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A posteriori error estimator for a strongly conservative finite element method of Stokes-Darcy coupling equation

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Abstract

In this paper, a posteriori error estimator for a strongly conservative finite element method will be presented for the coupling of Stokes flow with porous media flow in two dimensions. These flows are governed by Stokes and Darcy equations with the Beavers-Joseph-Saffman interface condition. We discretize using a divergence-conforming velocity space with matching pressure space (such as Raviart-Thomas spaces). A reliable and efficient residual-based a posteriori error estimator is derived for the coupled problem. Several numerical experiments are presented to validate the theoretical properties of this estimator and show the capability of the corresponding adaptive algorithm to localize the singularities of the solution.

Keywords:
a posteriori analysis, Divergence-conforming DG methods, Stokes flow, Darcy flow, Beavers-Joseph-Saffman transmissibility conditions.

1. Introduction

Computing the numerical solution of the coupling of fluid flow with porous media flow, in which the respective interface conditions are given by balance of normal forces, mass conservation and Beavers-Joseph-Saffman law, is a very active research area (see, e.g. [19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30] and references therein). These problems have many important applications such as the modeling of groundwater contamination through streams and filtration problems [32, 33]. In literature, there are two most popular approaches to solve the weak formulation of the Stokes-Darcy equation. In first approach, the normal continuity of the velocity field on the interface is imposed weakly through a Lagrange multiplier or a penalty (cf. e. g. [17, 18] and references therein). In second one, the normal continuity of the velocity field imposes on the interface strongly as a property of the function space chosen.

The strongly coupled formulation in [28, 30] is based on a uniform discretization of the Stokes and Darcy regions using the same finite element. This element is based on a formulation with velocities globally in $H^1$ with restrictions to the Stokes subdomain in $H^\text{div}$. Thus, a divergence-stable pair, for instance Raviart-Thomas elements with matching pressure space, is chosen on the whole domain and inconsistency in the Stokes subdomain is treated by a discontinuous Galerkin (DG) method. In [30], it was shown that this approach achieved mass conservation in the sense of $H^\text{div}(\Omega)$. Moreover, a-priori energy error estimates were derived. In [28], optimal $L^2$-error estimates were proven.

Recently, Chen et al. [29] derived an a posteriori error estimate for the $H^\text{div}$-conforming mixed finite element for the coupled Darcy-Stokes system using conforming triangular meshes without showing numerical results to validate the a posteriori error estimate. In this article on the other hand, we introduce a reliable and efficient residual-based a posteriori error estimator for rectangular meshes with hanging nodes. In addition, we use different interpolation operators to prove the reliability estimate. Our interpolation operator consists two parts. First, we define a Scott-Zhang interpolant for the whole domain. Then, we define an additional interpolant for Darcy-parts. Apart of this, the proof of efficiency bound is rigorously discussed with allowing hanging nodes. Several numerical experiments are given to validate the theoretical properties of this estimator and show the capability of the corresponding adaptive algorithm to localize the singularities of the solution.

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The remainder of this article is organized as follows. In Section 2, we introduce the strongly conservative method for Stokes-Darcy coupling. Section 3 is devoted to define a posteriori error estimator and present reliability and efficiency estimates. In Section 4, proofs of reliability and efficiency theorems are discussed. Section 5 is devoted to the validation of the theoretical results through numerical experiments.

2. A strongly conservative method for Stokes-Darcy coupling

2.1. Preliminaries

Let $\Omega$ be a bounded Lipschitz polygon in $\mathbb{R}^2$ with the boundary $\Gamma = \partial \Omega$. Let $H^s(\omega)$ denotes as the standard Sobolev space with the associated norm $\| \cdot \|_{s,\omega}$ for $s \geq 0$ (see [1]). In case of $\omega = \Omega$, we use $\| \cdot \|_s$ instead of $\| \cdot \|_{s,\omega}$. By boldface letters $H'(\omega) = H'(\omega; \mathbb{R}^2)$ we denote vector-valued Sobolev spaces. Furthermore, we introduce the spaces

$$L^2_s(\Omega) := \left\{ v \in L^2(\Omega) \mid \int_\Omega v \, dx = 0 \right\}$$

$$H^s(\Omega) := \left\{ v \in L^2(\Omega) = (L^2(\Omega))^2 \mid \nabla \cdot v \in L^2(\Omega) \right\}$$

$$H^s(\Omega) := \left\{ v \in H^s(\Omega) \mid v \cdot n = 0 \text{ on } \Gamma \right\}$$

$$H_0^s(\Omega) := \left\{ v \in H^s(\Omega) \mid \nabla \cdot v = 0 \right\}.$$

Here $\Omega$ split into Lipschitz subregions $\Omega_S$ and $\Omega_D$ of free and porous media flow, respectively. Let $\Gamma_S$, $\Gamma_D$, and $\Gamma_I$ be defined as follows:

$$\Gamma_S = \Gamma \cap \partial \Omega_S, \quad \Gamma_D = \Gamma \cap \partial \Omega_D, \quad \Gamma_I = \Gamma_S \cap \Gamma_D.$$

In the Stokes region $\Omega_S$, the fluid velocity $u_S$ and fluid pressure $p_S$ satisfy the Stokes equations:

$$-\nabla \cdot (2\mu u_S) + \nabla p_S = f_S \quad \text{in } \Omega_S,$$

$$\nabla \cdot u_S = 0 \quad \text{in } \Omega_S,$$

$$u_S = 0 \quad \text{on } \Gamma_S,$$

where $\nu > 0$ and $f_S$ are the dimensionless fluid viscosity and a prescribed external body force, respectively. Here $\varepsilon(u_S) = \frac{1}{2}(\nabla u_S + (\nabla u_S)^T)$ is the deformation tensor.

In the Darcy region $\Omega_D$, the fluid velocity $u_D$ and fluid pressure $p_D$ satisfy the Darcy equations:

$$u_D + K \nabla p_D = 0 \quad \text{in } \Omega_D,$$

$$\nabla \cdot u_D = f_D \quad \text{in } \Omega_D,$$

$$u_D \cdot n = 0 \quad \text{on } \Gamma_D,$$

where $K > 0$ is the dimensionless permeability of the porous medium. Here $f_D$ models sinks and sources in the porous medium.

Next we define the Beavers-Joseph-Saffman conditions at the interface $\Gamma_I$:

$$u_S \cdot n = u_D \cdot n \quad \text{on } \Gamma_I,$$

$$p_S - 2\nu \varepsilon(u_S)n \cdot n = p_D \quad \text{on } \Gamma_I,$$

$$\gamma K^{-1/2} u_S \cdot \tau = 2\nu \varepsilon(u_S)n \cdot \tau = 0 \quad \text{on } \Gamma_I,$$

where $n$ and $\tau$ are unit normal and tangential vectors to $\Gamma_I$, respectively. Here $\gamma > 0$ is a phenomenological friction coefficient. We use the following notations for the integrals over the $\Omega$, boundaries and subset of those, namely

$$(f, g)_\Omega := \int_\Omega f \odot g \, dx, \quad (f, g)_\Gamma := \int_\Gamma f \odot g \, ds,$$

where $\odot$ is a generic multiplication operator. Similarly, the semi-norms are as follows:

$$\| f \|_\Omega := \sqrt{(f, f)_\Omega} \quad \text{and} \quad \| f \|_\Gamma := \sqrt{(f, f)_\Gamma}.$$

Next, we define the function spaces for our weak formulation such that

$$V = \left\{ v \in H^1_0(\Omega) \mid v|_{\Omega_S} \in H^1(\Omega_S) \right\} \quad \text{and} \quad Q = L^2(\Omega).$$
From [28, 30], the weak formulation of the problem (2.2) is as follows: find \((u, p) \in V \times Q\) such that there holds
\[
\mathcal{A}(u, p; v, q) = (fs, v)_{\Omega_2} + (f_0, q)_{\partial_\Omega}, \quad \forall \ (v, q) \in V \times Q,
\]
where
\[
\mathcal{A}(u, p; v, q) = a_S(u, v) + a_D(u, v) + a_I(u, v) - (p, \nabla \cdot v) - (q, \nabla \cdot u).
\]
Here
\[
a_S(u, v) = 2\nu(\mathbf{e}(u), \mathbf{e}(v))|_{\Omega_2}, \quad \forall u, v \in V,
\]
\[
a_D(u, v) = (K^{-1}v, u)|_{\partial_0}, \quad \forall u, v \in V,
\]
\[
a_I(u, v) = \gamma K^{-1/2}(u_S \cdot \tau, v_S \cdot \tau), \quad \forall u, v \in V.
\]

Note that we use the notations \(v_S = v|_{\partial_2}\) and \(v_D = v|_{\partial_\infty}\) throughout the paper. Moreover, the tangential traces of \(u\) and \(v\) from the Stokes subdomain at the interface \(I\) are denoted by \(u_S \cdot \tau\) and \(v_S \cdot \tau\), respectively.

### 2.2. Discretization

First we divide the domain \(\Omega\) by a subdivision \(\mathcal{T}_h\) into a mesh of shape-regular, rectangular cells \(K\) such that \(\Gamma_I\) is the union of element edges. Let \(h_K\) and \(E(\mathcal{T}_h)\) be denoted as the diameter of an element \(K\) and the set of edges of \(\mathcal{T}_h\), respectively. For given mesh \(\mathcal{T}_h\), the notions of broken spaces for the continuous and differentiable function spaces are denoted as \(C(\mathcal{T}_h)\) and \(H^1(\mathcal{T}_h)\) which are the spaces such that the restriction to each mesh cell \(K \in \mathcal{T}_h\) is in \(C(K)\) and \(H^1(K)\), respectively. Now, we define \(Q_k(K), Q_k(K)^d\) and \(Q_k(K)^{div}\) as the space of scalar, vector and tensor valued polynomials of degree at most \(k\) in each coordinate direction on the reference cell \(\tilde{K}\), mapped to the actual mesh cell \(K\).

**Remark 2.1.** The inf-sup stability of discretizations with hanging nodes using RT elements is in part still an open question. In [10], there exists a stability proof only for the pair \(RT_1/Q_1\) defined in (2.7) and (2.8) with \(k \geq 2\) for quadrilaterals with one-irregular meshes. However, we conjecture from our computational results that stability also holds for \(k = 1\). Moreover, the stability result for the divergence-free elements proposed in [6] is not available for triangles with hanging nodes. On the other hand, locally refined triangular meshes without hanging nodes can be obtained using bisection. The results below are all to be read in view of the restrictions cited in this remark.

Let \(K_1\) and \(K_2\) be mesh cells which have a common face \(E \in E(\mathcal{T}_h)\). The traces of function \(u \in C(\mathcal{T}_h)\) on \(E\) from \(K_1\) and \(K_2\) are defined as \(u_1\) and \(u_2\), respectively. Then the average operator \(||\cdot||\) is as follows:
\[
||u|| = \frac{u_1 + u_2}{2}.
\]

Let \(n_1\) and \(n_2\) be the outward normal vector to \(K_1\) and \(K_2\), respectively and \(n_2 = -n_1\). Then the jumps \(||\cdot||\) of these functions across \(E\) are defined as follows:
\[
||u|| = u_1 n_1 + u_2 n_2 = (u_1 - u_2)n_1, \quad ||u|| = (u_1 - u_2).
\]
Next, we define a tangential vector \(\tau = (-n_2, n_1)\) for given normal vector \(n = (n_1, n_2)^T\). Thus \(\tau_2 = -\tau_1\) for \(n_2 = -n_1\).

For \(k \geq 1\) we define the velocity space \(V_h\) based on the Raviart-Thomas element of degree \(k\) as
\[
V_h = \{v \in H_0^{div} | \forall K \in \mathcal{T}_h : v|_K \in RT_k\}
\]
and
\[
V_h^0 = \{v \in V_h | \nabla \cdot v = 0\}.
\]
Let \(Q_h\) be the discrete sub-space of \(L_0^2(\Omega)\) such that
\[
Q_h = \{v \in L_0^2 \mid \forall K \in \mathcal{T}_h : v|_K \in Q_k(K)\}. \tag{2.8}
\]
On any mesh, these spaces are connected by
\[
\nabla \cdot V_h \subset Q_h.
\]
Now, the shorthand notation for the integrals over the meshes and set of the faces is denoted as
\[
(f, g)_{T_h} := \sum_{k \in T_h} (f, g)_k; \quad \langle f, g \rangle_{E(h)} := \sum_{E \in \partial T_h} \langle f, g \rangle_E = \sum_{E \in \partial T_h} \int_E f \circ g \, ds,
\]
where \(\circ\) is a generic multiplication operator. Similarly, the semi-norms are as follows:
\[
\|f\|_{T_h} := \sqrt{(f, f)_{T_h}} \quad \text{and} \quad \|f\|_{E(h)} := \sqrt{\langle f, f \rangle_{E(h)}}.
\]

2.3. A strongly conservative formulation for Stokes-Darcy coupling equation

From [28, 30], the discrete weak formulation of the problem (2.2) is as follows: find \((u, p) \in V_h \times Q_h\) such that there holds
\[
\mathcal{A}_h(u, p; v, q) = (f, v), \quad \forall \ (v, q) \in V_h \times Q_h,
\]
where
\[
\mathcal{A}_h(u, p; v, q) = a_{h,S}(u, v) + a_{h,D}(u, v) + a_I(u, v) - (p, \nabla \cdot v) - (q, \nabla \cdot u).
\]

Here
\[
a_{h,D}(u, v) = (K^{-1}u, v), \quad a_I(u, v) = \gamma K^{-1/2}(u, \tau, v, \tau), \quad a_{h,S}(u, v) = 2\nu(e(u), e(v))_{T_h} + a_I^z(u, v) + a_I^v(u, v),
\]
\[
a_I^z(u, v) = a_I^z(v, u) - a_I^z(v, u) - a_I^z(u, v).
\]
The interior face terms \(a_I^z(u, v), a_I^v(u, v)\) and Nitsche terms are defined as:
\[
a_I^z(u, v) = 2\nu(\langle e(u) \rangle, \|v \otimes n\|_{E(T_h)}), \quad a_I^v(u, v) = \langle \sigma^2_h u \otimes n, \|v \otimes n\|_{E(T_h)} \rangle, \quad a_I^v(u, v) = 2(\sigma^2_h u, v)_{E(T_h)},
\]
where \(\sigma^2_h = \frac{\sigma^2}{h_E} \). Here \(h_E\) is suitably defined mesh size on the edge \(E\) and \(\sigma\) is an penalty parameter which is chosen so as to guarantee the stability of the DG formulation.

3. A posteriori error estimates

3.1. A posteriori error estimator

First we define a local error indicators \(\eta_{S,K}\) and \(\eta_{D,K}\) for each \(K \in T_{S,h}\) and \(K \in T_{D,h}\), respectively. The local error indicators \(\eta_{S,K}\) and \(\eta_{D,K}\) are the sum of the following terms:
\[
\eta_{S,K}^2 = \eta_{S,RK}^2 + \eta_{S,EX}^2 + \eta_{S,JK}^2; \quad \eta_{D,K}^2 = \eta_{D,RK}^2 + \eta_{D,EX}^2,
\]
where
\[
\eta_{S,RK}^2 = h_K^2 \|f_{S,h} + \nabla(2\nu e(u_{S,h})) - \nabla p_{S,h} \|^2_{0,K},
\]
\[
\eta_{S,EX}^2 = \frac{1}{2} \sum_{E \in \partial K \cap F} h_E \|\left(p_{S,h} - 2\nu e(u_{S,h})\right) \cdot n \|^2_{0,E},
\]
\[
\eta_{S,JK}^2 = \frac{1}{2} \sum_{E \in \partial K \cap F} \sigma^2_h \|u_{S,h} \otimes n\|^2_{0,E} + \sum_{E \in \partial K \cap E} \sigma^2_h \|u_{S,h}\|^2_{0,E},
\]
\[
\eta_{D,RK}^2 = h_K^2 \|\text{curl}(K^{-1} u_{D,h})\|^2_{0,K} + h_K^2 \|K^{-1} u_{D,h} + \nabla p_D\|^2_{0,K} + \|\nabla \cdot u_{D,h} - f\|^2_{0,K},
\]
\[
\eta_{D,EX}^2 = \frac{1}{2} \sum_{E \in \partial K \cap F} h_E \|K^{-1} (u_{D,h} \otimes n)\|^2_{0,E}.
\]
Next we define the interface estimator
\[ \eta_{I,E_k}^2 = \sum_{E \in \Omega_k} h_E \| | p_{S,E} - 2 \nu \varepsilon(u_{S,E}) n + p_{D,E} n + \gamma K^{-1/2}(u_{S,E} \cdot \tau) \| |^2. \]

Now, we introduce a data oscillation term
\[ \Theta_{S,K}^2 = h_k^2 \| | \mathcal{S} - f_{S,K} \| |_{0,K}^2. \]

Finally, we introduce the a posteriori error estimator
\[ \eta = \left( \sum_{k \in F_{I,h}} \eta_{S,K}^2 + \sum_{k \in F_{D,h}} \eta_{D,K}^2 + \sum_{k \in F_{I,h}} \eta_{I,E_k}^2 \right)^{1/2}. \]  

and the data oscillation error
\[ \Theta_S = \left( \sum_{k \in F_{I,h}} \Theta_{S,K}^2 \right)^{1/2}. \]

3.2. Reliability and efficiency

In this section, we analyse a reliable and efficient energy norm error estimator for a strongly conservative finite element method of Stokes-Darcy coupling equation. Proofs of the following results are collected in the next section. The symbols \( \lesssim \) and \( \gtrsim \) are used to denote the bounds which are valid up to positive constants independent of the local mesh size.

In first main result, we present that the estimator (3.1) gives rise to a reliable a-posteriori error bound.

**Theorem 3.1.** Let \((u, p)\) be the solution of problem and \((u_h, p_h) \in V_h \times Q_h\) the DG approximation obtained by (2.9). Let then \( \eta \) and \( \Theta_S \) be the error estimator and the data approximation term in (3.1) and (3.2), respectively. Then we obtain the following a posteriori error bound
\[ \| | u - u_h, p - p_h \| | \lesssim \eta + \Theta_S. \]  

In next theorem, we present a lower bound for the error and show the efficiency of the error estimator (3.1).

**Theorem 3.2.** Let \((u, p)\) be the solution of problem and \((u_h, p_h) \in V_h \times Q_h\) the DG approximation obtained by (2.9). Let the \( \eta \) and \( \Theta_S \) be the error estimator and the data approximation term in (3.1) and (3.2), respectively. Then we obtain the efficiency estimate
\[ \eta \lesssim \| | u - u_h, p - p_h \| | + \Theta_S. \]  

4. Proofs

In this section, we discuss the detailed proofs of Theorems 3.1 and 3.2.

4.1. Auxiliary forms and their properties

In this subsection, we discuss the properties of the DG forms from [6, 11]. The DG form \( a_h(u, v) \) is not well defined for the function \( u, v \) which belong to \( V \). This difficulty has been addressed by the use of a suitable lifting operator, see for instance [11, 37]. Here, we discuss a different and new approach where we split the discontinuous Galerkin form into several parts and avoid the continuity estimate on the consistency and symmetrization terms completely.

First, we define the following form
\[ D_h(u, v) = \sum_{K \in T_{I,h}} \int_K 2 \nu \varepsilon(u) \cdot \varepsilon(v) \, dx + \sum_{K \in T_{D,h}} \int_K K^{-1} \mathbf{u} \cdot v \, dx, \]
\[ O_h(u, v) = \gamma K^{-1/2} \sum_{E \in T_{I,E}} \int_E (u_S \cdot \tau)(v_S \cdot \tau) \, ds \]
\[ K_h(u, v) = - \sum_{E \in T_{I,E}} 2 \nu \int_E \| | \varepsilon(u) \| | \cdot \| | v \otimes n \| | \, ds \]
\[ J_h(u, v) = \sum_{E \in T_{I,E}} \sigma_h \int_E \| | u \otimes n \| | \cdot \| | v \otimes n \| | \, ds \]
\[ B_h(u, p) = - \sum_{K \in T_{D,h}} \int_K (\nabla \cdot u) \cdot p \, dx. \]
Choose $\tilde{A}_h(u, v)$ as
\[
\tilde{A}_h(u, v) = D_h(u, v) + J_h(u, v) + O_h(u, v).
\] (4.1)
The above form $\tilde{A}_h(u, v)$ is well defined for all $u, v \in V(h) = V_h + V$. Moreover, it holds
\[
\mathcal{A}_h(u, p; v, q) = \tilde{A}_h(u, v) + \mathcal{K}_h(u, v) + B_h(v, p) + B_h(u, q)
\]
Next, we prove that the auxiliary forms defined in (4.1) are continuous.

**Lemma 4.1.** The estimate
\[
|D_h(u, v)| \lesssim ||u|| ||v||, \quad u, v \in V(h),
\]
\[
|J_h(u, v)| \lesssim ||u|| ||v||, \quad u, v \in V(h),
\]
\[
|O_h(u, v)| \lesssim ||u|| ||v||, \quad u, v \in V(h),
\]
holds.

**Proof.** All bounds directly follow from the Cauchy-Schwarz inequality and trace theorem. \hfill \Box

Now, we prove some results for the auxiliary forms $A_h(u, v)$ and $B_h(u, p)$.

**Lemma 4.2.** There holds
\[
\tilde{A}_h(u, v) \lesssim ||u|| ||v||, \quad u, v \in V(h),
\]
\[
\tilde{A}_h(u, u) \gtrsim ||u||^2, \quad u \in V,
\]
\[
|B_h(u, p)| \lesssim ||u|| ||p||, \quad u \in V(h), p \in L^2(\Omega).
\] (4.4)

**Proof.** The first bound directly follows from Lemma 4.1. Next we prove the second bound. Here
\[
\tilde{A}_h(u, u) = D_h(u, u) + J_h(u, u) + O_h(u, u), \quad \forall u \in V,
\]
\[
= D_h(u, u) + O_h(u, u), \quad \forall u \in V,
\]
\[
\gtrsim ||u||^2, \quad \forall u \in V,
\]
The third bound directly follows from the consequence of the Cauchy-Schwarz inequality. \hfill \Box

**Lemma 4.3.** Let $u \in V_h$ and $v \in V \cap V_h$, then it holds:
\[
\mathcal{K}_h(u, v) \lesssim \sigma^{-1} \left( \sum_{E \in \mathcal{E}(\Gamma_{1,h})} \frac{1}{h_E} \int_E |\nabla v|^2 \right)^{1/2} ||v||.
\]

**Proof.** Here $v \in V \cap V_h$, then
\[
\mathcal{K}_h(u, v) = -\sum_{E \in \mathcal{E}(\Gamma_{1,h})} 2v \int_E \langle [\varepsilon(v)], [[u \times n]] \rangle ds.
\]
Applying Cauchy-Schwarz inequality, implies
\[
\mathcal{K}_h(u, v) \lesssim \sigma^{-1} \left( \sum_{E \in \mathcal{E}(\Gamma_{1,h})} \frac{1}{h_E} \int_E |\nabla v|^2 \right)^{1/2} \left( \sum_{E \in \mathcal{E}(\Gamma_{1,h})} \int_E \sigma_h^2 ||u \otimes n||^2 ds \right)^{1/2}
\]
Using the inverse estimate
\[
||v||^2_{0,h,K} \lesssim h_K^{-1/2} ||v||_{0,K}, \quad v \in V \cap V_h
\]
and shape-regularity of mesh, it follows:
\[
\mathcal{K}_h(u, v) \lesssim \sigma^{-1} \left( \sum_{k \in \mathcal{T}_h} ||v||^2_{0,k} \right)^{1/2} \left( \sum_{k \in \mathcal{T}_h} \int_E \sigma_h^2 ||u \otimes n||^2 ds \right)^{1/2}
\]
\hfill \Box

**Lemma 4.4.** For any $(u, p) \in V \times L^2_0(\Omega)$, there is $(v, q) \in V \backslash \{0\} \times L^2_0(\Omega)$ with $||v|| < 1$ and
\[
\mathcal{A}_h(u, p; v, q) \gtrsim ||(u, p)||
\]

**Proof.** Proof directly follows from the inf-sup results of [11, 30]. \hfill \Box
4.2. Approximation operators

First we define the discontinuous RT space \( \overline{V}_h = \{ v \in L^2(\Omega) : v|_K \in RT_1(K), K \in T_{S,h} \} \). From [11], we define \( V_h^+ = \overline{V}_h \cap V \). The orthogonal complement of \( V_h^+ \) in \( \overline{V}_h \) with respect to the norm \( \| \cdot \| \) is defined by \( V_h^- \). Then we obtain \( \overline{V}_h = V_h^+ \oplus V_h^- \) with \( V_h^K|_{\partial \Omega} = V_h|_{\partial \Omega} \).

Next we define an approximation operator \( \omega_h : \overline{V}_h \to V_h^+ \) (for more details, see [15] and Appendix A of [29]). Then from Proposition 5.4 of [15] and Appendix A of [29], the following estimates hold.

**Lemma 4.5.** For any \( v \in V_h \subset \overline{V}_h \) with \( v_S \cdot n = v_D \cdot n \) on \( \Gamma_D \), the estimates

\[
\sum_{K \in T_{S,h}} \| v - \omega_h v \|_{0,K}^2 \lesssim \sum_{K \in E(T_{S,h})} h_K \int_E \| v \|^2\,ds
\]

\[
\sum_{K \in T_{S,h}} \| \nabla (v - \omega_h v) \|_{0,K}^2 \lesssim \sum_{K \in E(T_{S,h})} h_K^{-1} \int_E \| v \|^2\,ds
\]

holds.

Define \( v \in V \) with \( v|_{\partial \Omega} = v_S \) and \( v|_{\partial \Omega} = v_D \). Using a Helmholtz decomposition, we can write \( v_D \) as follows:

\[
v_D = w + \text{curl } \phi.
\]

From [35], \( w \in H^1(\Omega_D) \), which satisfies

\[
\nabla \cdot w = \nabla \cdot v_D, \quad w|_{\Gamma_D} = 0, \quad w|_{\Gamma_I} = v_S|_{\Gamma_I}.
\]

Moreover, it follows:

\[
\| w \|_{1,\Omega_D} \lesssim \| v \|_V.
\]

It is clear from equation (4.5) and (4.6) that \( w \cdot n|_{\Gamma_D} = 0 \) and \( w \cdot n|_{\Gamma_I} = v_S \cdot n|_{\Gamma_I} \). Moreover, we have \( w \cdot n|_{\partial \Omega} = v_D \cdot n|_{\partial \Omega} \) and the following compatibility condition holds:

\[
\int_{\partial \Omega} \nabla \cdot w \, ds = \int_{\partial \Omega} \nabla \cdot v_D \, ds = \int_{\partial \Omega} v_S \cdot n \, ds = \int_{\partial \Omega} w \cdot n \, ds
\]

Here \( (v_D - w)|_{\partial \Omega} \in \mathbf{H}^{0,\text{div}}(\Omega_D) \) is divergence-free. Hence there exists a function \( \phi \in H^1(\Omega_D) \) such that

\[
\text{curl } \phi = v_D - w \quad \text{and} \quad \| \phi \|_{1,\Omega_D} \lesssim \| v \|_V.
\]

Next we define \( \overline{v} \) such that

\[
\overline{v} = \begin{cases} 
  v_S & \text{in } \Omega_S, \\
  0 & \text{in } \Omega_D,
\end{cases}
\]

with \( \| \overline{v} \| = 0 \) on \( \Gamma_I \). (using (4.6))

Hence \( \overline{v} \in H^1(\Omega) \). Then there exists an operator Scott-Zhang interpolant \( I_h \) (see [9, 11]) such that

\[
\sum_{K \in T_{S,h}} (h_K^2 \| \overline{v} - I_h \overline{v} \|_{0,K}^2 + \| \overline{v} - I_h \overline{v} \|_{1,K}^2) \lesssim \| \nabla \overline{v} \|_{0,\Omega}^2 \lesssim \| v \|_V
\]

and

\[
\sum_{E \in E(T_{S,h})} h_E^{-1} \| \overline{v} - I_h \overline{v} \|_{0,E}^2 \lesssim \| \nabla \overline{v} \|_{0,\Omega}^2 \lesssim \| v \|_V.
\]

Moreover, we have

\[
\sum_{K \in T_{S,h}} (h_K^2 \| v_S - I_{S,h} v_S \|_{0,K}^2 + \| v_S - I_{S,h} v_S \|_{1,K}^2) \lesssim \| v \|_V,
\]

\[
\sum_{E \in E(T_{S,h})} h_E^{-1} \| v_S - I_{S,h} v_S \|_{0,E}^2 \lesssim \| v \|_V.
\]

where \( I_{S,h} = I_h|_{\partial \Omega} \) and \( I_{D,h} = I_h|_{\partial \Omega} \). Similarly, there exists an operator Scott-Zhang interpolant \( I_{D,h} \) such that

\[
\sum_{K \in T_{D,h}} (h_K^2 \| \phi - I_{D,h} \phi \|_{0,K}^2 + \| \phi - I_{D,h} \phi \|_{1,K}^2) \lesssim \| \nabla \phi \|_{0,\Omega}^2 \lesssim \| v \|_V.
\]
4.3. Proof of Theorem 3.1

For $\Omega_\delta$, we decompose the DG velocity approximation uniquely into

$$u_{S,h} = u_{S,h}^c + u_{S,h}^c,$$

where $u_{S,h}^c \in V_h^c$ and $u_{S,h}^c \in V_h^c$. Using triangle inequality, we can write

$$\|u_{S} - u_{S,h}\|_{1,h} \leq \|u_{S} - u_{S,h}^c\|_{1,h} + \|u_{S,h}^c\|_{1,h}$$

(4.15)

**Lemma 4.6.** The estimate

$$\|u_{S,h}^c\|_{1,h} \lesssim \eta$$

holds.

**Proof.** Here $\|u_{S,h}^c\| = \|u_{S,h}\|$, then we obtain

$$\|u_{S,h}^c\|_{1,h} = \nu \sum_{K \in T_{h \delta}} \|\nabla u_{S,h}^c\|^2_{0,K} + \sum_{E \in E(T_{h \delta})} \sigma h \|\nabla u_{S,h}\|_{0,E}^2 + \sum_{E \in E^D(T_{h \delta})} 2\sigma h \|u_{S,h}\|_{0,E}^2.$$  

Moreover, we have by definition of $\eta$:

$$\|u_{S,h}^c\|_{1,h} \lesssim \nu \sum_{K \in T_{h \delta}} \|\nabla u_{S,h}^c\|^2_{0,K} + \sum_{K \in T_{h \delta}} \eta^2_{S,J_k}.$$  

Using Lemma 4.5, the following estimates hold:

$$\nu \sum_{K \in T_{h \delta}} \|\nabla u_{S,h}^c\|^2_{0,K} \lesssim \nu \sum_{E \in E(T_{h \delta})} \frac{1}{h_E} \|\nabla u_{S,h}\|_{0,E}^2 = \sum_{E \in E(T_{h \delta})} \frac{\nu h}{h_E \sigma^2} \|u_{S,h}\|_{0,E}^2 \lesssim \sigma^{-2} \sum_{K \in T_{h \delta}} \eta^2_{S,J_k}. $$

**Lemma 4.7.** The estimate

$$\int_{\Omega_\delta} f_{S} (v - I_{h}v) dx - \tilde{A}_{h}(u_{h}, v - I_{h}v) - B_{h}(p_{h}, v - I_{h}v) \lesssim (\eta + \Theta_\delta) \|v\|_h,$$

holds for any $v \in V$. Here $I_{h}$ is defined as follows:

$$I_{h} = \begin{cases} 
I_{S,h}v & \text{in } \Omega_\delta, \\
I_{D,h}w + I_{D,h} \phi & \text{in } \Omega_D, 
\end{cases}$$

(4.18)

where $I_{S,h}, I_{D,h}$ and $I_{D,h}$ are defined in (4.10)-(4.14).

**Proof.** Define

$$T = \int_{\Omega_\delta} f_{S} (v - I_{h}v) dx - \tilde{A}_{h}(u_{h}, v - I_{h}v) - B_{h}(p_{h}, v - I_{h}v).$$

Using integration by part, we have

$$T = \sum_{K \in T_{h \delta}} \int_{\Omega_\delta} V_{h} (v - I_{h}v) dx - \sum_{K \in T_{h \delta}} \int_{\partial K} (2v S_{h}) \nabla u_{S,h} (v - I_{h}v) dx + \int_{T_{h \delta}} (p_{D,h}) \nabla u_{S,h} (v - I_{h}v) dx$$

$$- \sum_{K \in T_{h \delta}} \int_{\Omega_\delta} \nabla p_{D,h} (v - I_{h}v) dx.$$  

$$= T_1 + T_2 + T_3$$  

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Adding and subtracting the data approximation terms in $T_1$, then

$$T_1 = \sum_{k \in \mathcal{T}_{o,s}} \int K^{-1} u_{D,h}(v - I_h v) dx - \sum_{k \in \mathcal{T}_{o,s}} \int K^{-1} u_{D,h} \cdot \tau(\phi - I_{D,h} \phi) dx.$$

Using $v_D = w + \text{curl}(\phi)$ and $I_h v|_{\partial D} = I_{D,h} w + \text{curl}(I_{D,h} \phi)$, then

$$T_3 = - \sum_{k \in \mathcal{T}_{o,s}} \int_K (K^{-1} u_{D,h} + \nabla p_{D,h})(w - I_{D,h} w) dx - \sum_{k \in \mathcal{T}_{o,s}} \int_K K^{-1} u_{D,h} \text{curl}(\phi - I_{D,h} \phi) dx - \sum_{k \in \mathcal{T}_{o,s}} \int_K \nabla p_{D,h}(w - I_{D,h} w) dx.$$

Using integration by parts, we obtain

$$T_3 = - \sum_{k \in \mathcal{T}_{o,s}} \int_K (K^{-1} u_{D,h} + \nabla p_{D,h})(w - I_{D,h} w) dx - \sum_{k \in \mathcal{T}_{o,s}} \int_K K^{-1} u_{D,h} \cdot \tau(\phi - I_{D,h} \phi) dx + \sum_{k \in \mathcal{T}_{o,s}} \int_K \text{curl}(K^{-1} u_{D,h})(\phi - I_{D,h} \phi) dx.$$

Applying Cauchy-Schwarz inequality and (4.10)-(4.14), gives

$$T_3 \lesssim \left( \sum_{k \in \mathcal{T}_{o,s}} \eta_{3,R,k}^2 \right)^{1/2} \left\| \nabla \phi \right\|.$$

Combining the above estimate, imply the desired result.

**Lemma 4.8.** The estimate

$$\| (u - u_{D,h}^\circ, p - p_h) \| \lesssim \eta + \Theta_S$$

holds.
Proof. From Lemma 4.4, we have

\[ ||(u - u_h^i, p - p_h)|| \lesssim A_h(u - u_h^i, p - p_h; v, q) \]

with \( ||(v, q)|| \leq 1 \).

Here

\[ A_h(u - u_h^i, p - p_h; v, q) = \int_{\Omega_e} f v \, dx - \int_{\Omega_p} f q \, dx - A_h(u_h^i, p; v, q) \]

where

\[ A_h(u_h^i, p; v, q) = \tilde{A}_h(u_h^i, v) + B_h(p_h, v) + B_h(q_h, u_h^i) \]

Now, we decompose \( A_h(u - u_h^i, p - p_h; v, q) \) into following parts which are as follows:

\[ A_h(u - u_h^i, p - p_h; v, q) = T_1 + T_2 + T_3 \]

where

\[
\begin{align*}
T_1 &= \int_{\Omega_e} f_3 (v - Iv) \, dx - \tilde{A}_h(u_h, v - L_h v) - B_h(p_h, v - Iv), \\
T_2 &= D_h(u_{S,h}^i, v) + J_h(u_{S,h}^i, v) - B_h(q_h, u_{S,h}^i), \\
T_3 &= -K_h(u_{S,h}, Iv), \\
T_4 &= \int_{\Omega_e} f_5 q \, dx - B_h(q_h, u_{D,h}).
\end{align*}
\]

where \( Iv \) is defined in (4.18).

Using Lemma 4.7, we have

\[ T_1 \lesssim (\eta + \Theta_S)||v|| \]

Applying Continuity result from Lemma and Lemma 4.6, it follows

\[ T_2 \lesssim \eta||v|| \]

Using Lemma 4.3 for the bound of \( T_3 \), we have

\[ T_3 \lesssim \gamma^{-1/2} \left( \sum_{k \in I_{S,k}} \eta_{S,K}^2 \right)^{1/2} ||v||. \]

Using Cauchy-Schwarz inequality in \( T_4 \), we obtain

\[ T_4 \lesssim \left( \eta_{D,K}^2 \right)^{1/2} ||q||_{H^1}. \]

\[ \square \]

4.4. Proof of Theorem 3.2

In this section, we are proving the efficiency bounds for \( \eta_{S,R_k}, \eta_{D,R_k}, \eta_{S,J_k}, \eta_{S,E_k} \) and \( \eta_{I,E_k} \). The bubble function technique is used to prove the theorem.

Lemma 4.9. Let \( K \) be the element of \( T_h \). Let \( \chi_K \) denotes the standard polynomial bubble function on \( K \) (see [4, 2, 11]). Then the following estimates hold:

\[
\begin{align*}
||\chi_K v||_{L^0(K)} &\lesssim ||v||_{L^0(K)}, \\
||v||_{L^2(K)} &\lesssim ||\chi_K v||_{L^0(K)}, \\
||\nabla(\chi_K v)||_{L^0(K)} &\lesssim h_K^2 ||v||_{L^0(K)}, \\
||\chi_K v||_{L^1(K)} &\lesssim h_K^2 ||v||_{L^0(K)}.
\end{align*}
\]

where \( v \) denotes a vector valued polynomial function on \( K \).
Lemma 4.10. There holds

\[
\left( \sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \right)^{1/2} \lesssim \|(u - u_h, p - p_h)\|.
\]

Proof. Using \(\|u\| = 0\), then

\[
\eta_{S,K}^2 = \frac{1}{2} \sum_{E \in \partial K} \alpha_{E}^2 \|\|u_{S,h}\|\|_{0,E}^2 + \sum_{E \in \partial K} \alpha_{E}^2 \|\|u_{S,h}\|\|_{0,E}^2.
\]

Moreover, we have

\[
\left( \sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \right)^{1/2} \lesssim \|(u - u_h, p - p_h)\|.
\]

\qed

Lemma 4.11. There holds

\[
\left( \sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \right)^{1/2} \lesssim \|(u - u_h, p - p_h)\| + \Theta_S.
\]

Proof. Choose \(K\) as a element in \(T_{\text{S,h}}\), then

\[
R_{Ih} = (f_{S,h} + \nabla(2\psi(u_{S,h})) - \nabla p_{S,h})|_{K}
\]

Define \(W_{|_{K}} = h_{K}^2 R_{X,K}\). Using Lemma 4.9, we obtain

\[
\sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \lesssim \|(R, h_{K}^2 R_{X,K})|_{T_{\text{S,h}}} = (R, W)|_{T_{\text{S,h}}} = (f_{S,h} + \nabla(2\psi(u_{S,h})) - \nabla p_{S,h}, W)|_{T_{\text{S,h}}}.
\]

Note that \((f_s + \nabla(2\psi(u_s)) - \nabla p_s)|_{K} = 0\) for exact solution \((u, p)\). Using integration by parts and add the exact problem in (4.23), we have

\[
\sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 = (2\psi(u_s - u_{S,h}), \nabla W)|_{T_{\text{S,h}}} + (p_{S,h} - p_s, \nabla : W)|_{T_{\text{S,h}}} + (f_{S,h} - f_s, W)|_{T_{\text{S,h}}}.
\]

Applying Cauchy-Schwarz inequality in (4.24), implies

\[
\sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \lesssim \left( \|(u - u_h, p - p_h)\| + \Theta_S \right) \left( \|W\|^2 + h_{K}^2 \|W\|^2_{T_{\text{S,h}}} \right)^{1/2}.
\]

Using Lemma 4.9 in (4.25), it follows:

\[
\sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \lesssim \left( \|(u - u_h, p - p_h)\| + \Theta_S \right) \left( \sum_{K \in T_{\text{Ih}}} \eta_{S,K}^2 \right)^{1/2}.
\]

\qed

Lemma 4.12. There holds

\[
h_{K}^2 \|\text{curl}(K^{-1}u_{D,h})\|_{0,K}^2 \lesssim \|u_{D} - u_{D,h}\|_{0,K}^2
\]

Proof. See Lemma 4.8 of [17] or Lemma 6.1 of [34].

Lemma 4.13. The estimate holds:

\[
\left( \sum_{K \in T_{\text{Ih}}} \eta_{D,K}^2 \right)^{1/2} \lesssim \|K^{-1/2}(u_{D} - u_{D,h})\|_{T_{\text{S,h}}} + \|\nabla \cdot (u_{D,h} - u_D)\|_{T_{\text{S,h}}} + \|p_D - p_{D,h}\|_{T_{\text{S,h}}}.
\]

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Proof. Choose $K$ as an element in $T_{2,h}$, then

$$R|_K = (K^{-1}u_{D,h} + \nabla p_{D,h})|_K$$

Define $W|_K = h_{K}^2 R_K$. Using Lemma 4.9, we obtain

$$\sum_{K \in T_{2,h}} \eta_{D,Rx}^2 = \sum_{K \in T_{2,h}} h_K^2 ||R||^2_{L^2,0,K} \lesssim (R, h_K^2 R_K R_{T_{2,h}})_{T_{2,h}} = (R, W)_{T_{2,h}} = (K^{-1}u_{D,h} + \nabla p_{D,h}, W)_{T_{2,h}}. \tag{4.26}$$

Note that $(K^{-1}u_D + \nabla p_D)|_K = 0$ for exact solution $(u, p)$. Using integration by parts and add the exact problem in (4.26), we have

$$\sum_{K \in T_{2,h}} h_K^2 ||R||^2_{L^2,0,K} = (K^{-1}(u_D - u_{D,h}), W)_{T_{2,h}} + (p_{D,h} - p_D, \nabla \cdot W)_{T_{2,h}}. \tag{4.27}$$

using $W|_K = 0$. Applying Cauchy-Schwarz inequality in (4.27), implies

$$\sum_{K \in T_{2,h}} h_K^2 ||R||^2_{L^2,0,K} \lesssim (||K^{-1/2}(u_D - u_{D,h})||_{T_{2,h}} + ||p_{D,h} - p_D||_{T_{2,h}})(||K^{-1/2}W||_{T_{2,h}} + ||\nabla \cdot W||_{T_{2,h}}). \tag{4.28}$$

Using Lemma 4.9 in (4.28), it follows:

$$\sum_{K \in T_{2,h}} h_K^2 ||R||^2_{L^2,0,K} \lesssim \left( ||K^{-1/2}(u_D - u_{D,h})||_{T_{2,h}} + ||p_{D,h} - p_D||_{T_{2,h}} \right) \left( \sum_{K \in T_{2,h}} h_K^2 ||R||^2_{L^2,0,K} \right)^{1/2}. \tag{4.29}$$

Next, we obtain

$$||\nabla \cdot u_{D,h} - f||_{L^2,0,K} \lesssim ||\nabla \cdot (u_{D,h} - u_D)||_{L^2,0,K}. \tag{4.30}$$

Combining Lemma 4.12 and the above estimate, gives the desired result.

Let $E$ be denotes as the interior edge which is shared by two elements $K$ and $K'$. Now we define the standard polynomial bubble function $\chi_E$ on $E$. Next, we define $\delta_E = \{K, K'\}$. In case of regular edge $E$, we choose $K = K'$. When one vertex of $E$ is a hanging node, then we choose that $E$ is an entire edge of $K$. Moreover we define $\tilde{K} \subset K'$ as the largest rectangle contained in $K'$. Therefore $E$ is one of the entire edge of $\tilde{K}$.

Now we define $\delta_E = \{K, \tilde{K}\}$. While $\sigma$ is a vector-valued polynomial function on $E$, then

$$||\sigma||_{L^2,E} \lesssim ||\chi_E^{1/2} \sigma||_{L^2,0,E}$$

Moreover we can define an extension $\sigma_E \in H^1_0(K \cap \tilde{K})$ such that $\sigma_E|_E = \chi_E \sigma$ and from [4, 2, 11], we have

$$||\sigma||_{L^2,E} \lesssim h_E^{1/2} ||\sigma||_{L^2,0,E}, \tag{4.31}$$

$$||\nabla \sigma||_{L^2,E} \lesssim h_E^{1/2} ||\sigma||_{L^2,0,E}, \tag{4.32}$$

$$||\sigma||_{L^2(K \cap \tilde{K})} \lesssim h_E^{1/2} ||\sigma||_{L^2,0,E}. \tag{4.33}$$

Here $\sigma_E$ can extend by zero outside the patch formed by the union of $K$ and $\tilde{K}$.

Lemma 4.14. The estimate

$$\left( \sum_{K \in T_{3,h}} \eta_{S,E}^2 \right)^{1/2} \lesssim \|(u - u_h, p - p_h)\| + \Theta_5.$$

holds.

Proof. Define

$$\Lambda = \sum_{E \in \mathcal{T}(T_{3,h})} h_E \|p_{S,E} - 2 \alpha \sigma(u_{S,h})\|_{L^2,E}.$$

Proof. Define
Note that the solution \((u, p)\) of the Stokes-Darcy coupling problem satisfies
\[
\|p_S I - 2\nu e(u_S)\|_E = 0.
\]

Then we have
\[
\sum_{K \in T_h} \eta_{S, E_k}^2 \lesssim \sum_{E \in \{T_h\}} \|\| p_S I - 2\nu e(u_S) \| \Lambda_E = \sum_{E \in \{T_h\}} \|\| p_S I - 2\nu e(u_S) \| - \| p_S I - 2\nu e(u_S) \| \Lambda_E.
\]

Applying Green’s formula over each the two element of \(w_E\), implies
\[
\sum_{E \in \{T_h\}} \|\| p_S I - 2\nu e(u_S) \| - \| p_S I - 2\nu e(u_S) \| \Lambda_E
\]
\[
= \sum_{E \in \{T_h\}} \sum_{K \in \partial E} \left( \int_K (-\nabla \cdot (2\nu (e(u_S) - e(u_S))) + \nabla (p_S - p_S) \Lambda) \right) dx
\]
\[
+ \int_K (-2\nu (e(u_S) - e(u_S))) + (p_S - p_S) \mathbf{I} : \nabla \Lambda dx.
\]

Using the fact that \((u, p)\) solves the problem, gives
\[
\sum_{K \in \partial E} \sum_{K \in \partial E} \left( \int_K (f_S + \nabla (2\nu e(u_S))) - \nabla p_S) \Lambda \right) dx
\]
\[
+ \sum_{E \in \{T_h\}} \sum_{K \in \partial E} \int_K (f_S - f_S) \cdot \Lambda dx
\]
\[
+ \sum_{E \in \{T_h\}} \sum_{K \in \partial E} \left( \int_K (-2\nu (e(u_S) - e(u_S))) + (p_S - p_S) \mathbf{I} : \nabla \Lambda dx, \right)
\]
\[
\lesssim T_1 + T_2 + T_3.
\]

Using the Cauchy-Schwarz inequality, shape-regularity of the mesh, Lemma 4.11 and \(\| \cdot \|_{0, K} \lesssim \| \cdot \|_{0, K'}\), it follows:
\[
T_1 \lesssim (\|\| u - u_h, p - p_h \| \| + \Theta_S \left( \sum_{E \in \{T_h\}} \sum_{K \in \partial E} h_E^2 \| \Lambda \|_{0, K}^2 \right)^{1/2}.
\]

From inequality (4.31), we have
\[
\left( \sum_{E \in \{T_h\}} \sum_{K \in \partial E} h_E^2 \| \Lambda \|_{0, K}^2 \right)^{1/2} \lesssim \left( \sum_{K \in \partial E} \eta_{S, E_k}^2 \right)^{1/2}
\]

Hence, the following estimate holds
\[
T_1 \lesssim (\|\| u - u_h, p - p_h \| \| + \Theta_S \left( \sum_{K \in \partial E} \eta_{S, E_k}^2 \right)^{1/2}.
\]

Applying Cauchy-Schwarz inequality, shape-regularity of the mesh, \(\| \cdot \|_{0, K} \lesssim \| \cdot \|_{0, K'}\) and inequality (4.31) in \(T_2\), implies
\[
T_2 \lesssim \Theta_S \left( \sum_{K \in \partial E} \eta_{S, E_k}^2 \right)^{1/2}.
\]

Now we estimate \(T_4\) which is as follows:
\[
T_3 = \sum_{E \in \{T_h\}} \sum_{K \in \partial E} \left( \int_K (-2\nu (e(u_S) - e(u_S))) + (p_S - p_S) \mathbf{I} : \nabla \Lambda dx.
\]

Again using Cauchy-Schwarz inequality, shape-regularity of the mesh, \(\| \cdot \|_{0, K} \lesssim \| \cdot \|_{0, K'}\) and inequality (4.32) , we obtain
\[
T_3 \lesssim (\|\| u - u_h, p - p_h \| \| ( \sum_{K \in \partial E} \eta_{S, E_k}^2 \right)^{1/2}
\]

Combining the above estimates \(T_1, T_2\) and \(T_3\) the desire result holds. \(\Box\)
Lemma 4.15. The estimate holds:

$$h_E \| K^{-1}(u_{DA} \otimes n) \|_{0,E}^2 \lesssim \sum_{K \in \mathcal{T}_h} \| u_D - u_{DA} \|_{0,K}^2$$

Proof. See Lemma 4.9 of [17] or Lemma 6.2 of [34].

Lemma 4.16. The estimate holds:

$$\left( \sum_{K \in \mathcal{T}_h} \eta_{I,E}^2 \right)^{1/2} \lesssim \| u - u_h \| + \nu^{-1/2} \| p - p_h \|_{\mathcal{T}_h} + \Theta_e.$$

Proof. Define

$$\Lambda = \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} h_E (p_{S,h} n - 2\nu \varepsilon(u_{S,h}) n - p_{D,h} n + \gamma K^{-1/2}(u_{S,h} \cdot \tau) \xi_E).$$

Note that the solution \((u, p)\) of the problem satisfies

$$(p_S n - 2\nu \varepsilon(u_S) n - p_D n + \gamma K^{-1/2}(u_S \cdot \tau) \xi)_E = 0. \quad \forall E \in \mathcal{E}_h(\mathcal{T}_h).$$

Then we have

$$\sum_{K \in \mathcal{T}_h} \eta_{I,E}^2 \lesssim \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} ((p_{S,h} n - 2\nu \varepsilon(u_{S,h}) n - p_{D,h} n + \gamma K^{-1/2}(u_{S,h} \cdot \tau)^E), \Lambda)_{E}$$

$$= \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} ((p_{S,h} n - 2\nu \varepsilon(u_{S,h}) n - p_{D,h} n + \gamma K^{-1/2}(u_{S,h} \cdot \tau)^E)$$

$$- (p_S n - 2\nu \varepsilon(u_S) n - p_D n + \gamma K^{-1/2}(u_S \cdot \tau) \xi, \Lambda)_E$$

$$= \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} ((p_{S,h} n - 2\nu \varepsilon(u_{S,h}) n - (p_S n - 2\nu \varepsilon(u_S) n), \Lambda)_E + ((p_D n - p_{D,h} n), \Lambda)_E$$

$$- \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} ((\gamma K^{-1/2}(u_S \cdot \tau) \tau - (u_{S,h} \cdot \tau) \xi), \Lambda)_E$$

$$= T_1 + T_2 + T_3$$

Applying Green’s formula in \(T_1\) (But on those element which are parts of \(\mathcal{T}_{S,h}\), implies

$$T_1 = \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} \sum_{K \in \mathcal{T}_h} \int_K (f_{S,h} + \nabla (2\nu \varepsilon(u_{S,h}))) \cdot \nabla p_{S,h} \Lambda dx + \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} \sum_{K \in \mathcal{T}_h} \int_K (f_S - f_{S,h}) \cdot \Lambda dx$$

$$+ \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} \sum_{K \in \mathcal{T}_h} \int_K (-2\nu \varepsilon(u_S) - \varepsilon(u_{S,h})) + (p_S - p_{S,h}) I : \nabla \Lambda dx,$$

Then same as in Lemma 4.14, we obtain

$$T_1 \lesssim \left( \| (u - u_h, p - p_h) \| + \Theta_e \right) \left( \sum_{K \in \mathcal{T}_h} \eta_{I,E}^2 \right)^{1/2},$$

Using Cauchy-Schwarz inequality and shape-regularity of the mesh, we have

$$T_2 \lesssim \left( \sum_{E \in \mathcal{E}_h(\mathcal{T}_h)} \| p_D - p_{D,h} \|_{0,K}^2 \right) \left( \sum_{K \in \mathcal{T}_h} \eta_{I,E}^2 \right)^{1/2}.$$

Using trace inequality and the following inequality

$$h_K^2 \| p_D - p_{D,h} \|_{1,K}^2 \lesssim h_K^2 \| \nabla p_D - \nabla p_{D,h} \|_{0,K}^2 = h_K^2 \| K^{-1} u_D - \nabla p_{D,h} \|_{0,K}^2$$

$$\lesssim \left( h_K^2 \| u_D - u_{D,h} \|_{0,K}^2 + h_K^2 \| K^{-1} u_{D,h} + \nabla p_{D,h} \|_{0,K}^2 \right),$$

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it follows:

\[ T_2 \lesssim (\| u - u_h \|_{1,h} + \| p - p_h \|_{\mathbf{r}_h} + \Theta_S) \left( \sum_{K \in T_h} \eta^2_{h,k} \right)^{1/2}. \]

using Lemma 4.13.

Using Cauchy-Schwarz inequality and shape-regularity of the mesh, we have

\[ T_3 \lesssim (\| u - u_h \| \left( \sum_{K \in T_h} \eta_{h,k} \right)^{1/2}. \]

Combining the above estimate, the final result holds.

5. Numerical results

In this section, we show the practical applicability of the proposed numerical schemes and to validate the theoretical results obtained for the Stokes-Darcy coupling equation in two dimension. Moreover, we discussed the convergence behavior for uniform and adaptive meshes using a posteriori error estimator. Let \( N_\ell \) denotes the degrees of freedom (DOF) such that

\[ N_\ell := \dim(\mathbf{V}_h) + \dim(\mathbf{Q}_h). \]

For uniform meshes, \( O(N_\ell^{-r}) \approx O(h^{2r}), \ r > 0. \)

Our implementation of the proposed method is based on the dealii finite element library [38] and amandus [39].

In general, it is difficult to find analytic solution satisfying the interface conditions. To resolve this difficulty, we use the idea of generalizing the equations to include a nonhomogeneous term. Then

\[ p_{h,S} \mathbf{n} - 2\nu \mathbf{\varepsilon}(u_h) \mathbf{n} + p_{h,D} \mathbf{n} = -\gamma K^{-1/2}(u_1, \mathbf{\tau}) \mathbf{\tau} + g. \]

In all example, we choose \( \nu = 1, K = 1 \) and \( \gamma = 1. \)

5.1. Some simply constructed examples

In this example, we take \( \Omega_D := (0, 0.5) \times (0, 1) \) and \( \Omega_D := (0, 1)^2 \setminus \bar{\Omega}_D. \) To calculate the \( f_S \) and \( f_D, \) the exact solution is given by

**Case 1:**

\[ u_S = \begin{pmatrix} 0 \\ \frac{1}{2} - y_5 \end{pmatrix}, \quad u_D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p = 0. \]

This case is discussed in [19]. For \( \text{RT}_1 \times \mathbf{Q}_1 \) and \( \text{RT}_2 \times \mathbf{Q}_2 \) elements, the convergence results for uniform refinement meshes in Figure 1(a) and 1(b) show more than optimal order convergence for the error \( \| e_{\ell} \| \) and

![Figure 1](image-url)
the proposed estimator $\eta_\ell$ because of the structure of Raviart-Thomas polynomial and exact solution. In Figure 1(a) and 1(b), the convergence result based on adaptive refinement also achieve more the optimal convergence $O(N_{\ell}^{-k/2})$, $k = 1, 2$ for the error $|||e_\ell|||$ and in a posteriori error control $\eta_\ell$ because of the same reason. Figures 3 and 4 show the adaptivity refined meshes for $RT_k \times Q_k$ ($k = 1, 2$) elements. Moreover, Figures 3 and 4 show much refinement in Stokes part compare to Darcy part. It is expacted because we are calucating the error for Stokes Part in $|||u|||_{1,h}$ norm. Moreover, the graph of the estimates $\eta_\ell$ is parallel to the error in energy norm $|||e_\ell|||$ in adaptive mesh. Thus, it confirms that error estimator is numerically reliable and efficient. The graph of efficiency is given in Figure 2.

Case 2:

$$u_S = \begin{pmatrix} \cos(x_1 x_2) \\ e^{x_1 y} \end{pmatrix}, \quad u_D = \begin{pmatrix} \cos(x_1 x_2) \\ 0 \end{pmatrix}, \quad p = e^{x_1} \sin(x + y).$$
This case is also discussed in [19]. For $RT_1 \times Q_1$ and $RT_2 \times Q_2$ elements, the convergence results for

uniform refinement meshes in Figure 5(a) and 5(b) show the optimal order convergence for the error $\|\varepsilon\|$ and the proposed estimator $\eta$ because of smooth solution. In Figure 5(a) and 5(b), the convergence result based on adaptive refinement also achieve the optimal convergence $O(N^{-k/2})$, $k = 1, 2$ for the error $\|\varepsilon\|$ and in a posteriori error control $\eta$. Figures 3 and 4 show the adaptivity refined meshes for $RT_k \times Q_k (k = 1, 2)$ elements. Moreover, Figures 7 and 8 show much refinement in Stokes part compare to Darcy part. Moreover, the graph
of the estimates $\eta_\ell$ is parallel to the error in energy norm $\|e_\ell\|$ in adaptive mesh. Thus, it confirms that error estimator is numerically reliable and efficient. The graph of efficiency is given in Figure 6.

5.2. Porous medium completely surrounded by a fluid

In this example, we take $\Omega_S := (-0.5, 0.5)^2$ and $\Omega_D := (-1, 1)^2 \setminus \overline{\Omega}_S$. To calculate the $f_S$ and $f_D$, the exact solution is given by

$$u = \left( -2 \sin^2(\pi x_1) \sin(\pi x_2) \cos(\pi x_2), 2 \sin(\pi x_1) \sin^2(\pi x_2) \cos(\pi x_2) \right), \quad p = x_1^3 e^{x_2}. $$

This example is also discussed in [18]. For $RT_1 \times Q_1$ and $RT_2 \times Q_2$ elements, the convergence results for uniform refinement meshes in Figures 9(a) and 9(b) show the optimal order convergence for the error $\|e_\ell\|$ and the proposed estimator $\eta_\ell$ because of smooth solution. In Figures 9(a) and 9(b), the convergence result based on adaptive refinement also achieve the optimal convergence $O(N^{-k/2})$, $k = 1, 2$ for the error $\|e_\ell\|$ and in a posteriori error control $\eta_\ell$. Figures 3 and 4 show the adaptivity refined meshes for $RT_k \times Q_k$ ($k = 1, 2$) elements. Moreover, Figures 11 and 12 show much refinement around the corners. Moreover, the graph of the estimates $\eta_\ell$ is parallel to the error in energy norm $\|e_\ell\|$ in adaptive mesh. Thus, it confirms that error estimator is numerically reliable and efficient. The graph of efficiency is given in Figure 10.

5.3. Fluid completely surrounded by a porous medium

In this example, we take $\Omega_S := (-0.5, 0.5)^2$ and $\Omega_D := (-1, 1)^2 \setminus \overline{\Omega}_S$. To calculate the $f_S$ and $f_D$, the exact solution is given by

$$u = \left( -2 \sin^2(\pi x_1) \sin(\pi x_2) \cos(\pi x_2), 2 \sin(\pi x_1) \sin^2(\pi x_2) \cos(\pi x_2) \right), \quad p = x_1^3 e^{x_2}. $$
For $RT_1 \times Q_1$ and $RT_2 \times Q_2$ elements, the convergence results for uniform refinement meshes in Figures 13(a) and 13(b) show the optimal order convergence for the error $||e\|\|$ and the proposed estimator $\eta$ because of smooth solution. In Figures 13(a) and 13(b), the convergence result based on adaptive refinement also achieve the optimal convergence $O(N^{-k/2})$, $k = 1, 2$ for the error $||e\|\|$ and in a posteriori error control $\eta$. Figures 15 and 16 show the adaptivity refined meshes for $RT_1 \times Q_2(k = 1, 2)$ elements. Moreover, Figures 15 and 16 show much refinement around the corners. Moreover, the graph of the estimates $\eta$ is parallel to the error in energy norm $||e\|\|$ in adaptive mesh. Thus, it confirms that error estimator is numerically reliable and efficient. The graph of efficiency is given in Figure 14.

5.4. L-shape domain

In this example, we take $\Omega_D := (-1, 1) \times (-1, -0.5)$ and $\Omega_S := (-1, 1) \times (-0.5, 1) \setminus [0, 1]^2$. This example is based on the standard singular solution of the Stokes problem with $\nu = 1$ on an L-shaped domain (see [5, 15]).
For Darcy part, we choose same solution as Stokes part. The form of the singularity at the re-entrant corner are of $r^4$ and $r^{k-1}$ ($\lambda \approx 0.544$) for the velocity and pressure, respectively. For $RT_1 \times Q_1$ and $RT_2 \times Q_2$ elements, the convergence results for uniform refinement meshes in Figure 18(a) and 18(b) show low-order convergence $O(N^{-0.27})$ for the error $\|\varepsilon\|$ and the proposed estimator $\eta$. For uniform meshes, we can not expect convergence orders better than $\lambda$ in the energy norm $\|\cdot\|$. Moreover, the graph of the estimates $\eta$ is parallel to the error in energy norm $\|\varepsilon\|$. Thus, it confirms that error estimator is numerically reliable and efficient. In Figure 18(a) and 18(b), the convergence result based on adaptive refinement achieve the optimal convergence $O(N^{-k/2})$, $k = 1, 2$ for the error $\|\varepsilon\|$ and in a posteriori error control $\eta$. Figures 19(a) and 19(b) show the adaptivity refined meshes for $RT_k \times Q_k (k = 1, 2)$ elements. Moreover, Figures 19(a) and 19(b) show strong refinement toward the singularity.
5.5. Interface with corner

In this example, we take $\Omega_D := (-1,0)^2$ and $\Omega_S := (-1,1)^2 \setminus \overline{\Omega_D}$. To calculate the $f_S$ and $f_D$, the exact solution is given by

$$u = \text{curl}(0.1(x_2^2 - 1)^2 \sin^2(\pi x_1)), \quad p = \frac{1}{100(x_1^2 + x_2^2) + 0.1}. $$

This example is also discussed in [18]. For Darcy part, we choose same solution as Stokes part. Note that the pressure $p$ has high gradients around the origin. For $RT_1 \times Q_1$ and $RT_2 \times Q_2$ elements, the convergence results for uniform refinement meshes in Figure 21(a) and 21(b) show low-order convergence for the error $\|\|e\|\|$ and the
Figure 19: Adaptive refined meshes for $RT_1 \times Q_1$ with (a) 2356 DOF, (b) 7810 DOF, (c) 27606 DOF, (d) 102036 DOF.

Figure 20: Adaptive refined meshes for $RT_1 \times Q_1$ with (a) 1632 DOF, (b) 5193 DOF, (c) 30096 DOF, (d) 137388 DOF.

Figure 21: (a) $RT_1 \times Q_1$, (b) $RT_2 \times Q_2$.

proposed estimator $\eta_l$ because of high gradients of pressure around the origin. In Figure 21(a) and 21(b), the convergence result based on adaptive refinement achieve the optimal convergence $O(N^k/2)$, $k = 1, 2$ for the error $\|\epsilon\|$ and in a posteriori error control $\eta_l$. Figures 23 and 24 show the adaptivity refined meshes for $RT_k \times Q_k (k = 1, 2)$ elements. Moreover, Figures 23 and 24 show strong refinement towards the origin where the pressure $p$ has high gradients. Moreover, the graph of the estimates $\eta_l$ is parallel to the error in energy norm $\|\epsilon\|$ in adaptive mesh. Thus, it confirms that error estimator is numerically reliable and efficient.

6. Conclusions

In this paper, we have presented a posteriori error estimator for a strongly conservative finite element method for Stokes-Darcy coupling equation. Proofs of a reliable and efficient a posteriori error estimator is rigorously discussed.
Figure 22: Adaptive refined meshes for $RT_1 \times Q_1$ with (a) $RT_1 \times Q_1$, (b) $RT_2 \times Q_2$

Figure 23: Adaptive refined meshes for $RT_1 \times Q_1$ with (a) 770 DOF, (b) 4930 DOF, (c) 24048 DOF, (d) 90324 DOF

Figure 24: Adaptive refined meshes for $RT_2 \times Q_2$ with (a) 3831 DOF, (b) 12303 DOF, (c) 49731 DOF, (d) 229875 DOF

Specific numerical experiments have demonstrated to the effectiveness of a posteriori error estimator for a wide range of different coupling possibilities.

**References**


