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THE MATHEMATICS OF MOTION CAMOUFLAGE

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Abstract. Motion camouflage is a strategy whereby an aggressor moves towards a target whilst appearing stationary to the target except for the inevitable perceived change in size of the aggressor as it approaches. The strategy has been observed in insects, and mathematical models using discrete time or neural network control have been used to simulate the behaviour. Here the differential equations for motion camouflage are derived and some simple cases are analysed. These equations are easy to simulate numerically, and simulations indicate that motion camouflage is more efficient than the classical pursuit strategy ('move directly towards the target').

Keywords: motion camouflage, stealth strategy, pursuit path, dragonfly, chaotic pursuit

1. Introduction

Motivated by observations of mating hoverflies, Srinivasen & Davey (1995) describe a new form of stealth strategy which can be used by one creature – the shadower, or aggressor – to approach another – the shadowee, or target. In motion camouflage the aggressor moves so that it is always on the line segment from the target to a given fixed point. If the effect of size with distance is ignored then this means that the target is unable to tell that the aggressor is moving; the aggressor appears to be at its initial position, or is camouflaged by a stationary object in the background. There is now strong evidence that dragonflies use this strategy in territorial disputes (Mizutani *et al.*, 2003) and that humans can be tricked in the same way (Anderson & McOwan, 2003b). Anderson & McOwan (2003a) show that an aggressor can achieve a good approximation to motion camouflage using a neural net control system, and use this to reproduce motion camouflage trajectories. Srinivasen

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& Davey (1995) describe several algorithms by which approximate motion camouflage might be achieved, and present numerical simulations of these algorithms.

These approaches to simulating motion camouflage paths are approximate; they rely either on control methods or on a set of discrete time observations. It is clearly important to be able to determine the accuracy and efficiency of these simulations, and in section two a differential equation is derived which gives the ideal motion camouflage paths for an aggressor moving with constant speed. This differential equation makes it possible to compute accurate motion camouflage paths, and to compare these with other strategies or with other algorithms for motion camouflage. As a first step towards a better understanding of motion camouflage the standard test case of a target moving with constant velocity is treated mathematically at the end of section two. In section three this test case is investigated numerically and the solutions are compared with those of the classic pursuit strategy, 'travel at constant speed directly towards the target', which goes back (possibly) to da Vinci (Davis, 1962). In section four the strategy is applied to the pursuit of a chaotic target in three dimensions, and in section five variants of the ideal motion camouflage equations are discussed.

The simulations of sections three and four suggest that motion camouflage is more efficient than the classical pursuit strategy in the following sense. If the aggressor is quicker than the target then the motion camouflage strategy captures the target faster than the classical pursuit strategy. Whist if the aggressor is slower than the target then motion camouflage is more likely to capture the target than the classical pursuit strategy.

2. The ideal motion camouflage equations

Suppose that the position of the target is $\mathbf{z}(t)$ and that of the aggressor is $\mathbf{r}(t)$, where $\mathbf{z}(t)$ is given and $\mathbf{r}(t)$ is to be found and both lie either in a plane or three dimensional Euclidean space. If the aggressor uses motion camouflage then $\mathbf{r}(t)$ lies on the line connecting the target and some fixed reference point $\mathbf{r}_0$. This means that

$$\mathbf{r}(t) = \mathbf{r}(0) + u(\mathbf{z}(t) - \mathbf{r}_0) \qquad (1)$$

where $u(t)$ is a real function with $u(0) = 0$. An initial consistency condition must also hold:

$$\mathbf{r}(0) \times (\mathbf{z}(0) - \mathbf{r}_0) = \mathbf{r}_0 \times \mathbf{z}(0)$$

which ensures that the aggressor starts on the connecting line. This condition is automaticaly satisfied if $\mathbf{r}(0) = \mathbf{r}_0$, i.e. if the fixed reference point is the beginning of the aggressors attack. To simplify some manipulations this assumption will be made throughout the remainder of this paper. In particular, this assumption implies that the aggressor and target are at the same place if $u = 1$.

If u can be found then equation (1) determines $\mathbf{r}$, and any continuous function $u(t)$ which takes values less than one represents a motion camouflage path. If both the aggressor and the target move with constant speed (a standard assumption) then a unique aggressive path is determined, although there is also a defensive solution. The constant speed constraint is $|\dot{\mathbf{z}}| = v$ and $|\dot{\mathbf{r}}| = c$ with $v, c > 0$. Differentiating (1) gives $\dot{\mathbf{r}} = \dot{u}(\mathbf{z}(t) - \mathbf{r}_0) + u\dot{\mathbf{z}}$ so squaring gives

$$c^2 = \dot{u}^2|\mathbf{z}(t) - \mathbf{r}_0|^2 + 2u\dot{u}[(\mathbf{z}(t) - \mathbf{r}_0).\dot{\mathbf{z}}] + v^2u^2 \quad (2)$$

This is a quadratic equation for $\dot{u}$ and the standard quadratic formula with the plus sign on the discriminant gives the differential equation for u:

$$\dot{u} = \frac{-[(\mathbf{z}(t) - \mathbf{r}_0).\dot{\mathbf{z}}]u + \sqrt{[(\mathbf{z}(t) - \mathbf{r}_0).\dot{\mathbf{z}}]^2u^2 - (v^2u^2 - c^2)|\mathbf{z}(t) - \mathbf{r}_0|^2}}{|\mathbf{z}(t) - \mathbf{r}_0|^2} \quad (3)$$

with the initial condition $u(0) = 0$. Since we have assumed that the fixed point $\mathbf{r}_0$ is the initial position of the aggressor when the attack begins, $\mathbf{r}_0 = \mathbf{r}(0)$, the aggressor captures the target in the sense that their positions coincide if there exists a time $T > 0$ with $u(T) = 1$.

Taking the negative sign of the discriminant when solving (2) for $\dot{u}$ gives a defensive solution; an equation for a stealthy retreat.

Equation (3) is the general equation which determines the ideal motion camouflage path of the aggressor, and it is equally valid for motion in two or in three dimensions. The equation is simple to integrate numerically for arbitrary $\mathbf{z}$ (see below), but in the special case of the target moving with constant velocity in the plane it is possible to get a little further mathematically.

In this special case with $c = v = 1$ we may take $\mathbf{z} = (0, t)$, so setting $\mathbf{r}_0 = (x_0, y_0)$ equation (3) becomes

$$\dot{u} = \frac{-(t - y_0)u + \sqrt{x_0^2 + (t - y_0)^2 - x_0^2u^2}}{x_0^2 + (t - y_0)^2} \quad (4)$$

with initial condition $u(0) = 0$. Unfortunately this equation does not have a known solution. To see this set $s = t - y_0$ and define a new variable U by

$$u = \frac{(s^2 + x_0)^{\frac{1}{2}}U}{x_0(1 + U^2)^{\frac{1}{2}}} \quad (5)$$

Then (4) implies that U satisfies the differential equation

$$U' = (1 + U^2)\left(\tfrac{x_0}{(s^2+x_0^2)} - \tfrac{2s}{(s^2+x_0^2)}U\right) \quad (6)$$

where the prime denotes differentiation with respect to s and with $U(-y_0) = 0$. This is an Abel equation of the first kind (Murphy, 1962) for which no analytic solution is available in the literature A survey of what is known about solutions to Abel's equation can be found in Cheb-Terrab & Roche (2000). The best we can do is to write

$$u(t) = \frac{(x_0^2 + (t - y_0)^2)^{\frac{1}{2}}U(t - y_0)}{x_0(1 + [U(t - y_0)]^2)^{\frac{1}{2}}} \quad (7)$$

where $U(s)$ is the solution of (6) with $U(-y_0) = 0$. Of course, the lack of an analytic solution is no barrier to numerical simulations. Some solutions together with the corresponding motion camouflage paths are shown in Figure 1.

3. Motion camouflage and classical pursuit curves

In classical pursuit strategies predators move directly towards their prey at each instant, and the differential equations modelling this movement are well established. Davis (1962) ascribes the first mathematical treatment to Bouguer in 1732. If the prey has position $\mathbf{z}(t)$ then the predator moves on the curve $\mathbf{r}(t)$ so that at each instant its velocity is in the direction of the line from $\mathbf{r}(t)$ to $\mathbf{z}(t)$. If the predator has

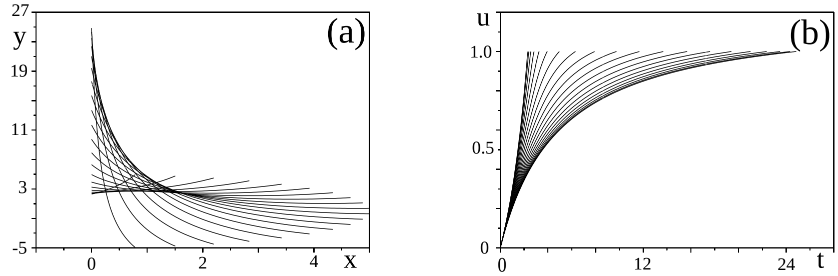

FIGURE 1. (a) Motion camouflage paths in the $(x,y)-$plane. The target is moving along the $y-$axis with $v=1$ and position $\mathbf{z}=(0,t)$ and the aggressor has speed $c=1.2$. The curves represent the paths of twenty aggressors with initial positions regularly spaced on the circle of radius 5 units centred initial position of the target. (b) The corresponding functions $u(t)$, which are the solutions of (4). These curves can be matched to those of Figure 1(a) by noting that at the point of capture, if $y=Y$ when $x=0$ in (a) then $u(Y)=1$ in (b).

(constant) speed, $c>0$, then the differential equation for the motion is

$$\dot{\mathbf{r}} = c\frac{\mathbf{z}-\mathbf{r}}{|\mathbf{z}-\mathbf{r}|} \tag{8}$$

If the prey is assumed to move in a straight line in the plane with unit speed and $c=1$ (Bouguer's problem) then the equation can be solved explicitly, although the paths are given in terms of special functions (e.g. Davis, 1962).

Figure 2(a) shows a solution of the classical pursuit problem (P) together with the corresponding motion camouflage curve (M) with $c=1.2$ and $v=1$, so the aggressor moves faster than the target. Although the classical pursuit path looks more direct, this is an illusion. Figure 2(b) shows that it is only in the final phase of the motion that there is a significant difference between the distance to the target in the two strategies, and that it is the motion camouflage path which captures the target first. (In the simulation shown, capture is interpreted as being within 0.001 units of the target, but the qualitative statement that motion camouflage captures first appears to be robust). Indeed, further numerical simulations suggest that this is true more generally: the distance between the aggressor and target decreases initially with the same linear behaviour in both the classical pursuit strategy and the

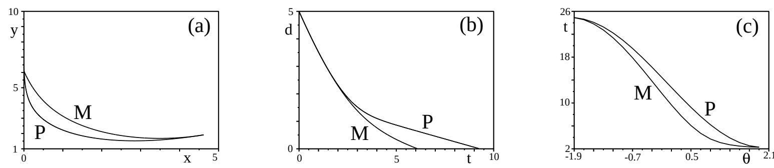

FIGURE 2. (a) A motion camouflage path M and a pursuit path P in the $(x, y)-$plane. The target is moving along the $y-$axis with $v = 1$ and position $\mathbf{z} = (0, t)$ and the aggressor has speed $c = 1.2$. The initial condition is $(5\cos\frac{\pi\theta}{4}, 5\sin\frac{\pi\theta}{4})$ with $\theta = 0.5$. (b) The distance from the target (d) as a function of time (t) on the two paths shown in (a). (c) The time (t) to capture as a function of angle if the target moves as in (a). The angular variable θ is in units of $\frac{\pi}{4}$ as in (a). The aggressor has speed $c = 1.2$ and capture is interpreted as being within 0.001 units of the target. The times for the motion camouflage strategy are labelled M and the times for the pursuit strategy are labelled P.

motion camouflage strategy, but as shown in Figure 2(c), the simulated motion camouflage paths are shorter (and hence more efficient) for all initial conditions equidistant from the starting point of the target.

If the speed of the aggressor is less than the speed of the target then motion camouflage appears to be more efficient than the pursuit strategy in the sense that capture is possible from a greater range of initial conditions which are equidistant from the initial position of the target but at different angles angles to the line of motion of the target. This observation is illustrated further in the next section.

4. Chaotic pursuit

The examples above model motion in the plane, and the motion of the target is linear. In this section the same models, motion camouglage given by (3) and classical pursuit by (8), are used to investigate more complicated three dimensional motion of the target.

We will assume that the target moves with constant speed v along the Rössler chaotic attractor (Rössler, 1976). In other words, $\mathbf{z} =$

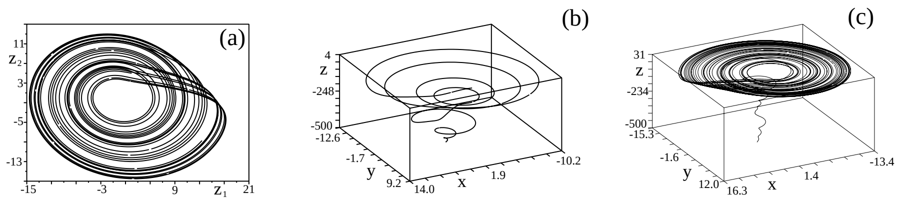

FIGURE 3. (a) The Rössler attractor: projection onto the (z_1, z_2)−plane of solutions to (9) with initial conditions $(2, 5, 7)$. The results of 100 time units with $v = 20$ are shown. (b) The motion camouflage solution with $v = 20$, $c = 18$. The initial conditions of the aggressor are $(3, -1, -500)$. (c) The corresponding classical pursuit path.

(z_1, z_2, z_3), with

$$\begin{aligned} \dot{z}_1 &= (-z_2 - z_3)/\Delta \\ \dot{z}_2 &= (z_1 + 0.15z_2)/\Delta \\ \dot{z}_3 &= (0.2 - 10z_3 + z_1z_3)/\Delta \end{aligned} \tag{9}$$

where $v^2\Delta^2 = (z_2+z_3)^2+(z_1+0.15z_2)^2+(0.2-10z_3+z_1z_3)^2$ and $\Delta \geq 0$. Δ has been chosen so that the target moves with speed v, and $\Delta = 0$ at the stationary points of the differential equation, so (9) is only valid on the attractor provided it does not contain any stationary points. No direct biological relevance is claimed for this choice of chaotic target path, which is shown in Figure 3(a), but it is natural to test pursuit strategies against more complicated target paths.

Figures 3(b) shows the path obtained by integrating the coupled set of four differential equations for motion camouflage, (3) and (9), with initial target position $(2, 5, 7)$ and initial aggressor position $(3, -1, -500)$ and with $v = 20$ and $c = 18$. Despite the fact that the target moves faster than the aggressor, the aggressor captures the target after approximately 26.4 time units. The classic pursuit problem, equations (8) and (9), results in a six dimensional set of differential equations and, given the same initial positions and speeds the pursuit path is shown in Figure 3(c). In this case the aggressor moves close to the attractor containing the target's path, but does not get within 0.1 units of the target in the 200 units of time shown here. Again, this illustrates that motion camouflage is a more efficient interception strategy than classical pursuit.

5. Related Models

The techniques described here can be used to construct other motion camouflage models incorporating different effects. For example, there is nothing in the derivation of (3) that requires the target to move with constant speed, and the equation could easily be modified to allow the target to change speed. The modification to (3) on the assumption that there is a time delay τ between the aggressor's observation of the position of the target and its motion can also be derived by replacing $\mathbf{z}(t)$ by $\mathbf{z}(t-\tau)$ in section two.

Whilst it is easy to relax the assumption of constant speed for the target, it is harder to do the same thing for the aggressor, since it must be replaced by another constraint in order to give a unique aggressive solution. One possible alternative would be to impose a maximum relative speed of approach. This is natural, since if the aggressor grows too rapidly as perceived by the target then the motion camouflage is more likely to be detected. It would be interesting to determine which strategy within the space of all motion camouflage strategies is adopted by dragonflies. The framework developed here should help to make this possible.

A related motion camouflage strategy which is also observed in dragonflies (Mizutani *et al.*, 2003) involves remaining stationary with respect to distant objects, i.e. remaining on the same bearing as seen by the target. As Mitzutani *et al.* (2003) note, this is equivalent to taking the fixed reference point at infinity, and is often cited in sailing manuals (and pilot training) as a criterion indicating a potential collision course. Again, this is easy to detect in experiments as the lines connecting the aggressor and the target at different times will be parallel. The differential equation for the motion of the aggressor can be derived by fixing a constant unit vector $\mathbf{e}$ and noting that the condition that the angle between the line from the target to the aggressor and the direction determined by $\mathbf{e}$ as seen from the target at $\mathbf{z}$ is

$$(\mathbf{z}+\mathbf{e}).(\mathbf{z}-\mathbf{r}) = |\mathbf{z}+\mathbf{e}||\mathbf{z}-\mathbf{r}|\cos\theta \tag{10}$$

which is the starting point for the derivation of another set of equations. In this case, linear motion by the target is met by linear motion from the aggressor if capture is possible.

6. Conclusion

Motion camouflage strategies are likely to be encountered in many circumstances; humans seem equally susceptible (Anderson & McOwan, 2003b). This note provides a simple modelling technique which can be used to assess the strategy numerically using widely available differential equation packages. This has also made it possible to provide accurate pictures of the ideal motion in standard and non-standard cases. Moreover, the theoretical ideas presented are easily modified to incorporate refinements of the constant speed solutions.

If the target moves in a straight line with constant speed, the motion camouflage equations do not have a closed solution in terms of standard special functions, but simulations show that the motion camouflage strategy is more efficient than the classical pursuit strategy (Figure 2(c)) although aggressors moving according to the two strategies initially start to close the gap between themselves and the target at the same linear rate. Furthermore, simulations indicate that even if the aggressor moves slower than the target capture is possible in cases where the classical pursuit path ends up following the trail of the target, and that this holds whether the target moves on a straight line or on a chaotic attractor.

This paper provides a coherent mathematical framework within which motion camouflage strategies can be analysed. It raises many questions, mathematical and biological: Which motion camouflage strategy is adopted by dragonflies and hoverflies? How is the transition between motion camouflage and other pursuit strategies determined in real situations (i.e. when does the illusion break down due to size, and how should the target and aggressor react to this discovery)? What is the relationship between the aggressors path and the strange attractor in Figure 3(c)? These questions and others can at least be given a clear mathematical formulation.

The theory presented here does not pretend to explain *how* an insect might follow an ideal motion camouflage path, but it does make it possible to compute these paths and to gain intuition about the strategy given different target movement. These ideal paths can also be compared with experimental measurements and theoretical models which do incorporate realistic biological control mechanisms, as well as other ideal strategies such as the classical pursuit paths discussed here.

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Figure Captions

Figure 1: (a) Motion camouflage paths in the $(x, y)-$plane. The target is moving along the $y-$axis with $v = 1$ and position $\mathbf{z} = (0, t)$ and the aggressor has speed $c = 1.2$. The curves represent the paths of twenty aggressors with initial positions regularly spaced on the circle of radius 5 units centred initial position of the target. (b) The corresponding functions $u(t)$, which are the solutions of (4). These curves can be matched to those of Figure 1(a) by noting that at the point of capture, if $y = Y$ when $x = 0$ in (a) then $u(Y) = 1$ in (b).

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Short Heading: MOTION CAMOUFLAGE