On the symmetry of epsilon factors for GL_n

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ON THE SYMMETRY OF EPSILON FACTORS FOR GL_n

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Abstract. We show how the epsilon factors for GL_n factor, as finite morphisms of algebraic varieties, through the corresponding extended quotients. The finite morphisms are, up to a constant, rational characters of complex tori.

Contents

1. Introduction

We begin with some background on the epsilon factors [L]. The epsilon factors are very important and central in the theory of Artin L-functions. If K denotes a global field, the completed L-function $L(s, V)$ of a representation $W_K \to \mathrm{GL}(V)$ of the Weil group W_K of the global field K defines a meromorphic function in the complex plane satisfying the functional equation

$$L(s, V) = \varepsilon(s, V)L(1 - s, V^*)$$

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where V^* is the dual of the representation V of W_K, and the epsilon factor $\varepsilon(s, V)$ is defined by the product

$$\varepsilon(s, V) = \prod \varepsilon_{K_\nu}(s, V_\nu, \psi_\nu). \tag{1}$$

Here, ψ_ν is the local component at a place ν of a non-trivial additive character ψ of $\mathbb{A}_K^+$ trivial on K, so that ψ_ν is a non-trivial additive character of the local field K_ν.

From now on, K will denote a local non-archimedean field. The epsilon factor $\varepsilon_K(s, V, \psi)$, with three variables, is related as follows to the epsilon factor $\varepsilon(V, \psi)$ with two variables:

$$\varepsilon_K(s, V, \psi) = \varepsilon_K(V \otimes \omega_{s-1/2}, \psi)$$

for all $s \in \mathbb{C}$, see [Tate, (3.6.4)]. For $s \in \mathbb{C}$, $\omega_s : W_K \to \mathbb{C}^\times$ is the quasicharacter defined by $\omega_s(w) = |\mathrm{Art}_K(w)|_K^s$ for all $w \in W_K$. In this Note, we will be concerned with the epsilon factor $\varepsilon(V, \psi)$.

If V is a 1-dimensional continuous complex representation of W_K, and $\chi : W_K \to \mathbb{C}^\times$ is the corresponding quasicharacter, then $\varepsilon_K(\chi, \psi)$ is the abelian local constant of Tate, see [Tate, (3.6.3)].

1.1. **The smooth dual of $\mathrm{GL}_n(K)$.** Let $G = \mathrm{GL}_n = \mathrm{GL}_n(K)$. Let $\mathfrak{B}(G)$ denote the Bernstein spectrum of G and let $\mathfrak{s}$ be a point in $\mathfrak{B}(G)$. Attached to the point $\mathfrak{s}$ are an algebraic variety $D^\mathfrak{s}$, which has the structure of a complex torus, and a finite group $W^\mathfrak{s}$, which acts on $D^\mathfrak{s}$. We have the quotient variety $D^\mathfrak{s}/W^\mathfrak{s}$ and the Bernstein variety $\Omega(G)$ is the disjoint union of the quotient varieties $D^\mathfrak{s}/W^\mathfrak{s}$.

Attached to $\mathfrak{s}$ we also have the extended quotient $D^\mathfrak{s}//W^\mathfrak{s}$. Let $\mathfrak{X}$ be an irreducible component in $D^\mathfrak{s}//W^\mathfrak{s}$. There is one irreducible component for each conjugacy class c in $W^\mathfrak{s}$, and we will write $\mathfrak{X}(c)$ for the irreducible component attached to c. So we have

$$D^\mathfrak{s}//W^\mathfrak{s} = \bigsqcup \mathfrak{X}(c)$$

Now the smooth dual $\mathbf{Irr}(G)$ can be realised as the disjoint union of the extended quotients $D^\mathfrak{s}//W^\mathfrak{s}$, see [BP] for the details for this assertion. So the domain of the epsilon factors for GL_n can be realised as the disjoint union of the extended quotients.

At the same time, the set $\mathcal{G}_n(K)$ of equivalence classes of n-dimensional Deligne representations can be organised as a disjoint union of varieties:

$$\mathcal{G}_n(K) = \bigsqcup \mathcal{O}(\rho')$$

We shall view each orbit $\mathcal{O}(\rho')$ as a pointed set, by choosing a Galois representative for each irreducible representation of W_K. We recall that, given an irreducible representation V of W_K, there exists an irreducible representation V^{Gal} of Galois type such that $V = V^{\mathrm{Gal}} \otimes \omega_s$ for some $s \in \mathbb{C}$, see [Tate, (2.2.1)].

Exploiting the local Langlands correspondence for GL_n, the authors of [BP] proved that, for each orbit $\mathcal{O}(\rho') \subset \mathcal{G}_n(K)$, there exists a point $\mathfrak{s} \in \mathfrak{B}(G)$ and an irreducible component $\mathfrak{X}(c) \subset D^{\mathfrak{s}}//W^{\mathfrak{s}}$ such that

$$\mathfrak{X}(c) \simeq \mathcal{O}(\rho')$$

as complex algebraic varieties. In this isomorphism, c is the conjugacy class in $W^{\mathfrak{s}}$ corresponding to the partition which is well-adapted to the Deligne representation ρ'.

1.2. **Epsilon factors.** The main claim of this Note is that the epsilon factors for GL_n factor in a simple way through the irreducible components $\mathfrak{X} = \mathfrak{X}(c)$. From this point of view, the epsilon factor is the composition of three maps:

The isomorphism

$$\mathcal{O}(\rho') \to \mathfrak{X} \tag{2}$$

a morphism

$$m(\mathfrak{X}, \psi) : \mathfrak{X} \to \mathfrak{X} \tag{3}$$

and the product map combined with a constant $e(\mathfrak{X}, \psi)$

$$\mathfrak{X}(c) \to \mathbb{C}^\times \tag{4}$$

We are claiming a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}(c) & \xrightarrow{m(\mathfrak{X},\psi)} & \mathfrak{X}(c) \\ \uparrow & & \downarrow \\ \mathcal{O}(\rho') & \xrightarrow[\varepsilon_K]{} & \mathbb{C}^\times \end{array}$$

The map $m(\mathfrak{X}, \psi)$ is induced by the the following map of complex tori:

$$(z_1, \ldots, z_k) \mapsto (z_1^{\beta_1}, \ldots, z_k^{\beta_k}) \tag{5}$$

where the z_j are torus coordinates,

$$\rho' = ((V, N), \psi) = \bigoplus_{j=1}^{k} \omega_{s_j} \otimes V_j^{\mathrm{Gal}} \otimes \mathrm{Sp}(d_j)$$

and

$$\beta_j = (d_j - 1) \dim V_j^I + d_j[a(V_j^{\mathrm{Gal}}) + n(\psi) \dim(V_j^{\mathrm{Gal}})]$$

where $a(-)$ denotes the Artin conductor exponent, and $n(\psi)$ denotes the conductor of ψ. Since the conductors are integers, the number β_j is an integer. So the map (5) is a morphism of complex tori as algebraic groups.

We have set

$$z_j = q_K^{-s_j} = \omega_{s_j}(\Phi_K)$$

The rather lengthy formula for the constant $e(\mathfrak{X}, \psi)$ appears later in this Note, see (22) and (23).

We emphasize that the morphism $m(\mathfrak{X},\psi)$ and the constant $e(\mathfrak{X},\psi)$ *depend only on the irreducible component* $\mathfrak{X}$, once the additive character ψ has been chosen and fixed. In this way, the theory of the extended quotient illuminates the epsilon factor formulas.

In writing this Note, we were greatly influenced by the preprint of Ikeda [Ikeda]. We thank Paul Baum for several valuable conversations, which led to major changes in the exposition of this Note.

2. Deligne representations

In preparation for all this, we need to recall some material, following closely the exposition in [BP]. Let K be a non-archimedean local field. The Weil group W_K fits into a short exact sequence

$$0 \to I_K \to W_K \xrightarrow{d} \mathbb{Z}$$

where I_K is the inertia group of K. A *Deligne representation* is a pair (ρ, N) consisting of a continuous representation $\rho : W_K \to \mathrm{GL}_n(V)$, $\dim_{\mathbb{C}}(V) = n$, together with a nilpotent endomorphism $N \in \mathrm{End}(V)$ such that

$$\rho(w)N\rho(w)^{-1} = ||w||N.$$

For any $n \geq 1$, the representation $\mathrm{Sp}(n)$ is defined by

$$V = \mathbb{C}^n = \mathbb{C}e_0 + \cdots + \mathbb{C}e_{n-1}$$

with $\rho(w)e_i = ||w||^i e_i$ and $Ne_i = e_{i+1}\,(0 \leq i \leq n-1)$, $Ne_{n-1} = 0$.

Let $\mathcal{G}_n(K)$ be the set of equivalence classes of semisimple n-dimensional Deligne representations. Let $\mathbf{Irr}(\mathrm{GL}_n(K))$ be the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(K)$.

We recall the local Langlands correspondence

$$\mathrm{rec}_K : \mathbf{Irr}(\mathrm{GL}_n(K)) \to \mathcal{G}_n(K)$$

which is unique subject to the conditions listed in [HT, p.2].

We identify the elements of the set $\mathcal{G}_1(K)$, the quasicharacters of W_K, with quasicharacters of $K^\times$ via the local Artin reciprocity map

$$\mathrm{Art}_K : W_K \to K^\times$$

The local Langlands correspondence is compatible with twisting by quasicharacters [HT, p.2].

A quasicharacter $\psi : W_K \to C^\times$ is (arithmetically) *unramified* if ψ is trivial on the inertia group I_K. In that case we have $\psi(w) = z^{d(w)}$ with $z \in \mathbb{C}^\times$. The group of unramified quasicharacters of W_K is denoted $\Psi(W_K)$. Let $\Phi = \Phi_K$ denote a geometric Frobenius element in W_K. The isomorphism $\Psi(W_K) \simeq \mathbb{C}^\times$ is secured by the map $\psi \mapsto \psi(\Phi_K)$.

Let now

$$\rho' = \rho_1 \otimes \mathrm{Sp}(r_1) \oplus \cdots \oplus \rho_m \otimes \mathrm{Sp}(r_m)$$

be a Deligne representation. The set

$$\{\psi\rho_1 \otimes \mathrm{Sp}(r_1) \oplus \cdots \oplus \psi_m\rho_m \otimes \mathrm{Sp}(r_m) : \psi_1, \ldots, \psi_m \in \Psi(W_K)\}$$

will be called the *orbit* of ρ' under the action of

$$\Psi(W_K) \times \cdots \times \Psi(W_K)$$

(m factors). This orbit will be denoted $\mathcal{O}(\rho')$. The orbits create a partition of $\mathcal{G}_n(K)$. The set $\mathcal{G}_n(K)$ is a disjoint union of orbits:

$$\mathcal{G}_n(K) = \bigsqcup \mathcal{O}(\rho')$$

We note that $\Psi(W_K)^m \simeq (\mathbb{C}^\times)^m$, a complex torus. To determine the structure of each orbit, we have to pay attention to the torsion numbers of $\rho_1, \ldots, \rho_m$ and to the action of $\mathrm{GL}_n(\mathbb{C})$ by conjugation. In this way, the set $\mathcal{G}_n(K)$ acquires (locally) the structure of complex algebraic variety. Each irreducible component in this variety is the quotient of a complex torus by a product of symmetric groups.

Let ΩG be the Bernstein variety of $G = \mathrm{GL}_n$. Each point in ΩG is a conjugacy class of cuspidal pairs (M, σ). A quasicharacter $\psi : M \to \mathbb{C}^\times$ is *unramified* of ψ is trivial on M°. The group of unramified quasicharacters of M is denoted $\Psi(M)$. We have $\Psi(M) \simeq (\mathbb{C}^\times)^\ell$ where ℓ is the parabolic rank of the Levi subgroup M. The group $\Psi(M)$ now creates orbits: the orbit of (M, σ) is $\{(M, \psi \otimes \sigma : \psi \in \Psi(M)\}$. Denote this orbit by D, and set $\Omega = D/W(M, D)$ where $W(M)$ is the Weyl group of M and $W(M, D)$ is the subgroup of $W(M)$ which leaves D globally invariant. The orbit D has the structure of a complex torus, and so Ω is a complex algebraic variety. We view Ω as an irreducible component in the algebraic variety ΩG.

We recall the *extended quotient*. Let the finite group Γ act on the space X. Let $\widetilde{X} = \{(x, \gamma) : \gamma x = x\}$, let Γ act on $\widetilde{X}$ by $\gamma_1(x, \gamma) = (\gamma_1 x, \gamma_1 \gamma \gamma_1^{-1})$. Then the extended quotient of X by Γ is

$$X/\!/\Gamma := \widetilde{X}/\Gamma.$$

Let X^γ denote the γ-fixed set, and let $Z(\gamma)$ be the Γ-centralizer of γ. Choose one γ in each Γ-conjugacy class, then we have

$$X/\!/\Gamma = \bigsqcup X^\gamma / Z(\gamma).$$

We will view each orbit $\mathcal{O}(\rho')$ as a pointed set, with base point

$$\rho' = \rho_1 \otimes \mathrm{Sp}(t_1) \oplus \cdots \oplus \rho_m \otimes \mathrm{Sp}(t_m).$$

In the notation of [Ikeda], we will choose ρ_j to be of Galois type, $\rho_j = V_j^{\mathrm{Gal}}$. Let $\mathfrak{s}$ be the point in $\mathfrak{B}(G)$ determined by the cuspidal pair

$$(\mathrm{GL}_{\dim(V_1)}(K) \times \cdots \times \mathrm{GL}_{\dim(V_m)}(K), \mathrm{rec}_K^{-1}\rho_1 \otimes \cdots \otimes \mathrm{rec}_K^{-1}\rho_m)$$

The orbit of this cuspidal pair in $\mathbf{Irr}(G)$ is also a pointed set. The map

$$\psi_1\rho_1 \otimes \mathrm{Sp}(t_1) \oplus \cdots \oplus \psi_m\rho_m \otimes \mathrm{Sp}(t_m) \mapsto (\psi_1(\varpi_K), \ldots, \psi_m(\varpi_K))$$

induces an isomorphism of algebraic varieties

$$\mathcal{O}(\rho') \simeq \mathfrak{X}(c)$$

where c is the conjugacy class in $W^{\mathfrak{s}}$ which is well-adapted to ρ'.

Example. Consider $\mathrm{GL}_n(K)$, suppose that a divides n. Let $r = n/a$, let $\rho : W_K \to \mathrm{GL}_a(\mathbb{C})$ be an irreducible representation of the Weil group W_K, and define

$$(M,\sigma) := (\mathrm{GL}_a(K) \times \cdots \times \mathrm{GL}_a(K), \mathrm{rec}_K^{-1}\rho \otimes \cdots \otimes \mathrm{rec}_K^{-1}\rho)$$

with $\mathrm{GL}_a(K)$ and $\mathrm{rec}_K^{-1}\rho$ each repeated r times. Then (M,σ) is a cuspidal pair for $\mathrm{GL}_n(K)$. Let $\mathfrak{s}$ be the point in $\mathfrak{B}(\mathrm{GL}_n(K))$ determined by (M,σ). Then we have

$$D^{\mathfrak{s}} \simeq (\mathbb{C}^\times)^r, \qquad W^{\mathfrak{s}} \simeq S_r$$

the symmetric group on r letters. Let now $d_1 + \cdots + d_k$ be a partition of r. This partition of r determines a conjugacy class c in the symmetric group S_r.

Let

$$\rho' = \rho \otimes \mathrm{Sp}(d_1) \oplus \cdots \oplus \rho \otimes \mathrm{Sp}(d_k) : W_K \to \mathrm{GL}_n(\mathbb{C})$$

Then we have

$$\mathfrak{X}(c) \simeq \mathcal{O}(\rho')$$

As we vary the conjugacy class c, we obtain all the irreducible components of the extended quotient $D^{\mathfrak{s}}//W^{\mathfrak{s}}$. The structure of the orbit $\mathcal{O}(\rho')$ depends of the structure of the partition, i.e. on the number of times each part of the partition is repeated. See [BP] for more details on this point.

3. The formulas

We recall that, if (V,N) is any Φ-semisimple Deligne representation, then we have a finite direct sum decomposition of (V,N) into indecomposable Deligne representations as follows:

$$(V,N) = V_1 \otimes \mathrm{Sp}(d_1) \oplus \cdots \oplus V_m \otimes \mathrm{Sp}(d_m) \tag{6}$$

We will write

$$V_j = V_j^{\mathrm{Gal}} \otimes \omega_{s_j}.$$

We need the following three items in order to compute epsilon factors.

3.1. The definition. First, the definition, see [Tate, 4.1.6]

$$\varepsilon((V,N),\psi) := \varepsilon(V,\psi)\det(-\Phi|V^I/V_N^I) \tag{7}$$

3.2. Additivity. Next, additivity with respect to V, see [Tate, 3.4.2]:

$$\varepsilon(V_1 \oplus \cdots \oplus V_k, \psi) = \varepsilon(V_1,\psi)\ldots\varepsilon(V_k,\psi) \tag{8}$$

3.3. Unramified twist. Finally, behaviour under unramified twist, see [Tate, 3.4.5]:

$$\varepsilon(V \otimes \omega_s, \psi) = \varepsilon(V,\psi)q^{-s[a(V)+n(\psi)\dim V]} \tag{9}$$

where $a(V)$ is the Artin conductor exponent of V, and $n(\psi)$ is the conductor of ψ.

3.4. **The term $\varepsilon(V)$.** A typical direct summand in (20) is

$$V_j^{\mathrm{Gal}} \otimes \omega_{s_j} \otimes \omega_k$$

with $1 \leq j \leq m,\ 0 \leq k \leq d_j - 1$. We have

$$\varepsilon(V_j^{\mathrm{Gal}} \otimes \omega_{s_j} \otimes \omega_k) = \varepsilon(V_j^{\mathrm{Gal}} \otimes \omega_{s_j+k})$$

For this summand, we have by (9)

$$\varepsilon(V_j^{\mathrm{Gal}} \otimes \omega_{s_j} \otimes \omega_k) = \varepsilon(V_j^{\mathrm{Gal}}) q^{-(s_j+k)[a(V_j^{\mathrm{Gal}})+n(\psi)\dim V_j^{\mathrm{Gal}}]} \tag{10}$$

and then the formula for $\varepsilon(V)$ follows from (8). We obtain

$$\varepsilon(V) = \prod_{j=1}^{m} \left(\varepsilon(V_j^{\mathrm{Gal}}, \psi)\right)^{d_j} \cdot q^{-[s_j d_j + \frac{(d_j-1)d_j}{2}][a(V_j^{\mathrm{Gal}})+n(\psi_+)\dim(V_j^{\mathrm{Gal}})]} \tag{11}$$

Note that Ikeda succeeds in describing the numbers $\varepsilon(V_j^{\mathrm{Gal}}, \psi)$ in terms of the non-abelian local class field theory of K, see [Ikeda, Theorem 5.4].

3.5. **The determinant.** A typical factor in (7) is

$$\det(-\Phi | (E_j)^I / (E_j)^I_{1\otimes N_j}) \tag{12}$$

where

$$\begin{aligned} E_j :&= V_j \otimes \mathrm{Sp}(d_j) \\ &= V_j \otimes (\mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{d_j-1}) \end{aligned}$$

We have

$$(E_j)_{1\otimes N_j} = V_j \otimes \mathbb{C}e_{d_j-1}$$

and so we have

$$E_j^I / (E_j)^I_{1\otimes N_j} = V_j^I \otimes (\mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{d_j-2}) \tag{13}$$

$$\simeq V_j^I \otimes (\omega_0 \oplus \omega_1 \oplus \cdots \oplus \omega_{d_j-2}) \tag{14}$$

We have

$$V_j = \omega_{s_j} \otimes V_j^{\mathrm{Gal}}$$

and

$$V_j^I = (V_j^{\mathrm{Gal}})^I$$

Recall that

$$\omega_s(\Phi) = ||\varpi_K||^s = q_K^{-s}$$

Now V_j is a W_K-module which we will denote as $\rho_j : W_K \to \mathrm{GL}(V_j)$. A typical direct summand in (13) is $V_j^I \otimes \omega_k$ with $0 \leq k \leq d_j - 2$. We have to determine the determinant of $(\omega_{s_j} \otimes \rho_j)(-\Phi)$ acting on the vector space

$V_j^I \otimes \mathbb{C}e_k$. This determinant will be $q^{-(s_j+k)\dim V_j^I} \cdot \det(-\rho_j(\Phi))|V_j^I$. There are $d_j - 1$ direct summands so the resulting determinant will be the product

$$\prod_{k=0}^{d_j-2} q^{-(s_j+k)\dim V_j^I} \cdot \det(-\rho_j(\Phi))|V_j^I) \tag{15}$$

$$= \det(-\rho_j(\Phi)|V_j^I)^{d_j-1} \cdot q^{-s_j(d_j-1)\dim V_j^I} \cdot q^{-(1+2+\ldots+d_j-2)\dim V_j^I} \tag{16}$$

$$= \det(-\rho_j(\Phi)|V_j^I)^{d_j-1} \cdot q^{-s_j(d_j-1)\dim V_j^I} \cdot q^{-\frac{1}{2}(d_j-2)(d_j-1)\dim V_j^I} \tag{17}$$

provided that $d_j \geq 3$. By inspection, this formula is also valid for $d_j = 1$ or 2.

3.6. **The term $\varepsilon((V,N)),\psi)$.** From the definition (7) we infer that $\varepsilon((V,N,\psi)$ is the product of (11) and (15).

We wish to isolate the term which is dependent on the variables $s_1, \ldots, s_m$. This term is of exponential type as follows:

$$const \cdot q^{\sum_{j=1}^m -s_j\beta_j} \tag{18}$$

where

$$\beta_j = (d_j - 1)\dim V_j^I + d_j[a(V_j) + n(\psi_+)\dim(V_j)]$$

for all $1 \leq j \leq m$. Note that β_j is an *integer*:

$$\beta_j \in \mathbb{Z}.$$

The constant can be read off from (11) and (15). The formula (18) is intricate, but, from our point of view, it has a simple form, namely

$$const \cdot z_1^{\beta_1} \cdots z_m^{\beta_m} \tag{19}$$

where

$$z_j = q^{-s_j}$$

Apart from the constant term, the formula (19) for the epsilon factor is a *rational character* of the complex torus $(\mathbb{C}^\times)^m$, i.e. the morphism

$$(\mathbb{C}^\times)^m \to \mathbb{C}^\times, \qquad (z_1, \ldots, z_m) \mapsto z_1^{\beta_1} \cdots z_m^{\beta_m}$$

of algebraic groups.

Consider the following set:

$$\{\omega_{s_1} \otimes V_1^{\mathrm{Gal}} \otimes \mathrm{Sp}(d_1) \oplus \cdots \oplus \omega_{s_m} \otimes V_m^{\mathrm{Gal}} \otimes \mathrm{Sp}(d_m) : s_1, \ldots, s_m \in \mathbb{C}\}. \tag{20}$$

After allowing for conjugacy in the Langlands dual group $\mathrm{GL}_n(\mathbb{C})$, this set has the structure of a complex algebraic variety $\mathfrak{X}$ in $\mathcal{G}_n$. In fact $\mathfrak{X}$ is an irreducible component in $\mathcal{G}_n(K)$:

$$\mathfrak{X} \subset \mathcal{G}_n(K)$$

Applying the local Langlands correspondence, we have, by transport of structure, an irreducible component in the smooth dual:

$$\mathrm{rec}_K^{-1}(\mathfrak{X}) \subset \mathbf{Irr}(\mathrm{GL}_n(K))$$

Looking carefully at the formulas (11) and (15), we see that the constant in (18) depends on the variety $\mathfrak{X}$, and on the additive character ψ. We will denote this constant by $e(\mathfrak{X}, \psi)$, so that (18) can be re-written

$$e(\mathfrak{X}, \psi) \cdot q^{-s_1\beta_1} \cdots q^{-s_m\beta_m} \tag{21}$$

To summarize: we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{m(\mathfrak{X},\psi)} & \mathfrak{X} \\ \uparrow & & \downarrow \\ \mathcal{O}(\rho') & \xrightarrow[\varepsilon_K]{} & \mathbb{C}^\times \end{array}$$

which reveals that the epsilon-factor ε_K factors through $\mathfrak{X}$. The right-hand vertical map is the product map times the constant $e(\mathfrak{X}, \psi)$. The constant $e(\mathfrak{X}, \psi)$ is itself the product of

$$\prod_{j=1}^{m} \Big(\varepsilon_K(V_j^{\mathrm{Gal}}, \psi)\Big)^{d_j} \cdot q^{-\frac{1}{2}(d_j-1)d_j[a(V_j^{\mathrm{Gal}})+n(\psi_+)\dim(V_j^{\mathrm{Gal}})]} \tag{22}$$

with

$$\prod_{j=1}^{m} \det(-\rho_j(\Phi)|V_j^I)^{d_j-1} \cdot q^{-\frac{1}{2}(d_j-2)(d_j-1)\dim V_j^I} \tag{23}$$

The terms $\varepsilon_K(V_j^{\mathrm{Gal}}, \psi)$ are the epsilon factors attached to irreducible representations of the local absolute Galois group G_K. These terms are defined in [Ikeda, p.15].

4. The arithmetically unramified representations of $\mathrm{GL}_n(K)$

Here, the underlying representation of the Weil group is the trivial n-dimensional representation $\rho : W_K \to \mathrm{GL}_n(\mathbb{C})$. So we have $V_j^{\mathrm{Gal}} = 1, 1 \leq j \leq n$.

Let W be the Weyl group $\mathfrak{S}_n$. The arithmetically unramified representations of $\mathrm{GL}_n(K)$ have, by definition, the following set of Langlands parameters (Deligne representations):

$$\{\omega_{s_1} \otimes \mathrm{Sp}(d_1) \oplus \cdots \oplus \omega_{s_k} \otimes \mathrm{Sp}(d_k) : s_j \in \mathbb{C}\} \tag{24}$$

where $d_1 + \cdots + d_k = n$. This set determines a complex algebraic variety $\mathfrak{X}$ in $\mathcal{G}_n(K)$.

We choose ψ to have conductor 0. In this case $\varepsilon(V) = 1$ and $\beta_j = d_j - 1$. We have

$$\begin{aligned} \varepsilon((V, N, \psi) &= e(\mathfrak{X}, \psi) \prod_{j=1}^{m} q^{-(d_j-1)s_j} \\ &= e(\mathfrak{X}, \psi) \prod_{j=1}^{m} z_j^{d_j-1} \end{aligned}$$

where

$$e(\mathfrak{X}, \psi) = \prod_{j=1}^{m} (-1)^{d_j - 1} q^{-(d_j-1)(d_j-2)/2}$$

and $z_j := q^{-s_j}$.

The epsilon factor records the dimensions d_j of the special representations $\mathrm{Sp}(d_j)$ which occur in the Deligne representation (V, N).

We will now re-organise the partition $d_1 + \cdots + d_k = n$. Suppose that this partition has distinct parts $t_1, \ldots, t_m$ with $t_1 < t_2 < \cdots < t_m$ and that t_j is repeated r_j times so that

$$r_1 t_1 + \cdots r_m t_m = n.$$

Then, as a function on the complex torus $(\mathbb{C}^\times)^{r_1 + \cdots + r_m}$, the epsilon factor is *invariant under the following product of symmetric groups*:

$$\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2} \times \cdots \times \mathfrak{S}_{r_m}$$

and therefore factors through the following quotient variety

$$(\mathbb{C}^\times)^{r_1} / \mathfrak{S}_{r_1} \times \cdots \times (\mathbb{C}^\times)^{r_m} / \mathfrak{S}_{r_m}$$

which is precisely an irreducible component of the extended quotient $T//W$. Every irreducible component is accounted for in this way. The epsilon factors have precisely the amount of symmetry required to factor through these quotient varieties.

Example 1. By way of illustration, we will consider $\mathrm{GL}_4(K)$. In the following table, the first column is a conjugacy class c in $\mathfrak{S}_4$, viewed as a partition of 4, the second column is the corresponding irreducible component $\mathfrak{X}(c)$ of the extended quotient $T//W$, and the third column is the associated epsilon factor.

Partition	$\mathfrak{X}(c)$	epsilon factor
$1+1+1+1$	$\mathrm{Sym}^4\, \mathbb{C}^\times$	1
$2+2$	$\mathrm{Sym}^2\, \mathbb{C}^\times$	$q^{-s_1} q^{-s_2}$
$1+3$	$(\mathbb{C}^\times)^2$	$q \cdot q^{-2s_2}$
4	$\mathbb{C}^\times$	$-q^3 \cdot q^{-3s_1}$
$1+1+2$	$\mathrm{Sym}^2\, \mathbb{C}^\times \times \mathbb{C}^\times$	$-q^{-s_3}$

This table may be clearer if we write $z_j := q^{-s_j}$:

Partition	$\mathfrak{X}(c)$	epsilon factor
$1+1+1+1$	$\mathrm{Sym}^4\, \mathbb{C}^\times$	1
$2+2$	$\mathrm{Sym}^2\, \mathbb{C}^\times$	$z_1 z_2$
$1+3$	$(\mathbb{C}^\times)^2$	$q \cdot z_2^2$
4	$\mathbb{C}^\times$	$-q^3 \cdot z_1^3$
$1+1+2$	$\mathrm{Sym}^2\, \mathbb{C}^\times \times \mathbb{C}^\times$	$-z_3$

Example 2. Here, we consider the following Deligne representation of $\mathrm{GL}_{19}(K)$:

$$\omega_{s_1}\mathrm{Sp}(2) \oplus \omega_{s_2}\mathrm{Sp}(2) \oplus \omega_{s_3}\mathrm{Sp}(2) \oplus \omega_{s_4}\mathrm{Sp}(3) \oplus \omega_{s_5}\mathrm{Sp}(3) \oplus \omega_{s_6}\mathrm{Sp}(7)$$

The epsilon factor of this representation is

$$const \cdot z_1 z_2 z_3 z_4^2 z_5^2 z_6^6$$

which will factor through the following irreducible component of the extended quotient $T//W$:

$$\mathrm{Sym}^3(\mathbb{C}^\times) \times \mathrm{Sym}^2(\mathbb{C}^\times) \times \mathbb{C}^\times$$

This perfectly illustrates the symmetry properties of the epsilon factors. Each epsilon factor has precisely the symmetry, *neither more nor less*, of the corresponding irreducible component in the extended quotient $T//W$. Each epsilon factor will therefore factor through the corresponding irreducible component in $T//W$.

5. Appendix

Lemma 5.1. *Let $a(V)$ denote the Artin conductor exponent of V. Then we have $a(V \otimes \omega_s) = a(V)$.*

Proof. The definition is

$$a(V) = \dim V - \dim V^I + \sum_{k \geq 1} \frac{1}{[I : I_k]} \cdot \dim V/V^{I_k} \tag{25}$$

where $I = I_0 \supset I_1 \supset \ldots \supset I_k \supset \ldots$ are the ramification subgroups of the inertia group I.

We have

$$\dim(V \otimes \omega_s) = \dim V$$

Now ω_s is an unramified quasi-character of W_K:

$$\omega_s(I) = ||\mathrm{Art}_K(I)||^s = ||U_K||^s = 1$$

and so

$$(V \otimes \omega_s)^{I_k} = V^{I_k}$$

for all $k \geq 0$. The result now follows from (25). □

Lemma 5.2. *Let ψ be an additive character $K \to \mathbb{C}^\times$. Then we have $\varepsilon_K(1, \psi) = 1$.*

Proof. We start with the classical formula in [Tate, 3.6.3]:

$$\varepsilon_K(\chi, \psi) = \chi(c) \frac{\int_{\mathcal{O}^\times} \chi^{-1}(u)\psi(u/c)\, du}{|\int_{\mathcal{O}^\times} \chi^{-1}(u)\psi(u/c)\, du|}$$

where c is an element of $K^\times$ of valuation $a(\chi) + n(\psi)$. Now set $\chi = 1$ and let $n(\psi) = k$. Then we take $c = \varpi^k$. Then $u \in \mathcal{O}^\times \implies u/c \in \varpi^{-k}\mathcal{O}^\times$. But we have $\psi(\varpi^{-k}\mathcal{O}) = 1$ since ψ has conductor k. Therefore we have

$$\begin{aligned}\varepsilon_K(1,\psi) &= \frac{\int_{\mathcal{O}^\times} \psi(u/c)\,du}{|\int_{\mathcal{O}^\times} \psi(u/c)\,du|}\\ &= \frac{\mathrm{vol}(\mathcal{O}^\times)}{|\mathrm{vol}(\mathcal{O}^\times)|}\\ &= 1\end{aligned}$$

□

5.1. **Comparison of notations.** We note that $\varepsilon_K(V,\psi)$ is denoted $\varepsilon_K^{\mathrm{Langlands}}(V,\psi)$ in [Ikeda] and $\varepsilon_L(V,\psi)$ in [Tate, 3.6].

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