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MILNOR ATTRACTORS AND TOPOLOGICAL ATTRACTORS OF A PIECEWISE LINEAR MAP JANUARY 2000 (REVISED OCTOBER 2000)

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ABSTRACT. A very simple two-dimensional map is discussed. It is shown that for appropriate values of the parameters there is a two dimensional subset of the plane on which the dynamics is transitive and periodic orbits are dense, but that this topological attractor contains a one dimensional set which attracts almost all points (i.e. it is a Milnor attractor). This arises naturally as a precursor to a blowout bifurcation to on-off intermittency in this system, and confirms a conjecture due to Pikovsky and Grassberger.

Key words: Milnor attractor, transitivity, blowout bifurcation.

1. Introduction

Since the pioneering work of Fujisaka and Yamada [11,12,13,29] the study of the stability of synchronized states, and especially the stability of these states to non-synchronized (transversal) perturbations, has been a subject of interest. Many of the models of this phenomenon involve coupling identical nonlinear systems. These models are particularly interesting when the nonlinear systems are chaotic in the absence of coupling due to the possibility of Milnor attractors and riddled basins appearing when the systems are coupled [1,3,21]. These exotic objects arise naturally as part of a blowout bifurcation: as a parameter is varied we may imagine that some orbits of the synchronized attractor lose stability in transverse directions although typical synchronized orbits remain transversally stable. At this stage the synchronized state is no longer Liapunov stable, although it is possible that almost all orbits are eventually attracted to the synchronized state (the precise details depend on global features of the system [2,3,23]) which is called a Milnor attractor. As a parameter is varied further the typical orbits in the synchronized state may lose transverse stability in a blowout bifurcation, leading to dynamics with intermittent characteristics – orbits spend a long time close to the synchronized state interspersed with larger fluctuations away from the synchronized state. This implies that there is a discontinuous change in the geometry of the attracting set as the parameter passes through the blowout bifurcation point. The aim of this paper is to show that in some examples this is part of a continuous change in the topological structure of orbits. In the example considered below we show that for parameter values for which the synchronized state is a Milnor attractor, there is a larger invariant set, containing a dense orbit and the synchronized state, which is the attractor of open sets in the system. This topological attractor is the object which becomes the attractor of almost all orbits after the blowout bifurcation. In this way, this example provides one way by which the apparently discontinuous jump in topology at the blowout bifurcation can be seen as a continuous phenomenon. It is hoped that this will be a more general feature of supercritical blowout bifurcations. The analysis also provides a concrete example of one of the possible structures for Milnor attractors described in [4].

Pikovsky and Grassberger [25] introduced a family of two dimensional non-invertible maps as a simple paradigm of synchronization:

$$\begin{aligned} x_{n+1} &= (1-\omega)f_a(x_n) + \omega f_a(y_n) \\ y_{n+1} &= \omega f_a(x_n) + (1-\omega) f_a(y_n) \end{aligned} \tag{1.1}$$

where $\omega \in (0, \frac{1}{2})$ and $f_a : \mathbf{R} \to \mathbf{R}$ is the skew tent map

$$f_a(z) = \begin{array}{ll} az & \text{if } z \leq a^{-1} \\ \frac{a}{a-1}(1-z) & \text{if } z \geq a^{-1} \end{array} \quad a > 1 \tag{1.2}$$

It should be obvious that this system has a synchronized state in which $x_n = y_n$ for all $n \in \mathbf{N}$. Geometrically, this corresponds to motion on the diagonal, and this motion is governed by the one-dimensional skew tent map (1.2) since if $(x_n, y_n) = (z_n, z_n)$ then $(x_{n+1}, y_{n+1}) = (f_a(z_n), f_a(z_n))$. It should be equally clear that the unit square $\mathcal{S} = [0,1]^2$ is invariant, and henceforth all remarks will be confined to the map restricted to $\mathcal{S}$. For this example it is possible to determine the transverse stability of any periodic orbit embedded in the chaotic synchronized state (see section two and [15,25]) and also find the natural invariant measure on the synchronized state (it is simply Lebesgue measure [5]). Thus, if a is regarded as fixed, as ω decreases from $\frac{1}{2}$ it is possible to determine the parameter at which the first synchronized orbit loses transverse stability (the point at which the synchronized state as a whole loses asymptotic stability), the point at which 'typical' trajectories lose transverse stability (the blowout bifurcation point in the language of [3,8]) and finally the point beyond which all synchronized states are transversely unstable. For the range of values of a considered below, there is on-off intermittency immediately after the blowout bifurcation [1,2,3,19,24,25,26], and between the loss of asymptotic stability and the blowout bifurcation the synchronized state is a Milnor attractor (i.e. it attracts almost all initial conditions locally in a measure theoretic sense [1,3,4,8,17,21,25]). Pikovsky and Grassberger [25] conjecture that for this range of parameter values this Milnor attractor is embedded in a closed quadrilateral inside which periodic points are dense, and this closed quadrilateral (in some sense the topological attractor before the blowout bifurcation) is effectively the object which becomes both the measure theoretic and the topological attractor after the blowout bifurcation. In this picture, then, there is no discontinuous jump in the topological attractor at the blowout bifurcation point – it is only the geometry of the measure theoretic description of the dynamics which changes radically. By generalizing the idea of the *locally eventually onto* property (which was used by Williams [28] to prove results about one-dimensional Lorenz maps [16]) to piecewise affine maps of the plane we prove this conjecture, showing that between the loss of asymptotic stability of the synchronized state and the point of blowout bifurcation there is a closed quadrilateral on which the dynamics is transitive (there is a dense orbit) and periodic points are dense. The proof works only for a limited range of values of a, and it will also become apparent that the cases $1 < a < 2$ and $a > 2$ lead to different results, particularly on the geometry of absorbing regions.

The remainder of this paper is organized as follows. In section two we review the results known for (1.1,2) [15,17,25] and state the main theorem, which was announced in [15]. In section three we describe an absorbing region in a subset of the case $1 < a < 2$, and in section four we give the formal definition of the affine locally eventually onto property and prove that the dynamics restricted to the absorbing region is transitive if $\frac{1}{2}(1+\sqrt{5}) < a < 2$. In section five we complete the proof of the theorem. In section six we turn attention to the absorbing region in the case $a > 2$. A brief appendix outlines the consequences of the affine locally eventually onto property used in the proof of the theorem.

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2. Preliminaries and statement of results

The system (1.1,2) was introduced because it is possible to compute stability exponents (Liapunov exponents) explicitly and the natural equilibrium (Sinai-Bowen-Ruelle) measure is known. It is therefore a very natural example to use in the exploration of synchronization and blowout bifurcations. A synchronized state may be described via the behaviour of either coordinate, since both coordinates evolve by (1.2): $z_{n+1} = f_a(z_n)$ (note that this equation is independent of the second parameter, ω, so ω is a *normal* parameter in the jargon of [3,8]). The dynamics of points under this one dimensional map can be described topologically using kneading theory [9,20,27]. Let $c(z) = 1$ if $z > a^{-1}$ and $c(z) = 0$ if $z < a^{-1}$. Then the kneading sequence of a point which is not a preimage of a^{-1} is the sequence of 0s and 1s, $k(z) = c(z)c(f_a(z))c(f_a^2(z))\dots$. If z is a preimage of a^{-1} then two kneading sequences (the *upper* and *lower* kneading sequences) are associated to z, one by approaching z from above through points which are not preimages of a^{-1} and one by approaching z from below through points which are not preimages of a^{-1}. Such sequences will be of the form $A10^\infty$ and $A0^\infty$ (not necessarily respectively) where A is a finite sequence of 0s and 1s and 0^∞ represents an infinite sequence of 0s. All possible sequences of 1s and 0s arise as the kneading sequence (possibly upper or lower) of some point in the unit interval, and every periodic sequence corresponds to a point on a periodic orbit of the same period as the sequence (a statement which is not true for general unimodal maps). It is not surprising, given the piecewise linear nature of f_a, that the stability of orbits is related to the proportion of $0s$ and 1s in its kneading sequence. The form of the coupling between the maps means that in the full system (1.1,2), the stability of a synchronized state can be read off from the stability of the corresponding one-dimensional map [10,11,12,13,14,18,29]. This stability in turn determines the blowout bifurcation point.

Let $z \in [0,1]$. Then the Liapunov exponent of z under f_a, $\lambda(z)$, is the limit

$$\lambda(z) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|f_a'(f_a^k(z))| \tag{2.1}$$

provided the limit exists. Similarly, if $\mathcal{A}$ is a uniquely ergodic invariant set with invariant measure m then the Liapunov exponent of $\mathcal{A}$ is

$$\lambda(\mathcal{A}) = [m(\mathcal{A})]^{-1}\int_{\mathcal{A}} \log|f_a'(x)|\, dm. \tag{2.2}$$

With these definitions it is easy to prove the following lemma.

Lemma 2.1. *The unit interval, $\mathcal{I} = [0,1]$ is uniquely ergodic and the invariant measure is Lebesgue measure. Furthermore,*

$$\lambda(\mathcal{I}) = \log a - (1-a^{-1})\log(a-1).$$

If $\lambda(z) = \ell$ for some $z \in \mathcal{I}$ then $\ell = \log a - \rho\log(a-1)$ for some $\rho \in [0,1]$. Conversely, for all $\rho \in [0,1]$ there exists $z \in [0,1]$ such that $\lambda(z)$ exists and equals $\log a - \rho\log(a-1)$.

Proof: The first statements follow immediately from [5] and direct integration (see also [15,17]). Now for $z \in \mathcal{I}$ let

$$\rho_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} c(f_a^k(z)) \tag{2.3}$$

with $c(z)$ as defined at the beginning of this section, and

$$\lambda_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} |f_a'(f_a^k(z))| \tag{2.4}$$

Then by direct calculation,

$$\lambda_n(z) = \log a - \rho_n(z)\log(a-1) \tag{2.5}$$

and so λ_n tends to a limit if and only if ρ_n tends to a limit. If ρ_n does tend to a limit then that limit is in $[0,1]$ and so in such a case $\lambda_n(z) \to \log a - \rho \log(a-1)$ for some $\rho \in [0,1]$ as required. To complete the proof we need only show that for each $\rho \in [0,1]$ there exists $z \in \mathcal{I}$ such that $\lim_{n\to\infty} \rho_n(z) = \rho$. But this is true since given any $\rho \in [0,1]$ there is certainly an infinite sequence of 0s and 1s such that the proportion of 1s in that sequence is asymptotically equal to ρ. But for any such sequence, $\mathbf{s}$ say, there is a point z such that $k(z) = \mathbf{s}$ and so $\lambda(z) = \log a - \rho \log(a-1)$.

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Lemma 2.1 gives the first indication that the cases $1 < a < 2$ and $a > 2$ will be different: If $1 < a < 2$ then the most unstable point (largest Liapunov exponent) is the non-trivial fixed point which corresponds to $\rho = 1$, i.e. $z = a/(2a-1)$, whilst the least unstable is the fixed point with $\rho = 0$, i.e. $z = 0$. These are reversed in the case $a > 2$.

If we now consider the stability of synchronized orbits of the full two dimensional system (1.1,2) it is not hard to see that the Jacobian matrix has an eigenvector of $(1,1)^T$, (the superscript T indicates the transpose of the vector) corresponding to perturbations in the synchronized direction, and an eigenvector of $(1,-1)^T$, corresponding to transverse perturbations. A synchronized orbit therefore has two Liapunov exponents: one, in the synchronized direction, is simply the Liapunov exponent under the one dimensional map f_a given by (2.1). The second, $\lambda_\perp(z)$, is in the transverse direction:

$$\lambda_\perp(z) = \log|1-2\omega| + \lambda(z) \tag{2.6}$$

which clearly exists if $\lambda(z)$ exists. Equation (2.6), together with Lemma 2.1 allows us to obtain the transversal bifurcation structure of synchronized states. Let

$$\omega_b = \frac{1}{2}\left(1 - \mathrm{e}^{-\lambda(\mathcal{I})}\right) \tag{2.7}$$

with $\lambda(\mathcal{I})$ given by Lemma 2.1 (ω_b is the blowout bifurcation value).

Lemma 2.2. *Let $(\alpha_1, \alpha_2) = (\frac{a-1}{2a}, \frac{1}{2a})$ if $a \in (1,2)$ and $(\alpha_1, \alpha_2) = (\frac{1}{2a}, \frac{a-1}{2a})$ if $a > 2$. If $a > 1$ and $a \neq 2$ then*
(i) if $\omega > \alpha_2$ then the synchronized state is asymptotically stable;
(ii) if $\omega \in (\omega_b, \alpha_2)$ then at least some of the synchronized orbits are transversely unstable, but $\mathcal{I}$ is transversely stable (i.e. $\lambda(\mathcal{I}) < 0$);
(iii) if $\omega \in (\alpha_1, \omega_b)$ then at least some of the synchronized orbits are transversely stable, but $\mathcal{I}$ is transversely unstable;
(iv) if $\omega \in (0, \alpha_1)$ then all synchronized orbits are transversely unstable.

Proof: See [15,17].

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It is the parameter regions characterized by Lemma 2.2(ii) with $a < 2$ which will be our main focus of attention below. Here, typical orbits in the synchronized subspace are transversely stable, but there are transversely unstable orbits embedded in $\mathcal{I}$. This means that the immediate basin of attraction of the synchronized state is locally riddled [1,2,3,4,8,19,23], i.e. the diagonal is a Milnor attractor [1,4,17].

The proof of the main result of this paper relies on a generalization of the idea of the locally eventually onto property exploited by Williams [28] in the context of one dimensional maps, see section 4 and the Appendix. The result is as follows.

Theorem A. *Fix* $a \in (\frac{1}{2}(1+\sqrt{5}), 2)$ *and let* $O = (0,0)$, $I = (1,1)$, $R = (2\omega, \frac{1-2\omega+2\omega^2}{1-\omega})$ *and* $R' = (\frac{1-2\omega+2\omega^2}{1-\omega}, 2\omega)$. *If* $\mathcal{D}$ *is the filled in quadrilateral* $ORIR'$ *and if* $\omega \in (\omega_b, \frac{1}{2a})$ *then*
(i) F *is transitive on* $\mathcal{D}$ *(i.e. there is a dense orbit in* $\mathcal{D}$*); and*
(ii) periodic points are dense in $\mathcal{D}$.

This confirms a conjecture of Pikovsky and Grassberger [25]. The next two sections contain the proof of this result. It will become apparent in the next two sections that $\mathcal{D}$ may be called the global attractor: $\mathcal{D}$ is a global attractor for $\mathcal{S}$ if $\mathcal{D}$ is invariant and for all open $U \subseteq \mathcal{S}$ there exists n_0 (which depends on U) such that $F^n(U) \subseteq \mathcal{D}$ for all $n > n_0$. Indeed, in our case

$$\mathcal{D} = \cup_{n>n_0} F^n(U)$$

for any such U (this result follows from the affinely locally eventually onto property proved in section 4). It is possible that this way of seeing the result provides a clue to a more general feature of systems undergoing supercritical [2,3,8] (or hysteretic [23]) blowout bifurcations. We conjecture that if a system undergoes a generic supercritical blowout bifurcation then for parameter values just before bifurcation (when the synchronized state is a Milnor attractor) then the system has a non-trivial global attractor which contains a continuation of the support of the invariant measure for dynamics in the regime of on-off intermittency after the blowout bifurcation.

3. Absorbing region: $a \in (1,2)$

The aim of this section is to establish the existence of an invariant region $\mathcal{D}$ which attracts all orbits in the unit square $\mathcal{S}$. The critical lines of the skew tent map (1.2) divide $\mathcal{S}$ into four regions on each of which the map (1.1) is affine, and $\mathcal{D}$ intersects each of these affine regions. The strategy of this section will be to define $\mathcal{D}$ in terms of iterates of the critical lines, then to show that the images of the four parts of $\mathcal{D}$ in the affine regions are in $\mathcal{D}$, thus establishing the invariance of $\mathcal{D}$. Finally it is shown that orbits in $\mathcal{S}$ are attracted to $\mathcal{D}$.

Consider (1.1,2) with $a \in (1,2)$. The critical lines $x = a^{-1}$ and $y = a^{-1}$ divide the square $\mathcal{S}$ into four regions on which the map acts as an affine map, and so given two

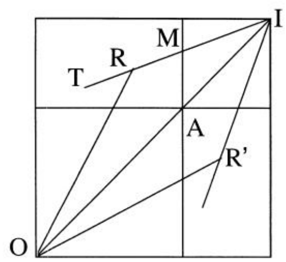

FIGURE 1. Sketch of the construction of the absorbing region $\mathcal{D} = ORIR'$. The points labelled are defined in the main text but for ease of reference recall: $A = (a^{-1}, a^{-1})$, $O = (0,0)$, $I = (1,1)$, $T = F(0, a^{-1})$, $F(O) = F(I) = 0$, $F(A) = I$ and $F(M) = R'$. Primed variables are obtained from unprimed variables by reflection in the diagonal.

points in the same region, the line between them is mapped to the line connecting their images. For convenience we shall give these four regions names:
Region 1: $\{(x,y) \in \mathcal{S} \mid x \leq a^{-1}, y \leq a^{-1}\}$
Region 2: $\{(x,y) \in \mathcal{S} \mid x \leq a^{-1}, y \geq a^{-1}\}$
Region 2$'$: $\{(x,y) \in \mathcal{S} \mid x \geq a^{-1}, y \leq a^{-1}\}$
Region 4: $\{(x,y) \in \mathcal{S} \mid x \geq a^{-1}, y \geq a^{-1}\}$.

In general, primed points and regions will denote the reflection of the unprimed point or region in the diagonal, reflecting the invariance of the equations under the symmetry

$$(x,y) \to (y,x) \tag{3.1}$$

and wherever possible we shall use primes to denote points which are below the diagonal, so sometimes a point (for example, R below) wil be defined via its prime point (R' below). By Lemma 2.2, the synchronized state is asymptotically stable if $2a\omega > 1$, so assume that

$$2a\omega < 1. \tag{3.2}$$

The absorbing region is constructed following [6,7,22] by considering the images of the critical lines. Figure 1 shows the geometry of these lines and their images for the parameter values considered below.

Let $I = (1,1)$ and $O = (0,0)$ and let $F : \mathbf{R}^2 \to \mathbf{R}^2$ denote the map (1.1,2), so by direct calculation, $F(I) = F(O) = O$, and if $A = (a^{-1}, a^{-1})$ then $F(A) = I$. The image of the line $y = a^{-1}$ is the union of the image of the line segment from A to

$(0, a^{-1})$ and the line segment from A to $(1, a^{-1})$, and by direct calculation again

$$F(1, a^{-1}) = F(0, a^{-1}) = (\omega, 1-\omega)$$

so, since $F(A) = I$ the image of $y = a^{-1}$ is IT, where

$$T = (\omega, 1-\omega). \tag{3.3}$$

By definition, $\omega < \frac{1}{2}$ (see immediately below (1.1)), and $a \in (1,2)$, so $\omega < a^{-1}$. Thus T lies either in region 1 or region 2. The line IT intersects the critical line $x = a^{-1}$ at M where

$$M = \left(a^{-1}, 1 - \frac{\omega}{1-\omega}\frac{a-1}{a}\right) \tag{3.4}$$

and so M lies above the critical line $y = a^{-1}$ provided

$$1 - \frac{\omega}{1-\omega}\frac{a-1}{a} > a^{-1}.$$

Although this inequality looks complicated, a little manipulation shows that it is equivalent to $a > 1$, and so M always lies above A. We now consider the image of IM. Since IM lies in region 4, the image of IM is the straight line between $F(I) = O$ and $F(M) = R'$. Since M is on the critical curve $x = a^{-1}$, R' is on IT' (recall that the primes denote reflection in the diagonal, and R' lies below the diagonal, so $R = F(M')$ is obtained from R' using the symmetry (3.1)). Now, by direct calculation,

$$R' = \left(\frac{1-2\omega+2\omega^2}{1-\omega}, 2\omega\right) \tag{3.5}$$

and we now wish to show that R' is in region $2'$ as shown in Figure 1. Clearly (3.2) implies that $2\omega < a^{-1}$, so R' lies in $y < a^{-1}$. It is therefore necessary to show that

$$\frac{1-2\omega+2\omega^2}{1-\omega} > a^{-1}$$

or equivalently

$$2\omega^2 - (2-a^{-1})\omega + 1 - a^{-1} > 0. \tag{3.6}$$

Fix $a \in (1,2)$.The left hand side of (3.6) has a minimum at $\omega = \frac{1}{4}(2-a^{-1})$ and so the minimum value of the left hand side is $\frac{1}{8}(8-(2+a^{-1})^2)$. Thus (3.6) holds provided

$$\frac{1}{2(\sqrt{2}-1)} < a < 2. \tag{3.7}$$

It is worth noting that this result is not improved if we include the constraint that $\omega \in (0, \frac{1}{2a})$. If (3.7) does not hold then there will be some values of $\omega < \frac{1}{2a}$ for which R lies in region 1; these would need to be treated separately if the analysis were to be extended to values of a close to 1.

Proposition 3.1. *If $1/[2(\sqrt{2}-1)] < a < 2$ and $0 < \omega < \frac{1}{2a}$ then the region $\mathcal{D} = ORIR'$ is invariant and if, in addition, $a > \frac{3}{2}$ then $F^2(\mathcal{S}) = \mathcal{D}$.*

Proof: We have already established that for these values of the parameter, R lies in region 2 and so the geometry is as shown in Figure 1. By definition,

$$F(AMIM') = ORIR' = \mathcal{D} \tag{3.8}$$

which establishes that a point inside the part of $\mathcal{D}$ in region 4 remains in $\mathcal{D}$ under one iteration of the map, and will also be useful in the transitivity argument later. Now consider the part of $\mathcal{D}$ which lies in region 1: this is $OSAS'$ where

$$S = \left(\frac{2\omega(1-\omega)}{a(1-2\omega+2\omega^2)}, a^{-1}\right) \tag{3.9}$$

Since S lies on the critical line $y = a^{-1}$, its image will lie on IT. Moreover, since $F(A) = I$ and $F(O) = O$ we see that $F(OSAS') \subseteq \mathcal{D}$ provided $F(S)$ lies to the right of R. By direct calculation again, this condition is

$$\frac{2\omega(1-\omega)^2}{1-2\omega+2\omega^2} + \omega > 2\omega \tag{3.10}$$

which, after some manipulation is seen to hold iff $\omega < \frac{1}{2}$. Hence $F(OSAS') \subseteq \mathcal{D}$.

To complete the proof of the invariance of $\mathcal{D}$ it is therefore sufficient to prove that $F(SAMR) \subseteq \mathcal{D}$. We already know that $F(A) = I$, $F(M) = R'$ and $F(S)$ is on the line segment RI, hence all that remains is to show that $F(R) \in \mathcal{D}$. We shall do this in a number of steps. First note that if R is in region 2 then

$$F(R) = \left(2a\omega(1-\omega) + \frac{a\omega^2(1-2\omega)}{(1-\omega)(a-1)}, 2a\omega^2 + \frac{a\omega(1-\omega)}{a-1}\right) \tag{3.11}$$

The first two steps establish that $F(R)$ lies inside the cone formed by the lines through OR and OR' by checking that the slope of the line $OF(R)$ is less than the slope of OR and greater than the slope of OR'. After a little tidying up the slope of $OF(R)$ is

$$\frac{2(a-1)\omega(1-\omega) + (1-\omega)(1-2\omega)}{2(a-1)(1-\omega)^2 + \omega(1-2\omega)} \tag{3.12}$$

Step 1: The slope of OR' is less than the slope of $OF(R)$, i.e.

$$\frac{2\omega(1-\omega)}{1-2\omega+2\omega^2} < (1-\omega)\left(\frac{2(a-1)\omega + (1-2\omega)}{2(a-1)(1-\omega)^2 + \omega(1-2\omega)}\right)$$

Cancelling the factors of $1-\omega$ and rearranging we find that this condition is equivalent to $2a\omega < 1$ and hence always holds.

Step 2: The slope of OR is greater than the slope of $OF(R)$, i.e.

$$\frac{1-2\omega+2\omega^2}{2\omega(1-\omega)} > \frac{2(a-1)\omega(1-\omega) + (1-\omega)(1-2\omega)}{2(a-1)(1-\omega)^2 + \omega(1-2\omega)}.$$

After some further manipulation this inequality can be seen to be equivalent to the condition

$$2a\omega^2 - (4a-3)\omega + 2(a-1) > 0. \tag{3.13}$$

Viewing (3.13) as a quadratic in ω with a fixed we see that it takes its minimum value if $\omega = \frac{1}{4a}(4a-3)$ and at this value the quadratic is

$$-\frac{1}{8a}(4a-3)^2 + 2(a-1) = \frac{1}{8a}(8a-9).$$

Thus, provided $a > \frac{9}{8}$, the quadratic is always positive, and since $\frac{9}{8} < \frac{1}{2(\sqrt{2}-1)}$, $F(R)$ is always below the line obtained by extending OR.

To complete the invariance proof for $\mathcal{D}$ we now repeat the process for the cone obtained by extending the lines RI and $R'I$. For future reference, the slope of RI is the same as TI, which from (3.3) is $\omega/(1-\omega)$. Similarly, the slope of $R'I$ is $(1-\omega)/\omega$.

Step 3: $F(R)$ lies to the left of the line $R'I$.
The line $R'I$ is the line

$$x = 1 - \frac{\omega}{(1-\omega)}(1-y). \tag{3.14}$$

Consider a horizontal line ($y =$ constant) through the point $F(R)$. This will intersect the line (3.14) at x_1 obtained by substituting the $y-$coordinate of $F(R)$ into (3.14). We require the $x-$component of $F(R)$ to be less than x_1, i.e.

$$2a\omega(1-\omega) + \frac{a\omega^2(1-2\omega)}{(a-1)(1-\omega)} < 1 - \frac{\omega}{(1-\omega)}\left(1 - 2a\omega^2 - \frac{a\omega(1-2\omega)}{a-1}\right).$$

After some miraculous cancellations this condition is seen to be equivalent to $2a\omega < 1$ and so it holds for all relevant parameter regions.

Step 4: $F(R)$ lies below the line RI.
The line RI is the line

$$y = 1 - \frac{\omega}{(1-\omega)}(1-x) \tag{3.15}$$

Consider a vertical line through the point $F(R)$ which meets (3.15) at y_2. We wish to show that the $y-$coordinate of $F(R)$ is smaller than y_2, i.e.

$$2a\omega^2 + \frac{a\omega(1-2\omega)}{a-1} < 1 - \frac{\omega}{(1-\omega)}\left(1 - 2a\omega(1-\omega) + \frac{a\omega^2(1-2\omega)}{(a-1)(1-\omega)}\right).$$

After some simplification, this condition is equivalent to

$$2a\omega^2 - (2a-1)\omega + (a-1) > 0$$

and, viewing this as a quadratic in ω with a fixed, the minimum value is $(2a-1)^2/(8a) > 0$, so this condition always holds.

Step 4 completes the proof that $\mathcal{D}$ is invariant for the stated parameter values, it only remains to show that it is attracting if $a > \frac{3}{2}$ in the sense that $F^2(\mathcal{S}) = \mathcal{D}$. By considering the image of the points $(1,0)$, $(0,a^{-1})$ and so on it is easy to see that

$F(\mathcal{S}) = OTIT'$. If $1-\omega > a^{-1}$ then T lies in region 2. But $1-\omega > a^{-1}$ iff $\omega < \frac{a-1}{a}$ and since $\frac{1}{2a} < \frac{a-1}{a}$ if $a > \frac{3}{2}$ this condition holds, so T is in region 2 and the line OT intersects the critical line at a point Q. Since $\mathcal{D}$ is invariant we need only show that the images of OSQ (a triangle in region 1) and $SRTQ$ (a quadrilateral in region 2) are in $\mathcal{D}$ to complete the proof. We know enough about all points except Q and T already, and it is an easy calculation to check that $F(Q)$ lies on the line RI between R and $F(S)$. Hence the image of the region OSQ lies in $\mathcal{D}$ as required.

To establish the position of $F(T)$ we need two small results. First, the slope of the line $OF(T)$ is

$$\frac{(1-\omega)+\omega(a-1)}{(1-\omega)(a-1)+\omega} \tag{3.15}$$

from which it is straightforward to show that the line $OF(T)$ lies between the lines OR and OR' provided $\omega < \frac{a-1}{a}$ and $\omega < \frac{1}{a}$ respectively, both of which hold in this region of parameter space. Second, if we compare the $x-$component of R with that of $F(T)$ we find that $F(T)$ lies to the right of R provided $\omega < \frac{a-1}{a}$ (these are straightforward calculations which will be omitted). Taken together, these facts imply that $F(T)$ is in $\mathcal{D}$ and so the image of $SRTQ$ also lies in $\mathcal{D}$ and the proof of the proposition is complete.

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We have not bothered to consider what happens if $a < \frac{3}{2}$ in detail, but expect that $\mathcal{D}$ remains an attracting region (unless a is close to 1), but it may take more steps before orbits enter the region.

4. The affinely locally eventually onto property for $\mathcal{D}$: $a \in ((1+\sqrt{5})/2, 2)$.

We now want to show that the dynamics on $\mathcal{D}$ is transitive and that periodic points are dense in $\mathcal{D}$. The proof of both of these properties is obtained by proving that the dynamics on $\mathcal{D}$ satisfies a stronger condition that we call *affinely locally eventually onto* or *a.l.e.o.* In the Appendix we show that this property implies that the dynamics in $\mathcal{D}$ is transitive and that periodic points are dense, so this property implies the two results of Theorem A in section 2.

Definition: An open set U in $\mathcal{D}$ has the a.l.e.o property iff there exists $n > 0$ and a set $W \subseteq U$ homeomorphic to a closed disc, such $F^n|_W$ is affine and invertible, and that $F^n(W) = \mathcal{D}$. The map F is a.l.e.o. on $\mathcal{D}$ iff every open set $U \subseteq \mathcal{D}$ has the a.l.e.o property.

The proof that the affine map (1.1,2) is a.l.e.o. on $\mathcal{D}$ if $a \in (\frac{1}{2}(1+\sqrt{5}), 2)$ and $\omega \in (\omega_c, \frac{1}{2a})$ follows the corresponding proof for one dimensional tent maps quite closely (cf. [27]). First, we show that every open set must eventually intersect one of the two critical lines $x = a^{-1}$ or $y = a^{-1}$, and use this to show that it must in fact eventually contain a piece of the diagonal. Then we can use the properties of the skew tent maps to show that any open region eventually contains a neighbourhood of the non-trivial synchronized fixed point, and then finally properties of this fixed point are used to establish the result. Throughout this process, if U is mapped across a critical line we choose to follow only one branch of the iterates so that the map (restricted to an appropriate subset of U) remains affine. Before going through this argument in more detail we need a couple of technical asides.

In each of the four regions of $\mathcal{S}$ separated by the critical lines, F is an affine map, and det DF is non-zero and has modulus greater than one if $\omega \in (0, \frac{1}{2a})$ and $a \in (\frac{1}{2}(1+\sqrt{5}), 2)$ (see below). Hence if x is not a preimage of a critical line, $|\det DF^n(x)| > 1$ for all $n \geq 0$ and so F^n is invertible at such points. In order to control the geometry of iterates of sets, we choose to work with convex polygons for much of this proof. Recall that U is a convex polygon if the boundary of U is a collection of line segments and if for all $x, y \in U$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in U$. Three properties of open convex polygons and their images will be important below. First, if U is an open convex polygon and $F|_U$ is affine and invertible, then both $F(U)$ and $F^{-1}(U)$ are open convex polygons (this is obvious as straight lines map to straight lines under affine maps). Second, if F and U are as before and L is a straight line, then $L \cap U \neq \emptyset$ implies that L divides U into two non-empty open convex polygons separated by the connected line segment $L \cap U$. The important point here is that the images of convex polygons are divided into two connected components and no more. Finally note that given any open subset $J \subseteq \mathcal{S}$ there exists an open convex polygon $K \subset J$, so if we can show that all open convex polygons eventually intersect the diagonal, then the result is also true for all open sets. Moreover, since we have already established that the images of any open set in $\mathcal{S}$ eventually lie in $\mathcal{D}$ we may restrict attention to open sets in $\mathcal{D}$.

Throughout the remainder of this section U_i and V_i will represent open convex polygons in $\mathcal{D}$, and $|U|$ will denote the area of U.

Suppose that U does not intersect a critical line. Then $F|_U$ is affine, and the area of $F(U)$ is the product of $|U|$ with the determinant of the relevant affine map. So if U lies in region 1 (se the beginning of section 3)

$$|F(U)| = a^2(1-2\omega)|U| \tag{4.1}$$

Similarly, if U lies entirely in region 2 or $2'$ then

$$|F(U)| = \frac{a^2}{a-1}(1-2\omega)|U| \tag{4.2}$$

whilst if U lies entirely in region 4 then

$$|F(U)| = \frac{a^2}{(a-1)^2}(1-2\omega)|U|. \tag{4.3}$$

If $a < 2$ then region 1 is the least expansive, but areas are increased under iteration provided $a^2(1-2\omega) > 1$. Now, $2\omega < a^{-1}$, so $1-2\omega > 1-a^{-1}$, and $a^2(1-2\omega) > a^2-a$. Hence $a^2(1-2\omega) > 1$ if $a^2-a-1 > 0$, i.e. if $a > (1+\sqrt{5})/2$.

Hence if $a > (1+\sqrt{5})/2$ there exists $\epsilon > 0$ such that if $F^k(U)$ lies entirely in one of the four regions separated by the critical lines defined in section three for each $k = \{0, 1, 2, \ldots, n\}$, then

$$|F^{n+1}(U)| \geq (1+\epsilon)^n |U| \tag{4.4}$$

If $U \subseteq \mathcal{D}$ then the left hand side is bounded by the area of $\mathcal{D}$ and so there must exist $N > 0$ such that the interior of $F^N(U)$ intersects one of the critical lines $x = a^{-1}$ or $y = a^{-1}$. In this case, the region is folded over itself at the next iteration, but it is still possible to obtain a lower bound on the expansion. Before giving these results note that we have used two very important inequalities which hold for our chosen range of parameters: $a^2 - a > 1$ and $1-2\omega > (a-1)/a$. Hence we may assume that there exists $\delta > 0$ such that

$$a(a-1) > 1+\delta \tag{4.5}$$

and

$$1-2\omega > \left(\frac{a-1}{a}\right)(1+\delta). \tag{4.6}$$

Suppose that U intersects SA and no other critical lines. Then (as U is an open convex polygon) SA divides U into two (open convex) components, U_1 in region 1 and U_2 in region 2, such that $U = U_1 \cup U_2 \cup (SA \cap U)$, and $F|_{U_i}$, $i = 1, 2$, is affine. Thus for some $\alpha \in (0,1)$ $|U_1| = \alpha|U|$ and $|U_2| = (1-\alpha)|U|$. Hence if $d_1 = a^2(1-2\omega)$ and $d_2 = a^2(1-2\omega)/(a-1)$, (cf. (4.1,2))

$$|F(U)| \geq \max(d_1\alpha|U|, d_2(1-\alpha)|U|).$$

Clearly, the worst possible case is if the two quantities are equal, i.e. if $\alpha = d_2/(d_1+d_2)$ giving a minimal expansion or contraction of $d_1d_2/(d_1+d_2)$ times $|U|$. Evaluating this expression we find that there exists $k \in \{1, 2\}$ such that

$$|F(U_k)| \geq a(1-2\omega)|U|. \tag{4.7}$$

The same expression holds if U intersects $S'A$ and no other critical curves of course. Similarly, if U intersects AM but no other critical line, U is divided into two components, U_3 in region 2 and U_4 in region 4 with $F|_{U_i}$ affine, $i = 3, 4$, and a similar argument leads to an expansion or contraction of areas by at least a factor of $d_2d_3/(d_2+d_3)$ where $d_3 = a^2(1-2\omega)/(a-1)^2$, cf. (4.3). Thus in this case, or in the equivalent case involving $M'A$, there exists $U_m \subset U$ such that $F|_{U_m}$ is affine and

$$|U_m| \geq \frac{a}{a-1}(1-2\omega)|U|. \tag{4.8}$$

If $\omega < \frac{1}{2a}$ then $a(1-2\omega)/(a-1) > 1+\delta$ so (4.8) represents an increase in area again. These expansion results are the main ingredients of the proof of Lemma 4.2, for which we will also need to know about the position of $F(R)$.

Lemma 4.1. *Let $F(R) = (F(R)_1, F(R)_2)$. If $a \in ((1+\sqrt{5})/2, 2)$ and $\omega \in (0, \frac{1}{2a})$ then $F(R)_1 > F(R)_2$, i.e. $F(R)$ lies below the diagonal.*

Proof: $F(R)$ is given by (3.11), so all that needs to be shown is that

$$2a\omega(1-\omega) + \frac{a\omega^2(1-2\omega)}{(1-\omega)(a-1)} > 2a\omega^2 + \frac{a\omega(1-\omega)}{a-1} \tag{4.9}$$

Multiplying through by $(1-\omega)(a-1) > 0$ and rearranging terms, this condition is equivalent to $2(a-1)(1-\omega) > 1-2\omega$ or

$$a > \frac{3}{2} - \frac{\omega}{2(1-\omega)} \tag{4.10}$$

which is clearly satisfied.

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Lemma 4.2. *Let U be an open convex polygon in $\mathcal{D}$. Then there exists $k > 0$ and $V \subseteq U$ such that $F^k|_V$ is affine and $F^k(V)$ intersects the diagonal, OI.*

Proof: Suppose not. We have already argued that an iterate of U must intersect a critical line, and since (by the argument below (4.8)) areas are increased even if an iterate falls on AM or AM' alone, either there exists n and $U_1 \subseteq U$ such that $F^n|_{U_1}$ is affine, $|F^n(U_1)| > (1+\delta)^n|U|$ (where δ is defined by (4.5,6)) and either
(a) $F^n(U_1)$ intersects SA alone; or
(b) (equivalently) $F^n(U_1)$ intersects $S'A$ alone; or
(c) $F^n(U_1)$ intersects at least two critical lines.

The easier case is (c), so we consider this first. Since (by hypothesis) $F^n(U_1)$ does not intersect the diagonal, it must intersect both SA and AM or (equivalently) $S'A$ and AM'. Take the unprimed case. Let U_2 be the component of $F^n(U_1)$ in region 2 (there is only one such connected component since $F^n(U_1)$ is a convex polygon), and note that there exists $U_3 \in U_1$ such that $F^n(U_3) = U_2$ and $F^{n+1}|_{U_3}$ is affine. Now, U_2 contains a segment of SA and a segment of AM in its boundary, and hence $F(U_2) = F^{n+1}(U_3)$ contains a segment of RI (i.e. the image of AS) and a segment of $R'I$ (i.e. the image of AM). But this implies that $F^{n+1}(U_3)$ contains a piece of the diagonal, contradicting the hypothesis.

Now take case (a), or equivalently (b). If $F^n(U_1)$ intersects SA alone the AS divides $F^n(U_1)$ into two components on which F is affine: U_4 in region 1 and U_5 in region 2, both of which have $F^n(U_1) \cap SA$ on their boundaries. By the argument at the beginning of this section we know that for at least one of these polygons, U_m say ($m \in \{4, 5\}$),

$$|F(U_m)| > a(1 - 2\omega)|F^n(U_1)|. \tag{4.11}$$

Now either $F(U_m)$ lies in region 2, or in region 4, or it intersects a critical line. In either of the first two cases the expansion is at least that of region 2, so

$$|F^2(U_m)| > \frac{a^3}{a-1}(1 - 2\omega)^2|F^n(U_1)|.$$

Since $1 - 2\omega > (a - 1)/a$, the growth factor here is larger than $a(a - 1)$ which is greater than one (cf. (4.5)). Moreover, by the choice of intervals, there exists $U_6 \subseteq U_1$ such that $F^{n+2}|_{U_6}$ is affine and $F^n(U_6) = U_m$, i.e.

$$|F^{n+2}(U_6)| > (1 + \delta)^{n+1}|U|. \tag{4.12}$$

This represents a continued increase in area, and so cannot happen infinitely often.

The second possibility for case (a) is that $F(U_m)$ intersects a critical line (but not two critical lines by the argument already rehearsed for case (c)). If $F(U_m)$ intersects AS we are done, since the connected component of $F(U_m)$ in region 2, U_7, contains a piece of RI on its boundary (the image of AS – if this did not lie in region 2 then we would have an intersection with AM leading to a contradiction) and a piece of AS. Hence the boundary of $F(U_7)$ contains a segment below the diagonal (the image of IR lies below the diagonal by Lemma 4.1) and a segment above the diagonal (the image of AS in RI) and hence $F(U_7)$ contains a piece of the diagonal: a contradiction.

This leaves only case (a) with U_m defined by (4.11) and $F(U_m)$ intersects AM and has piece of RI on its boundary. This is the beginning of a sequence of iterations which must be dealt with inductively. AM divides $F(U_m)$ into two pieces, one in region 2 and the other in region 4. Let U_8 denote the component which maximizes the area of its image. By (4.8),

$$|F(U_8)| > (1 + \delta)|F(U_m)| \tag{4.13}$$

and $F(U_8)$ contains a piece of IR' on its boundary. If $F(U_8)$ does not intersect a critical line then it lies either in region $2'$ or in region 4 and we have a continued increase in area by the same argument as that which gives (4.12), whilst if it intersects AS' we obtain a contradiction equivalent to that obtained in the previous paragraph. Hence, to avoid a continued net expansion (compared to $|U|$) we find that $F(U_8)$ intersects AM' and we are in the same situation as before but with U_m replaced by U_8 and primed symbols exchanged with unprimed line symbols. The only way to continue to avoid a net increase in area (compared with $|U|$) is for the largest image to intersect AM then AM', then AM again and to keep oscillating. But even this produces a small increase in area (by a factor of $1 + \delta$) and so it cannot continue indefinitely, and eventually there is an increase in area compared to $|U|$.

This implies that if images of U avoid the diagonal, the area of the images becomes arbitrarily large, a contradiction as the images remain in $\mathcal{D}$.

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Let k and V have the properties defined in Lemma 4.2, so $F^k(V)$ contains a piece of the diagonal. If $a<2$ then the non-trivial synchronized fixed point, G given by

$$G=\left(\frac{a}{2a-1},\frac{a}{2a-1}\right) \tag{4.14}$$

is unstable in both the synchronized direction and the transverse direction, see section 2. Moreover, since preimages of the fixed point $a/(2a-1)$ are dense for the one dimensional skew tent map (1.2), there exists $p\geq 0$ and $V_1\subseteq V$ such that such $F^{k+p}|_{V_1}$ is affine and $G\in F^{k+p}(V_1)$. Our final lemma ensures that every such neighbourhood of G contains a preimage of $\mathcal{D}$.

Lemma 4.3. *Suppose that $0<2a\omega<1$ with $a\in(1,2)$ and let $U\subseteq\mathcal{D}$ be an open neighbourhood of G. Then there exists $q\geq 0$ and a compact convex polygon $W\subseteq U$ such that $F^q|_W$ is affine and $F^q(W)=\mathcal{D}$.*

Proof: Since $F(AMIM')=\mathcal{D}$ (cf. (3.8) above), all we need to do is show that in any neighbourhood of G there is a compact convex polygon W such that $F^{q-1}(W)=AMIM'$. Now, G is in region 4 where F has Jacobian matrix

$$-\frac{a}{a-1}\begin{pmatrix}(1-\omega) & \omega\\ \omega & (1-\omega)\end{pmatrix}$$

with eigenvalues $\lambda_s=-a/(a-1)$ and $\lambda_\perp=-a(1-2\omega)/(a-1)$, which both have modulus greater than one. Hence the inverse of this map, F_4^{-1}, is an affine contraction mapping and the sets $F_4^{-n}(AMIM')$ are convex polygons which accumulate on the fixed point G. Hence given any open neighbourhood U of G there exists n such that $F_4^{-n}(AMIM')\subset U$ and we may set $W=F_4^{-n}(AMIM')$.

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Corollary 4.4. *If $a\in((1+\sqrt{5})/2,2)$ and $0<2a\omega<1$ then F is a.l.e.o. on $\mathcal{D}$.*

Proof: By Lemma 4.2, every open subset J in $\mathcal{D}$ contains an open convex polygon V_2 such that $F^n|_{V_2}$ is affine and $F^n(V_1)$ intersects the diagonal for some $n\geq 0$. By the argument above the statement of Lemma 4.3, this implies that there exists an open convex polygon $V_1\subseteq V_2$ and $p\geq 0$ such that $F^{n+p}|_{V_1}$ is affine and $G\in F^{n+p}(V_1)$. But then by Lemma 4.3 there exists $q\geq 0$ and a compact convex polygon $W\in F^{n+p}(V_1)$ such that $F^q|_W$ is affine and invertible, and $F^q(W)=\mathcal{D}$. By choosing the appropriate preimage of W in V_1 we find a compact convex polygon W' in V_1 (and hence in J) such that $F^{n+p+q}|_{W'}$ is affine and invertible, and $F^{n+p+q}(W')=\mathcal{D}$.

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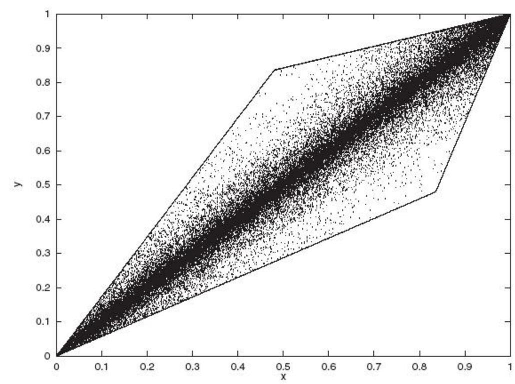

FIGURE 2. Numerical simulation of the attractor with $a = 1.8$ and $\omega = 0.24$ (just after the blowout bifurcation which is at $\omega_b \approx 0.24845$ but before the synchronized state becomes completely transversely stable at $\omega \approx 0.2222$). 50000 iterates are plotted with initial value $(x, y) = (0.2, 0.25)$. Note that they are concentrated close to the diagonal indicating on-off intermittency. The attractor is $\mathcal{D}$.

Statements (i) and (ii) of Theorem A now follow immediately from the a.l.e.o. property as shown in Theorem B of the Appendix. Figure 2 shows the numerically computed attractor of (1.1,2) just after the blowout bifurcation (i.e. with ω a little less than ω_b). The support of the attractor is clearly $\mathcal{D}$, and points seem to be concentrated close to the diagonal (the Milnor attractor when ω is a little larger than ω_b).

5. The Milnor attractor: $a \in ((1+\sqrt{5})/2, 2)$

If $a \in ((1+\sqrt{5})/2, 2)$ and $\omega \in (0, \frac{1}{2a})$ then the dynamics on $\mathcal{D}$ is transitive. If $\omega \in (\omega_b, \frac{1}{2a})$ then at least some of the synchronized orbits are transversely unstable, but 'typical' synchronized states are still transversely stable: the diagonal is a Milnor attractor with locally riddled basin, $\mathcal{B}(V)$, where, for any open neighbourhood V of the diagonal $\mathcal{T}$,

$$\mathcal{B}(V) = \{X \in V \mid F^n(X) \in V \text{ all } n > 0 \text{ and } F^n(X) \to \mathcal{T}\} \tag{5.1}$$

The relative measure of $\mathcal{B}(V)$ tends to one as the measure of V ($m(V)$) tends to zero [1,3,4,8,17]. Thus, $\mathcal{B}(V)$ is large in a measure-theoretic sense. On the other hand, $B(V)$ contains no open sets (every open set expands to cover the whole of $\mathcal{D}$) and is in this sense small from a topological point of view. These remarks, effectively proved in [1,17], establish the claims made in the Introduction.

6. Absorbing regions: $a > 2$

If $a > 2$ then the geometry of the absorbing regions can be quite different to that described above, and has an interesting bifurcation structure. We shall not go into full detail of the absorbing regions, but we can indicate some of the effects observed. Recall from section two that if $a > 2$ then the synchronized state loses asymptotic stability at $\omega = \frac{a-1}{2a}$ and becomes completely transversely unstable at $\omega = \frac{1}{2a}$, with the blowout bifurcation at $\omega_b \in (\frac{1}{2a}, \frac{a-1}{2a})$.

Recall the definition of the point R in section 3, equation (3.5). If $2\omega a < 1$ (as in section 3) R is in region 2, and the geometry is as before, with $\mathcal{D}$ an absorbing region provided $F(R) \in \mathcal{D}$. The interesting new case is $\omega \in (\frac{1}{2a}, \frac{a-1}{2a})$.

In this case, R is in region 4 and $\mathcal{D} = ORIR'$ remains an absorbing area, but is no longer the smallest such region (see Figure 3). Let S denote the intersection of OR with $y = a^{-1}$ and P denote the intersection of OR with $x = a^{-1}$. The image of OS is $OF(S)$, where $F(S)$ is on RI, and $OF(S)$ intersects $x = a^{-1}$ at a point we shall denote D, which is above A. Now, $F^2(S)$ is on OR' and $F(D)$ is on IR' (as it is on the critical line $x = a^{-1}$). These points allow us to build a closed region from the images of critical curves as follows.

Let $\mathcal{L}$ be the region shown in Figure 3a,b, bounded by the curves listed below (always starting with the line which lies above the diagonal)

- $F(S)I$ and $F(S')I$;
- $OF^2(S')$, which is the image of $F(S')I$, and $OF^2(S)$;
- the segment of $F^2(S')F(D')$ which lies to the left of $OF(S)$ (and which is part of the image of the line $OF(S')$), and the image of this segment under reflection in the diagonal;
- the segment of $OF(S)$ which lies above $F^2(S')F(D')$ and the image of this segment under reflection in the diagonal.

The region $\mathcal{L}$ is an absorbing area provided the obvious set of invariance conditions hold (cf. [6,7,22]): we shall not write them down as they are not illuminating, and furthermore, they hold in all of the numerical examples we have examined. As ω is decreased through $\frac{1}{2a}$, $\mathcal{L}$ loses invariance because the transverse unstable manifold of the non-trivial fixed point (which does not exist if $\omega > \frac{1}{2a}$) now pierces the boundary of $\mathcal{L}$, and $\mathcal{D}$ becomes the smallest absorbing region. At the bifurcation value, $\omega = \frac{1}{2a}$ there is an orbit of period two on the boundary of $\mathcal{L}$. In fact, the invariant transverse manifold of the non-trivial fixed point in region 4 consists of marginally stable points of period two since the corresponding eigenvalue of the Jacobian is -1. The change in nature of the fixed point in region 4 is clearly visible in Figures 4b,c. In Figure 4b the characteristic behaviour of a saddle is seen (the eigenvalues of the affine map at the fixed point can be read off from the calculation in the proof of Lemma 4.3 – they are approximately -1.14 and -0.914), whilst in Figure 4c the behaviour is characteristic of a node (unstable in this case, with eigenvalues of -1.14 and -1.028 approximately),

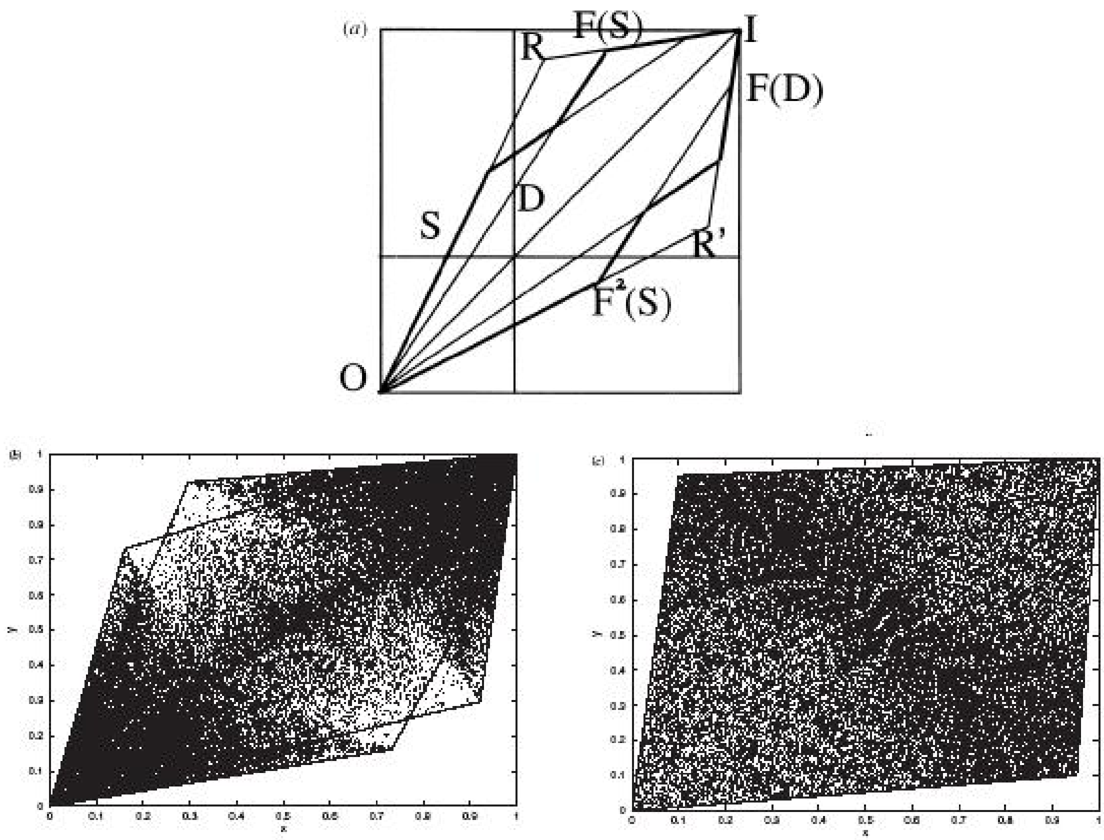

FIGURE 3. (a) Geometry of the critical lines in the construction of the region $\mathcal{L}$; (b) numerical simulation of the attractor with $a = 8$ and $\omega = 0.1$, which is after the blowout bifurcation $\omega_b \approx 0.15696$) but before the bifurcation at $\omega = \frac{1}{16} = 0.0625$ below which all synchronized states are transversely unstable; (c) numerical simulation of the attractor with $a = 8$ and $\omega = 0.05$ showing that the attractor is $\mathcal{D}$.
50000 iterates are plotted with initial conditions $(0.2, 0.25)$ in both (b)

where orbits are tangent to the weaker unstable direction, the transverse eigenvector in this case.

With more work a more detailed description of the absorbing regions and attractors for $a > 2$ could no doubt be established. The remarks above give some indication as to how the boundary of the attractor can evolve, and how it may differ from the cases studied in the main part of this paper.

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APPENDIX

The aim of this appendix is to prove two basic results which follow from the affine locally eventually onto property. I am grateful to Mark Muldoon for suggestions as to how to simplify some of the proofs below. Although the results are still nowhere near optimal, they are enough for the purposes of this paper.

Consider a piecewise affine map $f:\mathbf{R}^2\to\mathbf{R}^2$ and let Df^n denote the 2×2 Jacobian matrix of f^n. Recall that f is is affine locally eventually onto an invariant set $\mathcal{D}\subseteq\mathbf{R}^2$ ($a.l.e.o$) iff for every open set U in $\mathcal{D}$ there exists a compact $W\subseteq U$ homeomorphic to a disc, and $n>0$ such that $f^n|_V$ is affine and invertible, $f^n(V)=\mathcal{D}$.

Theorem B. *Suppose that the piecewise affine map $f:\mathbf{R}^2\to\mathbf{R}^2$ is a.l.e.o. on $\mathcal{D}$. Then*
(i) periodic points are dense in $\mathcal{D}$; and
(ii) f is transitive on $\mathcal{D}$ (i.e. there exists a dense orbit in $\mathcal{D}$).

To prove this result we need the following easy lemma.

Lemma. *Consider the linear map $x\to Bx$ where B is an invertible matrix. If W is a compact subset of $\mathbf{R}^2$ homeomorphic to a disc, and $W\subseteq BW$ then W contains a periodic point of the map.*

Proof: Since B is invertible, $B^{-1}W\subseteq W$ and the map B^{-1} is continuous on W. Thus we may apply the Brouwer fixed point theorem to deduce the existence of a fixed point in W.

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Proof of Theorem B: Part (i) is a trivial consequence of the above Lemma (after a translation of coordinates). Part (ii) follows from the standard style of argument, which is given below for completeness.

Let δ_i be a monotonic sequence of positive real numbers with $\delta_i\to 0$ as $i\to\infty$. Let $\mathcal{C}_i$ be a collection of N_i ($N_i<\infty$) δ_i-balls, B_{ij}, $0\leq j\leq N_i-1$, such that $\mathcal{D}\subseteq\cup_j B_{ij}$ and $B_{ij}\cap\mathcal{D}\neq\emptyset$. Choose this cover (adding an extra $\delta_{i+1}-$ball if necessary) so that $B_{i+1,0}\subset B_{i,N_i}$. By taking appropriate inverses it is easy to see that the a.l.e.o. property implies that for each i there exists compact connected $V_i\subset B_{i0}$ and increasing sequence $p(i,r)$, $1\leq r\leq N_i$ such that $f^{p(i,N_i)}|_{V_i}$ is affine, $f^{p(i,r)}(V_i)\subseteq B_{ir}$, $1\leq r<N_i$, and $f^{p(i,N_i)}(V_i)=\mathrm{c}\ell(B_{i+1,0})$.

Let $q_i=\sum_{k=1}^{i-1}p(k,N_k)$. By taking appropriate inverses, for each $m>0$ there exists a closed connected subset $U_m\subseteq B_{11}$ such that $f^{q_i+p(i,r)}(U_m)\subseteq B_{ir}$, for $1\leq i<m$, $0\leq r<N_m$, and $f^{q_{m+1}}(U_m)=B_{m+1,0}$ with $f^{q_{m+1}}|_{U_m}$ affine. Moreover, $U_{m+1}\subseteq U_m$. Hence the limit

$$U_\infty=\bigcap_{m=1}^{\infty}U_m$$

exists and is non-empty. Indeed, U_∞ must consist of a point (by the a.l.e.o. property there are no wandering domains) and the orbit of U_∞ is clearly dense in $\mathcal{D}$, as required.

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