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2016

MIMS EPrint: **2016.38**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

ISSN 1749-9097

Characterization of objects by electrosensing fish based on the first order polarization tensor

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Abstract. Weakly electric fish generate electric current and use hundreds of voltage sensors on the surface of their body to navigate and locate food. Experiments [G. von der Emde and S. Fetz, J. Exp Biol, 210, 3082–3095, 2007] show that they can discriminate between differently shaped conducting or insulating objects by using electrosensing. One approach to electrically identify and characterize the object with a lower computational cost rather than full shape reconstruction is to use the first order Polarization Tensor (PT) of the object.

In this paper, by considering experimental work on Peters' elephantnose fish *Gnathonemus petersii*, we investigate the possible role of the first order PT in the ability of the fish to discriminate between objects of different shape. We also suggest some experiments that might be performed to further investigate the role of the first order PT in electrosensing fish. Finally, we speculate on the possibility of electrical cloaking or camouflage in prey of electrosensing fish and what might be learnt from the fish in human remote sensing.

Keywords: Electrosensing, *Gnathonemus petersii*, weakly electric fish, polarization tensor, Pólya-Szegö tensor, electrical impedance tomography

1. Introduction

Weakly electric fish can be found in the rivers of South America and Africa. They have an electric discharge organ and hundreds of voltage sensing cells on their body to perform electrosensing, which is used for navigation as well as to characterize and locate prey [35, 36, 17]. Only one electric source is found in each fish. Furthermore, species such as Peters' elephantnose fish *Gnathonemus petersii* generates a broad spectrum pulse like signal while the black knife ghost fish *Apteronotus albifrons* uses a signal closer to a sine wave [26].

In biomedical engineering and industrial process monitoring, a sinusoidal current is applied to a body and the resulting voltages are measured using surface electrodes to

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determine the interior conductivity (and permittivity) distribution in a system called Electrical Impedance Tomography (EIT) (see [25, 1]). A similar systems known as Electrical Resistivity Tomography (ERT) and Induced Polarization Tomography (IPT) are used by geophysicists for examples to locate subsurface features and archaeological objects. Typically, a balanced pair of square wave pulses is used, although sine wave excitation can be used as well. In IPT, the transient response is recorded to locate polarizable minerals (see references in [1]).

When a weakly electric fish moves through the water approaching an object, its single but moving electric discharges organ acts in a similar way to switching between driven electrodes in an EIT/ERT/IPT system. In this case, the moving and changing surface of the fish is equivalent to a changing boundary shape and electrode positions in EIT.

Electrical imaging is a non-linear and illposed inverse problem of considerable computational complexity so it would be surprising if electrosensing fish are able to perform a complete three dimensional spatial reconstruction in real time. Computational methods employed include (i). optimization methods, in which a smoothness constrained conductivity is fitted to the measured data [25, 1] and (ii). shape based methods, where, the surface on which the conductivity has a jump discontinuity is parameterized [33, 1]. Other methods use *sampling* or *probe* methods to locate a surface of discontinuity [34, 16, 20]. In both types of algorithm, repeated solution of the forward problem, that is the calculation of solutions of a partial differential equation, are required. We hypothesise that a less computationally intense method involving fitting a small number of parameters without the need for the solution of partial differential equations is likely to be closer to the location and characterization methods used by the fish.

Experimental studies [36] have shown that the fish *Gnathonemus petersii* can be trained to recognize and discriminate between conductive and insulating objects with variety of shapes without seeing the objects. One possible mechanism to recognize shapes of objects from electrical data at a distance, independent of the orientation of the object, is to use the Generalized Polarization Tensors (GPT) of the object. This method requires some knowledge of the unperturbed electric fields to locate the object and essentially a pattern matching approach to characterization. The GPTs form the coefficients of an asymptotic expansion of the voltage perturbation as the object size tends to zero. The lowest order term is determined by a rank 2 Polarization Tensor (PT), which we called as the first order PT and also known as the Pólya-Szegö tensor. By choosing an orthogonal basis, the PT can be represented as a symmetric matrix and for the first order PT, the eigenvalues of this matrix have been described by the depolarization or demagnetization factors depending on the context. Recently, Ammari et. al [10] have modelled electrosensing fish in a two dimensional space. They then apply the PT to investigate a possible two dimensional method that could be used by electrosensing fish for shape recognition and classification [11].

On the other hand, by considering electrosensing fish in three dimensional space, we evaluated numerically the PT for some objects used during the experiments conducted on *Gnathonemus petersii* by [36]. Later, in [2], we examined the calculated PT and related it to the first part of the experiments (the training phase) of [36] to investigate the role of the first order PT in electrosensing. We then extended our investigation in [7] by considering another experiment (the discrimination tests) in [36]. Based on these studies, we have found that the fish can easily distinguish two objects when the first order PTs of the objects have a big difference. These findings are consistent with our hypothesis that weakly electric fish use the first order PT as part of their characterization algorithm. Our approaches in [2, 7] are also different with the works done by Ammari et. al [10, 11] as we calculate the real eigenvalues of the first order PT of objects used during some experiments in three dimensionsal space which include the fish and also a conductivity contrast, whereas Ammari et. al [10, 11] use the PT in two dimensions, dependent on frequency, to simulate discrimination and shape reconstruction.

More generally, the polarization (also called polariziability) tensor is widely used to describe the perturbation in potential and wave fields due to an object of contrasting material. One key advantage is that the PT depends on the shape and material, transforms as a tensor under rotation (see Theorem 1 in Section 2), and the perturbation can be expressed in an asymptotic expansion in which all the spatial dependence is represented by the unperturbed fields. This provides a method of object recognition and characterization. However, in other electromagnetic and acoustic problems, the leading order term in the asymptotic expansion of the perturbed field is not always associated with the Pólya-Szegö tensor and can have different forms depending on the chosen application. For a general introduction including potential fields and the acoustic case, see [14]. For far field electromagnetic methods, see [15, 22]. For near field inductive detection and characterization of conductive objects, see [23, 24].

In this paper, we will revisit our results in [2] by using a slightly different approach to demonstrate the possible role of the first order PT in electrosensing fish. The difference between our approach in this more detailed paper and our approach in the papers [2, 7] will be further explained in Section 3, but specifically, our earlier work used crude zeroth-order quadrature while our new approach builds on quadrature methods used in the Boundary Element Method (BEM). In addition, we also suggest some experiments to further investigate the hypothesis.

2. Mathematical Formulation of the First Order Polarization Tensor

Let the conductivity of the water or other objects in the water in the region exterior to a weakly electric fish be σ. The electrical voltage u due to electrical current generated by the fish in the region satisfies the equation

$$\nabla \cdot (\sigma \nabla u) = 0, \tag{1}$$

assuming no current source exterior to the fish. Consider the domain $\Omega = \mathbb{R}^3 - F$ where F is the fish. Suppose that there is an isolated object B which is assumed to be a Lipschitz bounded domain in $\mathbb{R}^3$ at some distance from the fish and for any point $\mathbf{x}$

$\in \mathbb{R}^3$

$$\sigma(\mathbf{x}) = \begin{cases} k & \text{for } \mathbf{x} \in B \\ 1 & \text{for } \mathbf{x} \in \Omega - B \end{cases} \tag{2}$$

where k is constant. Here we take a static approximation where u is assumed time invariant. Ammari and Kang [14] have shown that the perturbation in the voltage due to a small object in the region Ω can be approximated by an asymptotic expansion where the dominant term of the expansion is determined by the PT.

Let H be the voltage in the water without the object B such that $\nabla^2 H = 0$ from (1). Then, from [14], we have

$$(u - H)(\mathbf{x}) = -\nabla\Gamma(\mathbf{x}) \cdot M\nabla H(0) + O(1/|\mathbf{x}|^2) \text{ as } |\mathbf{x}| \to \infty \tag{3}$$

where the origin $O \in B$, $\Gamma(\mathbf{x}) = -(4\pi|\mathbf{x}|)^{-1}$ and M is the first order PT. Moreover, M of an object B is represented in the standard basis as 3×3 matrix (so, the juxtaposition with a vector on the right is matrix multiplication, equivalent to contraction of the rank 2 symmetric tensor over one index, while the dot product is contraction over the other index) and can be determined by evaluating the following moment integral over the boundary of B, ∂B [14]

$$M = \int_{\partial B} \mathbf{\Phi}(\mathbf{y})\mathbf{y}\, \mathrm{d}\sigma(\mathbf{y}) \tag{4}$$

where

$$\mathbf{\Phi}(\mathbf{y}) = (\lambda I - \mathcal{K}_B^*)^{-1}\boldsymbol{\nu}_{\mathbf{x}} \tag{5}$$

such that λ is defined as $\lambda = (k+1)/2(k-1)$ and $\boldsymbol{\nu}_{\mathbf{x}}$ is the unit outward normal vector to ∂B at $\mathbf{x} \in \partial B$. $\mathcal{K}_B^*$ is an integral operator defined with Cauchy principal value $P.V.$ as

$$\mathcal{K}_B^* \mathbf{\Phi}(\mathbf{x}) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\nu}_{\mathbf{x}}}{|\mathbf{x}-\mathbf{y}|^3} \mathbf{\Phi}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{6}$$

where $|\mathbf{x}-\mathbf{y}|$ is the distance between $\mathbf{x}$ and $\mathbf{y}$. Furthermore, M is independent of position B from the fish, F as given by (3) and it can also be shown from [14] that M is symmetric. Alternatively, M can also be determined by finding the solution of a transmission problem of the Laplace equation [14].

If B is an ellipsoid E, aligned with the coordinate axes, represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in Cartesian coordinates, where a, b and c are the lengths of semi principal axes of E, the first order PT of E at conductivity k denoted by $M(k, E)$ is known explicitly [14] as

$$M(k,E) = (k-1)|E| \begin{bmatrix} \frac{1}{(1-P)+kP} & 0 & 0 \\ 0 & \frac{1}{(1-Q)+kQ} & 0 \\ 0 & 0 & \frac{1}{(1-R)+kR} \end{bmatrix} \tag{7}$$

where $|E|$ is the volume of E and P, Q and R are constants defined by the elliptic integrals

$$P = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{t^2\sqrt{t^2 - 1 + (\frac{b}{a})^2}\sqrt{t^2 - 1 + (\frac{c}{a})^2}}\, \mathrm{d}t,$$

$$Q = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{(t^2 - 1 + (\frac{b}{a})^2)^{\frac{3}{2}} \sqrt{t^2 - 1 + (\frac{c}{a})^2}} \, \mathrm{d}t, \tag{8}$$

$$R = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{\sqrt{t^2 - 1 + (\frac{b}{a})^2} (t^2 - 1 + (\frac{c}{a})^2)^{\frac{3}{2}}} \, \mathrm{d}t.$$

The constants P, Q and R are related to what are known classically as demagnetization factors, from their use in magnetostatics (see [24, eq 60]). They date from at least as early as the work of Poisson [29].

The following theorem given in [14] is also useful to describe the first order PT for an object in this study.

Theorem 1 *Let R be an orthogonal matrix transformation of a domain B and R^T is the transpose of R such that $B' = RB$. If $M(k, B)$ and $M(k, B')$ are the first order PT associated to domains B and B' respectively for a conductivity $0 < k \neq 1 < +\infty$ then $M(k, B') = RM(k, B)R^T$.*

According to this theorem, the first order PT of an object rotates as the object rotates. For example, if B rotates $90°$ around z-axis, the first order PT for B also rotates and it is the first order PT for B after the rotation. If we know the first order PT for B, the first order PT for B after the rotation can be obtained by using Theorem 1 with

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

This suggests that, for a fixed k, the eigenvalues of the first order PT depend on the shape while the eigenvectors tell the orientation of the object in space. At the same time, this also means shape of two different objects might not be discriminated by this method if they both have the same first order PT.

3. Methodology

In order to investigate the role of the first order PT in electrosensing, we first review the experiments called the training phase conducted by von der Emde and Fetz [36] on eight fish of the species *Gnathonemus petersii*. These experiments have being conducted to test the ability of the fish to discriminate between two different objects. During their study, the fish were actually trained to accept or reject two different objects. Each fish was rewarded for choosing the correct object and punished for choosing the wrong object until they were able to choose the correct object with 75% succession rate in three consecutive days. Some controls were also used to ensure that the fish depended only on their electrical sense when making decisions.

In this paper, we will refer to the recorded period taken by each fish to complete the training in [36]. However, only five fish from the study will be considered as only

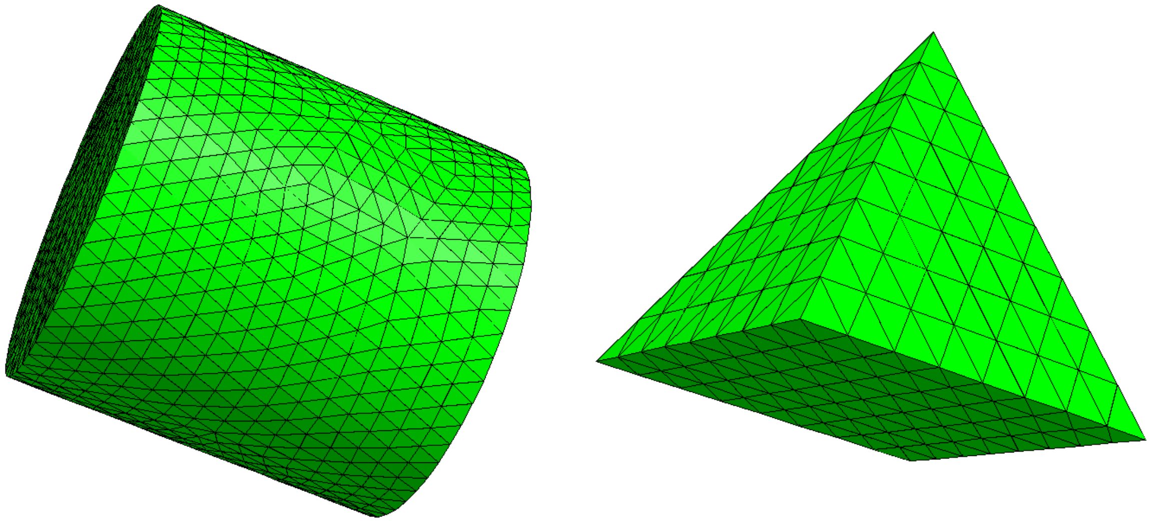

Figure 1. Triangulation of a cylinder and a pyramid

these objects that they were trained to recognise had sufficient dimensional information. For later convenience, we have renamed Fish 1, Fish 2, Fish 3, Fish 7 and Fish 8 in [36] as Fish B, Fish A, Fish E, Fish C and Fish D respectively.

We need to numerically determine the first order PT for the objects used during the experiment by using (4), (5) and (6). These are boundary integral operators so the boundary of the desired object B is firstly approximated by using N flat triangles (see Figure 1 for examples). We used *Netgen* [30] to create a surface mesh with N triangles for the objects.

During our investigations in [2] and [7], by using the mesh and a simple zeroth order quadrature formula (which is exact for functions that are constant on triangles), we have generated a discrete approximation for K_B^* in (6) to solve (5) numerically for Φ and the same quadrature is then used to numerically compute the first order PT from (4). Moreover, the triangularization of the surface can be refined in *Netgen* to increase the accuracy of the computation. We have explained in more detail about these procedures in [3].

Meanwhile, in this study, we will use some ideas from BEM and the software *BEM++* [32] to efficiently and quickly compute the first order PT. A brief guide to use *BEM++* for this purpose is presented in [4]. Similar to [2, 7], the number of triangles, N for the mesh of the object is choosen to be sufficiently large so that every coefficient of the approximated first order PT converges. Our approach using higher order quadrature with *BEM++* as presented in [4] provides better approximation to the first order PT than the approach described in [3] (see [8] for more examples). *BEM++* is actually an object-oriented code for BEM and is specifically developed by [32] to solve boundary value problems in the form of boundary integral equations.

Figure 2 shows the error when the first order PT for the sphere of radius 1 with $k = 1.5$ is approximated according to our approach in [3] using *Matlab* and also with

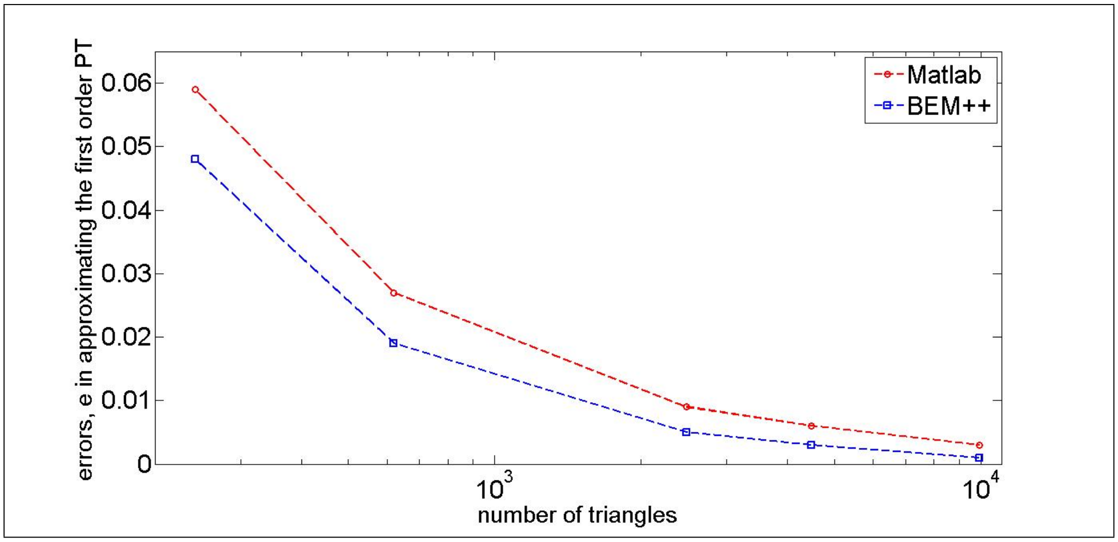

Figure 2. The error, e when the first order PT for the sphere of radius 1 is approximated at $k = 1.5$ by both *Matlab* and *BEM++* on the mesh consisting 242, 620, 2480, 4480 and 9920 triangles against the number of triangles

BEM++ as described in [4]. Note that we used *Matlab* for our previous studies in [2, 7] whereas, we are going to use *BEM++* in this particular study. By using the entry-wise norm for the 3×3 matrix A given by $\|A\|_2 = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} |A_{ij}|^2}$, the error in Figure 2 is defined by $e = \left\|M - \bar{M}\right\|_2 / \|M\|_2$. Here, M is the first order PT for the sphere computed based on (7) while $\bar{M}$ is the approximated first order PT computed by both *Matlab* and *BEM++* on five different meshes for the sphere. Based on the curves in Figure 2, we can see that e decreases as the number of triangles used for the mesh increases when approximating the first order PT for the sphere with $k = 1.5$ by both *Matlab* and *BEM++*. However, e are smaller when the first order PT for the sphere are approximated by *BEM++* for each mesh. This suggests that the approach by using *BEM++* gives better approximation to the first order PT.

After computation with *BEM++*, the approximated first order PT for all considered objects are then analysed. Here, for the first order PT of an object B, denoted by M_B, all three eigenvalues of M_B are determined. Each eigenvalue of M_B is then normalized by dividing with the largest eigenvalue of M_B. The average of the normalized eigenvalues is computed as a measure of the magnitude of M_B. We say that shape of two objects (say B_1 and B_2) are electrically similar if the average of the normalized eigenvalues for M_{B_1} is the same with the average of the normalized eigenvalues for M_{B_2}.

We also suggest as future work to test the ability of the fish to distinguish between objects with the same first order PT. One way to do this is to make an ellipsoid that has the same first order PT as that calculated for a given non-elliptical test object. This is possible because an analytical formula of the first order PT for ellipsoid, (7), is known. By fixing the conductivity k and setting (7) equal to the first order PT for the other

Table 1. The First Order PT for Several Objects

Object, B	Dimension (cm)	M_B	$\hat{e}_{M_B}$	$\hat{c}$
Cone	$d = h = 3$	$10^{-4} \times \begin{bmatrix} 0.28 & 0 & 0 \\ 0 & 0.28 & 0 \\ 0 & 0 & 0.31 \end{bmatrix}$	0.91 0.91 1	0.94
Cube	$l = w = h = 3$	$10^{-4} \times \begin{bmatrix} 0.98 & 0 & 0 \\ 0 & 0.98 & 0 \\ 0 & 0 & 0.98 \end{bmatrix}$	1 1 1	1
Pyramid	$l = w = h = 3$	$10^{-4} \times \begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 0.42 & 0 \\ 0 & 0 & 0.32 \end{bmatrix}$	0.75 1 1	0.92

note : d = diameter, h = height, l = length, w = width

object, three non-linear equations can be derived from (7) and (8). This system of non-linear equations can then be solved simultaneously to determine all semi-principal axes for the ellipsoid. Here, we will use our previous technique explained in [5] to determine the ellipsoid.

4. Results and Discussion

Now, we consider a selection of the objects presented in [36] where, each is made from a highly conducting metal such that the conductivity contrast is $k = 10^7$. By considering mesh with sufficiently large N (more than 15000 triangles), the first order PT for each object B considered, denoted by M_B, can be calculated accurately to six decimal places as shown in Table 1. The normalized eigenvalues of M_B, $\hat{e}_{M_B}$ and their average denoted by $\hat{c}$ are also included in the table. From Table 1, we can see that each M_B is a diagonal matrix. Furthermore, the cone and pyramid have two distinct eigenvalues while cube has only one.

Table 2 shows time taken by the fish in [36] to complete their training in accepting and rejecting two objects (denoted by $S+$ and S- respectively) based on the explanation in the previous section. In this table, Fish A and Fish B are asked to distinguish pyramid and cube, Fish C and Fish D distinguish cone and cube while Fish E distinguish cone and pyramid. As given by [36], Fish A, Fish B, Fish C, Fish D and Fish E are able to complete their task in 4, 7, 8, 10 and 19 days respectively. Now, we extend these results by finding the absolute difference of $\hat{c}$ for $S+$ and S- denoted by d to measure the difference between the first order PT for $S+$ and S-. According to the table, Fish

Table 2. Results for the Training in [36]

Fish	Accept, $S+$	Reject, S-	Period (Days)	d
A B	Pyramid	Cube	4 7	0.08
C D	Cone	Cube	8 10	0.06
E	Cone	Pyramid	19	0.02

E takes the longest time to discriminate cone ($S+$) and pyramid (S-) and we can see that this pair of $S+$ and S- has the smallest d among the other pair of $S+$ and S-. Furthermore, Fish A and Fish B are able to achieve their tasks in a shorter period than Fish C and Fish D while d between pyramid and cube is larger than between cone and cube. This suggests that the fish use the first order PT as part of their recognition algorithm where they need more time to complete the training when both $S+$ and S- have almost similar first order PT.

5. Future Work

In the introduction, we have remarked that PTs are also known for other problems. Ammari et al. have obtained an asymptotic expansion for the perturbed magnetic field generated in the presence of small conducting objects in the eddy current regime [12, 13], which is relevant for metal detection problems. The leading order term is expressed in terms of a new class of rank 4 PTs. In [23], Ledger and Lionheart have shown that for orthonormal coordinate, the expansion in [12, 13] reduces to a simpler result involving a new symmetric magnetic rank 2 PT and they have also obtained several interesting properties of the new PT in [24]. This new explicit formula provides a rigorous mathematical foundation to the PT previously used in inductive security screening [27, 28] and landmine detection [18]. In [9], we have numerically computed this magnetic rank 2 PT for several interesting threat objects. In these studies [9, 18, 27, 28], the magnetic PT is used to improve metal detection i.e. to distinguish between threat and non-threat objects for inductive security screening and to locate the metal component of buried landmines. The magnetic PT being derived from the eddy current approximation to Maxwells equation is different to the Pólya-Szegö PT (4), which is derived from the solution of a different set of integral equations. By contrast, the far field full Maxwell's equation case results in the same PT as for the static case [22]. As yet, a GPT expansion for inductive metal detector case is not known. We expect that from a better understanding of the characterization algorithms used by fish, we will be able to improve our methods for security screening and classifying buried objects using inductive and microwave measurements.

During this study, we have found results which are consistent with our hypothesis that the first order PT is used by the fish for object recognition and presented them in the previous section. However, our findings are not yet conclusive so, in order to decide whether the first order PT has a role in the way fish perform this task, further experiments must be carried out. First of all, as the expansion (3) is asymptotic in distance, any experiment should be performed in such a way that the fish are at least a certain distance from the object. According to Sicardi et. al [31], depending on the object, the signal received by the fish about an object also depends on the distance between the fish and the object. However, as stated before, the first order PT does not depend on the distance. In this case, we have to make sure the distance is not too close to ensure that formula (3) is mathematically valid. On the other hand, the object also must not be too far from the fish so that it can be electrically sensed.

Next, we may test the ability of the fish to distinguish between two objects with the same first order PT. If the fish can discriminate these two objects then we can conclude that it uses not just the first order PT. It might be that it uses higher order GPT as well for example. Table 3 shows ellipsoids E with semi principal axes a, b and c that have similar first order PT as given objects B where, the ellipsoids are obtained as explained in the last paragraph of Section 3. According to [36], electrosensing fish can discriminate between two objects of different materials. Therefore, to make sure that the choice of the fish is not based on the material of the object, the conductivity should be the same for all objects. In our computations, we choose $k = 10^7$, that is all of them are made by the same high conducting metal. A pair of objects with similar first order PT from this table could then be used during the experiment in the future.

In their study, [36] has also showed that electrosensing fish tend to choose objects with a smaller volume from a pair of objects. Thus, it might be important also for the pair B and E in Table 3 to have the same volume so that the fish will not make the choice based on the volume of the objects. Therefore, we also include in Table 3 the absolute difference between the volume of B and E (denoted by v (cm^3)) for all pairs. From these values, we can see that there is only a small difference between the volume of B and E for each pair B and E. In the future, we may assume that the volume of B is similar to the volume of E when testing whether electrosensing fish can discriminate between B and E.

Another straight forward experiment that can be performed is to give a pair of cones pointing up and down for the fish to discriminate. In this case, not only the eigenvalues but also the eigenvectors are the same, as can be seen from Theorem 1. Indeed, as the first order PT in some sense electrically best in fitting ellipsoid, it is clear that the size and orientation of the ellipsoid is the same for both orientations of the cone. As the full series of all GPTs determines the shape of object completely up to rotation [14], features such as the direction a cone is pointing are encoded in the higher order PTs. Beyond the first order PT, it is not yet known what features are encoded in the GPTs up to a certain order.

Table 3. Ellipsoids and Objects with Similar First Order PT

Object, B	Ellipsoid, E	First Order PT of B	v
Cone $d = h = 3$	$a = 1.28$ $b = 1.28$ $c = 1.40$	$10^{-4} \times \begin{bmatrix} 0.28 & 0 & 0 \\ 0 & 0.28 & 0 \\ 0 & 0 & 0.31 \end{bmatrix}$	2.54
Cube $l = w = h = 3$	$a = 1.98$ $b = 1.98$ $c = 1.98$	$10^{-4} \times \begin{bmatrix} 0.98 & 0 & 0 \\ 0 & 0.98 & 0 \\ 0 & 0 & 0.98 \end{bmatrix}$	5.52
Cylinder $d = h = 3$	$a = 1.69$ $b = 1.69$ $c = 1.99$	$10^{-4} \times \begin{bmatrix} 0.67 & 0 & 0 \\ 0 & 0.67 & 0 \\ 0 & 0 & 0.82 \end{bmatrix}$	2.60
Hemisphere $d = 3$	$a = 1.50$ $b = 1.50$ $c = 0.82$	$10^{-4} \times \begin{bmatrix} 0.31 & 0 & 0 \\ 0 & 0.31 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}$	0.66
Pyramid $l = w = h = 3$	$a = 1.56$ $b = 1.56$ $c = 1.24$	$10^{-4} \times \begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 0.42 & 0 \\ 0 & 0 & 0.32 \end{bmatrix}$	3.64

note : dimensions are in centimeter (cm),
d = diameter, h = height, l = length, w = width

6. Discussion and Conclusions

In this paper, we have shown that the weakly electric fish studied take a longer time to electrically discriminate two objects with a small difference between the average of the eigenvalues for their first order PT. This result suggests that the first order PT may play a role in object recognition but it is not conclusive. Therefore, a few future experiments that can be conducted to the fish to further investigate this have also been suggested here.

Moreover, we have considered only the static approximation, and it is possible that the fish use time domain and phase information to discriminate between, for example prey and inedible objects, which may well have a different dispersion relationship. This could be analysed using frequency dependent complex polarization tensors.

Given the possible selective advantage for prey species, it seems likely that prey of electrosensing fish have developed electrical camouflage or cloaking. For camouflage, if the hypothesis that the (complex) PT is used as important classifier by the fish, it would be interesting to test the PT of organisms for which electrosensing fish are important predators and compare with similar organisms that have evolved without such predators. Meanwhile, cloaking is a phenomena that has been widely explored in theoretical EIT [19], in which anisotropic meta materials can be used to make targets invisible to EIT measurement. It would be interesting to investigate using an EIT test cell if any prey species have evolved electrical cloaking. Our suggestion would be to look first in areas where electrosensing fish such as *Apteronotus albifrons* that use a relatively narrow bandwitdth are also the main predators. We suspect it would be easier to achieve effective cloaking in such a range over a narrow band of frequencies.

7. Acknowledgments

The authors would like to thank Dr Timo Betcke for his help and encouragement in using *BEM++*, and Professors Habib Ammari and Hyeonbae Kang for their patient explanation of polarization tensors. We would like to thank Professor Mark Nelson and Professor Gerhard von der Emde for helpful discussions on electrosensing fish, and the anonymous referees for their constructive suggestions. The authors are indebted to the Ministry of Education of Malaysia who provided financial support for this study. This work was also partially supported by EPSRC grants EP/G065047/1, EP/K039865/1 and EP/K00428X/1, and by the Royal Society Wolfson Research Merit Award.

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