Chamber Graphs of some Geometries that are Almost Buildings

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Abstract

The global structure of the chamber graph of certain rank 3 geometries that are almost buildings is determined. Computer files containing extensive details of these graphs accompany this paper.

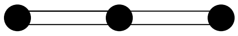
1 Introduction

The study of geometries that are almost buildings was instigated by Tits in [19]. The acronym, GAB's, was bestowed upon them by Kantor in [9], and they also go under the names of "geometries of type M" or "Tits geometries of type M". These geometries are Buekenhout–Tits geometries [3] all of whose rank 2 residue geometries are generalized polygons (though they are not required to satisfy the intersection property). That is, they are incidence geometries satisfying axioms (1) and (2) but not necessarily (3) of [3]. We recall that an incidence geometry over a set I is a triple $(\Gamma, *, \tau)$ where Γ is a set, τ an onto map from Γ to I and $*$ is an incidence relation on Γ such that if $x, y \in \Gamma$ and $x * y$ then $\tau(x) \neq \tau(y)$. The map τ is called the type map and $|I|$ the rank of Γ. As is customary, we shall abbreviate $(\Gamma, *, \tau)$ to Γ. A flag F of Γ is a subset of Γ such that $x * y$ for all $x, y \in F, x \neq y$ and the type of F is $\{\tau(x) \mid x \in F\}$. The residue of F in Γ, Γ_F, is the (subgeometry) given by $\{x \in \Gamma \mid y * x \quad \text{for all} \quad y \in F\}$. If $F = \{x\}$, then we write Γ_x instead of $\Gamma_{\{x\}}$. We shall call a maximal flag of Γ a chamber of Γ. Note that, by axiom (1) of [3], the type of a chamber of a GAB is I. The chamber graph $\mathcal{C}(\Gamma)$ is defined as follows. The vertices are the chambers of Γ with distinct chambers γ and γ' deemed adjacent in $\mathcal{C}(\Gamma)$ if $|\gamma \cap \gamma'| = |I| - 1$. We sometimes also say that γ and γ' are i-adjacent if $I = \{i\} \cup \{\tau(x) \mid x \in \gamma \cap \gamma'\}$. Let γ be a chamber of Γ. The the i^{th} disc of γ, denoted by $\Delta_i(\gamma)$, consists of all the chambers which are distance i from γ in the graph $\mathcal{C}(\Gamma)$. We shall use $d(\, , \,)$ for the distance metric on $\mathcal{C}(\Gamma)$ and $Diam(\mathcal{C}(\Gamma))$ for the diameter of $\mathcal{C}(\Gamma)$. For more on incidence geometries, consult [2], [4], while for GAB's the survey paper [11] by Kantor contains much interesting material.

The chamber graph of a building contains all the important geometric information about the building. For example, the (chambers of the) apartments of the building can be detected in the chamber graph. The sets $\Delta_i(\gamma)$, for γ a chamber, encode data relating to the Weyl group of the building. Further, if d is the diameter of the chamber graph and G is the automorphism group of the building, then G_γ, a Borel subgroup of G, acts transitively on $\Delta_d(\gamma)$. See [12], [18] and [19] for more on buildings. It is natural to wonder about chamber graphs of other geometries associated with groups which are, in some sense, close to buildings. This has prompted a number of papers which have focussed on analyzing the disc structure of such chamber graphs. See [6],

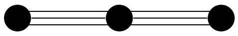
[14], [15], [16]. Most of the geometries of interest have a large number of chambers and so these investigations have necessarily involved extensive computation using packages such as MAGMA [5]. Here we continue this line of work, examining the chamber graphs of rank 3 GAB's. The examples we look at have been drawn from [1], [8], [9], [13] (see also [7], [10], and [20]). We now state our main results on the disc structure of these GAB's.

Theorem 1.1 *Let G denote one of the five groups $P\Omega_6^-(3)$, $G_2(3)$, $U_6(2)$, $\Omega_8^+(2)$ and Suz, and let Γ denote a GAB associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = Stab_G(\gamma_0)$.*

(i) If $G \cong P\Omega_6^-(3)$ and Γ has diagram *, then $\mathcal{C}$ has 25515 chambers, 196 B-orbits, diameter 10 and disc structure*

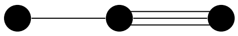
i^{th} DISC	1	2	3	4	5	6	7	8	9	10
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	64	176	416	1024	2432	5120	9088	7168
NUMBER OF B-ORBITS	3	5	8	12	15	19	27	35	43	28

.

(ii) If $G \cong G_2(3)$ and Γ has diagram *, then $\mathcal{C}$ has 66339 chambers, 1144 B-orbits, diameter 12 and disc structure*

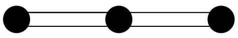
i^{th} DISC	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	64	208	600	1728	4640	10368	17920	20416	9472	896
NUMBER OF B-ORBITS	3	6	10	18	27	42	90	176	288	321	148	14

.

(iii) If $G \cong G_2(3)$ and Γ has diagram *, then $\mathcal{C}$ has 66339 chambers, 1144 B-orbits, diameter 13 and disc structure*

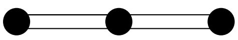
i^{th} DISC	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	56	144	384	960	2176	4864	10368	19072	21248	6976	64
NUMBER OF B-ORBITS	3	6	9	14	21	31	51	92	172	302	332	109	1

.

(iv) If $G \cong U_6(2)$ and Γ has diagram *, then $\mathcal{C}$ has 1576960 chambers, 505 B-orbits, diameter 8 and disc structure*

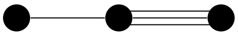
i^{th} DISC	1	2	3	4	5	6	7	8
$\lvert\Delta_i(\gamma_0)\rvert$	15	117	972	6075	35721	203391	875043	455625
NUMBER OF B-ORBITS	3	6	10	17	35	98	246	89

.

(v) If $G \cong \Omega_8^+(2)$ and Γ has diagram *, then $\mathcal{C}$ has 179200 chambers, 317 B-orbits, diameter 9 and disc structure*

i^{th} Disc	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\|\Delta_i(\gamma_0)\|$	*9*	*45*	*216*	*891*	*3159*	*11421*	*37098*	*80676*	*45684*
Number of B-orbits	*3*	*6*	*10*	*16*	*26*	*43*	*68*	*95*	*49*

.

(vi) If $G \cong Suz$ and Γ has diagram *, then $\mathcal{C}$ has 18243225 chambers, 1276 B-orbits, diameter 16 and disc structure*

i^{th} Disc	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\|\Delta_i(\gamma_0)\|$	*8*	*32*	*128*	*432*	*1216*	*3712*	*11008*	*29184*	*81920*
Number of B-orbits	*3*	*5*	*8*	*12*	*15*	*19*	*26*	*33*	*44*

10	*11*	*12*	*13*	*14*	*15*	*16*
229376	*598016*	*1576960*	*3595264*	*5410816*	*5304320*	*1400832*
66	*99*	*155*	*241*	*270*	*222*	*57*

.

The GAB associated with the Lyons sporadic simple group is beyond our computational reach having 207060716016 chambers. However, we can give bounds on the diameter of the chamber graph.

Theorem 1.2 *Let Γ be the GAB for Ly. Then $11 \leq Diam(\mathcal{C}(\Gamma)) \leq 16$.*

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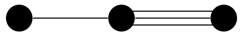
2 Properties of $\mathcal{C}(\Gamma)$

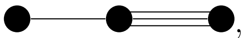
The information collated in Theorem 1.1 was obtained using the code available with [6] and employing Magma. In fact, much more intricate details about $\mathcal{C}(\Gamma)$ were obtained, and these are available in the accompanying computer files [17]. We give a brief summary of such things. The chambers of Γ are viewed as the right cosets of B. The panel stabilizers will be denoted by P_1, P_2 and P_3 (recall we are only looking at rank 3 geometries). The data obtained and program code is underpinned by DB, a sequence containing the (B, B) double coset representatives. So for $g = DB[j]$, the Bg coset is a representative for the B-orbits on the chambers of Γ. To minimise storage, we record j rather than $DB[j]$ whenever possible.The important output files are $BorbitsDiscs$ and $Neighbours$. The first is a sequence where $BorbitsDiscs[i]$ tells us the B-orbits making up $\Delta_i(\gamma_0)$ (where γ_0 is identified with the coset B). Here we give B-orbit representatives Bg, where $g = DB[k]$, by recording k. $Neighbours$ is also a sequence where $Neighbours[j]$ is giving information on the neighbours of Bg (where $g = DB[j]$). Suppose we have $[P_i : B] = 3$ for $i = 1, 2, 3$ (as happens for the GAB associated with $P\Omega_6^-(3)$, for example), so $\mathcal{C}(\Gamma)$ has valency 6. Returning to $Neighbours[j]$, in this case this would be a 6-tuple $[k_1, k_2, k_3, k_4, k_5, k_6]$. This is saying that the six neighbours of Bg are in the B-orbits of $B * DB[k_i]$ $(i = 1, \ldots 6)$. More than this we are also keeping track of the kind of adjacency. So $B * DB[k_1]$, $B * DB[k_2]$ are 1-adjacent to Bg, $B * DB[k_3]$, $B * DB[k_4]$ are 2-adjacent to Bg, and $B * DB[k_5]$, $B * DB[k_6]$ are 3-adjacent to Bg.

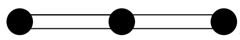
Proof of Theorem 1.2 Let $G = Ly$ and let γ_0 be a chamber of $\mathcal{C}(\Gamma)$, and put $B = Stab_G(\gamma_0)$. Recall that the diagram for Γ is . Let x be a point of Γ. Then by Section 6 of [9], Γ_x is a generalized hexagon dual to the usual $G_2(5)$

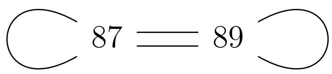
generalized hexagon. In particular, for any two chambers γ, γ' of Γ containing x we have $d(\gamma, \gamma') \leq 6$. Let the point, line and plane of γ_0 be respectively x_0, l_0, p_0 and γ_1 a chamber whose point, line and plane are respectively x_1, l_0, p_1. So x_0 and x_1 are collinear in Γ. Now $\gamma_0 = \{x_0, l_0, p_0\}, \{x_0, l_0, p_1\}, \{x_1, l_0, p_1\} = \gamma_1$ is a path in $\mathcal{C}(\Gamma)$, whence $d(\gamma_0, \gamma_1) \leq 2$. Since the point-line collinearity graph of Γ has diameter 2 (see Section 6 of [9] again), we infer that $Diam(\mathcal{C}(\Gamma)) \leq 2 + 6 + 2 + 6 = 16$.

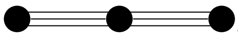
The number of chambers in the GAB associated with the Lyons group is $\frac{|G|}{N_G(S)} = \frac{|G|}{5^6 \cdot 2^4} = 207060716016$, where $S \in Syl_5(G)$. We find a lower bound for the diameter of the $\mathcal{C}(\Gamma)$ by working out the maximum number of chambers that can be in each disc. We have $[P_i : B] = 6$, $i = 1, 2, 3$, and so the valency of $\mathcal{C}(\Gamma)$ is 15. Therefore each chamber γ in $\Delta_1(\gamma_0)$ is joined to 5 chambers in $\Delta_1(\Gamma_0) \cup \{\gamma_0\}$. Hence $|\Delta_1(\gamma) \cap \Delta_2(\gamma)| = 10$. Of course for $i \geq 2$, a chamber in $\Delta_i(\gamma_0)$ can have at most 14 neighbours in $\Delta_{i+1}(\gamma_0)$. Thus, letting $k = Diam(\mathcal{C}(\Gamma)) - 2$,
$207060716016 \leq 15(1 + 10 + 10 \cdot 14 + \ldots + 10 \cdot 14^k) \leq 15 + 150\left(\frac{14^k - 1}{14 - 1}\right)$.
This gives $k \geq log_{14}(\frac{13}{150}(207060716001) + 1)$, whence $k \geq 8.947$. Consequently, $Diam(\mathcal{C}(\Gamma)) \geq 11$ which completes the proof of Theorem 1.2. □

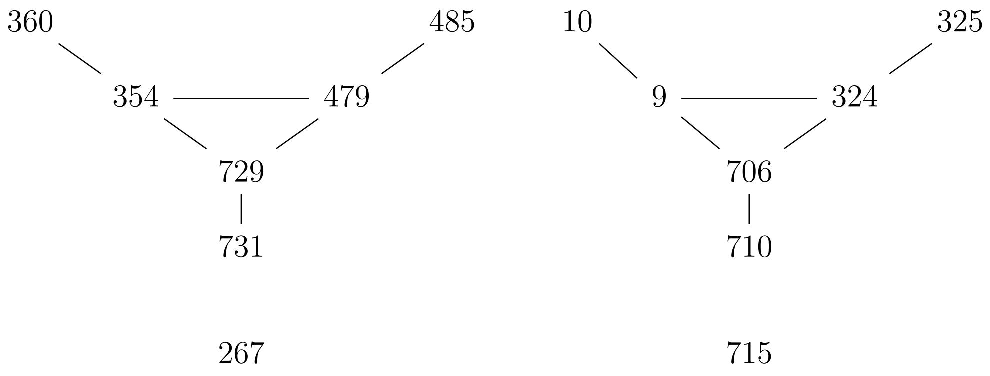
2.1 Collapsed adjacency graphs

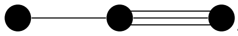
For a GAB with diameter of say d, we call $\Delta_d(\gamma_0)$ the last disc (of γ_0) of the chamber graph. When examining the number of B-orbits which comprise the last disc we see, from the point of the chamber graph, the appellation of "almost building" is something of a misnomer. Of the GAB's investigated here only the GAB associated with $G_2(3)$, diagram , has its last disc as a B-orbit. Because of this we have calculated the geodesic closure for this GAB, the results of which are in Section 2.2. All the others have the number of B-orbit ranging from 14 to 89. Indeed the more sporadic geometries studied in [15] and [6] come closer to buildings in this respect.
Notwithstanding the above comments on the last disc, we have looked at the induced graph on this disc. The most interesting (as far as we can see) are the GAB's from $G_2(3)$. Now we describe the collapsed adjacency graphs for the last disc of γ_0 ("collapsing to the B-orbits", $B = Stab_G \gamma_0$). We use j to stand for the B orbit of $B * DB[j]$ (where j is as given in the accompanying files).

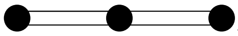
(i) If $G \cong P\Omega_6^-(3)$ and Γ has diagram , then the last disc has all the B-orbits apart from 89 and 87 connected, with 89 and 87 having the following collapsed adjacency graph.

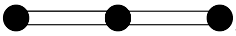


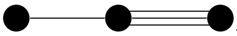
(ii) If $G \cong G_2(3)$ and Γ has diagram , then the 14 B-orbits in the last disc form the following collapsed adjacency graph.

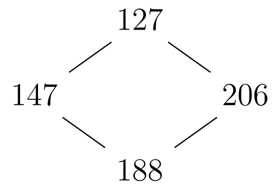


(iii) If $G \cong G_2(3)$ and Γ has diagram , then there is only one B-orbit in the last disc and $\Delta_{13}(\gamma_0)$ is a co-clique.

(iv) If $G \cong U_6(2)$ and Γ has diagram , then all the B-orbits are connected apart from 215 and 377 which have the following collapsed adjacency graph.

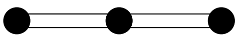
(v) If $G \cong \Omega_8^+(2)$ and Γ has diagram , then the collapsed adjacency graph of $\Delta_9(\gamma_0)$ is connected.

(vi) If $G \cong Suz$ and Γ has diagram , then all the B-orbits are connected apart from 127, 147, 188 and 206 which have the following collapsed adjacency graph.

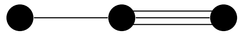


2.2 Geodesic Closure

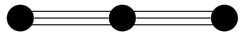
For $\gamma, \gamma' \in \mathcal{C}$ a shortest path between them in $\mathcal{C}$ is called a geodesic. The geodesic closure of a set of chambers X is defined to be the set $\overline{X}$ of all chambers lying on some geodesic of γ, γ' for any pair $\gamma, \gamma' \in X$. The motivation for geodesic closures comes from the fact that in the chamber graph of a building, the geodesic closure of two chambers at maximal distance apart yields (the chambers of) an apartment.

Suppose $G \cong P\Omega_6^-(3)$ and Γ has diagram *, and let $\gamma_i \in \Delta_{10}(\gamma_0)$, $i = 1, ..., 28$ be B-orbit representatives of $\Delta_{10}(\gamma_0)$. Set $n_{i,j} = |\overline{\{\gamma_0, \gamma_i\}} \cap \Delta_j(\gamma_0)|$. Then*

j	0	1	2	3	4	5	6	7	8	9	10
$n_{1,j}$, $n_{2,j}$	1	3	4	6	6	4	6	6	4	3	1
$n_{3,j}$, $n_{4,j}$, $n_{5,j}$, $n_{6,j}$	1	2	2	3	3	2	3	3	2	2	1
$n_{7,j}$, $n_{8,j}$, $n_{9,j}$, $n_{10,j}$	1	3	4	5	6	5	4	4	3	2	1
$n_{11,j}$, $n_{12,j}$	1	3	4	6	6	4	4	4	2	2	1
$n_{13,j}$, $n_{14,j}$	1	1	2	1	1	2	1	1	2	1	1
$n_{15,j}$, $n_{16,j}$, $n_{17,j}$, $n_{18,j}$	1	3	4	4	5	6	5	4	4	3	1
$n_{19,j}$, $n_{20,j}$, $n_{21,j}$, $n_{22,j}$	1	2	3	4	4	5	6	5	4	3	1
$n_{23,j}$, $n_{24,j}$, $n_{25,j}$, $n_{26,j}$	1	2	2	2	2	2	2	2	2	2	1
$n_{27,j}$, $n_{28,j}$	1	2	2	4	4	4	6	6	4	3	1

Suppose $G \cong G_2(3)$ and Γ has diagram *, and let $\gamma' \in \Delta_{13}(\gamma_0)$. Set $n_j = |\overline{\{\gamma_0, \gamma'\}} \cap \Delta_j(\gamma_0)|$. Then*

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
n_j	1	6	15	23	24	26	25	25	26	24	23	15	6	1

Suppose $G \cong G_2(3)$ and Γ has diagram *, and let $\gamma_i \in \Delta_{12}(\gamma_0)$, $i = 1, ..., 14$ be B-orbit representatives of $\Delta_{12}(\gamma_0)$. Set $n_{i,j} = |\overline{\{\gamma_0, \gamma_i\}} \cap \Delta_j(\gamma_0)|$. Then*

j	0	1	2	3	4	5	6	7	8	9	10	11	12
$n_{1,j}$, $n_{2,j}$	1	3	6	9	9	10	12	10	9	9	6	3	1
$n_{3,j}$, $n_{4,j}$	1	5	9	13	13	13	18	13	13	13	9	5	1
$n_{5,j}$, $n_{6,j}$	1	6	14	17	25	29	26	29	25	17	14	6	1
$n_{7,j}$, $n_{8,j}$	1	3	5	6	6	7	7	8	7	7	5	3	1
$n_{9,j}$, $n_{10,j}$	1	5	12	15	18	18	16	18	18	15	12	5	1
$n_{11,j}$, $n_{12,j}$	1	3	5	7	7	8	7	7	6	6	5	3	1
$n_{13,j}$, $n_{14,j}$	1	5	8	12	12	13	16	13	12	12	8	5	1

2.3 Apartments of GABs associated with $U_6(2)$ and $\Omega_8^+(2)$

The GAB's for $U_6(2)$ and $\Omega_8^+(2)$ possesses apartments (see [9]), viewed as the fixed chambers of T. For $U_6(2)$ we take T to be a cyclic group of order 4, and for $\Omega_8^+(2)$ we take T to be an elementary abelian group order 4, see [9]. In both cases the apartments are isomorphic and have diameter 8. They also have the property that the distance between any two chambers in the apartment (as measured in the apartment) is the same as in the chamber graph. So this is something one expects from a building. However, for $\Omega_8^+(2)$ the diameter of its chamber graph is 9, so not equal to the diameter of the apartment - unlike the situation in a building.

2.3.1 $\mathbf{G \cong \Omega_8^+(2)}$

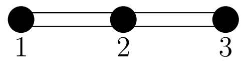
An apartment. $\mathcal{A}$, of the GAB associated with $\Omega_8^+(2)$ cuts the discs as follows.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{A} \cap \Delta_i(\gamma_0)\rvert$	1	3	5	8	11	13	13	8	2	0

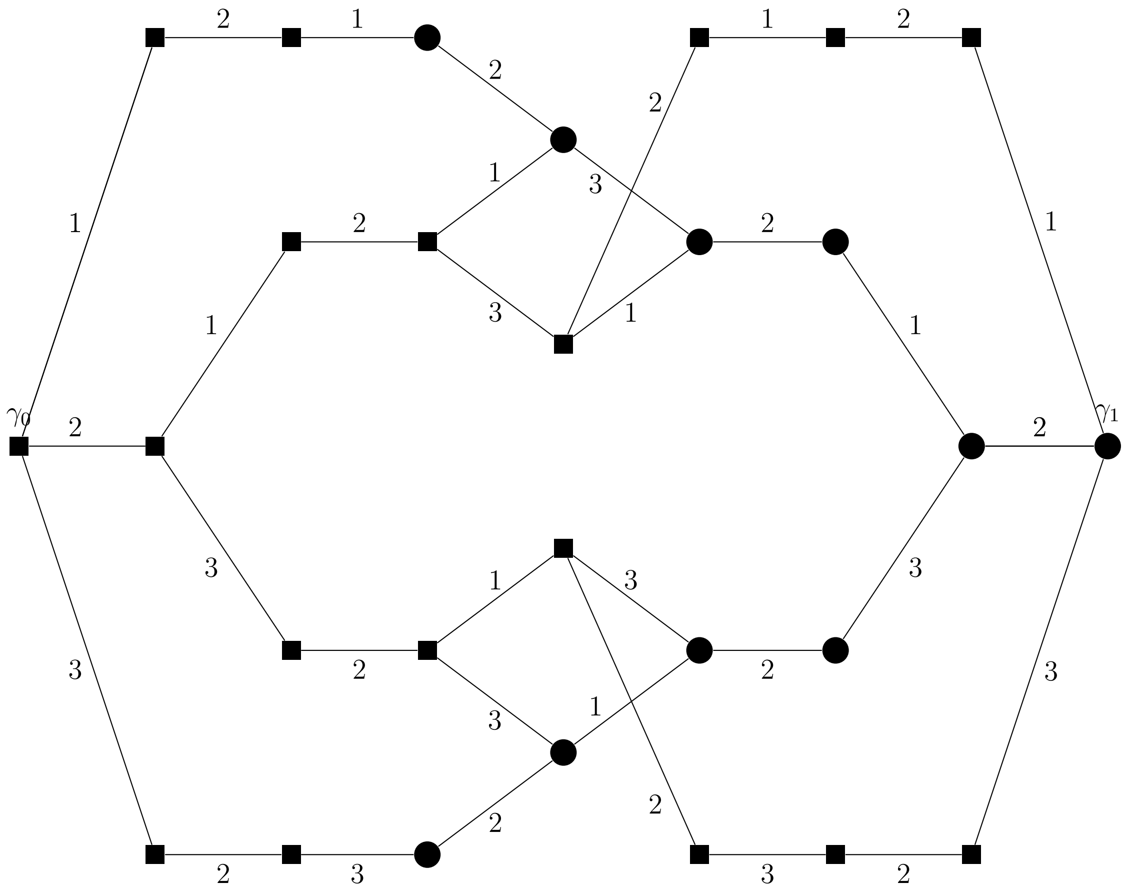
Let $\mathcal{A} \cap \Delta_8(\gamma_0) = \{\gamma_1, \gamma_2\}$. For $j = 1, 2$ the geodesic Closure of the γ_0, γ_j cuts the discs as follows.

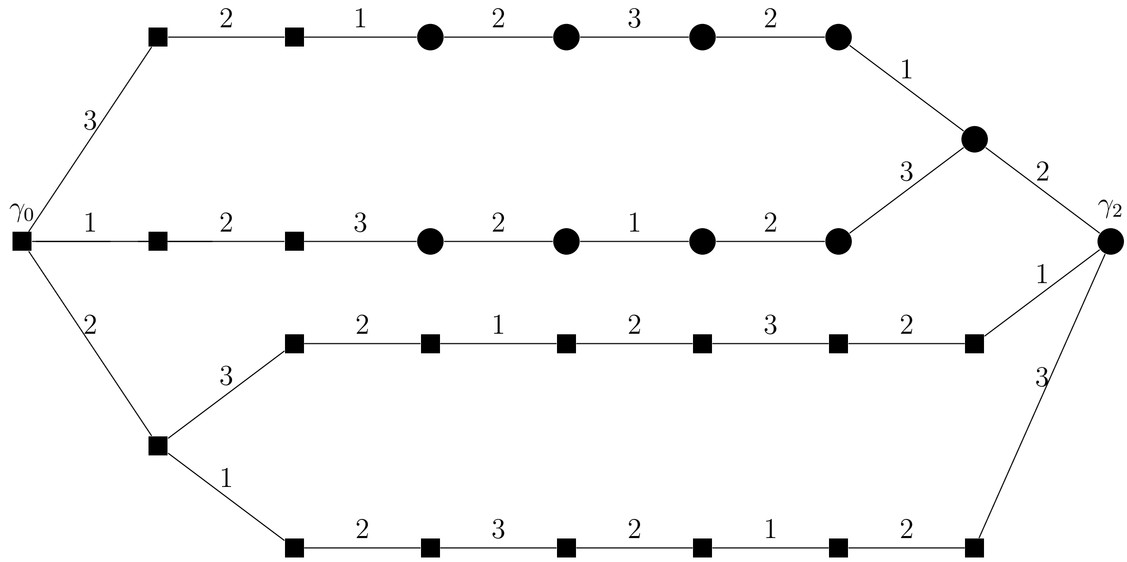
Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\overline{\{\gamma_0, \gamma_j\}} \cap \Delta_i(\gamma_0)\rvert$	1	3	4	4	4	4	4	3	1

The graphs below are the geodesic closures $\overline{\{\gamma_0, \gamma_1\}}$ and $\overline{\{\gamma_0, \gamma_2\}}$ where γ_i, $i = 1, 2$ are representatives of the two B-orbits in the apartment which are in $\Delta_8(\gamma_0)$ for the GAB of $\Omega_8^+(2)$. The type of adjacency between two connected chambers is shown by the labelling on the edges where



The set of chambers in both geodesic closures are subsets of the apartment. The intersection between $\overline{\{\gamma_0, \gamma_1\}}$ and $\overline{\{\gamma_0, \gamma_2\}}$ has size 18 and the chambers that lie in both geodesic closures are labelled with squares rather than circles.





2.3.2 $\mathbf{G \cong U_6(2)}$

Structure of the apartment of the GAB associated with $U_6(2)$ within the chamber graph. Let $\mathcal{A}$ be the chambers of an apartment.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\mathcal{A} \cap \Delta_i(\gamma_0)\rvert$	1	3	5	8	11	13	13	9	1

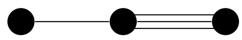
Geodesic Closure of the chamber in the last disc of the apartment of the GAB associated with $U_6(2)$. Let $\gamma' \in \mathcal{A} \cap \Delta_8(\gamma_0)$.

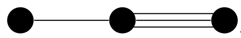
Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\overline{\{\gamma_0, \gamma'\}} \cap \Delta_i(\gamma_0)\rvert$	1	3	4	4	4	4	4	3	1

The graph for the geodesic closure of the only B-orbit in the last disc of the apartment in the GAB of $U_6(2)$ is identical to the first diagram in Subsection 2.3.1. Again, the set of chambers in the geodesic closure is a proper subset of the apartment (once more not very building like).

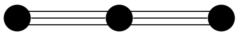
2.4 Maximal opposite sets

A maximal opposite set of chambers are a set of chambers of maximal size subject to having the property that any two chambers are opposite to each other, meaning that their distance apart is the diameter of the graph.

Theorem 2.1 *If $G \cong G_2(3)$ and Γ has diagram* *, then a maximal opposite set of chambers consists of three chambers.*

Proof Suppose $G \cong G_2(3)$ and Γ has diagram . Since G_{γ_0} is transitive on $\Delta_{13}(\gamma_0)$, we may assume our maximal opposite set contains $\{\gamma_0, \gamma_1\}$, where $\gamma_1 \in \Delta_{13}(\gamma_0)$ is the chamber corresponding to $B * DB[153]$ (the right coset of B containing $DB[153]$). We identify a chamber γ with the triple $\{F_1(\gamma), F_2(\gamma), F_3(\gamma)\}$ which corresponds to a point-line-quad triple. Using the action of B, we determine $\Delta_{13}(\gamma_0)$, and by applying $DB[153]$ to this set we obtain $\Delta_{13}(\gamma_1)$. We can then see that $|\Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1)| = 1$. If we take $\gamma_2 \in \Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1)$ we can see that

$|\Delta_{13}(\gamma_0) \cap \Delta_{13}(\gamma_1) \cap \Delta_{13}(\gamma_2)| = 0$, and so $\{\gamma_0, \gamma_1, \gamma_2\}$ is a maximal opposite set. $\square$

Theorem 2.2 *If $G \cong G_2(3)$ and Γ has diagram* *, then each choice of the B-orbits in the last disc gives rise to a maximal opposite set of chambers consisting of four chambers.*

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