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ON THE SPECTRA OF FINITE TYPE ALGEBRAS

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Contents

1. Introduction

Let X be a complex affine variety and k its coordinate algebra. Equivalently, k is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. A k-algebra is an algebra A over the complex numbers $\mathbb{C}$ which is a k-module (with an evident compatibility between the algebra structure of A and the k-module structure of A). A is not required to have a unit. A is not required to be commutative. A k-algebra A is of *finite type* if as a k-module A is finitely generated. This paper will review Morita equivalence for k-algebras and will then review — for finite type k-algebras — a weakening of Morita equivalence called *spectral equivalence*.

The *spectrum* of A is, by definition, the set of equivalence classes of irreducible A-modules. For any finite type k-algebra A, the spectrum of A is in bijection with the set of primitive ideals of A. The spectral equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A. However, the spectral equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita

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equivalence.

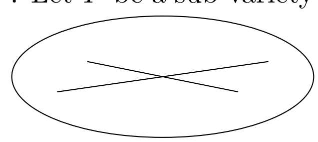
A key example illustrating the distinction between Morita equivalence and spectral equivalence relation is provided by affine Hecke algebras associated to affine Weyl groups. Let A be the group algebra, with coefficients $\mathbb{C}$, of of an affine Weyl group. For each non-zero complex number ζ there is the affine Hecke algebra (with equal parameters) A_ζ. Here $A_1 = A$ and $A_\zeta \simeq A_{1/\zeta}$. Except for ζ in a finite set of roots of unity, none of which is 1, the algebras A_ζ are spectrally equivalent. In §10, we give examples of affine Hecke algebras A_ζ which are spectrally equivalent, but not Morita equivalent, to A_1.

The ABPS (Aubert-Baum-Plymen-Solleveld) conjecture asserts that if G is a connected split reductive p-adic group, then the finite type algebra which Bernstein assigns to any given Bernstein component is spectrally equivalent to the coordinate algebra of the associated extended quotient — and that the spectral equivalence can be chosen so that the resulting bijection between the Bernstein component and the extended quotient has properties as in the statement of ABPS.

2. An example

If $X, Y, Z, \ldots$ are affine algebraic varieties over the complex numbers $\mathbb{C}$, then $\mathcal{O}(X), \mathcal{O}(Y), \mathcal{O}(Z), \ldots$ will denote the coordinate algebras of $X, Y, Z, \ldots$

Let X be a complex affine variety. Set $k = \mathcal{O}(X)$. Let Y be a sub-variety of X.



$\mathcal{I}_Y$ denotes the ideal in $\mathcal{O}(X)$ determined by Y.
$\mathcal{I}_Y = \{\omega \in \mathcal{O}(X) \mid \omega(p) = 0 \quad \forall\, p \in Y\}$

Let A be the algebra of all 2×2 matrices whose diagonal entries are in $\mathcal{O}(X)$ and whose off-diagonal entries are in $\mathcal{I}_Y$. Addition and multiplication in A are matrix addition and matrix multiplication. As a k-module, A is the direct sum of $\mathcal{O}(X) \oplus \mathcal{O}(X)$ with $\mathcal{I}_Y \oplus \mathcal{I}_Y$.

$$A = \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{O}(X) \end{pmatrix}$$

Set $B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$, so that B is the coordinate algebra of the disjoint union $X \sqcup Y$. We have $\mathcal{O}(Y) = \mathcal{O}(X)/\mathcal{I}_Y$. As a $k = \mathcal{O}(X)$-module, B is the direct sum $\mathcal{O}(X) \oplus (\mathcal{O}(X)/\mathcal{I}_Y)$. The algebras A and B are not Morita equivalent, but are equivalent in the new equivalence relation.

$$A \sim B \qquad A \underset{\text{Morita}}{\nsim} B$$

3. k-ALGEBRAS

k denotes the coordinate algebra of a complex affine variety X.

$$k = \mathcal{O}(X)$$

Equivalently, k is a unital algebra over the complex numbers which is unital, commutative, finitely generated, and nilpotent-free. The Hilbert Nullstellensatz implies that there is an equivalence of categories

$$\left(\begin{array}{l}\text{unital commutative} \\ \text{finitely generated} \\ \text{nilpotent-free } \mathbb{C}\text{-algebras}\end{array}\right) \sim \left(\begin{array}{l}\text{affine complex al-} \\ \text{gebraic varieties}\end{array}\right)^{op}$$

$$\mathcal{O}(X) \mapsto X$$

Here *op* denotes the opposite category.

Definition 3.1. A *k-algebra* is a $\mathbb{C}$-algebra A such that A is a unital (left) k-module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda\omega)a \quad \forall(\lambda, \omega, a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1 a_2) = (\omega a_1)a_2 = a_1(\omega a_2) \quad \forall(\omega, a_1, a_2) \in k \times A \times A.$$

Remark 3.2. A is not required to have a unit.

Notation. $Z(A)$ is the center of A. $\qquad Z(A) := \{c \in A \mid ca = ac \ \forall a \in A\}$.

Remark 3.3. Let A be a unital k-algebra. Denote the unit of A by 1_A.
$\omega \mapsto (\omega)1_A \qquad \omega \in k \qquad$ is then a unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A)$$

i.e. unital k-algebra = unital $\mathbb{C}$-algebra A with a given unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A)$$

Definition 3.4. Let A, B be two k-algebras. A *morphism of k-algebras* is a morphism of $\mathbb{C}$-algebras

$$f : A \to B$$

which is also a morphism of (left) k-modules,

$$f(\omega a) = \omega f(a) \quad \forall(\omega, a) \in k \times A.$$

Definition 3.5. Let A be a k-algebra. A *representation* of A [or a (left) A-*module*] is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$

$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$

such that

(1) $k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$ is unital.
(2) $(\omega a)v = \omega(av) = a(\omega v) \quad \forall(\omega,a,v) \in k \times A \times V$.

From now on in this article, A will denote a k-algebra.

A representation of A

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$

will often be denoted

$$A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$$

it being understood that the action of k on V

$$k \to \mathrm{Hom}_{\mathbb{C}}(V,V)$$

is part of the given structure.

Definition 3.6. A representation $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ is *non-degenerate* iff $AV = V$. i.e. for any $v \in V$, $\exists\ v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r.$$

Definition 3.7. A representation $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ is *irreducible* if $AV \neq \{0\}$ and $\nexists$ a sub-$\mathbb{C}$-vector space W of V with:

$$\{0\} \neq W \ , \ W \neq V$$

and

$$\omega w \in W \quad \forall(\omega,w) \in k \times W$$

and

$$aw \in W \quad \forall(a,w) \in A \times W$$

Definition 3.8. Two representations of the k-algebra A

$$\varphi_1\colon A \to \mathrm{Hom}_{\mathbb{C}}(V_1,V_1)$$
$$\varphi_2\colon A \to \mathrm{Hom}_{\mathbb{C}}(V_2,V_2)$$

are *equivalent* if $\exists$ an isomorphism of $\mathbb{C}$-vector spaces

$$T\colon V_1 \to V_2$$

with

$$T(av) = aT(v) \quad \forall\ (a,v) \in A \times V$$

and

$$T(\omega v) = \omega T(v) \quad \forall\ (\omega,v) \in k \times V$$

The spectrum of A, also denoted $\mathrm{Irr}(A)$, is the set of equivalence classes of irreducible representations of A.

$$\mathrm{Irr}(A) := \{\text{Irreducible representations of A}\}/\sim .$$

4. The "k-action for free" lemma

For a k-algebra A, $A_{\mathbb{C}}$ denotes the underlying $\mathbb{C}$ algebra of A.
$A_{\mathbb{C}}$ is obtained from A by forgetting the action of k on A.

For $A_{\mathbb{C}}$ there are the usual definitions :
A *representation* of $A_{\mathbb{C}}$ [or a (left) $A_{\mathbb{C}}$-*module*] is a $\mathbb{C}$-vector space V with a given morphism of $\mathbb{C}$-algebras

$$A_{\mathbb{C}} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$

An $A_{\mathbb{C}}$-module V is *irreducible* if $A_{\mathbb{C}}V \neq \{0\}$ and $\nexists$ a sub-$\mathbb{C}$-vector space W of V with:

$$\{0\} \neq W \ , \ W \neq V$$

and

$$aw \in W \quad \forall (a,w) \in A_{\mathbb{C}} \times W$$

Two representations of $A_{\mathbb{C}}$

$$\varphi_1 : A \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_1)$$
$$\varphi_2 : A \to \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if $\exists$ an isomorphism of $\mathbb{C}$-vector spaces

$$T : V_1 \to V_2$$

with

$$T(av) = aT(v) \quad \forall\, (a,v) \in A \times V$$

$\mathrm{Irr}(A_{\mathbb{C}})$:=$\{$Irreducible representations of $A_{\mathbb{C}}\}/\sim$.

An $A_{\mathbb{C}}$-module V for which the following two properties are valid is *strictly non-degenerate*

- $A_{\mathbb{C}}V = V$
- If $v \in V$ has $av = 0 \ \forall a \in A_{\mathbb{C}}$, then $v = 0$.

Lemma 4.1. *Any irreducible $A_{\mathbb{C}}$-module is strictly non-degenerate.*

Proof. Let V be an irreducible $A_{\mathbb{C}}$-module. First, consider $A_{\mathbb{C}}V \subset V$. $A_{\mathbb{C}}V$ is preserved by the action of $A_{\mathbb{C}}$ on V. Cannot have $A_{\mathbb{C}}V = \{0\}$ since this would contradict the irreducibility of V. Therefore $A_{\mathbb{C}}V = V$.
Next, set

$$W = \{v \in V | av = 0 \ \forall a \in A_{\mathbb{C}}\}$$

W is preserved by the action of $A_{\mathbb{C}}$ on V. W cannot be equal to V since this would imply $A_{\mathbb{C}}V = \{0\}$. Hence $W = \{0\}$.

□

Lemma 4.2. *Let A be a k-algebra, and let V be a strictly non-degenerate $A_{\mathbb{C}}$-module. Then $\exists$ a unique unital morphism of $\mathbb{C}$ algebras*

$$k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$

which makes V an A-module.

Proof. Given $v \in V$, choose $v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r$$

For $\omega \in k$, define ωv by :

$$\omega v = (\omega a_1) v_1 + (\omega a_2) v_2 + \cdots + (\omega a_r) v_r$$

The second condition in the definition of strictly non-degenerate implies that ωv is well-defined. □

Lemma 4.2 will be referred to as the "k-action for free lemma".

Notation. If V is an A-module, $V_{\mathbb{C}}$ will denote the underlying $A_{\mathbb{C}}$-module. $V_{\mathbb{C}}$ is obtained from V by forgetting the action of k on V.

Lemma 4.3. *If V is any irreducible A-module, then $V_{\mathbb{C}}$ is an irreducible $A_{\mathbb{C}}$-module.*

Proof. Suppose that $V_{\mathbb{C}}$ is not an irreducible $A_{\mathbb{C}}$-module. Then $\exists$ a sub-$\mathbb{C}$-vector space W of V with:

$$0 \neq W, \quad W \neq V$$

and

$$aw \in W \qquad \forall (a, w) \in A \times W$$

Consider $AW \subset W$. AW is preserved by both the A-action on V and the k-action on V. Thus if $AW \neq \{0\}$, then V is not an irreducible A-module. Hence $AW = \{0\}$. Consider $kW \supset W$. kW is preserved by the k-action on V and is also preserved by the A-action on V because A annihilates kW. Since A annihilates kW, cannot have $kW = V$. Therefore $\{0\} \neq kW$, $kW \neq V$, which contradicts the irreducibility of the A-module V. □

A corollary of Lemma 4.2 is :

Corollary 4.4. *For any k-algebra A, the map*

$$\mathrm{Irr}(A) \to \mathrm{Irr}(A_{\mathbb{C}})$$
$$V \mapsto V_{\mathbb{C}}$$

is a bijection.

Proof. Surjectivity follows from lemmas 4.1 and 4.2. For injectivity, let V, W be two irreducible A-modules such that $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are equivalent $A_{\mathbb{C}}$-modules. Let $T\colon V \to W$ be an isomorphism of $\mathbb{C}$ vector spaces with

$$T(av) = aT(v) \quad \forall\, (a, v) \in A \times V$$

Given $v \in V$ and $\omega \in k$, choose $v_1, v_2, \dots, v_r \in V$ and $a_1, a_2, \dots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r$$

Then

$$\begin{aligned} T(\omega v) &= T((\omega a_1)v_1 + (\omega a_2)v_2 + \cdots + (\omega a_r)v_r) \\ &= (\omega a_1)Tv_1 + (\omega a_2)Tv_2 + \cdots + (\omega a_r)Tv_r \\ &= \omega(a_1 Tv_1 + a_2 Tv_2 + \cdots + a_r Tv_r) \\ &= \omega(Tv). \end{aligned}$$

Hence $T\colon V \to W$ intertwines the k-actions on V, W and thus V, W are equivalent A-modules. □

5. Central character

An ideal I in a k-algebra A is *primitive* if I is the null-space of an irreducible representation of A, i.e. $\exists$ an irreducible representation of A $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ with

$$I = \{a \in A \mid \varphi(a) = 0\}$$

$\mathrm{Prim}(A)$ denotes the set of all primitive ideals in A. The evident map

$$\mathrm{Irr}(A) \to \mathrm{Prim}(A)$$

sends an irreducible representation to its null-space. On $\mathrm{Prim}(A)$ there is the Jacobson topology. If S is any subset of $\mathrm{Prim}(A)$, $S \subset \mathrm{Prim}(A)$, then the closure $\overline{S}$ of S is :

$$\overline{S} := \{I \in \mathrm{Prim}(A) \mid I \supset \cap_{L \in S} L\}$$

A k-algebra A is of finite type if, as a k-module, A is finitely generated. For any finite type k-algebra A, the following three statements are valid :

- If $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is any irreducible representation of A, then V is a finite dimensional $\mathbb{C}$ vector space and $\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is surjective.
- The evident map $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ is a bijection.
- Any primitive ideal in A is a maximal ideal.

Since $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ is a bijection, the Jacobson topology on $\mathrm{Prim}(A)$ can be transferred to $\mathrm{Irr}(A)$ and thus $\mathrm{Irr}(A)$ is topologized. Equivalently, $\mathrm{Irr}(A)$ is topologized by requiring that $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ be a homeomorphism.

For a finite type k-algebra A ($k = \mathcal{O}(X)$), the central character is a map

$$\mathrm{Irr}(A) \longrightarrow X$$

defined as follows. Let φ

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

be an irreducible representation of A. I_V denotes the identity operator of V

$$I_V(v) = v \quad \forall v \in V.$$

For $\omega \in k = \mathcal{O}(X)$, define

$$T_\omega : V \to V$$

by

$$T_\omega(v) = \omega v \quad \forall v \in V.$$

T_ω is an intertwining operator for $A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$. According to Lemma 4.3 plus Schur's Lemma T_ω is a scalar multiple of I_V.

$$T_\omega = \lambda_\omega I_V \qquad \lambda_\omega \in \mathbb{C}$$

The map

$$\omega \mapsto \lambda_\omega$$

is a unital morphism of $\mathbb{C}$ algebras $\mathcal{O}(X) \to \mathbb{C}$ and thus is given by evaluation at a unique ($\mathbb{C}$ rational) point p_φ of X.

$$\lambda_\omega = \omega(p_\varphi) \qquad \forall \omega \in \mathcal{O}(X)$$

The central character $\mathrm{Irr}(A) \longrightarrow X$ is

$$\varphi \mapsto p_\varphi$$

Remark. Corollary 4.4 states that $\mathrm{Irr}(A)$ depends only on the underlying $\mathbb{C}$ algebra $A_{\mathbb{C}}$. The central character $\mathrm{Irr}(A) \to X$, however, does depend on the structure of A as a k-module. A change in the action of k on $A_{\mathbb{C}}$ will change the central character.

The central character $\mathrm{Irr}(A) \to X$ is continuous where $\mathrm{Irr}(A)$ is topologized as above and X has the Zariski topology. For a proof of this assertion see [11, Lemma 1, p.326]. From a somewhat heuristic non-commutative geometry point of view, $A_{\mathbb{C}}$ is a non-commutative complex affine variety, and a given action of k on $A_{\mathbb{C}}$, making $A_{\mathbb{C}}$ into a finite type k-algebra A, determines a morphism of algebraic varieties $A_{\mathbb{C}} \to X$.

6. Morita equivalence for k-algebras

Definition 6.1. Let B be a k-algebra. A *right B-module* is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$B^{op} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

such that:

(1) $k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is unital
(2) $v(\omega b) = (v\omega)b = (vb)\omega \quad \forall (v, \omega, b) \in V \times k \times B$.

B^{op} is the opposite algebra of B. V is non-degenerate if $VB = V$.

Remark. "Right B-module" = "Left B^{op}-module."

With k fixed, let A, B be two k-algebras. An $A-B$ bimodule, denoted ${}_AV_B$, is a $\mathbb{C}$ vector space V such that :

(1) V is a left A-module.
(2) V is a right B-module.
(3) $a(vb) = (av)b \qquad \forall\, (a, v, b) \in A \times V \times B$.

(4) $\omega v = v\omega \qquad \forall\,(\omega, v) \in k \times V.$

An $A - B$ bimodule ${}_AV_B$ is non-degenerate if $AV = V = VB$. I_V is the identity map of V. ($I_V(v) = v \quad \forall v \in V$.) A is an $A - A$ bimodule in the evident way.

Definition 6.2. A k-algebra A has *local units* if given any finite set $a_1, a_2, \dots, a_r$ of elements of A, $\exists$ an idempotent $Q \in A$ ($Q^2 = Q$) with

$$Qa_j = a_jQ = a_j \qquad j = 1, 2, \dots, r.$$

An algebra with local units is referred to in [ ] as an "idempotented algebra".

Definition 6.3. Let A, B be two k-algebras with local units. A *Morita equivalence* (between A and B) is given by a pair of non-degenerate bimodules

$${}_AV_B \quad {}_BW_A$$

together with isomorphisms of bimodules

$$\alpha\colon V \otimes_B W \to A$$

$$\beta\colon W \otimes_A V \to B$$

such that there is commutativity in the diagrams:

$$\begin{array}{ccc} V \otimes_B W \otimes_A V & \xrightarrow{I_V \otimes \beta} & V \otimes_B B \\ \downarrow{\scriptstyle \alpha \otimes I_V} & & \downarrow{\scriptstyle \cong} \\ A \otimes_A V & \xrightarrow[\cong]{} & V \end{array}$$

$$\begin{array}{ccc} W \otimes_A V \otimes_B W & \xrightarrow{I_W \otimes \alpha} & W \otimes_A A \\ \downarrow{\scriptstyle \beta \otimes I_W} & & \downarrow{\scriptstyle \cong} \\ B \otimes_B V & \xrightarrow[\cong]{} & W \end{array}$$

<u>The linking algebra.</u> Let A, B two k-algebras with local units, and suppose given a Morita equivalence

$${}_AV_B \qquad {}_BW_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B$$

The linking algebra is

$$L({}_AV_B, {}_BW_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

i.e. $L({}_AV_B, {}_BW_A)$ consists of all 2×2 matrices having (1, 1) entry in A, (2, 2) entry in B, (2, 1) entry in W, and (1, 2) entry in V. Addition and multiplication are matrix addition and matrix multiplication. Note that α and β are used in the matrix multiplication.

$L({}_AV_B, {}_BW_A)$ is a k-algebra. With $\omega \in k$, the action of k on $L({}_AV_B, {}_BW_A)$ is given by

$$\omega \begin{pmatrix} a & v \\ w & b \end{pmatrix} = \begin{pmatrix} \omega a & \omega v \\ \omega w & \omega b \end{pmatrix}$$

<u>Remark.</u> A Morita equivalence between A and B determines an equivalence of categories between the category of non-degenerate left A-modules and the category of non-degenerate left B-modules. Similarly for right modules. Also, a Morita equivalence determines isomorphisms (between A and B) of Hochschild homology, cyclic homology, and periodic cyclic homology.

A Morita equivalence between two finite type k-algebras A, B preserves the central character i.e. there is commutativity in the diagram

$$\begin{array}{ccc} \mathrm{Irr}(A) & \longrightarrow & \mathrm{Irr}(B) \\ \downarrow & & \downarrow \\ X & \underset{I_X}{\longrightarrow} & X \end{array}$$

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters, and I_X is the identity map of X.

<u>Example.</u> For n a positive integer, let $M_n(A)$ be the k-algebra of all $n \times n$ matrices with entries in A. If A has local units, A and $M_n(A)$ are Morita equivalent as follows. For m, n positive integers, denote by $M_{m,n}(A)$ the set of all $m \times n$ (i.e. m rows and n columns) matrices with entries in A. Matrix multiplication then gives a map

$$M_{m,n}(A) \times M_{n,r}(A) \longrightarrow M_{m,r}(A)$$

With this notation, $M_{n,n}(A) = M_n(A)$ and $M_{1,1}(A) = M_1(A) = A$. Hence matrix multiplication gives maps

$$M_{1,n}(A) \times M_n(A) \longrightarrow M_{1,n}(A) \qquad M_n(A) \times M_{n,1} \longrightarrow M_{n,1}(A)$$

Thus $M_{1,n}(A)$ is a right $M_n(A)$-module and $M_{n,1}(A)$ is a left $M_n(A)$-module.

Similarly, $M_{1,n}(A)$ is a left A-module and $M_{n,1}(A)$ is a right A-module.

With $A = A$ and $B = M_n(A)$, the bimodules of the Morita equivalence are $V = M_{1,n}(A)$ and $W = M_{n,1}(A)$.

Note that the required isomorphisms of bimodules

$$\alpha: V \otimes_B W \to A$$

$$\beta: W \otimes_A V \to B$$

are obtained by observing that the matrix multiplication maps

$$M_{1,n}(A) \times M_{n,1}(A) \to A$$

$$M_{n,1}(A) \times M_{1,n}(A) \to M_n(A)$$

factor through the quotients $M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A)$, $M_{n,1}(A) \otimes_A M_{1,n}(A)$ and so give bimodule isomorphisms

$$\alpha: M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A) \to A$$
$$\beta: M_{n,1}(A) \otimes_A M_{1,n}(A) \to M_n(A)$$

If A has local units, then α and β are isomorphisms. Therefore A and $M_n(A)$ are Morita equivalent.

If A does not have local units, then α and β can fail to be isomorphisms, and there is no way to prove that A and $M_n(A)$ are Morita equivalent. In examples, this already happens with $n = 1$, and there is then no way to prove (when A does not have local units) that A is Morita equivalent to A. For more details on this issue see below where the proof is given that in the new equivalence relation A and $M_n(A)$ are equivalent even when A does not have local units.

A finite type k-algebra A has local units iff A is unital.

7. Spectrum preserving morphisms

Let A, B two finite type k-algebras, and let $f: A \to B$ be a morphism of k-algebras.

Definition 7.1. f is *spectrum preserving* if

(1) Given any primitive ideal $J \subset B$, $\exists$ a unique primitive ideal $I \subset A$ with $I \supset f^{-1}(J)$

and

(2) The resulting map
$$\mathrm{Prim}(B) \to \mathrm{Prim}(A)$$
is a bijection.

Example 7.2. *Let A, B two unital finite type k-algebras, and suppose given a Morita equivalence*

$$_AV_B \qquad _BW_A \qquad \alpha: V \otimes_B W \to A \qquad \beta: W \otimes_A V \to B$$

With the linking algebra $L(_AV_B, {_BW_A})$ as above, the inclusions

$$A \hookrightarrow L(_AV_B, {_VW_A}) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \leftarrow\!\shortmid b$$

are spectrum preserving morphisms of finite type k-algebras. The bijection

$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$

so obtained is the bijection determined by the given Morita equivalence.

Remark. If $f: A \to B$ is a spectrum preserving morphism of finite type k-algebras, then the resulting bijection

$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$

is a homeomorphism. For a proof of this assertion see [6, Theorem 3, p.342]. Consequently, if A, B are two unital finite type k-algebras, and

$$_AV_B \qquad _BW_A \qquad \alpha\colon V\otimes_B W \to A \qquad \beta\colon W\otimes_A V \to B$$

is a Morita equivalence, then the resulting bijection

$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$

is a homeomorphism.

Definition 7.3. An ideal I in a k-algebra A is a *k-ideal* if $\omega a \in I \ \forall\,(\omega, a) \in k \times I$.

<u>Remark.</u> Any primitive ideal in a k-algebra A is a k-ideal.

Given A, B two finite type k-algebras, let $f\colon A \to B$ be a morphism of k-algebras.

Definition 7.4. f is *spectrum preserving with respect to filtrations* if $\exists$ k-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A \qquad \text{in } A$$

and k ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{r-1} \subset J_r = B \qquad \text{in } B$$

with $f(I_j) \subset J_j$, $(j = 1, 2, \ldots, r)$ and $I_j/I_{j-1} \to J_j/J_{j-1}$, $(j = 1, 2, \ldots, r)$ is spectrum preserving.

8. Algebraic Variation of k-structure

Let A be a unital $\mathbb{C}$-algebra, and let

$$\Psi\colon k \to Z\left(A[t, t^{-1}]\right)$$

be a unital morphism of $\mathbb{C}$-algebras. Here t is an indeterminate,so $A[t, t^{-1}]$ is the algebra of Laurent polynomials with coefficients in A. As above Z denotes "center". For $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, $ev(\zeta)$ denotes the "evaluation at ζ" map:

$$ev(\zeta)\colon A[t, t^{-1}] \to A$$
$$\sum a_j t^j \mapsto \sum a_j \zeta^j$$

Consider the composition

$$k \xrightarrow{\Psi} Z\left(A[t, t^{-1}]\right) \xrightarrow{ev(\zeta)} Z(A).$$

Denote the unital k-algebra so obtained by A_ζ. $\forall \zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, the underlying $\mathbb{C}$-algebra of A_ζ is A.

$$(A_\zeta)_{\mathbb{C}} = A \qquad\qquad \forall \zeta \in \mathbb{C}^\times$$

Such a family $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, of unital k-algebras, will be referred to as *an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$.*

9. Spectral Equivalence

With k fixed, consider the collection of all finite type k-algebras. On this collection, *spectral equivalence* is, by definition, the equivalence relation generated by the two elementary steps :

Elementary Step 1. If $\exists$ a morphism of k-algebras $f : A \to B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.

Elementary Step 2. If $\exists$ $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, such that each A_ζ is a unital finite type k-algebra, then for any $\zeta, \eta \in \mathbb{C}^\times, A_\zeta \sim A_\eta$.

Thus, two finite type k-algebras A, B are equivalent iff $\exists$ a finite sequence $A_0, A_1, A_2, \ldots, A_r$ of finite type k-algebras with $A_0 = A, A_r = B$, and for each $j = 0, 1, \ldots, r-1$ one of the following three possibilities is valid :

- a morphism of k-algebras $A_j \to A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- a morphism of k-algebras $A_j \leftarrow A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, is given such that each A_ζ is a unital finite type k-algebra, and $\eta, \tau \in \mathbb{C}^\times$ have been chosen with $A_j = A_\eta, A_{j+1} = A_\tau$.

To give a spectral equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology [5, 6] are determined:

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B) \qquad \mathrm{HP}_*(A) \cong \mathrm{HP}_*(B)$$

Proposition 9.1. *If two unital finite type k-algebras A, B are Morita equivalent (as k-algebras) then they are spectrally equivalent.*

$$A \underset{Morita}{\sim} B \Longrightarrow A \sim B$$

Proof. Let A, B two unital finite type k-algebras, and suppose given a Morita equivalence

$$_AV_B \qquad _BW_A \qquad \alpha \colon V \otimes_B W \to A \qquad \beta \colon W \otimes_A V \to B$$

The linking algebra is

$$L(_AV_B, {}_BW_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

The inclusions

$$A \hookrightarrow L(_AV_B, {}_VW_A) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \leftarrow\!\shortmid b$$

are spectrum preserving morphisms of finite type k-algebras. Hence A and B are spectrally equivalent. □

According to the above, a Morita equivalence of A and B gives a homeomorphism

$$\mathrm{Prim}(A) \simeq \mathrm{Prim}(B)$$

However, the bijection

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B)$$

obtained from a spectral equivalence might not be a homeomorphism, as in the following example — which is the example in §2 revisited.

EXAMPLE. We recall the example in §2.

$$A = \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{O}(X) \end{pmatrix}$$

$$B = \mathcal{O}(X) \oplus \mathcal{O}(Y).$$

Let $M_2(\mathcal{O}(X)$ denote the algebra of all 2×2 matrices with entries in $\mathcal{O}(X)$. Consider the algebra morphisms

$$A \longrightarrow M_2(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \longleftarrow \mathcal{O}(X) \oplus \mathcal{O}(Y)$$

$$T \mapsto (T, t_{22}|Y) \qquad (T_\omega, \theta) \leftarrow\!\shortmid (\omega, \theta)$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \qquad T_\omega = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}$$

The filtration of A is given by

$$\{0\} \subset \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{I}_Y \end{pmatrix} \subset A$$

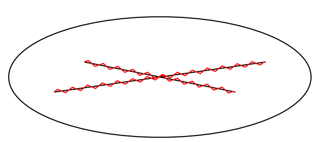
and the filtration of $M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y)$ is given by

$$\{0\} \subset M_2(\mathcal{O}(X) \oplus \{0\}) \subset M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y).$$

The rightward pointing arrow is spectrum preserving with respect to the indicated filtrations. The leftward pointing arrow is spectrum preserving (no filtrations needed). We infer that

$$A \sim B.$$

Note that

$\mathrm{Prim}(A) =$ 

$= X$ with each point of Y replaced by two points and

$$\begin{aligned}\mathrm{Prim}(B) &= \mathrm{Prim}\big(\mathcal{O}(X) \oplus \mathcal{O}(Y)\big)\\ &= X \sqcup Y\end{aligned}$$

The spaces Prim(A) and Prim(B) are not homeomorphic, and so we have

$$A \underset{\text{Morita}}{\nsim} B$$

Remark. Unlike Morita equivalence, spectral equivalence works well for finite type k-algebras whether or not the algebras are unital, e.g. A and $M_n(A)$ are spectrally equivalent even when A is not unital. See Proposition 9.3 below.

Remark. For any k-algebra A there is the evident isomorphism of k-algebras $M_n(A) \cong A \otimes_{\mathbb{C}} M_n(\mathbb{C})$. Hence, using this isomorphism, if W is a representation of A and U is a representation of $M_n(\mathbb{C})$, then $W \otimes_{\mathbb{C}} U$ is a representation of $M_n(A)$.

Notation. As in 6 above $M_{n,1}(\mathbb{C})$ denotes the $n \times 1$ (i.e. n rows, 1 column) matrices with entries in $\mathbb{C}$. Matrix multiplication gives the usual action of $M_n(\mathbb{C})$ on $M_{n,1}(\mathbb{C})$.

$$M_n(\mathbb{C}) \times M_{n,1}(\mathbb{C}) \longrightarrow M_{n,1}(\mathbb{C})$$

This is the unique irreducible representation of $M_n(\mathbb{C})$. For any k-algebra A, if W is a representation of A, then $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is a representation of $M_n(A)$.

Lemma 9.2. *Let A be a finite type k-algebra and let n be a positive integer. Then:*
(i) If W is an irreducible representation of A, $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is an irreducible representation of $M_n(A)$.
(ii) The resulting map $Irr(A) \to Irr(M_n(A))$ is a bijection.

Proof. For (i), suppose given an irreducible representation W of A. Let J be the primitive ideal in A which is the null space of W. Then the null space of $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is $J \otimes_{\mathbb{C}} M_n(\mathbb{C})$.Consider the quotient algebra
$A \otimes_{\mathbb{C}} M_n(\mathbb{C}) / J \otimes_{\mathbb{C}} M_n(\mathbb{C}) = (A/J) \otimes_{\mathbb{C}} M_n(\mathbb{C})$. This is isomorphic to $M_{rn}(\mathbb{C})$ where $A/J \cong M_r(\mathbb{C})$, and so $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is irreducible.

Proposition 9.3. *Let A be a finite type k-algebra and let n be a positive integer, then A and $M_n(A)$ are spectrally equivalent.*

Proof. Let $f\colon A \to M_n(A)$ be the morphism of k-algebras which maps $a \in A$ to the diagonal matrix

$$\begin{bmatrix} a & 0 & \dots & 0\\ 0 & a & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a\end{bmatrix}$$

It will suffice to prove that $f\colon A \to M_n(A)$ is spectrum preserving.
Let J be an ideal in A. Denote by $J^{\diamond}$ the ideal in $M_n(A)$ consisting of all $[a_{ij}] \in M_n(A)$ such that each a_{ij} is in J. Equivalently, $M_n(A)$ is $A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ and $J^{\diamond} = J \otimes_{\mathbb{C}} M_n(\mathbb{C})$. It will suffice to prove

(1) If J is a primitive ideal in A, then $J^{\diamondsuit}$ is a primitive ideal in $M_n(A)$.
(2) If L is any primitive ideal in $M_n(A)$, then $\exists$ a primitive ideal J in A with $L = J^{\diamondsuit}$.

For (1), J primitive $\Longrightarrow J^{\diamondsuit}$ primitive, because the quotient algebra $M_n(A)/J^{\diamondsuit}$ is $(A/J)\otimes_{\mathbb{C}} M_n(\mathbb{C})$ which is (isomorphic to) $M_{rn}(\mathbb{C})$ where $A/J \cong M_r(\mathbb{C})$.

For (2), since $\mathbb{C}$ is commutative, the action of $\mathbb{C}$ on A can be viewed as both a left and right action. Matrix multiplication then gives a left and a right action of $M_n(\mathbb{C})$ on $M_n(A)$

$$M_n(\mathbb{C}) \times M_n(A) \to M_n(A)$$
$$M_n(A) \times M_n(\mathbb{C}) \to M_n(A)$$

for which the associativity rule

$$(\alpha\theta)\beta = \alpha(\theta\beta) \qquad \alpha,\beta \in M_n(A) \quad \theta \in M_n(\mathbb{C})$$

is valid.

If V is any representation of $M_n(A)$, the associativity rule

$$(\alpha\theta)(\beta v) = \alpha[(\theta\beta)v] \qquad \alpha,\beta \in M_n(A) \quad \theta \in M_n(\mathbb{C}) \quad v \in V$$

is valid.

Now let V be an irreducible representation of $M_n(A)$, with L as its null-space. Define a (left) action

$$M_n(\mathbb{C}) \times V \to V$$

of $M_n(\mathbb{C})$ on V by proceeding as in the proof of Lemma 4.2 (the "k-action for free" lemma) i.e. given $v \in V$, choose $v_1, v_2, \ldots, v_r \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_r \in M_n(A)$ with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r$$

For $\theta \in M_n(\mathbb{C})$, define θv by :

$$\theta v = (\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \cdots + (\theta\alpha_r)v_r$$

The strict non-degeneracy, Lemmas 4.1 and 4.3, of V implies that θv is well-defined as follows. Suppose that $u_1, u_2, \ldots, u_s \in V$ and $\beta_1, \beta_2, \ldots, \beta_r \in M_n(A)$ are chosen with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_s u_s$$

If α is any element of $M_n(A)$, then

$\alpha[(\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \cdots + (\theta\alpha_r)v_r - (\theta\beta_1)u_1 - (\theta\beta_2)u_2 - \cdots - (\theta\beta_s)u_s] = (\alpha\theta)[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r - \beta_1 u_1 - \beta_2 u_2 - \cdots - \beta_s u_s] = (\alpha\theta)[v - v] = 0$

Use $f\colon A \to M_n(A)$ to make V into an A-module

$$av := f(a)v \qquad a \in A \quad v \in V$$

The actions of A and $M_n(\mathbb{C})$ on V commute. Thus for each $\theta \in M_n(\mathbb{C}), \theta V$ is a sub-A-module of V, where θV is the image of $v \mapsto \theta v$. Denote by E_{ij} the matrix in $M_n(\mathbb{C})$ which has 1 for its (i,j) entry and zero for all its other entries. Then, as an A-module, V is the direct sum

$$V = E_{11}V \oplus E_{22}V \oplus \cdots \oplus E_{nn}V$$

Moreover, the action of E_{ij} on V maps $E_{jj}V$ isomorphically (as an A-module) onto $E_{ii}V$. Hence as an $M_n(A) = A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ module, V is isomorphic to $(E_{11}V)\otimes_{\mathbb{C}}\mathbb{C}^n$ — where $\mathbb{C}^n$ is the standard representation of $M_n(\mathbb{C})$ i.e. is the unique irreducible representation of $M_n(\mathbb{C})$.

$$V \cong (E_{11}V)\otimes_{\mathbb{C}}\mathbb{C}^n$$

$E_{11}V$ is an irreducible A-module since if not $V=(E_{11}V)\otimes_{\mathbb{C}}\mathbb{C}^n$ would not be an irreducible $A\otimes_{\mathbb{C}} M_n(\mathbb{C})$-module.
If J is the null space (in A) of $E_{11}V$, then $J^{\diamond}=J\otimes_{\mathbb{C}} M_n(\mathbb{C})$ is the null space of $V=(E_{11}V)\otimes_{\mathbb{C}}\mathbb{C}^n$ and this completes the proof. □

10. Affine Hecke Algebras

Let G be a connected reductive complex Lie group with maximal torus T. W denotes the Weyl group

$$W = N_G(T)/T$$

and $X^*(T)$ is the character group of T. $N_G(T)$ is the normalizer (in G) of T. The semi-direct product $X^*(T)\rtimes W$ is the affine Weyl group of G. For each non-zero complex number q, there is the affine Hecke algebra $\mathcal{H}_q(G)$. This is an affine Hecke algebra with equal parameters and $\mathcal{H}_1(G)$ is the group algebra of the affine Weyl group:

$$\mathcal{H}_1(G) = \mathbb{C}[X^*(T)\rtimes W] = \mathcal{O}(T)\rtimes W.$$

$\mathcal{H}_q$ is the algebra generated by T_x, $x\in X^*(T)\rtimes W$, with relations

$$T_xT_y = T_{xy}, \quad \text{if } \ell(xy)=\ell(x)+\ell(y), \text{ and}$$
$$(T_s-q)(T_s+1)=0, \quad \text{if } s\in S.$$

ℓ is the length function on $X^*(T)\rtimes W$.
S is the set of order 2 generators of the finite Coxeter group W.
Using the action of W on T, form the quotient variety T/W and let k be its coordinate algebra,

$$k = \mathcal{O}(T/W)$$

For all $q\in\mathbb{C}^\times$, $\mathcal{H}_q(G)$ is a unital finite type k-algebra.

Theorem 10.1 (Lusztig). *Except for q in a finite set of roots of unity, none of which is 1, $\mathcal{H}_q(G)$ is spectrally equivalent to $\mathcal{H}_1(G)$:*

$$\mathcal{H}_q(G) \sim \mathcal{H}_1(G).$$

Proof. Let J be Lusztig's asymptotic algebra [16, 2.7]. As a $\mathbb{C}$-vector space, J has a basis $\{T_x : x\in X^*(T)\rtimes W\}$, and there is a canonical structure of associative $\mathbb{C}$-algebra on J. Except for q in a finite set of roots of unity (none of which is 1) Lusztig constructs a morphism of k-algebras

$$\phi_q\colon \mathcal{H}_q(G)\longrightarrow J$$

which is spectrum preserving with respect to filtrations. The algebra $\mathcal{H}_q(G)$ is viewed as a k-algebra via the canonical isomorphism

$$\mathcal{O}(T/W)\cong Z(\mathcal{H}_q(G)).$$

Lusztig's map ϕ_q maps $Z(\mathcal{H}_q(G))$ to $Z(J)$ and thus determines a unique k-structure for J such that the map ϕ_q is a morphism of k-algebras. J with this k-structure will be denoted J_q. $\mathcal{H}_q(G)$ is then spectrally equivalent to $\mathcal{H}_1(G)$ by the three elementary steps

$$\mathcal{H}_q(G)\rightsquigarrow J_q\rightsquigarrow J_1\rightsquigarrow \mathcal{H}_1(G).$$

The second elementary step (i.e. passing from J_q to J_1) is an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$. The first elementary step uses Lusztig's map ϕ_q, and the third elementary step uses Lusztig's map ϕ_1. Hence (provided q is not in the exceptional set of roots of unity—none of which is 1) $\mathcal{H}_q(G)$ is spectrally equivalent to

$$\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W.$$

□

As observed in section 11 below, $\mathrm{Irr}(\mathcal{H}_1(G)) = T/\!/W$. Thus the spectral equivalence of $\mathcal{H}_q(G)$ to $\mathcal{H}_1(G)$ determines a bijection

$$T/\!/W \longleftrightarrow \mathbf{Irr}(\mathcal{H}_q(G))$$

Here $T/\!/W$ is the extended quotient for the action of W on T. See section 11 below.

With $q = 1$, there is commutativity in the diagram

$$\begin{array}{ccc} T/\!/W & \longrightarrow & \mathrm{Irr}(\mathcal{H}_1(G)) \\ \downarrow & & \downarrow \\ T/W & \xrightarrow[I_{T/W}]{} & T/W \end{array}$$

where the left vertical arrow is the projection of the extended quotient on the ordinary quotient and the right vertical arrow is the central character for $\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W$.

Theorem 10.2. *Consider the affine Hecke algebra $\mathcal{H}_q := \mathcal{H}_q(\mathrm{SL}_3(\mathbb{C}))$. For $|q| \neq 1$, $\mathcal{H}_q$ is not Morita equivalent to $\mathcal{H}_1$.*

Proof. We consider Hochschild homology HH_*. We note that HH_* is Morita invariant. We have the isomorphism of Solleveld [14, Theorem 2(a)] onto the W-invariant algebraic forms on $\widetilde{T}$:

$$\mathrm{HH}_*(\mathcal{H}_q) \cong \Omega^*(\widetilde{T})^W, \tag{1}$$

$$\widetilde{T} := \{(w,t) \in W \times T : w(t) = t\}.$$

The right-hand-side is independent of q.

For every q under consideration there is a canonical isomorphism $Z(\mathcal{H}_q) \cong \mathcal{O}(T)^W$, and the resulting action of $\mathcal{O}(T)^W$ on (1) does depend on q. To be precise, the action on $\Omega(T_i^w)$ is the same as the action via the embedding

$$T_i^w \to T : t \mapsto c_{w,i} t$$

where T_i^w is a connected component of $T^w \cong (w, T^w) \subset \widetilde{T}$ and

$$c_{w,i} : X^*(T) \to \{q^n : n \in \mathbb{Z}\}$$

is defined in [14, Theorem 1(c)].

Since any Morita equivalence preserves the center of an algebra, its Hochschild homology and the action of the center on that, we can deduce a necessary condition for Morita equivalence $\mathcal{H}(R,q)$ and $\mathcal{H}(R,q')$. Namely, there must exist an automorphism of T/W that sends every sub-variety $c_{w,i}T_i^w/Z_W(w)$ to a sub-variety $c_{w',i'}T_{i'}^{w'}/Z_W(w')$.

For the affine Hecke algebra with (X,R) of type $\widetilde{A}_2$ it was shown in [17, §3] that this condition is only fulfilled if $q' = q$ or $q' = 1/q$. □

It appears that the above condition on subvarieties of T/W is rather strong, at least when R is not a direct product of root systems A_1. On this basis we conjecture that Theorem 10.2 holds for any affine Hecke algebra whose root system contains non-perpendicular roots.

11. Extended quotient

Let Γ be a finite group acting as automorphisms of a complex affine variety X.

$$\Gamma \times X \to X.$$

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Let $\mathbf{Irr}(\Gamma_x)$ be the set of (equivalence classes of) irreducible representations of Γ_x. These representations are on finite dimensional vector spaces over the complex numbers $\mathbb{C}$.

The *extended quotient*, denoted $X/\!/\Gamma$, is constructed by replacing the orbit of x (for the given action of Γ on X) by $\mathbf{Irr}(\Gamma_x)$. This is done as follows :

Set $\widetilde{X} = \{(x,\tau) \,\big|\, x \in X \text{ and } \tau \in \mathbf{Irr}(\Gamma_x)\}$. Endowed with the topology that sees only the first coordinate, this is an algebraic variety (in the sense of e.g. [9]), although it is usually not separated. Then Γ acts on $\widetilde{X}$ by

$$\Gamma \times \widetilde{X} \to \widetilde{X},$$

$$\gamma(x,\tau) = (\gamma x, \gamma_* \tau),$$

where $\gamma_* \colon \mathbf{Irr}(\Gamma_x) \to \mathbf{Irr}(\Gamma_{\gamma x})$. $X/\!/\Gamma$ is defined by :

$$X/\!/\Gamma := \widetilde{X}/\Gamma,$$

i.e. $X/\!/\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$.

The projection $\widetilde{X} \to X \quad (x,\tau) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to give the projection of $X/\!/\Gamma$ onto X/Γ.

$$\pi_1 \colon X/\!/\Gamma \longrightarrow X/\Gamma$$

Denote by triv_x the trivial one-dimensional representation of Γ_x. The inclusion

$$X \hookrightarrow \widetilde{X}$$

$$x \mapsto (x, \mathrm{triv}_x)$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion

$$X/\Gamma \hookrightarrow X/\!/\Gamma$$

This will be referred to as the inclusion of the ordinary quotient in the extended quotient.

Let $\mathcal{O}(X)$ be the coordinate algebra of the complex affine variety X and let $\mathcal{O}(X) \rtimes \Gamma$ be the crossed-product algebra for the action of Γ on $\mathcal{O}(X)$.

There are canonical bijections

$$\mathbf{Irr}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow \mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow (X/\!/\Gamma)$$

where $\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ is the set of primitive ideals in $\mathcal{O}(X) \rtimes \Gamma$ and $\mathbf{Irr}(\mathcal{O}(X) \rtimes \Gamma)$ is the set of (equivalence classes of) irreducible representations of $\mathcal{O}(X) \rtimes \Gamma$. The irreducible representation of $\mathcal{O}(X) \rtimes \Gamma$ associated to $x, \tau) \in (X/\!/\Gamma$ is

$$\mathrm{Ind}_{\mathcal{O}(X) \rtimes \Gamma_x}^{\mathcal{O}(X) \rtimes \Gamma} (\mathbb{C}_x \otimes \tau).$$

Here $\mathbb{C}_x \colon \mathcal{O}(X) \to \mathbb{C}$ is the irreducible representation of $\mathcal{O}(X)$ given by evaluation at $x \in X$. $\mathrm{Ind}_{\mathcal{O}(X) \rtimes \Gamma_x}^{\mathcal{O}(X) \rtimes \Gamma}$ is induction from $\mathcal{O}(X) \rtimes \Gamma_x$ to $\mathcal{O}(X) \rtimes \Gamma$.

$\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ is endowed with the Jacobson topology, which makes it a (not necessarily separated) algebraic variety. This structure can be transferred via the canonical bijection $\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow X/\!/\Gamma$ to $X/\!/\Gamma$. Hence $X/\!/\Gamma$ is a complex algebraic variety. In many examples $X/\!/\Gamma$ is not separated, and is not an affine variety.

12. A conjectural refinement of the Bernstein program

Let F be a non-archimedean local field, and let G be a connected reductive algebraic group over F. Let $\mathfrak{s}$ be a point in the Bernstein spectrum $\mathfrak{B}(G)$ of G. Attached to $\mathfrak{s}$ there is a complex torus $T_{\mathfrak{s}}$ and a finite group $W_{\mathfrak{s}}$ acting on $T_{\mathfrak{s}}$.

We denote the space of (equivalence classes of) irreducible smooth complex G-representations by $\mathbf{Irr}(G)$. We have the Bernstein decomposition

$$\mathbf{Irr}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \mathbf{Irr}^{\mathfrak{s}}(G)$$

and the (restriction of) the cuspidal support map

$$\mathbf{Sc} : \mathbf{Irr}^{\mathfrak{s}}(G) \to T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

see [13, VI.7.1.1]. The map $\mathbf{Sc}$ is finite-to-one and the quotient $T_{\mathfrak{s}}/W_{\mathfrak{s}}$ has the structure of a complex affine algebraic variety.

Bernstein constructs a finite type $k^{\mathfrak{s}}$-algebra $A^{\mathfrak{s}}$ with the property that $\mathbf{Irr}(A^{\mathfrak{s}})$ is in bijection with the Bernstein component $\mathbf{Irr}^{\mathfrak{s}}(G)$. Here $k^{\mathfrak{s}}$ is the coordinate algebra of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$:

$$k^{\mathfrak{s}} = \mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$$

The classical theory leaves open the geometric structure of each component $\mathbf{Irr}^{\mathfrak{s}}(G)$. Here is a conjectural refinement of the Bernstein program. The set $\mathbf{Irr}^{\mathfrak{s}}(G)$ has the structure of a scheme (possibly non-separated), and there is an algebraic family of finite morphisms $\{\pi_\zeta : \zeta \in \mathbb{C}^\times\}$ such that $\pi_q = \mathbf{Sc}, \pi_1 = \rho$:

$$\begin{array}{ccccc}
\mathbf{Irr}^{\mathfrak{s}}(G) & = & \mathbf{Irr}^{\mathfrak{s}}(G) & = & \mathbf{Irr}^{\mathfrak{s}}(G) \\
\downarrow{\scriptstyle \mathbf{Sc}} & & \downarrow{\scriptstyle \pi_\zeta} & & \downarrow{\scriptstyle \rho} \\
T^{\mathfrak{s}}/W^{\mathfrak{s}} & = & T^{\mathfrak{s}}/W^{\mathfrak{s}} & = & T^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{array}$$

This, for split groups, will follow from the following

Conjecture 12.1. *Let G be a connected reductive split algebraic group over F. For each Bernstein component in the smooth dual of G, Bernstein's finite type $k^{\mathfrak{s}}$-algebra $A^{\mathfrak{s}}$ is spectrally equivalent to the crossed-product algebra $\mathcal{O}(T^{\mathfrak{s}})\rtimes W^{\mathfrak{s}}$. In addition, the spectral equivalence between $A^{\mathfrak{s}}$ and $\mathcal{O}(T^{\mathfrak{s}})\rtimes W^{\mathfrak{s}}$ can be chosen such that the resulting bijection*

$$\mathbf{Irr}^{\mathfrak{s}}(G) \longleftrightarrow T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$$

satisfies a number of conditions itemized in [2].

Moussaoui [12], building on the work of many mathematicians e.g. [4], [7], [8], [10], has verified the ABPS conjecture (without the spectral equivalence) for all the split classical p-adic groups. A different proof of the ABPS conjecture (without the spectral equivalence) for split classical groups was first obtained by Solleveld [15]. Solleveld's approach is more general since it works as soon as we know that the algebra A_ζ is an extended affine Hecke algebra. The spectral equivalence is established for $\mathrm{GL}_n(F)$ in [7], and for the principal series of the exceptional group G_2, see [1]. A more involved version of the spectral equivalence is proved in our paper on the inner forms of $\mathrm{SL}_n(F)$, see [3].

An essential feature of Moussaoui's work is the compatibility of the extended quotient structure (for each Bernstein component) with the local Langlands correspondence (LLC). Moussaoui proves that, *independently* of what is happening in the smooth dual, an extended quotient structure is present in the enhanced Langlands parameters — and that the LLC consists of isomorphisms of extended quotients. This phenomenon was first observed in the special case of GL_n in [7].

A foundational issue in local Langlands is to make precise the properties that uniquely determine the correspondence. It appears, at the present time, that one of these properties should be (using enhanced Langlands parameters) that the correspondence should consist of isomorphisms of extended quotients.

Part of this is (as in [12] et al.) that the extended quotient structure should appear *independently* on the Galois side and the representation theory side.

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