Periodic orbits in some classes of Hamiltonian systems with symmetry

Alomair, Reem

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PERIODIC ORBITS IN SOME CLASSES OF HAMILTONIAN SYSTEMS WITH SYMMETRY

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Reem Alomair
School of Mathematics

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We study the existence of families of periodic orbits near a symmetric equilibrium point in different classes of Hamiltonian systems with symmetry. We center our attention to special types of symmetry less-studied in the literature, such as systems with (semi-)invariant Hamiltonian and reversible equivariant Hamiltonian systems, when the linearisation has two pairs of purely imaginary eigenvalues.

In each case, we provide normal forms for the symmetries, the linear structure map and the linearisation. Moreover, the existence of symmetric and non-symmetric periodic orbits is proved. Another result we found is the classification of Hamiltonian systems with dihedral symmetry, of order eight, with all different possible combinations of time-reversing and symplectic-reversing actions.

The method used in finding periodic orbits is the Liapunov-Schmidt reduction. The symmetry plays a vital role in determining the set of (semi-)invariants, in order to write the reduced problem and then to distinguish the solutions according to their symmetry type.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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To my beloved children, Norah, Abdullah and Abdulaziz I hope I have made you proud.

Chapter 1

Introduction

Periodic and circular motions are signs of regular and natural movements that have attracted many mathematicians for a long time. In dynamics, a periodic orbit arises when a state of the system repeats after some fixed interval of time. Interestingly, many physical examples are modeled by Hamiltonian systems and symmetry is a common theme in such models.

Studying the existence of periodic orbits in Hamiltonian systems is a major area of interest within the field of dynamical systems. In a linear Hamiltonian system, each purely imaginary eigenvalue possesses a family of periodic orbits, with constant period in its eigenspace. This family of solutions is called a normal mode. In a nonlinear Hamiltonian system, families of periodic solutions near an equilibrium that tend to the normal modes are called nonlinear normal modes. A leading figure in the study of nonlinear normal modes is Liapunov, who proved in 1895 his celebrated result: *the Liapunov center theorem* [1]. The theorem guarantees the existence of a one-parameter family of periodic orbits near an equilibrium corresponding to each simple non-resonant purely imaginary eigenvalue of the linearisation.

Later, there have been many extensions of this theorem by many authors. In [36, 31] Weinstein and Moser relaxed the non-resonance condition on eigenvalues. Their main hypothesis was that the quadratic part of the Hamiltonian is positive definite on each resonance space of a purely imaginary eigenvalue. They proved the existence of at least n periodic orbits, with period close to the period of the linear system on each energy level, where n is the dimension of the resonance space. However, this estimation for the number of solutions was not very accurate in some systems. One reason is that they did not consider the effect of symmetry on the possible number of periodic orbits. This was treated by Montaldi *et al.* [28] who took equivariance symmetry into account. They proved the existence of at least $\frac{1}{2}\dim \operatorname{Fix}\Sigma$ periodic orbits, where Σ is an isotropy subgroup. This result is known as *the equivariant Weinstein-Moser theorem.* Other useful results on periodic orbits in symmetric Hamiltonian systems can be found in [29, 30].

Other versions of the Liapunov center theorem assumed the time-reversing symmetry, such as Devaney [11] and Vanderbauwhede [35]. In [11], Devaney studied the existence of symmetric periodic orbits under the action of anti-symplectic time-reversing involutions. He proved under a non-resonance condition, the existence of a two-dimensional manifold consisting of a nested one-parameter family of symmetric periodic orbits. In more recent studies such as Buzzi and Teixeira [10] and Buzzi and Lamb [9], the time reversing involution is assumed to be symplectic. In particular, Buzzi and Lamb announced a remarkable result on the existence of a three-dimensional subset consisting of a two-parameter family of symmetric periodic orbits, with period close to 2π as they approach the equilibrium, if the linearisation has two pairs of eigenvalues $\pm i$.

Another consideration is given by Fadell and Rabinowitz [12], who did not assume the Hamiltonian to be positive definite as in the Weinstein-Moser theorem. In [12], they showed the number of periodic orbits to be at least $\frac{1}{2}|\nu|$, where ν is the signature of $D^2H_p|_{V_\lambda}$ i.e. the number of positive eigenvalues minus the number of negative ones and V_λ is the resonance space.

This thesis aims to prove the existence of families of periodic orbits near equilibria in Hamiltonian systems with presence of symmetry. In particular, we will focus on finite symmetry groups, with semi-invariant Hamiltonian property, which act on $\mathbb{C}^2$. In those cases, when the Hamiltonian is semi-invariant, and since we assume the linear system is periodic, the equilibrium will be in $1 : -1$ resonance. The methods mostly used in finding periodic orbits have a variational structure. In this work, we adapt the constrained Liapunov-Schmidt reduction, given in [9] and [16], to determine periodic orbits of the candidate Hamiltonian system. Although there is a theme that runs through each problem we investigate, different techniques are applied, in order to prove our results.

Most of the material presented in Chapter 2 and Chapter 3 is well-known and introduces the foundations needed in this thesis. This includes definitions and basic results related to the context of Hamiltonian systems with symmetry, and the theory of periodic orbits. In addition, in Chapter 3 we discuss, in some details, the Liapunov-Schmidt reduction, the main tool used in finding periodic orbits in our work.

In Chapter 4, we study the existence of periodic orbits in a purely reversible Hamiltonian system, where the reversing involution R acts symplectically. The problem was introduced and analysed by Buzzi and Lamb [9], a study which we will use extensively in this thesis. Motivated by their work, we looked at the problem using different coordinates, and therefore, a different set of invariants. We recover their result on the existence of symmetric periodic solutions, but obtain a different conclusion for the non-symmetric solutions. This difference is due to a sign error in one of their calculations in the proof of the non-symmetric solutions, which we correct in our approach.

In Chapter 5, we investigate the local dynamics of an equivariant Hamiltonian system, with an involutory symmetry S acting anti-symplectically. Bifurcations of equilibria in Hamiltonian systems with such symmetry have been considered recently by Bosschaert and Hanßmann [6]. The problem was discussed by Li and Shi [25], however, their approach has some serious errors and concerns. The main concern of the study is the failure to write the correct expression of the Hamiltonian, which affect their results significantly. Therefore, we decided to analyse the problem from a different point of view, and we obtain completely different results on the existence of symmetric and non-symmetric periodic solutions to those in [25]. We show that generically, there are no symmetric periodic orbits in a neighbourhood of the equilibrium, and between 2 and 12 non-symmetric orbits.

Chapter 6 highlights two new problems in the reversible equivariant Hamiltonian settings. We focus on this type of symmetry precisely because, to our knowledge, it has not been investigated much in the literature as can be seen in [8, 24, 26, 29, 30] for results on reversible equivariant systems. In the first problem, we consider a Hamiltonian system combined with a reversing symmetry group generated by the two involutions R and S which are studied in Chapter 4 and Chapter 5 respectively. The symmetry forced the Hamiltonian function to be a special case of that discussed in Chapter 4, and therefore, implies a similar result on the existence of periodic orbits.

The work presented in Chapter 4, Chapter 5 and Section 6.1 has been published in Alomair and Montaldi [2].

The second problem that we discuss in Chapter 6 can be seen as a generalisation of the result studied in Chapter 4. Here we increase the order of the cyclic group from 2 to any even number $2r$, where the generator is time-reversing symplectic. We construct the problem from the beginning, by defining the group action, writing the linearisation and finding possible isotropy subgroups. We prove the existence of different families of periodic orbits, depending on the choice of the number r.

Classifying Hamiltonian systems with dihedral symmetry D_4 is the core of Chapter 7. We centre our attention on describing all possible D_4 symmetries, using a representation theory argument. This will be done by illustrating all possible normal forms for the quadratic Hamiltonian, the linear structure map and therefore the linearisation of all D_4 symmetry types. The remainder of this chapter is devoted to studying the existence of families of periodic orbits in four different classes of D_4 Hamiltonian systems.

In Chapter 8, we conclude this thesis by recalling our main results which were introduced in the earlier chapters. In addition, we suggest other considerations, which may lead to further work in the future.

Chapter 2

Preliminaries

This chapter is devoted to introducing some definitions, concepts and well-known properties of Hamiltonian systems with symmetry, which will be relied on throughout the thesis. Most of the material presented in this chapter is standard, and can be found in the literature, for example [1, 4, 14, 18, 27]. In Section 2.1, we define Hamiltonian dynamical systems and their basic properties. In addition, we show the geometric interpretation of Hamiltonian systems. Section 2.2 introduces an overview of symmetry in Hamiltonian systems and its different types.

2.1 Hamiltonian systems

As the main interest of this thesis is to study the existence of periodic orbits in some classes of symmetric Hamiltonian systems, we start by recalling some basic facts about these systems.

2.1.1 Hamiltonian formulation

Many natural and physical systems are well modeled by Hamiltonian systems which are considered as a special kind of dynamical systems, in which the energy is conserved. In a sense, Hamiltonian systems are very essential in classical mechanics due to their direct relation to the energy concept. Moreover, the symmetry of Hamilton's equations leads to an elegant and rich geometric structure in Hamiltonian systems.

The idea of Hamiltonian systems is due to W.R. Hamilton, who observed that by means of a Legendre transformation, a Lagrangian system of n second order differential equations is converted to a symmetrical system of $2n$ first order differential equations, which is called a Hamiltonian system [4].

Consider a mechanical system with configuration space $\mathbb{R}^n$ and Lagrangian $\mathcal{L}(q, \dot{q})$. Recall that *the Lagrangian function* is often defined as the difference between the kinetic and potential energies. *Lagrange's equation* is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}, \quad i = 1, \dots, n,$$

where the q_i are configuration coordinates. One defines the conjugate momentum by

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad i = 1, \dots, n.$$

The Hamiltonian function is defined by

$$H = p \cdot \dot{q} - \mathcal{L}(q, \dot{q}).$$

Substituting $\dot{q}$ as a function of p yields $H = H(q, p)$. In simple mechanical systems, the Hamiltonian is the sum of the kinetic and potential energies of the system, i.e. presents the total energy. The equation of motion derived from the Hamiltonian and expressed by position and momentum coordinates, is called *a Hamiltonian system* and is defined as follows:

A Hamiltonian system is a system of $2n$ ordinary differential equations of the form

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \dots, n, \end{aligned} \tag{2.1}$$

where $q = (q_1, q_2, \dots, q_n)$ and $p = (p_1, p_2, \dots, p_n)$ are *position and momentum coordinates*, and $H : \mathbb{R}^{2n} \to \mathbb{R}$ is *the Hamiltonian or the energy function*. The integer n is called *the number of degrees of freedom* of the system.

Another useful formula of Hamiltonian systems is by using complex coordinates. This is due to the even dimensional property of such systems; the following lemma illustrates this formula:

Lemma 2.1.1 *For $z = q + ip$ the Hamiltonian system* (2.1) *takes the form*

$$\dot{z} = -2i\frac{\partial H}{\partial \bar{z}}.$$

Proof Using complex variables, one can write $z_j = (q_j + ip_j)$, for $j = 1, 2, \ldots, n$. Clearly,

$$\begin{aligned} q_j =& \frac{1}{2}(z_j + \bar{z}_j), \\ p_j =& \frac{-i}{2}(z_j - \bar{z}_j). \end{aligned}$$

By partial differentiation, we have

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_j} &= \frac{\partial q_j}{\partial \bar{z}_j}\frac{\partial}{\partial q_j} + \frac{\partial p_j}{\partial \bar{z}_j}\frac{\partial}{\partial p_j} \\ &= \frac{1}{2}(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}). \end{aligned}$$

By the Hamiltonian system (2.1) and for each $j = 1, \ldots, n$ we obtain

$$\dot{z}_j = \dot{q}_j + i\dot{p}_j = \left(\frac{\partial H}{\partial p_j} - i\frac{\partial H}{\partial q_j}\right) = -i\left(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}\right)H = -2i\frac{\partial H}{\partial \bar{z}_j}.$$

Thus, for $z = q + ip$ where $q = (q_1, q_2, \ldots, q_n)$ and $p = (p_1, p_2, \ldots, p_n)$ one can write the Hamiltonian system (2.1) as

$$\dot{z} = -2i\frac{\partial H}{\partial \bar{z}}.$$

□

2.1.2 Geometric approach to Hamiltonian systems

The anti-symmetry property of Hamilton's equations gave rise to the use of symplectic geometry in the Hamiltonian context. This means that the Hamiltonian vector field of a function H over a symplectic manifold is formed to have the same role of gradient of the Hamiltonian and the symplectic structure takes care of skew-symmetry and conservation properties. Before defining Hamiltonian systems over a symplectic manifold, we will introduce some basic definitions.

Let V be an n-dimensional vector space over a field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *bilinear form* B is a mapping $B : V \times V \to \mathbb{F}$ which is linear in both components, i.e.

- $B(u_1 + u_2, v) = B(u_1, v) + B(u_2, v)$,
- $B(u, v_1 + v_2) = B(u, v_1) + B(u, v_2)$,
- $B(\lambda u, v) = B(u, \lambda v) = \lambda B(u, v)$,

for all $u, u_1, u_2, v, v_1, v_2 \in V$ and $\lambda \in \mathbb{F}$. A bilinear form B is called *skew-symmetric*, if $B(u,v) = -B(v,u)$, for all $u, v \in V$. A bilinear form B is called *non-degenerate*, if

$$B(u,v) = 0, \forall v \in V \Rightarrow u = 0.$$

Definition 2.1.2 A *symplectic form* ω on a vector space V is a non-degenerate, skew-symmetric bilinear form. The pair (V, ω) is called *a symplectic space.*

Note that the linear space V should be even dimensional. That is because, any skew-symmetric matrix of odd size is always singular and therefore, a non-degenerate, skew-symmetric form is defined on an even dimensional space. For a more general picture, we shall define a symplectic manifold to be the phase space for Hamiltonian systems. Throughout, let M be a manifold of dimension $2n$. A 2-form ω on M is called closed, if $d\omega = 0$, where d is the exterior derivative.

Definition 2.1.3 A *symplectic form* ω on a manifold M is a non-degenerate, skew-symmetric, bilinear, closed differential 2-form. A *symplectic manifold* (M, ω) is a manifold M equipped with the symplectic form ω.

Now, we can define Hamiltonian vector fields with symplectic structure.

Definition 2.1.4 Let (M, ω) be a symplectic manifold, and let $H : M \to \mathbb{R}$ be a given C^k differentiable function where $k \geq 1$. The vector field X_H determined by the condition

$$\omega(X_H, u) = dH.u$$

is called *the Hamiltonian vector field with energy (Hamiltonian) function H*. The triple (M, ω, X_H) is called *a Hamiltonian system.*

By using canonical coordinates for the symplectic form ω given in Darboux's theorem [1], the Hamiltonian vector field can be written locally in the following, elegant way.

Proposition 2.1.5 (Proposition 3.3.2, [1]) *Let $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ be canonical coordinates for ω, so $\omega = \sum dq^i \wedge dp_i$. Then, in these coordinates,*

$$X_H = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}) = J.dH,$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Thus, $(q(t), p(t))$ *is an integral curve of* X_H *if and only if Hamilton's equations hold:*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2, \ldots, n.$$

The matrix J is called the *(linear) structure map*, and it is the skew-symmetric matrix associated to the symplectic form ω i.e.

$$\omega(u, v) = \langle u, Jv \rangle, \quad \forall u, v \in \mathbb{R}^{2n},$$

where $\langle ., . \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. Moreover, J satisfies

$$J^2 = -I_{2n},$$

where I_{2n} is the $2n \times 2n$ identity matrix. For more details, one can see [5, 27].

2.2 Hamiltonian systems with symmetry

Symmetry, similarity and repeated patterns are interesting and regular features in nature. Therefore, studying symmetric dynamical systems was of interest to many mathematicians past and present. Symmetries help us understand and describe pattern formation in such dynamical systems. Specifically, symmetries could be used to write down typical forms and structures independently of the dynamics, which gives a good amount of information about the system. In a sense, all systems with same type of symmetry can share the same pattern-forms, and exhibit the same behaviour deduced from a candidate system. We will begin this section by introducing some general concepts of symmetry groups and their actions. After that, we will demonstrate different symmetry types which may accrue in Hamiltonian systems.

2.2.1 Symmetry groups

Symmetries of a Hamiltonian system can be seen as a group of transformations that preserves the structure of the system, and especially its solutions. Symmetry groups are often assumed to be compact Lie groups that act on the phase space. However, in our work we focus on finite groups only. Let (M, ω) be a symplectic manifold and let G be a compact Lie group acting on M. Note that we choose M to be a symplectic manifold for the sake of generality, however, we often work with $\mathbb{R}^{2n}$ coordinates as our results are taken locally.

Definition 2.2.1 An action of G on M is a smooth map $\phi : G \times M \to M$ defined by $gx = \phi(g, x)$ that satisfies the two following conditions

1. $1x = x$, where 1 is the identity element of G,
2. $(gh)x = g(hx)$,

for all $g, h \in G$ and $x \in M$.

Now, we define isotropy subgroups, which are subgroups of G that include all symmetries of a certain point.

Definition 2.2.2 Given $x \in M$, *the isotropy subgroup of* x is

$$G_x = \{g \in G : gx = x\}.$$

Conversely, one can ask about points that respect a certain symmetry which leads to the definition of fixed point subspaces.

Definition 2.2.3 Let H be a subgroup of G. *The fixed point space of* H is defined by

$$\mathrm{Fix}\,(H) = \{x \in M : gx = x, \forall g \in H\}.$$

An important feature of group actions on symplectic manifolds is the semi-symplectic action. A Lie group G *acts semi-symplectically* on a symplectic manifold (M, ω) if $\omega(gx, gy) = \pm\omega(x, y)$ for $g \in G$ and $x, y \in M$. In this case, the choice of sign determines a homomorphism $\chi : G \to \mathbb{Z}_2$ which is called *the symplectic character*, and is defined by

$$\omega(gx, gy) = \chi(g)\omega(x, y).$$

Moreover,

1. If $\chi(g) = +1$, then g is said to *act symplectically*.

2. If $\chi(g) = -1$, then g *acts anti-symplectically*.

Similarly, we say that the Hamiltonian H is G *semi-invariant*, if $H(gx) = \pm H(x)$ for $g \in G$. The Hamiltonian sign is determined by a homomorphism $\alpha : G \to \mathbb{Z}_2$ such that

$$H(gx) = \alpha(g)H(x).$$

Therefore,

1. If $\alpha(g) = +1$, then H is g *invariant*.

2. If $\alpha(g) = -1$, then H is g *anti-invariant*.

After considering symplectic manifolds as the general phase space for Hamiltonian systems, we will assume from now on that our phase space will be $V = \mathbb{R}^{2n}$. That is because, we are interested in the local dynamics near equilibria and all problems discussed in this thesis are of this type.

As a result, it is useful to view the action of symmetry groups by group representations. In the following, we will introduce some basic definitions and results from the theory of group representations and characters, which will be used frequently in the following chapters.

Let V be a vector space of dimension n over the field $\mathbb{F}$.

Definition 2.2.4 A *linear representation* of the finite group G in V is a homomorphism $\rho : G \to GL(n, \mathbb{F})$, where $GL(n, \mathbb{F})$ is the group of invertible $n \times n$ matrices with entries in the field $\mathbb{F}$.

For a given representation ρ we often say that V is a representation of G. An irreducible representation V is a representation where the only invariant subspaces are 0 and V.

Definition 2.2.5 A *character* of a representation ρ is the function χ_ρ defined by

$$\chi_\rho(g) = Tr(\rho_g), \quad \forall g \in G.$$

The group character of a group representation is a class function, i.e. it is constant on a conjugacy class. Moreover, the number of conjugacy classes of a finite group is equal to the number of irreducible characters. Thus, the values of irreducible characters can be written as a square matrix (table), known as a character table.

Now, we consider some basic properties of characters, which will be used later, especially in Chapter 7.

Lemma 2.2.6 *If V,V_1 and V_2 are (real or complex) representations, with characters χ,χ_1 and χ_2, then*

1. *The character of the direct sum $V_1 \oplus V_2$ is $\chi_1 + \chi_2$.*

2. *The character of the tensor product $V_1 \otimes V_2$ is $\chi_1\chi_2$.*

3. *The character of the symmetric and anti-symmetric tensor products of a representation V are*

$$\chi_s(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right), \quad \chi_a(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right).$$

4. *The inner product of two characters χ_1 and χ_2 is defined to be*

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\chi_2^*(g),$$

where $\chi^*(g)$ is the complex conjugate. Consequently, the norm is defined by

$$\|\chi\|^2 = \langle \chi, \chi \rangle = \frac{1}{|G|}\sum_{g\in G} |\chi(g)|^2.$$

5. *If W is irreducible with character χ_W, and V is any representation with character χ, then the multiplicity of W in V is equal to $\langle \chi_W, \chi \rangle / \|\chi_W\|^2$.*

More information on representations and character theory can be found, for example, in [22] and [33].

2.2.2 Symmetry types in Hamiltonian systems

Let (V, ω) be a symplectic space, e.g. $V = \mathbb{R}^{2n}$, and $H : V \to \mathbb{R}$ a Hamiltonian function, that generates the Hamiltonian system

$$\dot{x} = f(x). \tag{2.2}$$

Suppose that G is a compact Lie group acting on V. The system (2.2) equipped with the symmetry group G can only be one of the following types:

- Equivariant Hamiltonian systems.
- Reversible Hamiltonian systems.
- Reversible equivariant Hamiltonian systems.

More details about each kind is given in the following.

Equivariant Hamiltonian systems

Equivariance in (Hamiltonian) dynamical systems is a classical type of symmetry, which has been studied in many publications, such as [13, 14, 18, 19, 28]. Precisely, the equivariance symmetry happens when a Hamiltonian vector field commutes with a specific linear map.

Let S be a transformation from V to itself. The Hamiltonian vector field f is called *S-symmetric* or *S-equivariant* if

$$f(Sx) = Sf(x), \forall x \in V.$$

In many applications, the vector field can have more than one symmetry. Moreover, these symmetries form a group and so gave rise to defining equivariant dynamical systems under the action of a group of symmetries.

The Hamiltonian vector field (2.2) is said to be equivariant, under the action of the group G, if

$$f(gx) = gf(x), \text{ for all } g \in G, x \in V.$$

Simply said, the vector field f is *G-equivariant*, if it commutes with all elements in G. A basic property of solutions of equivariant Hamiltonian systems is that, for any solution $x(t)$ either

1. $gx(t) = x(t), \forall t$, or
2. $gx(t) = y(t) \neq x(t), \forall t$.

The first solution is called a *g-symmetric solution.* Note that $y(t)$ is also a solution of the system. Thus, it is possible to determine the solutions of equivariant dynamical systems according to their symmetry properties.

Clearly, the previous argument fits for any dynamical system, not necessarily Hamiltonian. Now, we apply the Hamiltonian structure to specialise the definition.

Equivariance in Hamiltonian systems can accrue in two cases:

- The symmetry S acts symplectically, and the Hamiltonian H is S invariant. This case is classical and was discussed in many publications (for example, see [28, 29]).
- The symmetry S acts anti-symplectically, and H is S anti-invariant. This case is less discussed in publications, however, it has been discussed recently in [6] and we have considered a system with this property in Chapter 5.

Now we consider a simple example, that illustrates the definition.

Example 2.2.7 Consider the *Hénon-Heiles* Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}({p_1}^2 + {p_2}^2 + {q_1}^2 + {q_2}^2) + \frac{1}{3}{q_1}^3 - q_1{q_2}^2.$$

Accordingly, its Hamiltonian system takes the following form

$$\begin{aligned}
\dot{q_1} &= p_1, \\
\dot{q_2} &= p_2, \\
\dot{p_1} &= -(q_1 + {q_1}^2 - {q_2}^2), \\
\dot{p_2} &= -(q_2 - 2q_1q_2).
\end{aligned}$$

This system is D_3-equivariant, where $D_3 = \langle \rho, \kappa \rangle$ acts on the q-plane and the p-plane as follows:

$$\rho(q_1 + iq_2, p_1 + ip_2) = (e^{2\pi i/3}(q_1 + iq_2), e^{2\pi i/3}(p_1 + ip_2)),$$
$$\kappa(q_1 + iq_2, p_1 + ip_2) = (q_1 - iq_2, p_1 - ip_2).$$

For more details about this example see [18, 28].

Reversible Hamiltonian systems

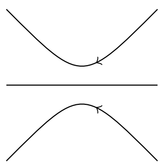
Reversing symmetry is one of most common types of symmetry that arises naturally in physically-motivated dynamical systems. Moreover, the majority of applications of time-reversing symmetry have been found in Hamiltonian systems. Simply put, the idea of a time-reversing symmetry is if we cannot decide whether an object of some mechanical system is moving forward or reverse the direction. The time-reversing symmetry in dynamical systems was discussed in many publications, for example, see [11, 23, 34]. Next, we give the formal definition of reversing symmetries and some basic properties.

Let R be a transformation on V. We say that a vector field f is *R-reversible*, or that f possesses a *time-reversing symmetry R* if

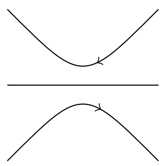
$$f(Rx) = -Rf(x), \forall x \in V.$$

This implies that the reversible Hamiltonian system (2.2) has a solution $x(t)$ if and only if $Rx(-t)$ is also a solution. If the vector field f does not have any other non-trivial symmetries, then $R^2 = I$ and we say that f is a *reversible Hamiltonian system*, or more precisely, a *purely reversible system*. Time-reversing symmetries in Hamiltonian systems are split into two types: symplectic and anti-symplectic, as we will discuss in more details later. A typical example of a time-reversing Hamiltonian system is the ideal pendulum.

Figure 2.1 shows the difference between equivariance and reversing symmetries, by plotting a planar flow, that is symmetric under a horizontal reflection. In Figure (a) trajectories are identical after reflection, where in (b) the direction is reversed.



(a) Equivariant symmetry



(b) Time-reversing symmetry

Figure 2.1: Phase portraits of planar flows that are symmetric with respect to a horizontal reflection. In (a) the reflection is an equivariant symmetry and in (b) the reflection is a reversing symmetry.

Reversible equivariant Hamiltonian systems

Reversible equivariant symmetry can be seen as a generalisation of the two previous types. It is known that the set of symmetries of a given system is closed under composition, while the set of reversing symmetries is not. But, the composition of two reversing symmetries is a symmetry. As a result, the set of symmetries and reversing symmetries of a vector field f forms a group G. There are only limited publications on this type of symmetry, especially in Hamiltonian systems, for example [8, 24, 29]. In the following, we will assume that G is a compact Lie group.

A Hamiltonian vector field f is called *G-reversible equivariant* if there exists a representation ρ of G i.e. a group homomorphism $\rho : G \to GL(2n, \mathbb{R})$ and a homomorphism (reversing sign) $\sigma : G \to \{\pm 1\}$ such that

$$f\rho(g) = \sigma(g)\rho(g)f, \forall g \in G. \tag{2.3}$$

Example 2.2.8 The dihedral group D_4 is defined by $D_4 = \langle \kappa, \kappa' | \kappa^2 = \kappa'^2 = (\kappa\kappa')^4 = e\rangle$. Let D_4 act on $\mathbb{C}^2$ by

$$\kappa(q_1 + ip_1, q_2 + ip_2) = (q_1 + ip_1, -q_2 - ip_2),$$

$$\kappa'(q_1 + ip_1, q_2 + ip_2) = (p_2 + iq_2, p_1 + iq_1).$$

In matrix form these actions are equivalent to

$$\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \kappa' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the Hamiltonian $H(q_1, p_1, q_2, p_2) = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2)$. It is easy to check that the Hamiltonian system generated by H is κ-equivariant and κ'-reversing and therefore, a D_4-reversible equivariant system.

It will be convenient to recall and collect together the following homomorphisms, which play a vital role in distinguishing between the previous kinds of symmetry. For a group of symmetries and reversing symmetries G of the Hamiltonian system (2.2), we define the following group homomorphisms.

- The reversing sign σ

$$\sigma : G \to \{\pm 1\}$$

$$\sigma(g) = \begin{cases} 1, & \text{if } f(gx) = gf(x), \\ -1, & \text{if } f(gx) = -gf(x), \end{cases}$$

  which is used to distinguish between symmetries and reversing symmetries.

- The symplectic sign χ

$$\chi : G \to \{\pm 1\}$$

$$\chi(g) = \begin{cases} 1, & \text{if } \omega(gx, gy) = \omega(x, y), \\ -1, & \text{if } \omega(gx, gy) = -\omega(x, y). \end{cases}$$

  In the first case, we say that g is acting symplectically, and in the second one, g is acting anti-symplectically.

- The Hamiltonian sign α

$$\alpha : G \to \{\pm 1\}$$

$$\alpha(g) = \begin{cases} 1, & \text{if } H(gx) = H(x), \\ -1, & \text{if } H(gx) = -H(x). \end{cases}$$

  In the first case, H is g invariant, and in the second one, H is g anti-invariant.

The following lemma identifies the relation between these homomorphisms.

Lemma 2.2.9 *For a G-reversible equivariant Hamiltonian vector field, the Hamiltonian function H satisfies*

$$H(\rho(g)u) = \sigma(g)\chi(g)H(u), \forall g \in G, \forall u \in V.$$

Proof Since the Hamiltonian vector field f is G-reversible equivariant, then

$$J\nabla H(\rho(g)u) = \sigma(g)\rho(g)J\nabla H(u),$$

where the representation ρ is chosen to be orthogonal, i.e. $\rho(g)\rho(g)^T = I$. It is clear that the matrix J satisfies

$$J\rho(g) = \chi(g)\rho(g)J.$$

Therefore,

$$J\nabla H(\rho(g)u) = \sigma(g)\chi(g)J\rho(g)\nabla H(u).$$

As $\sigma(g), \chi(g)$ are just numbers, and J is nonsingular, we get

$$\nabla H(\rho(g)u) = \sigma(g)\chi(g)\rho(g)\nabla H(u).$$

The chain rule yields

$$
\begin{aligned}
[\rho^{-1}(g)]^T \nabla_u H(\rho(g)u) &= \sigma(g)\chi(g)\rho(g)\nabla_u H(u), \\
\Leftrightarrow \rho(g)\nabla_u H(\rho(g)u) &= \sigma(g)\chi(g)\rho(g)\nabla_u H(u), \\
\Leftrightarrow \nabla_u H(\rho(g)u) &= \sigma(g)\chi(g)\nabla_u H(u).
\end{aligned}
$$

Integration with respect to u implies the result. □

The previous result can be written simply as $\alpha(g) = \sigma(g)\chi(g)$, for all $g \in G$.

For a Hamiltonian system, (anti-)commuting properties of a (reversing) symmetry g implies the following cases which will be used frequently in this thesis.

Type of symmetry	Notation	σ	χ	α
equivariant	SE	$+1$	$+1$	$+1$
equivariant	AE	$+1$	-1	-1
reversing	AR	-1	-1	$+1$
reversing	SR	-1	$+1$	-1

Table 2.1: Possible types of symmetries in Hamiltonian systems.

The symmetry type (SE) represents a symplectic equivariant symmetry. Similarly, (AE) stands for an anti-symplectic equivariant symmetry. A similar pattern is used for the time-reversing symmetries (AR) and (SR).

Chapter 3

Periodic orbits in Hamiltonian systems

Studying the existence of periodic orbits is a very common theme when analysing Hamiltonian systems, especially near equilibria. In this chapter, we will introduce some basic concepts and techniques used in the theory of periodic solutions in symmetric Hamiltonian systems. In Section 3.1, we will introduce some definitions related to periodic orbits in Hamiltonian systems and their symmetries. Section 3.2 will review the Liapunov-Schmidt reduction, which is a classical tool in proving the existence of periodic orbits in dynamical systems. Furthermore, in Section 3.3, we will state some well-known theorems in the context of periodic orbits in Hamiltonian systems, with and without symmetry.

3.1 Basic concepts

Alongside equilibria, periodic solutions are considered to be another simple set of solutions of a dynamical system. In the Hamiltonian context, finding periodic orbits was of wide interest especially after the celebrated Liapunov center theorem described in Chapter 1. Now, we will consider some key facts and definitions related to periodic orbits, which will be used throughout this thesis.

Consider a symplectic space (V, ω). Let $H : V \to \mathbb{R}$ be a Hamiltonian function that generates the Hamiltonian system

$$\dot{u} = f(u), \tag{3.1}$$

with the origin as an equilibrium point of this system. The eigenvalues of the linearised vector field L at 0 arise as quadruplets $\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$. If $\mathrm{Re}\lambda \neq 0$ for all eigenvalues λ of L, then, by Hartman-Grobman theorem the vector field (3.1) is *topologically equivalent* to its linear part $\dot{u} = Lu$ in a neighbourhood of the equilibrium. An equilibrium with this property is called a *hyperbolic equilibrium.* Therefore, in generic (non-Hamiltonian) systems, this theorem is enough to describe the local dynamics, as the eigenvalues do not generically lie on the imaginary axis. However, in Hamiltonian system, having purely imaginary eigenvalues is a generic property, so, we need more than a linearisation of the system to study the local dynamics [7].

Now, we introduce the symmetry settings of this chapter. Let G be a compact Lie group acting on V, so that, the vector field f is G-reversible equivariant as described before in Equation (2.3). This implies the existence of a representation $\rho : G \to GL(2n, \mathbb{R})$ and a reversing sign $\sigma : G \to \{\pm 1\}$ such that

$$f\rho(g) = \sigma(g)\rho(g)f, \forall g \in G.$$

Remark 3.1.1 Equivariance and time-reversing symmetries are the classical types of symmetry mostly discussed in the literature of Hamiltonian systems. However, we assume the group G to be reversible equivariant for the sake of generality. That is because, it covers both cases, when g is a symmetry or a reversing symmetry.

Let $u(t)$ be a 2π-periodic solution of the Hamiltonian system (3.1). For a (time-reversing) symmetry $g \in G$ we have that $gu(\sigma(g)t)$ is also a periodic solution, and by uniqueness of differential equations, either

$$\{u(t)\} = \{gu(\sigma(g)t)\} \text{ or } \{u(t)\} \cap \{gu(\sigma(g)t)\} = \emptyset.$$

So, for the first case we have

$$gu(t) = u(\sigma(g)t + \theta),$$

for some phase shift θ and $u(t)$ is called a *symmetric periodic solution.* This gives rise to defining the circle group S^1, which can be identified with $\mathbb{R}/2\pi\mathbb{Z}$, and its action on the space of periodic orbits. Accordingly, the symmetry group on the space of periodic orbits of the system (3.1) is $G \ltimes S^1$. In the following, we introduce some fundamental properties of the group $G \ltimes S^1$.

Definition 3.1.2 The multiplication of elements in the group $G \ltimes S^1$ is determined by the homomorphism σ and is given by

$$(g_1, \theta_1) \cdot (g_2, \theta_2) = (g_1 g_2, \sigma(g_2)\theta_1 + \theta_2),$$

for $g_1, g_2 \in G$ and $\theta_1, \theta_2 \in S^1$.

Lemma 3.1.3 *The following formula defines an action of the semidirect product $G \ltimes S^1$ on the space of loops:*

$$((g, \theta)u)(t) = gu(\sigma(g)t + \theta),$$

for $(g, \theta) \in G \ltimes S^1$.

PROOF By the definition of the group action stated in Definition 2.2.1, we need to check the identity and compatibility conditions. Obviously, the identity condition is satisfied. For the second condition, let $(g_1, \theta_1), (g_2, \theta_2) \in G \ltimes S^1$. Then

$$\begin{aligned}
(g_1, \theta_1)\left((g_2, \theta_2)u\right)(t) &= (g_1, \theta_1)\left(g_2 u(\sigma(g_2)t + \theta_2)\right) \\
&= \left((g_1, \theta_1)v\right)(t), \quad \text{where } v(t) = g_2 u(\sigma(g_2)t + \theta_2) \\
&= g_1 v(\sigma(g_1)t + \theta_1) \\
&= g_1 g_2 u(\sigma(g_2)(\sigma(g_1)t + \theta_1) + \theta_2) \\
&= g_1 g_2 u(\sigma(g_1 g_2)t + \sigma(g_2)\theta_1 + \theta_2) \\
&= \left((g_1 g_2, \sigma(g_2)\theta_1 + \theta_2)u\right)(t) \\
&= \left((g_1, \theta_1) \cdot (g_2, \theta_2)u\right)(t).
\end{aligned}$$

Thus, the formula $((g, \theta)u)(t) = gu(\sigma(g)t + \theta)$ indeed defines an action of the semi direct product $G \ltimes S^1$ on the loop space. □

Now, we define a spatio-temporal symmetry group, which is the group of symmetries that fixes an orbit as a set.

Definition 3.1.4 The *spatio-temporal symmetry group* of the periodic orbit $u(t)$ is

$$\Sigma_u = \{(g, \theta) \in G \ltimes S^1, gu(t) = u(\sigma(g)t + \theta)\} < G \ltimes S^1.$$

Thus, symmetries of periodic solutions are given by isotropy subgroups of $G \ltimes S^1$.

The following proposition characterises isotropy subgroups of $G \ltimes S^1$ depending on the subgroups of G. We will follow the settings given in Golubitsky *et al.* [19]. However, we will apply some changes according to the semidirect product structure.

Definition 3.1.5 Consider the semidirect product $G \ltimes S^1$. A mapping $\phi : G \to S^1$, satisfying the condition

$$\phi(g_1 g_2) = \sigma(g_2)\phi(g_1) + \phi(g_2),$$

is called a crossed homomorphism.

Definition 3.1.6 Let H be a subgroup of G and let $\theta : H \to S^1$ be a crossed group homomorphism. The set

$$H^\theta = \{(h, \theta(h)) \in G \ltimes S^1, h \in H\},$$

is called a *twisted subgroup* of $G \ltimes S^1$.

Proposition 3.1.7 *Let Σ be an isotropy subgroup of $G \ltimes S^1$ and S^1 acts freely outside the origin, with $\Sigma \neq G \ltimes S^1$. Let $\pi : G \ltimes S^1 \to G$ be the projection, and $H = \pi(\Sigma)$. Then,*

1. *$\pi : \Sigma \to H$ is an isomorphism.*

2. *There is a crossed homomorphism $\theta : H \to S^1$ such that $\Sigma = H^\theta$.*

Proof

1. By the fact that S^1 is acting freely outside the origin; the projection $\pi : \Sigma \to H$ is an isomorphism.

2. The isomorphism $\pi : \Sigma \to H$ guarantees the existence of a map, $\theta : H \to S^1$ such that each element in Σ can be uniquely written as $(h, \theta(h))$, $h \in H$. It remains to prove that θ is a crossed homomorphism, for that purpose, let $(h_1, \theta(h_1)), (h_2, \theta(h_2)) \in \Sigma$. By Definition 3.1.2, we have

$$\begin{aligned}(h_1, \theta(h_1)) \cdot (h_2, \theta(h_2)) &= (h_1 h_2, \sigma(h_2)\theta(h_1) + \theta(h_2)) \\ &= (h_1 h_2, \theta(h_1 h_2)).\end{aligned}$$

Therefore,

$$\theta(h_1 h_2) = \sigma(h_2)\theta(h_1) + \theta(h_2),$$

which proves that θ is a crossed homomorphism. □

3.2 The Liapunov-Schmidt reduction

The classical way of finding periodic orbits in Hamiltonian systems is by solving an equation on the loop space $\mathcal{C}_{2\pi}$ defined below. This equation is of infinite dimension and can be reduced by a procedure called the *Liapunov-Schmidt Reduction*. In other words, the problem of finding periodic orbits of a given system, with period close to the period of the linear system, which we rescale to be 2π, is equivalent to the problem of finding zeros of a suitable map Φ on the loop space, with a parameter τ representing the perturbed period. The Liapunov-Schmidt procedure reduced the original problem to finding zeros of the reduced bifurcation equation which is of finite dimension, and inherits symmetry properties of the original map Φ. Moreover, the bifurcation equation will be a Hamiltonian vector field. In this section, we will give an overview of that method, and how to use it in finding periodic orbits near an equilibrium point in a reversible equivariant Hamiltonian system . We will use the method introduced in [9] and [16]. In addition, some useful details can be found in [15].

Now, we give the general assumptions of the Liapunov-Schmidt reduction. Let $f : V \to V$ be the Hamiltonian vector field generated by the Hamiltonian function $H : V \to \mathbb{R}$ and $f(0) = 0$, where V is a symplectic space of dimension $2n$. Let G be a compact Lie group acting on V, and f a G-reversible equivariant Hamiltonian vector field. For simplicity, we divide the Liapunov-Schmidt reduction into few basic steps.

3.2.1 Defining the operator Φ

Let $\mathcal{C}_{2\pi}$ be the Banach space of $\mathbb{R}^{2n}$-valued, 2π-periodic mappings, and let $\mathcal{C}^1_{2\pi}$ be the space of $\mathcal{C}_{2\pi}$ functions that are continuously differentiable. Let $\dfrac{du}{ds} = f(u)$ be a Hamiltonian dynamical system with 0 as an equilibrium point. By introducing a scaling parameter τ let $t = (1+\tau)s$ so we have $\dfrac{d}{ds} = (1+\tau)\dfrac{d}{dt}$. Define the map

$$\Phi : \mathcal{C}^1_{2\pi} \times \mathbb{R} \to \mathcal{C}_{2\pi},$$

$$\Phi(u, \tau) = (1+\tau)\frac{du}{dt} - f(u). \tag{3.2}$$

It is readily seen that zeros of Φ are periodic solutions of the given Hamiltonian system with period $\frac{2\pi}{1+\tau}$. Now we can define the group action on the loop space $\mathcal{C}_{2\pi}$ or $\mathcal{C}^1_{2\pi}$ as follows:

$$T : \tilde{G} \times \mathcal{C}_{2\pi} \to \mathcal{C}_{2\pi},$$

$$(T_g u)(t) = \rho(\gamma)(u(\sigma(\gamma)t + \theta)),$$

where $g = (\gamma, \theta)$ is an element of $\tilde{G} = G \ltimes S^1$.

Lemma 3.2.1 *The map Φ is $\tilde{G}$-reversible equivariant, that is*

$$\Phi(T_g u, \tau) = \sigma(\gamma) T_g \Phi(u, \tau), \forall g = (\gamma, \theta) \in \tilde{G}.$$

Proof By the definition of Φ and the action of T_g one can write

$$\begin{aligned}
\Phi(T_g u, \tau)(t) &= (1+\tau)\frac{d}{dt}((T_g u)(t)) - f(T_g u)(t) \\
&= (1+\tau)\frac{d}{dt}\left(\rho(\gamma)(u(\sigma(\gamma)t+\theta)) - f(\rho(\gamma)(u(\sigma(\gamma)t+\theta))\right) \\
&= (1+\tau)\sigma(\gamma)\rho(\gamma)\frac{du(s)}{ds} - \rho(\gamma)\sigma(\gamma)f(u(s)), s = \sigma(\gamma)t + \theta \\
&= \sigma(\gamma)\left[(1+\tau)\rho(\gamma)\frac{du(s)}{ds} - \rho(\gamma)f(u(s))\right] \\
&= \sigma(\gamma)\rho(\gamma)\left[(1+\tau)\frac{du(s)}{ds} - f(u(s))\right] \\
&= \sigma(\gamma)\rho(\gamma)\Phi(u,\tau)(\sigma(\gamma)t+\theta) \\
&= \sigma(\gamma)T_g\Phi(u,\tau).
\end{aligned}$$

□

3.2.2 The linearisation of Φ

The linear part of Φ is defined by:

$$\mathcal{L} = (d\Phi)_{(0,0)}.$$

First, we calculate the directional derivative $(d\Phi(u,\tau))(v)$ at $(u,\tau) \in \mathcal{C}^1_{2\pi} \times \mathbb{R}$ in the direction of v,

$$\begin{aligned}
(d(u,\tau)\Phi)v &= \lim_{h\to 0}\frac{\Phi(u+hv,\tau) - \Phi(u,\tau)}{h} \\
&= \lim_{h\to 0}\frac{(1+\tau)\frac{d}{dt}(u+hv) - f(u+hv) - (1+\tau)\frac{du}{dt} + f(u)}{h} \\
&= (1+\tau)\frac{dv}{dt} - \lim_{h\to 0}\frac{f(u+hv) - f(u)}{h} \\
&= (1+\tau)\frac{dv}{dt} - (df(u))v.
\end{aligned}$$

Thus,

$$\mathcal{L}u = \frac{du}{dt} - Lu,$$

where $L = (df)_0$. Now let $u \in \ker \mathcal{L}$, this means

$$\frac{du}{dt} = Lu. \tag{3.3}$$

By solving the linear system (3.3), we get

$$u(t) = u(0) \exp Lt.$$

Therefore, $\ker \mathcal{L}$ corresponds to the periodic solutions of the linear system $\frac{du}{dt} = Lu$ with period 2π.

3.2.3 The splittings

We start by defining the Fredholm operator, as given in [17].

Definition 3.2.2 Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. A bounded linear operator $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is called *Fredholm* if

1. $\ker \mathcal{F}$ is a finite dimensional subspace of $\mathcal{X}$.

2. $\operatorname{range} \mathcal{F}$ is a closed subspace of $\mathcal{Y}$ of finite codimension.

Definition 3.2.3 If $\mathcal{F}$ is a Fredholm operator, the index *of* $\mathcal{F}$ is the integer

$$\operatorname{index} \mathcal{F} = \dim \ker \mathcal{F} - \operatorname{codim} \operatorname{range} \mathcal{F}.$$

The following proposition motivates the introduction of Fredholm operators.

Proposition 3.2.4 (Proposition 1.3, [17]) *If $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is Fredholm, then there exist closed subspaces M and N of $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that*

$$\mathcal{X} = \ker \mathcal{F} \oplus M, \qquad \mathcal{Y} = N \oplus \operatorname{range} \mathcal{F}. \tag{3.4}$$

By the Liapunov-Schmidt construction $\mathcal{L}$ is naturally $\tilde{G}$-reversible equivariant i.e.

$$\mathcal{L} \circ T_g = \sigma(\gamma) T_g \circ \mathcal{L}.$$

In the following lemma we prove the invariance of $\ker \mathcal{L}$ and $\operatorname{range} \mathcal{L}$.

Lemma 3.2.5 $\ker \mathcal{L}$ *and* $\operatorname{range} \mathcal{L}$ *are both invariant, under the action of* T_g.

PROOF First, let $u \in \ker \mathcal{L}$ and $g \in \tilde{G}$. By $\mathcal{L}$ reversing equivariance, we have

$$\mathcal{L}(T_g u) = \sigma(\gamma) T_g \mathcal{L}(u) = 0.$$

Thus, $(T_g u) \in \ker \mathcal{L}$. For the second part, assume $v \in \operatorname{range} \mathcal{L}$ i.e. $v = \mathcal{L}(u), u \in \mathcal{C}^1_{2\pi}$. We have

$$T_g v = T_g(\mathcal{L}(u)) = \sigma(\gamma)\mathcal{L}(T_g u),$$

which means that $T_g v \in \operatorname{range} \mathcal{L}$. □

For Fredholm operators of index zero, the subspaces M and N can be chosen to be the orthogonal complements

$$M = (\ker \mathcal{F})^{\perp}, \qquad N = (\operatorname{range} \mathcal{F})^{\perp}.$$

In the Liapunov-Schmidt procedure, the operator $\mathcal{L} : \mathcal{C}^1_{2\pi} \to \mathcal{C}_{2\pi}$ is a Fredholm operator of index zero by [17]. Therefore,

$$\dim \ker \mathcal{L} = \dim(\operatorname{range} \mathcal{L})^{\perp}.$$

Accordingly, the splittings in (3.4) take the form

$$\mathcal{C}^1_{2\pi} = \ker \mathcal{L} \oplus (\ker \mathcal{L})^{\perp}, \qquad \mathcal{C}_{2\pi} = (\operatorname{range} \mathcal{L})^{\perp} \oplus \operatorname{range} \mathcal{L}. \tag{3.5}$$

Note that the orthogonal complements are taken with respect to the natural inner product in $\mathcal{C}_{2\pi}$ and $\mathcal{C}^1_{2\pi}$

$$[u, v] = \int_{\tilde{G}} \langle T_g u, T_g v \rangle d\mu,$$

where $\langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} [u(t)]^t v(t) dt$ and μ is a normalised Haar measure for $\tilde{G}$. The inner product $[u, v]$ is T_g invariant, because for $g, g' \in \tilde{G}$ we have

$$\begin{aligned} [T_{g'} u, T_{g'} v] &= \int_{\tilde{G}} \langle T_g T_{g'} u, T_g T_{g'} v \rangle d\mu \\ &= \int_{\tilde{G}} \langle T_{gg'} u, T_{gg'} v \rangle d\mu \\ &= [u, v]. \end{aligned}$$

Therefore, the complements $(\ker \mathcal{L})^{\perp}$ and $(\operatorname{range} \mathcal{L})^{\perp}$ are T_g invariant, which makes the two decompositions in (3.5) T_g invariant. Now, define the projections

$$E : \mathcal{C}_{2\pi} \to \operatorname{range} \mathcal{L},$$

$$I - E : \mathcal{C}_{2\pi} \to (\operatorname{range} \mathcal{L})^{\perp}.$$

Clearly, solving the equation $\Phi(u, \tau) = 0$ is equivalent to solving the pair

$$E\Phi(u, \tau) = 0, \tag{3.6}$$

$$(I - E)\Phi(u, \tau) = 0. \tag{3.7}$$

Commuting properties of these projections are given in [9] and we recall them in the following lemma.

Lemma 3.2.6 *The projections E and $I - E$ commute with the action of T_g.*

PROOF By the decomposition (3.5), the element $u \in \mathcal{C}_{2\pi}$ can be written as

$$u = v + w, v \in (\operatorname{range} \mathcal{L})^{\perp} \text{ and } w \in \operatorname{range} \mathcal{L}.$$

Therefore,

$$T_g(E(u)) = T_g w = E(T_g w) = E(T_g v + T_g w) = E(T_g u).$$

Using a similar argument, one can write

$$T_g((I - E)u) = T_g v = (I - E)(T_g v) = (I - E)(T_g v + T_g w) = (I - E)(T_g u),$$

and the lemma follows. □

Our aim now is to solve Equation (3.6) and substitute its solution into (3.7), which will give the final equation for the desired periodic orbits. We shall make use of the implicit function theorem.

By the decomposition (3.5), $u \in \mathcal{C}^1_{2\pi}$ can be written as $u = v + w$ with $v \in \ker \mathcal{L}, w \in (\ker \mathcal{L})^{\perp}$. Therefore,

$$E\Phi(u, \tau) = E\Phi(v + w, \tau) = 0. \tag{3.8}$$

We define the map $B : \ker \mathcal{L} \times (\ker \mathcal{L})^{\perp} \times \mathbb{R} \to \operatorname{range} \mathcal{L}$ by

$$B(v, w, \tau) = E\Phi(v + w, \tau).$$

The differential of B with respect to w at the origin is

$$E(d\Phi)_{(0,0)} = E\mathcal{L} = \mathcal{L},$$

by the definition of E as the projection onto range $\mathcal{L}$. Consider the restriction

$$\mathcal{L} : (\ker \mathcal{L})^{\perp} \to \operatorname{range} \mathcal{L}.$$

Since $\mathcal{L}$ is a Fredholm operator of index zero, then it is invertible on $(\ker \mathcal{L})^{\perp}$. By the implicit function theorem, Equation (3.6) can be uniquely solved for $w = W(v, \tau)$. It is easily checked that W commutes with T_g.

3.2.4 Reduction

The Liapunov-Schmidt method reduces the original problem to the problem of finding zeros of the *bifurcation map* φ defined by

$$\varphi : \ker \mathcal{L} \times \mathbb{R} \to (\operatorname{range} \mathcal{L})^{\perp},$$

$$\varphi(v, \tau) = (I - E)\Phi(v + W(v, \tau), \tau),$$

where $W : \ker \mathcal{L} \times \mathbb{R} \to (\ker \mathcal{L})^{\perp}$ is the unique function that solves the equation $E\Phi(v + W(v, \tau), \tau) = 0$. The following proposition describes the symmetry of the bifurcation map.

Proposition 3.2.7 (Proposition 4.1, [9]) *If f is G-reversible equivariant, then the bifurcation map φ is $\tilde{G}$-reversible equivariant, i.e.*

$$\varphi(T_g v, \tau) = \sigma(\gamma) T_g \varphi(v, \tau), \forall g \in \tilde{G}.$$

This proposition shows that the reduced problem inherits the same symmetry properties as the original one.

3.2.5 The Hamiltonian structure

In this section, we discuss the consequences of the Hamiltonian structure. The main result here is that the reduced bifurcation map is also a Hamiltonian vector field, as stated in [9] and [16]. One can prove that Φ is a Hamiltonian vector field, with respect to the weak symplectic form

$$\Omega(u, v) = \frac{1}{2\pi} \int_0^{2\pi} \omega(u(s), v(s))\, ds,$$

and the Hamiltonian function

$$\mathcal{H}(u,\tau) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}\omega\left((1+\tau)\frac{du}{ds}, u\right) - H(u)\right)\, ds.$$

This is equivalent to

$$d_u\mathcal{H}\cdot v = \Omega(\Phi, v).$$

Moreover, the weak symplectic form Ω and the Hamiltonian function $\mathcal{H}$ have the same (anti-)invariance properties as ω and H, respectively. That is, for $g = (\gamma, \theta) \in \tilde{G}$ we have

$$\begin{aligned} \Omega(T_g u, T_g v) =& \chi(\gamma)\Omega(u, v), \\ \mathcal{H}(T_g u, T_g v) =& \sigma(\gamma)\chi(\gamma)\mathcal{H}(u, v). \end{aligned} \tag{3.9}$$

Theorem 6.2 in [16] states that if

$$\ker \mathcal{L} = \ker \mathcal{L}^*, \tag{3.10}$$

then, the bifurcation map is a Hamiltonian vector field. Equivalently, the condition (3.10) can be written as

$$J\left(\ker \mathcal{L}\right) = \ker \mathcal{L}. \tag{3.11}$$

In the following lemma, we illustrate a useful way of checking the condition (3.11).

Lemma 3.2.8 *If the matrix L is skew-symmetric, then $\ker \mathcal{L}$ is J invariant.*

Proof It is known from the Hamiltonian structure that $L = JS$, where J is skew-symmetric, and S is symmetric. Consequently, $L^T = -SJ$. Thus,

$$L^T = -L \Leftrightarrow SJ = JS.$$

Now, we study the J invariance condition on $\ker \mathcal{L}$.

$$\begin{aligned} u \in \ker \mathcal{L} \Rightarrow& \\ \dot{u}(t) &= Lu(t), \forall t \\ &= JSu(t). \end{aligned}$$

Accordingly,

$$
\begin{aligned}
\frac{d}{dt}\left(Ju(t)\right) &= J\dot{u}(t), \forall t \\
&= J(JS)u(t) \\
&= J(SJ)u(t) \\
&= (JS)Ju(t) \\
&= L(Ju(t)).
\end{aligned}
$$

Therefore, $\ker \mathcal{L}$ is J invariant. □

All problems discussed in this thesis are finite dimensional, and the condition (3.11) is satisfied by the previous lemma. As a result, the bifurcation map is a Hamiltonian vector field, and the corresponding Hamiltonian function satisfies the (anti-)invariance properties given in (3.9), with respect to the action of $\tilde{G}$ restricted to $\ker \mathcal{L}$.

These are the general steps of the Liapunov-Schmidt reduction with Hamiltonian symmetric structure. After applying the method, one needs to solve the bifurcation equation in order to find the periodic solutions of the studied problem.

3.3 General existence theorems

In this section, we introduce some remarkable theorems on the existence of periodic solutions in the Hamiltonian context. We will start with the *Liapunov center theorem*, which is considered as one of the most basic and fundamental results on the existence of periodic orbits near equilibria in Hamiltonian systems. The theorem states that for each non-resonant purely imaginary eigenvalue, there exists a family of periodic orbits parameterised by the energy level with period close to 2π in a neighbourhood of an equilibrium point, see [1, 18]. One main assumption in this theorem was the non-resonance in eigenvalues. This condition was relaxed later by Weinstein [36] and Moser [31] and the result is known as the *Weinstein- Moser theorem*. The last theorem to be considered in this section is the *equivariant Weinstein-Moser theorem* due to Montaldi *et al.* [28] which takes symmetry into account. Theorem statements are taken from [18].

Consider the following definition:

Definition 3.3.1 An eigenvalue $i\omega$ of L is *non-resonant* if it is simple, and there is no integer multiple of $i\omega$ which is also an eigenvalue. Otherwise, $i\omega$ is a *resonant* eigenvalue.

Now, we give assumptions of the Liapunov center theorem. Let $H : V \to \mathbb{R}$ be a Hamiltonian function defined on the symplectic space V, and let p be an equilibrium point.

Theorem 3.3.2 (The Liapunov center theorem, [18]) *If the linearised flow at an equilibrium has a simple, non-resonant, purely imaginary eigenvalue $i\omega$, then there exists a smooth two-dimensional submanifold of V, which passes through p and intersects every energy level near p in a periodic orbit, such that the period of that orbit approaches $\frac{2\pi}{|\omega|}$ for orbits near p.*

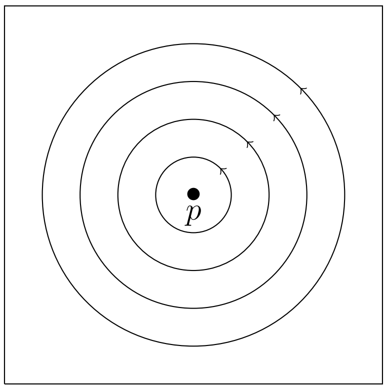


Figure 3.1: Liapunov center family of periodic orbits encircling an equilibrium p.

This simply means that for each simple non-resonant eigenvalue, there exists a one-parameter family of periodic orbits, with period close to that of the linear system (see Figure 3.1). Many authors tried to extend this result to more general systems; one of the most remarkable ones is due to Weinstein and Moser [36, 31], who allowed multiple eigenvalues. This led to defining the resonance space as follows:

Consider the linear system $\dot{x} = Lx$ with an initial condition $x(0) = x_0$. Solutions of the system are given by $x(t) = \exp(tL)x_0$. A periodic solution with period λ must satisfy $x(\lambda) = x_0$ and thus, $(\exp(\lambda L) - I)x_0 = 0$.

Definition 3.3.3 For $\lambda \in \mathbb{R}$, the *resonance space* V_λ for the linear system $\dot{x} = Lx$ is defined by

$$V_\lambda = \ker(\exp(\lambda L) - I).$$

In other words, the resonance space V_λ is a subspace of the phase space that consists of all periodic solutions of a period λ. Note that if $i\omega$ is non-resonant eigenvalue of L, then for $\lambda = 2\pi/\omega$, $\dim V_\lambda = 2$.

Let $i\omega$ be a nonzero purely imaginary eigenvalue of the linear system L. Assume $x = u + iv$ to be an eigenvector associated to the eigenvalue $i\omega$. Therefore,

$$\begin{aligned} Lx &= i\omega x, \\ Lu + iLv &= -\omega v + i\omega u. \end{aligned}$$

Taking the real and imaginary parts yield

$$\begin{aligned} Lu &= -\omega v, \\ Lv &= \omega u. \end{aligned}$$

It is readily verified that the function $x(t) = u\cos\omega t - v\sin\omega t$ is a periodic solution of the linear system, with period $\frac{2\pi}{\omega}$ and initial condition $x(0) = u$. An equivalent definition of the resonance space is given by the following.

Definition 3.3.4 Let $i\omega$ be a nonzero purely imaginary eigenvalue of the linear system L and $V = T_pV$. The *resonance space* $V_\omega \subseteq V$ is the unique subspace of V, maximal with respect to all eigenvalues of $L|V_\omega$ being integer multiples of $i\omega$. Equivalently, V_ω is the (real part of the) sum of the generalised eigenspaces of L for eigenvalues $ki\omega, k \in \mathbb{Z}$.

Weinstein proved in the celebrated paper [36] that even with resonance, there exist at least $\frac{1}{2}\dim V_\omega$ families of periodic solutions on each energy level near p. The proof was simplified later by Moser in [31], and subsequently, became known as the Weinstein-Moser theorem.

Theorem 3.3.5 (The Weinstein-Moser theorem, [18]) *If $i\omega$ is an eigenvalue of the linear system and*

(1) *d^2H_p is non-degenerate,*

(2) *$d^2H_p|_{V_\omega}$ is positive definite,*

then there exist at least $\frac{1}{2}\dim V_\omega$ periodic solutions on each energy level $\varepsilon^2 + H(p)$, for sufficiently small $\varepsilon \in \mathbb{R}$.

Although Weinstein-Moser was a strong result, it failed to predict the correct number of periodic solutions in equivariant Hamiltonian systems. Consequently, it needed to be modified to cover the symmetry settings. Montaldi *et al.* [28] applied symmetry conditions to Weinstein-Moser theorem, and proved their well-known result: the equivariant Weinstein-Moser theorem.

Let Γ be a compact Lie group acting symplectically on V, let $p \in V$ be a fixed point for Γ, and let the Hamiltonian H be Γ invariant. Furthermore, let $i\omega$ be a non-zero purely imaginary eigenvalue of the linear system L. Note that the symmetry often forces some eigenvalues to be resonant. As before, assume that

(1) d^2H_p is a non-degenerate quadratic form,

(2) $d^2H_p|_{V_\omega}$ is positive definite.

Theorem 3.3.6 (The equivariant Weinstein-Moser theorem, [18]) *Suppose that the Hamiltonian H satisfies (1) and (2). Then, for every isotropy subgroup Σ of the $\Gamma \times S^1$ action on V_ω, and for all sufficiently small $\varepsilon \in \mathbb{R}$, there exist at least $\frac{1}{2}\dim \mathrm{Fix}\,(\Sigma)$ periodic orbits with periods near $2\pi/\omega$ and symmetry group containing Σ, on the energy surface $H = H(p) + \varepsilon^2$.*

In the following chapters, we will prove the existence of families of periodic orbits near an equilibrium point in different classes of Hamiltonian systems with symmetry, using the Liapunov-Schmidt procedure illustrated in this chapter. By changing the symmetry groups we find that there exist different families of periodic orbits near the origin under some generic conditions. These results suggest that we can extend our results and analysis to more general settings e.g. a reversing equivariant Weinstein-Moser theorem.

Chapter 4

Symplectic time-reversing involution

In this chapter, we prove the existence of symmetric and non-symmetric periodic solutions in Hamiltonian systems, with a reversing involutory symmetry acting symplectically (type SR in Table 2.1). The problem was first studied by Buzzi and Lamb [9], but there is a minor sign error in the calculations in Lemma 6.4, which affects the statement in their Theorem 6.1. They (correctly) prove the existence of a three-dimensional conical subspace of symmetric periodic solutions in a neighbourhood of the origin if the linearisation has two pairs of purely imaginary eigenvalues rescaled to be $\pm i$. Also, they find that the origin is contained in two 2-dimensional manifolds, each containing a non-symmetric family of periodic solutions, with period close to 2π. Using our expressions for the (semi-)invariants, we first recover their result on the symmetric solutions, and then we correct their Theorem 6.1 to show that generically, there may or may not be two families of non-symmetric periodic orbits in a neighbourhood of the equilibrium point 0, depending on the coefficients of the Hamiltonian. Buzzi and Lamb also distinguish between two cases, called elliptic and hyperbolic, distinguishing between the possibilities of the period function on the three-dimensional family being monotonic or not. It turns out that this distinction coincides with the two cases of existence or non-existence of non-symmetric periodic orbits. The work presented in this chapter is published in Alomair and Montaldi [2].

The following sections will give some background materials and our new proofs.

4.1 Linear reversible Hamiltonian vector fields

Let $\dot{x} = f(x)$, $x \in \mathbb{R}^{2n}$ be a Hamiltonian system which is purely reversible under the action of the symplectic involution R. Also, assume that L is the linear system at 0 and J is the linear structure map. In [21], Hoveijn *et al.* give normal forms of linear systems in eigenspaces of (anti-)automorphisms of order two, which can be adapted to our problem. These normal forms are based on writing minimal $\langle J, R\rangle -$invariant subspaces. Since we are interested in generic systems with codimension zero, by [21] we can only focus on the case when L is semi-simple on minimal invariant subspaces. Also, we assume that L has at least one pair of purely imaginary eigenvalues $\pm i$ after rescaling. Normal forms of R, J and L are given in the following lemma.

We use the notation

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 4.1.1 (Lemma 3.1, [9]) *Let L be a linear Hamiltonian vector field on $\mathbb{R}^{2n}$. Suppose L is R-reversible, with R acting symplectically (symmetry type SR).*
Let V be a minimal (L, J, R)-invariant subspace, on which L has eigenvalues $\pm i$. Then, $\dim V = 4$ *and $R|_V, J|_V$ and $L|_V$ can take the following normal forms:*

$$R|_V = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad J|_V = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}, \quad \textit{and} \quad L|_V = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}.$$

4.2 The existence of periodic orbits

In this section, we prove the existence of symmetric and non-symmetric periodic solutions in Hamiltonian systems, with a reversing symmetry acting symplectically. We start by choosing the set of (semi-)invariants, according to the group action. After that, we write the formula of the reduced Hamiltonian deduced from the Liapunov-Schmidt procedure described in Chapter 3. Finally, we solve the bifurcation equation in order to find the desired periodic solutions, and classify them according to their symmetry.

By the normal forms given in Lemma 4.1.1, we have $\dim \ker \mathcal{L} = 4$, so we can write

$\ker \mathcal{L} \cong \mathbb{C}^2$. Therefore, the bifurcation map is given by

$$\varphi : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2,$$
$$\varphi = 2J\nabla_z h,$$

with Hamiltonian function

$$h : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}.$$

Denote by $\mathbb{Z}_2^R$ the cyclic group generated by R. Together with the circle action we have $\mathbb{Z}_2^R \ltimes S^1$. The reversing symmetry R acts on $\mathbb{C}^2$ by

$$R(z_1, z_2) = (z_2, z_1),$$

while the S^1 action is defined by

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2).$$

If $\mathcal{E}$ is the ring of S^1 invariants, then one can write

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-,$$

where $\mathcal{E}_+$ consists of $\mathbb{Z}_2^R$ invariants and $\mathcal{E}_-$ consists of $\mathbb{Z}_2^R$ anti-invariants.

Lemma 4.2.1 *If $\mathbb{Z}_2^R \ltimes S^1$ acts on $\mathbb{C}^2$ as above, then*

1. *$\mathcal{E}$ is the ring generated by A, B, C, D where $A = |z_1|^2, B = |z_2|^2, C + iD = 2z_1 z_2$.*

2. *$\mathcal{E}_+$ is the subring of $\mathcal{E}$ generated by N, C, D where $N = |z_1|^2 + |z_2|^2$, and $\mathcal{E}_-$ is the module over $\mathcal{E}_+$ generated by the function $\delta = |z_1|^2 - |z_2|^2$.*

3. *The orbit map $O : \mathbb{C}^2 \to \mathbb{R}^3$ defined by $(z_1, z_2) \to (N, C, D)$ has image $\{(N, C, D) \mid N^2 \geq C^2 + D^2\}$.*

Note that the functions N, C, D and δ satisfy the identity $\delta^2 = N^2 - C^2 - D^2$.

Before we prove Lemma 4.2.1 we will state the following theorem by Schwarz [32], which is used in writing the formula of the Hamiltonian function according to its invariance properties.

Theorem 4.2.2 (Schwarz,[32]) *Let G be a compact Lie group acting on the vector space V. Let $u_1, u_2, \cdots, u_s$ be a Hilbert basis for the G invariant polynomials $\mathcal{P}(G)$. Let f be a C^∞, G invariant germ. Then there exists a smooth germ $h \in \mathcal{E}_s$ such that*

$$f(x) = h(u_1(x), u_2(x), \cdots, u_s(x)).$$

Here $\mathcal{E}_s$ is the ring of C^∞ germs $\mathbb{R}^s \to \mathbb{R}$.

Now we give the sketch of the proof of Lemma 4.2.1.

Proof We will prove this lemma by standard algebraic computations, similar to those found for example in [19]. By the definition of the S^1 action given above, we have that the S^1 invariant generators on $\mathbb{C}^2$ are $A = |z_1|^2, B = |z_2|^2, C + iD = 2z_1z_2$. We select R invariant, real valued functions generated by A, B, C, D and they are N, C, D. On the other hand there is only one R anti-invariant generator δ. By the relation $\delta^2 = N^2 - C^2 - D^2$, solutions will have meaning only when $\delta^2 \geq 0$ which is equivalent to $N^2 \geq C^2 + D^2$. □

Now, we can apply Lemma 4.2.1 to our Hamiltonian. The function h is S^1 invariant, R anti-invariant and real valued. This implies

$$h = \delta g(N, C, D, \tau). \tag{4.1}$$

In order to find the periodic solutions, we need to solve the bifurcation equation given by

$$\nabla_z h = 0.$$

This can be written as

$$\begin{cases} \dfrac{\partial h}{\partial z_1} = \bar{z}_1 g + \delta \dfrac{\partial g}{\partial z_1} = 0, \\ \dfrac{\partial h}{\partial z_2} = -\bar{z}_2 g + \delta \dfrac{\partial g}{\partial z_2} = 0. \end{cases} \tag{4.2}$$

We now consider, in turn, the symmetric and non-symmetric periodic orbits.

4.2.1 Symmetric periodic orbits

In finding symmetric periodic orbits, defined in Section 3.1, we recover the result in [9].

Theorem 4.2.3 (Theorem 6.1, [9]) *Consider a symmetric equilibrium* 0 *of a reversible Hamiltonian vector field* f *with the reversing involution acting symplectically. Suppose that* $Df(0)$ *has two purely imaginary pairs of eigenvalues* $\pm i$ *with no other eigenvalues of the form* $\pm ki, k \in \mathbb{Z}$. *Then, the equilibrium is contained in a three-dimensional flow invariant conical subset, given by the equation* $\delta = 0$, *and this consists of a two-parameter family of symmetric periodic solutions whose period tends to* 2π *as they approach the equilibrium.*

Note that a subset A of $\mathbb{R}^n$ is conical, if $x \in A, \lambda \geq 0 \Rightarrow \lambda x \in A$.

Proof Since the Hamiltonian is R anti-invariant, then all symmetric solutions are zeros of the bifurcation equation that lie in the level set $h = 0$. For symmetric solutions, we have $\delta = 0$. Therefore, the bifurcation equation calculated in $\operatorname{Fix} R = \{(z, z) \mid z \in \mathbb{C}\}$ will take the form

$$\bar{z}g(z, \tau) = 0.$$

Non-zero solutions yield $g(z, \tau) = 0$. By the formula of the reduced Hamiltonian (4.1), the lowest order term of the variable τ is given by

$$h = (|z_1|^2 - |z_2|^2)\frac{\tau}{2} + \text{h.o.t.}$$

This implies that $\frac{\partial g}{\partial \tau}(0, 0) = \frac{1}{2} \neq 0$. By the implicit function theorem for each small non-zero z there exists a τ such that (z, z) lies in a periodic orbit with period $\frac{2\pi}{\tau+1}$. By reversing property, each R-symmetric solution intersects $\operatorname{Fix} R$ in two points. Since the conical subset $\delta = 0$ is three-dimensional, and all points in $\operatorname{Fix} R$ are solutions of the bifurcation equation, we conclude that the conical subspace completely consists of these periodic solutions, with period close to 2π as they approach the origin. $\square$

4.2.2 Non-symmetric periodic orbits

We prove the existence of two families of non-symmetric periodic solutions under suitable conditions on the coefficients of the Hamiltonian. This result is fairly different from the one in [9]. To prove the existence of non-symmetric solutions, one needs to solve the bifurcation equation without any symmetry conditions. By calculating the partial derivatives of g the bifurcation equation will be

$$\frac{\partial h}{\partial z_1} = \bar{z}_1(g + \delta g_N) + z_2\delta(g_C - ig_D) = 0,$$
$$\frac{\partial h}{\partial z_2} = \bar{z}_2(-g + \delta g_N) + z_1\delta(g_C - ig_D) = 0,$$

where $g_N = \dfrac{\partial g}{\partial N}$, $g_C = \dfrac{\partial g}{\partial C}$ and $g_D = \dfrac{\partial g}{\partial D}$. Multiplying the first equation by z_1 and the second one by z_2, we get

$$|z_1|^2(g + \delta g_N) + z_1 z_2\delta(g_C - ig_D) = 0, \tag{4.3}$$

$$|z_2|^2(-g + \delta g_N) + z_1 z_2\delta(g_C - ig_D) = 0. \tag{4.4}$$

By adding (4.3) and (4.4) we have

$$\delta(g + Ng_N + (C + iD)(g_C - ig_D)) = 0. \tag{4.5}$$

Taking the imaginary part of the above equation gives

$$Dg_C - Cg_D = 0. \tag{4.6}$$

Therefore, Equation (4.5) will be

$$\delta(g + Ng_N + Cg_C + Dg_D) = 0. \tag{4.7}$$

By subtracting (4.4) from (4.3) we have

$$Ng + \delta^2 g_N = 0,$$

this can also be written by the formula

$$\frac{g}{\delta^2} = -\frac{g_N}{N}. \tag{4.8}$$

Substituting (4.6) and (4.8) in (4.7) yields

$$\frac{g_N}{N} = -\frac{g_C}{C}. \tag{4.9}$$

Thus,

$$\frac{g}{\delta^2} = -\frac{g_N}{N} = \frac{g_C}{C} = \frac{g_D}{D},$$

which is equivalent to

$$\frac{N}{g_N} = -\frac{C}{g_C} = -\frac{D}{g_D}. \tag{4.10}$$

Note that if $C = 0$, then (4.6) implies $D = 0$. In this case, the bifurcation equation takes a simple form, and can be solved for $\tau = \tau(N)$. A similar argument can be used for the case $D = 0$.

In order to prove the existence of non-symmetric periodic solutions to the original Hamiltonian system, we need to prove the following lemma. Let

$$g_N(0) = n,\ g_C(0) = c,\ g_D(0) = d.$$

Lemma 4.2.4 *If n, c and d are not all zero, then there exists a unique solution in $\mathbb{R}^4 \cong (\tau, N, C, D)$-space for the system of equations*

$$g + Ng_N + Cg_C + Dg_D = 0, \tag{4.11}$$

$$Ng_C + Cg_N = 0, \tag{4.12}$$

$$Dg_C - Cg_D = 0, \tag{4.13}$$

$$Ng_D + Dg_N = 0. \tag{4.14}$$

Proof It is clear that the last three equations are not linearly independent, but we will use them all to make up for the special cases, when one of the numbers n, c or d is equal to zero. Suppose that $n \neq 0$. Then, we only need to solve (4.11),(4.12) and (4.14). In order to apply the implicit function theorem, we need to study the following Jacobian matrix, with respect to τ, C, D and N

$$\left(\begin{array}{ccc|c} \frac{1}{2} & 2c & 2d & 2n \\ 0 & n & 0 & c \\ 0 & 0 & n & d \end{array} \right) = \left(\begin{array}{c|c} X & Y \end{array} \right).$$

Since $n \neq 0$, the matrix X is non-singular. Therefore, by the implicit function theorem, there exists a unique curve $S = S(N)$, with $dS(0) = -X^{-1}Y$, that solves the system. If $n = 0$ but $c \neq 0$ we can choose Equations (4.11), (4.12) and (4.13). Solving by the implicit function theorem gives a unique solution $S = S(C)$. A similar argument can be used for the remaining case with $d \neq 0$ and $n = c = 0$. $\square$

Now, we state and prove the main theorem about the existence of non-symmetric periodic solutions for the given reversible Hamiltonian system.

Theorem 4.2.5 *Suppose that $n^2 \neq c^2 + d^2$, then there exist the symmetric Liapunov centre families of periodic solutions filling the set $\delta = 0$ described before. Moreover,* *i) If $n^2 > c^2 + d^2$ then there exist two families of non-symmetric periodic orbits for the Hamiltonian system distinguished by the sign of δ. The period of the periodic solutions converges to 2π as the solutions tend to the origin.*

ii) If $n^2 < c^2 + d^2$ then the only periodic orbits with period close to 2π in a neighbourhood of the origin are the symmetric ones.

Proof To prove the existence of non-symmetric periodic orbits, we have to solve the equations (4.11),(4.12),(4.13) and (4.14). By the condition $n^2 \neq c^2 + d^2$ we have that n, c and d cannot all be zero. Applying Lemma 4.2.4, we have a unique solution for those equations. Therefore, we can write

$$\frac{N}{g_N} = -\frac{C}{g_C} = -\frac{D}{g_D} = s,$$

which is equivalent to $N = g_N s$, $C = -g_C s$ and $D = -g_D s$. To get non-symmetric solutions, we should have $\delta^2 = N^2 - C^2 - D^2 > 0$. This implies

$$({g_N}^2 - {g_C}^2 - {g_D}^2)s^2 > 0, \text{ for } s \neq 0,$$

and therefore, ${g_N}^2 - {g_C}^2 - {g_D}^2 > 0$. Taking the limit at the origin gives $n^2 \geq c^2 + d^2$. We conclude that non-symmetric solutions exist when $n^2 > c^2 + d^2$ and split into two families according to δ being positive or negative. On the other hand, when $n^2 < c^2+d^2$ the only periodic orbits with period close to 2π in a neighbourhood of the origin are the symmetric ones. □

4.3 Period distribution within the family of symmetric periodic solutions

Following the argument given in [9], we describe the structure of period distribution for symmetric periodic solutions. Since Fix R is two-dimensional, the level sets of the period will be given by $\tau = \tau(x, y)$. If we change the coordinates in a neighbourhood of the origin, such that $\tau = \varepsilon_1\tilde{x}^2 + \varepsilon_2\tilde{y}^2$ with $\varepsilon_j = \pm 1$, where the sign depends on the details of h and H, one can give the following definition:

Definition 4.3.1 The level sets of the period τ can be of two types:

1. *Elliptic* when $\varepsilon_1\varepsilon_2 = 1$. In that case, the level sets of the period form approximate circles, and τ increases or decreases monotonically with increasing radius.

2. *Hyperbolic* when $\varepsilon_1\varepsilon_2 = -1$. Here, the level sets of the period form two families of approximate hyperbolae, one family with positive increasing τ and one with negative decreasing τ.

(a) Elliptic (b) Hyperbolic

Figure 4.1: Curves on which the period of the symmetric periodic orbits is constant.

Now we can prove the following proposition:

Proposition 4.3.2 *Depending on the quartic terms of the Hamiltonian function (and hence, quadratic terms of the function g), among the three-dimensional subspace of symmetric periodic solutions near the equilibrium point, the level sets of τ are elliptic when $n^2 > c^2 + d^2$ or hyperbolic when $n^2 < c^2 + d^2$.*

PROOF As discussed in the proof of the existence of symmetric periodic solutions, $\tau(x, y)$ can be calculated using the equation $g(z, \tau) = 0$, with $z = x + iy$. Using our variables N, C and D, and depending on the quadratic terms of that equation, we have $g(N, C, D, \tau) = 0$ which is equivalent to $nN + cC + dD + \cdots = -\frac{\tau}{2}$, or

$$2n(x^2 + y^2) + 2c(x^2 - y^2) - 4d(xy) = -\frac{\tau}{2}. \qquad (4.15)$$

By the Morse Lemma, the shape of $\tau(x, y)$ near the origin is given by the determinant

$$D = 4^2(n^2 - c^2 - d^2).$$

Therefore, the family of periodic orbits is elliptic when $(n^2 - c^2 - d^2) > 0$ or hyperbolic when $(n^2 - c^2 - d^2) < 0$. $\square$

Accordingly, one can easily deduce the following corollary.

Corollary 1 *The two-dimensional families of non-symmetric periodic orbits given in Theorem* 4.2.5 *exist if and only if the three-dimensional family of symmetric periodic orbits is of elliptic type.*

Comparing the expressions for τ in (4.7) and (4.15) shows that in the elliptic case, the period is increasing in the three-dimensional family if and only if it is also increasing in the two other modes (here increasing means increasing with increasing amplitude).

Chapter 5

Anti-symplectic involution

In this chapter, we analyse the problem of existence of periodic orbits in a Hamiltonian system, which is equivariant under the action of an anti-symplectic involution S (type AE in Table 2.1). This was studied by Li and Shi in [25], but that paper contains a number of errors. Firstly, the form of the Hamiltonian is not sufficiently general, for example, the polynomial function $h = DN$, where $N = |z_1|^2 + |z_2|^2$ and $D = -i(z_1 z_2 - \bar{z}_1 \bar{z}_2)$ satisfies the symmetry of the problem, but is not in the form assumed in [25]. This affects the results significantly, and the general form of the Hamiltonian makes the calculations more difficult. There is also a serious error in the proof of their Lemma 5.3. As a result, we consider the problem anew. We use a different basis from [25], so the invariants and anti-invariants are different, and we determine a general formula for the reduced Hamiltonian. Firstly, we find that no symmetric periodic orbits can occur generically (opposite to the result claimed in [25]). Secondly, we prove the existence of at least 2 and at most 12 families of non-symmetric periodic solutions near the equilibrium point. The work presented in this chapter is published in [2].

5.1 Linear equivariant Hamiltonian vector fields

In this problem, we study an equivariant Hamiltonian system under the action of an anti-symplectic involution S. In order to find periodic orbits of that system, it is essential to choose the appropriate formulas for the involution symmetry S, the structure map J and the linear system L, on their minimal invariant space V. In addition, we write the reduced Hamiltonian according to the symmetry properties.

As in the previous chapter, we use the notation,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By using the normal forms given in [21], we prove the following lemma.

Lemma 5.1.1 *Let L be a linear Hamiltonian vector field on $\mathbb{R}^{2n}$. Suppose L is S-equivariant, with S acting anti-symplectically (symmetry type AE).*
Let V be a minimal (L, J, S)-invariant subspace, on which L has eigenvalues $\pm i$. Then, $\dim V = 4$ *and $S|_V$, $J|_V$ and $L|_V$ can take the following normal forms:*

$$S|_V = \begin{pmatrix} 0 & S_2 \\ S_2 & 0 \end{pmatrix}, \quad J|_V = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}, \quad \textit{and} \quad L|_V = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}.$$

Proof Let W be a two-dimensional symplectic subspace, on which L has the pair of eigenvalues $\pm i$ and $S(W) = W$. It is known in the Hamiltonian context that L and J can take the same normal form on W, taking into account multiplication of time by a scalar. Equivariance property yields $SL = LS$. On W, L and J take the same form, which gives $SJ = JS$. The latter equation contradicts the fact that S is acting anti-symplectically. Thus, the minimal invariant subspace is four-dimensional, and is given by $V = W \oplus W'$, $W' = S(W)$. The anti-symplectic property implies $J|_{W'} = -J|_W$ while equivariance gives $L|_{W'} = L|_W = J|_W$. Therefore, normal forms given in [21] show

$$S|_V = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad J|_V = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}, \quad \text{and} \quad L|_V = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}.$$

Now, we apply the change of coordinates

$$z_1 = w_1, \; z_2 = \bar{w}_2.$$

In these new coordinates S, J and L take the forms given in the lemma above. □

An immediate consequence of our assumptions is that the Hamiltonian is S anti-invariant (as pointed in Table 2.1). By the normal forms given in Lemma 5.1.1, we have $\dim\ker\mathcal{L} = 4$ i.e. $\ker\mathcal{L} \cong \mathbb{C}^2$. The bifurcation map is given by the formula

$$\varphi : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2,$$

$$\varphi = 2J\nabla_z h,$$

with the Hamiltonian

$$h : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R},$$

where J is the structure map. Now, define the action of $\mathbb{Z}_2^S \times S^1$ on $\mathbb{C}^2$ by

$$S(z_1, z_2) = (\bar{z}_2, \bar{z}_1),$$
$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2).$$

In the following, we study the set of (anti-)invariants and find the appropriate formula for h.

Lemma 5.1.2 *For $\mathbb{Z}_2^S \times S^1$ acting on $\mathbb{C}^2$ as above, then*

1. *The $\mathbb{Z}_2 \times S^1$ invariant functions are generated by N, C, D^2 where*

$$N = |z_1|^2 + |z_2|^2, C + iD = 2z_1 z_2.$$

2. *The S^1 invariant but $\mathbb{Z}_2$ anti-invariant functions are generated by δ and D, where*

$$\delta = |z_1|^2 - |z_2|^2.$$

According to the lemma above, the Hamiltonian h will take the form

$$h = \delta g^1(N, C, D^2, \tau) + Dg^2(N, C, D^2, \tau).$$

The bifurcation equation is given by

$$\frac{\partial h}{\partial z_1} = \bar{z}_1 g^1 + \delta \frac{\partial g^1}{\partial z_1} - iz_2 g^2 + D \frac{\partial g^2}{\partial z_1} = 0, \tag{5.1}$$
$$\frac{\partial h}{\partial z_2} = -\bar{z}_2 g^1 + \delta \frac{\partial g^1}{\partial z_2} - iz_1 g^2 + D \frac{\partial g^2}{\partial z_2} = 0. \tag{5.2}$$

5.2 Symmetric periodic orbits

Symmetric periodic solutions of that equivariant Hamiltonian system lie in the set $\operatorname{Fix} S = \{(z, \bar{z}), z \in \mathbb{C}\}$. Moreover, by anti-invariance, that is $h \circ S = -h$, all symmetric solutions will be in the level set $h = 0$. In order to obtain the symmetric periodic solutions, we need to solve the bifurcation equation calculated in $\operatorname{Fix} S$. Consequently, one needs to solve (5.1) and (5.2), with conditions: $\delta = D = 0$ and $N = C$. Thus,

$$\bar{z}_1 g^1 - iz_2 g^2 = 0, \tag{5.3}$$
$$-\bar{z}_2 g^1 - iz_1 g^2 = 0. \tag{5.4}$$

By multiplying (5.3) by z_1 and (5.4) by z_2 we get

$$|z_1|^2 g^1 - iz_1 z_2 g^2 = 0,$$
$$-|z_2|^2 g^1 - iz_1 z_2 g^2 = 0.$$

Adding and subtracting these two equations yield

$$\delta g^1 - i(C + iD)g^2 = 0,$$
$$Ng^1 = 0.$$

With the conditions $\delta = D = 0$ we have

$$Cg^2 = 0,$$
$$Ng^1 = 0.$$

Since we are looking for nonzero solutions, then $N = C \neq 0$ and therefore, solutions are common zeros of g^1 and g^2 in a neighbourhood of the origin. But, g^1 and g^2 are independent functions, and generically, the only common zero in a neighbourhood of the origin is 0 itself. As a result, there are no symmetric periodic orbits for the given Hamiltonian system near the origin.

Remark 5.2.1 Another way to see the non-existence of symmetric solutions in that system is by using a Liapunov function. Consider the Hamiltonian given by the formula

$$H = \delta(a_1 + b_1 N + c_1 C + \cdots) + D(a_2 + b_2 N + c_2 C + \cdots).$$

Restricting the Hamiltonian system on the the two-dimensional invariant space $\operatorname{Fix} S$ gives

$$\dot{x} = 2y\left(a_1 + 2(b_1 + c_1)(x^2 + y^2) + \cdots\right) + 2x\left(a_2 + 2(b_2 + c_2)(x^2 + y^2) + \cdots\right),$$
$$\dot{y} = -2x\left(a_1 + 2(b_1 + c_1)(x^2 + y^2) + \cdots\right) + 2y\left(a_2 + 2(b_2 + c_2)(x^2 + y^2) + \cdots\right).$$

Easy computations show that the eigenvalues of the linear system are $\lambda = 2(a_2 \pm a_1 i)$. In order to get periodic orbits, we should have $a_2 = 0$, and the system would be written as

$$\dot{x} = 2y\left(a_1 + 2(b_1 + c_1)(x^2 + y^2) + \cdots\right) + 2x\left(2(b_2 + c_2)(x^2 + y^2) + \cdots\right),$$
$$\dot{y} = -2x\left(a_1 + 2(b_1 + c_1)(x^2 + y^2) + \cdots\right) + 2y\left(2(b_2 + c_2)(x^2 + y^2) + \cdots\right).$$

Consider as Liapunov function $V = x^2 + y^2$. Differentiating V in the direction of the Hamiltonian vector field yields

$$\begin{aligned}\dot{V} &= 2x\dot{x} + 2y\dot{y}\\ &= 8(x^2+y^2)^2(b_2+c_2).\end{aligned}$$

The number $b_2 + c_2$ is generically non-zero and therefore, $\dot{V}$ is non-zero. This means that the sign of $\dot{V}$ (either positive or negative) is constant along any trajectory, so that the trajectory cannot be closed. Thus, the system does not have any symmetric periodic orbits.

5.3 Non-symmetric periodic orbits

For the case of non-symmetric periodic orbits, we need to solve the pair (5.1) and (5.2) without any additional conditions. Multiplying (5.1) by z_1 and (5.2) by z_2 gives

$$\begin{aligned}|z_1|^2g^1 + \delta\left(g^1_N|z_1|^2 + g^1_C z_1z_2 + g^1_{D^2}2D(-iz_1z_2)\right) - iz_1z_2g^2&\\ + D\left(g^2_N|z_1|^2 + g^2_C z_1z_2 + g^2_{D^2}2D(-iz_1z_2)\right) &= 0, \quad (5.5)\end{aligned}$$

$$\begin{aligned}-|z_2|^2g^1 + \delta\left(g^1_N|z_2|^2 + g^1_C z_1z_2 + g^1_{D^2}2D(-iz_1z_2)\right) - iz_1z_2g^2&\\ + D\left(g^2_N|z_2|^2 + g^2_C z_1z_2 + g^2_{D^2}2D(-iz_1z_2)\right) &= 0. \quad (5.6)\end{aligned}$$

By adding theses two equations, we have

$$\begin{aligned}\delta\left(g^1 + Ng^1_N + (C+iD)g^1_C + 2(-iD)(C+iD)g^1_{D^2}\right) - i(C+iD)g^2&\\ + D\left(Ng^2_N + (C+iD)g^2_C + 2(-iD)(C+iD)g^2_{D^2}\right) &= 0. \quad (5.7)\end{aligned}$$

The real and imaginary parts of Equation (5.7) are

$$\delta\left(g^1 + Ng^1_N + Cg^1_C + 2D^2g^1_{D^2}\right) + Dg^2 + D\left(Ng^2_N + Cg^2_C + 2D^2g^2_{D^2}\right) = 0, \quad (5.8)$$

$$\delta\left(Dg^1_C - 2CDg^1_{D^2}\right) - Cg^2 + D\left(Dg^2_C - 2CDg^2_{D^2}\right) = 0. \quad (5.9)$$

The last equation to be considered comes from subtracting (5.6) from (5.5), and it will take the following form

$$Ng^1 + \delta^2 g^1_N + D\delta g^2_N = 0. \quad (5.10)$$

This means, finding non-symmetric periodic solutions of the Hamiltonian system will be done by solving the triple (5.8), (5.9) and (5.10). Clearly the system is singular at the origin, and so can be solved using a blow-up method. For that purpose, define the new coordinates (r, u, v, w, t, x) by setting

$$N = rv, \quad C = ru, \quad D = rw,$$

$$\tau = rt, \quad \delta = rx,$$

combined together by the relation $v^2 = u^2 + w^2 + x^2$ according to the relation $N^2 = \delta^2 + C^2 + D^2$. Substituting these new coordinates in (5.8), (5.9) and (5.10) gives

$$r\left(vg^1 + rx^2 g^1_N + rxwg^2_N\right) = 0,$$

$$r\left(x(g^1 + rvg^1_N + rug^1_C + 2r^2w^2g^1_{D^2}) + w(g^2 + rvg^2_N + rug^2_C + 2r^2w^2g^2_{D^2})\right) = 0, \quad (5.11)$$

$$r\left(x(rwg^1_C - 2r^2wug^1_{D^2}) - ug^2 + w(rwg^2_C - 2r^2uwg^2_{D^2})\right) = 0.$$

We are interested in the non-zero solutions, i.e. $r \neq 0$. The first step is to divide by the common power of r in these equations. For simplicity, we can write the Taylor series for the functions g^1 and g^2 as

$$g^1(\tau, N, C, D^2) = \frac{\tau}{2} + a_1 N + c_1 C + d_1 D^2 + \cdots,$$
$$g^2(\tau, N, C, D^2) = b_2\tau + a_2 N + c_2 C + d_2 D^2 + \cdots,$$

which, with the new coordinates, take the forms

$$g^1 = r\bar{g}^1(r, t, v, u, w^2) = r(\frac{t}{2} + a_1 v + c_1 u + d_1 r w^2 + \cdots),$$
$$g^2 = r\bar{g}^2(r, t, v, u, w^2) = r(b_2 t + a_2 v + c_2 u + d_2 r w^2 + \cdots).$$

Lemma 5.3.1 *For g^1, g^2 and $\bar{g}^1$, $\bar{g}^2$ defined above , we have*

$$g^i_N = \bar{g}^i_v, g^i_C = \bar{g}^i_u \text{ and } rg^i_{D^2} = \bar{g}^i_{w^2}, \text{ for } i = 1, 2.$$

PROOF Knowing that $\bar{g}^i(r, v, u, w^2, t) = \frac{1}{r} g^i(N, C, D^2, \tau)$, for $i = 1, 2$, one can write

$$d\bar{g}^i = \frac{-1}{r^2} dr g^i + \frac{1}{r} dg^i$$
$$\frac{\partial \bar{g}^i}{\partial r} dr + \frac{\partial \bar{g}^i}{\partial v} dv + \frac{\partial \bar{g}^i}{\partial u} du + \cdots = \frac{-1}{r^2} g^i dr + \frac{1}{r}[\frac{\partial g^i}{\partial N} dN + \frac{\partial g^i}{\partial C} dC + \cdots]$$
$$= \frac{-1}{r^2} g^i dr + \frac{1}{r}[\frac{\partial g^i}{\partial N}(vdr + rdv) + \frac{\partial g^i}{\partial C}(udr + rdu) + \cdots].$$

This implies

$$\frac{\partial \bar{g}^i}{\partial v} = \frac{\partial g^i}{\partial N}, \quad \frac{\partial \bar{g}^i}{\partial u} = \frac{\partial g^i}{\partial C} \quad \text{and} \quad \frac{\partial \bar{g}^i}{\partial w^2} = r\frac{\partial g^i}{\partial D^2}.$$

Using our previous notation $g^i_N = \frac{\partial g^i}{\partial N}, \cdots$,etc. yields the result. □

According to Lemma 5.3.1, the system (5.11) can be written as

$$\begin{aligned} r^2\left(v\bar{g}^1 + x^2\bar{g}^1_v + xw\bar{g}^2_v\right) &= 0,\\ r^2\left(x(\bar{g}^1 + v\bar{g}^1_v + u\bar{g}^1_u + 2w^2\bar{g}^1_{w^2}) + w(\bar{g}^2 + v\bar{g}^2_v + u\bar{g}^2_u + 2w^2\bar{g}^2_{w^2})\right) &= 0,\\ r^2\left(x(w\bar{g}^1_u - 2wu\bar{g}^1_{w^2}) - u\bar{g}^2 + w(w\bar{g}^2_u - 2uw\bar{g}^2_{w^2})\right) &= 0. \end{aligned} \tag{5.12}$$

Dividing (5.12) by r^2 and substituting $r = 0$ in the rest yields

$$\begin{aligned} v(\tfrac{t}{2} + a_1v + c_1u) + a_1x^2 + a_2xw &= 0,\\ x(\tfrac{t}{2} + 2a_1v + 2c_1u) + w(b_2t + 2a_2v + 2c_2u) &= 0,\\ c_1xw - u(b_2t + a_2v + c_2u) + c_2w^2 &= 0. \end{aligned} \tag{5.13}$$

Clearly, the system cannot be solved by the implicit function theorem at this point in the argument. As a result, we will use a different technique, as illustrated in the next section. We will show that (5.13) has non-degenerate solutions, then apply a continuation argument to show (5.12) has solutions, when $r > 0$. Adding the relation between the variables N, C, D and δ gives us the system

$$\begin{aligned} v(\frac{t}{2} + a_1v + c_1u) + a_1x^2 + a_2xw &= 0,\\ x(\frac{t}{2} + 2a_1v + 2c_1u) + w(b_2t + 2a_2v + 2c_2u) &= 0,\\ c_1xw - u(b_2t + a_2v + c_2u) + c_2w^2 &= 0,\\ u^2 + w^2 + x^2 - v^2 &= 0. \end{aligned} \tag{5.14}$$

First of all, we want to count the number of all solutions of the system (5.14). For that purpose, we need the following theorem.

Theorem 5.3.2 (Bezout's theorem) *Suppose n homogeneous polynomials on $\mathbb{C}$ in $n+1$ variables, of degrees $d_1, d_2, .., d_n$, that define n hypersurfaces in the projective space of dimension n. If the number of intersection points of the hypersurfaces is finite, then this number is $d_1d_2\cdots d_n$ if the points are counted with their multiplicity.*

For more details and proof, see for example [20].

The system (5.14) consists of four homogeneous equations, each of degree two with five variables. So, the equations are naturally viewed as equations on real projective space. According to Bezout's Theorem, we have 16 complex solutions for that system. Therefore, there are at most 16 real solutions, and these can be divided into two main types: solutions when $v = 0$ and solutions when $v \neq 0$.

5.3.1 Solutions when $v = 0$

In this case, algebraic calculations give a total of three different solutions:

1. $\{t \in \mathbb{R}, u = 0, w = 0, x = 0\}$
2. $\{t = \dfrac{\mp 2c_2wi}{b_2}, u = \pm wi, w \in \mathbb{R}, x = 0\}$.

Now, we want to study the multiplicity of each solution. Consider the Jacobian matrix for the system (5.14), with respect to v, t, u, w, x:

$$J = \begin{pmatrix} \frac{1}{2}t+2a_1v+c_1u & \frac{1}{2}v & c_1v & a_2x & 2a_1x+a_2w \\ 2a_1x+2a_2w & \frac{1}{2}x+b_2w & 2c_1x+2c_2w & 2a_2v+b_2t+2c_2u & \frac{1}{2}t+2a_1v+2c_1u \\ -a_2u & -b_2u & -a_2v-b_2t-2c_2u & c_1x+2c_2w & c_1w \\ -2v & 0 & 2u & 2w & 2x \end{pmatrix}.$$

Substituting the values of the first solution, and the condition $v = 0$ in J yields

$$J\mid_{v=0,\text{sol.}1} = \begin{pmatrix} \frac{1}{2}t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2t & \frac{1}{2}t \\ 0 & 0 & -b_2t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To get the appropriate square submatrix, we eliminate the second column as t is non-zero, and obtain

$$J_1 = \begin{pmatrix} \frac{1}{2}t & 0 & 0 & 0 \\ 0 & 0 & b_2t & \frac{1}{2}t \\ 0 & -b_2t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is of rank three, and therefore, the first solution is not simple. To study its multiplicity, we need to study the behaviour of the system (5.14) near a solution point, for example, say $(v,t,u,w,x) = (0,2,0,0,0)$. Consider the system

$$\begin{aligned} v(1+a_1v+c_1u)+a_1x^2+a_2xw &= \varepsilon_1,\\ x(1+2a_1v+2c_1u)+w(2b_2+2a_2v+2c_2u) &= \varepsilon_2,\\ c_1xw-u(2b_2+a_2v+c_2u)+c_2w^2 &= \varepsilon_3,\\ u^2+w^2+x^2-v^2 &= \varepsilon_4. \end{aligned} \tag{5.15}$$

Near the point $(v,t,u,w,x) = (0,2,0,0,0)$, the first equation can be solved by the implicit function theorem for v, the second for x and the third equation for u. As a result, we end up with solving the equation

$$w^2 + f(w) = \varepsilon_4,$$

where $f(w)$ is a function constructed by substituting the solutions from the implicit function theorem in the last equation of the system (5.15). Clearly, f is of degree greater than one. So, the least order coefficient is w^2 and the studied solution is of multiplicity two.

Regarding the multiplicity of the second and third solution, we should assume that $w \neq 0$ for a non-zero solution; for simplicity let $w = 1$. The matrix J will take the form

$$J\mid_{v=0,w=1,\text{sol}.2}= \begin{pmatrix} \mp c_2i/b_2 \pm c_1i & 0 & 0 & 0 & a_2\\ 2a_2 & b_2 & 2c_2 & 0 & \mp c_2i/b_2 \pm 2c_1i\\ \mp a_2i & \mp b_2i & 0 & 2c_2 & c_1\\ 0 & 0 & \pm 2i & 2 & 0 \end{pmatrix}.$$

Since $w = 1$, we can omit the w-column and get

$$J_2 = \begin{pmatrix} \mp c_2i/b_2 \pm c_1i & 0 & 0 & a_2\\ 2a_2 & b_2 & 2c_2 & \mp c_2i/b_2 \pm 2c_1i\\ \mp a_2i & \mp b_2i & 0 & c_1\\ 0 & 0 & \pm 2i & 0 \end{pmatrix},$$

$$\det J_2 = -2(a_2^2b_2^2 + b_2^2c_1^2 - 2b_2c_1c_2 + c_2^2)/b_2.$$

We can assume that this result is non-zero, and therefore, the second and the third solutions are simple. We conclude that the case $v = 0$ corresponds to four solutions, where the first solution is doubled, but the others are of multiplicity one. Note that

$v = 0$ implies $N = |z_1|^2 + |z_2|^2 = 0$. Thus, these four solutions would not be counted as periodic solutions of the given system, but will help us find out how many non-zero periodic solutions are there.

5.3.2 Solutions when $v \neq 0$

There remain 12 solutions for the case $v \neq 0$ according to Bezout's theorem. The following proposition guarantees a minimum of two real solutions for the system (5.14).

Proposition 5.3.3 *For any choice of coefficients* $\{a_1, a_2, b_2, c_1, c_2\}$ *the two points*

$$\{v \in \mathbb{R}^*, t = -4a_1 v, u = w = 0, x = \pm v\},$$

satisfy Equation (5.14).

PROOF Straightforward calculations yield the result. □

In order to find out more about the maximum number of real solutions we can find, we will use a numerical approach. We choose various values for the constants in the system (5.14), and then solve the equations using Maple. Since we are interested in solutions with $v \neq 0$, we put $v = 1$ for simplicity. These numerical calculations show that the system can have a maximum of at least eight real solutions, including the two analytic solutions given by Proposition 5.3.3. In addition, there are examples of systems with four or six real solutions. Our aim is to prove that for each of these cases, the solutions are non-degenerate. Then, under any perturbation of the set of coefficients, there still exist (nearby) real solutions (i.e. periodic solutions). In the following, we study an example of each set of coefficients that has two, four, six or eight real solutions for the studied system (5.14). Then, we check their non-degeneracy conditions. Note that all numbers are rounded to four decimal digits.

Example 5.3.4 (A system with two real solutions) Consider the set

$$R = \{a_1 = 1, a_2 = 5, b_2 = 1, c_1 = 2, c_2 = 2, v = 1\}.$$

The corresponding system has only two real solutions

$$\{t = -4, u = 0, w = 0, x = \pm v = \pm 1\},$$

which are those given in Proposition 5.3.3. The remaining 10 solutions are non-real and they are

1. $\{t = -4.2 \mp 2.4i, u = -0.2 \pm 0.1i, w = 0.1 \pm 0.2i, x = 1\}$
2. $\{t = -4.2 \pm 2.4i, u = -0.2 \mp 0.1i, w = -0.1 \pm 0.2i, x = -1\}$
3. $\{t = -4.7211 - 2.6884i, u = -0.6402 + 0.1912i, w = \pm 0.5228 \pm 0.3977i, x = \pm 0.7249 \mp 0.1180i\}$
4. $\{t = -7.1863, u = 1.1519, w = \pm 0.2152i, x = \mp 0.5297i\}$
5. $\{t = -4.7211 + 2.6884i, u = -0.6402 - 0.1912i, w = \pm 0.5228 \mp 0.3977i, x = \pm 0.7249 \pm 0.1180i\}$.

In order to check the non-degeneracy condition for the real solutions, we need to study the proper submatrix of J for each solution, and ensure that its determinant is non-zero. Substituting the values given in R and the two solutions in J yields

$$J_1 = \begin{pmatrix} 0 & 0.5 & 2 & \pm 5 & \pm 2 \\ \pm 2 & \pm 0.5 & \pm 4 & 6 & 0 \\ 0 & 0 & -1 & \pm 2 & 0 \\ -2 & 0 & 0 & 0 & \pm 2 \end{pmatrix}.$$

Since $t \neq 0$, we omit the $t-$column and we have the submatrix

$$J_{11} = \begin{pmatrix} 0 & 2 & \pm 5 & \pm 2 \\ \pm 2 & \pm 4 & 6 & 0 \\ 0 & -1 & \pm 2 & 0 \\ -2 & 0 & 0 & \pm 2 \end{pmatrix},$$

$$\det J_{11} = \pm 20.$$

Therefore, these two solutions are non-degenerate.

A similar argument is used in the remaining examples, to prove the non-degeneracy of solutions in each case.

Example 5.3.5 (A system with four real solutions) Let the set of coefficients in the system (5.14) be

$$R = \{a_1 = 1, a_2 = 5, b_2 = -2, c_1 = 2, c_2 = 2, v = 1\}.$$

The associated system has four real solutions and eight non-real ones given by

1. $\{t = -4, u = 0, w = 0, x = \pm v = \pm 1\}$
2. $\{t = 1.5602, u = -0.9681, w = \pm 0.0855, x = \pm 0.2354\}$
3. $\{t = 2.0735 \pm 0.0441i, u = -0.2132 \pm 0.5221i, w = 0.5221 \pm 0.2132i, x = -1\}$
4. $\{t = 2.0735 \mp 0.0441i, u = -0.2132 \mp 0.5221i, w = -0.5221 \pm 0.2132i, x = 1\}$
5. $\{t = 3.6488 + 0.5231i, u = -0.1494 - 0.1945i, w = \pm 0.8141 \pm 0.1651i, x = \mp 0.6638 \pm 0.2463i\}$
6. $\{t = 3.6488 - 0.5231i, u = -0.1494 + 0.1945i, w = \pm 0.8141 \mp 0.1651i, x = \mp 0.6638 \mp 0.2463i\}$.

Clearly, the first four solutions are real-valued. Substituting R and the first two solutions in the matrix J implies

$$J_1 = \begin{pmatrix} 0 & 0.5 & 2 & \pm 5 & \pm 2 \\ \pm 2 & \pm 0.5 & \pm 4 & 18 & 0 \\ 0 & 0 & -13 & \pm 2 & 0 \\ -2 & 0 & 0 & 0 & \pm 2 \end{pmatrix}.$$

Now, we can choose the submatrix J_{11} by omitting the second column, because t is non-zero, and we find its determinant to be $\det J_{11} = \pm 692 \neq 0$. In the same way, we can study the third and fourth solutions to get

$$J_{22} = \begin{pmatrix} 0.8439 & 2 & \pm 1.1772 & \pm 0.8985 \\ \pm 1.3262 & \pm 1.2839 & 3.0071 & -1.0924 \\ 4.8406 & 1.9929 & \pm 0.8130 & \pm 0.1711 \\ -2 & -1.9362 & \pm 0.1711 & \pm 0.4709 \end{pmatrix}.$$

We have $\det J_{22} = \pm 35.6351 \neq 0$.

Since the determinants are non-zero, all four solutions are non-degenerate, and we can find an open set of coefficients that give four real solutions.

Example 5.3.6 (A system with six real solutions) Let

$$R = \{a_1 = -2, a_2 = -11, b_2 = -5, c_1 = 1, c_2 = 2, v = 1\}.$$

The system (5.14) with those coefficients has the following solutions:

1. $\{t = 8, u = 0, w = 0, x = \pm v = \pm 1\}$

2. $\{t = -2.5592, u = 0.0346, w = \pm 0.4980, x = \mp 0.8665\}$

3. $\{t = -3.7663, u = 0.1529, w = \pm 0.8984, x = \mp 0.4118\}$

4. $\{t = -2.1607 \mp 0.1659i, u = 0.0491 \mp 0.4574i, w = 0.4574 \pm 0.0491i, x = -1\}$

5. $\{t = -2.1607 \pm 0.1659i, u = 0.0491 \pm 0.4574i, w = -0.4574 \pm 0.0491i, x = 1\}$

6. $\{t = -1.4887, u = 1.8444, w = \pm 0.2254i, x = \mp 1.5333i\}$.

The non-degeneracy of the 6 real-valued solutions can be studied in pairs. Firstly, we study the determinant of the appropriate matrix J_{11} associated to the first and second solutions.

$$J_{11} = \begin{pmatrix} 0 & 1 & \mp 11 & \mp 4 \\ \mp 4 & \pm 2 & -62 & 0 \\ 0 & 51 & \pm 1 & 0 \\ -2 & 0 & 0 & \pm 2 \end{pmatrix},$$
$$\det J_{11} = \mp 20816.$$

Similarly, for the rest of solutions we have

$$J_{22} = \begin{pmatrix} -5.2450 & 1 & \pm 9.5314 & \mp 2.0120 \\ \mp 7.4900 & \pm 0.2590 & -9.0658 & -5.2104 \\ 0.3804 & -1.9342 & \pm 1.1255 & \pm 0.4980 \\ -2 & 0.0692 & \pm 0.9960 & \mp 1.7330 \end{pmatrix},$$

$$\det J_{22} = \mp 164.8123,$$

$$J_{33} = \begin{pmatrix} -5.7303 & 1 & \pm 4.5299 & \mp 8.2346 \\ \mp 18.1165 & \pm 2.7698 & -2.5567 & -5.5774 \\ 1.6818 & -8.4433 & \pm 3.1816 & \pm 0.8984 \\ -2 & 0.3058 & \pm 1.7967 & \mp 0.8236 \end{pmatrix},$$
$$\det J_{33} = \pm 1827.2294.$$

As a result, all real solutions of this case are non-degenerate.

We end with an example of a system with eight real solutions, which is the largest number of real solutions we found using numerical calculations.

Example 5.3.7 (A system with eight real solutions) Let

$$R = \{a_1 = 1, a_2 = -4, b_2 = -1, c_1 = 1, c_2 = 2, v = 1\}.$$

The corresponding solutions are as follows:

1. $\{t = -4, u = 0, w = 0, x = \pm v = \pm 1\}$
2. $\{t = -4.9432, u = -0.2615, w = \pm 0.2274, x = \mp 0.9380\}$
3. $\{t = -2.8537, u = 0.8527, w = \pm 0.4155, x = \pm 0.3165\}$
4. $\{t = -6.4260, u = 0.2940, w = \pm 0.8063, x = \mp 0.5133\}$
5. $\{t = -4.32 \mp 0.76i, u = -0.08 \pm 0.06i, w = 0.06 \pm 0.08i, x = -1\}$
6. $\{t = -4.32 \pm 0.76i, u = -0.08 \mp 0.06i, w = -0.06 \pm 0.08i, x = 1\}$.

There are eight real solutions and their non-degeneracy conditions are

$$J_{11} = \begin{pmatrix} 0 & 1 & \mp 4 & \pm 2 \\ \pm 2 & \pm 2 & -4 & 0 \\ 0 & 0 & \pm 1 & 0 \\ -2 & 0 & 0 & \pm 2 \end{pmatrix},$$

$$\det J_{11} = \pm 4,$$

$$J_{22} = \begin{pmatrix} -0.7331 & 1 & \pm 3.7521 & \mp 2.7857 \\ \mp 3.6953 & \mp 0.9664 & -4.1029 & -0.9947 \\ -1.0461 & 0.1029 & \mp 0.0284 & \pm 0.2274 \\ -2 & -0.5231 & \pm 0.4548 & \mp 1.8760 \end{pmatrix},$$

$$\det J_{22} = \mp 13.8083,$$

$$J_{33} = \begin{pmatrix} 1.4259 & 1 & \mp 1.2659 & \mp 1.0293 \\ \mp 2.6915 & \pm 2.2951 & -1.7353 & 2.2786 \\ 3.4110 & -2.2647 & \pm 1.9787 & \pm 0.4155 \\ -2 & 1.7055 & \pm 0.8311 & \pm 0.6329 \end{pmatrix},$$

$$\det J_{33} = \mp 43.7450,$$

$$J_{44} = \begin{pmatrix} -0.9190 & 1 & \pm 2.0533 & \mp 4.2517 \\ \mp 7.4767 & \pm 2.1984 & -0.3979 & -0.6249 \\ 1.1761 & -3.6021 & \pm 2.7117 & \pm 0.8063 \\ -2 & 0.5881 & \pm 1.6125 & \mp 1.0266 \end{pmatrix},$$

$$\det J_{44} = \pm 111.6657.$$

Therefore, all eight solutions are non-degenerate.

5.4 Conclusion

Bezout's theorem guaranteed a total of 12 solutions for the case $v \neq 0$, but numerical calculations found at least two and at most eight of them to be real. The last thing to consider is the effect of the addition of higher order terms to the system (5.14), when solving by the implicit function theorem. We will choose one of the previous examples, and prove the existence of periodic orbits in that system, and the rest can be done in the same way.

We consider the solution point $(t, u, w, v, x, r) = (-4, 0, 0, 1, 1, 0)$ as a candidate. We want to apply the implicit function theorem on the system in a neighbourhood of that point. Note that the functions g^1, g^2 are given by

$$g^1(N, C, D^2, \tau) = \frac{\tau}{2} + a_1 N + c_1 C + d_1 D^2 + e_1 N^2 + f_1 NC + g_1 N\tau + \cdots, \tag{5.16}$$

$$g^2(N, C, D^2, \tau) = b_2\tau + a_2 N + c_2 C + d_2 D^2 + e_2 N^2 + f_2 NC + g_2 N\tau + \cdots. \tag{5.17}$$

In our new coordinates (5.16) and (5.17) will take the form

$$g^1(N, C, D^2, \tau) = r[\frac{t}{2} + a_1 v + c_1 u + d_1 r w^2 + e_1 r v^2 + f_1 r v u + g_1 r v t + \cdots],$$

$$g^2(N, C, D^2, \tau) = r[b_2 t + a_2 v + c_2 u + d_2 r w^2 + e_2 r v^2 + f_2 r v u + g_2 r v t + \cdots],$$

and therefore, the matrix formula associated to the implicit function theorem calculated at the point $(-4, 0, 0, 1, 1, 0)$ will be

$$\left(\begin{array}{cccc|c} 0 & 2 & 5 & 2 & 3e_1 - 8g_1 \\ 2 & 4 & 6 & 0 & 3e_1 - 8g_1 \\ 0 & -1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 \end{array}\right) = \left(X \,\middle|\, Y \right).$$

The matrix X is invertible, and by the implicit function theorem, we can solve v, u, w, x as functions of r. The linear part of the Taylor series of those solutions is determined by the matrix

$$X^{-1}Y = \begin{pmatrix} 7/5 & -9/10 & -4/5 & -7/5 \\ -2/5 & 2/5 & -1/5 & 2/5 \\ -1/5 & 1/5 & 2/5 & 1/5 \\ 7/5 & -9/10 & -4/5 & -9/10 \end{pmatrix} \begin{pmatrix} 3e_1 - 8g_1 \\ 3e_1 - 8g_1 \\ 0 \\ 0 \end{pmatrix}.$$

Those solutions can be written as functions of r as follows

$$\begin{aligned} v(r) &= 1 - \frac{1}{2}(3e_1 - 8g_1)r + \text{h.o.t.} \\ x(r) &= 1 - \frac{1}{2}(3e_1 - 8g_1)r + \text{h.o.t.} \\ v &= w = 0. \end{aligned}$$

Converting back to our basic coordinates N, C, D, δ gives

$$\begin{aligned} N &= rv = r - \frac{1}{2}(3e_1 - 8g_1)r^2 + \text{h.o.t.} \\ \delta &= rx = r - \frac{1}{2}(3e_1 - 8g_1)r^2 + \text{h.o.t.} \\ C &= D = 0. \end{aligned}$$

This curve of solutions gives a one-parameter family of periodic orbits for the equivariant Hamiltonian system. Similarly, one can prove the existence of a one-parameter family of periodic solutions for each case studied before, because of their non-degeneracy conditions.

It remains to investigate whether there are more periodic orbits associated to the higher order terms or not. By Theorem 5.1 in [29], which states if the Hamiltonian is written in the form $H = H_2 + H_4 + \tilde{H}$, where H_4 is homogeneous of degree 4 and the truncated Hamiltonian $H_2 + H_4$ satisfies a non-degeneracy condition, then there is a 1-1 correspondence which maps the nonlinear normal modes of H to those of $H_2 + H_4$. One can verify the non-degeneracy condition for our problem and then we conclude that there is no more periodic orbits coming from the higher order terms. We summarise the results of this chapter in the following theorem.

Theorem 5.4.1 *Consider an equilibrium point* 0 *of a* C^∞ *equivariant Hamiltonian vector field* f*, with the the symmetry* S *acting anti-symplectically and* $S^2 = I$*. Assume* *that the linear Hamiltonian vector field L has two pairs of purely imaginary eigenvalues $\pm i$ and no other eigenvalues of the form $\pm ki, k \in \mathbb{Z}$. The reduced Hamiltonian is in the form $h = \delta g^1(N, C, D^2, \tau) + Dg^2(N, C, D^2, \tau)$. Then,*

1. *For an open dense set of coefficients $\{a_1, a_2, b_2, c_1, c_2\}$ there exists a neighbourhood of 0 with no symmetric periodic orbits, and at least two and at most 12 one-parameter families of non-symmetric periodic solutions of the equivariant Hamiltonian system.*

2. *There exist open sets of coefficients U_j $(j = 1, 2, 3, 4)$, such that for coefficients in U_j there are precisely $2j$ one-parameter families of non-symmetric periodic orbits of period close to 2π as they tend to zero.*

Remark 5.4.2 Of course, it is perfectly possible there are open regions of the space of coefficients, for which, there are 10 or 12 real one-parameter families of periodic orbits through the origin: our numerical search was certainly not exhaustive. Although, by Arnold's principle of the fragility of all good things, one would expect relatively smaller regions with higher numbers of real solutions, see [3].

Chapter 6

Some classes of reversible equivariant Hamiltonian systems

This chapter aims to generalise the results on the existence of periodic orbits discussed in the previous two chapters. In particular, we study the existence of periodic orbits in two classes of reversible equivariant Hamiltonian systems. There is only a limited amount of published work under the reversible equivariant Hamiltonian settings, such as [29], which motivated us to study these problems.

The first problem we consider is a natural consequence of Chapter 4 and Chapter 5, that is analysing the reversible equivariant Hamiltonian system under the action of the group $G = \mathbb{Z}_2^R \times \mathbb{Z}_2^S$, where the involutions R and S are as defined in Chapter 4 and 5 respectively. This problem forms a section in [2].

Another generalisation that we discuss in this chapter is proving the existence of families of periodic orbits in a reversible equivariant Hamiltonian system with respect to the cyclic group $\mathbb{Z}_{2r}^R$, where R is a reversing symmetry acting symplectically (type SR in Table 2.1). We will denote reversible equivariant as (RE).

6.1 Periodic orbits in $\mathbb{Z}_2^R \times \mathbb{Z}_2^S$- RE Hamiltonian systems

The first simple example of a RE Hamiltonian system that we investigate in this chapter is the system possessing both involutions, R and S, studied in the two previous chapters. These two involutions commute and generate the reversing equivariant group $G = \mathbb{Z}_2^R \times \mathbb{Z}_2^S$. Recall that R is of type (SR) where S is (AE). In this section, we prove the existence of families of periodic solutions in a neighbourhood of the origin in that system.

By Lemma 4.1.1 and Lemma 5.1.1, the linearisation L and the involutions R and S take the following forms

$$L = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad S|_V = \begin{pmatrix} 0 & S_2 \\ S_2 & 0 \end{pmatrix},$$

where I_2, J_2 and S_2 are as defined in Lemma 5.1.1.

By the Liapunov-Schmidt reduction, the reduced Hamiltonian h should be R and S anti-invariant but S^1 invariant. Recall the actions

$$\begin{aligned} R(z_1, z_2) &= (z_2, z_1), \\ S(z_1, z_2) &= (\bar{z}_2, \bar{z}_1), \\ \theta(z_1, z_2) &= (e^{i\theta} z_1, e^{-i\theta} z_2). \end{aligned}$$

Therefore, on $\mathbb{C}^2$ the function h will be a special case of the Hamiltonian in Chapter 4, and it takes the form

$$h(z_1, z_2, \tau) = \delta g(N, C, D^2, \tau).$$

Accordingly, the bifurcation equation will be

$$\begin{cases} \dfrac{\partial h}{\partial z_1} = \bar{z}_1 g + \delta \dfrac{\partial g}{\partial z_1} = 0, \\ \dfrac{\partial h}{\partial z_2} = -\bar{z}_2 g + \delta \dfrac{\partial g}{\partial z_2} = 0. \end{cases} \tag{6.1}$$

After taking partial derivatives, we multiply the first equation of (6.1) by z_1 and the second one by z_2 to get

$$|z_1|^2 g + \delta[|z_1|^2 g_N + z_1 z_2 g_C - 2iD(z_1 z_2) g_{D^2}] = 0, \tag{6.2}$$

$$-|z_2|^2 g + \delta[|z_2|^2 g_N + z_1 z_2 g_C - 2iD(z_1 z_2) g_{D^2}] = 0. \tag{6.3}$$

Adding (6.2) to (6.3) gives

$$\delta[g + N g_N + (C + iD) g_C - 2iD(C + iD) g_{D^2}] = 0. \tag{6.4}$$

Subtracting (6.3) from (6.2) implies

$$Ng + \delta^2 g_N = 0.$$

Thus, the bifurcation equation finally takes the form

$$\begin{aligned} \delta[g + Ng_N + Cg_C + 2D^2 g_{D^2}] &= 0, \\ Ng + [N^2 - C^2 - D^2]g_N &= 0, \\ \delta D[g_C - 2Cg_{D^2}] &= 0. \end{aligned} \tag{6.5}$$

Now, we classify the solutions according to their symmetry type.

6.1.1 Periodic orbits in the conical subset $\delta = 0$

Substituting $\delta = 0$ in the system of equations (6.5) yields

$$Ng = 0. \tag{6.6}$$

For R-symmetric solutions, one needs to solve (6.6) in $\mathrm{Fix}\, R$. This implies

$$g(z, \tau) = 0,$$

which can be solved for $\tau = \tau(z), z \in \mathrm{Fix}\, R$ by the implicit function theorem. This means, any periodic orbit in the subspace $\delta = 0$ has symmetry R. Solving Equation (6.6) for $z \in \mathrm{Fix}\, S$ gives one periodic orbit of symmetry S, and it is therefore G-symmetric (that is, the periodic orbit is invariant under every element of G). Moreover, solving Equation (6.6) for $z \in \mathrm{Fix}\,(S, \pi)$ gives another orbit with symmetry S. Here $(S, \pi) \in G \ltimes S^1$ and $(S, \pi)(z_1, z_2) = (-\bar{z}_2, -\bar{z}_1)$.

6.1.2 Periodic orbits in $\delta \neq 0$

Since all R or S-symmetric solutions must lie in the cone $\delta = 0$, it remains to study the existence of solutions with symmetry SR lying in the open subset $\delta \neq 0$.

Clearly $\mathrm{Fix}\, RS = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{R}\}$ which implies $D = 0$ and therefore, the system (6.5) takes the form

$$\begin{aligned} g + Ng_N + Cg_C &= 0, \\ Ng + [N^2 - C^2]g_N &= 0. \end{aligned} \tag{6.7}$$

Eliminating g from both equations gives

$$C(Ng_C + Cg_N) = 0. \tag{6.8}$$

If $C = 0$, then by the fact $\frac{\partial g}{\partial \tau}(0) = \frac{1}{2}$ we can solve (6.7) using the implicit function theorem. Now, if $C \neq 0$, and $g_N(0) = n$, $g_C(0) = c$ are not both zero, then the system (6.7) can be solved by the implicit function theorem. By the argument used in Theorem 4.2.5, we conclude that SR-periodic solutions exist when $n^2 - c^2 > 0$.

Note that generically, we do not expect any non-symmetric solutions in the subset $\delta \neq 0$. That is because the bifurcation equation (6.5) in this case will take the form

$$\begin{aligned} g + Ng_N + Cg_C + 2D^2 g_{D^2} &= 0, \\ Ng + [N^2 - C^2 - D^2]g_N &= 0, \\ g_C - 2Cg_{D^2} &= 0. \end{aligned} \tag{6.9}$$

Clearly, the third equation implies $g_C(0) = 0$, which is not a generic condition. The following theorem describes the families of periodic solutions existing in this system.

Theorem 6.1.1 *Consider a symmetric equilibrium* 0 *of a* $\mathbb{Z}_2^R \times \mathbb{Z}_2^S$*-reversible equivariant Hamiltonian vector field* f*, where* R *is a reversing involution acting symplectically, and* S *is an involution acting anti-symplectically. Suppose that* $Df(0)$ *has two purely imaginary pairs of eigenvalues* $\pm i$ *with no other eigenvalues of the form* $\pm ki, k \in \mathbb{Z}$*. Also, denote* $g_N(0) = n$ *and* $g_C(0) = c$*. Then,*

1. *The conical subspace* $\delta = 0$ *consists of a two-parameter family of* R*-symmetric periodic solutions, with two of the periodic orbits having extra symmetry* S*.*

2. *There exist two Liapunov centre families of* SR*-symmetric periodic solutions in the open subset* $\delta \neq 0$ *provided that* $n^2 - c^2 > 0$*: one with* $\delta > 0$ *and one with* $\delta < 0$*. These two families are exchanged by both involutions* R *and* S*.*

The period of all such orbits tends to 2π *as they approach the equilibrium.*

Finally, we illustrate the relation between fixed point spaces of the involutions R, S, SR and (S, π) geometrically. Buzzi and Lamb [9] show that the intersection between the cone $\delta = 0$ and the unit sphere in $\mathbb{C}^2$ is a torus T parameterised by two angles (θ_1, θ_2) and draw $\operatorname{Fix} R$ on T. In addition to that, we show the intersection between $\operatorname{Fix} S$ and the torus T, which is given by the line $\theta_2 = -\theta_1$ and the intersection between $\mathrm{Fix}\,(S,\pi)$ and T, which is given by $\theta_1 + \theta_2 = \pi$. The last thing is to intersect $\mathrm{Fix}\,G$ with T, which gives a total of two points $(0,0)$ and (π,π) (shown as large dots in Figure 6.1).

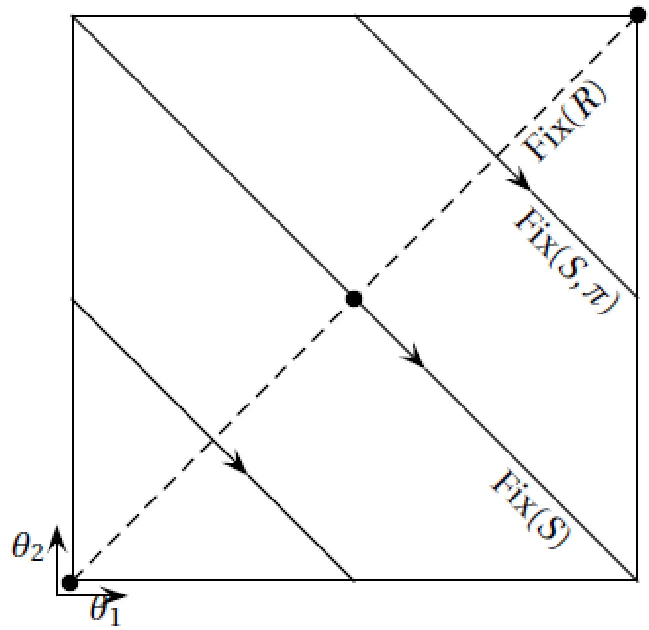


Figure 6.1: Fixed point spaces on the torus in the cone $\{\delta = 0\}$. The flow of the linear system (the S^1-action) restricted to the torus is parallel to the line Fix(S).

6.2 Periodic orbits in $\mathbb{Z}_{2r}^R$-RE Hamiltonian systems

Our goal in this section is to prove a general statement on the existence of periodic orbits in a $\mathbb{Z}_{2r}^R$-reversible equivariant Hamiltonian system, where the generator R is a reversing symmetry acting symplectically (type SR) on $\mathbb{C}^2$. This comes as a natural generalisation of our work on systems with symmetry group $\mathbb{Z}_2^R$ in Chapter 4. To the best of our knowledge, such generalisation does not appear in the literature, so we have constructed the problem from the beginning. This includes defining the group action and choosing the appropriate formulas for the Hamiltonian, the linear system, and therefore, the S^1 action. This will be done in several steps. Firstly, we will define the action of the cyclic group $\mathbb{Z}_{2r}^R$ on $\mathbb{C}^2$ taking into account reversing and symplectic properties of its generator. Secondly, we will deduce the S^1 action from the linear Hamiltonian vector field. The final step will be solving the bifurcation equation, according to the possible isotropy subgroups of $\mathbb{Z}_{2r}^R \ltimes S^1$.

6.2.1 The group action

The main ingredient of the group $\mathbb{Z}_{2r}^R = \langle R \rangle$ is the generator R, which is of order $2r$, symplectic and time-reversing. Note that the order of the group should be even because of the reversing property. We will use a representation theory argument to write the general form of this group action. On $\mathbb{R}^4 \simeq \mathbb{C}^2$ a symplectic action is an action that commutes with the matrix J or equivalently, an action that commutes with the complex number i. Thus, we will use complex representations to define the action. By the definition of the cyclic group $\mathbb{Z}_{2r}^R = \langle R \rangle$ we have $R^{2r} = 1$. Consequently, there is a one-to-one correspondence between $\mathbb{Z}_{2r}^R$ and the $2r$-roots of unity, i.e.

$$\begin{aligned} R &\to e^{i\theta_0}, \\ R^{2r} &\to e^{2ri\theta_0} = 1 = e^{2ki\pi}, \\ \theta_0 &= \frac{k\pi}{r}, k = 0, 1, \ldots, 2r-1. \end{aligned}$$

Since $\mathbb{Z}_{2r}^R$ is abelian, then all its irreducible representations are one-dimensional. Accordingly, $\mathbb{C}^2$ can be written as a direct sum of two one-dimensional irreducible complex representations of $\mathbb{Z}_{2r}^R$. Without loss of generality, we can assume that

$$\mathbb{C}^2 = A_1 \oplus A_k, \tag{6.10}$$

where A_1 is the irreducible representation of the rotation $e^{i\pi/r}$ and A_k is the irreducible representation of the rotation $e^{k\pi/r}$ and we will determine the value of k according to the properties of the action.

Another property to be considered is time reversing. The Hamiltonian function H should be R anti-invariant i.e. $H(Rx) = -H(x)$. Therefore, we need to choose all quadratic terms in $\mathbb{C}^2$ which are R anti-invariant. By (6.10) the $\mathbb{Z}_{2r}^R$ action on $\mathbb{C}^2$ can be written as

$$R(z_1, z_2) = (e^{i\pi/r} z_1, e^{ik\pi/r} z_2).$$

Possible R anti-invariance properties are described in Table 6.1

We deduce from Table 6.1 that the possible four-dimensional R anti-invariant quadratic generators must have $k = r \pm 1$. The last step is to check the possibility of the existence of periodic solutions, by evaluating the eigenvalues of the linear Hamiltonian system associated to each of the two cases $k = r \pm 1$.

$f(z)$	$Rf(z)$	anti-invariant
z_1^2	$e^{2\pi i/r}z_1^2$	yes, if $r=2$
$\bar{z}_1^2$	$e^{-2\pi i/r}\bar{z}_1^2$	yes, if $r=2$
$z_1\bar{z}_1$	$z_1\bar{z}_1$	no
z_2^2	$e^{2\pi ik/r}z_2^2$	yes, if $k=r/2$
$\bar{z}_2^2$	$e^{-2\pi ik/r}\bar{z}_2^2$	yes, if $k=r/2$
$z_2\bar{z}_2$	$z_2\bar{z}_2$	no
z_1z_2	$e^{i\pi(k+1)/r}z_1z_2$	yes, if $k=r-1$
$\bar{z}_1\bar{z}_2$	$e^{-i\pi(k+1)/r}\bar{z}_1\bar{z}_2$	yes, if $k=r-1$
$\bar{z}_1z_2$	$e^{i\pi(k-1)/r}\bar{z}_1z_2$	yes, if $k=r+1$
$z_1\bar{z}_2$	$e^{-i\pi(k-1)/r}z_1\bar{z}_2$	yes, if $k=r+1$

Table 6.1: R anti-invariant generators on $\mathbb{C}^2$.

Case 1. $k=r+1$

In this case, R anti-invariant generators are $z_1\bar{z}_2$ and $\bar{z}_1z_2$. Therefore, the quadratic part of the Hamiltonian, which is R anti-invariant can be written as

$$H_2(q_1,p_1,q_2,p_2)=a(q_1q_2+p_1p_2)+b(q_2p_1-q_1p_2),$$

where $z_1=q_1+ip_1, z_2=q_2+ip_2$ and $a,b\in\mathbb{R}$. The Hamiltonian system will be

$$\begin{aligned}\dot{q_1}&=ap_2+bq_2,\\ \dot{p_1}&=-(aq_2-bp_2),\\ \dot{q_2}&=ap_1-bq_1,\\ \dot{p_2}&=-(aq_1+bp_1).\end{aligned}$$

The linear system can be presented by the matrix

$$L=\begin{pmatrix}0&0&b&a\\0&0&-a&b\\-b&a&0&0\\-a&-b&0&0\end{pmatrix}, \tag{6.11}$$

which has two pairs of purely imaginary eigenvalues named $\lambda=\pm\sqrt{a^2+b^2}\,i$. Thus, the Hamiltonian system under the action $R(z_1,z_2)=(e^{i\pi/r}z_1,-e^{i\pi/r}z_2)$ can be considered while studying the existence of periodic solutions.

Case 2. $k = r - 1$

Here, R anti-invariant generators are $z_1 z_2$ and $\bar{z}_1 \bar{z}_2$. The quadratic part of the Hamiltonian function will take the form

$$H_2(q_1, p_1, q_2, p_2) = a(q_1 q_2 - p_1 p_2) + b(q_1 p_2 + q_2 p_1),$$

where $z_1 = q_1 + ip_1, z_2 = q_2 + ip_2$ and $a, b \in \mathbb{R}$. The associated Hamiltonian vector field is

$$\begin{aligned}
\dot{q_1} &= -ap_2 + bq_2, \\
\dot{p_1} &= -(aq_2 + bp_2), \\
\dot{q_2} &= -ap_1 + bq_1, \\
\dot{p_2} &= -(aq_1 + bp_1).
\end{aligned}$$

The eigenvalues of the linear system are $\lambda = \pm\sqrt{a^2 + b^2}$ which are always real-valued. Therefore, the case $k = r - 1$ cannot possess any periodic solutions.

We conclude that, in order to study the existence of periodic orbits in a $\mathbb{Z}_{2r}^R$-reversible equivariant Hamiltonian system, with R acting symplectically, we define the action on $\mathbb{C}^2$ by

$$R(z_1, z_2) = (e^{i\pi/r} z_1, -e^{i\pi/r} z_2).$$

We highlight two possibilities which we will consider in the coming sections:

- If r is odd, then R^r is a symplectic reversing involution, so it falls in the scope of Buzzi and Lamb symmetry, studied in Chapter 4.
- If r is even, then R^r is a symplectic involution (not reversing).

6.2.2 The S^1 action

We found that the linear Hamiltonian system (6.11) has eigenvalues $\pm\sqrt{a^2 + b^2}\, i$. For the basic case when eigenvalues are $\pm i$ we apply the following coordinate changes. Let $\sqrt{a^2 + b^2} = 1$, this is equivalent to saying $a = \sin t, b = \cos t$. Accordingly, L will take the form

$$L = \begin{pmatrix} 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \\ -\cos t & \sin t & 0 & 0 \\ -\sin t & -\cos t & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-it} \\ -e^{it} & 0 \end{pmatrix}.$$

If we choose a transformation T to be

$$T = \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix},$$

then T is symplectic and commutes with $\mathbb{Z}_{2r}$. Moreover, T satisfies

$$L' = T^{-1}LT = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Thus, the later matrix L' can be used to represent the linear Hamiltonian vector field with reversing equivariant group $\mathbb{Z}_{2r}^R$. Clearly, L' has eigenvalues $\pm i$.

It is known from the Liapunov-Schmidt reduction that the S^1 action can be deduced from the basis of $\ker \mathcal{L}$. The kernel condition is given by

$$\begin{aligned} \mathcal{L}v = 0 \Rightarrow \frac{dv}{ds} &= L'v \\ \Rightarrow v &= \exp(sL')\ v_0, \end{aligned}$$

for an initial condition v_0. According to the formula of L', $\ker \mathcal{L}$ is generated by the basis

$$e_1(s) = \begin{pmatrix} \cos s \\ 0 \\ -\sin s \\ 0 \end{pmatrix}, \quad e_2(s) = \begin{pmatrix} 0 \\ \cos s \\ 0 \\ -\sin s \end{pmatrix}, \quad e_3(s) = \begin{pmatrix} \sin s \\ 0 \\ \cos s \\ 0 \end{pmatrix}, \quad e_4(s) = \begin{pmatrix} 0 \\ \sin s \\ 0 \\ \cos s \end{pmatrix}.$$

Therefore, the action of $\theta \in S^1$ on $\mathbb{R}^4$ is given by the matrix

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

In complex coordinates, the S^1 action is given by

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

or shortly,

$$\theta(z_1, z_2) = (\cos\theta z_1 + \sin\theta z_2, -\sin\theta z_1 + \cos\theta z_2).$$

6.2.3 The bifurcation equation

A fundamental step in finding the periodic solutions for the given Hamiltonian system is writing the bifurcation equation, which is given by

$$\nabla_z h = 0,$$

where h is the reduced Hamiltonian. In order to write the bifurcation equation, one needs to choose the set of R (semi-)invariant generators. Recall that we defined the $\mathbb{Z}_{2r}^R \ltimes S^1$ action on $\mathbb{C}^2$ by

$$\begin{aligned} R(z_1, z_2) &= (e^{i\pi/r} z_1, -e^{i\pi/r} z_2), \\ \theta(z_1, z_2) &= (\cos\theta z_1 + \sin\theta z_2, -\sin\theta z_1 + \cos\theta z_2). \end{aligned}$$

The following lemma describes the (semi-)invariant functions of this action.

Lemma 6.2.1 *If $\mathbb{Z}_{2r}^R \ltimes S^1$ acts on $\mathbb{C}^2$ as above, then*

1. *The S^1 invariants are generated by $N = |z_1|^2 + |z_2|^2, F + iG = z_1^2 + z_2^2$ and $E = -i(\bar{z}_1 z_2 - z_1 \bar{z}_2)$.*

2. *The $\mathbb{Z}_{2r}^R \ltimes S^1$ invariant functions are generated by $N, A, B, F^2 + G^2$ where*

$$A + iB = (z_1^2 + z_2^2)^r = (F + iG)^r.$$

3. *The S^1 invariant but R anti-invariant functions are generated by*

$$E = -i(\bar{z}_1 z_2 - z_1 \bar{z}_2).$$

Furthermore, these generators satisfy the relation

$$E^2 = N^2 - (F^2 + G^2).$$

The reduced Hamiltonian h is R anti-invariant, S^1 invariant and real-valued. According to Lemma 6.2.1, h can be written as

$$h(z_1, z_2, \tau) = Eg(N, A, B, F^2 + G^2, \tau).$$

The bifurcation equation, thus, takes the form

$$\begin{aligned} \frac{\partial h}{\partial z_1} &= i\bar{z}_2 g + E[g_N \bar{z}_1 + g_A(\tfrac{\partial A}{\partial F} - i\tfrac{\partial A}{\partial G}) z_1 \\ &\quad + g_B(\tfrac{\partial B}{\partial F} - i\tfrac{\partial B}{\partial G}) z_1 + 2g_{F^2+G^2}(F - iG) z_1] = 0, \end{aligned} \tag{6.12}$$

$$\begin{aligned}\frac{\partial h}{\partial z_2} &= -i\bar{z}_1 g + E[g_N \bar{z}_2 + g_A(\tfrac{\partial A}{\partial F} - i\tfrac{\partial A}{\partial G})z_2 \\ &\quad + g_B(\tfrac{\partial B}{\partial F} - i\tfrac{\partial B}{\partial G})z_2 + 2g_{F^2+G^2}(F - iG)z_2] = 0,\end{aligned} \tag{6.13}$$

where $g_N, g_A, \ldots$ represent the partial derivatives of g with respect to $N, A, \ldots$. Multiplying (6.12) by z_1 and (6.13) by z_2 gives

$$\begin{aligned}&iz_1\bar{z}_2 g + E[|z_1|^2 g_N + g_A(\tfrac{\partial A}{\partial F} - i\tfrac{\partial A}{\partial G})z_1^2 \\ &+ g_B(\tfrac{\partial B}{\partial F} - i\tfrac{\partial B}{\partial G})z_1^2 + 2g_{F^2+G^2}(F - iG)z_1^2] = 0,\end{aligned} \tag{6.14}$$

$$\begin{aligned}&-i\bar{z}_1 z_2 g + E[|z_2|^2 g_N + g_A(\tfrac{\partial A}{\partial F} - i\tfrac{\partial A}{\partial G})z_2^2 \\ &+ g_B(\tfrac{\partial B}{\partial F} - i\tfrac{\partial B}{\partial G})z_2^2 + 2g_{F^2+G^2}(F - iG)z_2^2] = 0.\end{aligned} \tag{6.15}$$

Adding (6.14) and (6.15) and taking real and imaginary parts of the result yields

$$\begin{aligned}&E[g + Ng_N + (F\tfrac{\partial A}{\partial F} + G\tfrac{\partial A}{\partial G})g_A \\ &+ (F\tfrac{\partial B}{\partial F} + G\tfrac{\partial B}{\partial G})g_B + 2(F^2 + G^2)g_{F^2+G^2}] = 0,\end{aligned} \tag{6.16}$$

$$E[(G\tfrac{\partial A}{\partial F} - F\tfrac{\partial A}{\partial G})g_A + (G\tfrac{\partial B}{\partial F} - F\tfrac{\partial B}{\partial G})g_B] = 0. \tag{6.17}$$

Now we multiply (6.12) by z_2 and (6.13) by z_1

$$\begin{aligned}&i|z_2|^2 g + E[\bar{z}_1 z_2 g_N + (\tfrac{\partial A}{\partial F} - i\tfrac{\partial A}{\partial G})z_1 z_2 g_A \\ &+ (\tfrac{\partial B}{\partial F} - i\tfrac{\partial B}{\partial G})z_1 z_2 g_B + 2(F - iG)z_1 z_2 g_{F^2+G^2}] = 0,\end{aligned} \tag{6.18}$$

$$\begin{aligned}&-i|z_1|^2 g + E[z_1\bar{z}_2 g_N + (\tfrac{\partial A}{\partial F} - i\tfrac{\partial A}{\partial G})z_1 z_2 g_A \\ &+ (\tfrac{\partial B}{\partial F} - i\tfrac{\partial B}{\partial G})z_1 z_2 g_B + 2(F - iG)z_1 z_2 g_{F^2+G^2}] = 0.\end{aligned} \tag{6.19}$$

By subtracting (6.19) from (6.18) we obtain

$$Ng + E^2 g_N = 0.$$

Substituting $E^2 = N^2 - (F^2 + G^2)$ in the previous equation implies

$$Ng + (N^2 - (F^2 + G^2))g_N = 0. \tag{6.20}$$

Thus, the bifurcation equation is equivalent to the system of equations (6.16), (6.17) and (6.20). In order to simplify these equations, we will apply two facts. Firstly, since A and B are homogeneous in F and G of order r then, Euler's theorem for homogeneous functions implies $F\frac{\partial A}{\partial F} + G\frac{\partial A}{\partial G} = rA$ and similarly $F\frac{\partial B}{\partial F} + G\frac{\partial B}{\partial G} = rB$. Substituting these two formulas in (6.16) gives

$$E[g + Ng_N + rAg_A + rBg_B + 2(F^2 + G^2)g_{(F^2+G^2)}] = 0. \tag{6.21}$$

Secondly, for any complex number $z = F + iG$ the partial derivative operator $\frac{\partial}{\partial z}$ is defined by $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial F} - i\frac{\partial}{\partial G})$. Therefore,

$$(F+iG)\frac{1}{2}\left(\frac{\partial}{\partial F} - i\frac{\partial}{\partial G}\right)(F+iG)^r = r(F+iG)^r = r(A+iB), \tag{6.22}$$

since $\frac{\partial}{\partial z}(z)^r = rz^{r-1}$. On the other hand, by expanding the left hand side of Equation (6.22) we have

$$\begin{aligned}
&\frac{1}{2}[(F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G}) + i(G\frac{\partial}{\partial F} - F\frac{\partial}{\partial G})](A+iB) = \\
&\frac{1}{2}[(F\frac{\partial A}{\partial F} + G\frac{\partial A}{\partial G} - G\frac{\partial B}{\partial F} + F\frac{\partial B}{\partial G}) + i(F\frac{\partial B}{\partial F} + G\frac{\partial B}{\partial G} + G\frac{\partial A}{\partial F} - F\frac{\partial A}{\partial G})] = \\
&r(A+iB).
\end{aligned}$$

Taking real and imaginary parts of the latter equation implies

$$\begin{aligned}
\frac{1}{2}[rA - G\frac{\partial B}{\partial F} + F\frac{\partial B}{\partial G})] &= rA, \\
\frac{1}{2}[rB + G\frac{\partial A}{\partial F} - F\frac{\partial A}{\partial G})] &= rB,
\end{aligned}$$

and therefore

$$\begin{aligned}
F\frac{\partial B}{\partial G} - G\frac{\partial B}{\partial F} &= rA, \\
G\frac{\partial A}{\partial F} - F\frac{\partial A}{\partial G} &= rB.
\end{aligned}$$

Accordingly, (6.17) will take the form

$$E[Bg_A - Ag_B] = 0. \tag{6.23}$$

As a result, to prove the existence of periodic solutions of the studied $\mathbb{Z}_{2r}^R$-RE Hamiltonian system one needs to solve equations (6.20), (6.21) and (6.23).

6.2.4 Isotropy subgroups of $\mathbb{Z}_{2r}^R \ltimes S^1$

Another important step in finding symmetric periodic solutions of this problem is determining their possible symmetries. In this subsection, we seek possible isotropy subgroups of $\mathbb{Z}_{2r}^R \ltimes S^1$ with two-dimensional fixed point spaces. For this purpose, let

$$(R^k, \theta)(z_1, z_2) = (z_1, z_2), \text{ for } 1 \le k \le 2r.$$

In matrix form, and by applying the R^k action first, we have

$$\begin{pmatrix} \cos\theta e^{k\pi i/r} & (-1)^k \sin\theta e^{k\pi i/r} \\ -\sin\theta e^{k\pi i/r} & (-1)^k \cos\theta e^{k\pi i/r} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

This is equivalent to having $\lambda = 1$ as an eigenvalue of the matrix on the left hand side. Let $\alpha = e^{k\pi i/r}$ then, the characteristic equation for $\lambda = 1$ is given by

$$\begin{vmatrix} \alpha\cos\theta - 1 & (-1)^k\alpha\sin\theta \\ -\alpha\sin\theta & (-1)^k\alpha\cos\theta - 1 \end{vmatrix} = 0. \tag{6.24}$$

To solve Equation (6.24), we consider two cases:

1. k is an even number.

2. k is an odd number.

Isotropy subgroups when k is an even number

If k is an even number then (6.24) will take the form

$$(\alpha\cos\theta - 1)^2 + \alpha^2\sin^2\theta = 0$$
$$\Leftrightarrow \alpha^2 - 2\cos\theta\ \alpha + 1 = 0.$$

Solving for θ gives

$$\cos\theta = \frac{\alpha^2+1}{2\alpha} = \cos\frac{k\pi}{r}$$

$$\Leftrightarrow \theta = \pm\frac{k\pi}{r}.$$

Now we calculate the fixed point space for $(R^k, \frac{k\pi}{r})$. Let

$$(R^k, \frac{k\pi}{r})(z_1, z_2) = (z_1, z_2).$$

This is equivalent to the pair

$$\alpha\cos\frac{k\pi}{r}z_1 + \alpha\sin\frac{k\pi}{r}z_2 = z_1, \tag{6.25}$$
$$-\alpha\sin\frac{k\pi}{r}z_1 + \alpha\cos\frac{k\pi}{r}z_2 = z_2. \tag{6.26}$$

Multiplying (6.25) by $\sin\frac{k\pi}{r}$ and (6.26) by $\cos\frac{k\pi}{r}$ and adding the results gives

$$(\alpha - \cos\frac{k\pi}{r})z_2 = \sin\frac{k\pi}{r}z_1$$

$$\Leftrightarrow z_2 = -iz_1.$$

Thus,

$$\mathrm{Fix}\,(R^k, \frac{k\pi}{r}) = \{(z, -iz) \mid z \in \mathbb{C}\},$$

which is clearly two-dimensional. Similar calculations can be done for the case $(R^k, -\frac{k\pi}{r})$ with

$$\mathrm{Fix}\,(R^k, -\frac{k\pi}{r}) = \{(z, iz) \mid z \in \mathbb{C}\}.$$

Since the fixed point spaces are independent from the choice of k, we conclude that the isotropy subgroups have come down to one group, that is $\langle R^2 \rangle \cong \mathbb{Z}_r$. Accordingly, the symmetry group to be considered while studying the existence of periodic solutions in this case will be

$$\tilde{\mathbb{Z}}_r := \langle (R^2, \pm\frac{2\pi}{r}) \rangle.$$

Isotropy subgroups when k is an odd number

For k odd (6.24) can be written as

$$\alpha^2 - 1 = 0$$

$$\Leftrightarrow \alpha = \pm 1.$$

Since $\alpha = e^{k\pi i/r}$, we have $k = 0, r$. But $k = 0$ is not an odd number, therefore, we only consider the case $k = r$, where r is an odd number. Now, we seek (R^r, θ) fixed point spaces. Suppose

$$(R^r, \theta)(z_1, z_2) = (z_1, z_2).$$

This equation splits into two equations as follows:

$$-\cos\theta z_1 + \sin\theta z_2 = z_1, \tag{6.27}$$

$$\sin\theta z_1 + \cos\theta z_2 = z_2. \tag{6.28}$$

By multiplying (6.27) by $\sin\theta$ and (6.28) by $\cos\theta$ and adding them up, we get

$$(1 - \cos\theta) z_2 = \sin\theta z_1.$$

Therefore,

$$\mathrm{Fix}\,(R^r, \theta) = \{((1 - \cos\theta)\, z, \sin\theta\, z) \mid z \in \mathbb{C}\}.$$

We conclude that for each $\theta \in S^1$ and r being an odd number, (R^r, θ) has a two-dimensional fixed point space.

6.2.5 The existence of periodic solutions

After considering all possible symmetries, we proceed to the final step in finding periodic solutions of the chosen $\mathbb{Z}_{2r}^R$-reversible equivariant Hamiltonian system, which is solving the bifurcation equation. For symmetric solutions, we solve the bifurcation equation, i.e. the triple (6.20), (6.21) and (6.23), taking into account isotropy subgroups discussed in Subsection 6.2.4. In addition, we study the existence of non-symmetric periodic solutions of this system. The following theorem presents the main result of this chapter, as it describes all families of periodic orbits, with period close to 2π which we found near the origin in this RE Hamiltonian system.

Theorem 6.2.2 *Consider a $\mathbb{Z}_{2r}$-reversible equivariant vector field with a symmetric equilibrium 0, where $\mathbb{Z}_{2r}$ is generated by the reversing symmetry R acting symplectically. Suppose that $Df(0)$ has two purely imaginary pairs of eigenvalues $\pm i$, with no other eigenvalues of the form $\pm ki, k \in \mathbb{Z}$. Then, generically*

1. *If r is an odd number, there exists a three-dimensional conical subset consisting of a two-parameter family of (R^r, θ)-symmetric periodic solutions, with $\theta \in S^1$, whose period tends to 2π as they approach the equilibrium point.*

2. *For any choice of r, there exist two one-dimensional families of $(R^2, \pm\frac{2\pi}{r})$-symmetric periodic orbits, distinguished by the sign of E. These orbits are just Liapunov modes, whose period tends to 2π as they approach the origin. Furthermore, there exist $2r$ one-parameter families of non-symmetric periodic orbits, with period close to 2π in a neighbourhood of the origin.*

Proof We divide the proof according to the symmetry type.

Periodic solutions with symmetry $\tilde{\mathbb{Z}}_r$

It is readily verified that R^2 is a symmetry acting symplectically. Therefore, periodic solutions with symmetry $\tilde{\mathbb{Z}}_r$ will lie in $\mathrm{Fix}\,(R^2, \pm\frac{2\pi}{r}) = \{(z, \mp iz) \mid z \in \mathbb{C}\}$. In order to get the wanted symmetric periodic solutions, we will solve the system of equations (6.20), (6.21) and (6.23), evaluated in $\mathrm{Fix}\,(R^2, \pm\frac{2\pi}{r})$. We start with periodic solutions with symmetry $(R^2, \frac{2\pi}{r})$. In this case we have

$$E = -2|z|^2, N = 2|z|^2, F = G = A = B = 0.$$

Substituting these values in (6.20), (6.21) and (6.23) result in one simple equation

$$|z|^2(g + 2|z|^2 g_N) = 0.$$

For nonzero z, we must have $g + 2|z|^2 g_N = 0$. Since $\frac{\partial g}{\partial \tau}(0,0) = \frac{1}{2} \neq 0$, the latter equation can be solved uniquely for $\tau = \tau(|z|^2)$ by the implicit function theorem. A similar argument can be used for orbits with symmetry $(R^2, -\frac{2\pi}{r})$ and will give a similar result. This proves the existence of two families of symmetric periodic solutions in a neighbourhood of the origin with symmetry $\tilde{\mathbb{Z}}_r$.

Periodic solutions with symmetry (R^r, θ), where r is an odd number

In this case, the involution R^r is a reversing symmetry acting symplectically, just as the group generator. On $\operatorname{Fix}(R^r, \theta)$ we have $E = 0$. Thus, we end up with solving

$$Ng(N, A, B, F^2 + G^2, \tau) = 0.$$

Since we are interested in nonzero z, we need to solve $g = 0$. By the implicit function theorem, in a neighbourhood of the origin there exists a unique solution τ such that $((1 - \cos\theta)\, z, \sin\theta\, z))$ lies on a periodic solution, with period $\frac{2\pi}{\tau+1}$. This proves the existence of a two-parameter family of reversing periodic solutions in a neighbourhood of the origin. Also, by the previous section, there exist two families of periodic solutions with symmetry $(R^2, \pm\frac{2\pi}{r})$. This result is similar to that given by Buzzi and Lamb [9], and discussed in details in Chapter 4. This is not surprising, because R^r is a reversing involution that acts symplectically, same as the reversing symmetry in their problem.

Non-symmetric periodic solutions

Now we would focus on studying the existence of non-symmetric periodic orbits in this $\mathbb{Z}_{2r}^R$-reversible equivariant Hamiltonian system. In this setting, the system of equations (6.20), (6.21) and (6.23) takes the form

$$\begin{aligned} Ng + (N^2 - (F^2 + G^2))g_N &= 0, \\ g + Ng_N + rAg_A + rBg_B + 2(F^2 + G^2)g_{(F^2+G^2)} &= 0, \\ Bg_A - Ag_B &= 0. \end{aligned} \tag{6.29}$$

Let $g_N(0) = n$, $g_A(0) = a$, $g_B(0) = b$ and $g_{(F^2+G^2)}(0) = f$. The matrix of partial derivatives, with respect to $A, B, F^2 + G^2, N$ and τ associated to the system (6.29) and calculated at the origin is given by

$$\left(\begin{array}{ccc|cc} 0 & 0 & -n & 0 & 0 \\ (r+1)a & (r+1)b & 3f & 2n & \frac{1}{2} \\ -b & a & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} X & Y \end{array} \right).$$

Clearly, $\det X = -(r+1)n(a^2+b^2)$. Thus, the systems (6.29) can be solved uniquely by the implicit function theorem if $n \neq 0$ and a, b are not both zero. Such a solution presents a periodic orbit of the studied system. By non-symmetric property, applying each element of the group to that orbit will produce another periodic solution of the system. $\square$

Chapter 7

Hamiltonian systems with D_4 symmetry

In this chapter, we will give an overview of all possible D_4 symmetries in Hamiltonian systems with two degrees of freedom. This will be done by introducing a systemic way of distinguishing these symmetries according to their symplectic and Hamiltonian signs. Then, we will demonstrate the appropriate formulas for the quadratic Hamiltonian H_2 and the linear structure map J on $\mathbb{C}^2$ in order to write the linearisation of each type of D_4 symmetry. In addition to finding these normal forms, we will study the existence of periodic orbits in some interesting types that links to our work in the former chapters.

7.1 D_4 symmetry types

The standard definition of the dihedral group D_4 on the plane is given by the symmetries of the square, with composition as an operation. This group includes four rotations and four reflections, which preserve the square and are generated by the two reflections κ and κ', illustrated in Figure 7.1.

The dihedral group D_4 is defined algebraically as follows:

$$D_4 = \langle \kappa, \kappa' | \kappa^2 = \kappa'^2 = (\kappa'\kappa)^4 = e \rangle.$$

We often write $\kappa'\kappa = \rho$, a rotation of order 4.

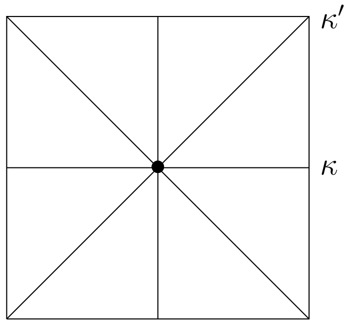


Figure 7.1: D_4 symmetry in the plane.

A Hamiltonian system with symmetry group D_4, can be either equivariant or reversible equivariant. However, these two main kinds can be split into various types. We can construct these types, by allowing all possible signs of the symplectic character $\chi : D_4 \to \{\pm 1\}$ and the Hamiltonian sign $\alpha : D_4 \to \{\pm 1\}$ for both generators κ and κ'. The following character table summarises the possible cases for a Hamiltonian system with D_4 symmetry.

type	$\alpha(\kappa)$	$\alpha(\kappa')$	$\chi(\kappa)$	$\chi(\kappa')$
(1)	1	1	1	1
(2)	1	-1	1	1
(3)	1	1	1	-1
(4)	1	-1	1	-1
(5)	-1	-1	1	1
(6)	-1	-1	1	-1
(7)	-1	1	1	-1
(8)	1	1	-1	-1
(9)	-1	-1	-1	-1
(10)	-1	1	-1	-1

Table 7.1: Possible types of D_4 symmetries.

Remark 7.1.1 Note that there is a total of 16 cases. However, because of the automorphism from D_4 to itself, which maps κ to κ', the number is decreased to 10 only. The 6 remaining cases will be equivalent to some other cases illustrated in Table 7.1.

7.2 The group action

For the sake of generality, we will fix a D_4 action on $\mathbb{C}^2$ among all types in Table 7.1. Let

$$\begin{aligned} \kappa(z_1, z_2) &= (z_1, -z_2), \\ \kappa'(z_1, z_2) &= (z_2, z_1). \end{aligned} \tag{7.1}$$

It is easy to check that κ and κ' defined in (7.1) generate a D_4 action on $\mathbb{C}^2$.

Now, we consider the D_4 real character table, which will be used throughout this chapter. Table 7.2 shows the D_4 real character table with conjugacy classes e, ρ, ρ^2, κ and κ' and irreducible representations A_0, A_1, B_1, B_2 and E. The $\sharp$ row gives the number of elements in each conjugacy class. Note that all of these representations are one-dimensional, except E which is two-dimensional.

D_4	e	ρ	ρ^2	κ	κ'
$\sharp$	1	2	1	2	2
A_0	1	1	1	1	1
A_1	1	1	1	-1	-1
B_1	1	-1	1	1	-1
B_2	1	-1	1	-1	1
E	2	0	-2	0	0

Table 7.2: The D_4 real character table.

7.3 Formulas for H and J

This section aims to write the appropriate formulas for the quadratic part of the Hamiltonian function H_2 and the linear structure map J, according to all D_4 symmetry types illustrated in Table 7.1. It is known that H_2 can be determined by a 4×4 symmetric matrix S, whereas J is a 4×4 skew-symmetric matrix satisfying $J^2 = -I$. For that purpose, we will choose two bases of the space of symmetric matrices and the space of skew-symmetric matrices in $gl(4, \mathbb{R})$ that respects the action (7.1). Respecting the action simply means that the basis elements are either invariant or anti-invariant under the action.

7.3.1 Basis of symmetric matrices

Let V be a representation of a group G and let $Q(V)$ denote the space of quadratic forms on V. The character $\chi_{Q(V)}$ is just the symmetric tensor product character χ_s defined in Lemma 2.2.6, (3).

In our setting, we have $V = \mathbb{R}^4$. Using the character table 7.2, and by the definition of our D_4 action we can write

$$V = \mathbb{R}^4 = E \oplus E.$$

Accordingly, we calculate the symmetric tensor product χ_s as follows:

D_4	e	ρ	ρ^2	κ	κ'
$\chi_V(g)$	4	0	-4	0	0
$\chi_V^2(g)$	16	0	16	0	0
$\chi_V(g^2)$	4	-4	4	4	4
$\chi_s(g)$	10	-2	10	2	2

Table 7.3: The character table of the symmetric tensor product for $V = E \oplus E$.

Note that the dimension of the space of quadratic forms is given by $\dim(Q(V)) = \frac{n(n+1)}{2}$, where n is the dimension of V. In our case, we have $\dim(Q(\mathbb{R}^4)) = 10$. In order to choose the appropriate basis for $Q(\mathbb{R}^4)$, we first write it as a decomposition of D_4 irreducible representations,

$$Q(\mathbb{R}^4) = a_1 A_0 \oplus a_2 A_1 \oplus a_3 B_1 \oplus a_4 B_2 \oplus a_5 E. \tag{7.2}$$

Now, we calculate the constants $a_1, \ldots, a_5$ by Lemma 2.2.6, (5) as follows:

1) $a_1 = \langle \chi_s, \chi_{A_0} \rangle = \frac{1}{8}[10 - 4 + 10 + 4 + 4] = 3.$

2) $a_2 = \langle \chi_s, \chi_{A_1} \rangle = \frac{1}{8}[10 - 4 + 10 - 4 - 4] = 1.$

3) $a_3 = \langle \chi_s, \chi_{B_1} \rangle = \frac{1}{8}[10 + 4 + 10 + 4 - 4] = 3.$

4) $a_4 = \langle \chi_s, \chi_{B_2} \rangle = \frac{1}{8}[10 + 4 + 10 - 4 + 4] = 3.$

5) $a_5 = \langle \chi_s, \chi_E \rangle = \frac{1}{8}[20 - 20] = 0.$

Thus, the decomposition (7.2) takes the following form

$$Q(\mathbb{R}^4) = 3A_0 \oplus A_1 \oplus 3B_1 \oplus 3B_2.$$

The second step in finding a suitable basis for $Q(\mathbb{R}^4)$ is writing the basis of quadratic forms, corresponding to the standard basis of symmetric matrices in $gl(4, \mathbb{R})$ using the relation $Q(x) = x^T Sx$. By choosing the coordinates $(q_1, p_1, q_2, p_2) \in \mathbb{R}^4$, the standard basis of quadratic forms is given by

$$q_1^2,\ p_1^2,\ q_2^2,\ p_2^2,\ 2q_1p_1,\ 2q_1q_2,$$

$$2q_1p_2,\ 2q_2p_1,\ 2p_1p_2,\ 2q_2p_2.$$

It is readily checked that this basis is not (semi-)invariant under the D_4 action given in (7.1). Therefore, we seek another basis, that is (semi-)invariant under that action. In the following table, we apply κ and κ' on our proposed basis to check its invariance properties.

notation	basis element	κ	κ'	rep. type
s_1	$q_1^2 + q_2^2$	1	1	A_0
s_2	$p_1^2 + p_2^2$	1	1	A_0
s_3	$2(q_1p_1 + q_2p_2)$	1	1	A_0
s_4	$2q_1q_2$	-1	1	B_2
s_5	$2(q_1p_2 + q_2p_1)$	-1	1	B_2
s_6	$2p_1p_2$	-1	1	B_2
s_7	$q_1^2 - q_2^2$	1	-1	B_1
s_8	$p_1^2 - p_2^2$	1	-1	B_1
s_9	$2(q_1p_1 - q_2p_2)$	1	-1	B_1
s_{10}	$2(q_1p_2 - q_2p_1)$	-1	-1	A_1

Table 7.4: A basis for $Q(\mathbb{R}^4)$. The numbers ± 1 denote the action being invariant or anti-invariant, respectively.

We conclude that the basis described above is (semi-)invariant under the chosen D_4 action. Equivalently, we can write a basis for the 4×4 symmetric matrices associated to the basis of quadratic forms, described in Table 7.4. The new basis for symmetric matrices is given by

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ S_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ S_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ S_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$S_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ S_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ S_9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$S_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, each matrix $S_i\,, i = 1, \ldots, 10$ carries the same representation type, for example $A_0, A_1, \ldots$ etc., as the corresponding quadratic form $s_i\,, i = 1, \ldots, 10$ illustrated in Table 7.4.

7.3.2 Basis of skew-symmetric matrices

In this subsection we apply the same analysis used in Subsection 7.3.1 in order to write a suitable basis for skew-symmetric 4×4 real-valued matrices. This basis will be used in writing the linear structure map J according to the properties of the chosen D_4 action. For $V = \mathbb{R}^4$ as before, we have $\mathbb{R}^4 = E \oplus E$ but $\dim A(\mathbb{R}^4) = 16 - 10 = 6$. We start by writing the decomposition of the space of skew-symmetric forms $A(V)$ in terms of D_4 irreducible representations. For that purpose, we calculate the character of the anti-symmetric tensor product χ_a of the representation $E \oplus E$. Following the formula of χ_a given in Lemma 2.2.6, (3) we have

D_4	e	ρ	ρ^2	κ	κ'
$\chi_V(g)$	4	0	-4	0	0
$\chi_V^2(g)$	16	0	16	0	0
$\chi_V(g^2)$	4	-4	4	4	4
$\chi_a(g)$	6	2	6	-2	-2

Table 7.5: The character table of the anti-symmetric tensor product for $V = E \oplus E$.

Now, assume the following decomposition

$$A(\mathbb{R}^4) = b_1 A_0 \oplus b_2 A_1 \oplus b_3 B_1 \oplus b_4 B_2 \oplus b_5 E. \tag{7.3}$$

We calculate the constants b_i as follows:

1) $b_1 = \langle \chi_a, \chi_{A_0} \rangle = \dfrac{1}{8}[6 + 4 + 6 - 4 - 4] = 1.$

2) $b_2 = \langle \chi_a, \chi_{A_1} \rangle = \dfrac{1}{8}[6 + 4 + 6 + 4 + 4] = 3.$

3) $b_3 = \langle \chi_a, \chi_{B_1} \rangle = \dfrac{1}{8}[6 - 4 + 6 - 4 + 4] = 1.$

4) $b_4 = \langle \chi_a, \chi_{B_2} \rangle = \dfrac{1}{8}[6 - 4 + 6 + 4 - 4] = 1.$

5) $b_5 = \langle \chi_a, \chi_E \rangle = \dfrac{1}{8}[12 - 12] = 0.$

This leads to

$$A(\mathbb{R}^4) = A_0 \oplus 3A_1 \oplus B_1 \oplus B_2.$$

We begin with a standard basis of 4×4 real-valued skew-symmetric matrices. Let $u = (q_1 \quad p_1 \quad q_2 \quad p_2)^T, v = (q_1' \quad p_1' \quad q_2' \quad p_2')^T$ be two column vectors in $\mathbb{R}^4$. The skew-symmetric form $B(u, v)$ corresponding to a skew-symmetric matrix A is given by the formula $u^T A v$. Thus, we write the basis of skew-symmetric forms, corresponding to the standard basis of skew-symmetric matrices by

$$q_1 p_1' - p_1 q_1', \ q_1 q_2' - q_2 q_1', \ q_1 p_2' - p_2 q_1',$$

$$p_1 q_2' - q_2 p_1', \ p_1 p_2' - p_2 p_1', \ q_2 p_2' - p_2 q_2'.$$

This basis is not (semi-)invariant under the action (7.1). We therefore suggest the following basis

notation	basis	κ	κ'	rep. type
j_1	$(q_1p_1' - q_1'p_1) + (q_2p_2' - q_2'p_2)$	1	1	A_0
j_2	$(q_1p_1' - q_1'p_1) - (q_2p_2' - q_2'p_2)$	1	-1	B_1
j_3	$(q_1p_2' - q_1'p_2) - (q_2'p_1 - q_2p_1')$	-1	1	B_2
j_4	$(q_1p_2' - q_1'p_2) + (q_2'p_1 - q_2p_1')$	-1	-1	A_1
j_5	$q_1q_2' - q_1'q_2$	-1	-1	A_1
j_6	$p_1p_2' - p_1'p_2$	-1	-1	A_1

Table 7.6: A basis for $A(\mathbb{R}^4)$. The numbers ± 1 denote the action being invariant or anti-invariant, respectively.

Table 7.6 illustrates a basis for the skew-symmetric forms in $\mathbb{R}^4$ that respects our D_4 action. Consequently, the corresponding matrix basis takes the form

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \; J_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; J_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Clearly, the matrices $J_1, J_2, \ldots, J_6$ carry the same representation type as the skew-symmetric forms $j_1, j_2, \ldots, j_6$ illustrated in Table 7.6. Another property to be considered while writing the formula of J is the condition $J^2 = -I$. Note that all basis elements satisfy this condition, apart from J_5 and J_6. This will be discussed in more details in the following section.

7.4 The D_4 linear Hamiltonian vector fields

Let $\dot{x} = f(x) = J\nabla H(x)$ be a D_4 Hamiltonian vector field, generated by the Hamiltonian H and the linear structure map J. It is known from the Hamiltonian context that the linear vector field L is given by the formula $L = JS$, where S is the symmetric matrix associated to the quadratic Hamiltonian i.e. $H_2(x) = \frac{1}{2}x^T Sx$. Therefore, according to the bases introduced in Subsection 7.3.1 and Subsection 7.3.2, one can write the linear Hamiltonian vector field due to its symmetry properties.

Using the D_4 character table 7.2, we can summarise the symmetry cases described in Table 7.1 by the following

type	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
S	A_0	B_1	A_0	B_1	A_1	A_1	B_2	A_0	A_1	B_2
J	A_0	A_0	B_1	B_1	A_0	B_1	B_1	A_1	A_1	A_1

Table 7.7: H and J irreducible representations corresponding to D_4 symmetry types.

In the following, we will write the appropriate formulas for S and J and therefore, the linearisation at the origin L of each type of symmetry displayed in Table 7.7. As we are interested in finding periodic orbits in Hamiltonian systems, we will calculate the eigenvalues of L in each symmetry type, and check their ability to be purely imaginary. By being purely imaginary we mean a strictly non-zero complex number in the form ai.

• Type 1

This is the classical type of equivariant symmetry, with an invariant Hamiltonian and a symplectic D_4 action. The symmetric matrix S lies in the A_0 space and can be written as

$$S = \begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. Also, J is of A_0 type and takes the form

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Consequently, the linear vector field will be

$$L = JS = \begin{pmatrix} c & b & 0 & 0 \\ -a & -c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & -a & -c \end{pmatrix},$$

with two pairs of eigenvalues $\lambda = \pm\sqrt{c^2 - ab}$. Clearly, it is possible for λ to be purely imaginary, and this agrees with the known results on the existence of periodic solutions in equivariant Hamiltonian systems with D_4 symmetry, such as [28].

- **Type 2**

This system is reversible equivariant, with the generator κ being (SE) but κ' being (SR). In this case, the matrix S is of B_1 representation type, while J is of type A_0. Using the bases described in Subsection 7.3.1 and Subsection 7.3.2, we have

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & -a & -c \\ 0 & 0 & -c & -b \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. We write the linear system L as follows

$$L = JS = \begin{pmatrix} c & b & 0 & 0 \\ -a & -c & 0 & 0 \\ 0 & 0 & -c & -b \\ 0 & 0 & a & c \end{pmatrix}.$$

Eigenvalues of L are a doubled pair of the form $\lambda = \pm\sqrt{c^2 - ab}$ which can be purely imaginary under suitable conditions on the constants a, b and c.

• Type 3

Now, we study a RE system with κ and κ' being (SE) and (AR), respectively. Accordingly, S and J are of types A_0 and B_1, respectively. We obtain

$$L = JS = \begin{pmatrix} c & b & 0 & 0 \\ -a & -c & 0 & 0 \\ 0 & 0 & -c & -b \\ 0 & 0 & a & c \end{pmatrix},$$

which is the same as the linear system in the previous case, and it has the same eigenvalues.

• Type 4

Although this system is equivariant, we will give it some attention because κ' is (AE) while κ is the classical (SE) symmetry, a combination not studied before. The linear Hamiltonian vector field takes the form

$$L = \begin{pmatrix} c & b & 0 & 0 \\ -a & -c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & -a & -c \end{pmatrix}.$$

Similarly, the eigenvalues are $\lambda = \pm\sqrt{c^2 - ab}$.

• Type 5

In this case, both generators are symplectic reversing. The Hamiltonian is of representation type A_1 and the matrix J is of type A_0. Thus,

$$L = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}.$$

This system has always purely imaginary eigenvalues of the form $\pm ai$.

• Type 6

We assumed κ to be (SR) and κ' to be (AE). The linear system therefore takes the form

$$L = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \\ -a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix}.$$

The eigenvalues of L are $\lambda = \pm a$ which are always real valued. This means that periodic orbits will not occur in this system.

• Type 7

Both generators in this case are reversing, but κ is acting symplectically, while κ' acts anti-symplectically. The linear vector field is written as

$$L = \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & -a & -b \\ -b & -c & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix}.$$

Eigenvalues of L are two pairs of the form $\lambda = \pm\sqrt{ac - b^2}$ for $a, b, c \in \mathbb{R}$. Clearly, it is possible to choose the numbers a, b and c, so that λ is purely imaginary.

• Type 8

For this D_4-reversible equivariant Hamiltonian system, both generators are of (AR) type. One issue that needs to be highlighted here is that the matrix J has representation type A_1 and therefore, it needs to be written as the linear combination $J = xJ_4 + yJ_5 + zJ_6$ provided that $J^2 = -I$. Now, we solve these two equations in order to find possible formulas of J. Using formulas of J_4, J_5 and J_6 we obtain

$$J = \begin{pmatrix} 0 & 0 & y & x \\ 0 & 0 & x & z \\ -y & -x & 0 & 0 \\ -x & -z & 0 & 0 \end{pmatrix}.$$

By applying the condition $J^2 = -I$ we have

$$\begin{pmatrix} x^2+y^2 & x(y+z) \\ x(y+z) & x^2+z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which leads to the system of nonlinear equations

$$\begin{aligned} x^2+y^2 &= 1, \\ x(y+z) &= 0, \\ x^2+z^2 &= 1. \end{aligned} \tag{7.4}$$

Solutions of the system (7.4) are

$$(x,y,z) = (0, \pm 1, \pm 1), \tag{7.5}$$

$$(x,y,z) = (\pm\sqrt{1-z^2}, -z, z),\ |z| < 1. \tag{7.6}$$

As we are interested in finding eigenvalues of the linear system, it will be enough to study the following expressions for J, namely J_{11}, J_{12} and J_{13}:

$$J_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_{12} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$J_{13} = \begin{pmatrix} 0 & 0 & -z & x \\ 0 & 0 & x & z \\ z & -x & 0 & 0 \\ -x & -z & 0 & 0 \end{pmatrix}, \ x^2+z^2 = 1.$$

Now, we show that J_{13} can be transformed to take the same form as J_{12}. Using the relation $x^2+z^2 = 1$ one can write

$$\begin{pmatrix} -z & x \\ x & z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

which represents a reflection over the line with angle $\dfrac{\theta}{2}$. After rotating this line by the angle $-\dfrac{\theta}{2}$, it will lie on the horizontal axis and J_{13} will be transformed to

$$J_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = J_{12}.$$

As a result, it is enough to consider the formulas J_{11} and J_{12} in analysing systems with matrix J of representation type A_1.

Based on these two formulas, the problem of finding eigenvalues of the linear system of type 8 splits into the following two cases.

(1) systems with $\boldsymbol{J = J_{11}}$

Knowing that S takes the form

$$S = \begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix},$$

we obtain

$$L = \begin{pmatrix} 0 & 0 & a & c \\ 0 & 0 & c & b \\ -a & -c & 0 & 0 \\ -c & -b & 0 & 0 \end{pmatrix}.$$

The eigenvalues of L are given by

$$\lambda = \pm\frac{1}{2}\sqrt{-2a^2 - 2b^2 - 4c^2 \pm 2\sqrt{a^4 - 2a^2b^2 + 4a^2c^2 + 8abc^2 + b^4 + 4b^2c^2}}.$$

We need to simplify this formula, to check for purely imaginary eigenvalues. Let $X = \sqrt{a^4 - 2a^2b^2 + 4a^2c^2 + 8abc^2 + b^4 + 4b^2c^2}$. One can write

$$a^4 - 2a^2b^2 + 4a^2c^2 + 8abc^2 + b^4 + 4b^2c^2 = (a^2 - b^2)^2 + 4c^2(a+b)^2 \geq 0.$$

For purely imaginary eigenvalues, we put

$$a^2 + b^2 + 2c^2 \mp X > 0.$$

Clearly, $a^2 + b^2 + 2c^2 + X > 0$. It remains to check the inequality $a^2 + b^2 + 2c^2 > X$. Squaring each side yields

$$(ab - c^2)^2 > 0,$$

which is satisfied, unless $ab = c^2$. Thus, in this case, eigenvalues are purely imaginary if $ab \neq c^2$.

(2) systems with $\boldsymbol{J = J_{12}}$

The linear system corresponding to this case is

$$L = \begin{pmatrix} 0 & 0 & a & c \\ 0 & 0 & -c & -b \\ -a & -c & 0 & 0 \\ c & b & 0 & 0 \end{pmatrix}.$$

Eigenvalues of L are given by

$$\lambda = \pm\frac{1}{2}\sqrt{-2a^2 - 2b^2 + 4c^2 \pm 2\sqrt{a^4 - 2a^2b^2 - 4a^2c^2 + 8abc^2 + b^4 - 4b^2c^2}}.$$

The first condition for λ to be purely imaginary is

$$a^4 - 2a^2b^2 - 4a^2c^2 + 8abc^2 + b^4 - 4b^2c^2 \geq 0.$$

Simplifying this formula implies

$$\begin{aligned} (a^2 - b^2)^2 - 4c^2(a-b)^2 &\geq 0 \\ \Leftrightarrow\ (a+b)^2 - 4c^2 &\geq 0. \end{aligned} \tag{7.7}$$

The second condition is given by

$$-2a^2 - 2b^2 + 4c^2 \pm 2X < 0,$$

where $X = \sqrt{a^4 - 2a^2b^2 - 4a^2c^2 + 8abc^2 + b^4 - 4b^2c^2}$. We will discus both cases in the following.

1. For $a^2 + b^2 - 2c^2 > X$, squaring both sides, and using the formula for X yield,

   $$(ab - c^2)^2 > 0,$$

   provided that the values of a, b, c satisfy (7.7).

2. For $a^2+b^2-2c^2 > -X$, we need to use the condition (7.7). Therefore,

$$\begin{aligned} a^2+b^2-2c^2 &\geq a^2+b^2-\frac{1}{2}(a+b)^2 \\ &= \frac{1}{2}(a-b)^2 \geq 0. \end{aligned}$$

Accordingly, $a^2+b^2-2c^2+X>0$ is true for a, b, c, satisfying (7.7) and $ab \neq c^2$.

We conclude that λ can be purely imaginary if and only if $|a+b| \geq 2|c|$ and $ab \neq c^2$.

• Type 9

This system is equivariant of a special type. That is because, both involutions κ, κ' are symmetries that act anti-symplectically. Consequently, S and J have representation type A_1. The symmetric matrix S takes the form

$$S = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}.$$

Since J of type A_1 has two different forms, we will treat each one separately.

(1) systems with $\boldsymbol{J = J_{11}}$

We start by giving the form of J as

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The linear system L is given by

$$L = \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \end{pmatrix},$$

with purely imaginary eigenvalues of the form $\lambda = \pm ai$.

(2) systems with $\boldsymbol{J = J_{12}}$

In this case, we have

$$L = \begin{pmatrix} 0 & -a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & -a & 0 \end{pmatrix}.$$

In contrast with the previous case, all eigenvalues here are real-valued and have the form $\lambda = \pm a$.

These results suggest that in studying the existence of periodic orbits in a D_4-equivariant Hamiltonian system of type 9, we only need to consider the formula J_{11} for the linear structure map.

• Type 10

As highlighted previously, we can divide our calculations according to the form of the matrix J.

(1) systems with $\boldsymbol{J = J_{11}}$

The linear system L takes the form

$$L = \begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & 0 & -b & -c \end{pmatrix}.$$

The eigenvalues of L are

$$\lambda = \pm\frac{1}{2}(c + a) \pm \frac{1}{2}\sqrt{(a - c)^2 + 4b^2},$$

which are always real-valued as the square root cannot be negative.

(2) systems with $\boldsymbol{J = J_{12}}$

The linear vector field is given by

$$L = \begin{pmatrix} a & b & 0 & 0 \\ -b & -c & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & 0 & b & c \end{pmatrix}.$$

The eigenvalues of L are

$$\lambda = \pm\frac{1}{2}(c-a) \pm \frac{1}{2}\sqrt{(a+c)^2 - 4b^2}.$$

These eigenvalues can be purely imaginary if and only if $a = c$ and $a^2 < b^2$. Note, this is not an open condition on the coefficients.

In Table 7.8, we summarise all normal forms and results that were found in this section. The first column indicates the D_4 symmetry type of the Hamiltonian system. The second column introduces the irreducible representation of S and J associated to the chosen symmetry type. Accordingly, normal forms of S and J are given. The last column determines the ability of having purely imaginary eigenvalues of the linear system $L = JS$.

type	(S, J) rep. type	S	J	λ imaginary
(1)	(A_0, A_0)	$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	possible
(2)	(B_1, A_0)	$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & -a & -c \\ 0 & 0 & -c & -b \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	possible
(3)	(A_0, B_1)	$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	possible
(4)	(B_1, B_1)	$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & -a & -c \\ 0 & 0 & -c & -b \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	possible
(5)	(A_1, A_0)	$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	always
(6)	(A_1, B_1)	$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	impossible
(7)	(B_2, B_1)	$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ b & c & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	possible
(8)	(A_0, A_1)	$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	possible possible
(9)	(A_1, A_1)	$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	always impossible
(10)	(B_2, A_1)	$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & c \\ a & b & 0 & 0 \\ b & c & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	impossible possible (not open)

Table 7.8: For all D_4 symmetry types, formulas of S and J are given according to their representation type described in the second column. The λ column indicates the possibility of having purely imaginary eigenvalues of the linear system L.

Following the main interest of this thesis, in the remaining few sections, we will study the existence of periodic orbits in some special types illustrated in Table 7.8. We choose these types specifically, because they were not studied before, and they extend our work from the earlier chapters.

7.5 The existence of periodic orbits in a D_4-equivariant Hamiltonian system of type 4

In this section, we will study in details the existence of periodic orbits in a D_4-equivariant Hamiltonian system, where κ is (SE) but κ' is (AE). Although the system is equivariant, we cannot apply classical results like [28]. That is because of the anti-Hamiltonian property of κ' which wasn't considered in [28]. The method of finding periodic orbits, as discussed in Chapter 3, depends on solving the bifurcation equation coming from the Liapunov-Schmidt reduction. This involves writing the reduced Hamiltonian, that inherits the symmetry properties of the original system. We will start by determining the S^1 action. Secondly, we will choose the appropriate set of (semi-)invariants, in order to write the reduced Hamiltonian, and therefore the bifurcation equation. After that, we will analyse all isotropy subgroups of $D_4 \times S^1$. Finally, we will prove the existence of families of symmetric periodic orbits, according to the isotropy subgroups we found.

7.5.1 The S^1 action

In order to write the appropriate set of (semi-)invariants of the studied system, we need to determine the circle action of S^1 according to the linear system L. As discussed in the previous section, the linear vector field of this case has eigenvalues $\lambda = \pm\sqrt{c^2 - ab}$, assuming that $c^2 - ab < 0$. The quadratic part of the Hamiltonian takes the form

$$H_2 = a(q_1^2 - q_2^2) + b(p_1^2 - p_2^2) + 2c(q_1 p_1 - q_2 p_2).$$

Now, we apply a symplectic change of coordinates to simplify this formula. We diagonalise the symmetric matrix S using the change of coordinates:

$$\begin{aligned} Q_1 &= q_1 + \tfrac{c}{a} p_1, \quad P_1 = p_1, \\ Q_2 &= q_2 + \tfrac{c}{a} p_2, \quad P_2 = p_2. \end{aligned} \tag{7.8}$$

The symplectic form $\omega = dq_1 \wedge dp_1 - dq_2 \wedge dp_2$ is invariant under this change of coordinates. Consequently, the quadratic part of the Hamiltonian can be transformed to

$$H_2 = a(Q_1^2 - Q_2^2) + \frac{ab - c^2}{a}(P_1^2 - P_2^2).$$

Define the symplectic transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$T(Q_1, P_1, Q_2, P_2) = (\alpha Q_1, \frac{1}{\alpha} P_1, \alpha Q_2, \frac{1}{\alpha} P_2), \tag{7.9}$$

where α is chosen to satisfy the equality $\alpha^4 = \dfrac{ab - c^2}{a^2}$. By assuming $\beta = a\alpha^2 = \dfrac{ab - c^2}{a\alpha^2}$, one can write

$$H_2 \circ T = \beta(Q_1^2 + P_1^2 - Q_2^2 - P_2^2).$$

After these coordinate changes, the linear vector field takes the form

$$L = \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}.$$

By choosing $\beta = -1$ we obtain a doubled pair of eigenvalues in the form $\lambda = \pm i$. Therefore, the S^1 action is given by

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

7.5.2 The bifurcation equation

It is known from the Liapunov-Schmidt procedure, that the periodic orbits of a given Hamiltonian system are the zeros of the bifurcation equation. In order to write this equation, for this case of study, we will write the reduced Hamiltonian, which is S^1 and κ invariant but κ' anti-invariant. Using an analogous analysis to Lemma 5.1.2 we obtain

$$h(z_1, z_2, \tau) = \delta g(N, A^2, B^2, \tau),$$

where $N = |z_1|^2 + |z_2|^2$, $\delta = |z_1|^2 - |z_2|^2$ and $A + iB = 2\bar{z}_1 z_2$. These functions satisfy the identity $\delta^2 = N^2 - (A^2 + B^2)$. By differentiating h partially with respect to z_1 and z_2, the bifurcation equation takes the form

$$\bar{z}_1 g + \delta[\bar{z}_1 g_N + 2A\,\bar{z}_2\,g_{A^2} + 2iB\,\bar{z}_2\,g_{B^2}] = 0, \tag{7.10}$$

$$-\bar{z}_2 g + \delta[\bar{z}_2 g_N + 2A\,\bar{z}_1\,g_{A^2} - 2iB\,\bar{z}_1\,g_{B^2}] = 0. \tag{7.11}$$

Multiplying (7.10) by z_1 and (7.11) by z_2 implies

$$|z_1|^2 g + \delta[|z_1|^2 g_N + 2A(z_1\bar{z}_2)g_{A^2} + 2iB(z_1\bar{z}_2)g_{B^2}] = 0, \tag{7.12}$$

$$-|z_2|^2 g + \delta[|z_2|^2 g_N + 2A(\bar{z}_1 z_2)g_{A^2} - 2iB(\bar{z}_1 z_2)g_{B^2}] = 0. \tag{7.13}$$

We first add (7.12) to (7.13)

$$\delta[g + Ng_N + 2A^2 g_{A^2} + 2B^2 g_{B^2}] = 0. \tag{7.14}$$

Now we multiply (7.10) by z_2 and (7.11) by z_1

$$\bar{z}_1 z_2 g + \delta[\bar{z}_1 z_2 g_N + 2A|z_2|^2 g_{A^2} + 2iB|z_2|^2 g_{B^2}] = 0, \tag{7.15}$$

$$-z_1\bar{z}_2 g + \delta[z_1\bar{z}_2 g_N + 2A|z_1|^2 g_{A^2} - 2iB|z_1|^2 g_{B^2}] = 0. \tag{7.16}$$

Subtracting (7.16) from (7.15) yields

$$A[g - 2\delta^2 g_{A^2}] = 0, \tag{7.17}$$

$$\delta B[g_N + 2Ng_{B^2}] = 0. \tag{7.18}$$

Thus, the bifurcation equation is given by the system of equations

$$\begin{aligned} \delta[g + Ng_N + 2A^2 g_{A^2} + 2B^2 g_{B^2}] &= 0, \\ A[g - 2\delta^2 g_{A^2}] &= 0, \\ \delta B[g_N + 2Ng_{B^2}] &= 0. \end{aligned} \tag{7.19}$$

Before solving the system (7.19), we discuss the possible isotropy subgroups of $D_4 \times S^1$ in the following subsection.

7.5.3 Isotropy subgroups of $D_4 \times S^1$

In this section, we analyse all isotropy subgroups of $D_4 \times S^1$ together with their fixed point spaces. Our $D_4 \times S^1$ action on $\mathbb{C}^2$ is defined by

$$\begin{aligned} \kappa(z_1, z_2) &= (z_1, -z_2), \\ \kappa'(z_1, z_2) &= (z_2, z_1), \\ \theta(z_1, z_2) &= (e^{i\theta} z_1, e^{i\theta} z_2). \end{aligned} \tag{7.20}$$

Depending on Proposition 7.2 in [19], we start seeking isotropy subgroups of $D_4 \times S^1$ by listing all the subgroups of D_4. It is known that up to conjugacy, subgroups of D_4 are

$$D_4,\ \mathbb{Z}_4,\ \mathbb{Z}_2^{\rho^2},\ \mathbb{Z}_2^{\kappa},\ \mathbb{Z}_2^{\kappa'},\ \mathbb{Z}_2^{\kappa} \times \mathbb{Z}_2^{\rho^2},\ \mathbb{Z}_2^{\kappa'} \times \mathbb{Z}_2^{\rho^2},\ \{e\}.$$

Then, we write possible homomorphisms from these subgroups to S^1 and after that, we compute isotropy subgroups of $D_4 \times S^1$ up to conjugacy. Isotropy subgroups of $D_4 \times S^1$ according to the group action (7.20) are described in Table 7.9. Note that $\mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)} = \{(0,0), (\kappa,0), (\rho^2,\pi), (\kappa\rho^2,\pi)\} \leq D_4 \times S^1$ and $\tilde{\mathbb{Z}}_4 = \langle \rho, \frac{3\pi}{2} \rangle$.

notation	isotropy subgroup	fixed point space	dimension
Σ_1	$D_4 \times S^1$	$\{(0,0)\}$	0
Σ_2	$\mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$	$\{(z,0) \mid z \in \mathbb{C}\}$	2
Σ_3	$\mathbb{Z}_2^{\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$	$\{(z,z) \mid z \in \mathbb{C}\}$	2
Σ_4	$\tilde{\mathbb{Z}}_4$	$\{(z,-iz) \mid z \in \mathbb{C}\}$	2
Σ_5	$\mathbb{Z}_2^{(\rho^2,\pi)}$	$\mathbb{C}^2$	4

Table 7.9: Isotropy subgroups of $D_4 \times S^1$ acting on $\mathbb{C}^2$ and their fixed point spaces in type 4 systems. The fourth column presents the dimension of the corresponding fixed point space.

Table 7.9 shows that there exist three isotropy subgroups, with two-dimensional fixed point spaces, namely Σ_2, Σ_3 and Σ_4. Moreover, the subgroup Σ_5 fixes the whole space $\mathbb{C}^2$ and therefore, any periodic orbit found from the bifurcation equation will have this type of symmetry. Note, Σ_5 is called the principal symmetry subgroup, as it is the smallest isotropy subgroup of $D_4 \times S^1$.

7.5.4 The existence of periodic orbits

In the following theorem, we summarise the families of periodic orbits that exist in the D_4-equivariant Hamiltonian system of type 4.

Theorem 7.5.1 *Consider a generic D_4-equivariant Hamiltonian system of type 4, i.e. κ is a symmetry acting symplectically, and κ' is a symmetry acting anti-symplectically. Suppose that* 0 *is a symmetric equilibrium, and the linearisation has two purely imaginary pairs of eigenvalues $\pm i$ with no other eigenvalues of the form $\pm ki, k \in \mathbb{Z}$. Then,* *in a neighbourhood of the origin, there exist precisely 8 one-parameter families of periodic solutions, with period close to 2π. These orbits have the following symmetries:*

1. *Two families with symmetries Σ_2 and its conjugate $\Sigma_2' = \mathbb{Z}_2^{-\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$, respectively.*

2. *Two families with symmetries Σ_3 and its conjugate $\Sigma_3' = \mathbb{Z}_2^{-\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$, respectively.*

3. *Two families with symmetry Σ_4 and its conjugate $\Sigma_4' = \langle \rho, \frac{\pi}{2} \rangle$, respectively.*

4. *Two families with principal symmetry Σ_5.*

Proof We shall prove this theorem by solving the system (7.19) in the fixed point spaces of Σ_2, Σ_3 and Σ_4. A similar argument can be used for their conjugates. Furthermore, we seek periodic solutions with principal symmetry only. Details are as follows.

Periodic orbits with symmetry group $\Sigma_2 = \mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$

In order to find periodic solutions of the studied D_4-equivariant Hamiltonian system with symmetry Σ_2, we need to solve the system (7.19) in the fixed point space $\operatorname{Fix}\Sigma_2$. By Table 7.9, we have

$$\operatorname{Fix}\Sigma_2 = \{(z,0) \mid z \in \mathbb{C}\}.$$

In this space, we have $A = B = 0$ and $N = \delta = |z|^2$, which simplifies (7.19) to

$$|z|^2[g(|z|^2,\tau) + |z|^2 g_N(|z|^2,\tau)] = 0.$$

For non-zero z this equation can be solved for $\tau = \tau(|z|^2)$ by the implicit function theorem as $\frac{\partial g}{\partial \tau}(0,0) = \frac{1}{2} \neq 0$.

Periodic orbits with symmetry group $\Sigma_3 = \mathbb{Z}_2^{\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$

In this case, we have

$$\operatorname{Fix}\Sigma_3 = \{(z,z) \mid z \in \mathbb{C}\}.$$

Therefore, system (7.19) takes the simple form

$$|z|^2\, g(|z|^2,\tau) = 0. \tag{7.21}$$

Here $\delta = B = 0$ which agrees with the fact that any periodic solution with symmetry group Σ_3 should lie in the level set $H = 0$ by κ' anti-Hamiltonian property. Equation (7.21) can be solved by the implicit function theorem for τ i.e. for each non-zero (z, z) there exists a periodic orbit that passes this point and lies in $\mathrm{Fix}\,\Sigma_3$.

Periodic orbits with symmetry group $\Sigma_4 = \tilde{\mathbb{Z}}_4$

In a similar way, the bifurcation equation on $\mathrm{Fix}\,\Sigma_4$ is given by

$$\bar{z}\, g(|z|^2, \tau) = 0,$$

which is solvable by the implicit function theorem for $\tau = \tau(|z|^2)$ as $\frac{\partial g}{\partial \tau}(0, 0) = \frac{1}{2}$.

Periodic orbits with principal symmetry $\Sigma_5 = \mathbb{Z}_2^{(\rho^2,\pi)}$

Periodic orbits with this type of symmetry do not possess any symmetries apart from the principal symmetry $\mathbb{Z}_2^{(\rho^2,\pi)}$. Therefore, the functions N, A, B and δ are not necessarily zero and the system (7.19) can be written as

$$\begin{aligned} g + N g_N + 2A^2 g_{A^2} + 2B^2 g_{B^2} &= 0, \\ g - 2(N^2 - A^2 - B^2) g_{A^2} &= 0, \\ g_N + 2N g_{B^2} &= 0. \end{aligned} \tag{7.22}$$

Assume the following derivatives:

$$\begin{gathered} g_N(0) = n_1, \quad g_{NN}(0) = n_2, \quad g_{N\tau}(0) = n_3, \\ g_{NA^2}(0) = n_4, \quad g_{A^2}(0) = a, \quad g_{B^2}(0) = b. \end{gathered}$$

The matrix of partial derivatives with respect to N, τ, A^2 computed at the origin is given by

$$\begin{pmatrix} 2n_1 & \frac{1}{2} & 3a \\ n_1 & \frac{1}{2} & 3a \\ (n_2 + 2b) & n_3 & n_4 \end{pmatrix}.$$

The determinant of the above matrix is $n_1[\frac{n_4}{2} - 3an_3]$ which is non-zero if and only if $n_1 \neq 0$ and $\frac{n_4}{2} - 3an_3 \neq 0$. Since these two quantities are generically non-zero, we deduce that by the implicit function theorem, there exists a unique curve $S = S(B^2)$ that solves the system (7.22). $\square$

7.6 The existence of periodic orbits in a D_4-equivariant Hamiltonian system of type 9

Here, we study a D_4-equivariant Hamiltonian system, where both involutions κ and κ' are of the (AE) type. Consequently, the Hamiltonian function H is both κ and κ' anti-invariant. Following the same organisation as in Section 7.5, we prove the existence of families of periodic orbits in this system in the coming steps.

7.6.1 The S^1 action

As we fixed the D_4 action for all cases by (7.1), it remains to deduce the S^1 action from the linear vector field. By Table 7.8, the quadratic part of the Hamiltonian is given by

$$H_2 = 2(q_1p_2 - q_2p_1),$$

and the linear vector field L with eigenvalues $\pm i$ takes the form

$$L = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the S^1 action is given by

$$\theta(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2).$$

Clearly, the $D_4 \times S^1$ action is identical to the one in Subsection 7.5.1. However, the set of (semi-)invariants is different, due to the difference in symmetry, as we will see in the following section.

7.6.2 The bifurcation equation

The generators of the S^1 real-valued invariants on $\mathbb{C}^2$, as studied in Subsection 7.5.2, are $N = |z_1|^2 + |z_2|^2, \delta = |z_1|^2 - |z_2|^2$, $A = \bar{z}_1z_2 + z_1\bar{z}_2$ and $B = -i(\bar{z}_1z_2 - z_1\bar{z}_2)$. These functions are related by the identity $\delta^2 = N^2 - (A^2 + B^2)$. The reduced Hamiltonian h should be both κ and κ' anti-invariant and S^1 invariant. Thus,

$$h(z_1, z_2, \tau) = Bg(N, A^2, B^2, \tau).$$

Recall the bifurcation equation

$$\frac{\partial h}{\partial z_1} = 0, \quad \frac{\partial h}{\partial z_2} = 0.$$

Equivalently, this can be written as

$$i\bar{z}_2 g + B[\bar{z}_1 g_N + 2A\bar{z}_2 g_{A^2} + 2iB\bar{z}_2 g_{B^2}] = 0, \tag{7.23}$$

$$-i\bar{z}_1 g + B[\bar{z}_2 g_N + 2A\bar{z}_1 g_{A^2} - 2iB\bar{z}_1 g_{B^2}] = 0. \tag{7.24}$$

In the following, we will simplify these equations. Multiplying (7.23) by z_1 and (7.24) by z_2 implies

$$iz_1\bar{z}_2 g + B[|z_1|^2 g_N + 2Az_1\bar{z}_2 g_{A^2} + 2iBz_1\bar{z}_2 g_{B^2}] = 0, \tag{7.25}$$

$$-i\bar{z}_1 z_2 g + B[|z_2|^2 g_N + 2A\bar{z}_1 z_2 g_{A^2} - 2iB\bar{z}_1 z_2 g_{B^2}] = 0. \tag{7.26}$$

By Adding (7.25) and (7.26) we have

$$B[g + Ng_N + 2A^2 g_{A^2} + 2B^2 g_{B^2}] = 0. \tag{7.27}$$

On the other hand, we multiply (7.23) by z_2 and (7.24) by z_1 to obtain

$$i|z_2|^2 g + B[\bar{z}_1 z_2 g_N + 2A|z_2|^2 g_{A^2} + 2iB|z_2|^2 g_{B^2}] = 0, \tag{7.28}$$

$$-i|z_1|^2 g + B[z_1\bar{z}_2 g_N + 2A|z_1|^2 g_{A^2} - 2iB|z_1|^2 g_{B^2}] = 0. \tag{7.29}$$

Subtracting (7.29) from (7.28) gives

$$Ng + B^2 g_N + 2B^2 N g_{B^2} = 0, \tag{7.30}$$

$$2AB\delta g_{A^2} = 0. \tag{7.31}$$

Therefore, solving the bifurcation equation is equivalent to solving Equation (7.27), (7.30) and (7.31). Before solving this system of equations, we consider the isotropy subgroups of $D_4 \times S^1$ in the following section.

7.6.3 Isotropy subgroups of $D_4 \times S^1$

Since the $D_4 \times S^1$ action is identical to the one in Section 7.5, it turns out that it has the same isotropy subgroups illustrated in Table 7.9.

7.6.4 The existence of periodic orbits

Theorem 7.6.1 *Consider a generic D_4-equivariant Hamiltonian system of type 9, i.e. κ and κ' are anti-symplectic symmetries. Suppose that 0 is a symmetric equilibrium, and the linearisation has two purely imaginary pairs of eigenvalues $\pm i$ with no other eigenvalues of the form $\pm ki, k \in \mathbb{Z}$. Then, in a neighbourhood of the origin, there exist precisely 6 one-parameter families of periodic solutions, with period close to 2π. These orbits have the following symmetries:*

1. *Two families with symmetries Σ_2 and its conjugate $\Sigma_2' = \mathbb{Z}_2^{-\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$, respectively.*
2. *Two families with symmetries Σ_3 and its conjugate $\Sigma_3' = \mathbb{Z}_2^{-\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$, respectively.*
3. *Two families have symmetry Σ_4 and its conjugate $\Sigma_4' = \langle \rho, \frac{\pi}{2} \rangle$, respectively.*

PROOF We investigate the existence of periodic orbits with symmetries Σ_2, Σ_3 and Σ_4 respectively in the following sections.

Periodic orbits with symmetry group $\Sigma_2 = \mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$

To find periodic orbits with symmetry Σ_2, we solve the system of equations (7.27), (7.30) and (7.31) in $\operatorname{Fix}\Sigma_2$. In this case, we have $A = B = 0$ and $N = \delta = |z|^2$. Thus, we need to solve

$$g(|z|^2, \tau) = 0,$$

as we are interested in $z \neq 0$. By the implicit function theorem, there exists a unique function $\tau = \tau(|z|^2)$ that solves the latter equation. Note that by κ and κ' anti-invariance, solutions with symmetries Σ_2 and Σ_3 lie in the level set $h = 0$.

Periodic orbits with symmetry group $\Sigma_3 = \mathbb{Z}_2^{\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$

Here, we have $\delta = B = 0$ but $N = A = 2|z|^2$. Similarly, we end up solving a simple equation

$$|z|^2 g(|z|^2, \tau) = 0,$$

which can be solved for τ by the implicit function theorem near the origin.

Periodic orbits with symmetry group $\Sigma_4 = \tilde{\mathbb{Z}}_4$

On $\mathrm{Fix}\,\Sigma_4$ we have $A = \delta = 0$ and $N = -B = 2|z|^2$. Substituting these values in the bifurcation equation yields

$$g + Ng_N + 2N^2 g_{N^2} = 0,$$

which can be, as well, solved for $\tau = \tau(|z|^2)$ near the origin, by the implicit function theorem. □

Remark 7.6.2 With regards to the existence of periodic orbits with principal symmetry, a quick look at Equation (7.31) shows that when there are not extra conditions on the equation, then A, B and δ are not necessarily zero. Therefore, $g_{A^2}(0) = 0$, which is not a generic condition. As a result, generically, there are no periodic solutions with principal symmetry only.

7.7 The existence of periodic orbits in a D_4-RE Hamiltonian system of type 3

In this section, we analyse the D_4-reversible equivariant Hamiltonian system of type 3. Here, the generator κ is assumed to be a symmetry that acts symplectically. Oppositely, we assume κ' to be a reversing symmetry that acts anti-symplectically.

7.7.1 The S^1 action

As illustrated in Table 7.8, the quadratic Hamiltonian associated to this symmetry type is given by

$$H_2 = a(q_1^2 + q_2^2) + 2c(q_1 p_1 + q_2 p_2) + b(p_1^2 + p_2^2).$$

The function H_2 can be simplified, using the coordinate changes (7.8) and then (7.9) to take the form

$$H_2 = \beta(Q_1^2 + Q_2^2 + P_1^2 + P_2^2).$$

Thus, the corresponding linear vector field with eigenvalues $\pm i$ is

$$L = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Accordingly, the S^1 action on $\mathbb{C}^2$ is defined by

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2).$$

7.7.2 The bifurcation equation

Now, we write the formula of the reduced Hamiltonian h deduced from the Liapunov-Schmidt reduction. According to the symmetry of the problem, h must be a real-valued function defined on $\mathbb{C}^2 \times \mathbb{R}$, that is invariant under the actions of S^1, κ and κ'. By a similar argument to Lemma 4.2.1, we obtain

$$h(z_1, z_2, \tau) = g(N, C^2, D^2, \tau).$$

Thus, the bifurcation equation is given by

$$\bar{z}_1 g_N + 2z_2 C g_{C^2} - 2iz_2 D g_{D^2} = 0, \tag{7.32}$$

$$\bar{z}_2 g_N + 2z_1 C g_{C^2} - 2iz_1 D g_{D^2} = 0. \tag{7.33}$$

We simplify these equations by multiplying (7.32) by z_1 and (7.33) by z_2 to get

$$|z_1|^2 g_N + C(C + iD) g_{C^2} - iD(C + iD) g_{D^2} = 0, \tag{7.34}$$

$$|z_2|^2 g_N + C(C + iD) g_{C^2} - iD(C + iD) g_{D^2} = 0. \tag{7.35}$$

Adding (7.34) to (7.35) implies

$$N g_N + 2C^2 g_{C^2} + 2D^2 g_{D^2} = 0, \tag{7.36}$$

$$CD(g_{C^2} - g_{D^2}) = 0. \tag{7.37}$$

On the other hand, subtracting (7.35) from (7.34) yields

$$\delta g_N = 0. \tag{7.38}$$

As a result, in order to study the existence of periodic orbits of the D_4-reversible equivariant Hamiltonian system of type 3, one needs to solve the triple (7.36), (7.37) and (7.38).

7.7.3 Isotropy subgroups of $D_4 \ltimes S^1$

By Proposition 3.1.7, we can list the isotropy subgroups of $D_4 \ltimes S^1$ in the following table:

notation	isotropy subgroup	fixed point space	dimension
Σ_1	$D_4 \ltimes S^1$	$\{(0,0)\}$	0
Σ_2	$\mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$	$\{(z,0) \mid z \in \mathbb{C}\}$	2
Σ_3	$\mathbb{Z}_2^{\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$	$\{(z,z) \mid z \in \mathbb{C}\}$	2
Σ_4	$\mathbb{Z}_2^{(\rho^2,\pi)}$	$\mathbb{C}^2$	4

Table 7.10: Isotropy subgroups of $D_4 \ltimes S^1$ acting on $\mathbb{C}^2$ and their fixed point spaces in type 3 systems.

7.7.4 The existence of periodic orbits

The following theorem describes the families of periodic orbits which arise in a D_4-reversible equivariant Hamiltonian system of type 3.

Theorem 7.7.1 *Consider a generic D_4-reversible equivariant Hamiltonian system of of type 3, i.e. κ is a symmetry acting symplectically, and κ' is a reversing symmetry acting anti-symplectically. Suppose that* 0 *is a symmetric equilibrium, and the linearisation has two purely imaginary pairs of eigenvalues $\pm i$ with no other eigenvalues of the form $\pm ki, k \in \mathbb{Z}$. Then, in a neighbourhood of the origin, there exist precisely 4 one-parameter families of periodic solutions, with period close to 2π. These orbits have the following symmetries:*

1. *Two families have symmetries Σ_2 and its conjugate $\Sigma_2' = \mathbb{Z}_2^{-\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$, respectively.*

2. *Two families have symmetries Σ_3 and its conjugate $\Sigma_3' = \mathbb{Z}_2^{-\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$, respectively.*

Proof In the following, we will solve the bifurcation equation given by the system of equations (7.36), (7.37) and (7.38) with respect to the non-trivial symmetry groups illustrated in Table 7.10.

Periodic orbits with symmetry group $\Sigma_2 = \mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$

Since the involution κ is a symmetry, then any periodic orbit with symmetry Σ_2 should lie in $\operatorname{Fix}\Sigma_2$. Clearly, in $\operatorname{Fix}\Sigma_2$ the bifurcation equation is given by

$$Ng_N(N,\tau) = 0,$$

with $N = |z|^2$ in this subspace. Since $g_N(0)$ is generically non-zero, then by the implicit function theorem, for each non-zero z there is a unique solution to the bifurcation equation. This solution corresponds to a periodic orbit of the studied system with symmetry Σ_2.

Periodic orbits with symmetry group $\Sigma_3 = \mathbb{Z}_2^{\kappa'} \oplus \mathbb{Z}_2^{(\rho^2,\pi)}$

For periodic orbits of symmetry Σ_3, we need to solve

$$\begin{aligned} Ng_N + 2C^2 g_{C^2} + 2D^2 g_{D^2} &= 0, \\ CD(g_{C^2} - g_{D^2}) &= 0, \end{aligned}$$

where $N = 2|z|^2, C + iD = 2z^2$ in $\operatorname{Fix}\Sigma_3$. If C and D are both non-zero, the latter equation will take the form

$$g_{C^2} = g_{D^2}, \tag{7.39}$$

which is not a generic condition in a neighbourhood of the origin. If $C = 0$, we get

$$Ng_N + 2D^2 g_{D^2} = 0.$$

If $g_N(0) \neq 0$, then by the implicit function theorem the previous equation can be uniquely solved for $N = N(D^2,\tau)$. Similar argument can be used if $D = 0$.

Note that, by Equation (7.38), we do not expect any extra orbits with principal symmetry only. □

7.8 The existence of periodic orbits in a D_4-RE Hamiltonian system of type 5

We sum up our study on the existence of periodic orbits in D_4 Hamiltonian systems by a system of type 5. Interestingly, both generators κ and κ' are symplectic reversing.

We aim to extend our results in Chapter 4 and Chapter 6, from the cyclic to the dihedral case.

7.8.1 The S^1 action

The linearisation of the Hamiltonian vector field takes the form

$$L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Therefore, the circle action on $\mathbb{C}^2$ is given by

$$\theta(z_1, z_2) = (\cos\theta z_1 + \sin\theta z_2, -\sin\theta z_1 + \cos\theta z_2).$$

7.8.2 The bifurcation equation

It is clear that the S^1 action is identical to the one defined in Subsection 6.2.2. Thus, the S^1 invariants, as described in Lemma 6.2.1 (1), are

$$N = |z_1|^2 + |z_2|^2, \quad F + iG = z_1^2 + z_2^2, \quad E = -i(\bar{z}_1 z_2 - z_1 \bar{z}_2).$$

Taking into account the κ and κ' anti-invariance, the Hamiltonian h is defined by

$$h(z_1, z_2, \tau) = Eg(N, F, G, \tau).$$

Accordingly, the bifurcation equation takes the form

$$i\bar{z}_2 g + E[\bar{z}_1 g_N + z_1 g_F - i z_1 g_G] = 0, \tag{7.40}$$

$$-i\bar{z}_1 g + E[\bar{z}_2 g_N + z_2 g_F - i z_2 g_G] = 0. \tag{7.41}$$

Now we multiply (7.40) by z_1 and (7.41) by z_2

$$i z_1 \bar{z}_2 g + E[|z_1|^2 g_N + z_1^2 g_F - i z_1^2 g_G] = 0, \tag{7.42}$$

$$-i\bar{z}_1 z_2 g + E[|z_2|^2 g_N + z_2^2 g_F - i z_2^2 g_G] = 0. \tag{7.43}$$

Adding Equation (7.42) and Equation (7.43) yields

$$E[g + N g_N + F g_F + G g_G] = 0, \tag{7.44}$$

$$E[G g_F - F g_G] = 0. \tag{7.45}$$

On the other hand, we multiply (7.40) by z_2 and (7.41) by z_1 to obtain

$$i|z_2|^2 g + E[\bar{z}_1 z_2 g_N + z_1 z_2 g_F - i z_1 z_2 g_G] = 0, \tag{7.46}$$

$$-i|z_1|^2 g + E[z_1 \bar{z}_2 g_N + z_1 z_2 g_F - i z_1 z_2 g_G] = 0. \tag{7.47}$$

Subtracting (7.47) from (7.46) gives

$$Ng + E^2 g_N = 0, \tag{7.48}$$

which is equivalent to

$$Ng + (N^2 - (F^2 + G^2))g_N = 0. \tag{7.49}$$

Thus, the bifurcation equation is given by the system of equations (7.44), (7.45) and (7.49).

7.8.3 Isotropy subgroups of $D_4 \ltimes S^1$

In the following table, we illustrate the possible isotropy subgroups in the D_4-reversible equivariant Hamiltonian system of type 5.

notation	isotropy subgroup	fixed point space	dimension
Σ_1	$D_4 \ltimes S^1$	$\{(0,0)\}$	0
Σ_2	$\mathbb{Z}_2^{\kappa} \oplus \mathbb{Z}_4^{(\rho,\frac{\pi}{2})}$	$\{(z,0) \mid z \in \mathbb{C}\}$	2
Σ_3	$\mathbb{Z}_2^{\kappa'} \oplus \mathbb{Z}_4^{(\rho,\frac{\pi}{2})}$	$\{(z,z) \mid z \in \mathbb{C}\}$	2
Σ_4	$\tilde{\mathbb{Z}}_4 = \langle \rho, \frac{\pi}{2} \rangle$	$\mathbb{C}^2$	4

Table 7.11: Isotropy subgroups of $D_4 \ltimes S^1$ acting on $\mathbb{C}^2$ and their fixed point spaces in type 5 systems.

7.8.4 The existence of periodic orbits

It remains to solve the bifurcation equation given by the system of equations (7.44), (7.45) and (7.49). Results are given in the following theorem.

Theorem 7.8.1 *Consider a generic D_4-reversible equivariant Hamiltonian system of type 5 i.e. κ and κ' are both reversing symplectic involutions. Suppose that* 0 *is a symmetric equilibrium, and the linearisation has two purely imaginary pairs of eigenvalues*

$\pm i$ with no other eigenvalues of the form $\pm ki, k \in \mathbb{Z}$. Then, in a neighbourhood of the origin, there exists a two-parameter family of D_4-symmetric periodic solutions, with period close to 2π in the subspace $E = 0$. Moreover, there exist, under a suitable condition on the coefficients of the Hamiltonian, two one-parameter families of periodic orbits with symmetry $\tilde{\mathbb{Z}}_4$ near the origin.

Proof We will divide the proof into two cases: periodic orbits in the subspaces $E = 0$ and in $E \neq 0$.

periodic orbits in the subset $E = 0$

Clearly, $\mathrm{Fix}\,\Sigma_2$ and $\mathrm{Fix}\,\Sigma_3$ lie in the subspace $E = 0$. Substituting $E = 0$ in Equation (7.44), (7.45) and (7.49) implies

$$Ng(z_1, z_2, \tau) = 0. \tag{7.50}$$

This equation can be solved for τ by the implicit function theorem on both fixed point spaces. To see this property, let us assume $(z_0, z_0) \in \mathrm{Fix}\,\Sigma_3$. Rotating this point by $\frac{\pi}{4}$ implies

$$\frac{\pi}{4}\,(z_0, z_0) = \left(\sqrt{2}z_0, 0\right) \in \mathrm{Fix}\,\Sigma_2.$$

In other words, any orbit passing a point in $\mathrm{Fix}\,\Sigma_3$ will pass another point in $\mathrm{Fix}\,\Sigma_2$ and therefore will have symmetry D_4.

periodic orbits in the subset $E \neq 0$

The bifurcation equation here takes the form

$$\begin{aligned} g + Ng_N + Fg_F + Gg_G &= 0, \\ Ng + (N^2 - (F^2 + G^2))g_N &= 0, \\ Gg_F - Fg_G &= 0. \end{aligned} \tag{7.51}$$

Through a similar argument to that in Subsection 4.2.2, we found that there can be two families of periodic orbits, with principal symmetry Σ_2, in the subspace $E \neq 0$ distinguished by the sign of E. □

Chapter 8

Conclusions

Throughout this thesis, we have investigated different types of symmetry in Hamiltonian systems. Mainly, we were interested in non-classical symmetry types, which involve a semi-invariant Hamiltonian defined on $\mathbb{R}^4$. We made several statements on the existence of non-linear normal modes (periodic orbits) in such systems. We introduced a full description on possible families of symmetric and non-symmetric periodic orbits, with period close to 2π near a symmetric equilibrium point. Another consideration was to build a representation theoretical argument, to classify all possible types of dihedral symmetry (D_4) in Hamiltonian systems, with two degrees of freedom.

In the following part we summarise our contributions presented in this thesis:

1. In Theorem 4.2.5 we set a generic condition in which two families of non-symmetric solutions can arise, in addition to the symmetric family found in [9], in a purely reversible Hamiltonian system where the reversing symmetry R acts symplectically.

2. A general description of the families of periodic solutions in an equivariant Hamiltonian system, under the action of an anti-symplectic involution S is given in Theorem 5.4.1.

3. The existence of a two-parameter family of symmetric periodic orbits and two families of non-symmetric solutions in a $\mathbb{Z}_2^R \times \mathbb{Z}_2^S$-reversible equivariant Hamiltonian system is shown in Theorem 6.1.1.

4. Theorem 6.2.2 provides all possible families of periodic solutions in a $\mathbb{Z}_{2r}$-reversible

equivariant Hamiltonian system. These families are distinguished by the choice of r being even or odd.

5. Normal forms for J and S and therefore the linearisation L of all Hamiltonian systems with D_4 symmetry are obtained in Section 7.3 and Section 7.4. These forms are summarised in Table 7.8.

6. Existence theorems in D_4 Hamiltonian systems of type 4, 9, 3 and 5 are given in Theorem 7.5.1, Theorem 7.6.1, Theorem 7.7.1 and Theorem 7.8.1, respectively.

Notably, studying the effect of symmetry in Hamiltonian/dynamical systems is a rich area of research. In this work, we tried to highlight and solve some of the problems related to the context of symmetry in Hamiltonian systems. However, there are always some open problems which can be considered in further work. For example, one can:

- Consider other reversible equivariant groups, such as the tetrahedral symmetry group, and find possible families of periodic orbits.
- Extend our results from $\mathbb{R}^4$ to higher dimensions.
- Apply the analysis used in Chapter 7 to the general case D_{2n} and even for different groups.
- Establish more general results and normal forms for reversible equivariant Hamiltonian systems as the published work in this field is limited.
- Study the effect of having some bifurcation parameters on our results.

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