symmetry breaking phenomena from different points of view. We can for example consider the body without the fluid. The fluid density is then understood as a "parameter". The $O(2)$-symmetry of the reduced Hamiltonian breaks into a D_2-symmetry when this parameter varies.

Extending Theorem 4.1.2 and Corollary 4.1.3 to the case of relative equilibria is a bit more challenging because we must take into account the conservation of momentum. This case is treated in Section 4.2. Whereas equilibria are just critical points of the Hamiltonian function h, relative equilibria are critical points of the restriction of this same function to a level set $\Phi_G^{-1}(\mu)$ of the momentum map. Let $m \in M$ be one of those critical points. The element $\xi \in \mathfrak{g}$ playing the role of a Lagrange multiplier is called the velocity of m, which is in general not unique when the action is not free. For that reason, we refer to a relative equilibrium as a pair $(m, \xi) \in M \times \mathfrak{g}$. We denote the underlying Lagrange function associated to ξ by h^ξ. A standard definition says that a relative equilibrium (m, ξ) of h is non-degenerate if the Hessian of h^ξ at m is a non-singular quadratic form when restricted to some symplectic subspace $N_1 \subset T_m M$, called the symplectic slice at m. If the perturbations h_λ are invariant with respect to the full symmetry group G, this notion of non-degeneracy is enough to guarantee the persistence of a relative equilibrium. This is no longer the case if h_λ has a smaller symmetry group than the one of h. In [23] a step in that direction is taken, when the symmetry group is a torus that breaks into a subtorus. In addition, all the group actions in consideration are assumed to be free. We extend their result to non-free actions and non-abelian symmetry groups.

A necessary condition for a relative equilibrium of h to persist under an H-perturbation is that the velocity ξ belongs to $\mathfrak{h}$, the Lie algebra of H. Another problem is that relative equilibria of an H-perturbation h_λ are critical points of the restriction of this same function to a level set $\Phi_H^{-1}(\alpha)$, where $\Phi_H : M \to \mathfrak{h}^*$ is the momentum map associated to the H-action on M. This momentum map can be constructed as the composition of the projection $i_\mathfrak{h}^* : \mathfrak{g}^* \to \mathfrak{h}^*$ with $\Phi_G : M \to \mathfrak{g}^*$, where $i_\mathfrak{h}^*$ is given by restriction of the linear forms on $\mathfrak{g}$ to the Lie subalgebra $\mathfrak{h}$. If μ is a regular value of the momentum map Φ_G and $\alpha = i_\mathfrak{h}^*(\mu)$, then the level set $\Phi_H^{-1}(\alpha)$ contains $\Phi_G^{-1}(\mu)$ as a submanifold. We show that

$$\ker\left(D\Phi_H(m)\right) = \ker\left(D\Phi_G(m)\right) \oplus \mathcal{M}$$

for some linear subspace $\mathcal{M} \subset T_m M$, described in Proposition 4.2.1. In Section 5.1.1 a splitting of $\mathfrak{g}$ is introduced, giving a more precise description of $\mathcal{M}$ (cf. Proposition 5.1.3). We say that a relative equilibrium (m, ξ) of h with momentum $\mu = \Phi_G(m)$ is α-nondegenerate if the Hessian of h^ξ at m is a non-singular quadratic form when restricted to the subspace

$$N_1 \oplus \mathcal{M}.$$

Another condition that we require is a regularity condition on the stabilizers. Explicitly we require $\mathfrak{g}_\mu \subset \mathfrak{g}_\xi$, which are respectively the Lie algebras of the stabilizers G_μ and G_ξ. This condition is based on Lemma 4.2.3 that we prove in Section 6.2, using the machinery of root systems.

Under these assumptions on a relative equilibrium $(m, \xi) \in M \times \mathfrak{h}$ of h, and modulo some technicalities, the least number of H_μ-orbits of relative equilibria with velocity close to ξ, which persist under a small H-perturbation in some neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$, is the positive integer

$$\mathrm{Cat}_{H_\mu}\left(G_\mu/G_m\right).$$

This is the content of Theorem 4.2.5 and Corollary 4.2.6. As an example, the Hamiltonian of the spherical pendulum can be thought as an S^1-perturbation of the $SO(3)$-invariant Hamiltonian h on T^*S^2 governing the co-geodesic motions on the sphere. The relative equilibria of h project to the great circles on S^2. Only one of those circles persists in the good neighbourhood under this S^1-perturbation, provided the gravity encoded in the parameter is sufficiently small. This example is discussed in Section 4.2.4.

Because of the importance of the equivariant Lyusternik-Schnirelmann category in this study, we dedicate a chapter to this notion where we obtain some new results. Let M be a topological space, acted on continuously and properly by a topological group G. We define a G-categorical open subset of M to be a G-invariant open subset of M admitting a G-deformation retract onto a G-orbit (cf. Section 3.1.1). The topological invariant $\mathrm{Cat}_G(M)$ is the least number (possibly infinite) of G-categorical open subsets that are required to cover M. If M and G have an additional smooth structure, a class of G-categorical open subsets consists of G-tubular open subsets, which are essentially tubular neighbourhoods of group orbits (cf. Definition 3.2.1). This fact is a direct application of the Tube Theorem 2.1.5.

This topological invariant is in general difficult to compute and we are usually only able to know an estimation of it, in term of the cup length of M. We obtain a new formula to reduce the calculation of $\mathrm{Cat}_G(M)$ to the calculation of the equivariant Lyusternik-Schnirelmann category of the minimal orbit-type strata of M. In general, any topological space M can be written as a disjoint union of smaller subsets M_β, called strata, indexed on some strictly partially ordered set $(\mathcal{B}, \prec)$. Those strata are required to fit in a specific way and form themselves a strictly partially ordered set (cf. Section 2.2). A stratum is minimal if it is minimal with respect to the strict partial order defined on them. If M is a proper G-manifold, the strata M_β are generally defined as the connected components of the orbit-type submanifolds. We use a modified definition of orbit-type stratum (cf. Definition 2.2.1). We say that an orbit-type stratum is a G-orbit of a connected component of the subset of M of all the points having the same stabilizer.

On a large class of proper G-manifolds M, including symplectic toric manifolds, we observe that M can be entirely covered by a subcover of its minimal orbit-type strata, made of G-tubular open subsets. Besides this cover is the smallest cover, made of G-categorical open subsets, that we can take. Such covers are called minimal G-tubular covers and are discussed in Section 3.2. However those covers do not exist in general. We present some non-examples in Section 3.2.1, when M is a non-Hamiltonian compact S^1-manifold. By using the natural stratification of the moment polytope, we show in Section 3.2.2 that every symplectic toric manifold admits a minimal G-tubular cover, where G in this case is a torus having half the dimension of M and acting effectively on it (cf. Theorem 3.2.6). In the case where M admits a minimal G-tubular cover, we show that the calculation of $\mathrm{Cat}_G(M)$ is intrinsically reduced to those of the minimal orbit-type strata of M. Explicitly, we obtain the localization formula

$$\mathrm{Cat}_G(M) = \sum \mathrm{Cat}_G\left(M_\beta\right)$$

where the summation is taken over the minimal orbit-type strata M_β (cf. Theorem 3.3.1 and Corollary 3.3.2). The result of Bayeh and Sarkar (cf. [6] Theorem 5.1), which states that the equivariant Lyusternik-Schnirelmann category of a quasitoric manifold is precisely the number of fixed points of the torus action, is a consequence of Theorem 3.2.6 and of our localization formula. After this work was completed we found that the result of Theorem 3.3.1, from which the localization formula follows under an additional assumption, had already been obtained by Hurder and Töben (cf. [28] Theorem 3.7).

Another question raised in this thesis concerns the choice of symplectic slices for subgroup actions. Since the questions of stability and of bifurcations of relative equilibria rely on the positive (or negative) definiteness of the Hessian of h^ξ (repectively h^ξ_λ) on a symplectic slice N_1 for the G-action (respectively a symplectic slice $\widetilde{N}_1$ for the H-action), knowing a way to compare N_1 and $\widetilde{N}_1$ is of particular interest. If μ is a regular value of the momentum map $\Phi_G : M \to \mathfrak{g}^*$, the orbit $G \cdot m$, of some point $m \in M$ with momentum μ, is transverse to the level set $\Phi_G^{-1}(\mu)$. As constructed by Roberts, Wulff and Lamb [61], a subspace of T_mM transverse to $G \cdot m$ is isomorphic to $\mathfrak{m}^* \times N_1$ where $\mathfrak{m}^*$ is isomorphic to $(\mathfrak{g}_\mu/\mathfrak{g}_m)^*$. The notation $\mathfrak{g} \cdot m$ denotes the tangent space at m of $G \cdot m$. A symplectic slice N_1 at m is a choice of G_m-invariant complement

$$N_1 := \ker(D\Phi_G(m))/\mathfrak{g}_\mu \cdot m.$$

It is endowed with a symplectic structure and a linear Hamiltonian action of the stabilizer G_m. Given a closed subgroup $H \subset G$ we can construct a momentum map $\Phi_H : M \to \mathfrak{h}^*$ as described above. For $\alpha = i^*(\mu)$, a symplectic slice at m is an H_m-invariant complement

$$\widetilde{N}_1 := \ker(D\Phi_H(m))/\mathfrak{h}_\alpha \cdot m.$$

It is used implicitly in [23] that, whenever G is a torus and H is a subtorus, both acting freely on M, a symplectic slice $\widetilde{N}_1$ at m can be chosen of the form

$$\widetilde{N}_1 = N_1 \oplus X_m,$$

for some symplectic subspace $X_m \subset T_mM$ isomorphic to $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$. We generalize this observation to non-abelian Lie groups and non-free actions with the assumption $G_m \subseteq N_G(H)$, where $N_G(H)$ denotes the normalizer of H in G. In this case we show that a symplectic slice $\widetilde{N}_1$ for H can be chosen of the form

$$\widetilde{N}_1 = N_1 \oplus X_m \oplus \mathfrak{s}(G, H, \mu) \cdot m,$$

for some subspace $X_m \subset T_mM$ symplectomorphic to a canonical cotangent bundle $\mathfrak{b} \times \mathfrak{b}^*$, and where $\mathfrak{s}(G, H, \mu) \cdot m$ is some symplectic vector subspace of $(\mathfrak{g}/\mathfrak{g}_\mu) \cdot m$ (cf. Theorem 5.1.4 and Lemma 5.1.5). We also give the associated splitting of the symplectic form on $\widetilde{N}_1$ (cf. Theorem 5.1.6), and the associated splitting of the momentum map associated to the linear Hamiltonian H_m-action on $\widetilde{N}_1$ in terms of the momentum map on N_1 (cf. Proposition 5.1.7).

The subspaces $\mathfrak{b}$ and $\mathfrak{s}(G, H, \mu)$ are constructed explicitly in Section 5.1.1, by using a splitting of the Lie algebra $\mathfrak{g}$. Although the notations can be misleading, those two subspaces are not Lie subalgebras of $\mathfrak{g}$ in general. When G is a torus and H is a subtorus, both acting freely on M, the subspace $\mathfrak{b}$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$ whereas the subspace $\mathfrak{s}(G, H, \mu)$ is trivial (cf. Example 5.1.8). In Perlmutter, Rodríguez-Olmos and Sousa-Dias [59], the subspace

$$\mathfrak{s}(G, H, \mu) \subset \mathfrak{g}$$

is obtained by a different construction. It is isomorphic to a symplectic slice at μ for the H-action on the coadjoint orbit $G \cdot \mu$. We show in Proposition 5.1.2 that our space, as constructed, coincides with their construction. This construction strongly depends on the choice of momentum μ as shown in Example 5.1.9, in the case where $G = SO(3)$ and $H = SO(2)$.

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PRELIMINARIES

We work with smooth manifolds and, except stated otherwise, the term *submanifold* refers to an embedded submanifold. A *Lie group* G is a set endowed with both a smooth structure and a group structure that are compatible in the sense that the group operations of inversion and multiplication are smooth. A (smooth) *action* of a Lie group G on a smooth manifold M is a group homomorphism $G \to \mathrm{Diff}(M)$ such that the action map $(g, m) \in G \times M \mapsto g \cdot m \in M$ is smooth. A *G-manifold* is a pair (M, G) where M is a smooth manifold acted on by a Lie group G. A smooth map $f : M \to N$ between two G-manifolds is *G-equivariant* if $f(g \cdot m) = g \cdot f(m)$ for all $g \in G$ and $m \in M$. Two G-manifolds (M, G) and (N, G) are *isomorphic* if there exists a G-equivariant diffeomorphism $f : M \to N$ between them. A real-valued smooth function $f : M \to \mathbb{R}$ defined on a G-manifold M is *G-invariant* if $f(g \cdot m) = f(m)$ for all $g \in G$ and $m \in M$. We denote by $C^\infty(M)^G$ the space of smooth real-valued G-invariant smooth functions on M. The *stabilizer* of $m \in M$ is the subgroup $G_m = \{g \in G \mid g \cdot m = m\}$. We say that the action of G on M is *free* if all the stabilizers G_m are equal to the trivial group $\mathbb{1}$. The *group orbit* or *G-orbit* of a point $m \in M$ is the set $G \cdot m = \{g \cdot m \mid g \in G\}$. We use the notation $\mathfrak{g} \cdot m$ to mean the tangent space to $G \cdot m$ at m. The space $\mathfrak{g}$ stands for the Lie algebra of G with Lie bracket $[\cdot, \cdot]$, obtained by identifying $\mathfrak{g}$ with the left invariant vector fields on G.

2.1 Proper G-manifolds

Let (M, G) be a G-manifold. Although group orbits are manifolds in their own right in the sense that they are injectively immersed submanifolds of M, they are not in general submanifolds for the subspace topology. However if the group action is proper then the group orbits are embedded submanifolds. The action of G on M is *proper* if the map

$$(g, m) \in G \times M \mapsto (m, g \cdot m) \in M \times M \tag{2.1}$$

is a proper map i.e. the preimage of any compact set is compact. A direct consequence is that all the stabilizers G_m are compact since they are identified with the preimage of a compact set. In this case we refer to (M, G) as a *proper G-manifold*.

Example 2.1.1. Clearly any G-manifold (M, G) where G is compact is a proper G-manifold. Other examples include:

(i) (G, G) is a proper G-manifold where G acts on itself by left or right multiplication. Indeed in this case the map (2.1) is a diffeomorphism.

(ii) If (M, G) is a proper G-manifold then so is (M, H) for any subgroup H of G acting on M as G does.

Proposition 2.1.2. *All the group orbits in a proper G-manifold (M, G) are embedded closed submanifolds.*

Proof. Let $G \cdot m$ be the group orbit of some $m \in M$. The orbit map

$$\tau_m : g \in G \mapsto g \cdot m \in M$$

is G-equivariant where G acts on itself by left multiplication. In particular τ_m has constant rank which equals the dimension of the orbit space G/G_m. Since τ_m is G_m-invariant, where G_m acts on G by right multiplication, it descends to a well-defined injective smooth map $\widetilde{\tau}_m : G/G_m \to M$. Since the quotient map $\pi : G \to G/G_m$ has maximal rank and τ_m has constant rank, it follows that $\widetilde{\tau}_m$ has constant rank. A smooth injective map of constant rank is an immersion (cf. [37] Theorem 4.14). We still have to show that $\widetilde{\tau}_m$ is a topological embedding. By continuity of π and properness of τ_m, the map $\widetilde{\tau}_m$ is also proper. Since every proper map whose codomain is metrizable is closed [56], $\widetilde{\tau}_m$ is a topological embedding (cf. [37] Theorem A.38). In particular the image $G \cdot m = \widetilde{\tau}_m(G/G_m)$ is a closed embedded submanifold of M. ■

A direct consequence of the closedness of the orbits for a proper G-action is that M/G is Hausdorff. If (M, G) is a proper G-manifold and if G acts freely on M then the orbit space M/G admits a smooth structure such that the quotient map $\pi : M \to M/G$ is a smooth submersion (cf. [44] Proposition 9.3.2).

Example 2.1.3 (Non-proper $\mathbb{R}$-manifold)**.** Think of S^1 as the complex numbers of length one and let $\mathrm{T} = S^1 \times S^1$ be the two-torus equipped with the $\mathbb{R}$-action

$$(t, (z, w)) \in \mathbb{R} \times \mathrm{T} \mapsto \left(e^{2\pi i t} z, e^{2\pi i \alpha t} w\right) \in \mathrm{T}$$

where α is an irrational number. This map defines an irrational flow on the torus. Each group orbit for this action is an injectively immersed submanifold diffeomorphic to $\mathbb{R}$. Since the orbit maps of Proposition 2.1.2 are not homeomorphisms onto their image, the group orbits are not embedded submanifolds of T. In fact, it can be shown that the orbits are dense in T (cf. [37] Examples 4.20 and 5.19). In particular $(\mathrm{T}, \mathbb{R})$ is an example of non-proper $\mathbb{R}$-manifold.

2.1.1 Existence of slices on proper G-manifolds

An important feature about proper G-manifolds is the existence of slices. Let (M, G) be a proper G-manifold. Given a subgroup K of G together with a K-manifold (S, K), there is a (left) K-action on the product $G \times S$ given by

$$k \cdot (g, s) = (gk^{-1}, k \cdot s). \tag{2.2}$$

This action is free and proper by freeness and properness of the action on the first factor. The orbit space $G \times_K S$ is thus a smooth manifold whose points are equivalence classes of the form $[(g, s)]$, and the orbit map $\rho : G \times S \to G \times_K S$ is a smooth surjective submersion. Moreover the group G acts smoothly and properly on $G \times_K S$, by left multiplication on the first factor.

Definition 2.1.4. Let (M, G) be a proper G-manifold and set $K = G_m$ for some $m \in M$. A K-manifold (S, K) is called a *slice at m* if

(i) S is an embedded submanifold of M containing m.

(ii) There is a G-invariant open subset $U \subset M$ containing m such that the G-equivariant map

$$\tau : [(g, s)] \in G \times_K S \mapsto g \cdot s \in U \tag{2.3}$$

is a diffeomorphism.

In particular, a slice at m is transverse at m to the group orbit $G \cdot m$ (cf. [52] Theorem 2.3.26). Palais proved [1] that when (M, G) is a proper G-manifold, there is a slice at every point $m \in M$ (cf. [53] Theorem 2.3.3). Previously, this result had been obtained by Koszul [33] for compact Lie group actions. When (M, G) is a proper G-manifold, the existence of slices follows from the Tube Theorem, which is here stated as in [52] (cf. Theorem 2.3.28). In the statement below, a *K-vector space* is a pair (N, K), where N is a vector space on which a Lie group K acts linearly.

Theorem 2.1.5 (Tube Theorem)**.** *Let (M, G) be a proper G-manifold and set $K = G_m$ for some $m \in M$. Let (N, K) be a K-vector space which is K-equivariantly isomorphic to $T_mM/\mathfrak{g} \cdot m$ and let $N_0 \subset N$ be an open K-invariant neighbourhood of 0. Then, there exists a G-invariant neighbourhood $U \subset M$ of m and a G-equivariant diffeomorphism*

$$\varphi : G \times_K N_0 \to U \tag{2.4}$$

sending $[(e, 0)]$ on m.

[1] In fact, Palais assumes that G is a Lie group and M is a completely regular Hausdorff topological space. He introduces the terminology "Palais-proper". If M is locally compact, this definition is equivalent to ours.

The triplet $(\varphi, G\times_K N_0, U)$ is called a *G-tube* at m and we also say that $G\times_K N_0$ is a *local model* for U. As a corollary, it is easy to verify that $S = \varphi\left([(e, N_0)]\right)$ is a slice at m. We thus get

Theorem 2.1.6 (Slice Theorem). *Let (M, G) be a proper G-manifold. Then there is a slice at every point of M.*

2.1.2 G-invariant metrics on proper G-manifolds

An important consequence of the existence of slices is that any proper G-manifold (M, G) admits a G-invariant Riemannian metric [57]. To show it, we recall the standard construction of K-invariant inner products on a finite dimensional K-vector space (V, K), when K is a compact Lie group. We call it the *averaging method.* Let (V, K) be a (finite dimensional) K-vector space with inner product $(\cdot, \cdot)$. If K is compact we can construct a new inner product on V which is K-invariant. To do that, let $\mathfrak{k}$ be the Lie algebra of K and we fix a basis $e_1, \dots, e_n$ of $\mathfrak{k}$, with dual basis $\varepsilon^1, \dots, \varepsilon^n$ of $\mathfrak{k}^*$. We define an n-form $\sigma \in \Omega^n(K)$ by

$$\sigma(g) = (L_{g^{-1}})^* \left(\varepsilon^1 \wedge \cdots \wedge \varepsilon^n\right)$$

which is in fact a volume form. By construction it satisfies the K-invariance property $\sigma(hg) = (L_{h^{-1}})^*\sigma(g)$ for all $h \in K$. Since K is compact, the integral $\int_K \sigma$ is finite and then $\omega = \frac{\sigma}{\int_K \sigma}$ is such that $\omega(K) = 1$. It thus defines a K-invariant measure on K, which is called the Haar-measure. Define an inner product by

$$\langle x, y\rangle = \int_K (g\cdot x, g\cdot y)\, \omega(g^{-1}). \tag{2.5}$$

Henceforth this inner product is K-invariant since, for all $h \in K$,

$$\begin{aligned}
\langle h \cdot x, h \cdot y \rangle &= \int_K (gh \cdot x, gh \cdot y)\, \omega((gh)^{-1}) \\
&= \int_K (k \cdot x, k \cdot y)\, (L_h)^* \omega(hk^{-1}) \\
&= \int_K (k \cdot x, k \cdot y)\, \omega(k^{-1}) \\
&= \langle x, y \rangle.
\end{aligned}$$

Proposition 2.1.7 (cf. [53] Theorem 4.3.1)**.** *Any proper G-manifold (M, G) admits a G-invariant Riemannian metric.*

Proof. Endow M with the Riemannian metric coming from the ambient Euclidean space. Let $m \in M$ with stabilizer $K = G_m$, compact by properness of the action. We know there is a slice S at m and let $\tau : G \times_K S \to U$ as in (2.3). Since τ is a diffeomorphism, the slice S embedds in M as $s \mapsto \tau([e, s])$. We pullback on S the Riemannian metric on M by mean of this embedding. This induces an inner product $\langle \cdot, \cdot \rangle_s$ on each tangent space $T_s S$. By compacity of K we can assume that this inner product is K-invariant using the averaging method if necessary. Since the orbit map $\rho : G \times S \to G \times_K S$ is a surjective submersion, any tangent vector in $T_{[g,s]}(G \times_K S)$ is of the form $T_{(g,s)}\rho \cdot (v, \alpha)$ for some $(v, \alpha) \in T_g G \times T_s S$. We define a Riemannian metric h on $G \times_K S$ by

$$h([g, s]) \Big(T_{(g,s)}\rho \cdot (v, \alpha), T_{(g,s)}\rho \cdot (w, \beta) \Big) := \langle \alpha, \beta \rangle_s. \tag{2.6}$$

It is well-defined since $\langle \cdot, \cdot \rangle_s$ is K-invariant. Furthermore, it is G-invariant by construction. The pullback of h along τ^{-1} defines a G-invariant Riemannian metric on U. Therefore since a slice exists at each $m \in M$, each G-orbit has a neighbourhood on which we can define a G-invariant metric. Using a partition of unity, they can all be patched together in order to get a G-invariant metric on the whole M. ■

2.2 Stratifications and orbit-type strata

A *partition* of a topological space M is a cover of M by pairwise disjoint subsets. Clearly every topological space admits a partition into its connected components. If our topological space is endowed with a group action, we can choose a partition which also encodes the information about the group action. For example, a proper G-manifold (M, G) can be partitioned into locally closed (connected) submanifolds called the orbit-type strata, each of them being a union of group orbits with the same orbit-type.

2.2.1 Orbit-type strata

Let G be a Lie group and $H \subset G$ be a closed subgroup. The *conjugacy class* of H is the set $(H) = \{L \subset G \mid L = gHg^{-1} \text{ for some } g \in G\}$. Given a G-manifold (M, G), we define the set

$$M_{(H)} := \{m \in M \mid G_m \in (H)\}$$

which is the union of all the G-orbits in M with orbit-type (H). Using the definitions and the G-invariance of $M_{(H)}$, it is shown in [52] (Proposition 2.4.4) that $M_{(H)} = G \cdot M_H$ where

$$M_H = \{m \in M \mid G_m = H\}.$$

Note that the biggest subgroup of G which leaves M_H invariant is the normalizer $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Furthermore this action induces a well-defined free action of the quotient group $N_G(H)/H$ on M_H. Write

$$M_H = \coprod_{b \in \mathcal{B}_H} M_{H,b}$$

as the disjoint union of its connected components, indexed on some set $\mathcal{B}_H$. Given $b \in \mathcal{B}_H$, we define the equivalence class (b) to be the set of indices $a \in \mathcal{B}_H$ such that $G \cdot M_{H,a} = G \cdot M_{H,b}$.

Definition 2.2.1. An *orbit-type stratum* $M_{(H),(b)}$ is the G-orbit of the connected component $M_{H,b}$ of M_H.

We use here a modified definition of the standard definition which states that an orbit-type stratum is a connected component of $M_{(H)}$. If the G-action on M is proper, the connected components $M_{H,b}$ are locally closed embedded submanifolds of M and so are their G-orbits (cf. [52] Proposition 2.4.7).

The example below illustrates the difference between the standard definition of orbit-type strata and ours. With our definition, the orbit-type strata might not be connected.

Example 2.2.2. Think of $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ as a multiplicative group and let it act on $M = \mathbb{R}^2$ by $t \cdot (x, y) = (x, ty)$. The stabilizers of points of M are either equal to the trivial group $\mathbb{1}$, or equal to $\mathbb{R}^*$. Then $M_{(\mathbb{R}^*)} = M_{\mathbb{R}^*}$ is the x-axis, and $M_{(\mathbb{1})} = M_{\mathbb{1}} = H_+ \cup H_-$ where $H_\pm = \{(x, y) \in M \mid \pm y > 0\}$. According to the standard definition of orbit-type strata, there are two strata with orbit-type $(\mathbb{1})$, namely the connected components H_+ and H_-; and one stratum with orbit-type $(\mathbb{R}^*)$, the x-axis.

With our definition, there is one stratum with orbit-type $(\mathbb{R}^*)$ which is the x-axis; but there is only one stratum with orbit-type $(\mathbb{1})$ which is $H_+ \cup H_-$. Indeed, $M_{\mathbb{1}}$ has two connected components, H_+ and H_-. The $\mathbb{R}^*$-orbits of each of them coincide. There is thus only one stratum with orbit-type $(\mathbb{1})$ and it is not connected.

Before giving some examples, we introduce the notion of G-connectivity. We say that a G-manifold (M, G) is *G-connected* if it can be written as the G-orbit of a connected submanifold. Although M might not be connected we still have the following fundamental result:

Proposition 2.2.3. *Let (M, G) be a G-connected G-manifold and let $A \subset M$ be a non-empty G-invariant subset. If A is open and closed then $A = M$.*

Proof. Since M is G-connected, $M = G \cdot N$ for some connected submanifold N. Let $A \subset M$ be a non-empty G-invariant subset. Assume by contradiction that $A^c = M \setminus A \neq \varnothing$. Then M can be written as the disjoint union of two non-empty open subsets, A and A^c. By G-invariance, the orbit spaces A/G and A^c/G are open for the quotient topology in $N = M/G$. Since there are non-empty and N is connected we have a contradiction. Consequently $A^c = \varnothing$ and then $A = M$. ■

Example 2.2.4. Given an equivalence class (H), the corresponding orbit-type strata might not all have the same dimension, as shown in the following example, appearing in Delzant [16] and Sjamaar and Lerman [66].

(i) Let $M = \mathbb{C}P^2$ endowed with the S^1-action $\theta \cdot [z_0 : z_1 : z_2] = [e^{i\theta} z_0 : z_1 : z_2]$. The set M_{S^1} has two connected components namely, the point $[1 : 0 : 0]$ and a copy of $\mathbb{C}P^1$, which consists of the points of the form $[0 : z_1 : z_2]$. Since S^1 acts trivially on each of these components, they form themselves two orbit-type strata, which are closed submanifolds of M. Since the action is free anywhere else, the last orbit-type stratum is $M \setminus (\{[1 : 0 : 0]\} \cup \mathbb{C}P^1)$. It has orbit-type $(\mathbb{1})$ and is an open dense submanifold of M.

(ii) Let $M = S^2$ be the 2-sphere embedded in $\mathbb{R}^3$, equipped with the S^1-action which rotates the sphere about the z-axis. There are three orbit-type strata namely, $M_{(\mathbb{1})}$ which is diffeomorphic to $S^1 \times (-1, 1)$ and the two closed connected components of M_{S^1} that are the North and South pole.

(iii) The rotations of a tetrahedron form a group $\mathbb{T}$ of order 12, which is a zero-dimensional Lie subgroup of $SO(3)$. In particular $\mathbb{T}$ acts on $M = S^2$. This group contains a copy of the cyclic group of order three $C_3 \simeq \mathbb{Z}_3$ for each vertex, one copy of $\mathbb{Z}_2$ for each axis joining the middle point of an edge and the middle point of the opposite edge, and the identity (cf. Figure 2.1).

There are two minimal strata with orbit-type $(\mathbb{Z}_3)$, one minimal stratum with orbit-type $(\mathbb{Z}_2)$, and one open dense stratum with orbit type $(\mathbb{1})$ (cf. Figure 2.2). Indeed, when $H = \mathbb{Z}_3$, the eight points forming $M_{(H)}$ are a union of two $\mathbb{T}$-orbits. There are thus two strata with orbit-type $(\mathbb{Z}_3)$. For $H = \mathbb{Z}_2$, the six points forming $M_{(H)}$ are a single $\mathbb{T}$-orbit and form a single stratum.

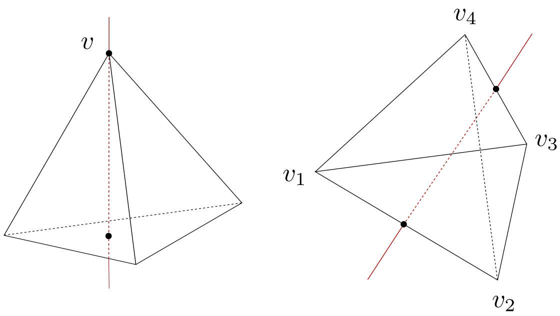


Figure 2.1: On the left hand side we fix a vertex v and permute the three other vertices. As a subgroup it is isomorphic to C_3. On the right hand side we permute v_1, v_2 and v_3, v_4. This subgroup is isomorphic to $\mathbb{Z}_2$.

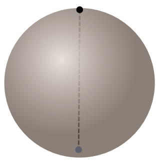
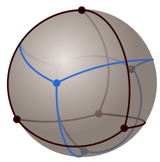
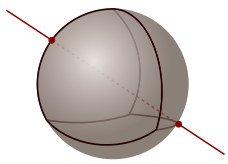
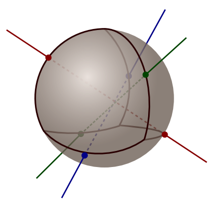
H	M_H	$M_{(H)}$
$\mathbb{T}$	$\varnothing$	$\varnothing$
$\mathbb{Z}_3$		
$\mathbb{Z}_2$		
$\mathbb{1}$	$M \setminus \{14 \text{ points }\}$	$M \setminus \{14 \text{ points }\}$

Figure 2.2: Orbit-type strata of $(M, \mathbb{T})$ where $M = S^2$.

2.2.2 Stratification by orbit-type strata

There are several ways to define stratifications. The one we present here is the definition used by Kirwan in her thesis [32]. It is more flexible than the standard definition of Duistermaat and Kolk [17] (Definition 2.7.3), especially for applications to algebraic geometry. Recall that a *partial order* $\preceq$ on a set $\mathcal{B}$ is a binary relation which is reflexive, antisymmetric and transitive. The pair $(\mathcal{B}, \preceq)$ is called a *partially ordered set*. A *strict partial order* $\prec$ on $\mathcal{B}$ is a binary relation which is irreflexive (an element cannot be compared with itself) and transitive. Note that in this case if $\alpha, \beta \in \mathcal{B}$ are such that $\alpha \prec \beta$, then $\beta \not\prec \alpha$. For example the set of conjugacy classes of subgroups of G admits the strict partial order $\prec_{conj}$, where we say that $(K) \prec_{conj} (H)$ if and only if H is conjugate to a proper subgroup of K. This relation is clearly irreflexive and transitive.

Example 2.2.5. The group $\mathbb{T}$ has four conjugacy classes, namely $(\mathbb{T}), (\mathbb{Z}_3), (\mathbb{Z}_2)$ and $(\mathbb{1})$. There are partially ordered with respect to $\prec_{conj}$ as shown in Figure 2.3.

$$\begin{tikzcd}
 & \mathbb{T} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Z}_3 \arrow[dr, dash] & & \mathbb{Z}_2 \arrow[dl, dash] \\
 & \mathbb{1} &
\end{tikzcd}$$

Figure 2.3: Conjugacy classes of subgroups of $\mathbb{T}$ where the order goes up to down i.e. $(\mathbb{T})$ is minimal with respect to $\prec_{conj}$.

Definition 2.2.6. A collection $\{M_\beta \mid \beta \in \mathcal{B}\}$ of subsets of a topological space M is *locally finite* if each compact set of M meets only finitely many M_β. A locally finite collection $\{M_\beta \mid \beta \in \mathcal{B}\}$ of locally closed (non-empty) subsets of M form a $\mathcal{B}$-*stratification* of M if M is the disjoint union of the *strata* M_β, and there is a strict partial order $\prec$ on the indexing set $\mathcal{B}$ such that

$$\overline{M}_\beta \subset \bigcup_{\alpha \preceq \beta} M_\alpha \tag{2.7}$$

for every $\beta \in \mathcal{B}$. We say that the $\mathcal{B}$-stratification is *smooth* if M is a smooth manifold and every M_β is a locally closed submanifold.

Given a $\mathcal{B}$-stratification of M, a strict partial order can be defined on the strata in the following way

$$M_\alpha < M_\beta \iff \alpha \prec \beta. \tag{2.8}$$

We say that a stratum M_β is *minimal* with respect to (2.8) if there is no $\alpha \in \mathcal{B}$ such that $M_\alpha < M_\beta$. Of course minimal strata are not unique because we just have a partial ordering. If M_β is a minimal stratum, (2.7) implies that $\overline{M}_\beta \subset M_\beta$. In particular $\overline{M}_\beta = M_\beta$ i.e. M_β is closed in M. Given a strict partial order on the strata, we can associate to it an oriented graph whose vertices are the indices $\beta \in \mathcal{B}$. Two vertices $\alpha, \beta \in \mathcal{B}$ are linked by an oriented edge $\alpha \longrightarrow \beta$ if and only if $M_\beta < M_\alpha$ (cf. Example 2.2.7).

We show now that orbit-type strata of a proper G-manifold form a $\mathcal{B}$-stratification, for some indexing set $\mathcal{B}$ that has to be determined. Let (M, G) be a proper G-manifold. For each closed subgroup $H \subset G$, we associate an indexing set $\mathcal{B}_H$ such that $M_{(H)}$ can be written as the disjoint union of the orbit-type strata $M_{(H),(b)}$ with $b \in \mathcal{B}_H$ (cf. Definition 2.2.1). In order to partition M, we define $\mathcal{B}$ to be the set of pairs $\beta = ((H), (b))$ where (H) is the conjugacy class of some closed subgroup of G, and $b \in \mathcal{B}_H$. We write $M_\beta = M_{(H),(b)}$ when $\beta = ((H), (b))$. We define a strict partial order $\prec$ on $\mathcal{B}$ as follows: for $\alpha = ((K), (a))$ and $\beta = ((H), (b))$,

$$\alpha \prec \beta \quad \Longleftrightarrow \quad \alpha \neq \beta \quad \text{and} \quad M_\alpha \cap \overline{M}_\beta \neq \varnothing. \tag{2.9}$$

By $\alpha \neq \beta$ we mean that the associated orbit-type strata M_α and M_β are distinct.

Example 2.2.7. Let $\mathrm{T} = S^1 \times S^1$ be a two-torus acting on $M = \mathbb{C}P^2 \times \mathbb{C}P^2$ by

$$(\theta, \phi) \cdot \Big([e^{i\theta} z_0 : z_1 : z_2], [e^{i\phi} w_0 : w_1 : w_2]\Big).$$

In what follows, $\mathbb{C}P^1$ denotes the copy of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ made of points of the form $[0 : z_1 : z_2]$. We denote by $\{m\}$ the set consisting of the single point $m = [1 : 0 : 0]$, and by U the (open) submanifold $\mathbb{C}P^2 \setminus (\{m\} \cup \mathbb{C}P^1)$. In this example we have four orbit-types (H), with H being either $\mathbb{1}$, $S^1 \times \mathbb{1}$, $\mathbb{1} \times S^1$ or the full torus $S^1 \times S^1$. Since $M_{(\mathbb{1})}$ is connected, there is only one stratum M_γ, with orbit-type $(\mathbb{1})$, which is open and dense in M. The other strata are given in the tables below.

$\alpha_a = ((S^1 \times \mathbb{1}), (a))$	M_{α_a}	$\overline{M}_{\alpha_a}$
α_1	$\{m\} \times U$	$\{m\} \times \mathbb{C}P^2$
α_2	$\mathbb{C}P^1 \times U$	$\mathbb{C}P^1 \times \mathbb{C}P^2$

$\alpha'_\ell = ((\mathbb{1} \times S^1), (\ell))$	$M_{\alpha'_\ell}$	$\overline{M}_{\alpha'_\ell}$
α'_1	$U \times \{m\}$	$\mathbb{C}P^2 \times \{m\}$
α'_2	$U \times \mathbb{C}P^1$	$\mathbb{C}P^2 \times \mathbb{C}P^1$

$\beta_b = ((S^1 \times S^1), (b))$	M_{β_b}	$\overline{M}_{\beta_b}$
β_1	$\{m\} \times \mathbb{C}P^1$	$\{m\} \times \mathbb{C}P^1$
β_2	$\{m\} \times \{m\}$	$\{m\} \times \{m\}$
β_3	$\mathbb{C}P^1 \times \mathbb{C}P^1$	$\mathbb{C}P^1 \times \mathbb{C}P^1$
β_4	$\mathbb{C}P^1 \times \{m\}$	$\mathbb{C}P^1 \times \{m\}$

The oriented graph associated to the strict partial order (2.9) is pictured below.

$$\begin{tikzcd}
\beta_3 & \alpha_2 \arrow[l] \arrow[r] & \beta_4 \\
\alpha'_2 \arrow[u] \arrow[d] & \gamma \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \alpha'_1 \arrow[u] \arrow[d] \\
\beta_1 & \alpha_1 \arrow[l] \arrow[r] & \beta_2
\end{tikzcd}$$

In general, using the strict partial order $\prec_{conj}$ on the conjugacy classes of subgroups of G is not enough to guarantee that we have a good stratification. For instance, in Example 2.2.4 (iii), we have $(\mathbb{T}) \prec_{conj} (\mathbb{Z}_2)$ but there are no strata with orbit-type $(\mathbb{T})$. However we have the following lemma:

Lemma 2.2.8. *If $\alpha = ((K), (a))$ and $\beta = ((H), (b))$ then*

$$\alpha \prec \beta \quad \Longrightarrow \quad (K) \prec_{conj} (H).$$

Proof. By definition $\alpha \prec \beta$ implies that there exists some $x \in M_\alpha \cap \overline{M}_\beta$. In particular $x \in M_\alpha$ and then $G_x \in (K)$. By the Tube Theorem 2.1.5, there is a G-invariant open neighbourhood $U \subset M$ of x, locally modelled by an associated bundle $G \times_{G_x} N_0$, in which x reads $[(e, 0)]$.

By definition of the adherence, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset M_\beta$ converging to x in M, with stabilizers $G_{x_n} \in (H)$. For n big enough, $x_n \in U$ and it can thus be identified with some point $[(g_n, \nu_n)] \in G \times_{G_x} N_0$. The stabilizer of $[(g_n, \nu_n)]$ is

$$G_{[(g_n,\nu_n)]} = g_n (G_x)_{\nu_n} g_n^{-1}$$

and is thus conjugate to a proper subgroup of G_x, because by assumption M_α and M_β are disjoint. Since $G_x \in (K)$ and $G_{x_n} \in (H)$, it follows that $(K) \prec_{conj} (H)$. ∎

Proposition 2.2.9. *Let (M, G) be a proper G-manifold and let $(\mathcal{B}, \prec)$ as above with partial order (2.9). Then the orbit-type strata $\{M_\beta \mid \beta \in \mathcal{B}\}$ form a smooth $\mathcal{B}$-stratification of M.*

Proof. By definition, the orbit-type strata form a disjoint cover of M. Therefore, any closure of an orbit-type stratum $\overline{M}_\beta$ is included in a union of strata M_α (including M_β), having non-empty intersection with $\overline{M}_\beta$. To show (2.7), we may show that

$$M_\alpha \cap \overline{M}_\beta \neq \varnothing \quad \Longrightarrow \quad M_\alpha \subset \overline{M}_\beta.$$

Assume $M_\alpha \cap \overline{M}_\beta \neq \varnothing$ for some (possibly equal) indices $\alpha = ((K), (a))$ and $\beta = ((H), (b))$ in $\mathcal{B}$. Note that $M_\alpha \cap \overline{M}_\beta$ is G-invariant. Indeed if $x \in M_\alpha \cap \overline{M}_\beta$, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset M_\beta$ converging to x. Given $g \in G$ the sequence $(g \cdot x_n)_{n\in\mathbb{N}}$ belongs to M_β by G-invariance of M_β, and converges to $g \cdot x$ by continuity of the action. By closedness of $\overline{M}_\beta$ and G-invariance of M_α, we have $g \cdot x \in M_\alpha \cap \overline{M}_\beta$.

The strategy now is to show that $M_\alpha \cap \overline{M}_\beta$ is closed and open in M_α. By G-connectedness of M_α we will get $M_\alpha \cap \overline{M}_\beta = M_\alpha$ and we are done. The closedness condition is immediate since, given a sequence $(x_n)_{n\in\mathbb{N}} \subset M_\alpha \cap \overline{M}_\beta$ converging to $x \in M_\alpha$, the limit point x must also belong to $\overline{M}_\beta$ because this set is closed.

Let us show that $M_\alpha \cap \overline{M}_\beta$ is open in M_α. Let $x \in M_\alpha \cap \overline{M}_\beta$ such that $G_x = K$. As in Lemma 2.2.8, we use the Tube Theorem 2.1.5 to get a G-invariant open neighbourhood $U \subset M$ of x, locally modelled by an associated bundle $G \times_K N_0$, in which x reads $[(e,0)]$. Then $V = U \cap M_\alpha$ is open in M_α for the subset topology. This subset locally reads

$$(G \times_K N_0)_{(K)} = G \times_K (N_0)_K.$$

Let $y \in V$ be an arbitrary point corresponding to some $[g,\nu]$ in the local model. By construction $\nu \in (N_0)_K$ and then $K_\nu = K$.

By definition of the adherence, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset M_\beta$ converging to x in M, with stabilizers $G_{x_n} \in (H)$. For $n \in \mathbb{N}$ big enough, $x_n \in U \cap M_\beta$. It can thus be identified with some $[(g_n,\nu_n)] \in (G \times_K N_0)_{(H)}$ whose stabilizer is conjugate to H. Besides, $G_{[(g_n,\nu_n)]} = g_n K_{\nu_n} g_n^{-1}$. Hence $K_{\nu_n} \in (H)$.

Let $N \in \mathbb{N}$ big enough such that $\nu + \nu_n \in N_0$. Then the sequence $(y_n)_{n\geq N} \subset U$ whose terms correspond to $[gg_n, \nu+\nu_n]$ in the local model converges to y. By linearity of the K-action on N_0, we have

$$K_{\nu+\nu_n} = K_\nu \cap K_{\nu_n} = K_{\nu_n} \in (H).$$

In particular $G_{y_n} \in (H)$. This shows that $(y_n)_{n\in\mathbb{N}} \subset M_\beta$ and thus $y \in \overline{M_\beta}$. Hence $V \subset M_\alpha \cap \overline{M}_\beta$ and then $M_\alpha \cap \overline{M}_\beta$ is open in M_α. Since M_α is G-connected,

$$M_\alpha = M_\alpha \cap \overline{M}_\beta \subset \overline{M}_\beta.$$

In particular, if $\alpha \neq \beta$, we have $\alpha \prec \beta$. We thus proved (2.7). This cover is a locally finite cover of locally closed submanifolds. The fact that the strata are locally closed embedded submanifolds is a consequence of Proposition 2.4.7 in [52]. That such a cover is locally finite is a consequence of Proposition 2.7.1 in [17]. Both facts require the group action to be proper. ■

2.2.3 Stratification of a convex polytope by open faces

There is a natural stratification of a convex polytope into vertices, edges and higher dimensional faces. Let $\Delta \subset (\mathbb{R}^n)^*$ be a n-dimensional convex polytope. Let $X_1, \ldots, X_d$ in $\mathbb{R}^n$ be the outward-pointing normal vectors to the facets. Then there exists real numbers $\lambda_1, \dots, \lambda_d$ such that Δ reads

$$\Delta = \bigcap_{i=1}^{d} \{\mu \in (\mathbb{R}^n)^* \mid \langle \mu, X_i \rangle \leq \lambda_i\}\,.$$

Let $\mathcal{B}$ be the set of subsets (possibly empty) $\beta \subset \{1, \dots, d\}$. For each $\beta \in \mathcal{B}$ we consider the intersection

$$F_\beta = \bigcap_{i \in \beta} \{\mu \in \Delta \mid \langle \mu, X_i \rangle = \lambda_i\}\,.$$

If $F_\beta \neq \varnothing$, its relative interior $\mathring{F}_\beta$ is called a l-dimensional open face of Δ where l is equal to n minus the cardinality of β. We equip $\mathcal{B}$ with the strict partial order

$$\alpha \prec \beta \quad \Longleftrightarrow \quad \alpha \neq \beta \quad \text{and} \quad \mathring{F}_\alpha \cap \overline{\mathring{F}_\beta} \neq \varnothing. \tag{2.10}$$

With this strict partial order, the collection $\{\mathring{F}_\beta \mid \beta \in \mathcal{B}\}$ forms a $\mathcal{B}$-stratification of Δ. A strict partial order is defined on the set of faces by

$$\mathring{F}_\alpha < \mathring{F}_\beta \quad \Longleftrightarrow \quad \alpha \prec \beta.$$

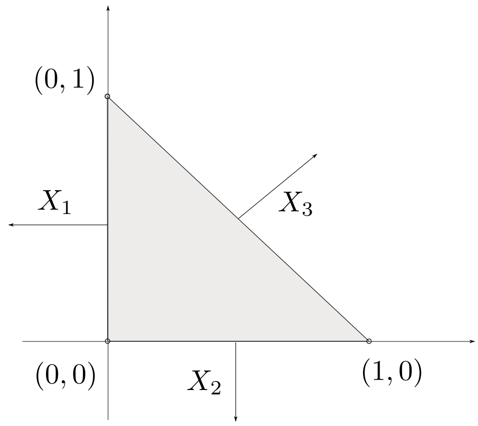
Finally note that, if $\alpha \prec \beta$ then $\beta \subset \alpha$.

Example 2.2.10. Let $\Delta \subset (\mathbb{R}^2)^*$ be the polytope with vertices $(1,0), (0,0)$ and $(0,1)$. According to our previous notations, set

$$X_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1$. The elements of Δ are the $\mu \in (\mathbb{R}^2)^*$ which satisfy

$$\begin{aligned} \langle \mu, X_1 \rangle &\leq \lambda_1, \\ \langle \mu, X_2 \rangle &\leq \lambda_2, \\ \langle \mu, X_3 \rangle &\leq \lambda_3. \end{aligned}$$



The faces corresponding to the subsets $\alpha = \{2, 3\}$ and $\beta = \{2\}$ are respectively the single vertex $\mathring{F}_\alpha = \{(1, 0)\}$, and $\mathring{F}_\beta$ which is the (open) edge joining $(1, 0)$ and $(0, 0)$. We clearly have $\mathring{F}_\alpha \subset \overline{\mathring{F}_\beta}$ and then $\alpha \prec \beta$.

2.3 Symplectic manifolds and Hamiltonian actions

In the Hamiltonian formulation of classical mechanics, the phase space of a dynamical system is a cotangent bundle whose zero section is a smooth manifold describing all the different configurations of the system. A cotangent bundle is a particular example of symplectic manifold.

2.3.1 Symplectic manifolds

A *symplectic manifold* is a pair (M, ω) where M is a $2n$-dimensional smooth manifold and $\omega \in \Omega^2(M)$ is a closed 2-form which is non-degenerate in the sense that the map

$$\omega^\flat : TM \to T^*M \tag{2.11}$$

defined by $\langle \omega^\flat(v), w\rangle = \omega(m)(v, w)$ for $v, w \in T_mM$, is a fiberwise isomorphism. In particular, ω^n obtained by wedging ω with itself n times is a volume form (nowhere vanishing top-degree form). Any symplectic manifold is therefore naturally oriented and has a canonical measure $\frac{\omega^n}{n!}$ called the *Liouville measure*. The 2-form ω is called a *symplectic form* or a *symplectic structure*.

Example 2.3.1 (The sphere S^2)**.** The $2n$-dimensional sphere S^{2n} is symplectic only when $n = 1$. There is in fact a cohomological reason for that. We start by proving a general fact about compact symplectic manifolds:

Proposition 2.3.2. *If (M, ω) is $2n$-dimensional and compact (closed without boundary), then the de Rham cohomology class $[\omega] \in H^2(M; \mathbb{R})$ is non-vanishing.*

Proof. By contradiction we assume that $[\omega^n] = 0$. This implies that there exists $\alpha \in \Omega^{2n-1}(M)$ such that $\omega^n = d\alpha$. Since M is compact, we can use Stokes Theorem to deduce

$$Vol(M) = \int_M \omega^n = \int_M d\alpha = \int_{\partial M} \alpha = 0.$$

Hence M has no volume which is a contradiction. Consequently $[\omega]$ is non-vanishing since otherwise $\omega^n = d(f \wedge \omega^{n-1})$ where $\omega = df$. In particular, exact symplectic forms only exist on non-compact manifolds. ∎

If we go back to our example, the only non-trivial de Rham cohomology groups of S^{2n} are $H^0(S^{2n}; \mathbb{R})$ and $H^{2n}(S^{2n}; \mathbb{R})$. Since $[\omega]$ belongs to $H^2(S^{2n}; \mathbb{R})$, this class is non-vanishing only when $n = 1$. Therefore S^2 is the only even-dimensional sphere which is symplectic. To find the symplectic form, we identify

$$TS^2 = \left\{ v = (x, y) \in S^2 \times \mathbb{R}^3 \mid x \cdot y = 0 \right\}.$$

The symplectic form $\omega \in \Omega^2(S^2)$ is defined pointwise by

$$\omega(x)\,(v, w) = x \cdot (y \times z).$$

where $v = (x, y)$ and $w = (x, z)$ are in $T_x S^2$. This form is closed because it is of top degree and is non-degenerate because if $v = (x, y)$ is such that $y \neq 0$, the form is non-vanishing. Indeed by choosing $w = (x, x \times y)$, we get $\omega(x)\,(v, w) = \|y\|^2 \neq 0$.

Example 2.3.3. Let $\mathbb{C}^n$ with complex coordinates $z = (z_0, \dots, z_{n-1})$ endowed with the standard hermitian metric $H = \sum_{i=0}^{n-1} dz_i \otimes d\bar{z}_i$ where

$$H(u,v) = \sum_{i=0}^{n-1} u_i \bar{v}_i \quad \text{for} \quad v, u \in T_z\mathbb{C}^n \simeq \mathbb{C}^n.$$

The standard symplectic form $\omega_0 \in \Omega^2(\mathbb{C}^n)$ is the imaginary part of H modulo a sign

$$\omega_0(u,v) = \frac{i}{2}\left(H(u,v) - \overline{H(u,v)}\right) = \frac{i}{2}\left(H(u,v) - H(v,u)\right).$$

In a chart, it has the expression $\omega_0 = \frac{i}{2}\sum_{i=0}^{n-1} dz_i \wedge d\bar{z}_i$.

Example 2.3.4 (Complex projective spaces)**.** Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the complex torus. The complex projective space $\mathbb{C}P^{n-1}$ is defined as the orbit space of the principal $\mathbb{C}^\times$-bundle $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ where the (right) action of $\mathbb{C}^\times$ on $\mathbb{C}^n \setminus \{0\}$ is

$$\lambda \cdot (z_0, \dots, z_{n-1}) = (z_0\lambda, \dots, z_{n-1}\lambda).$$

We denote by $\underline{z} = [z_0 : \cdots : z_{n-1}] = \pi(z)$ the class corresponding to the point $z = (z_0, \dots, z_{n-1})$. Setting $\|z\|^2 = H(z,z)$, the unit sphere in $\mathbb{C}^n$ is

$$S^{2n-1} = \{z \in \mathbb{C}^n \mid \|z\|^2 = 1\}.$$

The Lie group S^1, identified with the complex numbers of length one, acts on S^{2n-1} by right multiplication on each factor. Since this action is free and proper, the orbit map $\rho : S^{2n-1} \to S^{2n-1}/S^1$ is a principal S^1-bundle. Clearly the map $\mathbb{C}P^{n-1} \to S^{2n-1}/S^1$ given by

$$[z_0 : \cdots : z_{n-1}] \longmapsto \rho\left(\frac{z_0}{\|z\|}, \dots, \frac{z_{n-1}}{\|z\|}\right)$$

is a well-defined diffeomorphism. The complex projective space $\mathbb{C}P^{n-1}$ is equipped with the Fubini-Study symplectic form

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log\left(\|z\|^2\right) \tag{2.12}$$

which is defined on a copy of $\mathbb{C}^{n-1}$ in $\mathbb{C}P^{n-1}$, obtained by removing an hyperplane at infinity. The term $\log(\|z\|^2)$ is a Kähler potential and is not globally defined. In a coordinate chart on $\mathbb{C}P^{n-1}$, say

$$\varphi : (z_0, \dots, z_{n-2}) \in \mathbb{C}^{n-1} \longmapsto [z_0 : \cdots : z_{n-2} : 1] \in \mathbb{C}P^{n-1}, \tag{2.13}$$

the Fubini-Study symplectic form reads

$$\begin{aligned}
\varphi^*\omega_{FS} &= \varphi^*\left(\frac{i}{2}\partial\overline{\partial}\log\left(\|z\|^2\right)\right) \\
&= \frac{i}{2}\partial\left(\sum_{k=0}^{n-2}\frac{\partial}{\partial\overline{z}_k}\left(\log\left(\|z\|^2\right)\right)d\overline{z}_k\right) \\
&= \frac{i}{2}\partial\left(\sum_{k=0}^{n-2}\frac{z_k}{\|z\|^2}d\overline{z}_k\right) \\
&= \frac{i}{2}\sum_{j=0}^{n-2}\frac{\partial}{\partial z_j}\left(\sum_{k=0}^{n-2}\frac{z_k}{\|z\|^2}\right)dz_j\wedge d\overline{z}_k \\
&= \frac{i}{2}\sum_{j=0}^{n-2}\frac{1}{\|z\|^2}dz_j\wedge d\overline{z}_j - \frac{i}{2}\sum_{j,k=0}^{n-2}\frac{z_k\overline{z}_j}{\|z\|^4}dz_j\wedge d\overline{z}_k.
\end{aligned}$$

Example 2.3.5 (Coadjoint orbits)**.** Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ (thought as the left invariant vector fields on G). The *Adjoint representation* of G on $\mathfrak{g}$ is the representation $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$

$$\mathrm{Ad}_g(x) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} g\exp(tx)g^{-1}$$

where $\exp : \mathfrak{g} \to G$ is the group exponential. Denote by $\langle\cdot,\cdot\rangle$ the natural pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The *coAdjoint representation* is the dual representation $\mathrm{Ad}^* : G \to \mathrm{Gl}(\mathfrak{g}^*)$ given by

$$\langle \mathrm{Ad}^*_{g^{-1}}\mu, x\rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}x\rangle \quad \text{for} \quad \mu \in \mathfrak{g}^*, x \in \mathfrak{g}.$$

A group orbit $G\cdot\mu \subset \mathfrak{g}^*$ for this representation is called a *coadjoint orbit*. Coadjoint orbits $G\cdot\mu$ carry a symplectic structure

$$\omega(\mu)\left(x_{\mathfrak{g}^*}(\mu), y_{\mathfrak{g}^*}(\mu)\right) = -\langle \mu, [x,y]\rangle$$

where $x_{\mathfrak{g}^*}(\mu), y_{\mathfrak{g}^*}(\mu) \in T_\mu (G \cdot \mu)$. To define those tangent vectors, we consider the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathrm{Gl}(\mathfrak{g})$ defined by $\mathrm{ad}_x y = [x, y]$. The dual representation $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{Gl}(\mathfrak{g}^*)$ is called the *coadjoint representation.* Then elements of $T_\mu (G \cdot \mu)$ are of the form $x_{\mathfrak{g}^*}(\mu) = -\mathrm{ad}^*_x \mu \in \mathfrak{g}^*$.

2.3.2 Hamiltonian actions

Let (M, ω) be a symplectic manifold acted on by a Lie group G. The action is *canonical* or *symplectic* if the diffeomorphisms generated by the action are symplectic maps i.e. diffeomorphisms preserving the symplectic structure. Every $x \in \mathfrak{g}$ induces a vector field $x_M \in \mathfrak{X}(M)$, an *infinitesimal generator*, defined pointwise by

$$x_M(m) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(tx) \cdot m. \tag{2.14}$$

If the action is symplectic, $\pounds_{x_M} \omega = 0$. In this case Cartan's magic formula and the closedness of ω imply that $\iota_{x_M} \omega \in \Omega^1(M)$ is closed, and then locally exact. A canonical action is *Hamiltonian* if this 1-form is exact for all $x \in \mathfrak{g}$. In this case, the vector field x_M is Hamiltonian in the sense that

$$\iota_{x_M} \omega = -d\phi^x_G \tag{2.15}$$

for some $\phi^x_G \in C^\infty(M)$ depending linearly on x. A map $\Phi_G : M \to \mathfrak{g}^*$ satisfying $\langle \Phi_G(m), x\rangle = \phi^x_G(m)$ is called a *momentum map.* A momentum map is said *equivariant* if $\Phi_G(g \cdot m) = \mathrm{Ad}^*_{g^{-1}} \Phi_G(m)$ for all $m \in M$ and $g \in G$.

Definition 2.3.6. Let (M, ω) equipped with a Hamiltonian action of a (connected) Lie group G with equivariant momentum map $\Phi_G : M \to \mathfrak{g}^*$. We call the quadruple (M, ω, G, Φ_G) a *Hamiltonian G-manifold.* If in addition the group action is proper, this quadruple is called a *Hamiltonian proper G-manifold.*

Example 2.3.7 (Canonical non-Hamiltonian action)**.** An example of action which is not Hamiltonian is the action of $G = \mathbb{R}$ on $M = S^1 \times \mathbb{R}$, acting by translations on the second factor. Let $(\theta, z) \in S^1 \times \mathbb{R}$ be the coordinates on M. Any element $x \in \mathbb{R}$ induces a vector field $x_M = x\frac{\partial}{\partial z}$. The action preserves the symplectic form $\omega = d\theta \wedge dz$, but $\iota_{x_M} \omega = -x d\theta$ is not globally exact as θ is multivalued on M.

Momentum maps are not unique in the sense that, if $\Phi_1, \Phi_2 : M \to \mathfrak{g}^*$ arise from the same canonical Lie group action, then $\phi_1^x - \phi_2^x$ is a Casimir function on M for any $x \in \mathfrak{g}$ i.e. the associated Hamiltonian vector field $X_{\phi_1^x - \phi_2^x}$ vanishes identically. Indeed $X_{\phi_1^x} = X_{\phi_2^x} = x_M$ by construction. If M is connected, the only Casimir functions are the constants and then, in this case, a momentum map is determined up to a constant in $\mathfrak{g}^*$. The reader is referred to [52] for further details. An obstruction for the existence of a momentum map is the non-vanishing of the first de Rham cohomology group $H^1(M, \mathbb{R})$.

Proposition 2.3.8 ([52] Proposition 4.5.17)**.** *A canonical action of a Lie group G on (M, ω) is Hamiltonian if and only if the map below vanishes identically.*

$$\begin{aligned} \rho : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] &\longrightarrow H^1(M, \mathbb{R}) \\ [x] &\longmapsto [i_{x_M}\omega] \end{aligned}$$

This theorem gives a necessary and sufficient condition for the existence of a momentum map but it is not enough to guarantee than such a momentum map is equivariant. In general, a momentum map is equivariant modulo a cocycle which vanishes if, for example, the group G is compact or if M is a cotangent bundle (cf. [52] Section 4.5 for further details). We now give some well-known examples of (equivariant) momentum maps. Many other examples are presented in [44] (Section 11.4).

Example 2.3.9. The unitary group $U(n)$ consists of complex matrices $B \in \mathrm{M}_n(\mathbb{C})$ satisfying $BB^* = B^*B = I_n$, where B^* is the conjugate transpose of B. This Lie group acts on $\mathbb{C}^n$ by isometries. Its Lie algebra $\mathfrak{u}(n)$ consists of skew-hermitian matrices i.e. matrices $A \in \mathrm{M}_n(\mathbb{C})$ such that $H(Au, v) + H(u, Av) = 0$, or equivalently matrices such that $A = -A^*$. Since $U(n)$ preserves the hermitian structure of $\mathbb{C}^n$, it also preserves the symplectic form ω_0 on $\mathbb{C}^n$. It is in fact Hamiltonian with momentum map $\Phi : \mathbb{C}^n \to \mathfrak{u}(n)^*$ defined by

$$\langle \Phi(z), A \rangle = -\frac{1}{2}\omega_0(Az, z). \tag{2.16}$$

To show this is indeed a momentum map, let $v \in T_z\mathbb{C}^n \simeq \mathbb{C}^n$ and observe that

$$\begin{aligned}
\langle D\Phi(z) \cdot v, A \rangle &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \Phi(z + tv), A \rangle \\
&= -\frac{1}{2}\left(\omega_0(Av, z) + \omega_0(Az, v)\right) \\
&= -\frac{i}{2}\left(H(Az, v) - H(v, Az)\right) \\
&= -\omega_0(Az, v)
\end{aligned}$$

where Az is the infinitesimal generator for the $U(n)$-action on $\mathbb{C}^n$, at the point z.

Example 2.3.10. By linearity, the action of $U(n)$ on $\mathbb{C}^n$ preserves the set of lines in $\mathbb{C}^n \setminus \{0\}$ and thus descends to an action on $\mathbb{C}P^{n-1}$ making the orbit map $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ equivariant. This action preserves ω_{FS} and is Hamiltonian. A momentum map $\widetilde{\Phi} : \mathbb{C}P^{n-1} \to \mathfrak{u}(n)^*$ is given by

$$\langle \widetilde{\Phi}(\underline{z}), A \rangle = -\frac{i}{2}\frac{H(Az, z)}{\|z\|^2}. \tag{2.17}$$

To construct it in this form, we start by working in the chart $\varphi : \mathbb{C}^{n-1} \to \mathbb{C}P^{n-1}$ as in (2.13). This coordinate chart factors through the sphere S^{2n-1} as

$$\varphi = \left.\pi\right|_U \circ i$$

where $i : \mathbb{C}^{n-1} \hookrightarrow S^{2n-1}$ is the embedding

$$(z_0, \ldots, z_{n-2}) \longmapsto \frac{(z_0, \ldots, z_{n-2}, 1)}{\sqrt{1 + \sum_{i=0}^{n-2} |z_i|^2}}, \tag{2.18}$$

and $U \subset S^{2n-1}$ is the image of this embedding. We define $\widetilde{\Phi} : \mathbb{C}P^{n-1} \to \mathfrak{u}(n)^*$ in the chart as

$$\varphi^*\widetilde{\Phi} = i^*\left(\left.\Phi\right|_U\right)$$

with Φ as in (2.16). Explicitly, for $A \in \mathfrak{u}(n)$ and $\underline{z} = [z_0 : \cdots : z_{n-2} : 1]$ in this chart, we have

$$\begin{aligned}
\langle \widetilde{\Phi}(\underline{z}), A\rangle &= \langle \Phi\left(\frac{z}{\|z\|}\right), A\rangle \\
&= -\frac{1}{2}\frac{\omega_0(Az, z)}{\|z\|^2} \\
&= -\frac{i}{2}\frac{H(Az, z)}{\|z\|^2}.
\end{aligned}$$

Since this result does not depend on the chart, $\widetilde{\Phi}$ extends globally as in (2.17).

2.3.3 Symplectic reduction

The well-known Noether's theorem states that, to each (smooth) symmetry of a dynamical system corresponds a conserved quantity. For Hamiltonian systems this quantity is expressed in term of a momentum map. More explicitly assume that h is a G-invariant Hamiltonian defined on a Hamiltonian G-manifold (M, ω, G, Φ_G). In this case, Noether's Theorem says essentially that the flow lines φ_t of the G-equivariant Hamiltonian vector field X_h, are confined to level sets of the momentum map (cf.[44] Theorem 11.4.1). In particular, the number of variables of the system can be reduced. In a more abstract language, it means that the quotient space obtained by quotienting a level set $\Phi_G^{-1}(\mu)$ of the momentum map by the subgroup of G preserving it, and the reduced Hamiltonian h_μ defined on it, form a Hamiltonian system in their own right, provided that the group action is nice enough. This is the process of symplectic reduction of Marsden and Weinstein [41].

Theorem 2.3.11 (Symplectic Reduction)**.** *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold on which G acts freely. Then the reduced space $M_\mu = \Phi_G^{-1}(\mu)/G_\mu$ admits a symplectic form ω_μ, uniquely defined by the relation $\pi_\mu^*\omega_\mu = i_\mu^*\omega$. The maps $\pi_\mu : \Phi_G^{-1}(\mu) \to M_\mu$ and $i_\mu : \Phi_G^{-1}(\mu) \to M$ are the quotient map and the inclusion map, respectively.*

In the example below we show that the Fubini-Study symplectic form can be understood as the reduced symplectic form arising from a symplectic reduction process. The reader is invited to consult the excellent lecture notes of Berline and Vergne [7] (Section 2.3) for a more general exposition in the case of projective varieties.

Example 2.3.12. The Fubini-Study symplectic $\omega_{FS} \in \Omega^2(\mathbb{C}P^{n-1})$ of Example 2.3.4 arises from symplectic reduction. We first show that there is a unique symplectic form $\omega_{red} \in \Omega^2(\mathbb{C}P^{n-1})$ such that

$$\pi_a^*\omega_{red} = \omega_0\Big|_{S^{2n-1}}$$

where S^{2n-1} is a level set $\phi^{-1}(a)$ of a momentum map $\phi : \mathbb{C}^n \to \mathfrak{u}(1)^*$ arising from a Hamiltonian action of $U(1)$ on $\mathbb{C}^n$, and $\pi_a : \phi^{-1}(a) \to \phi^{-1}(a)/U(1)$ is the orbit map.

As before S^1 is identified with $U(1)$, the complex numbers of the form $e^{i\theta}$ with $\theta \in \mathbb{R}$. This group acts on $\mathbb{C}^n$ by multiplying each component on the right by $e^{i\theta}$. Since this action is hermitian, it also preserves the canonical symplectic form ω_0. Moreover this action is Hamiltonian with momentum map $\phi : \mathbb{C}^n \to \mathfrak{u}(1)^*$. If we think of $U(1)$ as embedded in $U(n)$, the momentum map ϕ can be computed using the formula derived in (2.16). Explicitly, we identify both $\mathfrak{u}(1)$ and its dual $\mathfrak{u}(1)^*$ with the pure imaginary complex numbers. Given $x \in \mathfrak{u}(1)$, we have

$$\langle \phi(z), x \rangle = -\frac{1}{2}\omega_0(xz, z) = -\frac{i}{4}\left(H(xz, z) - H(z, xz)\right) = -\frac{i}{2}x\sum_{i=0}^{n-1}|z_i|^2.$$

Therefore $\phi(z) = \frac{i}{2}\|z\|^2$. For $a = \frac{i}{2}$ we have that $\phi^{-1}(a) = S^{2n-1}$ and this level set is invariant by the $U(1)$-action. We can apply the symplectic reduction process to get the existence of a unique 2-form ω_{red} on the reduced space $\phi^{-1}(a)/U(1)$, uniquely defined by the relation

$$\pi_a^*\omega_{red} = \omega_0\Big|_{S^{2n-1}}.$$

It remains to show that ω_{red} coincides with the Fubini-Study symplectic form defined in (2.12). It is sufficient to show that they coincide in the chart φ :

$\mathbb{C}^{n-1} \to \mathbb{C}P^{n-1}$ defined in (2.13). This chart factors through the sphere S^{2n-1} via the embedding $i : \mathbb{C}^{n-1} \hookrightarrow S^{2n-1}$ defined in (2.18). Let $U \subset S^{2n-1}$ be the image of this embedding. Then

$$\varphi = \pi_a\big|_U \circ i.$$

By looking at the coordinates $(w_0, \ldots, w_{n-2}, 1)$ on U with

$$w_j \circ i = \frac{z_j}{\sqrt{1 + \sum_{i=0}^{n-2} |z_i|^2}},$$

it is a straightforward calculation to show that

$$i^* \left(\omega_0\big|_U\right) = i^* \left(\frac{i}{2} \sum_{j=0}^{n-2} dw_j \wedge d\overline{w}_j\right)$$

coincides with $\varphi^*\omega_{FS}$ as derived above. Using Proposition 2.3.12 we obtain

$$\varphi^*\omega_{red} = (\pi_a\big|_U \circ i)^*\omega_{red} = i^*(\pi_a\big|_U)^*\omega_{red} = i^* \left(\omega_0\big|_U\right) = \varphi^*\omega_{FS}$$

which shows that ω_{FS} coincides with the reduced symplectic form.

2.4 Symplectic tubular neighbourhoods of group orbits

In this section we introduce the Symplectic Tube Theorem (Theorem 2.4.1) which is a fundamental result to study both, the local dynamics and the local geometry of a Hamiltonian proper G-manifold (M, ω, G, Φ_G). It states essentially that every $m \in M$ admits a G-invariant neighbourhood, which is G-equivariantly symplectomorphic to a neighbourhood of the zero section of a symplectic associated bundle. This contruction provides tractable semi-global coordinates for M near G-orbits. Those coordinates are sometimes referred as *slice coordinates*. This theorem was obtained by Guillemin and Sternberg [26] and by Marle [40], for canonical Lie group actions with equivariant momentum map. It has been extended independantly by Ortega and Ratiu [52] and by Bates and Lerman [5], for general canonical Lie group actions. Schmah [64] and Perlmutter, Rodríguez-Olmos and Sousa-Dias [59] studied the case when M is a cotangent bundle.

2.4.1 Witt-Artin decomposition

We briefly recall the construction underlying the Symplectic Tube Theorem. The reader is referred to Ortega and Ratiu [52] (Chapter 7) or Cushman and Bates [52] (Appendix B Section 3.2) for details. Let $m \in M$ with momentum $\mu = \Phi_G(m)$. Denote by G_m and G_μ the stabilizers of m and μ respectively, and by $\mathfrak{g}_m$ and $\mathfrak{g}_\mu$ their respective Lie algebras. The stabilizer G_m is compact by properness of the G-action on M. We can thus split $\mathfrak{g}_\mu$ and $\mathfrak{g}$ into a direct sum of G_m-invariant subspaces

$$\mathfrak{g}_\mu = \mathfrak{g}_m \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{n}.$$

The tangent space T_mM can be decomposed into a direct sum of four G_m-invariant subspaces

$$T_mM = T_0 \oplus T_1 \oplus N_0 \oplus N_1 \tag{2.19}$$

with respect to which the skew-symmetric matrix associated to $\omega(m)$ has a specific normal form. This decomposition was first introduced by Witt [70] for symmetric bilinear forms. The part $T_0 \oplus T_1$ corresponds to the directions tangent to the group orbit whereas $N_0 \oplus N_1$ is a specific choice of normal space. Those subspaces are defined as follows:

(i) $T_0 := \ker(D\Phi_G(m)) \cap \mathfrak{g} \cdot m = \mathfrak{g}_\mu \cdot m$.

(ii) $T_1 := \mathfrak{n} \cdot m$ which is a symplectic vector subspace of $(T_mM, \omega(m))$.

(iii) N_1 is a choice of G_m-invariant complement to T_0 in $\ker(D\Phi_G(m))$. It is a symplectic subspace of $(T_mM, \omega(m))$ and is called the *symplectic slice*. The linear action of G_m on N_1 is globally Hamiltonian with momentum map $\Phi_{N_1} : N_1 \to \mathfrak{g}_m^*$ given by

$$\langle \Phi_{N_1}(\nu), x \rangle = \frac{1}{2}\omega(x_{N_1}(\nu), \nu)$$

for all $\nu \in N_1$ and $x \in \mathfrak{g}_m$.

(iv) N_0 is a G_m-invariant Lagrangian complement to T_0 in the symplectic orthogonal $(T_1 \oplus N_1)^{\omega(m)}$. Moreover, there is an isomorphism $f : N_0 \to \mathfrak{m}^*$ given by $\langle f(w), y \rangle = \omega(m)(y_M(m), w)$ for all $w \in N_0$ and $y \in \mathfrak{m}$.

A splitting (2.19) is called a *Witt-Artin decomposition* of T_mM, relative to the G-action. Note that the symplectic form $\omega(m)$ restricted to T_1 coincides with the Kostant-Kirillov-Souriau symplectic form. Moreover, the symplectic form $\omega(m)$ restricted to $T_0 \oplus N_0$ takes the form

$$\omega(m)(x_M(m) + w, x'_M(m) + w') = \langle f(w'), x\rangle - \langle f(w), x'\rangle \tag{2.20}$$

for every $x, x' \in \mathfrak{m}$, $w, w' \in N_0$, and f as in (iv). Indeed since $y_M(m) = 0$ for every $y \in \mathfrak{g}_m$, the elements of T_0 are of the form $x_M(m)$ with $x \in \mathfrak{m}$. Let $x, x' \in \mathfrak{m}$ and $w, w' \in N_0$. As both T_0 and N_0 are Lagrangian in $T_0 \oplus N_0$,

$$\omega(m)(x_M(m) + w, x'_M(m) + w') = \omega(m)(x_M(m), w') + \omega(x'_M(m), w)$$

which coincides with $\langle f(w'), x\rangle - \langle f(w), x'\rangle$.

Since N_1 is a G_m-invariant subspace, there is a well-defined action of G_m on the product $G \times \mathfrak{m}^* \times N_1$ given by

$$k \cdot (g, \rho, \nu) = (gk^{-1}, \mathrm{Ad}^*_{k^{-1}}\rho, k \cdot \nu). \tag{2.21}$$

This action is free and proper by freeness and properness of the action on the first factor. The orbit space Y is thus a smooth manifold whose points are equivalence classes of the form $[(g, \rho, \nu)]$. The group G acts smoothly and properly on Y, by left multiplication on the first factor. Let $\mathfrak{m}^*_0 \subset \mathfrak{m}^*$ and $(N_1)_0 \subset N_1$ be G_m-invariant neighbourhoods of zero in $\mathfrak{m}^*$ and N_1, respectively. Then

$$Y_0 := G \times_{G_m} (\mathfrak{m}^*_0 \times (N_1)_0) \tag{2.22}$$

is a neighbourhood of the zero section in Y. It comes with a symplectic structure ω_{Y_0} if it is chosen small enough ([52] Proposition 7.2.2). Define the *Chu map* $\Psi : M \to Z^2(\mathfrak{g})$ associated to the G-action by

$$\Psi(m)(x, y) := \omega(m)(x_M(m), y_M(m)). \tag{2.23}$$

Note that $\Psi(m)(x, y) = -\langle\mu, [x, y]\rangle$, and thus $\Psi(m)$ coincides with the Kostant-Kirillov-Souriau symplectic form on the coadjoint orbit $G \cdot \mu$ whenever $x, y \in \mathfrak{n}$.

2.4.2 The Symplectic Tube Theorem

We can now state the symplectic analogue of the Tube Theorem 2.1.5.

Theorem 2.4.1 (Symplectic Tube Theorem). *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold. Let $m \in M$ with momentum $\mu = \Phi_G(m)$. If the neighbourhood Y_0 defined in (2.22) is sufficiently small, it admits a symplectic structure ω_{Y_0}. In this case, there exists a G-invariant neighbourhood $U \subset M$ of m and a G-equivariant symplectomorphism*

$$\varphi : (Y_0, \omega_{Y_0}) \to (U, \omega\big|_U)$$

such that $\varphi([e, 0, 0]) = m$.

We call the triplet (φ, Y_0, U) a *symplectic G-tube* at m and we also say that (Y_0, ω_{Y_0}) is a *symplectic local model* for $(U, \omega\big|_U)$. Besides the momentum map $\Phi_G : M \to \mathfrak{g}^*$ can be expressed in terms of the slice coordinates:

Theorem 2.4.2 (Marle-Guillemin-Sternberg Normal Form Theorem). *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold and let (φ, Y_0, U) be a symplectic G-tube at $m \in M$. Then the G-action on Y_0 is globally Hamiltonian with associated momentum map $\widetilde{\Phi}_G : Y_0 \to \mathfrak{g}^*$ defined by*

$$\widetilde{\Phi}_G([g, \rho, \nu]) = Ad^*_{g^{-1}}(\Phi_G(m) + \rho + \Phi_{N_1}(\nu)). \tag{2.24}$$

It coincides with $\Phi_G\big|_U$ when pulled back along φ^{-1}.

THE EQUIVARIANT LYUSTERNIK-SCHNIRELMANN CATEGORY

The *Lyusternik-Schnirelmann category* or *LS-category* of a topological space X is the homotopical invariant $\mathrm{Cat}(X)$ defined to be the least number of open subsets $U \subset X$, whose inclusion is nullhomotopic, that are required to cover X. Although it is now the subject of a full theory in connection with algebraic topology, it was originally introduced by Lyusternik and Schnirelmann in a course on the global calculus of variations, when X is a smooth compact manifold without boundary [39]. In this case they show that any $f \in C^\infty(X)$ has at least $\mathrm{Cat}(X)$ critical points. The difference with Morse theory is that f is allowed to have degenerate critical points. However this is in no way a generalization of Morse theory because a Morse function determines entirely the topological structure of the underlying manifold. Indeed in this case X has the homotopy type of a CW-complex and each cell is determined by exactly one critical point in the sense that its dimension is the Morse index of the critical point [49]. Rewiews on the Lyusternik-Schnirelmann theory are for instance [29, 13, 4].

3.1 Terminologies

A topological space X is said to be *completely regular* if, for any closed subset Y of X and any $y \notin Y$, there is a continuous map $f \in C(X)$ sending y to 0, and Y to 1. In this section a pair (X, G), where X is a completely regular topological space on which a topological group G acts continuously, is called a *G-space.* If the action is proper, we refer to it as a *proper G-space.* The equivariant analogue of the LS-category has been introduced by Fadell [19] and Marzantowicz [46] for compact groups, and by Colman [12] for finite groups. A substantial part of the theory has been extended for non-compact Lie groups when (X, G) is a proper G-space by Ayala, Lasheras and Quintero [3], thanks to the result of Palais [53] on the existence of slices for proper Lie group actions.

3.1.1 G-categorical open subsets

Let (X, G) be a proper G-space. A homotopy $H : X \times [0, 1] \to X$ which satisfies $H(g \cdot x, t) = g \cdot H(x, t)$ for every $g \in G$, $x \in X$ and $t \in [0, 1]$ is called a *G-homotopy*. We write $H_t(x) = H(x, t)$. Let $A, B \subset X$ be two G-invariant subsets. A *G-deformation retract of A onto B* is a G-homotopy $H : A \times [0, 1] \to A$ such that $H_0(x) = x$, $H_1(x) \in B$ for every $x \in A$, and $H_1(b) = b$ for every $b \in B$.

Definition 3.1.1. A G-invariant subset $U \subset X$ is called *G-categorical* if there exists a G-deformation retract of U onto the orbit $G \cdot x$ of some $x \in U$.

Definition 3.1.2. Given a G-invariant subset $A \subset X$, the *equivariant LS-category of A in X*, denoted $\mathrm{Cat}_G(A, X)$, is the least number of G-categorical open subsets $U \subset X$ that are required to cover A. If no such cover exists, we set $\mathrm{Cat}_G(A, X) = \infty$. Furthermore we write $\mathrm{Cat}_G(X) = \mathrm{Cat}_G(X, X)$ and $\mathrm{Cat}_G(X) = \infty$ if such a cover does not exist. The non-equivariant LS-categories $\mathrm{Cat}(A, X)$ and $\mathrm{Cat}(X)$ are obtained by setting $G = \mathbb{1}$.

Observe that in particular a G-invariant subset $A \subset X$ is G-categorical if and only if $\mathrm{Cat}_G(A, X) = 1$.

Proposition 3.1.3 (Marzantowicz [46])**.** *Let (X, G) be a proper G-space and let A, B be G-invariant open subsets of X.*

(i) (Subadditivity) $Cat_G\,(A \cup B, X) \leq Cat_G\,(A, X) + Cat_G\,(B, X)$.

(ii) (Invariance) If $\varphi : X \to X$ is a G-equivariant homeomorphism, then $Cat_G\,(A, X) = Cat_G\,(\varphi(A), X)$.

Example 3.1.4. The equivariant version of the LS-category is in general different from its non-equivariant analogue, as shown in the examples below.

(i) Let $X = S^1 \times \mathbb{R}$ with cylindrical coordinates (θ, z). Define an S^1-action on X by $\phi \cdot (\theta, z) = (\theta + \phi, z)$. The cylinder itself is an S^1-categorical open subset with S^1-deformation retract $H : X \times [0,1] \to X$ given by $H((\theta, z), t) = (\theta, (1-t)z)$. Therefore, $\mathrm{Cat}_{S^1}(X) = 1$. However we require two contractible open subsets to cover X, which yields

$$1 = \mathrm{Cat}_{S^1}(X) < \mathrm{Cat}(X) = 2.$$

(ii) Consider the complex projective space $X = \mathbb{C}P^2$ with the S^1-action

$$\theta \cdot [z_0 : z_1 : z_2] = [e^{i\theta} z_0 : z_1 : z_2].$$

For $i = 0, 1, 2$, the open subsets $U_i = \{[z_0 : z_1 : z_2] \mid z_i \neq 0\}$ are S^1-invariant. On U_0, an S^1-deformation retract onto an orbit is given by

$$H([z_0 : z_1 : z_2], t) = [z_0 : (1-t)z_1 : (1-t)z_2].$$

The image $H_1(U_0)$ is the single point $[1 : 0 : 0]$ which is a fixed point of the action, hence an S^1-orbit. Similar homotopies can be found on U_1 and U_2, respectively. Therefore $\mathrm{Cat}_{S^1}(X)$ is at most three. The fact that we have an equality follows from Proposition 3.2.3 below. We conclude that

$$\mathrm{Cat}_{S^1}(X) = \mathrm{Cat}(X) = 3.$$

(iii) The group $\mathbb{T}$ acts on $X = S^2$ as in Example 2.2.4 (iii). We construct a cover of X by three $\mathbb{T}$-categorical open subsets as follows:

Pick a point $x_1 \in X$ and its opposite point $y_1 \in X$. The $\mathbb{T}$-orbit of x_1 forms a spherical tetrahedron with vertices x_1, x_2, x_3, x_4. Similarly the $\mathbb{T}$-orbit of y_1 forms another spherical tetrahedron with vertices y_1, y_2, y_3, y_4. For each $i < j$ denote by p_{ij} the middle point of the geodesic arc joining x_i and x_j.

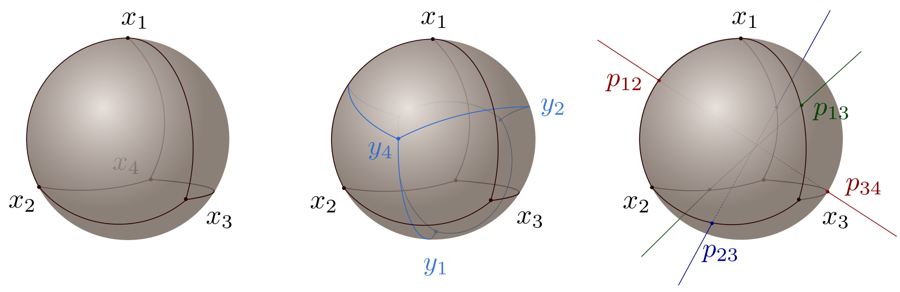


Figure 3.1: Spherical tetrahedrons on the sphere.

For each i, let $D_i \subset X$ be an open disk centered at x_i such that

$$\overline{D_i} \cap \overline{D_j} = \{p_{ij}\} \quad \forall i < j.$$

In the same way, let for each i, an open disk $E_i \subset X$ centered at y_i with the property

$$\forall i < j \quad \overline{E_i} \cap \overline{E_j} = \{p_{kl}\} \quad \text{where} \quad k, l \notin \{i, j\} \quad k < l$$

as shown in Figure 3.2.

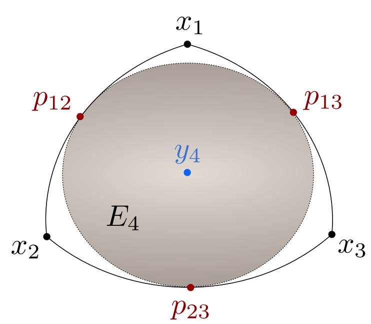


Figure 3.2: Disk E_4 centered at y_4.

Finally we define for each $i < j$, an open subset $B_{ij} \subset X$ containing p_{ij} such that

$$
\begin{aligned}
x_k &\in \overline{B_{ij}} \setminus B_{ij} \qquad \forall k = i, j \\
y_k &\in \overline{B_{ij}} \setminus B_{ij} \qquad \forall k \neq i, j
\end{aligned}
$$

We obtain the following $\mathbb{T}$-categorical open subsets:

$$D = \bigcup_{i=1}^{4} D_i$$

which retracts in a $\mathbb{T}$-equivariant way onto the orbit $\mathbb{T} \cdot x_1$,

$$E = \bigcup_{i=1}^{4} E_i$$

which retracts in a $\mathbb{T}$-equivariant way onto the orbit $\mathbb{T} \cdot y_1$, and

$$B = \bigcup_{i<j} B_{ij}$$

which retracts in a $\mathbb{T}$-equivariant way onto the orbit $\mathbb{T} \cdot p_{12}$. Those three subsets form a cover of X. This cover is in fact the smallest that we can take, by Proposition 3.2.3 below. Hence

$$3 = \mathrm{Cat}_{\mathbb{T}}(X) > \mathrm{Cat}(X) = 2.$$

3.1.2 LS-categories and orbit spaces

In this section, (X, G) is a proper G-space. In [3] (Proposition 2.4), Ayala, Lasheras and Quitero extended the result of Marzantowicz [46] for compact groups stating that

$$\mathrm{Cat}_G(X) \geq \mathrm{Cat}(X/G)$$

with equality under the additional assumptions that X is metrizable and G acts with only one orbit type, in particular freely.

Example 3.1.5. Let $S^5 \subset \mathbb{C}^3$ be given by

$$S^5 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 \mid \sum_{i=0}^{2} |z_i|^2 = 1\}.$$

We identify S^1 with the complex numbers of length one and we let it act on S^5 by right multiplication on each factor. This action is free and proper. The orbit space $\mathbb{C}P^2$ is therefore a smooth manifold and the orbit map is a principal S^1-bundle $\pi : S^5 \to \mathbb{C}P^2$. We thus have

$$\mathrm{Cat}_{S^1}(S^5) = \mathrm{Cat}(\mathbb{C}P^2) = 3.$$

Example 3.1.6 (Real projective spaces)**.** Let $G = \mathbb{Z}_2$ acting on S^n by the antipodal map. The orbit space for this action is homeomorphic to $\mathbb{R}P^n$ and the quotient map $\pi : S^n \to \mathbb{R}P^n$ is a covering map. We make use of the following result:

Theorem 3.1.7 ([46] Corollary 1.17)**.** *If G is a finite group acting freely on the n-dimensional sphere S^n then $Cat_G(S^n) = n + 1$.*

Since G is a finite group acting freely on S^n, Theorem 3.1.7 yields

$$\mathrm{Cat}(\mathbb{R}P^n) = \mathrm{Cat}_G(S^n) = n + 1.$$

Example 3.1.8 (Lens Spaces)**.** Let $S^{2n-1} \subset \mathbb{C}^n$ be given by

$$S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |z_i|^2 = 1\}.$$

Let $\varepsilon = e^{2\pi i/p}$ be a primitive pth root of unity and let $q_1, \ldots, q_n$ be integers relatively prime to p. We let $\mathbb{Z}_p = \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1}\}$ act on S^{2n-1} by

$$\varepsilon \cdot (z_1, \ldots, z_n) = (\varepsilon^{q_1} z_1, \ldots, \varepsilon^{q_n} z_n). \tag{3.1}$$

The orbit map $\pi : S^{2n-1} \to S^{2n-1}/\mathbb{Z}_p$ is a covering map as a consequence of [9] (Proposition 7.2). The orbit space $S^{2n-1}/\mathbb{Z}_p$ denoted $L(p; q_1, \ldots, q_n)$ is called the $(2n-1)$-dimensional *lens space*. By Theorem 3.1.7 we get

$$\operatorname{Cat}\left(L(p; q_1, \ldots, q_n)\right) = \operatorname{Cat}_{\mathbb{Z}_p}\left(S^{2n-1}\right) = 2n.$$

3.2 G-tubular covers

If (M, G) is a proper G-manifold, the Tube Theorem 2.1.5 allows us to produce G-categorical open subsets in the following way: any $m \in M$ admits a G-invariant neighbourhood $U \subset M$ such that the map $\varphi : Y_0 \to U$ defined in (2.4) is a G-equivariant diffeomorphism. Here

$$Y_0 = G \times_{G_m} N_0$$

where N_0 is a fixed neighbourhood of zero in some subspace $N \subset T_m M$, complementary to $\mathfrak{g} \cdot m$ in $T_m M$, on which G_m acts linearly. The proper G-manifold Y_0 is a local model for U, in which m reads $\varphi^{-1}(m) = [e, 0]$. The G-homotopy $F : Y_0 \times [0,1] \to Y_0$ defined by

$$F([(g, \nu)], t) = [(g, (1-t)\nu)].$$

is a G-deformation retract of Y_0 onto the orbit $G \cdot [e, 0]$. By using the fact that φ is a G-equivariant diffeomorphism, the open subset $U = \varphi(Y_0)$ is G-categorical since the G-homotopy $H : U \times [0,1] \to U$ given by

$$H(p, t) = \varphi\left(F(\varphi^{-1}(p), t)\right) \tag{3.2}$$

is a G-deformation retract of U onto $G \cdot m$.

Definition 3.2.1. A G-categorical open subset $U \subset M$ as above, with associated G-deformation retract as in (3.2), is called a *G-tubular open subset* of M. A cover of M made of G-tubular open subsets is called a *G-tubular cover* of M.

Clearly, every $m \in M$ admits a neighbourhood which is a G-tubular open subset of M. Consequently, G-tubular covers of M always exist. The question is whether they can be refined. Let $\mathcal{U}$ be any G-tubular cover of M. We know that M can be decomposed into the disjoint union of its orbit-type strata $\{M_\beta \mid \beta \in \mathcal{B}\}$, which form themselves a smooth $\mathcal{B}$-stratification of M. Let $\mathcal{B}' \subset \mathcal{B}$ be the biggest subset of indices $\beta \in \mathcal{B}$ such that M_β is minimal with respect to (2.8). Consider the disjoint union A of all the strata M_β with $\beta \in \mathcal{B}'$. From $\mathcal{U}$ we extract a subcover $\mathcal{U}'$, chosen as small as possible such that $\mathcal{U}'$ covers A. In particular $\mathcal{U}'$ is a refinement of $\mathcal{U}$. We ask the following:

(Q) *Does it exist $\mathcal{U}' \subset \mathcal{U}$, obtained as above, which is still a cover of M?*

The answer is in general negative (cf. Section 3.2.1). However it is positive for all the proper G-manifolds listed in Example 3.1.4, where $\mathcal{U}'$ is constructed explicitly.

Definition 3.2.2. Let (M, G) be a proper G-manifold. The subcover $\mathcal{U}'$ defined above is called a *minimal G-tubular cover* if the following occur:

(i) $\mathcal{U}'$ is a cover of M.

(ii) For each minimal orbit-type stratum M_β, the set

$$\mathcal{V}'_\beta = \{V_\beta = U \cap M_\beta \mid U \in \mathcal{U}'\}$$

is the smallest cover by G-categorical open subsets of M_β, where the topology of M_β is the subset topology.

We discuss the simplest example where such a cover exists. Let $S^2 \subset \mathbb{R}^3$, on which S^1 acts by rotations about the z-axis. This action has two minimal orbit-type strata, namely the North and South pole. Two small disks centered at those points are S^1-tubular open subsets and can be taken sufficiently big so that they form a minimal S^1-tubular cover of S^2. In this example, a disk centered at the North pole can be extended until its closure meets the South pole. The impossibility to extend it further relies on the fact that such neighbourhoods are constructed by mean of the Riemannian exponential map. This map is no longer injective if the disk contains two opposite points on the sphere. The next proposition gives another answer to this fact by using the properties of G-tubular open subsets.

Proposition 3.2.3. *Let (M, G) be a proper G-manifold. If $U \subset M$ is a G-tubular open subset which intersects a minimal orbit-type stratum M_β, then U retracts onto the orbit $G \cdot x$ of some $x \in M_\beta$. In particular G-tubular open subsets intersect at most one minimal orbit-type stratum.*

Proof. Let $\beta = ((H), (b)) \in \mathcal{B}$ such that M_β is a minimal orbit-type stratum. Let $U \subset M$ be a G-tubular open subset of M such that $U \cap M_\beta \neq \varnothing$, and let $H : U \times [0, 1] \to U$ be a G-deformation retract of U onto $G \cdot x$ for some $x \in M$. By contradiction, assume that $x \in M_\alpha$ for some $\alpha = ((G_x), (a)) \neq \beta$.

Each point $y \in U \cap M_\beta$ has stabilizer $G_y \in (H)$. By G-equivariance of the homotopy, G_y is a subgroup of $G_{H_1(y)}$ which is itself conjugate to G_x, as $H_1(y)$ and x lie on the same orbit. In particular $(G_x) \prec_{conj} (H)$. Two cases occur:

(i) If $M_\beta \cap \overline{M}_\alpha \neq \varnothing$, then $\beta \prec \alpha$ since $\beta \neq \alpha$. By Lemma 2.2.8 we get $(H) \prec_{conj} (G_x)$ which is a contradiction.

(ii) If $M_\beta \cap \overline{M}_\alpha = \varnothing$ we must use the assumption that U is G-tubular. Let $G \times_{G_x} N_0$ be the local model for U. Given $y \in M_\beta$ we define the G-equivariant path $y(t) = H_t(y)$, where $t \in [0, 1]$. In the local model, y reads $[g, \nu]$ and $y(t)$ reads $[g, \nu_t]$ where $\nu_t = (1 - t)\nu$. We can assume without lost of generality that $(G_x)_\nu = H$. Observe that, by linearity of the G_x-action on N_0, we have $(G_x)_{\nu_t} = (G_x)_\nu = H$ for all $t \neq 1$. Hence $G_{[g,\nu_t]} = g(G_x)_{\nu_t} g^{-1} = gHg^{-1}$ for every $t \neq 1$. In particular, $y(t) \in M_\beta$ for all $t \neq 1$. Since the path $y(t)$ starts at $y \in M_\beta$ and ends on $G \cdot x \subset M_\alpha$, there is some $t_0 \in [0, 1]$ such that $y(t_0) \in \overline{M_\alpha}$. The parameter t_0 is chosen the smallest such that this occurs. If $t_0 \neq 1$, the previous argument shows that $y(t_0) \in M_\beta \cap \overline{M_\alpha}$, which is a contradiction. Otherwise, since $y(t) \in M_\beta$ for all $t < t_0$, there is a sequence $(y_n)_{n \in \mathbb{N}} \subset M_\beta$ which converges to $y(t_0)$. By closedness of M_β, this yields $y(t_0) \in M_\beta \cap \overline{M_\alpha}$, which is again a contradiction. We conclude that $x \in M_\beta$.

■

3.2.1 Non-examples

The answer to question (Q) is in general negative. In the examples below, (M, G) is a proper G-manifold with M compact, and the action admits only one closed orbit-type stratum M_β.

Example 3.2.4. The first example discussed below is a compact S^1-manifold with only one closed orbit-type stratum, which is an S^1-orbit.

(i) Think of $M = S^3$ as the set of unit vectors $(z_1, z_2) \in \mathbb{C}^2$ equipped with the S^1-action

$$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{2i\theta} z_2).$$

This action has only one minimal orbit-type stratum M_β with $\beta = ((\mathbb{Z}_2), (b))$ for some index $b \in \mathcal{B}_{\mathbb{Z}_2}$. Explicitly

$$M_\beta = \left\{(0, z_2) \in \mathbb{C}^2 \mid |z_2|^2 = 1\right\}$$

which is diffeomorphic to a circle. In particular M_β is the S^1-orbit of the point $(0, 1) \in S^3$. The S^1-invariant open subset

$$U = \left\{(z_1, z_2) \in S^3 \mid |z_1|^2 < \frac{2}{3}\right\}$$

is an S^1-invariant tubular neighbourhood of the minimal orbit-type stratum, and is diffeomorphic to a solid torus (cf. Figure 3.3). Since M_β is an S^1-orbit, U is an S^1-tubular open subset. We may choose $\mathcal{U}' = \{U\}$. This cover satisfies (ii) of Definition 3.2.2 but it does not satisfy (i), since it does not cover M. To cover M we require the additional open subset

$$V = \left\{(z_1, z_2) \in S^3 \mid |z_1|^2 > \frac{1}{3}\right\}$$

which is also a solid torus, understood as an S^1-invariant tubular neighbourhood of the S^1-orbit of $(1, 0)$ (cf. Figure 3.3). It is therefore S^1-categorical and then $\mathrm{Cat}_{S^1}(M) \leq 2$. There is in fact equality because otherwise it would mean that S^3 is contractible onto a circle, which is untrue.

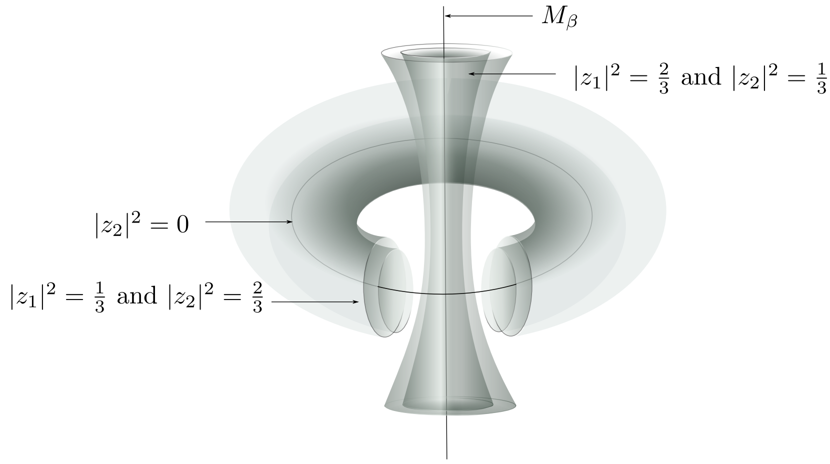


Figure 3.3: Representation in $\mathbb{R}^3$ of the sphere S^3 with a point removed. The stratum M_β is a circle closing at infinity and the tori around it form a solid torus, which is a tubular neighbourhood.

(ii) We discuss a more complicated example of an orientable compact smooth manifold equipped with a canonical S^1-action with only one closed orbit-type stratum, which is not an S^1-orbit. This example was found and suggested to us by Eckhard Meinrenken.

Let $S^2 \subset \mathbb{R}^3$ with cartesian coordinates (x, y, z). Consider the diffeomorphism $\tau_1 : S^2 \to S^2$ sending a point $m = (x, y, z) \in S^2$ onto $\tau_1(m) = (x, y, -z) \in S^2$. Similarly let $S^1 \subset \mathbb{R}^2$ with cartesian coordinates (u, v) and $\tau_2 : S^1 \to S^1$ sending a point $w = (u, v) \in S^1$ onto $\tau_2(w) = (u, -v) \in S^1$. Let

$$M = S^2 \times S^1$$

and the orientation-preserving diffeomorphism $f : M \to M$ defined by

$$f(m, w) = (\tau_1(m), \tau_2(w)) \quad \forall (m, w) \in S^2 \times S^1.$$

The mapping torus of f is the 4-dimensional smooth manifold

$$M_f = \frac{M \times [0, 1]}{((m, w), 0) \sim (f(m, w), 1)}.$$

It can be viewed as the total space of a smooth fiber bundle over S^1 with fiber M. Elements of M_f are equivalence classes of the form $[(m,w),t]$. We define an S^1-action on M_f as follows:

$$\theta \cdot [(m,w),t] = [(R_\theta(m),w),t]$$

where R_θ denotes the rotation in $\mathbb{R}^3$ of angle θ about the z-axis. This is well-defined since such rotations fix the z-axis and thus f above is S^1-equivariant. The minimal stratum has orbit-type (S^1) and is the S^1-orbit of

$$(M_f)_{S^1} = \left\{ [(N,w),t] \in M_f \mid N = (0,0,1) \in S^2, w \in S^1, t \in [0,1] \right\}.$$

The latter is diffeomorphic to a Klein bottle. We set $\beta = ((S^1),(b))$. Since the circle acts trivially on $(M_f)_{S^1}$, the minimal stratum is $M_\beta = (M_f)_{S^1}$.

Let $V_1 \subset S^2$ be a disk centered at the North pole, which does not meet the equator. Define V to be the union of V_1 and $\tau_1(V_1)$. The subset $U = V \times S^1 \times [0,1]$ is open in $M \times [0,1]$ and the corresponding mapping torus U_f is an S^1-invariant tubular neighbourhood of M_β. Hence there is a projection $p : U_f \to M_\beta$. Let $\mathcal{V}'_\beta$ be the smallest cover by G-tubular open subsets of M_β, for the subset topology. We can choose $\mathcal{U}' = \left\{ p^{-1}(V_\beta) \mid V_\beta \in \mathcal{V}'_\beta \right\}$. Then $\mathcal{U}'$ satisfies (ii) of Definition 3.2.2 but it does not satisfies (i).

3.2.2 Tubular covers of symplectic toric manifolds

In this section we show that symplectic toric manifolds admit a minimal tubular cover. Such a cover is constructed explicitly in Theorem 3.2.6. Symplectic toric manifolds form a particular class of algebraic toric varieties and their relations are discussed in the book of Cannas da Silva (cf. [10] Section 6.6).

Definition 3.2.5. Let T be an n-dimensional torus with Lie algebra $\mathfrak{t}$ and dual Lie algebra $\mathfrak{t}^*$. A Hamiltonian T-manifold $(M, \omega, \mathrm{T}, \Phi_{\mathrm{T}})$ is called a *symplectic toric manifold* if (M,ω) is a $2n$-dimensional compact connected symplectic manifold and the Hamiltonian action of T on M is effective.

For symplectic toric manifolds, the image $\Phi_{\mathrm{T}}(M)$ of the momentum map is a

Delzant polytope i.e. a convex polytope $\Delta \subset (\mathbb{R}^n)^*$ which is *simple* i.e. each vertex x meets exactly n edges, *rational* i.e. the edges meeting at a vertex x are of the form $x + t\alpha_{x,i}$ where $\alpha_{x,i} \in (\mathbb{Z}^n)^*$, *smooth* i.e. for each vertex x the isotropy weights $\alpha_{x,1}, \dots, \alpha_{x,n}$ form a $\mathbb{Z}$-basis of $(\mathbb{Z}^n)^*$.

This observation is due to Delzant (cf. [16] Lemmas 2.2 and 2.4). Delzant also proved that Δ determines entirely the symplectic toric manifold $(M, \omega, \mathrm{T}, \Phi_{\mathrm{T}})$, up to T-equivariant symplectomorphisms (cf. [16] Theorem 2.1). His proof relies on a well-known result of convexity obtained independently by Atiyah [2] and Guillemin and Sternberg [25], which states that the image of a momentum map for the action of a torus (not necessarily effective) on a compact symplectic manifold is a convex polytope.

We recall some standard facts about Morse theory applied to a symplectic toric manifold $(M, \omega, \mathrm{T}, \Phi_{\mathrm{T}})$. The reader is referred to the book of Guillemin and Sjamaar (cf. [24] Section 3.6) for details. Let M_{T} be the fixed point set of T. For every $m \in M_{\mathrm{T}}$, the torus acts on the tangent space at m. There is a T-invariant complex structure on M such that $T_m M$ is a complex T-representation with weight space decomposition

$$\mathbb{C}_{\alpha_{m,1}} \oplus \cdots \oplus \mathbb{C}_{\alpha_{m,n}}$$

where $\alpha_{m,1}, \dots, \alpha_{m,n} \in \mathfrak{t}^*$ are the weights of the representation. A *generic component* of the momentum map $\Phi_{\mathrm{T}} : M \to \mathfrak{t}^*$ is a component $\phi^\xi = \langle \Phi_{\mathrm{T}}(\cdot), \xi \rangle$ where $\xi \in \mathfrak{g}$ is generic i.e. $\alpha_{m,i}(\xi) \neq 0$ for every $m \in M_{\mathrm{T}}$ and $i = 1, \dots, n$. In this case, the critical points of ϕ^ξ are isolated and ϕ^ξ is a Morse function whose critical set is precisely M_{T}. Moreover every critical point of ϕ^ξ has even index. Therefore symplectic toric manifolds possess an extra structure given by the properties of the T-action. This structure is used to construct a minimal T-tubular cover of M.

Before proving this, we recall a standard fact of algebraic topology. Let X be a topological spaces and let $I = [0, 1]$. We denote by $C(I, X)$ the set of continuous maps from I to X. The *compact-open topology* on $C(I, X)$ is the topology generated by the subsets of the form

$$\mathcal{O}_{K,U} = \{f \in C(I, X) \mid f(K) \subset U\}$$

where $K \subset I$ is compact and $U \subset X$ is open. Then the map

$$\begin{aligned} C(X \times I, X) &\longrightarrow C(X, C(I, X)) \\ f &\longmapsto \{x \mapsto f_x : t \mapsto f(x,t)\} \end{aligned}$$

is a bijection (cf. [72] Proposition 1.2.3.2).

Theorem 3.2.6. *Let (M, ω, T, Φ_T) be a symplectic toric manifold. Then M admits a minimal T-tubular cover.*

Proof. Let $\{M_\beta \mid \beta \in \mathcal{B}\}$ be the $\mathcal{B}$-stratification of M into orbit-type strata, with strict partial order (2.9). Since T is compact, there are only finitely many minimal orbit-type strata $M_{\beta_1}, \dots, M_{\beta_\ell}$. By assumption on the T-action, each M_{β_i} is an isolated fixed point $m_i \in M_{\mathrm{T}}$. Then there is $\xi_i \in \mathfrak{t}$ such that ϕ^{ξ_i} is a generic component of the momentum map, which takes its minimum at m_i. Let $-\nabla\phi^{\xi_i}$ be the gradient vector field associated to this component, with corresponding flow φ_t. Since the image of the momentum map $\Phi_{\mathrm{T}}(M)$ is a Delzant polytope Δ, the $\mathcal{B}'$-stratification $\{\mathring{F}_{\beta'} \mid \beta' \in \mathcal{B}'\}$ of Δ by open faces (cf. Section 2.2.3) coincides with the $\mathcal{B}$-stratification by orbit-type of M. In other words, for every $i = 1, \dots, \ell$, we can associate to $\beta_i \in \mathcal{B}$ a unique index $\beta_i' \in \mathcal{B}'$ such that $\Phi_{\mathrm{T}}(M_{\beta_i})$ is precisely the zero-dimensional face $\mathring{F}_{\beta_i'}$. For each other index $\alpha \in \mathcal{B}$ there is a unique $\alpha' \in \mathcal{B}'$ such that $\Phi_{\mathrm{T}}(M_\alpha) = \mathring{F}_{\alpha'}$. Define an open subset $V_{\beta_i'} \subset \mathfrak{t}^*$ by

$$V_{\beta_i'} = \bigcup_{\beta_i' \preceq \alpha'} \mathring{F}_{\alpha'}.$$

By continuity and T-invariance of Φ_{T}, the subset $U_{\beta_i} = \Phi_{\mathrm{T}}^{-1}(V_{\beta_i'})$ is a T-invariant open neighbourhood of m_i in M. It reads

$$U_{\beta_i} = \bigcup_{\beta_i \preceq \alpha} M_\alpha.$$

For every $m \in U_{\beta_i} \setminus \{m_i\}$, the flow line $\varphi_t(m)$ is defined for every $t \in \mathbb{R}$, by compactity of M. By construction of U_{β_i}, the point m belongs to some orbit-type stratum M_α with $\beta_i \prec \alpha$. Since φ_t is stratum-preserving, $\varphi_t(m) \in U_{\beta_i}$ for every $t \in \mathbb{R}$. Moreover the only critical point of ϕ^{ξ_i} in U_{β_i} is m_i, and it is a minimum. Hence $\varphi_t(m)$ tends to m_i as t tends to infinity. Therefore the continuous map

$$\begin{aligned} f_m : [0,1[\;&\longrightarrow\; U_{\beta_i} \\ t \;&\longmapsto\; \varphi_{\frac{t}{1-t}}(m) \end{aligned}$$

extends by continuity into a map $\widetilde{f}_m : [0,1] \to U_{\beta_i}$ with $\widetilde{f}_m(1) = m_i$. Then the map

$$\begin{aligned} H : U_{\beta_i} \times [0,1] \;&\longrightarrow\; U_{\beta_i} \\ (m,t) \;&\longmapsto\; \widetilde{f}_m(t) \end{aligned}$$

is a T-deformation retract of U_{β_i} onto the orbit $\mathrm{T} \cdot m_i = m_i$. In particular U_{β_i} is T-categorical for every $i = 1, \ldots, \ell$. It is clear that $\mathcal{U} = \{U_{\beta_i}\}_{i=1}^{\ell}$ is a cover of M made of T-tubular open subsets, which are themselves tubular neighbourhoods of the closed strata. By Proposition 3.2.3, this cover is the smallest that we can take and then $\mathcal{U}$ is minimal. ■

As a corollary we obtain the result of Bayeh and Sarkar (cf. [6] Theorem 5.1). This result is also a direct consequence of the Localization Formula (Corollary 3.3.2) that we obtain in Section 3.3 below.

Corollary 3.2.7 ([6] Theorem 5.1)**.** *Let (M, ω, T, Φ_T) be a symplectic toric manifold. Then $Cat_T(M)$ coincides with the cardinality of M_T.*

Our choice to consider symplectic toric manifolds makes the proof of Theorem 3.2.6 relatively straightforward for two reasons. The first reason is that the fixed points of the T-action are isolated, and the second reason is that the stratification by orbit-type strata of M coincides with the stratification by open faces of the polytope. Whether this approach can be generalized to Hamiltonian manifolds equipped with an arbitrary Hamiltonian torus action is very likely to be true. The conjecture below is the result of a discussion with Yael Karshon and Eckhard Meinrenken, and its proof is still under construction.

Conjecture 3.2.8. *Any Hamiltonian T-manifold (M, ω, T, Φ_T) where T is a torus admits a minimal T-tubular cover.*

To close this section, we illustrate the steps of Theorem 3.2.6 on the standard example of a symplectic toric manifold.

Example 3.2.9. Consider the symplectic manifold $M = \mathbb{C}P^2$ on which the torus $\mathrm{T} = S^1 \times S^1$ acts effectively by

$$(\theta, \phi) \cdot [e^{i\theta} z_0 : e^{i\phi} z_1 : z_2].$$

A momentum map for this action can be chosen of the form

$$\Phi_{\mathrm{T}}([z_0 : z_1 : z_2]) = \left(\frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right)$$

where the factor $-\frac{1}{2}$ is ignored. The image of the momentum map is the polytope $\Delta \subset (\mathbb{R}^2)^*$ whose vertices are the points $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (0, 0)$ (cf. Figure 3.4). Those vertices are precisely the images under Φ_{T} of the fixed points of the T-action, namely $m_1 = [1 : 0 : 0]$, $m_2 = [0 : 1 : 0]$ and $m_3 = [0 : 0 : 1]$. The weight space decomposition at a fixed point $m_i \in M_{\mathrm{T}}$ is

$$\mathbb{C}_{\alpha_{m_i,1}} \oplus \mathbb{C}_{\alpha_{m_i,2}}$$

with weights $\alpha_{m_i,1}, \alpha_{m_i,2} \in (\mathbb{R}^2)^*$ given in Figure 3.4.

It is easily checked that the vector

$$\xi = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

m_i	$\alpha_{m_i,1}$	$\alpha_{m_i,2}$
m_1	$(-1,0)$	$(-1,1)$
m_2	$(0,-1)$	$(1,-1)$
m_3	$(1,0)$	$(0,1)$

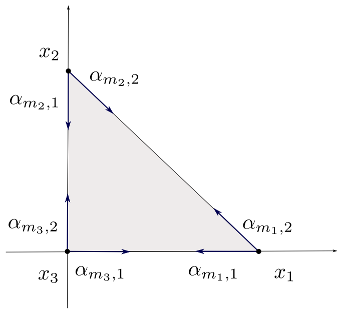


Figure 3.4: Weights of the torus action and moment polytope.

satisfies $\alpha_{m_i,j}(\xi) \neq 0$ for every $i = 1,2,3$ and $j = 1,2$. Therefore the component ϕ^ξ of the momentum map given by

$$\phi^\xi([z_0 : z_1 : z_2]) = \frac{2|z_0|^2 + |z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}$$

is generic and takes its minimum at $m_3 = [0 : 0 : 1]$.

In this example we have five orbit-types (H), with H being either the full torus T, the trivial group $\mathbb{1}$, or a subgroup diffeomorphic to a circle, namely $S^1 \times \mathbb{1}$, $\mathbb{1} \times S^1$ and the diagonal subgroup $S^1_{diag} = \{(\theta,\theta) \in \mathrm{T} \mid \theta \in S^1\}$. There is only one orbit-type stratum M_ε with orbit-type $(\mathbb{1})$, which is open and dense in M. All the strata with orbit-type diffeomorphic to S^1 are different copies of $\mathbb{C}P^1$ in $\mathbb{C}P^2$, namely

- For $\alpha = ((S^1 \times \mathbb{1}), (a))$ we have $M_\alpha = \{[0 : z_1 : z_2] \in \mathbb{C}P^2\}$.
- For $\delta = ((\mathbb{1} \times S^1), (d))$ we have $M_\delta = \{[z_0 : 0 : z_2] \in \mathbb{C}P^2\}$.
- For $\gamma = ((S^1_{diag}), (c))$ we have $M_\gamma = \{[z_0 : z_1 : 0] \in \mathbb{C}P^2\}$.

The three minimal orbit-type strata have orbit type (T). They form the fixed point set of the torus T and are given by

- For $\beta_1 = ((\mathrm{T}), (b_1))$ we have $M_{\beta_1} = \{[1 : 0 : 0] \in \mathbb{C}P^2\}$.
- For $\beta_2 = ((\mathrm{T}), (b_2))$ we have $M_{\beta_2} = \{[0 : 1 : 0] \in \mathbb{C}P^2\}$.
- For $\beta_3 = ((\mathrm{T}), (b_3))$ we have $M_{\beta_3} = \{[0 : 0 : 1] \in \mathbb{C}P^2\}$.

The oriented graph associated to the strict partial order (2.9) is given below (cf. Figure 3.5).

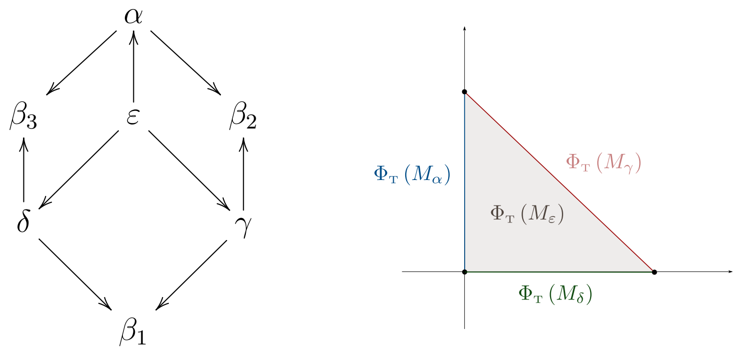


Figure 3.5: The oriented graph associated to the strict partial order (2.9) on the orbit-type strata is given on the left hand side. The image of the strata are shown on the right hand side.

As in Example 2.2.10, we take $\mathcal{B}'$ to be the set of subsets $\beta' \subset \{1, 2, 3\}$. For every $i = 1, 2, 3$ we define $\beta'_i = \{k, j\} \in \mathcal{B}'$ where $k < j$ and $k, j \neq i$. The corresponding zero-dimensional open face is the vertex $\mathring{F}_{\beta'_i} = x_i$. Following Example 2.2.10, we set $\alpha' = \{1\}$, $\delta' = \{2\}$ and $\varepsilon' = \varnothing$. Then for instance $\Phi_{\mathrm{T}}(M_\alpha) = \mathring{F}_{\alpha'}$. With the notations of Theorem 3.2.6 the open subset $V_{\beta'_3} \subset (\mathbb{R}^2)^*$ is

$$V_{\beta'_3} = \mathring{F}_{\beta'_3} \cup \mathring{F}_{\alpha'} \cup \mathring{F}_{\delta'} \cup \mathring{F}_{\varepsilon'}$$

and the corresponding T-tubular open subset $U_{\beta_3} = \Phi_{\mathrm{T}}^{-1}(V_{\beta'_3})$ is

$$U_{\beta_3} = M_{\beta_3} \cup M_\alpha \cup M_\delta \cup M_\varepsilon.$$

3.3 Localization Formula

In this section we obtain a localization formula (cf. Corollary 3.3.2) for proper G-manifolds which admit a minimal G-tubular cover. This formula says in particular that the equivariant LS-category of a proper G-manifold is intrinsic to the equivariant LS-category of its minimal orbit-type strata. The theorem below holds in general, without any assumption on the proper G-manifold.

Theorem 3.3.1. *Let (M, G) be a proper G-manifold and write M as the disjoint union of its orbit-type strata $\{M_\beta \mid \beta \in \mathcal{B}\}$. Let $\mathcal{B}'$ be the biggest subset of $\mathcal{B}$ such that M_β is minimal for every $\beta \in \mathcal{B}'$. Then*

$$Cat_G(M) \geq \sum_{\beta \in \mathcal{B}'} Cat_G(M_\beta).$$

Proof. Let $\mathcal{U}$ be a G-tubular cover of M. Choose $U \in \mathcal{U}$ such that $U \cap M_\beta \neq \varnothing$ for some $\beta \in \mathcal{B}'$, say $\beta = ((H), (b))$. By Proposition 3.2.3, U does not intersect any other minimal stratum and the G-deformation retract $H : U \times [0, 1] \to U$ retracts onto an orbit $G \cdot x$ of some $x \in M_\beta$. The set $V_\beta = U \cap M_\beta$ is open in M_β for the subset topology, and it is G-invariant because so are U and M_β.

Let $G \times_{G_x} N_0$ be the local model for U. Given $y \in V_\beta$ we define the G-equivariant path $y(t) = H_t(y)$, where $t \in [0, 1]$. In the local model, y reads $[g, \nu]$, and $y(t)$ reads $[g, \nu_t]$ where $\nu_t = (1 - t)\nu$. Since $(G_x)_\nu \in (H)$, we use the linearity of the G_x-action on N_0, to obtain $(G_x)_{\nu_t} = (G_x)_\nu = (H)$ for all $t \in [0, 1]$. Hence

$$G_{[g,\nu_t]} = g(G_x)_{\nu_t} g^{-1} \in (H) \quad \text{for all} \quad t \in [0, 1].$$

In particular, $y(t) \in M_\beta$ for all $t \in [0, 1]$. Because $y \in V_\beta$ is arbitrary and $[0, 1]$ is compact, the map $F : V_\beta \times [0, 1] \to V_\beta$ given by $F_t(y) = y(t)$ is a homotopy. It is clearly G-equivariant by construction and defines a G-deformation retract of V_β onto $G \cdot x$. It follows that V_β is G-categorical.

Let $\mathcal{U}_\beta \subset \mathcal{U}$ be the subset of all $U \in \mathcal{U}$ such that $U \cap M_\beta \neq \varnothing$. Then

$$\mathcal{V}_\beta = \{V_\beta = U \cap M_\beta \mid U \in \mathcal{U}_\beta\}$$

is a cover of M_β by G-categorical open subsets, which is not necessarily a minimal cover. This procedure associates to each $\beta \in \mathcal{B}'$ a cover $\mathcal{V}_\beta$ of M_β.

Proposition 3.2.3 says that, if $\alpha, \beta \in \mathcal{B}'$ are distinct, then $\mathcal{U}_\alpha \cap \mathcal{U}_\beta = \varnothing$. In particular, each $V_\beta \in \mathcal{V}_\beta$ is determined by a unique $U \in \mathcal{U}_\beta$. Therefore

$$\mathrm{Cat}_G(M) \geq \sum_{\beta \in \mathcal{B}'} \mathrm{Cat}_G(M_\beta).$$

■

After this work was completed we found that the result of Theorem 3.3.1 had already been obtained by Hurder and Töben (cf. [28] Theorem 3.7), by using a method similar to ours. However with our construction involving minimal G-tubular covers, we obtain the following:

Corollary 3.3.2 (Localization Formula). *Let (M, G) be a proper G-manifold which admits a minimal G-tubular cover. Decompose M into its orbit-type strata $\{M_\beta \mid \beta \in \mathcal{B}\}$. Let $\mathcal{B}'$ be the biggest subset of $\mathcal{B}$ such that M_β is minimal for every $\beta \in \mathcal{B}'$. Then*

$$Cat_G(M) = \sum_{\beta \in \mathcal{B}'} Cat_G(M_\beta).$$

Proof. By Theorem 3.3.1, $\mathrm{Cat}_G(M) \geq \sum_{\beta \in \mathcal{B}'} \mathrm{Cat}_G(M_\beta)$. The other inequality is a direct consequence of the properties of a minimal G-tubular cover (cf. Definition 3.2.2). ■

Proposition 3.3.3. *Let (M, G) be a proper G-manifold which admits a minimal G-tubular cover. Assume M_β is a minimal orbit-type stratum with $\beta = ((H), (b))$. Then*

$$Cat_G\left(M_\beta\right) = Cat_{N_G(H)}\left(M_{H,b}\right).$$

Proof. Let $\mathcal{U}'$ be a minimal G-tubular cover of M and let M_β be a minimal orbit-type stratum. By definition of $\mathcal{U}'$, the set $\mathcal{V}_\beta = \{V_\beta = U \cap M_\beta \mid U \in \mathcal{U}'\}$ is the smallest cover by G-categorical open subsets of M_β, where the topology of M_β is the subset topology.

For every $V_\beta \in \mathcal{V}_\beta$, let $\widetilde{V}_\beta = V_\beta \cap M_{H,b}$. Then $\widetilde{V}_\beta$ is an $N_G(H)$-invariant open subset of $M_{H,b}$, for the subset topology. Let $H : V_\beta \times [0,1] \to V_\beta$ be a G-deformation retract of V_β onto some orbit $G \cdot x$ of $x \in M_\beta$. Then the $N_G(H)$-homotopy $F : \widetilde{V}_\beta \times [0,1] \to \widetilde{V}_\beta$ defined by

$$F_t = H_t\Big|_{\widetilde{V}_\beta} \quad \text{for each} \quad t \in [0,1]$$

is an $N_G(H)$-deformation retract of $\widetilde{V}_\beta$ onto the orbit $N_G(H) \cdot x$. Therefore the set $\widetilde{\mathcal{V}}_\beta = \{\widetilde{V}_\beta = V_\beta \cap M_{H,b} \mid V_\beta \in \mathcal{V}_\beta\}$ is a cover of $M_{H,b}$ made of $N_G(H)$-categorical open subsets. This cover is minimal by assumption and because $M_\beta = G \cdot M_{H,b}$. We thus get

$$\mathrm{Cat}_G\left(M_\beta\right) = \mathrm{Cat}_{N_G(H)}\left(M_{H,b}\right).$$

■

The reader is invited to compare the above result with [46] (Proposition 2.1). By Theorem 3.2.6, every symplectic toric manifold satisfies the assumptions of Corollary 3.3.2 and Proposition 3.3.3. Therefore Corollary 3.2.7 is a direct consequence of the Localization Formula.

Example 3.3.4. We verify Theorem 3.3.2 on the other examples discussed in this chapter.

(i) Let $M = S^2 \subset \mathbb{R}^3$ on which S^1 acts by rotations about the z-axis. The minimal strata have orbit-type (H) where $H = S^1$, namely

$$M_{\beta_b} = \left\{(0,0,(-1)^{i-b})\right\}, \quad \beta_b = ((S^1),(b)) \quad \text{and} \quad b = 1,2.$$

Then

$$\mathrm{Cat}_{S^1}\left(M_{\beta_1}\right) + \mathrm{Cat}_{S^1}\left(M_{\beta_2}\right) = 1 + 1 = \mathrm{Cat}_{S^1}\left(M\right).$$

(ii) Let $M = \mathbb{C}P^2$ equipped with the action of S^1

$$\theta \cdot [z_0 : z_1 : z_2] = [e^{i\theta} z_0 : z_1 : z_2].$$

The minimal strata M_{β_1} and M_{β_2} have orbit-type $(H) = (S^1)$. There is a $\mathbb{C}P^1$ and the single point $[1 : 0 : 0]$, respectively. Therefore

$$\mathrm{Cat}_{S^1}(M_{\beta_1}) + \mathrm{Cat}_{S^1}(M_{\beta_2}) = 2 + 1 = \mathrm{Cat}_{S^1}(M).$$

(iii) Let $M = S^2$ acted on by the group $\mathbb{T}$ as in Example 3.1.4 (iii). There are three minimal orbit-type strata. Two of them, M_{β_1} and M_{β_2}, have orbit-type $(\mathbb{Z}_3)$. The last minimal stratum M_α has orbit-type $(\mathbb{Z}_2)$. We find

$$\mathrm{Cat}_{\mathbb{Z}_3}(M_{\beta_1}) + \mathrm{Cat}_{\mathbb{Z}_3}(M_{\beta_2}) + \mathrm{Cat}_{\mathbb{Z}_2}(M_\alpha) = 1 + 1 + 1 = \mathrm{Cat}_{\mathbb{T}}(M)$$

3.4 Critical point theory

In their original paper [39], Lyusternik and Schnirelmann showed that if M is a compact Riemannian C^2-manifold, then any function $f \in C^1(M)$ has at least $\mathrm{Cat}(M)$ critical points. The infinite dimensional case has been studied by Schwartz [65] when M is a complete C^2-manifold without boundary modeled on a separable Hilbert space, i.e. each point of M has a neighbourhood homeomorphic to an infinite dimensional Hilbert space, and f satisfies a suitable compactness condition. Motivated by existence theorems in the calculus of variations, Palais extended Schwartz's result for complete Finsler C^2-manifolds (cf. [54] Theorem 7.2) where there is no Riemannian metric to define the gradient. The equivariant analogue has been proved by Fadell [19] and Marzantowicz [46] in the case when G is compact. The extension to proper Lie group actions can be found in Ayala, Lasheras and Quintero [3]. In this section, we present Schwartz's version of the Lyusternik-Schnirelmann Theorem in the non-equivariant case (cf. Theorem 3.4.3) and explain how to extend it to the equivariant case following the lines of [3]. Complete proofs of the results presented below can be found in the book of Palais and Terng [57].

3.4.1 Gradient vector field

Let (M, g) be a connected Riemannian manifold without boundary. For $m_1, m_2 \in M$ we define

$$d(m_1, m_2) = inf \int_0^1 g(\gamma(t)) \left(\dot{\gamma}(t), \dot{\gamma}(t)\right)^{1/2} dt \tag{3.3}$$

where the infimum is taken over all the C^1 paths γ joining m_1 and m_2. The metric topology induced by d coincides with the original topology of M. We say that M is *geodesically complete* if the Riemannian exponential $Exp(m) : T_m M \to M$ is defined on the whole $T_m M$ for all m. By the Hopf-Rinow Theorem, M is geodesically complete if and only if it is complete as a metric space i.e. if the metric (3.3) is complete. Given $f \in C^1(M)$ the *gradient* of f is the unique vector field $\nabla f \in \mathfrak{X}(M)$ defined pointwise by

$$g(m)\left(v, \nabla f(m)\right) = df(m) \cdot v \quad \text{for all} v \in T_m M. \tag{3.4}$$

For $v \in T_m M$ we set $\|v\|^2 = g(m)(v, v)$. Recall that $m \in M$ is a *critical point* of f is $df(m) = 0$ or equivalently if $\nabla f(m) = 0$. Otherwise, we say that m is a *regular point*. The image by f of a critical point (resp. regular point) is called a *critical value* (resp. *regular value*). If c is a regular value then $f^{-1}(c)$ is a codimension one embedded submanifold of M. Denote by $\mathcal{C}(f)$ the set of critical points of f and by $\mathcal{C}_c(f)$ the set of critical points m such that $f(m) = c$. Away from $\mathcal{C}(f)$, we can define a normalized C^1 vector field X_f by

$$X(m) = \frac{-\nabla f(m)}{\|\nabla f(m)\|^2} \tag{3.5}$$

called the *gradient vector field*. Let φ_t be the flow generated by X. Since the derivative of $f(\varphi_t(m))$ with respect to t is identically minus one, f is monotonically decreasing along the flow lines of X i.e.

$$f(\varphi_t(m)) = f(m) - t.$$

The *time-one map* φ_1 is the homeomorphism on $M \setminus \mathcal{C}(f)$ that sends $m \in M \setminus \mathcal{C}(f)$ onto $\varphi_1(m)$, obtained from m by flowing down along the flow φ_t of the gradient vector field for the time $t = 1$. To deal with the non-compactness of M, Schwartz introduced the following condition on the manifold M and on the function f:

Definition 3.4.1 (Palais-Smale condition). We say that a pair (M, f) satisfies the *Palais-Smale condition (PS)* if the following holds.

(i) M is a complete C^2-Riemannian manifold without boundary modeled on a separable Hilbert space and $f \in C^1(M)$ is bounded below.

(ii) If $(x_n)_{n\in\mathbb{N}} \subset M$ is a sequence of points such that the sequence of images $f(x_n)$ is bounded and $\|\nabla f(x_n)\|$ converges to zero, then $(x_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in M.

Condition (ii) is a compactness condition which implies that f restricted to its critical set $\mathcal{C}(f)$ is a proper map. In particular for any real number c, the subset $\mathcal{C}_c(f)$ is a closed bounded subset of M. Since M is complete, the Hopf-Rinow Theorem implies that $\mathcal{C}_c(f)$ is compact in M.

3.4.2 Deformation Lemma

For $c \in \mathbb{R}$ we set $M_c = f^{-1}(]-\infty, c])$, the piece of the manifold below the level $f^{-1}(c)$. If c is a regular value of f then M_c is a smooth submanifold of M with boundary $f^{-1}(c)$. The hard part part of the Lyusternik-Schnirelmann Theorem embodies in the First Deformation Lemma which states essentially that, away form the critical points, the manifold M is a serie of submanifolds with boundary that look like products that can be retracted onto the lower pieces by flowing down along the flow of X. In other word, the topological information of the manifold is essentially confined to the critical points of f.

Lemma 3.4.2 (First Deformation Lemma [57](Theorem 9.2.3)). *Let U be any neighbourhood of $\mathcal{C}_c(f)$ in M. Then for $\varepsilon > 0$ sufficiently small, $\varphi_1(M_{c+\varepsilon} \setminus U) \subset M_{c-\varepsilon}$.*

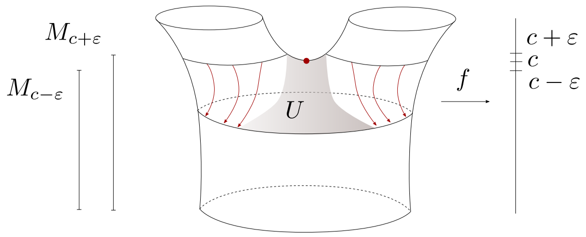


Figure 3.6: Outside of the open set U, the manifold is decomposed into products that can be retracted onto the lower piece $M_{c-\varepsilon}$ by flowing down along φ_1. The red lines indicate the flow lines of X.

3.4.3 Lyusternik-Schnirelmann Theorem

We may use the First Deformation Lemma to prove the Lyusternik-Schnirelmann Theorem . We first define

$$c_n(f) = inf \{c \in \mathbb{R} \mid \mathrm{Cat}\,(M_c, M) \geq n\}\,. \tag{3.6}$$

It is proved in [57] (Proposition 9.2.8) that $c_n(f)$ is a critical value of f for $n = 0, 1, \dots, \mathrm{Cat}(M)$ and that

$$c_n(f) \leq c_{n+1}(f) \tag{3.7}$$

with possible equality. The proof below is taken from [57] but the original paper of Palais is [55].

Theorem 3.4.3 (Lyusternik-Schnirelmann Theorem [65])**.** *Let (M, f) be a pair satisfying the (PS) condition. Then the number of critical points of f is greater than or equal to Cat(M).*

Proof. Since the theorem is trivial if f has an infinity of critical points, we assume that there are only a finite number of critical points $m_1, \dots, m_k$ in $f^{-1}(c)$ where $c = c_{n+1}(f) = \dots c_{n+l}(f)$ for some positive integer l. In particular, the critical points are isolated. Take neighbourhoods U_i of m_i whose respective closure are disjoint closed disks. Setting $U = \cup_{i=1}^{k} U_i$ we obtain using the monotony condition and contractibility of U_i,

$$\mathrm{Cat}(U, M) \leq \sum_{i=1}^{k} \mathrm{Cat}(U_i, M) = k.$$

By the First Deformation Theorem, there is $\varepsilon > 0$ such that $M_{c+\varepsilon} \setminus U$ can be deformed onto $M_{c-\varepsilon}$ by mean of the time-one map. Then

$$\mathrm{Cat}\left(M_{c+\varepsilon} \setminus U, M\right) \leq \mathrm{Cat}\left(M_{c-\varepsilon}, M\right).$$

Since $c - \varepsilon < c = c_{n+1}(f)$ and (3.7), we must have

$$\mathrm{Cat}\left(M_{c-\varepsilon}, M\right) < n+1 \quad \text{and} \quad \mathrm{Cat}\left(M_{c+\varepsilon} \setminus U, M\right) \leq n.$$

By the subadditivity property

$$\mathrm{Cat}\left(M_{c+\varepsilon}, M\right) \leq \mathrm{Cat}\left(M_{c+\varepsilon} \setminus U, M\right) + \mathrm{Cat}\left(U, M\right) \leq n + k. \tag{3.8}$$

Since $c_{n+l}(f) = c < c+\varepsilon < c_{n+k+1}(f)$ and (3.7) we conclude that $n+l \leq n+k+1$ and thus $k \geq l$. Then there are at least l critical points in $f^{-1}(c)$. In particular if $1 \leq n \leq \mathrm{Cat}(M)$, f has at least n critical point at or below the level $c_n(f)$. In total, f has at least $\mathrm{Cat}(M)$ critical points. ■

3.4.4 Critical points on proper G-manifolds

The above discussion generalizes when (M, G) is a proper G-manifold. The reader is refered to [3] for details. In this case we know by Proposition 2.1.7 that we can construct a G-invariant Riemannian metric on M, by applying the averaging method on each tangent space. Given a function $f \in C^\infty(M)^G$ the associated gradient vector field ∇f is G-equivariant. Note that by G-invariance of f, if $m \in C_c(f)$ then $G \cdot m \subset \mathcal{C}_c(f)$. The Palais-Smale condition is replaced by the orbitwise Palais-Smale condition.

Definition 3.4.4 (Orbitwise Palais-Smale condition [3]). A proper G-manifold (M, G) and a function $f \in C^1(M)^G$ satisfy the *orbitwise Palais-Smale condition (OPS)* if the following holds:

(i) M is a complete Riemannian proper G-manifold of class C^2, without boundary, modelled on a separable Hilbert G-space.

(ii) f is bounded below.

(iii) If $(x_n)_{n\in\mathbb{N}} \subset M$ is a sequence such that the associated sequence of images $f(x_n)$ is bounded and $\|\nabla f(x_n)\|$ converges to zero, then there exists a sequence

$$(g_n)_{n\in\mathbb{N}} \subset G$$

such that the sequence $(g_n \cdot x_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in M.

Similarly to the non-equivariant case, condition (iii) is a compactness condition which implies that f restricted to its set of critical orbits modulo G is a proper map. Note that if G is compact [19], condition (PS) implies condition (OPS). In [3], Alaya, Lasheras and Quintero proved an equivariant version of the First Deformation Lemma and obtained:

Theorem 3.4.5 ([3, 4]). *If a proper G-manifold (M, G) and a function $f \in C^1(M)^G$ satisfy condition (OPS), then f has at least $Cat_G(M)$ critical orbits.*

EXPLICIT SYMMETRY BREAKING

This chapter embodies the central topic of this thesis. As already mentioned in the introduction, a broad class of symmetric Hamiltonian systems can be viewed as a perturbation of another Hamiltonian system having a bigger symmetry group. For example, the motion of a spherical pendulum is governed by a Hamiltonian defined on T^*S^2, which is invariant with respect to a circle action. It can be viewed as a perturbation of the Hamiltonian governing the co-geodesic flow on T^*S^2, which is invariant with respect to the full group of rotations in the three dimensional Euclidean space. We start by specifying what we mean by explicit symmetry breaking perturbations. Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold. The non-degeneracy of ω implies that, associated to any Hamiltonian $h \in C^\infty(M)^G$, there is a unique vector field X_h defined by $\iota_{X_h}\omega = -dh$. Since the action of G on M is canonical and h is G-invariant, the integral curve $\varphi_t(m)$ of X_h starting at $m \in M$ satisfies $\varphi_t(g \cdot m) = g \cdot \varphi_t(m)$ for all $g \in G$. The resulting Hamiltonian equations

$$\frac{d}{dt}\varphi_t(m) = X_h(\varphi_t(m)) \tag{4.1}$$

are thus G-equivariant and we say that G is the *symmetry group* of (4.1). We study the effect of a small Hamiltonian perturbation of these equations, which is invariant with respect to a subgroup of G.

Definition 4.0.1. Let $h \in C^\infty(M)^G$ and $H \subset G$ be a closed subgroup. An *H-pertubation* of h is a family of functions $h_\lambda \in C^\infty(M)^H$ such that the map $(m, \lambda) \in M \times \mathbb{R} \mapsto h_\lambda(m) \in M$ is smooth, and $h_0 = h$.

4.1 Symmetry breaking for equilibria

The aim of this section is to give an estimate of the number of H-orbits of equilibria that persist under a small H-perturbation of some G-invariant Hamiltonian. This is the content of Corollary 4.1.3. A point $m \in M$ is an *equilibrium* of $h \in C^\infty(M)^G$ if $dh(m) = 0$, or equivalently if $X_h(m) = 0$. Assume N is some vector space and $f : G \times N \to \mathbb{R}$ is a smooth function. We denote by $d_N f$ the partial derivative(s) of f with respect to the N-variables. By abuse of notations we write $D^2 f$ for the Hessian of f and $D^2_N f$ for the Hessian with respect to the N-variables.

Definition 4.1.1. A *G-nondegenerate equilibrium* of $h \in C^\infty(M)^G$ is a point $m \in M$ such that

(i) $dh(m) = 0$,

(ii) $D^2_N h(m)$ is non-singular where N is any subspace of $T_m M$ complementary to $\mathfrak{g} \cdot m$. In other words, the Hessian is non-singular in the directions normal to the group orbit.

If $m \in M$ is a G-nondegenerate equilibrium of h then so is any $p \in G \cdot m$, by G-invariance. For the same reason, the tangent space $T_p(G \cdot m)$ is contained in $\ker(D^2 h(p))$ for any $p \in G \cdot m$. Definition 4.1.1 is a particular case of Morse-Bott non-degeneracy when $G \cdot m$ is the critical manifold of h (cf. [14] Appendix $E.2$). Note that Condition (ii) implies that the critical manifold $G \cdot m$ is isolated in the sense that there exists a tubular neighbourhood of $G \cdot m$ that does not contain any other critical points of h.

4.1.1 Persistence of equilibria

We say that a closed subgroup $H \subset G$ is *co-compact* (in G) if the left multiplication of H on G is co-compact i.e. the orbit space $H \setminus G$ under this action is compact (as a topological space). We can now state the main result of this section.

Theorem 4.1.2. *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold and let $H \subset G$ be a co-compact closed subgroup. Assume that $h_\lambda \in C^\infty(M)^H$ is an H-pertubation of some $h \in C^\infty(M)^G$ and that $m \in M$ is a G-nondegenerate equilibrium of h.*

Then there is a G-invariant neighbourhood $\widetilde{U} \subset M$ of m such that, if λ is sufficiently small, there exists a function $f_\lambda \in C^\infty(G/G_m)^H$ whose critical points are in one-to-one correspondence with those of h_λ in $\widetilde{U}$.

Proof. Let $m \in M$ be a G-nondegenerate equilibrium of h whose stabilizer is denoted by $K := G_m$. Following the notations of Section 2.4 the product space $N := \mathfrak{m}^* \times N_1$ is a K-vector space and it is isomorphic to some K-vector space complementary to $\mathfrak{g} \cdot m$ in $T_m M$. By the Symplectic Tube Theorem 2.4.1 and G-nondegeneracy of m, we can choose a K-invariant neighbourhood $N_0 \subset N$ of $0 \in N$, such that

(i) The associated bundle $G \times_K N_0$ is a symplectic local model of some G-invariant neighbourhood $U \subset M$ of m.

(ii) The only critical points of h in U are on $G \cdot m$.

In that model the point m reads $[(e, 0)]$ and the H-pertubation is identified with $h_\lambda : G \times_K N_0 \to \mathbb{R}$. Let $\rho : G \times N_0 \to G \times_K N_0$ be the orbit map. We define the lift of h_λ by

$$\widetilde{h}_\lambda := \rho^* h_\lambda : G \times N_0 \to \mathbb{R}$$

where ρ^* is the pullback map. The critical points of h_λ coincide with those of the lift $\widetilde{h}_\lambda$. Indeed since $\rho : G \times N_0 \to G \times_K N_0$ is a surjective submersion, we have

$$dh_\lambda\left([(g,\nu)]\right) = 0 \quad \Longleftrightarrow \quad \left(\rho^* dh_\lambda\right)\left((g,\nu)\right) = 0 \quad \Longleftrightarrow \quad d\widetilde{h}_\lambda\left((g,\nu)\right) = 0.$$

We may thus work with $\widetilde{h}_\lambda$ instead of h_λ.

We define a (left) action of the direct product $G \times K$ on $G \times N_0$ by

$$(h, k) \cdot (g, \nu) = (hgk^{-1}, k \cdot \nu).$$

By hypothesis, the lift $\widetilde{h}$ is $G \times K$-invariant whereas the perturbation $\widetilde{h}_\lambda$ is only $H \times K$-invariant. Since $(e, 0) \in G \times N_0$ is a G-nondegenerate critical point of $\widetilde{h}$,

$$d\widetilde{h}(e, 0) = 0 \quad \text{and} \quad D_N^2 \widetilde{h}(e, 0) \quad \text{is non-singular.} \tag{4.2}$$

In particular the map

$$d_N \widetilde{h} : G \times N_0 \to N_0^* \simeq N_0$$

satisfies $d_N \widetilde{h}(e, 0) = 0$ and its derivative with respect to the N_0-variables, evaluated at $(e, 0)$, is non-vanishing. The Implicit Function Theorem implies the existence of a neighbourhood $V_1 \times W_1$ of $(0, e)$ in $\mathbb{R} \times G$ such that, for any $(\lambda, g) \in V_1 \times W_1$, there is a unique $\phi^1_\lambda(g) \in N_0$ satisfying

$$d_N \widetilde{h}_\lambda(g, \phi^1_\lambda(g)) = 0. \tag{4.3}$$

By H-invariance[1] of $d_N \widetilde{h}_\lambda$, we can choose W_1 to be H-invariant. This procedure defines an H-invariant smooth function

$$\begin{aligned} \phi^1 : V_1 \times W_1 &\longrightarrow N_0 \\ (\lambda, g) &\longmapsto \phi^1_\lambda(g). \end{aligned}$$

By G-invariance of $\widetilde{h}$, (4.2) holds when replacing $(e, 0)$ by any $(g, 0) \in G \times N_0$. We apply the previous argument for every $(g, 0)$ with $g \notin H$ and use the compacity of $H \setminus G$ to extract a finite collection of open subsets $\{V_i \times W_i\}_{i=2}^n$ with associated H-invariant smooth functions $\phi^i : V_i \times W_i \to N_0$ satisfying (4.3). Let $V \subset \cap_{i=1}^n V_i$ be an open interval containing $0 \in \mathbb{R}$. By uniqueness of each ϕ^i, we can glue them together to define an H-invariant smooth function

$$\begin{aligned} \phi : V \times G &\longrightarrow N_0 \\ (\lambda, g) &\longmapsto \phi_\lambda(g) \end{aligned}$$

such that

$$d_N \widetilde{h}_\lambda(g, \phi_\lambda(g)) = 0. \tag{4.4}$$

[1] When we say that, we think of H as a subgroup of $H \times K$.

For every fixed parameter $\lambda \in V$,

$$\widetilde{h}_\lambda(hgk^{-1}, \phi_\lambda(hgk^{-1})) = \widetilde{h}_\lambda(g, \phi_\lambda(g)) \quad \text{for any} \quad (h,k) \in H \times K.$$

It thus descends to a function $f_\lambda \in C^\infty(G/K)^H$ given by

$$f_\lambda\left([g]_{\mathrm{K}}\right) := \widetilde{h}_\lambda\left([g]_{\mathrm{K}}, \phi_\lambda([g]_{\mathrm{K}})\right) \tag{4.5}$$

where $[g]_{\mathrm{K}}$ denotes a coset in G/K. For any pair $\beta = (\lambda, [g]_{\mathrm{K}}) \in V \times G/K$, we define the shift $\bar{h}_\beta : N_0 \to \mathbb{R}$ by

$$\bar{h}_\beta(\nu) := \widetilde{h}_\lambda([g]_{\mathrm{K}}, \nu + \phi_\lambda\left([g]_{\mathrm{K}}\right)) - f_\lambda([g]_{\mathrm{K}}). \tag{4.6}$$

This function has a non-degenerate[2] critical point at $0 \in N_0$. Indeed by (4.4) and K-invariance[3] of $\widetilde{h}_\lambda$,

$$d\bar{h}_\beta(0) = d_N\widetilde{h}_\lambda([g]_{\mathrm{K}}, \phi_\lambda([g]_{\mathrm{K}})) = 0.$$

Moreover, its Hessian $D^2\bar{h}_\beta(0)$ is non-singular, because non-degeneracy is a stable condition[4]. By the Morse Lemma (cf. Lemma 2.2 in [49]) there is a local coordinate system $\nu_\beta = (\nu^1, \dots, \nu^\ell)$, defined in a neighbourhood $N_\beta \subset N_0$ of 0 with $\nu_\beta(0) = 0$, such that

$$\bar{h}_\beta(\nu) = \bar{h}_\beta(0) + \sum_{i=1}^{\ell} \varepsilon_i(\nu_i)^2 = \sum_{i=1}^{\ell} \varepsilon_i(\nu_i)^2 \quad \text{for all} \quad \nu \in N_\beta. \tag{4.7}$$

where $\varepsilon_i = \pm 1$ and $\nu_\beta(\nu) = (\nu_1, \dots, \nu_\ell)$.

The Morse chart (N_β, ν_β) depends on $\beta = (\lambda, [g]_{\mathrm{K}})$. Since the functions defining (4.6) are H-invariant, the identity (4.7) holds on (N_β, ν_β) when replacing $\bar{h}_\beta$ by $\bar{h}_{\beta'}$ where $\beta' = (\lambda, [hg]_{\mathrm{K}})$ with $h \in H$. We repeat the previous argument for every $\beta = (\lambda, [g]_{\mathrm{H,K}}) \in V \times (H\backslash G/K)$, where $[g]_{\mathrm{H,K}}$ denotes the double coset of g. We thus obtain a collection of Morse charts (N_β, ν_β) indexed on $V \times (H\backslash G/K)$. The compacity of $H\backslash G$ is used next to extract a finite number of Morse charts $(N_{\beta_i}, \nu_{\beta_i})$

[2] in the Morse sense ([49] Section 2).
[3] When we say that we think of K as a subgroup of $H \times K$.
[4] We might have to take V smaller.

for $i = 1, \ldots, r$. Using a partition of unity, we construct a local coordinate system $\widetilde{\nu} = (\widetilde{\nu}^1, \ldots, \widetilde{\nu}^\ell)$, defined in a neighbourhood $\widetilde{N}_0 \subset \bigcap_{i=1}^r N_{\beta_i}$ of 0 with $\widetilde{\nu}(0) = 0$, such that

$$\bar{h}_\beta(\nu) = \sum_{i=1}^{\ell} \varepsilon_i (\widetilde{\nu}_i)^2 \quad \text{for every} \quad \beta \in V \times G/K, \quad \text{when} \quad \nu \in \widetilde{N}_0. \tag{4.8}$$

We may define a smooth map $\psi : V \times G/K \times \widetilde{N}_0 \to \widetilde{N}_0$ by

$$\psi(\lambda, [g]_{\mathrm{K}}, \nu) =: \psi_\lambda([g]_{\mathrm{K}}, \nu) = \nu + \phi_\lambda([g]_{\mathrm{K}}). \tag{4.9}$$

Replacing (4.9) in (4.6) yields

$$\widetilde{h}_\lambda([g]_{\mathrm{K}}, \psi_\lambda([g]_{\mathrm{K}}, \nu)) = \sum_i \varepsilon_i (\widetilde{\nu}_i)^2 + f_\lambda([g]_{\mathrm{K}}) \quad \text{whenever} \quad \nu \in \widetilde{N}_0 \tag{4.10}$$

where $\varepsilon_i = \pm 1$ and $\widetilde{\nu}(\nu) = (\widetilde{\nu}_1, \ldots, \widetilde{\nu}_\ell)$. Therefore $([g]_{\mathrm{K}}, \nu) \in G/K \times \widetilde{N}_0$ is a critical point of (4.10) if and only if

$$\left(\sum_{i=1}^{\ell} \varepsilon_i \widetilde{\nu}^i d\widetilde{\nu}^i \right)(\nu) = 0 \quad \text{and} \quad df_\lambda([g]_{\mathrm{K}}) = 0.$$

Let $\widetilde{U} \subset M$ be the G-invariant neighbourhood of m whose symplectic local model is $G \times_K \widetilde{N}_0$. In particular if $\lambda \in V$, the critical points of h_λ in $\widetilde{U}$ are in one-to-one correspondence with those of the function $f_\lambda \in C^\infty(G/K)^H$ defined in (4.5). ■

Corollary 4.1.3 (Persistence of Equilibria). *If the manifold G/G_m and the function $f_\lambda \in C^\infty(G/G_m)^H$ of Theorem 4.1.2 satisfy condition (OPS), then the number of H-orbits of equilibria that persist near $G \cdot m$ under a small H-perturbation is bounded below by $Cat_H(G/G_m)$.*

Proof. If λ is sufficiently small, Theorem 4.1.2 implies that the H-orbits of equilibria of h_λ in some neighbourhood of $G \cdot m$ are in one-to-one correspondence with those of the function $f_\lambda \in C^\infty(G/G_m)^H$ defined as in (4.5). By Theorem 3.4.5, the number of H-orbits of equilibria of h_λ is at least $\mathrm{Cat}_H(G/G_m)$. ■

Note that if G is compact, the (OPS) condition is automatically satisfied. Indeed, any compact manifold is automatically complete and the compactness condition on f_λ is fulfilled.

Example 4.1.4. Think of the cylinder $M = S^1 \times \mathbb{R}$ as embedded in $\mathbb{R}^3$ with coordinates (θ, z) and endow it with the standard symplectic form $\omega = d\theta \wedge dz$. The Lie group $G = O(2)$ acts on M by $R_\varphi \cdot (\theta, z) = (\theta + \varphi, z)$, if $R_\varphi \in O(2)$ is a rotation of angle φ; and by $r_\alpha \cdot (\theta, z) = (2\alpha - \theta, z)$, if $r_\alpha \in O(2)$ is a reflection about the line forming an angle α with the x-axis in $\mathbb{R}^3$. The action of G on M is Hamiltonian with momentum map $\Phi_G : (\theta, z) \in M \mapsto z \in \mathbb{R}$. Consider the 1-parameter family $h_\lambda : S^1 \times \mathbb{R} \to \mathbb{R}$ defined by

$$h_\lambda(\theta, z) = z^2 + \lambda \cos(n\theta).$$

Then $h = h_0$ is G-invariant and $m = (0, 0)$ is a G-nondegenerate equilibrium of h whose stabilizer is $G_m = \langle r_0 \rangle$. The perturbation h_λ is invariant[5] by $H = D_n$, where D_n is the dihedral group of order $2n$. The perturbed Hamiltonian h_λ has $2n$ critical points whose coordinates are $(\frac{\pi}{n}k, 0)$ for $k = 0, \ldots, 2n-1$, which form a regular $2n$-gone as shown in Figure 4.1 for the case $n = 3$. Since $G/G_m = O(2)/\langle r_0 \rangle$ is topologically a circle, we find $\mathrm{Cat}_H(G/G_m) = 2$, by Theorem 3.1.7. Since G is compact, condition (OPS) of Corollary 4.1.3 is automatically satisfied. There are thus two H-orbits of equilibria of h that will persist, and each of them is an n-gone (cf. Figure 4.2).

[5] In fact, the full symmetry group should be $D_n \times \mathbb{Z}_2$ since $\mathbb{Z}_2$ acts on the z-component by swapping the sign. However such an action is not canonical in the sense that it does not preserve the symplectic form. Since this discrete part does not contribute in the further application, we do not take it into account.

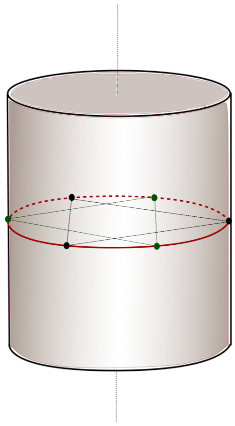


Figure 4.1: When $n = 3$, h has a G-orbit (red circle) consisting of G-nondegenerate equilibria, on which the six equilibria of h_λ lie.

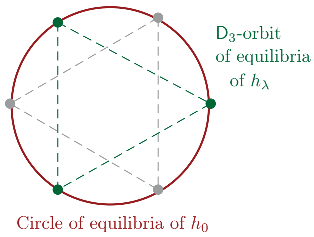


Figure 4.2: At the level of coordinate $z = 0$, the six equilibria of h_λ form two different D_3-orbits. One orbit is stable and one is unstable.

4.1.2 Dynamics of a 2D rigid body in a potential flow

We apply the result of Corollary 4.1.3 to the problem of a planar rigid body $\mathcal{B}$ of mass m moving in a planar irrotational, incompressible fluid with zero vorticity and zero circulation around the body. The motion is governed by Kirchhoff equations [31]. Classical treatments of the problem can be found in Lamb [35] and Milne-Thomson [48]. The configuration space of the body-fluid system is a submanifold Q of the product $SE(2) \times \mathrm{Emb}_{vol}(\mathcal{F}_0, \mathbb{R}^2)$, where $SE(2)$ is the special Euclidean group describing the motion of the body, and $\mathrm{Emb}_{vol}(\mathcal{F}_0, \mathbb{R}^2)$ is the space of volume-preserving embeddings of the fluid reference space $\mathcal{F}_0$ in $\mathbb{R}^2$. The symmetry group of this system is the direct product of $SE(2)$ (group of uniform body-fluid translations and rotations) and the particle relabeling symmetry group (volume-preserving diffeomorphisms of $\mathcal{F}_0$). Since these actions commute, the system can be reduced by the process of symplectic reduction by stages (cf. Marsden et al. [43]).

The Hamiltonian of the system is invariant under the particle relabeling symmetry group. Geometrically, eliminating the fluid variables amounts to carry out a symplectic reduction by this group. The particle relabeling symmetry group acts on T^*Q in a Hamiltonian fashion. The associated momentum map has two components corresponding to the vorticity and the circulation. The reduction at zero momentum corresponds to a fluid with zero circulation and zero vorticity. In this case, the symplectic reduced space is identified with $T^*SE(2)$, endowed with the canonical symplectic form and the $SE(2)$-invariant reduced Hamiltonian is the sum of the kinetic energy of the body-fluid system by the addition of the so-called "added masses", and the kinetic energy of the body. Those added masses depend only on the body's shape and not on the mass distribution. The reader is refered to Kanso et al. [30] and Vankerschaver et al. [68] for details. Since $SE(2)$ acts symplectically on $T^*SE(2)$, the dynamics can be reduced a second time using Poisson reduction and thereby the reduced motion is governed by the Kirchhoff equations that are the Lie-Poisson equations on the dual Lie algebra $\mathfrak{se}(2)^*$ (cf. Appendix A for details).

For the sake of simplicity we will assume that the body $\mathcal{B}$ is shaped as an ellipse with semi-axes of length $A > B > 0$. We will use the formulae and follow the notations of Fedorov et al. [20]. At the center of mass of $\mathcal{B}$ we attach a frame $\{E_1, E_2\}$ that is aligned with the symmetry axes of the body. Its position is related at any time to a fixed space frame $\{e_1, e_2\}$ by an element of $SE(2)$. An element of the Lie algebra $\xi \in \mathfrak{se}(2)$ is identified with a vector

$$(\dot{\theta}, v_1, v_2) \in \mathbb{R}^3 \tag{4.11}$$

where $\dot{\theta} \in \mathbb{R}$ is the angular velocity of $\mathcal{B}$ and $(v_1, v_2)^T \in \mathbb{R}^2$ is the linear velocity of its center of mass, expressed in the body's frame. In this setting the body has kinetic energy

$$T_{\mathcal{B}} = \frac{1}{2}\xi \cdot \mathbb{I}_{\mathcal{B}}\xi \tag{4.12}$$

with $\mathbb{I}_{\mathcal{B}} := \operatorname{diag}(I_{\mathcal{B}}, m, m)$, where $I_{\mathcal{B}}$ is the moment of inertia of the body about its center of mass. The kinetic energy of the fluid is given by

$$T_{\mathcal{F}} = \frac{1}{2}\xi \cdot \mathbb{I}_{\mathcal{F}}\xi \tag{4.13}$$

where $\mathbb{I}_{\mathcal{F}} = \frac{\rho\pi}{4}\operatorname{diag}((A^2 - B^2)^2, B^2, A^2)$ is the tensor of added masses, and ρ is the fluid density. In the absence of external forces, the Lagrangian of the body-fluid system $\mathcal{L} : TSE(2) \to \mathbb{R}$ is given by $\mathcal{L} = T_{\mathcal{B}} + T_{\mathcal{F}}$. It defines a Riemannian metric on $SE(2)$ with respect to which the motion of the body $\mathcal{B}$ is geodesic. Since $\mathcal{L}$ does not depend on the group variables, it is $SE(2)$-invariant and can thus be reduced to the function $\ell : \mathfrak{se}(2) \to \mathbb{R}$ given by

$$\ell(\xi) = \frac{1}{2}\xi \cdot (\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})\xi \tag{4.14}$$

with ξ as in (4.11). An element ν of the dual Lie algebra $\mathfrak{se}(2)^*$ is identified with a one by three matrix (x, α_1, α_2). The dual pairing $\langle\cdot,\cdot\rangle$ between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$ is thus given by

$$\langle \nu, \xi \rangle := (x, \alpha_1, \alpha_2)(\dot{\theta}, v_1, v_2)^T = x\dot{\theta} + \alpha_1 v_1 + \alpha_2 v_2. \tag{4.15}$$

We perform the Legendre transform $\mathbb{F}L : \xi \in \mathfrak{se}(2) \mapsto ((\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})\xi)^T \in \mathfrak{se}(2)^*$ to obtain the reduced Hamiltonian $h : \mathfrak{se}(2)^* \to \mathbb{R}$ defined by

$$h(\nu) = \frac{1}{2}\nu \cdot (\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})^{-1}\nu^T.$$

The Lie-Poisson equations on $\mathfrak{se}(2)^*$ that describe the motion of the body-fluid system are

$$\dot{\nu} = \mathrm{ad}^*_{\frac{\delta h}{\delta \nu}}\nu. \tag{4.16}$$

where $\mathrm{ad}^*_\xi \nu$ is identified with $(\alpha_1 v_2 - \alpha_2 v_1, \dot{\theta}\alpha_2, -\dot{\theta}\alpha_1)$ as computed in Appendix A. This problem turns out to exhibit symmetry breaking phenomena from different points of view:

(i) One point of view consists in looking at the body $\mathcal{B}$ without the fluid ($\rho = 0$). Adding the fluid amounts to seeing the fluid density ρ as a "parameter". The $O(2)$-symmetry of the kinetic reduced Hamiltonian breaks into a D_2-symmetry, where D_2 is the symmetry group of an ellipse.

(ii) On the other hand we can consider the original system as being a circular planar rigid body ($A = B$) in a fluid and the symmetry can be broken by deforming the body into an elliptical shaped body. This case exhibits the same pattern of symmetry breaking from $O(2)$ to the subgroup D_2.

These two approaches are the same from a group theoretical point of view. Contrary to Example 4.1.4, the Hamiltonian in consideration will not be perturbed by adding some potential energy. In this case, there is no potential energy involved, only the metric is perturbed giving rise to a modified kinetic energy. Let us now discuss the two cases mentioned above.

(i) The unperturbed system on the Poisson reduced space $\mathfrak{se}(2)^*$ is governed by the Hamiltonian

$$h(\nu) = \frac{1}{2}\nu \cdot \mathbb{I}\nu = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}}} + \frac{\alpha_1^2 + \alpha_2^2}{m}\right) \tag{4.17}$$

where $\nu := (x, \alpha_1, \alpha_2)$ and $\mathbb{I} := \mathbb{I}_{\mathcal{B}}^{-1}$. The Hamiltonian is invariant with respect to the group $G = O(2)$ [6]. In particular, for each $c \in \mathbb{R}$, the level sets $h(\nu) = c$ describe spheroids in $\mathbb{R}^3$.

Adding a fluid to the system amounts to look at the variation of the parameter[7]

$$\lambda = d\rho \quad \text{where} \quad d := \frac{A^2 - B^2}{m} > 0 \quad \text{is fixed.}$$

This gives rise to the perturbed Hamiltonian $h_\lambda(\nu) = \frac{1}{2}\nu \cdot \mathbb{I}_\lambda \nu$ with

$$\mathbb{I}_\lambda = \operatorname{diag}\left(\frac{1}{I_{\mathcal{B}} + \lambda c_1}, \frac{1}{m + \lambda c_2}, \frac{1}{m + \lambda c_3}\right). \tag{4.18}$$

where $c_1 = \frac{m^2 d\pi}{4}$, $c_2 = \frac{\pi(A^2 - md)}{4d}$ and $c_3 = \frac{\pi(B^2 + md)}{4d}$ are fixed constants encoding the datas of the system. The perturbed Hamiltonian reads

$$h_\lambda(\nu) = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}} + \lambda c_1} + \frac{\alpha_1^2}{m + \lambda c_2} + \frac{\alpha_2^2}{m + \lambda c_3}\right) \tag{4.19}$$

[6] In fact, the full symmetry group should be $O(2) \times \mathbb{Z}_2$ since $\mathbb{Z}_2$ acts on the x-component by swapping the sign. However since this discrete part does not contribute in the further application, we do not take it into account.

[7] We could simply consider ρ as being the parameter, but in this case the parameter would not be dimensionless and we want to avoid this.

and has symmetry $H = D_2$, the dihedral group of order four[8]. This perturbation coincides with h when $\lambda = 0$ and the function $(\lambda, \nu) \mapsto h_\lambda(\nu)$ is smooth. Therefore, h_λ is an H-pertubation of h. The symmetry is broken because the fluid influences the motion of the body if it is elliptical. If the body is circular ($A = B$), or if it moves in the vacuum, its center of mass would move at constant velocity and it would rotate at constant angular speed.

(ii) We carry out another kind of perturbation: rather than perturbing the rigid body motion by adding a fluid to the system, we start with a circular planar rigid body ($A = B$) in a fluid and break the symmetry by changing the body shape into an ellipse. The unperturbed Hamiltonian is given by

$$h(\nu) = \frac{1}{2}\nu \cdot \mathbb{I}\nu = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}}} + \frac{\alpha_1^2 + \alpha_2^2}{m + d_2}\right) \tag{4.20}$$

where $d_2 = \frac{\rho\pi B^2}{4}$, $\nu := (x, \alpha_1, \alpha_2)$, $\mathbb{I} := (\mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}})^{-1}$ and $A = B$ in the definition of $\mathbb{I}_{\mathcal{F}}$. The Hamiltonian is invariant with respect to $G = O(2)$. For each $c \in \mathbb{R}$, the level sets $h(\nu) = c$ also describe spheroids in $\mathbb{R}^3$.

We perturb the body shape by setting $\lambda = \frac{A^2 - B^2}{B^2}$ where $B > 0$ is fixed and $A \geq B > 0$ varies. This gives rise to the perturbed Hamiltonian $h_\lambda(\nu) = \nu \cdot \mathbb{I}_\lambda \nu$ with

$$\mathbb{I}_\lambda = \operatorname{diag}\left(\frac{1}{I_{\mathcal{B}} + \lambda^2 d_1}, \frac{1}{m + d_2}, \frac{1}{m + (\lambda + 1)d_2}\right) \tag{4.21}$$

where $d_1 = \frac{\rho\pi B^4}{4}$. The perturbed Hamiltonian is thus given by

$$h_\lambda(\nu) = \frac{1}{2}\left(\frac{x^2}{I_{\mathcal{B}} + \lambda^2 d_1} + \frac{\alpha_1^2}{m + d_2} + \frac{\alpha_2^2}{m + (\lambda + 1)d_2}\right) \tag{4.22}$$

and is again symmetric with respect to the action of $H = D_2$. In this case, if there was no fluid ($\rho = d_2 = 0$), no symmetries would have been broken.

[8] The group D_2 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ which leaves h_λ invariant when acting on α_1 and α_2 by swapping the signs.

Since the reduced motion is governed by the Lie-Poisson equations (4.16), it is constrained to the coadjoint orbits of $SE(2)$. As shown in [44] (Chapter 14.6), almost all of them are cylinders (the singular orbits consist of points on the vertical dashed line in Figure 4.3). In both cases, the level sets of h_λ are ellipsoids and those of $h = h_0$ are spheroids. Their intersections with a coadjoint orbit are shown in Figure 4.3. In particular, the circle of equilibria of h (in red in Figure 4.3) breaks into four fixed points of h_λ, two of which are connected by four heteroclinic cycles.

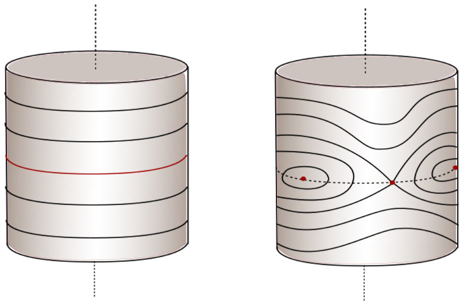


Figure 4.3: The flow lines are given by intersecting the level sets of h_λ (the ellipsoids) and the coadjoint orbits. On the left hand side, we see the flow lines of h on a coadjoint orbit. On the right hand side, the flow has been perturbed.

Let us go back to the first case we discussed above with h_λ as in (4.19). We will apply Corollary 4.1.3 to predict the existence of the four fixed points that persist (cf. Figure 4.3). The Fréchet derivative of h_λ is

$$\frac{\delta h_\lambda}{\delta \nu} = \left(\frac{x}{I_{\mathcal{B}} + \lambda c_1}, \frac{\alpha_1}{m + \lambda c_2}, \frac{\alpha_2}{m + \lambda c_3} \right). \tag{4.23}$$

Therefore, the Lie-Poisson equations (4.16) reduce to

$$\begin{cases} \dot{x} &= \frac{\lambda(c_2 - c_3)}{(m+\lambda c_3)(m+\lambda c_2)} \alpha_1 \alpha_2 \\ \dot{\alpha_1} &= \frac{x\alpha_2}{m+\lambda c_1} \\ \dot{\alpha_2} &= -\frac{x\alpha_1}{m+\lambda c_1} \end{cases} \tag{4.24}$$

Setting $\lambda = 0$ in (4.24), we see that the fixed points of $h = h_0$ are either of the form $(0, \alpha_1, \alpha_2)$ with $(\alpha_1, \alpha_2) \in (\mathbb{R}^2)^*$, or of the form $(x, 0, 0)$ which correspond to points on the singular coadjoint orbit.

Let $\mu := (0, \alpha_1, \alpha_2)$ with $\alpha_1^2 + \alpha_2^2 = 1$ be a fixed point of the unperturbed hamiltonian h. The stabilizer of μ is $G_\mu = \langle r_\vartheta \rangle$ where r_ϑ is a reflection in the plane. The quotient $G/G_\mu = O(2)/\langle r_\vartheta \rangle$ is topologically a circle yielding $\mathrm{Cat}_{D_2}(S^1) = 2$. The four fixed points appearing in Figure 4.3 are the two H-orbits that persist.

4.2 Symmetry breaking for relative equilibria

In this section, we extend Theorem 4.1.2 and Corollary 4.1.3 to the case of relative equilibria which is more subtle for two reasons: firstly we must take into account the conservation of momentum, and secondly for a non-zero velocity the so-called augmented Hamiltonian no longer has symmetry G.

We start by briefly recalling some standard facts about relative equilibria, the reader is invited to consult the book of Marsden [42] (Chapter 4) for a more detailed exposition. Given a Hamiltonian proper G-manifold (M, ω, G, Φ_G), a *relative equilibrium* of a Hamiltonian $h \in C^\infty(M)^G$ is a pair $(m, \xi) \in M \times \mathfrak{g}$ such that $X_h(m) = \xi_M(m)$. Equivalently, if (m, ξ) is a relative equilibrium of h, then m is a critical point of the *augmented Hamiltonian*

$$h^\xi := h - \phi_G^\xi \in C^\infty(M)^{G_\xi}$$

where $\phi_G^\xi(m) := \langle \Phi_G(m), \xi \rangle$, which is a G_ξ-invariant function which depends linearly on ξ. A standard fact about relative equilibria is that the velocity ξ and the momentum $\mu = \Phi_G(m)$ commute i.e. $\xi \in \mathfrak{g}_\mu$. Note that, if the stabilizer G_m is non trivial and (m, ξ) is a relative equilibrium of h, then $(m, \xi + \eta)$ is also a relative equilibrium of h, for any $\eta \in \mathfrak{g}_m$. Moreover if (m, ξ) is a relative equilibrium of h then so is $(g \cdot m, \mathrm{Ad}_g \xi)$ for every $g \in G$. In general a relative equilibrium is said to be non-degenerate if the Hessian $D^2 h^\xi(m)$ is a non-singular quadratic form, when restricted to the symplectic slice N_1 at m relative to the G-action. However, this definition of non-degeneracy is not enough to guarantee that a relative equilibrium of some $h \in C^\infty(M)^G$ persists under an H-perturbation. For that reason, one shall cook up a slightly stronger definition of non-degeneracy.

4.2.1 Induced momentum map

Let H be a closed subgroup of G. The dual of the inclusion of Lie algebras $i_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the projection $\mathfrak{i}_{\mathfrak{h}}^* : \mathfrak{g}^* \to \mathfrak{h}^*$ and is given by $i_{\mathfrak{h}}^*(\mu) = \mu\big|_{\mathfrak{h}}$, which is the restriction of the linear form μ to the subalgebra $\mathfrak{h}$. The action of H on M is still both canonical and Hamiltonian. A momentum map for this action is given by $\Phi_H = i_{\mathfrak{h}}^* \circ \Phi_G : M \to \mathfrak{h}^*$ and is called the *induced momentum map* for the H-action.

Proposition 4.2.1. *Consider the decomposition of T_mM as in (2.19), and define the subspace $\mathcal{M} := \{z_M(m) + w \in T_1 \oplus N_0 \mid -ad_z^*\mu + f(w) \in \mathfrak{h}^\circ\}$ where f denotes the isomorphism between N_0 and $\mathfrak{m}^*$, and $\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Then* $\ker(D\Phi_H(m)) = \ker(D\Phi_G(m)) \oplus \mathcal{M}$.

Proof. It is clear from the definitions that there is an inclusion of subspaces

$$\ker(D\Phi_G(m)) \subset \ker(D\Phi_H(m)) \,. \tag{4.25}$$

Let $(\varphi, G \times_{G_m} (\mathfrak{m}_0^* \times (N_1)_0), U)$ be a symplectic G-tube at m as in Theorem 2.4.1. Linearising φ^{-1} at m yields a linear symplectomorphism

$$T_m\varphi^{-1} : T_0 \oplus T_1 \oplus N_0 \oplus N_1 \to T_{\varphi^{-1}(m)}\left(G \times_{G_m} (\mathfrak{m}^* \times N_1)\right).$$

For $x + y \in \mathfrak{g}_m \oplus \mathfrak{m}$ and $z \in \mathfrak{n}$ we have

$$T_m\varphi^{-1} \cdot ((x+y)_M(m) + z_M(m) + w + \nu) = T_{(e,0,0)}\rho \cdot (x + y + z, f(w), \nu)$$

where $\rho : G \times \mathfrak{m}^* \times N_1 \to G \times_{G_m} (\mathfrak{m}^* \times N_1)$ is the orbit map. By definition, the subspace $\ker(D\Phi_H(m))$ consists of the elements

$$((x+y)_M(m) + z_M(m) + w + \nu) \in T_0 \oplus T_1 \oplus N_0 \oplus N_1$$

satisfying $D(\Phi_H \circ \varphi \circ \rho)(m) \cdot (x+y+z, f(w), \nu) = 0$. Equivalently

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Phi_H \circ \varphi\left([(\exp(t(x+y+z)), tf(w), t\nu)]\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} i_{\mathfrak{h}}^* \left(\mathrm{Ad}^*_{\exp(-t(x+y+z))}\left(\mu + tf(w) + \Phi_{N_1}(t\nu)\right)\right) \\ &= i_{\mathfrak{h}}^*\left(-\mathrm{ad}^*_z \mu + f(w)\right) \end{aligned}$$

where the normal form for the momentum map is given by Theorem 2.4.2. As required $-\mathrm{ad}^*_z\mu + f(w) \in \mathfrak{h}^\circ$ since the kernel of $i_{\mathfrak{h}}^*$ is equal to $\mathfrak{h}^\circ$. ■

4.2.2 Non-degeneracy condition and regularity condition

We now state a stronger version of non-degeneracy of a relative equilibrium.

Definition 4.2.2. Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold, $H \subset G$ be a closed subgroup, and $\Phi_H : M \to \mathfrak{h}^*$ be the induced momentum map. Setting $\alpha := \Phi_H(m)$, a relative equilibrium $(m, \xi) \in M \times \mathfrak{g}$ of $h \in C^\infty(M)^G$ is said to be *α-nondenegerate* if $D^2 h^\xi(m)$ is a non-singular quadratic form on $N_1 \oplus \mathcal{M}$ with $\mathcal{M}$ as in Proposition 4.2.1.

Definition 4.2.2 only depends on α and not on the underlying Witt-Artin decomposition of $T_m M$. If G is non-abelian, the space $\mathcal{M}$ might have an non-trivial intersection with $\mathfrak{g} \cdot m$. This intersection is the subspace $\mathfrak{q} \cdot m \subset \mathfrak{g} \cdot m$ where $\mathfrak{q}$ is an H_m-invariant complement to $\mathfrak{g}_\mu$ in the "symplectic orthogonal"

$$\mathfrak{h}^{\perp_\mu} := \left\{ x \in \mathfrak{g} \mid x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)} \right\}.$$

The non-singularity of $D^2 h^\xi(m)$ along $\mathfrak{g} \cdot m$ depends only on that of $D^2 \phi^\xi_G(m)$ which has symmetry group G_ξ. In the last chapter, we prove the following lemma:

Lemma 4.2.3. *Let (M,ω,G,Φ_G) be a Hamiltonian proper G-manifold. Let $m \in M$ with momentum $\mu = \Phi_G(m)$ and an element $\xi \in \mathfrak{g}_\mu$. If $\mathfrak{g}$ is semi-simple then the Hessian $D^2\phi_G^\xi(m)$ restricted to $\mathfrak{g}\cdot m$ is singular only along $(\mathfrak{g}_\xi + \mathfrak{g}_\mu)\cdot m$.*

Therefore if an equilibrium $(m,\xi) \in M \times \mathfrak{g}$ of some $h \in C^\infty(M)^G$ with momentum $\mu = \Phi_G(m)$ is α-nondegenerate in the sense of Definition 4.2.2, then $\mathfrak{g}_\xi$ has trivial intersection with $\mathfrak{q}$. In Theorem 4.2.5 we show that a number of orbits of relative equilibria of h persist under H-perturbation. Such relative equilibria must have their velocity ξ in $\mathfrak{h}_\mu$. We assume an additional regularity assumption

$$\mathfrak{g}_\mu \subset \mathfrak{g}_\xi \tag{R}$$

This says essentially that μ needs to be "more regular" than ξ in the sense of Definition 6.2.2, introduced in the last chapter. However if $\xi \in \mathfrak{h}_\mu$, this assumption depends on the embedding of $\mathfrak{h} \hookrightarrow \mathfrak{g}$ as shown in the example below. This is not a problem for us because isomorphic Lie algebras have different underlying Lie groups.

Example 4.2.4. In this example we show when condition (R) holds for $\mathfrak{g} = \mathfrak{so}(4)$ and a subalgebra isomorphic to $\mathfrak{h} = \mathfrak{so}(3)$. The Lie algebra $\mathfrak{g}$ consists of the matrices of the form

$$\begin{pmatrix} \hat{x} & a \\ -a^T & 0 \end{pmatrix} =: (x,a)$$

where $x, a \in \mathbb{R}^3$. We use the hat notation $\hat{x} \in \mathfrak{h}$ to mean the skew-symmetric matrix

$$\hat{x} := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Given $(x,a),(y,b) \in \mathfrak{g}$, the expression for the Lie bracket is

$$[(x,a),(y,b)] = (x \times y + a \times b, x \times b + a \times y). \tag{4.26}$$

Coadjoint action. The dual $\mathfrak{g}^*$ is computed using the standard pairing $\frac{1}{2}\mathrm{Tr}(A^T B)$ for square matrices A, B. The dual Lie algebra consists of pairs $(\chi, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3$ which satisfy

$$\langle (\chi, \rho), (x, a) \rangle = \chi \cdot x + \rho \cdot a.$$

With this identification the paring reduces to the standard dot product in $\mathbb{R}^3$. The linearized coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by

$$\mathrm{ad}^*_{(x,a)}(\chi, \rho) = (\chi \times x + \rho \times a, \chi \times a + \rho \times x) \tag{4.27}$$

Indeed, for $(x, a), (y, b) \in \mathfrak{g}$ and $(\chi, \rho) \in \mathfrak{g}^*$, we compute

$$\begin{aligned} \langle \mathrm{ad}^*_{(x,a)}(\chi, \rho), (y, b) \rangle &= \langle (\chi, \rho), [(x, a), (y, b)] \rangle \\ &= \langle (\chi, \rho), (x \times y + a \times b, x \times b + a \times y) \rangle \\ &= \chi \cdot (x \times y + a \times b) + \rho \cdot (x \times b + a \times y) \\ &= (\chi \times x + \rho \times a) \cdot y + (\chi \times a + \rho \times x) \cdot b \\ &= \langle (\chi \times x + \rho \times a, \chi \times a + \rho \times x), (y, b) \rangle. \end{aligned}$$

Lie subalgebras isomorphic to $\mathfrak{so}(3)$. Elements of $\mathfrak{h} = \mathfrak{so}(3)$ are identified with vectors $x \in \mathbb{R}^3$ using the inverse of the hat map $x \in \mathbb{R}^3 \mapsto \hat{x} \in \mathfrak{h}$. There are different ways to embed $\mathfrak{h}$ into $\mathfrak{g}$ as a Lie subalgebra. We restrict to the case when the inclusion reads

$$i_{\mathfrak{h}} : x \in \mathfrak{h} \mapsto (tx, sx) \in \mathfrak{g} \tag{4.28}$$

for some constants $t, s \in \mathbb{R}$ that have to be determined. In order for the image $i_{\mathfrak{h}}(\mathfrak{h})$ to have a Lie subalgebra structure, the closedness under the bracket (4.26) has to be satisfied. Calculating

$$[(tx, sx), (ty, sy)] = ((t^2 + s^2)x \times y, 2ts(x \times y)),$$

this should satisfy $((t^2 + s^2)x \times y, 2ts(x \times y)) = (tx \times y, sx \times y)$, leading us to solve the equations $t^2 + s^2 = t$ and $2st = s$. The only non-trivial solutions are $(t, s) = (1, 0)$ and $(t, s) = (\frac{1}{2}, \pm\frac{1}{2})$. We conclude that $\mathfrak{g}$ has three subalgebras isomorphic to $\mathfrak{so}(3)$ with inclusion as in (4.28), namely

(i) The Lie algebra of rotations in $\mathbb{R}^3$ denoted $\mathfrak{so}(3)_r = \{(x,0) \in \mathbb{R}^6 \mid x \in \mathbb{R}^3\}$ with Lie bracket $[(x,0),(y,0)] = (x \times y, 0)$.

(ii) The diagonal elements denoted $\mathfrak{so}(3)_d = \left\{\left(\frac{x}{2}, \frac{x}{2}\right) \in \mathbb{R}^6 \mid x \in \mathbb{R}^3\right\}$ with Lie bracket $[(\frac{x}{2},\frac{x}{2}),(\frac{y}{2},\frac{y}{2})] = (\frac{x\times y}{2}, \frac{x \times y}{2})$.

(iii) The anti-diagonal elements denoted $\mathfrak{so}(3)_{ad} = \left\{\left(\frac{x}{2}, -\frac{x}{2}\right) \in \mathbb{R}^6 \mid x \in \mathbb{R}^3\right\}$ with Lie bracket $[(\frac{x}{2},-\frac{x}{2}),(\frac{y}{2},-\frac{y}{2})] = (\frac{x\times y}{2}, -\frac{x \times y}{2})$.

Question: How many different embeddings $\mathfrak{h} \hookrightarrow \mathfrak{g}$ do we have in total?

Note that the diagonal elements $(\frac{x}{2},\frac{x}{2}) \in \mathfrak{so}(3)_d$ commute with the anti-diagonal elements $(\frac{x}{2},-\frac{x}{2}) \in \mathfrak{so}(3)_{ad}$ with respect to the bracket (4.26). Therefore there is an isomorphism of Lie algebras $(x,a) \in \mathfrak{so}(4) \mapsto \left(\left(\frac{x}{2},\frac{x}{2}\right), \left(\frac{a}{2},-\frac{a}{2}\right)\right) \in \mathfrak{so}(3)_d \times \mathfrak{so}(3)_{ad}$.

Regularity condition. Given a fixed momentum $\mu := (\chi, \rho) \in \mathfrak{g}^*$, the stabilizer Lie subalgebra is

$$\mathfrak{g}_\mu = \{(x,a) \in \mathfrak{g} \mid \chi \times x + \rho \times a = 0 \quad \text{and} \quad \chi \times a + \rho \times x = 0\}$$

by (4.27). We show below whether condition (R) is satisfied for different choices of Lie subalgebras isomorphic to $\mathfrak{h}$.

(i) Let $\mathfrak{h} = \mathfrak{so}(3)_r$ with inclusion map

$$i_{\mathfrak{h}} : x \in \mathfrak{h} \mapsto (x,0) \in \mathfrak{g}.$$

To compute the dual of this inclusion $i_{\mathfrak{h}}^* : \mathfrak{g}^* \to \mathfrak{h}^*$, we take $(\chi,\rho) \in \mathfrak{g}^*$ and $x \in \mathfrak{h}$ and we compute

$$\langle i_{\mathfrak{h}}^*(\chi,\rho), x\rangle = \langle (\chi,\rho), i_{\mathfrak{h}}(x)\rangle = \langle (\chi,\rho),(x,0)\rangle = \chi \cdot x.$$

Then

$$i_{\mathfrak{h}}^*((\chi,\rho)) = \chi \in \mathfrak{h}^*.$$

The symplectic orthogonal is $\mathfrak{h}^{\perp_\mu} = \{(x,a) \in \mathfrak{g} \mid \chi \times x + \rho \times a = 0\}$. Since the velocity $\xi \in \mathfrak{h}$ must commute with μ, it has to belong to the subspace $\mathfrak{h}_\mu = \mathfrak{g}_\mu \cap \mathfrak{h}$. Using equation (4.27),

$$\mathfrak{h}_\mu = \{(x,0) \in \mathfrak{so}(3)_r \mid \chi \times x = 0 \text{ and } \rho \times x = 0\}.$$

There are three cases to consider:

(a) If $\chi = \rho = 0$ then $\mathfrak{g}_\mu = \mathfrak{g}$ and $\mathfrak{h}_\mu = \mathfrak{h}$. We choose

$$\xi = (y,0) \in \mathfrak{h}$$

where $y \in \mathbb{R}^3$ is arbitrary. Using (4.26) we get

$$\mathfrak{g}_\xi = \{(\lambda_1 y, \lambda_2 y) \in \mathfrak{g} \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$$

and clearly (R) does not hold.

(b) If χ and ρ are not collinear, $\mathfrak{h}_\mu = \{(0,0)\}$. In this case, the only available velocity is $\xi = 0$ and thus $\mathfrak{g}_\xi = \mathfrak{g}$. In particular (R) holds.

(c) If $\mu = (\chi, \rho)$ is such that $\chi = s\rho$ for some $s \in \mathbb{R}$, we choose ξ of the form

$$\xi := (\lambda\chi, 0) \in \mathfrak{h}_\mu \quad \text{for some} \quad \lambda \in \mathbb{R}$$

and thus $\mathfrak{g}_\xi = \{(x,a) \in \mathfrak{g} \mid x \times \chi = 0 \text{ and } a \times \chi = 0\}$. Note that in particular, $\mathfrak{g}_\xi \subset \mathfrak{g}_\mu$. To see whether $\mathfrak{g}_\mu \subset \mathfrak{g}_\xi$, pick an element $(x,a) \in \mathfrak{g}_\mu$. By definition, it satisfies

$$x \times \chi = \rho \times a \quad \text{and} \quad \chi \times a = x \times \rho. \tag{4.29}$$

Using (4.29) and the fact that $\chi = s\rho$ we get,

$$x \times \chi = s(x \times \rho) = s(\chi \times a) = s^2(\rho \times a) = s^2(x \times \chi).$$

Similarly

$$a \times \chi = s(a \times \rho) = s(\chi \times x) = s^2(\rho \times x) = s^2(a \times \chi).$$

Therefore, $(x, a) \in \mathfrak{g}_\xi$ as long as $s^2 \neq 1$ i.e. (R) holds as long as $\mu \neq (\chi, \pm\chi)$, as shown in Figure 4.4.

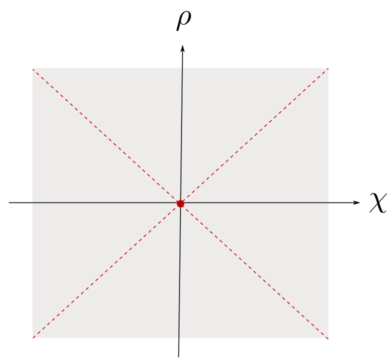


Figure 4.4: Condition (R) holds as long as μ is away form the red dashed lines that are subspaces of codimension three in $\mathbb{R}^6$.

(ii) Let $\mathfrak{h} = \mathfrak{so}(3)_d$ with inclusion map

$$i_{\mathfrak{h}} : x \in \mathfrak{h} \mapsto \left(\frac{x}{2}, \frac{x}{2}\right) \in \mathfrak{g}.$$

To compute the dual of this inclusion $i_{\mathfrak{h}}^* : \mathfrak{g}^* \to \mathfrak{h}^*$, we take $(\chi, \rho) \in \mathfrak{g}^*$ and $x \in \mathfrak{h}$ and we compute

$$\langle i_{\mathfrak{h}}^*(\chi, \rho), x\rangle = \langle(\chi, \rho), i_{\mathfrak{h}}(x)\rangle = \langle(\chi, \rho), \left(\frac{x}{2}, \frac{x}{2}\right)\rangle = \frac{\chi + \rho}{2} \cdot x,$$

Then

$$i_{\mathfrak{h}}^*((\chi, \rho)) = \frac{\chi + \rho}{2} \in \mathfrak{h}^*.$$

Set $\mu := (\chi, \rho) \in \mathfrak{g}^*$ and $\alpha := i_{\mathfrak{h}}^*(\mu) = \frac{\chi+\rho}{2} \in \mathfrak{h}^*$. Using Equation (4.27) we get

$$\mathfrak{h}_\mu = \left\{\left(\frac{x}{2}, \frac{x}{2}\right) \in \mathfrak{so}(3)_d \mid \alpha \times x = 0\right\}.$$

We thus choose a velocity of the form

$$\xi := (\lambda\alpha, \lambda\alpha) \in \mathfrak{h}_\mu$$

for some $\lambda \in \mathbb{R}$. By (4.26) the stabilizer Lie algebra of ξ is

$$\mathfrak{g}_\xi = \{(x, a) \in \mathfrak{g} \mid x \times \alpha + a \times \alpha = 0\} . \tag{4.30}$$

In particular, $\mathfrak{g}_\mu \subset \mathfrak{g}_\xi$ and (R) is automatic for any choice of μ.

(iii) The case $\mathfrak{h} = \mathfrak{so}(3)_{ad}$ is similar to the previous one.

4.2.3 Persistence of relative equilibria

We are now ready to state an equivalent version of Theorem 4.1.2 for relative equilibria. The proof follows the same steps as Theorem 4.1.2. For that reason some details have been skipped.

Theorem 4.2.5. *Let (M, ω, G, Φ_G) be a Hamiltonian proper G-manifold. Assume $h \in C^\infty(M)^G$ has a relative equilibrium $(m, \xi) \in M \times \mathfrak{h}$ with momentum $\mu = \Phi_G(m)$. Let α be the restriction of μ to $\mathfrak{h}$. We assume that*

(i) $\Phi_H^{-1}(\alpha)$ is a smooth manifold,

(ii) (m, ξ) is α-nondegenerate and (R) is satisfied,

(iii) $G_\mu \subset H_\alpha$,

(iv) $H_\mu \subset G_\mu$ is co-compact.

Then there is a G_μ-invariant neighbourhood $\widetilde{U} \subset \Phi_H^{-1}(\alpha)$ of m and a neighbourhood $V \subset \mathbb{R} \times \mathfrak{h}$ of $(0, \xi)$ such that, for each $(\lambda, \eta) \in V$, there is a function $f_\lambda^\eta \in C^\infty(G_\mu/G_m)^{H_\mu}$ whose critical points are in one-to-one correspondence with those of h_λ^η in $\widetilde{U}$.

Proof. Let $(m, \xi) \in M \times \mathfrak{h}$ be an α-nondegenerate relative equilibrium h, where α is the restriction of the momentum $\mu = \Phi_G(m)$ to $\mathfrak{h}$. By assumption $\Phi_H^{-1}(\alpha)$ is a smooth manifold on which $G_\mu \subset H_\alpha$ acts canonically and properly.

Let $K = G_m$ and consider the K-vector space $N := N_1 \oplus \mathcal{M}$, where N_1 is a symplectic slice at m relative to the G-action, and $\mathcal{M}$ is as in Proposition 4.2.1. By construction N is isomorphic to some K-vector space complementary to $\mathfrak{g}_\mu \cdot m$ in $T_m\left(\Phi_H^{-1}(\alpha)\right)$. By the Tube Theorem 2.1.5 there is a K-invariant neighbourhood $N_0 \subset N$ of $0 \in N$, such that

(i) The associated bundle $G_\mu \times_K N_0$ is a local model for some G_μ-invariant neighbourhood $U \subset \Phi_H^{-1}(\alpha)$ of m.

(ii) The only critical points of h^ξ in U are on $G_\mu \cdot m$.

In that model, the point m corresponds to $[(e, 0)]$ and the augmented Hamiltonian of an H-perturbation $h_\lambda \in C^\infty(M)^H$ of h is identified with $h_\lambda^\xi : G_\mu \times_K N_0 \to \mathbb{R}$. According to the proof of Theorem 4.1.2, the critical points of h_λ^ξ are in bijective correspondence with those of the lift

$$\widetilde{h}_\lambda^\xi := \rho^* h_\lambda^\xi : G_\mu \times N_0 \to \mathbb{R}$$

where $\rho : G_\mu \times N_0 \to G_\mu \times_K N_0$ is the orbit map. We may thus work with $\widetilde{h}_\lambda^\xi$ instead of h_λ^ξ.

We define a (left) action of the direct product $G_\mu \times K$ on $G_\mu \times N_0$ by

$$(h, k) \cdot (g, \nu) = (hgk^{-1}, k \cdot \nu).$$

As $G_\mu \subset G_\xi$ by the (R) assumption, the lift $\widetilde{h}^\xi$ is $G_\mu \times K$-invariant whereas $\widetilde{h}_\lambda^\xi$ is only $H_\mu \times K$-invariant. By α-nondegeneracy of (m, ξ),

$$d\widetilde{h}^\xi(e, 0) = 0 \quad \text{and} \quad D_N^2 \widetilde{h}^\xi(e, 0) \quad \text{is non-singular.}$$

As in the proof of Theorem 4.1.2, we can use the Implicit Function Theorem and the compacity of $H_\mu \backslash G_\mu$ to get an H_μ-invariant smooth function $\phi_\lambda^\eta : G_\mu \to N_0$, depending on parameters (λ, η) taken in a neighbourhood $V \subset \mathbb{R} \times \mathfrak{h}$ of $(0, \xi)$, satisfying

$$d_N\widetilde{h}^\eta_\lambda(g,\phi^\eta_\lambda(g))=0 \quad \text{for every} \quad g\in G_\mu.$$

For every fixed parameters $(\lambda,\eta)\in V$, the $H_\mu\times K$-invariance of $\widetilde{h}^\eta_\lambda$ allows us to define a function $f^\eta_\lambda\in C^\infty(G_\mu/K)^{H_\mu}$ by

$$f^\eta_\lambda([g]_{\mathrm{K}}):=\widetilde{h}^\eta_\lambda([g]_{\mathrm{K}},\phi^\eta_\lambda([g]_{\mathrm{K}}))$$

where $[g]_{\mathrm{K}}$ is a coset in G_μ/K. An application of the Morse Lemma with parameters gives us a Morse chart $(\widetilde{N_0},\widetilde{\nu})$ centered at $0\in N_0$, where $\widetilde{N_0}\subset N_0$ and $\widetilde{\nu}=(\widetilde{\nu}^1,\ldots,\widetilde{\nu}^n)$. Then there is a smooth map $\psi^\eta_\lambda: G_\mu/K\times\widetilde{N_0}\to\widetilde{N_0}$, depending on $(\lambda,\eta)\in V$ such that

$$\widetilde{h}^\eta_\lambda([g]_{\mathrm{K}},\psi^\eta_\lambda([g]_{\mathrm{K}},\nu))=\sum_{i=1}^n\varepsilon_i\widetilde{\nu}_i^2+f^\eta_\lambda([g]_{\mathrm{K}}) \tag{4.31}$$

where $\varepsilon_i=\pm1$ and $\widetilde{\nu}(\nu)=(\widetilde{\nu}_1,\ldots,\widetilde{\nu}_n)$. Therefore $([g]_{\mathrm{K}},\nu)\in G_\mu/K\times\widetilde{N_0}$ is a critical point of (4.31) if and only if

$$\left(\sum_{i=1}^n\varepsilon_i\widetilde{\nu}^i d\widetilde{\nu}^i\right)(\nu)=0 \quad\text{and}\quad df^\eta_\lambda([g]_{\mathrm{K}})=0.$$

Let $\widetilde{U}\subset\Phi_H^{-1}(\alpha)$ be the G_μ-invariant neighbourhood of m whose local model is $G_\mu\times_K\widetilde{N_0}$. In particular for $(\lambda,\eta)\in V$, the critical points of h^η_λ in $\widetilde{U}$ are in one-to-one correspondence with those of the function f^η_λ. ■

Corollary 4.2.6 (Persistence of relative equilibria)**.** *If the manifold G_μ/G_m and the function $f^\eta_\lambda\in C^\infty(G_\mu/G_m)^{H_\mu}$ of Theorem 4.2.5 satisfy condition (OPS) then the number of H_μ-orbits of relative equilibria of h with velocity close to ξ, that persist under a small H-perturbation in a neighbourhood of $G_\mu\cdot m$ in $\Phi_H^{-1}(\alpha)$, is bounded below by $Cat_{H_\mu}(G_\mu/G_m)$.*

Proof. We apply Theorem 3.4.5 to $f^\eta_\lambda\in C^\infty(G_\mu/G_m)^{H_\mu}$ and we obtain that the number of H_μ-orbits of critical points of f^η_λ is bounded below by $\mathrm{Cat}_{H_\mu}(G_\mu/G_m)$.

In other words, as long as $(\lambda, \eta) \in V$, the number of H_μ-orbits of relative equilibria of h_λ with velocity η in a neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$ is at least $\mathrm{Cat}_{H_\mu}(G_\mu/G_m)$. ■

Example 4.2.7 (Torus action)**.** As a first application, we recover the result of Grabsi, Montaldi and Ortega [23] for compact abelian groups and free actions. Let $(M, \omega, \mathrm{T}^n, \Phi_{\mathrm{T}^n})$ be a Hamiltonian T^n space where T^n is a n-torus acting freely on M and let T^r be a subtorus of T^n. Assume $h \in C^\infty(M)^{\mathrm{T}^n}$ has an α-nondegenerate relative equilibrium $(m, \xi) \in M \times \mathfrak{t}^r$ with momentum $\mu = \Phi_{\mathrm{T}^n}(m)$ and where $\alpha = \mu\big|_{\mathfrak{t}^r}$. As T^n and T^r are abelian, condition (R) always hold. By compactness of T^n, condition (OPS) is automatic and then any T^r-perturbation h_λ with λ small enough has at least

$$\mathrm{Cat}_{\mathrm{T}^r}(\mathrm{T}^n)$$

T^r-orbit of relative equilibria with velocity closed to ξ in a neighbourhood of $\mathrm{T}^n \cdot m$ in $\Phi_{\mathrm{T}^r}^{-1}(\alpha)$. Since T^n acts freely on T^r by left multiplication,

$$\mathrm{Cat}_{\mathrm{T}^r}(\mathrm{T}^n) = \mathrm{Cat}(\mathrm{T}^n/\mathrm{T}^r) = \mathrm{Cat}(\mathrm{T}^{n-r}).$$

Hence $\mathrm{Cat}(\mathrm{T}^{n-r}) = (n-r)+1$.

4.2.4 The spherical pendulum

As an application of Corollary 4.2.6, we consider the case of the spherical pendulum whose Hamiltonian is viewed as a perturbation of the Hamiltonian governing the motion of an unit mass point constrained to move on the surface of S^2. Endow $\mathbb{R}^3$ with the standard inner product $\langle \cdot, \cdot \rangle$ and let e_1, e_2, e_3 be the standard basis. The phase space for the spherical pendulum is the Hamiltonian proper G-manifold $(T^*S^2, \omega, G, \Phi_G)$ where $G = SO(3)$ acts on

$$T^*S^2 = \Big\{ (x, y) \in S^2 \times \mathbb{R}^3 \mid \langle x, y \rangle = 0 \Big\}$$

by matrix multiplication $A \cdot (x, y) = (Ax, Ay)$. The associated momentum map $\Phi_G : T^*S^2 \to \mathbb{R}^3$ is

$$\Phi_G(x, y) = x \times y.$$

Let $H = SO(2)$ be the subgroup of rotations about the e_3-axis with Lie algebra $\mathfrak{h}$, the one-dimensional vector space generated by e_3. We can think about the Hamiltonian of the spherical pendulum

$$h_\lambda(x, y) = \frac{1}{2}\|y\|^2 + \lambda\langle x, e_3\rangle$$

as an H-perturbation of the G-invariant Hamiltonian $h(x, y) = \frac{1}{2}\|y\|^2$.

The relative equilibria of h_λ are the pairs $((x, y), \xi) \in T^*S^2 \times \mathfrak{h}$ such that

$$dh_\lambda(x, y) = d\phi^\xi_H(x, y) \tag{4.32}$$

where $\phi^\xi_H(x, y) := \langle x \times y, \xi\rangle$. We show that these satisfy

$$\lambda + \|\xi\|^2\langle x, e_3\rangle = 0 \text{ and } y = \xi \times x. \tag{4.33}$$

Indeed the tangent space at $(x, y) \in T^*S^2$ is identified with

$$T_{(x,y)}T^*S^2 = \{(\dot{x}, \dot{y}) \mid \langle x, \dot{x}\rangle = 0 \text{ and } \langle x, \dot{y}\rangle + \langle y, \dot{x}\rangle = 0\}.$$

Equation 4.32 holds if, for all $(\dot{x}, \dot{y}) \in T_{(x,y)}T^*S^2$, there is some non zero $\xi \in \mathfrak{h}$ such that

$$\langle y, \dot{y}\rangle + \lambda\langle \dot{x}, e_3\rangle = \langle \dot{x} \times y + x \times \dot{y}, \xi\rangle. \tag{4.34}$$

By setting $\dot{x} = 0$, we obtain $\langle(\xi \times x) - y, \dot{y}\rangle = 0$ for all $\dot{y}$ perpendicular to x. In particular we choose $\dot{y} = (\xi \times x) - y$ and thus $\|(\xi \times x) - y\|^2 = 0$ which implies that

$$y = \xi \times x. \tag{4.35}$$

Replacing (4.35) in (4.34) yields

$$\lambda\langle \dot{x}, e_3\rangle = \langle \dot{x} \times y, \xi\rangle$$

which is equivalent to say that

$$\langle \lambda e_3 - (y \times \xi), \dot{x}\rangle = 0 \text{ for all } \dot{x} \text{ perpendicular to } x.$$

In other words, $\lambda e_3 - y \times \xi$ must be colinear to x. Using (4.35),

$$\lambda e_3 - y \times \xi = -\|\xi\|^2 x + \left(\lambda + \|\xi\|^2 \langle x, e_3\rangle\right) e_3$$

and thus $(\lambda + \|\xi\|^2\langle x, e_3\rangle)\, e_3$ must be colinear to x. One possibility is $x = \pm e_3$ which gives $y = 0$ by (4.35). The other possibility is

$$\lambda + \|\xi\|^2\langle x, e_3\rangle = 0 \text{ and } y = \xi \times x.$$

The relative equilibria $((x, y), \xi))$ of the unperturbed Hamiltonian h are such that x moves along a great circle on S^2 and those of h_λ are such that x describes a circular trajectory at fixed height in the lower hemisphere of S^2. Therefore the only H_μ-orbit that has a chance to persist under h_λ is the one when x moves along the equator. Indeed the point $m = (e_1, se_2)$ and $\xi = se_3$, with $s > 0$, define a G-nondegenerate relative equilibrium of the unperturbed Hamiltonian h, with momentum $\mu = se_3$. Observe that $G_\mu = SO(2)$ is the group of rotations about the e_3-axis. Embedding S^1 in S^2 as the equator allow us to view the orbit $G_\mu \cdot m$ as the set of perpendicular pairs $(x, y) \in S^1 \times S^1_s$ where S^1_s is the equator of the sphere of radius s. The (stable) relative equilibria of h_λ are of the form $((x, y), \eta)$ with $\eta = re_3$ for some $r \in \mathbb{R}$ and with $x \in S^2$ having uniform circular motion at constant negative height $-\frac{\lambda}{r^2}$. Since $x \in S^2$ we must have $|\lambda| < r^2$. Given x we can calculate y from (4.33). Setting $\alpha = s$ we get

$$(x, y) \in \Phi_H^{-1}(\alpha) \quad \Leftrightarrow \quad r(1 - \frac{\lambda^2}{r^4}) = s \quad \Leftrightarrow \quad r^3(r - s) = \lambda^2.$$

The last equation can be solved for r and we find two solutions, one positive and one negative. The condition $|\lambda| < r^2$ implies that the only valid solution is the positive one. Since the distance $|r - s|$ is controlled by λ, the velocity η is close to ξ as λ is sufficiently small. We conclude that for λ small enough, h_λ has exactly one H_μ-orbit of relative equilibria in a neighbourhood of $G_\mu \cdot m$ in $\Phi_H^{-1}(\alpha)$ with velocity close to ξ. For this example, the assumptions of Theorem 4.2.5 are all satisfied. As expected, we have

$$\mathrm{Cat}_{H_\mu}(G_\mu/G_m) = \mathrm{Cat}_{SO(2)}(S^1) = 1.$$

Bifurcation diagram If (x, y) satisfies (4.33) for some ξ, then it is a critical point of rank one of the energy momentum map $F_\lambda = (h_\lambda, \Phi_H)$. If we let ξ varies, s is viewed as a parameter and the boundary of $F_\lambda(T^*S^2)$ is the convex region in $\mathbb{R}^2$ defined by the curve

$$\gamma_\lambda(s) = \left(\frac{s^4 - 3\lambda^2}{2s^2}, s - \frac{\lambda^2}{s^3} \right). \tag{4.36}$$

The critical points of rank zero (when $\xi = 0$) are sent on $F_\lambda(\pm e_3, 0) = (\pm\lambda, 0)$. The point $(-e_3, 0)$ is of type elliptic-elliptic whereas the point $(e_3, 0)$ is of type focus-focus (cf. Cushman and Duistermaat [15] and Vu Ngoc and Sepe [69]).

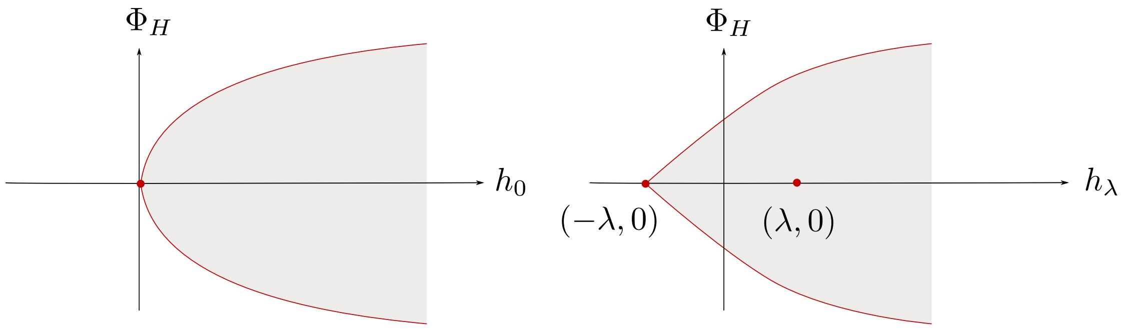


Figure 4.5: Bifurcation diagrams of the energy-momentum map F_λ when $\lambda = 0$ (left hand side) and when $\lambda > 0$ (right hand side).

SYMPLECTIC SLICE FOR ACTIONS OF SUBGROUPS

Stability properties and bifurcations of relative equilibria can be determined by a method developed by Krupa [34], which states that the dynamics of an equivariant vector field in a neighbourhood of a group orbit is entirely governed by the dynamics transverse to that group orbit by using the the so-called slice coordinates introduced by Palais [53]. While Krupa proved this result for compact Lie groups, Fiedler et al. [21] extended it to proper group actions. The Hamiltonian analogue has been studied by Mielke [47] as well as Roberts and Sousa Dias [60], and it was expanded in Roberts, Wulff and Lamb [61]. By "dynamics transverse to the group orbit", we mean that the vector field in question can be split into two parts, one part is defined along the group orbit and the other part belongs to a choice of normal subspace transverse to that group orbit. For Hamiltonian systems defined on (M, ω, G, Φ_G), the flow lines of a Hamiltonian vector field are confined to level sets of the momentum map, reflecting the conservation of momentum. Therefore the choice of normal space is more restrictive than for general dynamical systems (Section 2.4). As constructed in Roberts, Wulff and Lamb [61], a space transverse to $G \cdot m$ is isomorphic to $\mathfrak{m}^* \times N_1$ where $\mathfrak{m}^*$ is isomorphic to $(\mathfrak{g}_\mu/\mathfrak{g}_m)^*$. A *symplectic slice* N_1 at m is a G_m-invariant subspace of $(T_mM, \omega(m))$ defined by

$$N_1 := \ker(D\Phi_G(m))/\mathfrak{g}_\mu \cdot m. \tag{5.1}$$

It is endowed with both a natural symplectic structure ω_{N_1} coming from $\omega(m)$ and a canonical linear Hamiltonian action of G_m. This subspace is of particular interest for the study of stability, persistence and bifurcations of relative equilibria (cf. for instance Patrick et al. [58], Lerman and Singer [38] and Ortega and Ratiu [51], Montaldi and Rodriguez-Olmos [50]).

5.1 The symplectic slice of a subgroup

Given a closed subgroup $H \subset G$ we have seen that (M, ω, H, Φ_H) is a Hamiltonian H-space where $\Phi_H : M \to \mathfrak{h}^*$ is the induced momentum map. In this case, we can also consider a Witt-Artin decomposition of $T_m M$ relative to the H-action:

$$T_m M = \widetilde{T}_0 \oplus \widetilde{T}_1 \oplus \widetilde{N}_0 \oplus \widetilde{N}_1. \tag{5.2}$$

In particular, the H_m-invariant subspace $\widetilde{N}_1$ is a symplectic slice for the H-action. It is chosen such that

$$\widetilde{N}_1 := \ker(D\Phi_H(m))/\mathfrak{h}_\alpha \cdot m \tag{5.3}$$

where $\alpha := \Phi_H(m)$. In general two arbitrary decompositions (2.19) and (5.2) cannot be compared. In the study of explicit symmetry breaking phenomena, the Hamiltonian equations are perturbed in a way that the symmetry group G breaks into one of its subgroup H. The stability properties of the perturbed system rely on a symplectic slice relative to the H-action on M, which is bigger than a slice relative to the G-action. This leads us to find explicit relations between N_1 and $\widetilde{N}_1$. It has been implicitly used in [23] that, when G is a torus and H is a subtorus both acting freely on M, a symplectic slice $\widetilde{N}_1$ at m can be chosen of the form

$$\widetilde{N}_1 = N_1 \oplus X_m, \tag{5.4}$$

for some subspace $X_m \subset T_m M$ isomorphic to $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$. We generalize this observation for non-abelian Lie groups and non-free actions with the assumption $G_m \subseteq N_G(H)$, where $N_G(H)$ is the normalizer of H in G. We show that, under this assumption, there is a Witt-Artin decomposition (2.19) at m relative to the G-action and a Witt-Artin decomposition (5.2) at m for the H-action that are compatible in the sense that the symplectic slice $\widetilde{N}_1$ for H can be expressed in terms of the symplectic slice N_1 for G, and other subspaces of (2.19). Explicitly,

$$\widetilde{N}_1 = N_1 \oplus X_m \oplus \mathfrak{s}(G, H, \mu) \cdot m, \tag{5.5}$$

for some subspace $X_m \subset T_1 \oplus N_0$ symplectomorphic to a canonical cotangent bundle $\mathfrak{b} \times \mathfrak{b}^*$, and where $\mathfrak{s}(G, H, \mu) \cdot m$ is some symplectic vector subspace of T_1 (cf. Theorem 5.1.4 and Theorem 5.1.6). When G is a torus and H is a subtorus both acting freely on M, we recover the equality (5.4). In this case, the vector subspace $\mathfrak{b}$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$ whereas the subspace $\mathfrak{s}(G, H, \mu) \cdot m$ is trivial (cf. Example 5.1.8). In Perlmutter et al. [59], $\mathfrak{s}(G, H, \mu) \cdot m$ arises as a symplectic slice at μ for the H-action on the coadjoint orbit $G \cdot \mu$. We give another proof of this fact in Proposition 5.1.2. In our construction, $\mathfrak{s}(G, H, \mu)$ is defined as a subspace of the "symplectic orthogonal"

$$\mathfrak{h}^{\perp_\mu} := \{x \in \mathfrak{g} \mid x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)}\} \tag{5.6}$$

which is present in the context of geometric quantization (cf. for instance Duval and al. [18]). The construction holds only if we assume that $\mathfrak{h}^{\perp_\mu}$ is G_m-invariant which is automatic if we assume that $G_m \subseteq N_G(H)$.

5.1.1 Lie algebra splittings

Let $m \in M$ with momentum $\mu = \Phi_G(m)$ and assume that $G_m \subseteq N_G(H)$. In particular G_m acts on the stabilizer subalgebras $\mathfrak{h}_m$ and $\mathfrak{h}_\mu$ by mean of the Adjoint action. We start by splitting the Lie algebra $\mathfrak{g}$ into three parts

$$\mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{n}. \tag{5.7}$$

for some G_m-invariant subspaces $\mathfrak{m}$ and $\mathfrak{n}$ that we shall choose in a specific way. We use the following notations: if (V, ω) is a symplectic vector space and $W \subset V$ is a subspace, the *symplectic orthogonal* W^ω of W in V is the set of vectors $v \in V$ such that $\omega(v, w) = 0$ for all $w \in W$. Furthermore, if W, U are two subspaces such that $U \subset W \subset V$, then $U^{\perp_W}$ denotes a *complement* of U in W so that $U \oplus U^{\perp_W} = W$ is a direct sum.

By properness of the G-action, the stabilizer G_m is compact. The Lie subalgebra $\mathfrak{g}_m$ can thus be decomposed into a direct sum of G_m-invariant subspaces $\mathfrak{g}_m = \mathfrak{h}_m \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}}$. Similarly $\mathfrak{h}_\mu = \mathfrak{h}_m \oplus \mathfrak{p}$ for some G_m-invariant complement $\mathfrak{p}$.

Then,

$$\mathfrak{g}_m + \mathfrak{h}_\mu = \mathfrak{h}_m \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}} \oplus \mathfrak{p}.$$

Since $\mathfrak{g}_m + \mathfrak{h}_\mu \subset \mathfrak{g}_\mu$, we can choose a G_m-invariant complement

$$\mathfrak{b} := (\mathfrak{g}_m + \mathfrak{h}_\mu)^{\perp_{\mathfrak{g}_\mu}}$$

so that

$$\mathfrak{g}_\mu = \underbrace{\mathfrak{h}_m \oplus \mathfrak{p}}_{\mathfrak{h}_\mu} \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}} \oplus \mathfrak{b} = \underbrace{\mathfrak{h}_m \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}}}_{\mathfrak{g}_m} \oplus \mathfrak{p} \oplus \mathfrak{b}. \tag{5.8}$$

In particular we choose the G_m-invariant subspace $\mathfrak{m}$ of (5.7) to be

$$\mathfrak{m} := \mathfrak{p} \oplus \mathfrak{b}.$$

By (5.8), it satisfies $\mathfrak{g}_\mu = \mathfrak{g}_m \oplus \mathfrak{m}$. To define the space $\mathfrak{n}$ in the decomposition (5.7), we introduce the "symplectic orthogonal"

$$\mathfrak{h}^{\perp_\mu} := \{x \in \mathfrak{g} \mid x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)}\}. \tag{5.9}$$

Since $G_m \subset N_G(H)$, this subspace is G_m-invariant. It is characterized as follows:

Proposition 5.1.1. *The conditions below are equivalent:*

(i) $x \in \mathfrak{h}^{\perp_\mu}$

(ii) $\langle \mu, [x, \eta] \rangle = 0$ *for all* $\eta \in \mathfrak{h}$

(iii) $ad_x^* \mu \in \mathfrak{h}^\circ$ *where* $\mathfrak{h}^\circ := \{\lambda \in \mathfrak{g}^* \mid \lambda\big|_{\mathfrak{h}} = 0\}$ *is the annihilator of* $\mathfrak{h}$ *in* $\mathfrak{g}^*$

Proof. Let $m \in M$ and $\mu = \Phi_G(m)$. We first show that $(i) \iff (ii)$.

$$\begin{aligned} x \in \mathfrak{h}^{\perp_\mu} \quad &\iff \quad x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)} \\ &\iff \quad \omega(m)(x_M(m), \eta_M(m)) = 0 \text{ for all } \eta \in \mathfrak{h} \\ &\iff \quad \langle \mu, [x, \eta] \rangle = 0 \text{ for all } \eta \in \mathfrak{h}. \end{aligned}$$

Indeed,

$$\begin{aligned}
\omega(m)(x_M(m), \eta_M(m)) &= -d\phi_G^x(m) \cdot \eta_M(m) \\
&= -\langle D\Phi_G(m) \cdot \eta_M(m), x\rangle \\
&= -\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \Phi_G(\exp(t\eta) \cdot m), x\rangle \\
&= -\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \mathrm{Ad}^*_{\exp(-t\eta)} \Phi_G(m), x\rangle \\
&= -\langle \Phi_G(m), \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \mathrm{Ad}_{\exp(-t\eta)} x\rangle \\
&= -\langle \mu, [x, \eta]\rangle.
\end{aligned}$$

Finally, $(ii) \iff (iii)$ since

$$\begin{aligned}
\langle \mu, [x, \eta]\rangle = 0 \text{ for all } \eta \in \mathfrak{h} &\iff \langle \mathrm{ad}^*_x \mu, \eta\rangle = 0 \text{ for all } \eta \in \mathfrak{h} \\
&\iff \mathrm{ad}^*_x \mu \in \mathfrak{h}^\circ.
\end{aligned}$$

In particular, with the notations of Example 2.3.5, $x_{\mathfrak{g}^*}(\mu) \in T_\mu (G \cdot \mu) \cap \mathfrak{h}^\circ$. ■

Consider the projection $\alpha := \mu\big|_{\mathfrak{h}} \in \mathfrak{h}^*$ and let

$$\mathfrak{h}_\alpha = \{x \in \mathfrak{h} \mid \mathrm{ad}^*_x \alpha = \alpha\}.$$

Let us show that $\mathfrak{g}_\mu \cap \mathfrak{h}_\alpha = \mathfrak{h}_\mu$. Since $\mathfrak{g}_\mu \cap \mathfrak{h} = \mathfrak{h}_\mu$, the only non-trivial inclusion is $\mathfrak{g}_\mu \cap \mathfrak{h}_\alpha \supset \mathfrak{h}_\mu$. Let $x \in \mathfrak{h}_\mu$ and $y \in \mathfrak{h}$. Then

$$\begin{aligned}
\langle \mathrm{ad}^*_x \alpha, y\rangle &= \langle \alpha, [x, y]\rangle \\
&= \langle \mu, [x, y]\rangle \quad \text{since } [x, y] \in \mathfrak{h} \\
&= \langle \mathrm{ad}^*_x \mu, y\rangle \\
&= 0 \quad \text{since } x \in \mathfrak{h}_\mu
\end{aligned}$$

which shows that $x \in \mathfrak{g}_\mu \cap \mathfrak{h}_\alpha$. We now choose a G_m-invariant complement $\mathfrak{a}$ such that

$$\mathfrak{g}_\mu + \mathfrak{h}_\alpha = \underbrace{\mathfrak{h}_\mu \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}} \oplus \mathfrak{b}}_{\mathfrak{g}_\mu} \oplus \mathfrak{a} = \underbrace{\mathfrak{h}_\mu \oplus \mathfrak{a}}_{\mathfrak{h}_\alpha} \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}} \oplus \mathfrak{b}. \tag{5.10}$$

Furthermore observe that $\mathfrak{h}_\alpha \subset \mathfrak{h}^{\perp_\mu}$. Indeed, pick some element $x \in \mathfrak{h}_\alpha$, some $\eta \in \mathfrak{h}$, and notice that

$$\langle \mathrm{ad}_x^* \mu, \eta \rangle = \langle \mu, [x, \eta] \rangle = \langle \alpha, [x, \eta] \rangle = \langle \mathrm{ad}_x^* \alpha, \eta \rangle = 0. \tag{5.11}$$

Clearly $\mathfrak{g}_\mu \subset \mathfrak{h}^{\perp_\mu}$ since $\mathfrak{g}_\mu = \{x \in \mathfrak{g} \mid \mathrm{ad}_x^* \mu = 0\}$. We conclude that $\mathfrak{g}_\mu + \mathfrak{h}_\alpha \subset \mathfrak{h}^{\perp_\mu}$. Choose $\mathfrak{s}(G, H, \mu)$ to be a G_m-invariant complement to $\mathfrak{g}_\mu + \mathfrak{h}_\alpha$ in $\mathfrak{h}^{\perp_\mu}$. We can thus express (5.9) as the direct sum of G_m-invariant subspaces

$$\mathfrak{h}^{\perp_\mu} = \mathfrak{g}_\mu \oplus \mathfrak{a} \oplus \mathfrak{s}(G, H, \mu). \tag{5.12}$$

In particular,

$$\mathfrak{q} := \mathfrak{a} \oplus \mathfrak{s}(G, H, \mu) \tag{5.13}$$

is a G_m-invariant complement to $\mathfrak{g}_\mu$ in $\mathfrak{h}^{\perp_\mu}$. Finally, choosing a G_m-invariant complement $(\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}$ of $\mathfrak{h}^{\perp_\mu}$ in $\mathfrak{g}$ yields the decomposition

$$\mathfrak{g} = \underbrace{\mathfrak{h}_\mu \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}} \oplus \mathfrak{b}}_{\mathfrak{g}_\mu} \oplus \underbrace{\mathfrak{q} \oplus (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}}_{\mathfrak{n}}. \tag{5.14}$$

By (5.8) and (5.14), the G_m-invariant subspaces $\mathfrak{m}$ and $\mathfrak{n}$ of (5.7) can thus be chosen as follows:

$$\mathfrak{m} := \mathfrak{p} \oplus \mathfrak{b} \quad \text{and} \quad \mathfrak{n} = \mathfrak{q} \oplus (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}} \tag{5.15}$$

A Witt-Artin decomposition of M relative to the G-action can be chosen with the subspaces in (5.7) taken as in (5.15). This yields

$$T_m M = T_0 \oplus T_1 \oplus N_0 \oplus N_1 \tag{5.16}$$

with $T_0 = (\mathfrak{g}_m \oplus \mathfrak{p} \oplus \mathfrak{b}) \cdot m$ and $T_1 = (\mathfrak{q} \oplus (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}) \cdot m$. Note that we have some freedom in the choice of G_m-invariant normal subspaces N_0 and N_1. As we did previously we set $\alpha := \Phi_H(m) = \mu\big|_{\mathfrak{h}}$ and we define $\widetilde{T}_0 = \mathfrak{h}_\alpha \cdot m$. We shall give a specific choice of subspaces $\widetilde{T}_1, \widetilde{N}_0, \widetilde{N}_1$ such that the tangent space of M at m decomposes as

$$T_m M = \widetilde{T}_0 \oplus \widetilde{T}_1 \oplus \widetilde{N}_0 \oplus \widetilde{N}_1 \tag{5.17}$$

which is compatible with (5.16).

5.1.2 Symplectic slice construction

In this section, we explain how to choose the symplectic slice $(\widetilde{N}_1, \omega_{\widetilde{N}_1})$ at m appearing in (5.17). Explicitly we choose

$$\widetilde{N}_1 = \mathfrak{s}(G, H, \mu) \cdot m \oplus X_m \oplus N_1, \tag{5.18}$$

where $\mathfrak{s}(G, H, \mu)$ was previously defined as a G_m-invariant complement to $\mathfrak{g}_\mu + \mathfrak{h}_\alpha$ in $\mathfrak{h}^{\perp_\mu}$ and $X_m \subset T_m M$ is some subspace symplectomorphic to $\mathfrak{b} \times \mathfrak{b}^*$ endowed with the canonical symplectic form. We show in Lemma 5.1.5 that $\mathfrak{s}(G, H, \mu) \cdot m$ is a symplectic subspace of $(T_m M, \omega(m))$. This subspace depends on the choice of group, subgroup, and on the momentum μ. However the space $\mathfrak{b} = (\mathfrak{g}_m + \mathfrak{h}_\mu)^{\perp_{\mathfrak{g}_\mu}}$ also depends on the dynamics of the G-action on M as it involves the stabilizer subalgebra $\mathfrak{g}_m$. The next Proposition is a geometric description of the subspace $\mathfrak{s}(G, H, \mu) \cdot m$.

Proposition 5.1.2. *The subspace $\mathfrak{s}(G, H, \mu) \cdot m$ is identified with a symplectic slice at μ for the H-action on the coadjoint orbit $\mathcal{O}_\mu$.*

Proof. The subgroup H acts on the coadjoint orbit $\mathcal{O}_\mu = G \cdot \mu$ in the obvious way. Since the momentum map for the standard G-action on $\mathcal{O}_\mu$ is just the inclusion $\mathcal{O}_\mu \hookrightarrow \mathfrak{g}^*$, the momentum map $\Phi : \mathcal{O}_\mu \to \mathfrak{h}^*$ for the H-action is given by $\Phi(\mathrm{Ad}^*_{g^{-1}}\mu) = i^*_{\mathfrak{h}}(\mathrm{Ad}^*_{g^{-1}}\mu)$. The kernel of its differential is $\ker(D\Phi(\mu)) = (\mathfrak{a} \oplus \mathfrak{s}(G, H, \mu)) \cdot \mu$. Indeed, a straightforward calculation shows that

$$x_{\mathfrak{g}^*}(\mu) \in \ker\left(D\Phi(\mu)\right) \quad \Longleftrightarrow \quad \mathrm{ad}^*_x \mu \in \mathfrak{h}^\circ.$$

Proposition 5.1.1 implies that $x \in \mathfrak{h}^{\perp_\mu}$. Because of the identification

$$\mathfrak{g}^* = \mathfrak{n}^\circ \oplus T_\mu(\mathcal{O}_\mu)$$

and (5.12), we must have $x \in \mathfrak{a} \oplus \mathfrak{s}(G, H, \mu)$. The momentum of μ is $\Phi(\mu) = i_\mathfrak{h}^*(\mu) = \alpha$ and thus a symplectic slice for the H-action on $\mathcal{O}_\mu$ is a complement to $\mathfrak{h}_\alpha \cdot \mu$ in $\ker(D\Phi(\mu))$. By construction, this complement is $\mathfrak{s}(G, H, \mu) \cdot \mu$ which can be identified with $\mathfrak{s}(G, H, \mu) \cdot m$ as $\mathfrak{s}(G, H, \mu)$ has trivial intersection with $\mathfrak{g}_m$ and $\mathfrak{g}_\mu$. ■

Proposition 5.1.3. *Let (M, ω, G, Φ_G) be a Hamiltonian G-manifold, H be a closed subgroup of G and $\Phi_H : M \to \mathfrak{h}^*$ be the induced momentum map. Then*

$$\ker\left(D\Phi_H(m)\right) = \ker\left(D\Phi_G(m)\right) \oplus \mathcal{M},$$

where $\mathcal{M} \subset T_m M$ is isomorphic to $\mathfrak{q} \cdot m \times \mathfrak{b}^*$ as defined in (5.15).

Proof. By Proposition 4.2.1

$$\mathcal{M} = \{z_M(m) + w \in T_1 \oplus N_0 \mid -\mathrm{ad}_z^* \mu + f(w) \in \mathfrak{h}^\circ\}. \tag{5.19}$$

It remains to show that $\mathcal{M}$ is isomorphic to $\mathfrak{q} \cdot m \times \mathfrak{b}^*$. By construction

$$T_1 = \mathfrak{n} \cdot m = (\mathfrak{q} \oplus (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}) \cdot m$$

and N_0 is isomorphic to

$$\mathfrak{m}^* = \mathfrak{p}^* \oplus \mathfrak{b}^*.$$

An element $z_M(m) + w \in \mathcal{M}$ can thus be written uniquely as

$$u_M(m) + v_M(m) + w$$

for some unique elements $u \in \mathfrak{q}$ and $v \in (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}$. In addition, we set $f(w) = \pi + \beta$

for $\pi \in \mathfrak{p}^*$ and $\beta \in \mathfrak{b}^*$. By definition of $\mathcal{M}$ the following relation holds:

$$\langle -\mathrm{ad}^*_{u+v}\mu + \pi + \beta, \eta \rangle = 0 \quad \text{for all} \quad \eta \in \mathfrak{h}. \tag{5.20}$$

From the decomposition

$$\mathfrak{g}^*_\mu = \mathfrak{h}^*_\mu \oplus (\mathfrak{h}_m^{\perp_{\mathfrak{g}_m}})^* \oplus \mathfrak{b}^*,$$

we see that $\langle \beta, \eta \rangle = 0$ for all $\eta \in \mathfrak{h}$ since $\mathfrak{g}_\mu \cap \mathfrak{h} = \mathfrak{h}_\mu$ on which β vanishes. In addition, $\langle -\mathrm{ad}^*_{u+v}\mu, \eta \rangle = \langle -\mathrm{ad}^*_v\mu, \eta \rangle$ for all $\eta \in \mathfrak{h}$ as $u \in \mathfrak{q} \subset \mathfrak{h}^{\perp_\mu}$. Hence (5.20) reduces to

$$\langle -\mathrm{ad}^*_v\mu + \pi, \eta \rangle = 0 \quad \text{for all} \quad \eta \in \mathfrak{h}. \tag{5.21}$$

In particular, if $\eta \in \mathfrak{h}_\mu$, we are left with $\langle \pi, \eta \rangle = 0$ and thus $\pi = 0$. Since $\langle -\mathrm{ad}^*_v\mu, \eta \rangle = 0$ for all $\eta \in \mathfrak{h}$, this implies that $v \in \mathfrak{h}^{\perp_\mu} \cap (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}} = \{0\}$. Therefore the element $z_M(m) + w$ we started with is such that $z = u \in \mathfrak{q}$ and $f(w) = \beta \in \mathfrak{b}^*$. Conversely, it is straightforward to check from the argument above that an element

$$z_M(m) + w \in \mathfrak{q} \cdot m \oplus N_0$$

such that $f(w) = \beta \in \mathfrak{b}^*$ satisfies $-\mathrm{ad}^*_z\mu + \beta \in \mathfrak{h}^\circ$. We showed that

$$\mathcal{M} = \{u_M(m) + w \in \mathfrak{q} \cdot m \oplus N_0 \mid f(w) \in \mathfrak{b}^*\}.$$

The isomorphism is $F : u_M(m) + w \in \mathcal{M} \mapsto (u_M(m), f(w)) \in \mathfrak{q} \cdot m \times \mathfrak{b}^*$. ■

Theorem 5.1.4 (Symplectic Slice Reconstruction). *Given the decomposition (5.16), a symplectic slice $\widetilde{N_1}$ at m relative to the H-action can be chosen of the form*

$$\widetilde{N_1} = \mathfrak{s}(G, H, \mu) \cdot m \oplus X_m \oplus N_1, \tag{5.22}$$

where $X_m = \mathfrak{b} \cdot m \oplus Y_m$ with $Y_m \subset N_0$ isomorphic to $\mathfrak{b}^*$.

Proof. Let a Witt-Artin decompositin of M as in (5.16). Then by (5.8)

$$\begin{aligned}\ker\left(D\Phi_G(m)\right) &= \mathfrak{g}_\mu \cdot m \oplus N_1 \\ &= (\mathfrak{h}_\mu \oplus \mathfrak{h}_m^{\perp_{\mathfrak{g}_m}} \oplus \mathfrak{b}) \cdot m \oplus N_1 \\ &= \mathfrak{h}_\mu \cdot m \oplus \mathfrak{b} \cdot m \oplus N_1.\end{aligned} \tag{5.23}$$

By Proposition 5.1.3, there is a subspace $Y_m \subset N_0$ isomorphic to $\mathfrak{b}^*$ such that

$$\begin{aligned}\ker\left(D\Phi_H(m)\right) &= \ker\left(D\Phi_G(m)\right) \oplus \mathfrak{q} \cdot m \oplus Y_m \\ &= \mathfrak{h}_\mu \cdot m \oplus \mathfrak{b} \cdot m \oplus N_1 \oplus \mathfrak{q} \cdot m \oplus Y_m \quad \text{from (5.23)}.\end{aligned}$$

In (5.10) and (5.13) we obtained $\mathfrak{h}_\alpha = \mathfrak{h}_\mu \oplus \mathfrak{a}$ and $\mathfrak{q} = \mathfrak{a} \oplus \mathfrak{s}(G, H, \mu)$. Therefore

$$\mathfrak{h}_\mu \cdot m \oplus \mathfrak{q} \cdot m = \mathfrak{h}_\alpha \cdot m \oplus \mathfrak{s}(G, H, \mu) \cdot m.$$

Setting $X_m = \mathfrak{b} \cdot m \oplus Y_m$, we conclude that

$$\ker\left(D\Phi_H(m)\right) = \mathfrak{h}_\alpha \cdot m \oplus \mathfrak{s}(G, H, \mu) \cdot m \oplus X_m \oplus N_1. \tag{5.24}$$

A symplectic slice $\widetilde{N_1}$ at m for the H-action must satisfy

$$\ker\left(D\Phi_H(m)\right) = \mathfrak{h}_\alpha \cdot m \oplus \widetilde{N_1}.$$

Hence we choose $\widetilde{N_1} = \mathfrak{s}(G, H, \mu) \cdot m \oplus X_m \oplus N_1$. ∎

Lemma 5.1.5. *The subspace*

$$\mathfrak{s}(G, H, \mu) \cdot m = \{x_M(m) \mid x \in \mathfrak{s}(G, H, \mu)\}$$

is a symplectic vector subspace of $(T_m M, \omega(m))$. *The restriction of* $\omega(m)$ *on* $\mathfrak{s}(G, H, \mu) \cdot m$ *coincides with the Kostant-Kirillov-Souriau symplectic form.*

Proof. Using (5.13), the complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ defined in (5.15) reads

$$\mathfrak{n} = \underbrace{\mathfrak{a} \oplus \mathfrak{s}(G, H, \mu)}_{\mathfrak{q}} \oplus (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}. \tag{5.25}$$

To show that $\mathfrak{s}(G, H, \mu) \cdot m$ is symplectic, we use that

$$\mathfrak{n} \cdot m = \{z_M(m) \mid z \in \mathfrak{n}\}$$

is a symplectic vector subspace of $(T_m M, \omega(m))$. The restriction of $\omega(m)$ on $\mathfrak{n} \cdot m$ is non-degenerate and takes the form

$$\Psi(m)(x, y) = -\langle \mu, [x, y] \rangle.$$

Therefore $\omega(m)$ restricted to $\mathfrak{n} \cdot m$ coincides with the Kostant-Kirillov-Souriau symplectic form. Let us show that it is also non-degenerate when restricted to $\mathfrak{s}(G, H, \mu) \cdot m$. Assume $x \in \mathfrak{s}(G, H, \mu)$ is such that $\Psi(m)(x, y) = 0$ for all $y \in \mathfrak{s}(G, H, \mu)$. To show non-degeneracy we must show that $x_M(m) = 0$. By (5.25) any $z \in \mathfrak{n}$ can be written uniquely as $z = u + y + v$ with $u \in \mathfrak{a}, y \in \mathfrak{s}(G, H, \mu)$ and $v \in (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}$. This yields

$$\Psi(m)(x, z) = \Psi(m)(x, u) + \Psi(m)(x, v) \tag{5.26}$$

as the term $\Psi(m)(x, y)$ vanishes by assumption. Note that

$$\Psi(m)(x, u) = -\langle \mu, [x, u] \rangle = 0$$

since $x \in \mathfrak{h}^{\perp_\mu}$ by (5.12) and $u \in \mathfrak{a} \subset \mathfrak{h}$ by (5.10). Moreover the last term of (5.26) vanishes. To see this we construct a Witt-Artin decomposition at m relative to the H-action:

$$T_m M = \widetilde{T}_0 \oplus \widetilde{T}_1 \oplus \widetilde{N}_0 \oplus \widetilde{N}_1 \tag{5.27}$$

with $\widetilde{N}_1$ as in Theorem Theorem 5.1.4. Recall that

$$\ker(D\Phi_H(m)) = \widetilde{T}_0 \oplus \widetilde{N}_1.$$

Furthermore since $\ker(D\Phi_H(m)) = (\mathfrak{h}\cdot m)^{\omega(m)}$, we can write

$$\mathfrak{h}^{\perp_\mu} = \{x \in \mathfrak{g} \mid x_M(m) \in \widetilde{T}_0 \oplus \widetilde{N}_1\}. \tag{5.28}$$

There are two possibilities:

(i) If $v \in \mathfrak{h}$ then $v_M(m) \in \widetilde{T}_1$ since $v \in (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}$. The subspaces $\widetilde{T}_1$ and $\widetilde{N}_1$ are symplectically orthogonal. Hence $\Psi(m)(x, v) = 0$.

(ii) Otherwise $v_M(m) \in \widetilde{N}_0$. Indeed since $v \in (\mathfrak{h}^{\perp_\mu})^{\perp_\mathfrak{g}}$, it cannot belong to $\widetilde{T}_0 \oplus \widetilde{N}_1$ by (5.28). Since $x_M(m) \in \mathfrak{s}(G,H,\mu)\cdot m \subset \widetilde{N}_1$ and $\widetilde{T}_0 \oplus \widetilde{N}_0$ and $\widetilde{N}_1$ are symplectically orthogonal, we conclude that $\Psi(m)(x, v) = 0$.

Therefore (5.26) reduces to

$$\Psi(m)(x, z) = 0 \quad \text{for all} \quad z \in \mathfrak{n}.$$

Since $\mathfrak{n}\cdot m$ is symplectic we get $x_M(m) = 0$ and we are done. ∎

Theorem 5.1.6. *With respect to the splitting of Theorem 5.1.4, the symplectic form $\omega_{\widetilde{N}_1}$ reads*

$$\Psi(m) \oplus \omega_{X_m} \oplus \omega_{N_1}$$

with $\Psi(m)$ as in Lemma 5.1.5 and

$$\omega_{X_m}(b_M(m) + w, b'_M(m) + w') = \langle f(w'), b\rangle - \langle f(w), b'\rangle$$

for all $b, b' \in \mathfrak{b}$ and $w, w' \in Y_m$.

Proof. We already know by Lemma 5.1.5 that the symplectic form on $\mathfrak{s}(G,H,\mu)\cdot m$ is given by $\Psi(m)$. Denote by ω_{X_m} the restriction of $\omega(m)$ on X_m. By 2.20 it coincides with the pullback of the canonical symplectic form on $\mathfrak{b}\times\mathfrak{b}^*$ along the isomorphism

$$b_M(m) + w \in X_m = \mathfrak{b}\cdot m \oplus Y_m \mapsto (b, f(w)) \in \mathfrak{b}\times\mathfrak{b}^*.$$

Therefore

$$\omega_{X_m}(b_M(m)+w, b'_M(m)+w') = \langle f(w'), b\rangle - \langle f(w), b'\rangle$$

for all $b, b' \in \mathfrak{b}$ and $w, w' \in Y_m$. As stated, we obtain the decomposition

$$\omega_{\widetilde{N}_1}(m) = \Psi(m) \oplus \omega_{X_m}(m) \oplus \omega_{N_1}.$$

■

Proposition 5.1.7. *With respect to the splitting of Corollary 5.1.4, the momentum map $\Phi_{\widetilde{N}_1} : \widetilde{N}_1 \to \mathfrak{h}_m^*$ associated to the linear H_m-action on $\widetilde{N}_1$ decomposes as*

$$\langle \Phi_{\widetilde{N}_1}(\widetilde{\nu}), \eta\rangle = -\frac{1}{2}\langle (ad_x^*)^2\mu, \eta\rangle + \langle -ad_b^* f(w), \eta\rangle + \frac{1}{2}\omega_{N_1}(\eta_{N_1}(\nu), \nu)$$

for all $\eta \in \mathfrak{h}_m$, where $\widetilde{\nu} = x_M(m) + (b_M(m) + w) + \nu \in \widetilde{N}_1$ with $x \in \mathfrak{s}(G, H, \mu), b \in \mathfrak{b}, w \in Y_m$ and $\nu \in N_1$.

Proof. By linearity of the Hamiltonian H_m-action on $\widetilde{N}_1$, the momentum map $\Phi_{\widetilde{N}_1}$ takes the form

$$\langle \Phi_{\widetilde{N}_1}(\widetilde{\nu}), \eta\rangle = \frac{1}{2}\omega_{\widetilde{N}_1}\left(\eta_{\widetilde{N}_1}(\widetilde{\nu}), \widetilde{\nu}\right) \tag{5.29}$$

for all $\widetilde{\nu} \in \widetilde{N}_1$ and $\eta \in \mathfrak{h}_m$. With respect to the decomposition of $\widetilde{N}_1$ of Corollary 5.1.4, we write

$$\widetilde{\nu} = x_M(m) + (b_M(m) + w) + \nu \in \widetilde{N}_1$$

where $x \in \mathfrak{s}(G, H, \mu), b \in \mathfrak{b}, w \in Y_m$ and $\nu \in N_1$. For $\eta \in \mathfrak{h}_m$ we get

$$\begin{aligned} \eta_{\widetilde{N}_1}(x_M(m)) &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(t\eta) \cdot x_M(m) \\ &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \left(\mathrm{Ad}_{e^{t\eta}} x\right)_M (m) \\ &= [\eta, x]_M(m). \end{aligned}$$

Similary $\eta_{\widetilde{N}_1}(b_M(m)) = [\eta, b]_M(m)$. By Corollary 5.1.4, the symplectic form on $\widetilde{N}_1$ decomposes as $\omega_{\widetilde{N}_1}(m) = \Psi(m) \oplus \omega_{X_m}(m) \oplus \omega_{N_1}$ and then (5.29) reads

$$\begin{aligned} \frac{1}{2}\omega_{\widetilde{N}_1}\left(\eta_{\widetilde{N}_1}(\widetilde{\nu}), \widetilde{\nu}\right) &= \frac{1}{2}\Psi(m)([\eta, x], x) \\ &+\frac{1}{2}\omega_{X_m}(m)\left([\eta, b]_M(m) + \eta_{\widetilde{N}_1}(w), b_M(m) + w\right) \\ &+\frac{1}{2}\omega_{N_1}\left(\eta_{N_1}(\nu), \nu\right). \end{aligned}$$

By definition the second term of the above is $\frac{1}{2}\Big(\langle f(w), [\eta, b]\rangle - \langle f(\eta_{\widetilde{N}_1}(w)), b\rangle\Big)$. Since the linear map f is H_m-equivariant,

$$\langle f(\eta_{\widetilde{N}_1}(w)), b\rangle = \langle -\mathrm{ad}^*_\eta f(w), b\rangle = -\langle f(w), [\eta, b]\rangle.$$

Finally

$$\begin{aligned} \Psi(m)([\eta, x], x) &= -\langle \mu, [[\eta, x], x]\rangle \\ &= -\langle \mathrm{ad}^*_x \mu, [x, \eta]\rangle \\ &= -\langle (\mathrm{ad}^*_x)\mu, \eta\rangle. \end{aligned}$$

We thus obtain

$$\langle \Phi_{\widetilde{N}_1}(\widetilde{\nu}), \eta\rangle = -\frac{1}{2}\langle (\mathrm{ad}^*_x)^2\mu, \eta\rangle + \langle -\mathrm{ad}^*_b f(w), \eta\rangle + \frac{1}{2}\omega_{N_1}\left(\eta_{N_1}(\nu), \nu\right).$$

■

Example 5.1.8 (Abelian groups)**.** Let (M, ω, G, Φ_G) be a Hamiltonian G-manifold where G is abelian and let H be a subgroup of G. For simplicity we assume that this action is free i.e. all the stabilizers G_m are trivial. If $m \in M$ has momentum $\mu = \Phi_G(m)$, we then $\mathfrak{g}_\mu = \mathfrak{g}$ and $\mathfrak{h}_\alpha = \mathfrak{h}_\mu = \mathfrak{h}$. In particular $\mathfrak{g}_\mu + \mathfrak{h}_\alpha = \mathfrak{g}$. Since G is abelian $\mathfrak{h}^{\perp_\mu} = \mathfrak{g}$, and thus $\mathfrak{s}(G, H, \mu) = 0$ as it is the orthogonal complement of $\mathfrak{g}_\mu + \mathfrak{h}_\alpha$ in $\mathfrak{h}^{\perp_\mu}$. On the other hand, $\mathfrak{b} = \mathfrak{h}^{\perp_\mathfrak{g}}$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$. Corollary 5.1.4 implies

$$\widetilde{N_1} = N_1 \oplus X_m \tag{5.30}$$

where X_m is isomorpic to $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$.

Example 5.1.9. Let (M, ω, G, Φ_G) where $G = SO(3)$ is the group of rotations in $\mathbb{R}^3$. Assume that this action is free. Let $H = SO(2)$ be the subgroup of rotations about the axis defined by a vector $x \in \mathbb{R}^3$. The Lie algebra $\mathfrak{g}$ is the space of 3×3 skew-symmetric matrices. It is identified with $\mathbb{R}^3$ and so is its dual $\mathfrak{g}^*$ by using the standard dot product. Let $m \in M$ be a point with momentum $\Phi_G(m) = \mu \in \mathfrak{g}^*$ where $\mu := \chi \in \mathbb{R}^3$. Similarly an element $y \in \mathfrak{g}$ is identified with $y := y \in \mathbb{R}^3$. Clearly

$$\mathfrak{g}_\mu := \mathrm{span}(\chi) \subset \mathbb{R}^3 \quad \text{and} \quad \mathfrak{h} := \mathrm{span}(x) \subset \mathbb{R}^3.$$

Since $\mathrm{ad}_y^*\mu := \chi \times y \in \mathbb{R}^3$ the symplectic orthogonal $\mathfrak{h}^{\perp_\mu}$ is the subspace of $\mathbb{R}^3$ defined by

$$\mathfrak{h}^{\perp_\mu} := \{y \in \mathbb{R}^3 \mid (\chi \times x) \cdot y = 0\}. \tag{5.31}$$

There are three cases to be considered: when χ and x are not collinear, when they are collinear, and when $\chi = 0$.

(i) If χ and x are not collinear then there are no elements in H fixing χ. Therefore $\mathfrak{h}_\mu = 0$. As H is abelian, $\mathfrak{h}_\alpha = \mathfrak{h}$ and thus $\mathfrak{g}_\mu + \mathfrak{h}_\alpha := \mathrm{span}(\chi, x)$. Furthermore

$$\mathfrak{h}^{\perp_\mu} := \mathrm{span}(\chi, x)$$

is a two-dimensional plane. We conclude that $\mathfrak{s}(G, H, \mu) = 0$. The other subspace of interest is $\mathfrak{b} = (\mathfrak{g}_m + \mathfrak{h}_\mu)^{\perp_{\mathfrak{g}_\mu}}$. In this case, as $\mathfrak{g}_m + \mathfrak{h}_\mu = 0$, we deduce that $\mathfrak{b} = \mathfrak{g}_\mu$. By applying Corollary 5.1.4, the symplectic slice for the

H-action is given by

$$\widetilde{N_1} = N_1 \oplus X_m, \tag{5.32}$$

where X_m is isomorphic to $\mathfrak{g}_\mu \times \mathfrak{g}_\mu^*$.

(ii) When χ and x are collinear, all the elements of H will fix $\chi \in \mathbb{R}^3$. Consequently $\mathfrak{h}_\mu = \mathfrak{h} = \mathfrak{g}_\mu$ and

$$\mathfrak{h}^{\perp_\mu} = \mathfrak{g} := \mathbb{R}^3.$$

In this case $\mathfrak{s}(G, H, \mu) = \mathfrak{n}$ which is the orthogonal complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$. In $\mathbb{R}^3$, it is identified with the plane through the origin that is orthogonal to χ. It is also naturally isomorphic to the tangent space at χ of the 2-sphere of radius $\|\chi\|$ which corresponds to the coadjoint orbit $\mathcal{O}_\mu$. However, as $\mathfrak{h}_\mu = \mathfrak{h} = \mathfrak{g}_\mu$, we find that $\mathfrak{b} = 0$. Therefore,

$$\widetilde{N_1} = N_1 \oplus \mathfrak{s}(G, H, \mu) \cdot m \tag{5.33}$$

where $\mathfrak{s}(G, H, \mu) \cdot m = \mathfrak{n} \cdot m$.

(iii) When $\mu := \chi = 0$ we have $\mathfrak{g}_\mu = \mathfrak{g}$ and $\mathfrak{h}_\mu = \mathfrak{h}$. As H is abelian, $\mathfrak{h}_\alpha = \mathfrak{h}$ and we find $\mathfrak{g}_\mu + \mathfrak{h}_\alpha = \mathfrak{g} := \mathbb{R}^3$. This implies that $\mathfrak{s}(G, H, \mu) = 0$. The subspace $\mathfrak{b}$ is just $\mathfrak{h}^{\perp_\mathfrak{g}} \simeq \mathfrak{g}/\mathfrak{h}$ which is the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. Therefore

$$\widetilde{N_1} = N_1 \oplus X_m, \tag{5.34}$$

where X_m is isomorphic to $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$.

ROOT SYSTEMS AND HESSIAN DEGENERACY

This section is devoted to the proof of Lemma 4.2.3 which states that if (M, ω, G, Φ_G) is a Hamiltonian G-manifold with $\mathfrak{g}$ semi-simple, $m \in M$ is a point with momentum $\mu = \Phi_G(m)$ and $\xi \in \mathfrak{g}_\mu$, then the restriction of the Hessian $D^2\phi_G^\xi(m)$ to $\mathfrak{g} \cdot m$ is degenerate only along

$$(\mathfrak{g}_\xi + \mathfrak{g}_\mu) \cdot m.$$

Our proof relies on the machinery of root systems and will be a straightforward calculation using a specific basis of $\mathfrak{g}$ known as the *Weyl-Chevalley basis* (Theorem 6.1.11). To understand a Lie algebra $\mathfrak{g}$ with vector space basis $\{x_1, \ldots, x_n\}$, we need to know the *structure constants* which are the coefficient c_{ij}^k such that

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k.$$

In particular, the Weyl-Chevalley basis of $\mathfrak{g}$ is a basis with respect to which most of the brackets $[x_i, x_j]$ are zero. The idea is to first consider a basis of a Cartan subalgebra of $\mathfrak{g}$ which is in particular a maximal abelian subalgebra. Furthermore this basis is extended to the whole $\mathfrak{g}$ so that the associated root system of $\mathfrak{g}$ encodes entirely the structure constants. The references for this chapter are the books of Taylor [67], Sattinger and Weaver [63] and Zhelobenko [71].

6.1 Weyl-Chevalley Normal Form Theorem

In this thesis we always assumed that $\mathfrak{g}$ was a finite dimensional real Lie algebra. However the result we need to prove relies on results that hold for complex Lie algebras. This is not a big deal because any real Lie algebra can be complexified as shown below.

Definition 6.1.1. A *complex Lie algebra* is a complex vector space $\mathfrak{g}$ equipped with a bilinear Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is antisymmetric and satisfies the Jacobi identity.

A real Lie algebra $\mathfrak{k}$ with Lie bracket $[\cdot,\cdot]$ can be turned into a complex Lie algebra by the process of *complexification.* This means considering the complex vector space $\mathfrak{k}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}$ whose elements are abbreviated $1 \otimes_{\mathbb{R}} x = x$ and $i \otimes_{\mathbb{R}} x = ix$. We endow $\mathfrak{k}_{\mathbb{C}}$ with the Lie bracket $[\cdot,\cdot]_{\mathbb{C}}$ defined by

$$[x_1 + iy_1, x_2 + iy_2]_{\mathbb{C}} = [x_1, x_2] - [y_1, y_2] + i\left([x_1, y_2] + [y_1, x_2]\right) \tag{6.1}$$

for all $x_1, x_2, y_1, y_2 \in \mathfrak{k}$.

Example 6.1.2. The complexification of the Lie algebras of the real compact matrix Lie groups $SU(3)$ and $U(n)$ are $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$ respectively.

From now we assume that our Lie algebras are complex with Lie bracket as above. The subscript $\mathbb{C}$ is omitted except stated otherwise.

6.1.1 Reductive Lie algebras

Generally a Lie algebra is called reductive if its adjoint representation is completely reducible. Theorem 6 of Section 96 in [71] implies that all complex reductive Lie algebra are obtained by complexification of a compact Lie group. We thus take this characterization as a definition.

Definition 6.1.3. A complex Lie algebra $\mathfrak{g}$ is called *reductive* if there exists a real compact Lie group K with Lie algebra $\mathfrak{k}$ such that $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$. In this case $\mathfrak{k}$ is called a *compact real form.*

Proposition 6.1.4 (Inner product on reductive Lie algebras). *Let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be a reductive Lie algebra. There exists an inner product $\langle \cdot,\cdot \rangle$ on $\mathfrak{g}$ with the following properties:*

(i) $\langle \cdot, \cdot \rangle$ *restricted to* $\mathfrak{k} \times \mathfrak{k}$ *is real-valued.*

(ii) $\langle ad_x y, z \rangle + \langle y, ad_x z \rangle = 0$ *for all* $x \in \mathfrak{k}$*,* $y, z \in \mathfrak{g}$*. In particular the endomorphisms* $ad_x \in End(\mathfrak{g})$ *are skew-symmetric for all* $x \in \mathfrak{k}$*.*

(iii) $\langle ad_x y, z \rangle = \langle y, ad_{x^*} z \rangle$ *for all* $x, y, z \in \mathfrak{g}$ *with* $x^* = -\bar{x}$ *where* $\bar{x}$ *is the complex conjugate of* x *i.e. the operators* $ad_x \in End(\mathfrak{g})$ *are skew-hermitian.*

Proof. Firstly we construct a K-invariant inner product on $\mathfrak{g}$. As $\mathfrak{k}$ is the Lie algebra of some compact Lie group K we can take any real valued inner product on $\mathfrak{k}$ and average it to obtain a new inner product $\langle \cdot, \cdot \rangle$, invariant with respect to the Adjoint representation $\mathrm{Ad} : K \to \mathrm{Gl}(\mathfrak{k})$. We extend this inner product on $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ so that it defines an hermitian inner product

$$\langle x_1 + iy_1, x_2 + iy_2 \rangle := (\langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle) - i\,(\langle x_1, y_2 \rangle + \langle y_1, x_2 \rangle)$$

where $x_1, x_2, y_1, y_2 \in \mathfrak{k}$. It satisfies the following properties:

(i) Setting $y_1 = y_2 = 0$ in the above yields $\langle x_1, x_2 \rangle \in \mathbb{R}$. Therefore the restriction of $\langle \cdot, \cdot \rangle$ on $\mathfrak{k} \times \mathfrak{k}$ is real.

(ii) Pick any $x \in \mathfrak{k}$, $y, z \in \mathfrak{g}$. The result follows by differentiating the relation $\langle \mathrm{Ad}_{e^{tx}} y, \mathrm{Ad}_{e^{tx}} z \rangle = \langle y, z \rangle$ with respect to $t \in \mathbb{R}$ and evaluating it at $t = 0$.

(iii) Set $x = x_1 + ix_2$ and $x^* = -x_1 + ix_2$ with $x_1, x_2 \in \mathfrak{k}$. Then

$$\begin{aligned}
\langle \mathrm{ad}_x y, z \rangle &= \langle [x_1, y], z \rangle + i \langle [x_2, y], z \rangle \\
&= -\langle y, [x_1, z] \rangle - i \langle y, [x_2, z] \rangle \quad \text{as } x_1, x_2 \in \mathfrak{k} \\
&= -\langle y, [x_1, z] \rangle - \langle y, \bar{i}[x_2, z] \rangle \quad \text{by } \mathbb{C}\text{-bilinearity of } \langle \cdot, \cdot \rangle \\
&= \langle y, [-x_1, z] \rangle + \langle y, [ix_2, z] \rangle \\
&= \langle y, \mathrm{ad}_{x^*} z \rangle.
\end{aligned}$$

∎

6.1.2 Semi-simple Lie algebras and the Killing form

Complex semi-simple Lie algebras are a particular case of complex reductive Lie algebras for which there is a canonical choice of inner product as in Proposition 6.1.4. This inner product is given by the negative of the Killing form.

Definition 6.1.5. A reductive complex Lie algebra $\mathfrak{g}$ is *semi-simple* if its center $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \quad \forall y \in \mathfrak{g}\}$ is trivial.

The *Killing form* of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by

$$\kappa(x, y) = \mathrm{Tr}\left(\mathrm{ad}_x \mathrm{ad}_y\right)$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of a linear operator. Cartan's second criterion ([67] Theorem 14.4.7) states that $\mathfrak{g}$ is semi-simple if and only if the Killing form κ is non-degenerate. We show in Proposition 6.1.6 below that the negative $-\kappa$ defines a K-invariant inner product on the compact real form $\mathfrak{k}$ of $\mathfrak{g}$. The negative of the Killing form is then extended into an hermitian inner product on the whole $\mathfrak{g}$ as in Proposition 6.1.4.

Proposition 6.1.6. *The Killing κ restricted to the compact real form $\mathfrak{k}$ of $\mathfrak{g}$ has the following properties:*

(i) κ *is symmetric.*

(ii) $\kappa(Ad_k y, Ad_k z) = \kappa(y, z)$ *for every* $y, z \in \mathfrak{k}$ *and* $k \in K$.

(iii) $\kappa([x, y], z) + \kappa(y, [x, z]) = 0$ *for every* $x, y, z \in \mathfrak{k}$.

(iv) κ *is negative definite.*

Proof. The first statement (i) follows from the symmetry of the trace operator.

To show (ii) notice that, given $k \in K$ and $y, z \in \mathfrak{k}$, we have

$$\begin{aligned}
[Ad_k y, z] &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} Ad_{e^{tAd_k y}} z \\
&= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} Ad_k Ad_{e^{ty}} Ad_{k^{-1}} z \\
&= Ad_k [y, Ad_k^{-1} z].
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa(Ad_k y, Ad_k z) &= \mathrm{Tr}(\mathrm{ad}_{Ad_k y} \mathrm{ad}_{Ad_k z}) \\
&= \mathrm{Tr}(Ad_k \mathrm{ad}_y \mathrm{ad}_z Ad_k^{-1}) \\
&= \mathrm{Tr}(\mathrm{ad}_y \mathrm{ad}_z) \\
&= \kappa(y, z).
\end{aligned}$$

To prove (iii) we use (ii) with $k = e^{tx}$ for $t \in \mathbb{R}$ and $x \in \mathfrak{k}$. Differentiating (ii) with respect to t yields $\kappa([x, y], z) + \kappa(y, [x, z]) = 0$. Finally (iv) follows because, by (iii), the operators ad_x are skew-symmetric with respect to κ and the trace of the square of a skew-symmetric operator is negative. In particular the negative of the Killing form is a real inner product on $\mathfrak{k}$. ■

If $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate by Cartan's second criterion. It thus induces an isomorphism

$$\kappa^\sharp : \mathfrak{g}^* \to \mathfrak{g} \tag{6.2}$$

where $\kappa^\sharp(\lambda) =: t_\lambda$ is uniquely defined such that $\kappa(t_\lambda, x) = \lambda(x)$ for every $x \in \mathfrak{g}$. This isomorphism satisfies the equivariance property $\kappa^\sharp(\mathrm{ad}_y^* \lambda) = [y^*, t_\lambda]$ where $y^* = -\bar{y}$ for every $y \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$. Indeed by (iii) of Proposition 6.1.4,

$$\begin{aligned}
\kappa\left(\kappa^\sharp(\mathrm{ad}_y^* \lambda), x\right) &= \mathrm{ad}_y^* \lambda(x) \\
&= \lambda([y, x]) \\
&= \kappa\left(t_\lambda, [y, x]\right) \\
&= \kappa\left([y^*, t_\lambda], x\right) \quad \text{where} \quad y^* = -\bar{y}
\end{aligned}$$

for any $x, y \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$. The non-degeneracy condition implies that

$$\kappa^\sharp(\mathrm{ad}_y^*\lambda) = [y^*, t_\lambda]. \tag{6.3}$$

6.1.3 Cartan subalgebras and root systems

In all generality, a Cartan subalgebra of a Lie algebra is a nilpotent subalgebra which is its own normalizer. However for complex semi-simple Lie algebras, a Cartan subalgebra is a maximal abelian subalgebra whose elements are semi-simple (cf. (iii) below).

Definition 6.1.7. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. A complex Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* if

(i) $[h_1, h_2] = 0$ for all $h_1, h_2 \in \mathfrak{h}$,

(ii) If some $x \in \mathfrak{g}$ satisfies $[x, h] = 0$ for all $h \in \mathfrak{h}$ then $x \in \mathfrak{h}$,

(iii) The endomorphisms ad_h are diagonalizable for all $h \in \mathfrak{h}$.

Theorem 6.1.8 (Existence of Cartan subalgebras [67])**.** *Assume $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is a complex semi-simple Lie algebra for some compact real form $\mathfrak{k}$ and let $\mathfrak{t} \subset \mathfrak{k}$ be any maximal commutative Lie subalgebra. Then $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra. Furthermore every Cartan subalgebra arises that way.*

Fix a Cartan subalgebra $\mathfrak{h}$ of some complex semi-simple Lie algebra $\mathfrak{g}$. By (i) of Definition 6.1.7 and the Jacobi identity of the Lie bracket, the operators ad_h are pairwise commuting for all $h \in \mathfrak{h}$. Indeed if $x \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$ we have

$$\mathrm{ad}_{h_1}\mathrm{ad}_{h_2}x = [h_1, [h_2, x]] = -[x, [h_1, h_2]] + [h_2, [h_1, x]] = \mathrm{ad}_{h_2}\mathrm{ad}_{h_1}x.$$

Since in addition they are diagonalizable, there is a single basis of $\mathfrak{g}$ consisting of vectors which are simultaneously eigenvectors for all ad_h with $h \in \mathfrak{h}$, and whose corresponding eigenvalues depend linearly on h.

Definition 6.1.9. An element $\alpha \in \mathfrak{h}^*$ is called a *root* if there exists a non-zero vector $x \in \mathfrak{g}$ such that $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}$. The set of roots is denoted by $\mathcal{R}$ and is viewed as a subset of $\mathfrak{h}^*$. Given a root $\alpha \in \mathcal{R}$, we call the subspace

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \mathrm{ad}_h x = \alpha(h)x,\ \forall h \in \mathfrak{h}\}$$

the *root space* of α.

Theorem 6.1.10 (Cartan decomposition)**.** *A complex semi-simple Lie algebra $\mathfrak{g}$ admits a decomposition of the form*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha \right) \tag{6.4}$$

with the properties

(i) Two root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form, as long as $\alpha + \beta \neq 0$.

(ii) If $\alpha, \beta \in \mathcal{R}$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Proof. Since $\mathfrak{g}$ is semi-simple, elements $h \in \mathfrak{h}$ give rise to linear operators $\mathrm{ad}_h \in \mathrm{End}(\mathfrak{h})$ that are simultaneously diagonalizable. The Lie algebra thus admits the eigenspaces decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha \right). \tag{6.5}$$

The root space $\mathfrak{g}_0$ corresponding to the trivial root is the centralizer

$$\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [h, x] = 0 \quad \text{for all} \quad h \in \mathfrak{h}\}.$$

By the maximality assumption of the Cartan subalgebra, $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

(i) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Write $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$ and let $h \in \mathfrak{t}$. Hence

$$\begin{aligned} 0 &= \kappa(\mathrm{ad}_h x, y) + \kappa(x, \mathrm{ad}_h y) \\ &= \kappa(\alpha(h)x, y) + \kappa(x, \beta(h)y) \\ &= (\alpha(h) + \beta(h))\kappa(x, y). \end{aligned}$$

We use the $\mathbb{C}$-linearity of α and β to obtain that $(\alpha(h) + \beta(h))\kappa(x, y) = 0$ for all $h \in \mathfrak{h}$. Since $\alpha + \beta \neq 0$ we get $\kappa(x, y) = 0$.

(ii) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. If $\alpha + \beta \in \mathcal{R}$ then we can use the Jacobi identity to show that, for all $h \in \mathfrak{h}$,

$$\begin{aligned} \mathrm{ad}_h([x, y]) &= [h, [x, y]] \\ &= [[h, x], y] + [x, [h, y]] \\ &= [\alpha(h)x, y] + [x, \beta(h)y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

This shows that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. In particular, if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, the calculation above yields $\mathrm{ad}_h([x, y]) = 0$ and thus

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}. \tag{6.6}$$

If $\alpha + \beta$ is neither zero nor a root, then $[x, y] = 0$.

■

There are standard facts about root spaces that we do not prove. One of them is that for each root α, the only multiples of α that are roots are α itself and $-\alpha$. Another fact is that each $\mathfrak{g}_\alpha$ is one dimensional as a complex vector space (cf. [63] Theorem 7.23).

Theorem 6.1.10 above can be used to show that the Killing form κ restricted to a Cartan subalgebra $\mathfrak{h}$ is non-degenerate. As $\mathfrak{g}$ is semi-simple, κ is non-degenerate on $\mathfrak{g}$. Let $h \in \mathfrak{h}$ such that $\kappa(h, x) = 0$ for all $x \in \mathfrak{h}$. Since $\mathfrak{h}$ is κ-orthogonal to all the root spaces $\mathfrak{g}_\alpha$ such that $\alpha \neq 0$, we conclude that $\kappa(h, y) = 0$ for all $y \in \mathfrak{g}$ using decomposition (6.4). Thus $h = 0$ by non-degeneracy of κ on $\mathfrak{g}$. The isomorphism (6.2) reduces to an isomorphism $\kappa^\sharp : \mathfrak{h}^* \to \mathfrak{h}$. In particular to each root $\alpha \in \mathcal{R}$ is associated a unique element $t_\alpha \in \mathfrak{h}$ which is a basis element of the one dimensional subspace $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ (cf. [67] Theorem 14.5.7).

6.1.4 The Weyl-Chevalley Normal Form Theorem

In this section, we construct a basis of a complex semi-simple Lie algebra $\mathfrak{g}$ with respect to which the Lie bracket has a specific normal form. Let $\mathfrak{h}$ be a Cartan subalgebra with basis $\{H_1, \cdots, H_k\}$ orthogonal with respect to the Killing form. In particular

$$[H_i, H_j] = 0 \quad \text{for all} \quad i, j.$$

Consider the decomposition into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha \right).$$

Since each $\mathfrak{g}_\alpha$ is one-dimensional as a complex vector space, we fix a basis element X_α for each of them such that $\kappa(X_\alpha, X_{-\alpha}) = 1$. Such a basis always exists. Indeed if $X_\alpha \in \mathfrak{g}_\alpha$ is a non-trivial element, then $\kappa(X_\alpha, y) = 0$ for every $y \in \mathfrak{g}_\beta$ such that $\beta \in \mathcal{R} \setminus \{-\alpha\}$. By non-degeneracy of the Killing form, the root space $\mathfrak{g}_{-\alpha}$ contains some non-trivial element $X_{-\alpha}$ element satisfying $\kappa(X_\alpha, X_{-\alpha}) \neq 0$. The claim follows from a normalization process. We choose

$$\{H_1, \ldots, H_k\} \cup \{X_\alpha \mid \alpha \in \mathcal{R}\}$$

as a basis of $\mathfrak{g}$. Let us find the structure constants of the Lie bracket with respect to this basis. By definition of $\mathfrak{g}_\alpha$,

$$[H_i, X_\alpha] = \alpha(H_i) X_\alpha$$

and by (6.6),

$$[X_\alpha, X_{-\alpha}] = \sum_{i=1}^{k} \lambda_i H_i$$

with coefficients

$$\begin{aligned}\lambda_i &= \kappa([X_\alpha, X_{-\alpha}], H_i)\\ &= \kappa([H_i, X_\alpha], X_{-\alpha})\\ &= \alpha(H_i)\kappa(X_\alpha, X_{-\alpha})\\ &= \alpha(H_i)\end{aligned}$$

since we chose an orthogonal basis. There is still to work out what is $[X_\alpha, X_\beta]$ when $\alpha+\beta \neq 0$. Of course, if $\alpha+\beta$ is not a root, $[X_\alpha, X_\beta] = 0$ by (ii) of Theorem 6.1.10. For the same reason, if $\alpha+\beta \in \mathcal{R}$,

$$[X_\alpha, X_\beta] = \lambda_{\alpha\beta} X_{\alpha+\beta}$$

for some coefficients $\lambda_{\alpha\beta}$ sometimes referred to the *Cartan integers*, after a suitable normalization of each X_α. Working out the coefficients $\lambda_{\alpha\beta}$ is a bit tedious and is done along the lines of [62] (Section 2.5) and [63] (Theorem 10.1). We partially proved the following:

Theorem 6.1.11 (Weyl-Chevalley Normal Form)**.** *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra and $\mathcal{R}$ be its set of roots. Then there is a basis of $\mathfrak{g}$*

$$\{H_1, \ldots, H_k\} \cup \{X_\alpha \mid \alpha \in \mathcal{R}\}$$

with respect to which the structure constants read

$$\begin{array}{ll}[H_i, H_j] = 0 & [X_\alpha, X_{-\alpha}] = \sum_{i=1}^k \alpha(H_i) H_i\\ [H_i, X_\alpha] = \alpha(H_i) X_\alpha & [X_\alpha, X_\beta] = \lambda_{\alpha\beta} X_{\alpha+\beta}\end{array}$$

where $\lambda_{\alpha\beta} = 0$ unless $\alpha+\beta$ is a root. Furthermore there is a normalization of X_α such that all the structure constants are integers and $\lambda_{\alpha\beta} = -\lambda_{(-\alpha)(-\beta)}$.

6.2 Momentum map degeneracy along an orbit

This section is devoted to the proof of Lemma 4.2.3 where we assume that $\mathfrak{g}$ is semi-simple. For each $\xi \in \mathfrak{g}$ a momentum map $\Phi_G : M \to \mathfrak{g}^*$ defines a smooth function $\phi_G^\xi : M \to \mathbb{R}$ depending linearly on ξ

$$\phi_G^\xi(m) := \langle \Phi_G(m), \xi \rangle.$$

Assume that $(m, \xi) \in M \times \mathfrak{g}$ is a relative equilibrium of some Hamiltonian $h \in C^\infty(M)^G$ with momentum $\mu = \Phi_G(m)$. By definition of a relative equilibrium, ξ and μ commute i.e. $\mathrm{ad}_\xi^* \mu = 0$. We would like to describe the space of degeneracy of the Hessian $D^2\phi_G^\xi(m)$ along the orbit $\mathfrak{g} \cdot m$.

Proposition 6.2.1. *If $y_M(m), x_M(m) \in \mathfrak{g} \cdot m$ then*

$$D^2\phi_G^\xi(m)\,(y_M(m), x_M(m)) = \langle \Phi_G(m), [x, [y, \xi]] \rangle. \tag{6.7}$$

Proof. Let $y_M(m) \in \mathfrak{g} \cdot m$, then

$$\begin{aligned} d\phi_G^\xi(m) \cdot y_M(m) &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \phi_G^\xi(e^{ty} \cdot m) \\ &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \Phi_G(e^{ty} \cdot m), \xi \rangle \\ &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \Phi_G(m), Ad_{e^{-ty}}\xi \rangle \\ &= \langle \Phi_G(m), [\xi, y] \rangle. \end{aligned}$$

For another element $x_M(m) \in \mathfrak{g} \cdot m$, we get

$$\begin{aligned}
D^2\phi^\xi_G(m)\,(y_M(m),x_M(m)) &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} d\phi^\xi_G(e^{tx}\cdot m)\cdot y_M(e^{tx}\cdot m)\\
&= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \Phi_G(e^{tx}\cdot m),[\xi,y]\rangle\\
&= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \langle \Phi_G(m),Ad_{e^{-tx}}[\xi,y]\rangle\\
&= \langle \Phi_G(m),[x,[y,\xi]]\rangle.
\end{aligned}$$

■

Set $\mu = \Phi_G(m)$ and note that the Jacobi identity of the Lie bracket and the fact that $\mathrm{ad}^*_\xi\mu = 0$ imply that $\langle\mu,[x,[y,\xi]]\rangle = \langle\mu,[y,[x,\xi]]\rangle$, reflecting the symmetric property of the Hessian. The non-degeneracy space of $D^2\phi^\xi_G(m)$ along $\mathfrak{g}\cdot m$ consists of the elements $y\in\mathfrak{g}$ such that

$$\langle\mu,[y,[x,\xi]]\rangle = 0 \quad \text{for all} \quad x\in\mathfrak{g}. \tag{6.8}$$

Since μ and ξ commute, we can fix a maximal commutative Lie algebra $\mathfrak{t}\subset\mathfrak{g}$ such that $\xi\in\mathfrak{t}$ and $\mu\in\mathfrak{t}^*$. We complexify both of them as in Definition 6.1.1

$$\mathfrak{g}_\mathbb{C} = \mathbb{C}\otimes_\mathbb{R}\mathfrak{g} \quad \text{and} \quad \mathfrak{t}_\mathbb{C} = \mathbb{C}\otimes_\mathbb{R}\mathfrak{t}$$

and the Lie bracket extends into the bracket $[\cdot,\cdot]_\mathbb{C}$ as in (6.1). After this step the velocity and momentum read $\xi = 1\otimes_\mathbb{R}\xi$ and $\mu = 1\otimes_\mathbb{R}\mu$. There respective stabilizer subalgebras are given respectively by

$$\mathfrak{g}_\xi := \{x\in\mathfrak{g}_\mathbb{C} \mid [x,\xi]_\mathbb{C} = 0\} \quad \text{and} \quad \mathfrak{g}_\mu := \{x\in\mathfrak{g}_\mathbb{C} \mid \mathrm{ad}^*_x\mu = 0\}\,.$$

Consider the Cartan Lie subalgebra $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$. Since $\xi\in\mathfrak{h}$ and $\mu\in\mathfrak{h}^*$, it is clear that $\mathfrak{h}$ is a subspace of both $\mathfrak{g}_\xi$ and $\mathfrak{g}_\mu$. We thus write

$$\mathfrak{g}_\xi = \mathfrak{h}\oplus\left(\bigoplus_{\beta\in S_f}\mathfrak{g}_\beta\right) \quad \text{and} \quad \mathfrak{g}_\mu = \mathfrak{h}\oplus\left(\bigoplus_{\alpha\in D_f}\mathfrak{g}_\alpha\right) \tag{6.9}$$

for some finite subsets S_f and D_f of $\mathcal{R}$ with the property:

$$\alpha \in S_f \text{ (resp. } D_f) \quad \Longrightarrow \quad -\alpha \in S_f \text{ (resp. } D_f).$$

Definition 6.2.2. ξ (resp. μ) is *regular* if $S_f = \varnothing$ (resp. $D_f = \varnothing$).

Since $\mathfrak{g}_{\mathbb{C}}$ is semi-simple, the Killing form induces an isomorphism $\kappa^\sharp : \mathfrak{h}^* \to \mathfrak{h}$. Let $t_\mu \in \mathfrak{h}$ be the image of μ by this isomorphism and let $\mathcal{O}_{t_\mu}$ be the adjoint orbit of t_μ. According to (6.9) and (6.4), there is an identification

$$T_{t_\mu}\mathcal{O}_{t_\mu} = \sum_{\alpha \in \mathcal{R}\setminus D_f} \mathfrak{g}_\alpha.$$

By (6.3) the problem stated in (6.8), after complexification of the Lie algebra $\mathfrak{g}$, reduces to find all the $y \in \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\kappa\left([y^*, t_\mu]_{\mathbb{C}}, [x, \xi]_{\mathbb{C}}\right) = 0 \quad \text{for all} \quad x \in \mathfrak{g}_{\mathbb{C}}. \tag{6.10}$$

Let $\{H_1, \ldots, H_k\} \cup \{X_\alpha \mid \alpha \in \mathcal{R}\}$ be a Weyl-Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$ given by Theorem 6.1.11. Let $y \in \mathfrak{g}_{\mathbb{C}}$ be an arbitrary element and let $y^* = -\bar{y}$. With respect to the Weyl-Chevalley basis, this element is expressed as

$$y^* = \sum_{i=1}^{k} a_i H_i + \sum_{\alpha \in \mathcal{R}} \mu_\alpha X_\alpha \quad \text{for some unique} \quad a_i, \mu_\alpha \in \mathbb{C}. \tag{6.11}$$

Hence

$$\begin{aligned}
[y^*, t_\mu]_{\mathbb{C}} &= [\sum_{i=1}^{k} a_i H_i + \sum_{\alpha \in \mathcal{R}} \mu_\alpha X_\alpha, t_\mu]_{\mathbb{C}} \\
&= \sum_{\alpha \in \mathcal{R}} \mu_\alpha [X_\alpha, t_\mu]_{\mathbb{C}} \quad \text{as} \quad t_\mu \in \mathfrak{h} \\
&= -\sum_{\alpha \in \mathcal{R}} \mu_\alpha \alpha(t_\mu) X_\alpha \\
&= -\sum_{\alpha \in \mathcal{R}\setminus D_f} \mu_\alpha \alpha(t_\mu) X_\alpha
\end{aligned}$$

where the last equality follows because

$$[y^*, t_\mu]_{\mathbb{C}} \in T_{t_\mu}\mathcal{O}_{t_\mu}.$$

Similarly (6.9) allows us to write an element $[x, \xi]_{\mathbb{C}} \in T_\xi \mathcal{O}_\xi$ as

$$[x, \xi]_{\mathbb{C}} = \sum_{\beta \in \mathcal{R} \setminus S_f} \lambda_\beta X_\beta \quad \text{with} \quad \lambda_\beta \in \mathbb{C}.$$

Solving (6.10) is equivalent to solve

$$\sum_{\alpha \in \mathcal{R} \setminus D_f} \sum_{\beta \in \mathcal{R} \setminus S_f} \mu_\alpha \lambda_\beta \alpha(t_\mu) \kappa \left(X_\alpha, X_\beta\right) = 0 \quad \text{for any} \quad \lambda_\beta \in \mathbb{C}.$$

Using the fact that the $\mathfrak{g}_\alpha$'s appearing in the decomposition (6.4) are mutually orthogonal with respect to κ (except for those corresponding to the same root with opposite sign), we get

$$\begin{aligned}
0 &= \sum_{\alpha \in \mathcal{R} \setminus D_f} \sum_{\beta \in \mathcal{R} \setminus S_f} \mu_\alpha \lambda_\beta \alpha(t_\mu) \kappa \left(X_\alpha, X_\beta\right) \\
&= \sum_{\alpha, \beta \in \mathcal{R} \setminus (D_f \cup S_f)} \mu_\alpha \lambda_\beta \alpha(t_\mu) \kappa \left(X_\alpha, X_\beta\right) \\
&= \sum_{\alpha \in \mathcal{R} \setminus (D_f \cup S_f)} \mu_\alpha \lambda_\alpha \alpha(t_\mu) \kappa \left(X_\alpha, X_\alpha\right) \\
&\quad + \sum_{\alpha \in \mathcal{R} \setminus (D_f \cup S_f)} \mu_\alpha \lambda_{-\alpha} \alpha(t_\mu) \kappa \left(X_\alpha, X_{-\alpha}\right) \\
&= \sum_{\alpha \in \mathcal{R} \setminus (D_f \cup S_f)} \mu_\alpha \alpha(t_\mu) \left(\lambda_\alpha \kappa \left(X_\alpha, X_\alpha\right) + \lambda_{-\alpha} \kappa \left(X_\alpha, X_{-\alpha}\right)\right).
\end{aligned}$$

This is true for any $\lambda_\alpha \in \mathbb{C}$ if and only if $\mu_\alpha = 0$ for all $\alpha \in \mathcal{R} \setminus (D_f \cup S_f)$ as such roots satisfy $\alpha(t_\mu) \neq 0$ and both $\kappa(X_\alpha, X_\alpha)$ and $\kappa(X_\alpha, X_{-\alpha})$ do not vanish. We conclude that $y \in \mathfrak{g}_{\mathbb{C}}$ fulfils (6.10) for all $x \in \mathfrak{g}_{\mathbb{C}}$ if and only if y^* decomposes as

$$y^* = \sum_{i=1}^{k} a_i H_i + \sum_{\alpha \in D_f \cup S_f} \mu_\alpha X_\alpha. \tag{6.12}$$

Therefore,

$$y^* \in \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in D_f \cup S_f} \mathfrak{g}_\alpha \right) = \mathfrak{g}_\xi + \mathfrak{g}_\mu.$$

In particular this shows that the degeneracy set of the Hessian $D^2\Phi_G(m)$ along $\mathfrak{g} \cdot m$ belongs to $\mathfrak{g}_\xi + \mathfrak{g}_\mu$, by considering only the elements $y \in \mathfrak{g}_\mathbb{C}$ which are real. This proves Lemma 4.2.3 because the other inclusion is clear.

Appendix

This section is a complement to the example of Section 4.1.2 on the dynamics of a 2D Rigid body in a potential flow. Full details of what follows are available in the book of Marsden and Ratiu [44] in Chapters 13 and 14.6.

The special Euclidean group of the plane $SE(2)$ consists of pairs

$$(R_\theta, a) := \begin{pmatrix} R_\theta & a \\ 0 & 1 \end{pmatrix}$$

where $a = (a_1, a_2)^T \in \mathbb{R}^2$ is a vector, and R_θ is a rotation in the plane about the origin of angle θ. The group multiplication is just matrix multiplication. In particular, the inverse of $(R_\theta, a) \in SE(2)$, is given by

$$(R_\theta, a)^{-1} = (R_{-\theta}, -R_{-\theta} a).$$

To obtain the Lie algebra $\mathfrak{se}(2)$, we just differentiate paths in this group which start at identity. The velocities of those paths are the elements of the Lie algebra. Therefore $\mathfrak{se}(2)$ is made of the pairs

$$(\dot\theta, v) := \begin{pmatrix} -\dot\theta J & v \\ 0 & 0 \end{pmatrix}$$

where $\dot\theta \in \mathbb{R}$, $v = (v_1, v_2)^T \in \mathbb{R}^2$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The Lie bracket is then obtained by a direct calculation

$$[(\dot\theta, v), (\dot\phi, w)] = (0, \dot\phi J v - \dot\theta J w).$$

To figure out what the coadjoint representation of $\mathfrak{se}(2)$ on $\mathfrak{se}(2)^*$ is, we need to specify the elements of the dual $\mathfrak{se}(2)^*$. We claim that they are pairs of the form

$$(x, \alpha) := \begin{pmatrix} \frac{x}{2}J & 0 \\ \alpha & 0 \end{pmatrix}$$

where $x \in \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}^2)^*$. Since $\mathfrak{se}(2)$ is a semi-simple Lie algebra, the Killing form given by the trace of the produce of matrices defines a non-degenerate pairing between $\mathfrak{se}(2)$ and $\mathfrak{se}(2)^*$ and thus

$$\mathrm{Tr}\left(\begin{pmatrix} \frac{x}{2}J & 0 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} -\dot{\theta}J & v \\ 0 & 0 \end{pmatrix}\right) = x\dot{\theta} + \alpha_1 v_1 + \alpha_2 v_2.$$

Using our identifications, the pairing becomes

$$\langle (x, \alpha), (\dot{\theta}, v)\rangle = x\dot{\theta} + \alpha_1 v_1 + \alpha_2 v_2$$

which is the standard dot product in $\mathbb{R}^3$. It is now straightforward to compute the coadjoint representation. Given $(x, \alpha) \in \mathfrak{se}(2)^*$ and $(\dot{\theta}, v), (\dot{\phi}, w) \in \mathfrak{se}(2)$, we get

$$\begin{aligned} \langle \mathrm{ad}^*_{(\dot{\theta},v)}(x, \alpha), (\dot{\phi}, w)\rangle &= \langle (x, \alpha), [(\dot{\theta}, v), (\dot{\phi}, w)]\rangle \\ &= \langle (x, \alpha), (0, \dot{\phi}Jv - \dot{\theta}Jw)\rangle \\ &= \alpha_1(\dot{\phi}v_2 - \dot{\theta}w_2) + \alpha_2(\dot{\theta}w_1 - \dot{\phi}v_1) \\ &= (\alpha_1 v_2 - \alpha_2 v_1)\dot{\phi} + \alpha_2\dot{\theta}w_1 - \alpha_1\dot{\theta}w_2 \\ &= \langle (\alpha J v, -\dot{\theta}\alpha J), (\dot{\phi}, w)\rangle. \end{aligned}$$

Therefore $\mathrm{ad}^*_{(\dot{\theta},v)}(x, \alpha) = (\alpha J v, -\dot{\theta}\alpha J) \in \mathfrak{se}(2)^*$.

We now show that the Lie-Poisson equations are nothing else than coadjoint motion on $\mathfrak{se}(2)^*$. This is an application of the Lie-Poisson reduction Theorem obtained by Marsden, Ratiu and Weinstein [45] (cf. also [44] Chapter 13). Consider the cotangent bundle $T^*SE(2)$, equipped with the canonical symplectic structure. The group $SE(2)$ acts on $T^*SE(2)$ by cotangent lift of left multiplication and this action is canonical, free and proper. The left trivialization induces an $SE(2)$-equivariant symplectomorphism

$$T^*SE(2) \longrightarrow SE(2) \times \mathfrak{se}(2)^*$$

where the action of $SE(2)$ on the right hand side is given by left multiplication on the first factor (cf. [45] Section 2). In particular the orbit space

$$T^*SE(2)/SE(2)$$

is canonically isomorphic to $\mathfrak{se}(2)^*$, in the sense that there exists a Poisson diffeomorphism between them. Recall that $T^*SE(2)$ admits a canonical Poisson structure coming from the symplectic structure and that the action of $SE(2)$ is canonical, free and proper. By the Lie-Poisson Reduction Theorem (cf. [45] Theorem 2.1), the reduced space $\mathfrak{se}(2)^*$ admits a unique Poisson bracket[1] such that the quotient map $T^*SE(2) \to \mathfrak{se}(2)^*$ is a Poisson map. Given $\nu \in \mathfrak{se}(2)^*$ and two smooth real-valued functions f, g defined on $\mathfrak{se}(2)^*$, this Poisson bracket takes the form

$$\{f, g\}(\nu) = -\left\langle \nu, \left[\frac{\delta f}{\delta \nu}, \frac{\delta g}{\delta \nu}\right]\right\rangle \tag{A.1}$$

where $\frac{\delta f}{\delta \nu} \in \mathfrak{se}(2)$ is the functional derivative $Df(\nu)$ regarded as an element of $\mathfrak{se}(2)$ rather than $\mathfrak{se}(2)^{**}$. Explicitly

$$Df(\nu) \cdot \delta\nu = \left\langle \delta\nu, \frac{\delta f}{\delta \nu}\right\rangle \quad \text{where} \quad \delta\nu \in \mathfrak{se}(2)^*.$$

If we assume that ν in (A.1) depends on the time t, differentiating $f(\nu)$ with respect to t yields

$$Df(\nu) \cdot \dot{\nu} = \left\langle \dot{\nu}, \frac{\delta f}{\delta \nu}\right\rangle.$$

[1]If we consider the cotangent lift of right multiplication rather than left multiplication, the displayed Poisson bracket comes with a positive sign, rather than a negative sign.

On the other hand, (A.1) also reads

$$\{f,g\}(\nu) = \left\langle \mathrm{ad}^*_{\frac{\delta g}{\delta \nu}} \nu, \frac{\delta f}{\delta \nu} \right\rangle.$$

Therefore the general Lie-Poisson equations, determined by $\dot{f} = \{f,g\}$ are given by

$$\dot{\nu} = \mathrm{ad}^*_{\frac{\delta g}{\delta \nu}} \nu.$$

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