On the symmetric square of quaternionic projective space

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ON THE SYMMETRIC SQUARE OF QUATERNIONIC PROJECTIVE SPACE

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Engineering and Physical Sciences

2016

Yumi Boote
School of Mathematics
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The University of Manchester

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The main purpose of this thesis is to calculate the integral cohomology ring of the symmetric square of quaternionic projective space, which has been an open problem since computations with symmetric squares were first proposed in the 1930’s. The geometry of this particular case forms an essential part of the thesis, and unexpected results concerning two universal $Pin(4)$ bundles are also included. The cohomological computations involve a commutative ladder of long exact sequences, which arise by decomposing the symmetric square and the corresponding Borel space in compatible ways. The geometry and the cohomology of the configuration space of unordered pairs of distinct points in quaternionic projective space, and of the Thom space $MPin(4)$, also feature, and seem to be of independent interest.
Declaration

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At the time of writing this thesis, the world is facing great difficulties; I would like to quote the following, written by Jean Vanier.

Faced by social norms and laws,
is there not a personal conscience
within each of us
inviting us
to be agents of peace,
truth and justice?
Chapter 1

Introduction

The symmetric products of a space have attracted many researchers in mathematics over the last 80 years. The symmetric products of an algebraic curve or a space are subjects in algebraic geometry and topology, whilst the physicists may work on symmetric products orbifolds in relation to string theory. One of the classical contributions in this area of mathematical research is the work of Minoru Nakaoka [44–47] in the 1950’s, who was especially interested in the symmetric products and the cyclic products of $n$-spheres. In the 1960’s the homology groups of the symmetric products of a space were described by R. James Milgram [41], in which he completed the calculations begun by Morse, Smith and Richardson in the 1930’s. It is believed that the cohomology rings of the symmetric products, on the other hand, are very difficult to compute in general, and fully known only for a few special cases. Nevertheless, most recently, in 2015, Dmitry Gugnin submitted a paper [21] on the integral cohomology of the symmetric product of a space $X$ modulo torsion $H^*(SP^n(X);\mathbb{Z})/\text{Tor}$, where $X$ is connected and homotopy equivalent to a CW complex. One of the main results of this thesis is a computation of the whole cohomology ring, including torsion, of the symmetric square of quaternionic projective space; our method also works for the complex case, which we plan to discuss in a forthcoming paper [7]. The underlying geometry is beautiful and plays a vital role for our cohomological calculation.

The structure of the thesis is as follows. Chapter 1 contains notation, basics of quaternionic geometry, and a brief introduction to the symmetric square $SP^2(\mathbb{HP}^n)$ of quaternionic projective space, followed by the formulation of the main theorems in cohomology. In Chapter 2, we decompose the symmetric square $SP^2(\mathbb{HP}^n)$ and the
Borel space $B_n = S^\infty_{\mathbb{Z}/2}(\mathbb{HP}^n \times \mathbb{HP}^n)$ into subspaces, and explain their relationships; we also describe the infinite dimensional cases when $n = \infty$, which are considered as colimits $X = \bigcup_n X_n$. Chapter 3 mainly focuses on the group $Pin(4)$; we define the group action on $\mathbb{R}^4$, two models for the classifying space $BPin(4)$, and two universal vector bundles over $BPin(4)$. In Chapter 4, we compute the integral cohomology rings of $BPin(4)$, the Borel space $B_n$, and the configuration space of unordered pairs of distinct points in $\mathbb{HP}^n$. The main aim of Chapter 5 is to compute the homomorphism induced by the map $\mathbb{Z}/2 \times S^3 \to Pin(4)$. Chapter 6 is concerned with the geometry and the cohomology of certain Thom spaces, including $MPin(4)$. In the last two Chapters, 7 and 8, we justify the main theorems in cohomology which are formulated in Chapter 1. In the final part of Chapter 8, we also compare our cohomological results with Nakaoka’s work on the cyclic product of a finite connected simplicial complex. Some background materials, lemmas and examples, are included in Appendices.

Most of the geometry described in Chapters 1–6 comes from combining known results in Algebraic Topology. However, we believe that our study of the two universal $Pin(4)$ bundles, together with the Thom space $MPin(4)$ and its restrictions, are original; as are our applications of symmetric orthogonalisation. Chapters 4–8 contain original computations for the integral cohomology rings of the symmetric squares of quaternionic projective space (both finite and infinite), the configuration subspaces of unordered pairs of distinct points, and the associated Borel spaces for $\mathbb{HP}^n$.

## 1.1 Notation

We begin by listing some notation that we shall often use below.

- $\mathbb{HP}^n$ = the quaternionic projective space of dimension $n$
- $S^n$ = the standard $n$-sphere
- $SP_n^2$ = the symmetric square of $\mathbb{HP}^n$
- $B_n$ = the Borel space $S^\infty_{\mathbb{Z}/2}(\mathbb{HP}^n \times \mathbb{HP}^n)$
- $SP^2$ = the infinite union $\bigcup_n SP_n^2$
- $CP_p(X)$ = the $p$-fold cyclic product of a space $X$, for any prime $p$
CHAPTER 1. INTRODUCTION

\( C_n(X) = \) the \( n \)-th unordered configuration space of distinct points in \( X \)

\( V_{\mathbb{H}}(k, m) = \) the quaternionic Steifel manifold of orthonormal \( k \)-frames in \( \mathbb{H}^m \)

\( V_{n+1} = V_{\mathbb{H}}(2, n + 1) \)

\( Gr_{\mathbb{H}}(k, m) = \) the quaternionic Grassmannian of \( k \)-dimensional subspaces of \( \mathbb{H}^m \)

\( G_A = \) the compact Lie group \( \mathbb{Z}/2 \times S^3 \)

\( (G_i)^m = \bigoplus_{i=1}^n G_i, \) the direct sum of abelian groups \( G_i \)

\( M_{\mathbb{H}}(n) = \) the space of \( n \times n \) quaternionic matrices

\( M_{\mathbb{R}}(n) = \) the space of \( n \times n \) real matrices

\( \cong \) homotopic (of maps) or homotopy equivalence (of spaces)

\( \cong \) homeomorphism (of spaces) or isomorphism (of groups or rings)

\( H^*(X) = \) the reduced integral cohomology of \( X \)

\( H^*(X_+) = \) the unreduced integral cohomology of \( X \)

\( R[x_1, \ldots, x_n] = \) the polynomial ring in variables \( x_1, \ldots, x_n \) over \( R = \mathbb{Z} \) or \( \mathbb{Z}/2 \)

\( R\langle x_1, \ldots, x_n \rangle = \) the free \( R \)-module generated by \( x_1, \ldots, x_n \) over \( R = \mathbb{Z} \) or \( \mathbb{Z}/2 \)

\( \text{UCT} = \) universal coefficient theorem

\( \text{LES} = \) long exact sequence

\( \text{NDR} = \) neighbourhood deformation retract

\( \text{SSS} = \) Serre spectral sequence

1.2 The Quaternionic Geometry Essentials

In this section we review some basics of quaternionic geometry, which has a long history. There are many references on the subject, and we have mainly used [3, 4, 20, 40, 50, 58].
The quaternions $\mathbb{H} \cong \mathbb{R}^4$ form a non-commutative real algebra; its elements are expressed as $q = a1 + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$. For any quaternion $q \in \mathbb{H}$, the norm is the real number

$$|q| = |a1 + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2} \geq 0,$$

and satisfies the rule $|q_1 q_2| = |q_1| |q_2|$; $\mathbb{H}$ is a normed division algebra so that $q_1 q_2 = 0$ implies $q_1 = 0$ or $q_2 = 0$. For $q = a1 + bi + cj + dk \in \mathbb{H}$, its (quaternionic) conjugate is $\overline{q} = a1 - bi - cj - dk$, and conjugation satisfies $\overline{pq} = \overline{p} \overline{q}$, $q\overline{q} = \overline{q}q = |q|^21$ where $p, q \in \mathbb{H}$. It is often the case that 1 is omitted from these expressions for convenience, for example by writing $a1 + bi + cj + dk$ as $a + bi + cj + dk$. Then $a = \text{Re}(q)$ is the real part of $q$, and quaternions with $\text{Re}(q) = 0$ are called pure; we write the set of all pure quaternions as $\text{Pure}(\mathbb{H})$. When dealing with quaternions, care must be taken due to their non-commutative multiplication. A right $\mathbb{H}$-module $X$ means an abelian group with a right scalar multiplication $X \times \mathbb{H} \to X$, $(x, q) \mapsto xq$ which satisfies $(x+y)q = xq + yq$, $xp + q = x(p + q)$, $x(qp) = (xq)p$ for any $x, y \in X$ and $p, q \in \mathbb{H}$. In this thesis the right $\mathbb{H}$-linear structure is always used, unless otherwise stated. We sometimes call such an $X$ a right $\mathbb{H}$-vector space, and its elements can be written as column vectors, with scalars acting on the right. A right linear map $T; \mathbb{H}^n \to \mathbb{H}^m$ satisfies $T(uq) = (T(u))q$, for any $u \in \mathbb{H}^n$ and $q \in \mathbb{H}$, and $T$ can be represented by an $m \times n$ quaternionic matrix acting on the left.

We now list some further quaternionic notions.

**Definition 1.1.** The $\mathbb{H}$-inner product $\langle \ , \ \rangle$ is given by

$$\langle \ , \ \rangle: \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}, \quad (u, v) \mapsto \langle u, v \rangle = u^*v,$$

where $u$ and $v$ are column vectors and $u^*$ is the transpose-conjugate of $u$.

The $\mathbb{H}$-inner product has the following properties:

- $\langle u, u \rangle = \|u\|^2 \geq 0$, and $u = 0$ if $\langle u, u \rangle = 0$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$
• \[ \langle u + u', v \rangle = \langle u, v \rangle + \langle u', v \rangle \]

• \[ \langle ua, v \rangle = \bar{a} \langle u, v \rangle, \quad \langle u, vb \rangle = \langle u, v \rangle b \quad (\mathbb{H}\text{-linear in the second argument}) \]

for \( a, b \in \mathbb{H}, \ u, u', v \in \mathbb{H}^n \). The standard norm on \( \mathbb{H}^n \), or the length of \( u \in \mathbb{H}^n \), is the non-negative real number given by

\[ \|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\bar{u}u} \geq 0. \]

The norm \( \|\cdot\| : \mathbb{H}^n \to \mathbb{R} \geq 0 \) satisfies \([20]\) the triangle inequality \( \|u + v\| \leq \|u\| + \|v\| \).

Given some subspace \( W \subset \mathbb{H}^n \), the orthogonal complement is

\[ W^\perp = \{ v \in \mathbb{H}^n \mid \langle u, v \rangle = 0, \ \forall u \in W \}, \]

and \( u \) and \( v \) are called orthogonal if \( \langle u, v \rangle = 0 \).

Recall that the quaternionic projective space \( \mathbb{HP}^n \) is the space of lines in \( \mathbb{H}^{n+1} \) passing through the origin, with the following metric.

**Definition 1.2** (\([13]\)). The chordal distance between two points of \( \mathbb{HP}^n \) is

\[ d([u], [v]) \overset{\text{def}}{=} \sqrt{1 - |\langle u, v \rangle|^2} \quad \text{with} \quad 0 \leq d([u], [v]) \leq 1 \]

where \( u \) and \( v \) are unit vectors in \( \mathbb{H}^{n+1} \).

This metric is well-defined because

\[ \sqrt{1 - |\langle u, v \rangle|^2} = \sqrt{1 - |\langle ua, vb \rangle|^2} \]

for any unit quaternions \( a \) and \( b \), and clearly satisfies

\[ d([u], [v]) = d([v], [u]). \]

A matrix \( A \in M_{\mathbb{H}}(n) \) is said to be invertible if \( AB = BA = I \) for some \( B \in M_{\mathbb{H}}(n) \). Let \( GL_{\mathbb{H}}(n) \subset M_{\mathbb{H}}(n) \) be the subgroup of invertible \( n \times n \) matrices with respect to matrix multiplication.

**Definition 1.3.** For any \( n \geq 1 \), the (quaternionic) symplectic group is the compact Lie group

\[ Sp(n) = \{ A \in GL_{\mathbb{H}}(n) \mid A^*A = AA^* = I \} \]

where \( I \) is the \( n \times n \) identity matrix and \( A^* \) is the conjugate-transpose of \( A \).
Example 1.4. For \( n = 1 \),

\[
Sp(1) = \{ q \in \mathbb{H} \mid |q| = 1 \}
\]

is the group of unit quaternions, which is naturally identified with the 3 dimensional sphere \( S^3 \subset \mathbb{R}^4 \cong \mathbb{H} \).

A matrix \( A \in M_{\mathbb{H}}(n) \) is called Hermitian if \( A^* = A \). We list below some facts about quaternionic matrices.

**Theorem 1.5 ([58]).** For the quaternionic matrices \( A \) and \( B \), the following hold.

1. \( (A)^t = (A^t) \)
2. \( (AB)^* = B^*A^* \)
3. \( AB \neq BA \) in general
4. \( (AB)^t \neq B^tA^t \) in general
5. \( (AB)^{-1} = B^{-1}A^{-1} \) if \( A \) and \( B \) are invertible
6. \( (A^*)^{-1} = (A^{-1})^* \) if \( A \) is invertible
7. \( (A)^{-1} \neq (A^{-1})^t \) in general
8. \( (A^t)^{-1} \neq (A^{-1})^t \) in general

where \( A^t \) is the transpose of \( A \) and \( (A)^t = A^* \).

### 1.3 Symmetric Squares

In this section we introduce the symmetric square \( SP^2(M) \) of a topological space \( M \).

In our case, \( M \) is always a CW complex, and usually a manifold; so \( SP^2(M) \) is also CW [2, Section 5.2] [47].

The \( n \)-th symmetric product \( SP^n(M) \) of a space \( M \) is the quotient of the \( n \)-fold Cartesian product \( M^n = M \times \cdots \times M \) by the action of the symmetric group of degree \( n \), which permutes the coordinates. An element of \( SP^n(M) \) is an unordered \( n \)-tuple \([x_1, x_2, \cdots, x_n]\) of points \( x_i \in M \). Even if \( M \) is a manifold, however, \( SP^2(M) \) is usually not. Exceptions include the examples [28]

\[
SP^n(S^1) \cong S^1, \quad SP^n(\mathbb{C}) \cong \mathbb{C}^n, \quad SP^n(S^2) \cong \mathbb{C}P^n, \quad SP^n(\mathbb{R}P^2) \cong \mathbb{R}P^{2n}.
\]
CHAPTER 1. INTRODUCTION

The \( n \)-th symmetric product \( SP^n(M) \) of a manifold \( M \) is an orbifold \([1, \text{Example } 1.13]\); our main interest is when \( n = 2 \) and \( M = HP^n \).

**Definition 1.6.** The symmetric square \( SP^2(HP^n) \) of quaternionic projective space \( HP^n \) is the quotient

\[
SP^2(HP^n) = HP^n \times HP^n / \sim
\]

of the Cartesian square, where \((x, y) \sim (y, x)\) for any \( x, y \in HP^n \); that is

\[
SP^2(HP^n) = HP^n \times HP^n / \mathbb{Z}/2.
\]

The case \( n = \infty \) may also be considered, and is important.

Following \([2, \text{Section } 5.2]\), the sequence of inclusions \( HP^n \subset HP^{n+1} \) for \( n \geq 1 \) gives inclusions \( SP^2(HP^n) \subset SP^2(HP^{n+1}) \), whose colimit is

\[
SP^2(HP^\infty) \overset{\text{def}}{=} \bigcup_n SP^2(HP^n).
\]

So \( SP^2(HP^\infty) \) has the union or colimit topology.

### 1.4 The Main Results in Cohomology

In this section, we formulate our main theorem, which describes the integral cohomology ring of the symmetric square of quaternionic projective space.

For some infinite unions \( X = \bigcup_n X_n \), the cohomology \( H^*(X_n) \) is better described in terms of \( H^*(X) \). For example, \( H^*(HP^n) \) is usually expressed as the truncation \( \mathbb{Z}[z]/(z^{n+1}) \), \( |z| = 4 \) of \( H^*(HP^\infty^n) \cong \mathbb{Z}[z] \). We shall apply the same approach to the symmetric square, by first computing the cohomology ring \( H^*(SP^2(HP^\infty^n))_+ \), and then expressing \( H^*(SP^2(HP^n))_+ \) as its truncation.

The same procedure gives \( H^*(CP^n) \) as \( \mathbb{Z}[c_1]/(c_1^{n+1}) \).

In these descriptions, there are two possible choices of \( z \), which differ by sign. In this thesis we follow \([40, \text{Theorem } 5.6]\), so the natural map \( q: CP^\infty \to HP^\infty \) induces \( q^*(z) = c_1^2 \) (not \(-c_1^2\)) in integral cohomology.

From now on, we often write:

- \( SP^2 \) for \( SP^2(HP^\infty) = \bigcup_n SP^2(HP^n) \)
- \( SP^2_n \) for \( SP^2(HP^n) \).
Our description of \( H^*(SP^2_+):= H^*(SP^2_+;\mathbb{Z}) \) is as follows.

**Theorem 1.7.** The ring \( H^*(SP^2_+) \) is isomorphic to
\[
\mathbb{Z} \left[ (\frac{1}{2})^{s-1}h^s, \frac{1}{2}mg^f h^m, t_{i,j} : s, \ell, i \geq 1, m \geq 0, 1 \leq j < 2i \right] / \mathcal{I}
\]
where \(|g|=4, |h|=8, |t_{i,j}| = 4i + 2j + 1, \) and
\[
\mathcal{I} = (2t_{i,j}, t_{i,j}t_{k,l}, t_{i,j}(\frac{1}{2})^{s-1}h^s, t_{i,j}(\frac{1}{2})^m g^f h^m : k \geq 1, 1 \leq l < 2k).
\]

The torsion-free product structure is indicated by the notation \((\frac{1}{2})^{s-1}h^s, \frac{1}{2}mg^f h^m\) as follows.

For any \(s, s', \ell, \ell' \geq 1\) and \(m, m' \geq 0,\)
- \((\frac{1}{2})^{s-1}h^s \cdot (\frac{1}{2})^{s'-1}h^{s'} = 2 \cdot (\frac{1}{2})^{s+s'-1}h^{s+s'}\)
- \((\frac{1}{2})^m g^f h^m \cdot (\frac{1}{2})^{s-1}h^s = 2 \cdot (\frac{1}{2})^{m+s} g^f h^{m+s}\)
- \((\frac{1}{2})^m g^f h^m \cdot (\frac{1}{2})^{m'} g^f h^{m'} = (\frac{1}{2})^{m+m'} g^f h^{m+m'}\).

For example, \(g \cdot h = 2 \cdot \frac{1}{2} gh\) in \(H^{12}(SP^2_+)\) and \(h \cdot h = 2 \cdot \frac{1}{2} h^2\) in \(H^{16}(SP^2_+)\).

To truncate the above description in Theorem 1.7, we consider the homomorphisms induced by the inclusions of subspaces \(\mathbb{HP}^n \subset \mathbb{HP}^\infty\), to obtain the following.

**Theorem 1.8.** For \(n \geq 1\), the integral cohomology ring is
\[
H^*((SP^2_n)_+) \cong H^*((SP^2_+)\,(\theta_{n,1}, \theta_{m,2}, t_{i,j} : m \geq n, i \geq n + 1))
\]
where polynomials \(\theta_{m,1}\) and \(\theta_{m,2}\) are given by
\[
\begin{align*}
\theta_{1,1} &= g^2 - h \\
\theta_{1,2} &= \frac{1}{2}gh \\
\text{for } m = 1, \text{ and} \quad \begin{cases} 
\theta_{m,1} = g \theta_{m-1,1} - \theta_{m-1,2} & \text{for } m \geq 2. \\
2 \theta_{m,2} &= h \theta_{m-1,1}
\end{cases}
\end{align*}
\]

If \(n > 8n\), then the cohomology group \(H^*((SP^2_n)_+)\) is zero for cellular reasons [47], and we will confirm this in Section 8.1.1.

As a typical example, we look at the case \(n = 3\).

**Example 1.9.** We have
\[
H^*((SP^2_3)_+) \cong H^*((SP^2_3)_+)\,(\theta_{3,1}, \theta_{m,2}, t_{i,j} : m \geq 3, i \geq 4).
\]

Table 1.1 displays this information in an alternative form.
Table 1.1: $H^* := H^*((SP^2_3)^+_1; \mathbb{Z})$ with generators

<table>
<thead>
<tr>
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<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$t_{1,1}$</td>
<td>$g^2$, $h$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H^*$</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$t_{2,1}$</td>
<td>$g^3$, $\frac{1}{2}gh$</td>
<td>$t_{2,2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H^*$</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td></td>
<td>$t_{2,3}$, $t_{3,1}$</td>
<td>$\frac{1}{2}g^2h$, $\frac{1}{2}h^2$</td>
<td>$t_{3,2}$</td>
<td>$t_{3,3}$</td>
<td></td>
</tr>
<tr>
<td>$H^*$</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}gh^2$</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td></td>
<td>$t_{3,4}$</td>
<td>$t_{3,5}$</td>
<td>$\frac{1}{4}h^3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For $* \geq 25$ the cohomology is 0. The relations are

\[ g^4 = 4 \cdot \frac{1}{2}g^2h - \frac{1}{2}h^2, \quad \frac{1}{2}g^3h = 3 \cdot \frac{1}{4}gh^2, \]

\[ \frac{1}{2}g^4h - 4 \cdot \frac{1}{4}g^2h^2 + \frac{1}{4}h^3 = 0, \]

\[ t_{i,j} \alpha = 0 \text{ for any generator } \alpha \neq 1. \]
Chapter 2

Decompositions and Borel Space

The existence of a rational isomorphism $H^*(B_n; \mathbb{Q}) \cong H^*(SP^2(\mathbb{H}P^n); \mathbb{Q})$, where $B_n$ is the Borel space $B_n := S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}P^n \times \mathbb{H}P^n)$ is well known [1, page 38]. This will play an important part in our computation of the integral cohomology ring $H^*(SP^2(\mathbb{H}P^n))$. In this Chapter we will show how the Borel space $B_n$ and the symmetric square $SP^2(\mathbb{H}P^n)$ are related geometrically. We decompose these spaces into subspaces, which yield a commutative ladder of cofiber sequences. Our cohomological computation of the symmetric square $SP^2(\mathbb{H}P^n)$ is based on these commutative ladders.

2.1 The Decomposition of $SP^2(\mathbb{H}P^n)$

In this section we describe subspaces of the symmetric square $SP^2_n$ through the chordal distance on $\mathbb{H}P^n$,

$$d([u], [v]) = \sqrt{1 - |\langle u, v \rangle|^2}, \quad 0 \leq d([u], [v]) \leq 1$$

of Definition 1.2, where $u$ and $v$ are unit vectors in $\mathbb{H}^{n+1}$.

We shall often write an element of $SP^2_n$ as $[x, y]$ where $x, y \in \mathbb{H}P^n$.

**Definition 2.1.** There are subspaces of the symmetric square $SP^2_n$ given by

- $\triangle_n = \{[u, v] \in SP^2_n \mid d([u], [v]) = 0\}$
- $N_n = \{[u, v] \in SP^2_n \mid 0 \leq d([u], [v]) \leq 1/2\}$
- $A_n = \{[u, v] \in SP^2_n \mid d([u], [v]) = 1/2\}$
- $L_n = \{[u, v] \in SP^2_n \mid 1/2 \leq d([u], [v]) \leq 1\}$
- $\Gamma_n = \{[u, v] \in SP^2_n \mid d([u], [v]) = 1\}$.
CHAPTER 2. DECOMPOSITIONS AND BOREL SPACE

Having defined these subspaces, we may decompose $SP^2_n$ as

$$SP^2_n = L_n \cup A_n \cap N_n \tag{2.2}$$

with

$$\Gamma_n \subset L_n, \quad \Delta_n \subset N_n, \quad A_n = L_n \cap N_n,$$

and the decomposition in turn yields a commutative diagram

$$
\begin{array}{ccc}
A_n & \longrightarrow & L_n \\
\downarrow & & \downarrow \\
N_n & \longrightarrow & SP^2_n \\
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccc}
L_n & \longrightarrow & L_n/A_n \\
\downarrow \cong & & \downarrow \cong \\
SP^2_n & \longrightarrow & SP^2_n/N_n \\
\end{array}
\tag{2.3}
$$

with a canonical homeomorphism $L_n/A_n \cong SP^2_n/N_n$.

We should note that:

- there is a homeomorphism $\mathbb{H}P^n \cong \Delta_n$ given by $z \mapsto [z, z] \in \Delta_n$ for any $z \in \mathbb{H}P^n$; we call the subspace $\Delta_n \subset SP^2_n$ the diagonal

- the complement $SP^2_n \setminus \Delta_n$ consists of unordered pairs of distinct points on $\mathbb{H}P^n$.

Some additional properties and descriptions of subspaces of the symmetric square $SP^2_n$ come from [29, page 585], as follows. We refer the reader to Appendix B.1 for definition of NDR.

**Proposition 2.4** ([29], Corollary 4.2). Assume that $M$ is a closed smooth manifold. A neighbourhood deformation retract NDR of the diagonal $M$ in the symmetric square $SP^2(M)$ is homeomorphic to the fibrewise cone on the projectivised tangent bundle of $M$.

The point about this result is that the $\mathbb{Z}/2$ action on $M \times M$ restricts to the fibrewise antipodal action on the normal disk bundle to the diagonal, which is isomorphic to the tangent disk bundle of $M$.

From [29, Proposition 4.1, Corollary 4.2], we describe below the case $M = \mathbb{H}P^n$. Let $\tau$ be the tangent bundle of $\mathbb{H}P^n$, and let $D(\tau)$ and $S(\tau)$ be the total spaces of the unit disk and sphere bundles. Then
(i) the subspace $N_n \subset SP^2_n$ is homeomorphic to the total space of the fibrewise quotient bundle
\[
C\mathbb{RP}^{4n-1} \rightarrow D(\tau)/\mathbb{Z}/2 \rightarrow \mathbb{HP}^n, \quad (N_n \cong D(\tau)/\mathbb{Z}/2 \simeq \mathbb{HP}^n \cong \triangle_n)
\]
with fibre the cone $C\mathbb{RP}^{4n-1}$, the projection is the retraction of Proposition 2.4, and the section given by the cone points is its homotopy inverse,

(ii) the subspace $A_n \subset N_n$ is homeomorphic to the boundary $S(\tau)/\mathbb{Z}/2$ of $D(\tau)/\mathbb{Z}/2$, which is the total space $\mathbb{RP}(\tau)$ of the real projectivisation of $\tau$ with fibre $\mathbb{RP}^{4n-1}$
\[
\mathbb{RP}^{4n-1} \rightarrow \mathbb{RP}(\tau) \rightarrow \mathbb{HP}^n.
\]

**Proposition 2.5** ([14], 5.4.4, page 114). A pair $(X,A)$ is an NDR if and only if the inclusion $A \hookrightarrow X$ is a closed cofibration.

We then deduce that the inclusion $\triangle_n \hookrightarrow SP^2_n$ is a closed cofibration by Proposition 2.4 and Proposition 2.5. We shall confirm this property below by applying Illman’s triangulation results [26, Theorem 3.6–3.8], which are useful in several ways.

**Theorem 2.6** (Illman, 1978). Let $G$ be a finite group, $M$ a smooth $G$-manifold, and $M'$ a closed smooth $G$-submanifold. Then there exists a smooth equivariant triangulation of the pair $(M,M')$, and any triangulation is unique up to subdivision.

We always use this theorem with $G = \mathbb{Z}/2$. It give us, for example

**Lemma 2.7.** For any $n \geq 1$, the inclusions (1) $\triangle_n \hookrightarrow N_n$, and (2) $N_n \hookrightarrow SP^2_n$ are both cofibrations.

*Proof.* (1) Given $N_n = N'_n/\mathbb{Z}/2$, where $N'_n \cong D(\tau)$. Apply Theorem 2.6 with $M = N'_n \subset \mathbb{HP}^n \times \mathbb{HP}^n$ and $M' = \triangle_n$. Then $\triangle_n$ may be realised as a $\mathbb{Z}/2$ equivariant simplicial subcomplex of $N'_n$. Factoring out $\mathbb{Z}/2$, which acts trivially on $\triangle_n$, gives $\triangle_n$ as a simplicial (or CW) subcomplex of $N_n$, so (1) is true.

(2) Apply Theorem 2.6 with $M = \mathbb{HP}^n \times \mathbb{HP}^n$ and $M' = N'_n$. The rest of the proof is then the same as (1). \qed

We deduce from Lemma 2.7 that the composition $\triangle_n \hookrightarrow SP^2_n$ is also a cofibration [2, page 100].
Lemma 2.8. For \( n \geq 1 \), the map \( s: SP^2_n/\Delta_n \to SP^2_n/N_n \) is a homotopy equivalence.

Proof. The Gluing Lemma (Lemma A.1) \([9,10,37]\) applies to the diagram

\[
\begin{array}{ccc}
\{\bullet\} & \xrightarrow{p} & \Delta_n \xrightarrow{i} SP^2_n \\
\downarrow & \simeq & \downarrow \text{id} \\
\{\bullet\} & \xleftarrow{p'} & N_n \xrightarrow{i'} SP^2_n
\end{array}
\]  

(2.9)

since \( i \) and \( i' \) are closed cofibrations, where \( p \) and \( p' \) are projections onto one-point spaces, so the corresponding map of pushouts (or colimits) \( s: SP^2_n/\Delta_n \to SP^2_n/N_n \) is a homotopy equivalence. \( \square \)

Remark 2.10. The homotopy equivalence \( s: SP^2_n/\Delta_n \to SP^2_n/N_n \) combined with the homeomorphism of diagram (2.3) gives a canonical homotopy equivalence

\[
SP^2_n/\Delta_n \xrightarrow{\sim} SP^2_n/N_n \xleftarrow{\sim} L_n/A_n.
\]  

(2.11)

Basically, this map involves collapsing \( N_n/\Delta_n \subset SP^2_n/\Delta_n \), which is a contractible subcomplex.

2.2 Comparison of \( B_n \) and \( SP^2_n \)

In this section we compare the Borel space \( B_n = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}^n \times \mathbb{H}^n) \) and the symmetric square \( SP^2_n \). It is known that the space \( B_n \) and \( SP^2_n \) are closely related, and the correspondence is well studied. For example, the following properties come from \([8, \text{ page 371, Proposition 1.1}]\).

Let \( \Delta_n \cong \mathbb{H}^n \) be the diagonal \( \Delta_n \subset SP^2_n \). Then the projection of pairs

\[
(S^\infty \times (\mathbb{H}^n \times \mathbb{H}^n), S^\infty \times \Delta_n) \longrightarrow (\mathbb{H}^n \times \mathbb{H}^n, \Delta_n)
\]

induces a map of orbit spaces

\[
(S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}^n \times \mathbb{H}^n), \mathbb{R}P^\infty \times \Delta_n) \longrightarrow (SP^2_n, \Delta_n),
\]  

(2.12)

and the induced homomorphism in Čech cohomology

\[
\tilde{H}^*(SP^2_n, \Delta_n) \longrightarrow \tilde{H}^*(S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}^n \times \mathbb{H}^n), \mathbb{R}P^\infty \times \Delta_n)
\]  

(2.13)
is an isomorphism for any coefficients. Since we are working with CW complexes, where Čech and singular cohomology agree, our approach in this section is to describe the geometry of $B_n$ and $SP^2_n$ in enough detail to understand why the map of quotient spaces induced by (2.12) is a homotopy equivalence. Then (2.13) may be rewritten as an isomorphism
\[ H^*(SP^2_n/\triangle_n) \longrightarrow H^*(S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}P^n \times \mathbb{H}P^n)/\triangle_n \times \mathbb{R}P^\infty) \] (2.14)
in singular cohomology. We will apply this in Chapter 7 and Chapter 8.

Our description of $B_n$ and $SP^2_n$ includes the decompositions of these spaces as follows.

The Borel space $B_n$ is related to the symmetric square $SP^2_n$ by a projection
\[ \pi : B_n \rightarrow SP^2_n \quad \text{given by } \quad [v, (z_1, z_2)] \mapsto [z_1, z_2] \] (2.15)
where $v \in S^\infty$ and $z_1, z_2 \in \mathbb{H}P^n$. This lifts the decomposition of $SP^2_n = L_n \cup A_n \ N_n$ to a decomposition $B_n = \hat{L}_n \cup \hat{A}_n \hat{N}_n$, where
\[ \hat{\triangle}_n = \pi^{-1}(\triangle_n), \quad \hat{N}_n = \pi^{-1}(N_n), \quad \hat{A}_n = \pi^{-1}(A_n), \quad \text{and} \quad \hat{L}_n = \pi^{-1}(L_n). \]

So the following properties hold:

- $\hat{A}_n = \hat{L}_n \cap \hat{N}_n$
- $\hat{N}_n$ is a closed neighbourhood of the diagonal $\hat{\triangle}_n := \{[v, (z, z)]\} \subset B_n$
- $\hat{L}_n$ is the complement of the interior of $\hat{N}_n$ in $B_n$
- the decomposition yields the commutative diagram
\[ \begin{array}{ccc}
\hat{A}_n & \longrightarrow & \hat{L}_n \\
\downarrow & & \downarrow \\
\hat{N}_n & \longrightarrow & B_n \\
\end{array} \xrightarrow{\cong} \begin{array}{ccc}
\hat{A}_n & \longrightarrow & \hat{L}_n \\
\downarrow & & \downarrow \\
\hat{N}_n & \longrightarrow & B_n \\
\end{array} \]
(2.16)
with a canonical homeomorphism $\hat{L}_n/\hat{A}_n \cong B_n/\hat{N}_n$, which lifts the homeomorphism of (2.3).

We shall use the disk bundle $D^{4n} \rightarrow D(\tau) \rightarrow \mathbb{H}P^n \cong \triangle_n$ to identify $\hat{N}_n$ with $S^\infty \times_{\mathbb{Z}/2} D(\tau) \subset B_n$. The projection of the disk bundle
\[ D^{4n} \rightarrow S^\infty \times_{\mathbb{Z}/2} D(\tau) \rightarrow S^\infty \times_{\mathbb{Z}/2} \triangle_n \] (2.17)
therefore gives a homotopy equivalence \( \hat{N}_n \simeq \hat{\triangle}_n \), which is compatible with \( N_n \simeq \triangle_n \) under projection \( \pi \).

**Remark 2.18.** The map

\[
\mathbb{R}P^\infty \times \mathbb{H}P^n \to \hat{\triangle}_n, \quad ([v], z) \mapsto [v, (z, z)]
\]

is a homeomorphism, where \([v] \in \mathbb{R}P^\infty\), \(z \in \mathbb{H}P^n\) and \([v, (z, z)] \in \hat{\triangle}_n\).

Analogous to Lemma 2.7, there are some additional properties as follows.

**Lemma 2.19.** For any \( n \geq 1 \), the inclusions (1) \( \hat{\triangle}_n \hookrightarrow \hat{N}_n \) and (2) \( \hat{N}_n \hookrightarrow B_n \) are both cofibrations.

**Proof.** (1) For each \( n \geq 1 \), there is a standard equivariant cell decomposition of \( S^n \) with respect to the antipodal action. These are compatible for increasing \( n \), and their union is an equivariant CW structure on \( S^\infty \). Now take the subcomplex \( \triangle_n \subset N'_n \) from the proof of Lemma 2.7 and form the subcomplex \( S^\infty \times \mathbb{Z}/2 \triangle_n \subset S^\infty \times \mathbb{Z}/2 N'_n \). These are homeomorphic to \( \hat{\triangle}_n \subset \hat{N}_n \), so the inclusion map is a cofibration.

(2) Take the subcomplex \( N'_n \subset \mathbb{H}P^n \times \mathbb{H}P^n \), and repeat the method of (1). \( \square \)

We deduce from Lemma 2.19 that the composition \( \hat{\triangle}_n \hookrightarrow B_n \) is also a cofibration.

By making an analogy with Lemma 2.8, it follows that the map \( b_n : B_n/\hat{\triangle}_n \to B_n/\hat{N}_n \) is a homotopy equivalence. As in Remark 2.10, \( b_n \) combines with the homeomorphism of diagram (2.16) to give a canonical homotopy equivalence

\[
B_n/\hat{\triangle}_n \xrightarrow{\simeq} B_n/\hat{N}_n \xrightarrow{\simeq} \hat{L}_n/\hat{A}_n.
\]

This is compatible with the equivalence (2.11) under the projection \( \pi \), and comes from collapsing the contractible subcomplex \( \hat{N}_n/\hat{\triangle}_n \subset B_n/\hat{\triangle}_n \).

Notice that \( \pi \) is a double covering away from the diagonal, so \( \mathbb{Z}/2 \) acts freely on \( \hat{A}_n \) and \( \hat{L}_n \). This leads to the following additional facts.

**Remark 2.21.** The restricted projections \( \pi : \hat{A}_n \to A_n \) and \( \pi : \hat{L}_n \to L_n \), which are described by \([v, (z_1, z_2)] \mapsto [z_1, z_2] \) for \( z_1 \neq z_2 \), are homotopy equivalences, since they appear in fibrations \( S^\infty \to \hat{A}_n \to A_n \) and \( S^\infty \to \hat{L}_n \to L_n \), and \( S^\infty \) is contractible.
Having described the decompositions of both $SP^2_n$ and $B_n$, we now display a very important commutative diagram

\[
\begin{array}{cccc}
\Delta_n & \xrightarrow{i} & SP^2_n & \xrightarrow{\pi} \frac{SP^2_n}{\Delta_n} \\
\downarrow{\pi} & & \downarrow{\pi} & \downarrow{\pi_n} \\
\hat{\Delta}_n & \xrightarrow{f} & B_n & \xrightarrow{\pi} \frac{B_n}{\hat{\Delta}_n}
\end{array}
\]

where we now know that each row is a cofibre sequence.

**Proposition 2.23.** For any $n \geq 1$, the map $\pi_n : B_n/\hat{\Delta}_n \rightarrow \frac{SP^2_n}{\Delta_n}$ is a homotopy equivalence.

**Proof.** By combining the equivalences (2.11) and (2.20), we need only prove that the quotient projection $\pi : \hat{L}_n/\hat{A}_n \rightarrow L_n/A_n$ is a homotopy equivalence. So consider the commutative diagram

\[
\begin{array}{ccc}
\{\bullet\} & \xleftarrow{\hat{p}} & \hat{A}_n & \xrightarrow{\pi} \hat{L}_n \\
\downarrow{\pi} & & \downarrow{\pi} \\
\{\bullet\} & \xleftarrow{\hat{p}} & A_n & \xrightarrow{\pi} L_n
\end{array}
\]

where $p$ and $\hat{p}$ are projections onto one-point spaces. By Remark 2.21, both vertical maps $\pi$ are homotopy equivalences. Also, the inclusions $A_n \hookrightarrow L_n$ and $\hat{A}_n \hookrightarrow \hat{L}_n$ are cofibrations, by the same arguments as Lemma 2.7 and 2.19 \footnote{Or, alternatively, because $A_n$ is the boundary of the smooth manifold $L_n$, and $\hat{A}_n = \pi^{-1}(A_n)$ under the fibration $\pi|_{L_n}$ [14, page 117].}. So we can apply the gluing Lemma to the diagram, and deduce that the induced map of pushouts (or colimits) is a homotopy equivalence, as we needed. \qed

### 2.3 Infinite Cases

So far, our main focus has been on sequences of spaces $X_n$ that are constructed from $\mathbb{HP}^n$. Analogous to the case $SP^2 = \bigcup_n SP^2_n$ [2, Section 5.2], it is convenient to study versions that arise in the same way from $\mathbb{H}P^\infty$, by forming $X = \bigcup_n X_n$ with the colimit topology. Examples include

\[
\hat{\Delta} = \bigcup_n \hat{\Delta}_n \cong \mathbb{RP}^\infty \times \mathbb{HP}^\infty \quad \text{and} \quad B = \bigcup_n B_n.
\]

Each of these is based on the definition of $\mathbb{H}P^\infty$ as $\bigcup_n \mathbb{H}^n$ where $\mathbb{H}^n \subset \mathbb{H}^{n+1}$ comes from including $(q_1, q_2, \ldots, q_n)$ as $(q_1, q_2, \ldots, q_n, 0)$ for any $q_i \in \mathbb{H}$.\footnote{Or, alternatively, because $A_n$ is the boundary of the smooth manifold $L_n$, and $\hat{A}_n = \pi^{-1}(A_n)$ under the fibration $\pi|_{L_n}$ [14, page 117].}
**Proposition 2.25.** The inclusion $\triangle \hookrightarrow SP^2$ is a cofibration, where $\triangle \cong \mathbb{H}P^\infty$.

**Proof.** Illman’s triangulation results may be applied to the $\mathbb{Z}/2$-equivariant pairs 

$$(\triangle_n, \mathbb{H}P^n \times \mathbb{H}P^n) \subset (\triangle_{n+1}, \mathbb{H}P^{n+1} \times \mathbb{H}P^{n+1}),$$

for any $n \geq 1$. After subdivision, if necessary, the commutative diagram

$$
\begin{array}{ccc}
\triangle_n & \overset{\subset}{\longrightarrow} & \triangle_{n+1} \\
\downarrow & & \downarrow \\
SP^2_n & \overset{\subset}{\longrightarrow} & SP^2_{n+1}
\end{array}
$$

of subcomplexes arises; so all four inclusions are cofibrations. Also, the square is a pullback because $\triangle_{n+1} \cap SP^2_n$ in $SP^2_{n+1}$ is homeomorphic to $\triangle_n$. So by [33, Lemma 3.2], the map of colimits $\triangle \hookrightarrow SP^2$ is also a cofibration. \hfill $\square$

For future use, note from the considerations above that the diagram

$$
\begin{array}{ccc}
B_n/\hat{\triangle}_n & \overset{\pi}{\rightarrow} & SP^2_n/\triangle_n \\
\downarrow & & \downarrow \\
B_{n+1}/\hat{\triangle}_{n+1} & \overset{\pi}{\rightarrow} & SP^2_{n+1}/\triangle_{n+1}
\end{array}
$$

is also commutative for every $n$.

**Remark 2.28.** The inclusion $f: \hat{\triangle} \hookrightarrow B$ is also a cofibration, by starting with the pairs

$$S^\infty \times (\triangle_n, \mathbb{H}P^n \times \mathbb{H}P^n) \subset S^\infty \times (\triangle_{n+1}, \mathbb{H}P^{n+1} \times \mathbb{H}P^{n+1}),$$

and following the method of Lemma 2.19.

Therefore, the maps $i$ and $f$ of the commutative diagram

$$
\begin{array}{ccc}
\triangle & \overset{i}{\longrightarrow} & SP^2 \\
\downarrow & & \downarrow \\
\hat{\triangle} & \overset{f}{\longrightarrow} & B
\end{array}
\quad \begin{array}{ccc}
SP^2/\triangle & \overset{\pi}{\longrightarrow} & SP^2/\hat{\triangle} \\
\downarrow \pi & & \downarrow \pi \\
\pi_\infty
\end{array}
$$

are closed cofibrations. This diagram is the colimit of the diagrams (2.22) as $n \rightarrow \infty$.

So using the gluing lemma one more time yields

**Corollary 2.30.** The map $\pi_\infty: B/\hat{\triangle} \rightarrow SP^2/\triangle$ in diagram (2.29) is a homotopy equivalence.
Alternatively, this also follows from Proposition 2.25, Ladder Lemma A.2, and [33, Proposition 2.5].

We have now shown that $\pi^*_\infty$ is an isomorphism in the commutative ladder of long exact sequences induced by (2.29) with any coefficients.

As we shall see, the relationship between $B$ and $SP^2$ is interesting in other ways. For example, there exist homotopy equivalences $B \simeq \Gamma = \bigcup_n \Gamma_n$ and $\widehat{\Delta} \simeq A = \bigcup_n A_n$; also, the quotient space $SP^2/\Delta$ can be viewed as a Thom space over $\Gamma$.  

Chapter 3

The Classifying Space of $Pin(4)$

This chapter is mainly concerned with the classifying space $BPin(4)$. We will see that both the Borel space $B$ and the space $\Gamma$ are models for $BPin(4)$. Over the last 50 years, the $Pin$ group and its subgroup $Spin$ have attracted many mathematicians and physicists; as a result there exist different approaches to the $Pin$ and $Spin$ groups depending on their research interests. The standard definitions of these groups are usually given in terms of Clifford algebras; physicists may also work with gamma matrices, otherwise known as Dirac matrices. We shall exploit different interpretations and descriptions of the same group $Pin(4)$. We shall also describe two actions of $Pin(4)$ on $\mathbb{R}^4$, and consider the associated vector bundles over $BPin(4)$.

3.1 $Pin$ and $Spin$ Groups

In this section we review certain properties of the groups $Pin(n)$ and $Spin(n)$. These groups act naturally on $\mathbb{R}^n$, in a way that is usually defined in terms of multiplication in the Clifford algebra associated with $\mathbb{R}^n$. There are many references on these topics, which include [3,19,34,49].

We shall recall some classical low dimensional results as follows.
Proposition 3.1. [50, Cor.13.60] There are isomorphisms

\[ \begin{align*}
\text{Spin}(1) &\cong O(1) \cong S^0 \cong \mathbb{Z}/2 \\
\text{Spin}(2) &\cong U(1) \cong S^1 \\
\text{Spin}(3) &\cong Sp(1) \cong S^3 \\
\text{Spin}(4) &\cong Sp(1) \times Sp(1) \cong S^3 \times S^3 \\
\text{Spin}(5) &\cong Sp(2)
\end{align*} \]

of compact Lie groups.

From [5], we have some useful and important facts:

- the inclusion \( \text{Spin}(4) \hookrightarrow \text{Spin}(5) \) can be described as the embedding of the diagonal matrices \( Sp(1) \times Sp(1) \rightarrow Sp(2) \)

- the normaliser of \( Sp(1) \times Sp(1) \) in \( Sp(2) \) consists of all matrices

\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & a \\
b & 0
\end{pmatrix},
\]

where \( a \) and \( b \) are unit quaternions

- the normaliser of \( Sp(1) \times Sp(1) \) in \( Sp(2) \) is the wreath product of \( Sp(1) \) and \( \mathbb{Z}/2 \); that is, the semidirect product of \( Sp(1) \times Sp(1) \) with \( \mathbb{Z}/2 \) (Appendix B.2).

In this context, \( \mathbb{Z}/2 \) is generated by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

The \( \text{Pin} \) and \( \text{Spin} \) groups are often described in terms of Clifford algebra. The real Clifford algebra \( \text{Cl}_n \) associated to \( \mathbb{R}^n \) with the orthonormal basis \( \{e_1, \ldots, e_n\} \) is spanned by

\[
\{e_{i_1}e_{i_2} \cdots e_{i_k} \mid 0 \leq k \leq n \text{ and } i_1 < \cdots < i_k\}
\]

so the dimension of \( \text{Cl}_n \) is \( 2^n \). More precisely [19], given a vector space \( \mathbb{R}^{n=p+q} \) with a symmetric bilinear form \( Q_b : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R} \) and associated non-degenerate quadratic form \( Q(v) = Q_b(v, v) \) where \( Q \) is given by

\[
Q(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),
\]

then the Clifford algebra associated with \( \mathbb{R}^{p+q} \) and \( Q \) is a real algebra \( \text{Cl}_{p,q} \) together with a linear map \( i : \mathbb{R}^{p+q} \rightarrow \text{Cl}_{p,q} \) satisfying the condition\(^1\)

\[
i(v)^2 = Q(v) \cdot 1 \text{ for any } v \in \mathbb{R}^{p+q}
\]

\(^1\text{Lawson and Michelson [32] use the condition } i(v)^2 = -Q(v) \text{ to define the Clifford algebras, and our } \text{Cl}_{p,q} \text{ is their } \text{Cl}_{q,p} \text{ as a consequence of using the opposite sign convention.} \)
In this section we describe standard actions of Spin groups.

**Remark 3.2.** The normaliser of $Sp(1) \times Sp(1)$ in $Sp(2)$ (or equivalently, of Spin(4) in Spin(5)) is isomorphic to Pin(4).

### 3.2 The Actions of Spin(4), Pin(4) and Spin(5)

In this section we describe standard actions of Spin(4) $\subset$ Pin(4) $\subset$ Spin(5) on $\mathbb{R}^4$ and $\mathbb{R}^5$. The references include [5], which refers to the normaliser of Sp(1) $\times$ Sp(1) in Sp(2), or equivalently, to the wreath product of Sp(1) and $\mathbb{Z}/2$.

- Using the isomorphism $Spin(4) \cong Sp(1) \times Sp(1)$, the action of Spin(4) on $\mathbb{R}^4 \cong \mathbb{H}$ can be given by

  $$h \mapsto ahb^{-1} = ah\bar{b}$$
CHAPTER 3. THE CLASSIFYING SPACE OF PIN(4)

where \((a, b) \in Sp(1) \times Sp(1)\) and \(h \in \mathbb{H}\)

- using the isomorphism \(Spin(5) \cong Sp(2)\), the action of \(Spin(5)\) on \(\mathbb{R}^5\) can be given by conjugation \(AMA^{-1} = AMA^*\) on quaternionic Hermitian matrices of trace 0, \(M = \begin{pmatrix} r & h \\ \bar{h} & -r \end{pmatrix}\), where \(A \in Sp(2)\) and \((r, h) \in \mathbb{R}^5 \cong \mathbb{R} \oplus \mathbb{H}\)

- using the isomorphism \(Spin(4) \cong Sp(1) \times Sp(1)\) and the fact that \(Pin(4)\) is the normaliser of \(Sp(1) \times Sp(1)\) in \(Sp(2)\), the action of \(Pin(4)\) on \(\mathbb{R}^4 \cong \mathbb{H}\) can be given by writing \(Pin(4) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} | a, b \in Sp(1) \cong S^3 \}.\) Calculating gives

\[
\begin{pmatrix}
    a & 0 \\
    0 & b
\end{pmatrix}
\begin{pmatrix}
    r & h \\
    \bar{h} & -r
\end{pmatrix}
\begin{pmatrix}
    \bar{a} & 0 \\
    0 & \bar{b}
\end{pmatrix}
= \begin{pmatrix}
    r & a\bar{b} \\
    b\bar{a} & -r
\end{pmatrix},
\]

\[
\begin{pmatrix}
    0 & a \\
    b & 0
\end{pmatrix}
\begin{pmatrix}
    r & h \\
    \bar{h} & -r
\end{pmatrix}
\begin{pmatrix}
    0 & \bar{b} \\
    \bar{a} & 0
\end{pmatrix}
= \begin{pmatrix}
    -r & a\bar{b} \\
    b\bar{a} & r
\end{pmatrix},
\]

where \(r \in \mathbb{R}, h \in \mathbb{H} \cong \mathbb{R}^4\) and \(a, b\) are unit quaternions. So \(\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\) acts on \(h\) by \(h \mapsto a\bar{h}b\), where \(h \in \mathbb{H}\), and in particular, \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) acts on \(h\) by \(h \mapsto \bar{h}\). As required, the action of \(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\) is the action of \(Spin(4)\), that is, \(h \mapsto ah\bar{b}\).

3.3 Stiefel Manifolds and Grassmannians

In this section we summarise some basic properties of the quaternionic Steifel manifolds and Grassmannians; references include [22].

The quaternionic Steifel manifold \(V_H(k, m)\) is the space of orthonormal \(k\)-frames in \(\mathbb{H}^m\), and the quaternionic Grassmannian \(Gr_H(k, m)\) is the space of \(k\)-dimensional subspaces of \(\mathbb{H}^m\). These spaces have the structures of CW complexes. The infinite dimensional cases \(m = \infty\) are considered as the colimits \(V_H(k, \infty) = \bigcup_m V_H(k, m)\) and \(Gr_H(k, \infty) = \bigcup_m Gr_H(k, m)\). Related to these spaces we have fibre bundles

\[
Sp(k) \to V_H(k, m) \to Gr_H(k, m)
\]

\[
Sp(k) \to V_H(k, \infty) \to Gr_H(k, \infty)
\]

where \(Gr_H(k, -)\) is identified with \(V_H(k, -)/Sp(k)\). We shall describe the action of \(Sp(2)\) on \(V_H(2, -)\) in Section 3.4. The infinite dimensional Steifel manifold \(V_H(k, \infty)\) is contractible [22], so the fibre bundle (3.4) is a model for the universal \(G\)-bundle \(G \to EG \to BG\) where \(G = Sp(2)\) acts freely on a contractible space \(EG\).
CHAPTER 3. THE CLASSIFYING SPACE OF PIN(4)

Definition 3.5 ([40]). The base space $BG$ of the universal $G$-bundle $G \to EG \to BG$ is called a classifying space of $G$.

For all our examples, $BG$ can be chosen to be a CW complex, so it is unique up to homotopy equivalence [40, Cor 6.11]. In certain cases, it is necessary to consider more than one model at a time, and we need to take care with their relationship.

3.4 $BSpin(4)$, $BPin(4)$ and $BSpin(5)$

In this section we describe models for the classifying spaces of the compact Lie groups $Spin(4) \cong Sp(1) \times Sp(1)$, $Pin(4)$ and $Spin(5) \cong Sp(2)$.

For convenience we sometimes write:

- $V_{n+1}$ for $V_H(2, n + 1)$
- $V$ for $V_H(2, \infty) = \bigcup_m V_H(2, m)$.

The inclusions $Sp(1) \times Sp(1) \hookrightarrow Pin(4) \hookrightarrow Sp(2)$ induce the commutative diagrams

\[
\begin{array}{ccc}
Sp(1) \times Sp(1) & \longrightarrow & Pin(4) \longrightarrow Sp(2) \\
V_{n+1} & \longrightarrow & V_{n+1} \\
V_{n+1}/Sp(1) \times Sp(1) & \longrightarrow & V_{n+1}/Pin(4) \longrightarrow V_{n+1}/Sp(2)
\end{array}
\]

and

\[
\begin{array}{ccc}
Sp(1) \times Sp(1) & \longrightarrow & Pin(4) \longrightarrow Sp(2) \\
V & \longrightarrow & V \\
V/Sp(1) \times Sp(1) & \longrightarrow & V/Pin(4) \longrightarrow V/Sp(2)
\end{array}
\]

of bundle maps. The second diagram is the union of the first, over all $n$.

We write

\[
\begin{align*}
Sp(2) &= \{ A \in GL_H(2) \mid AA^* = A^*A = I \} \\
V &= \{(u,v) \mid u,v \in \mathbb{H}^\infty, \langle u,v \rangle = 0, |u| = |v| = 1 \},
\end{align*}
\]
so \((u v)\) is in \(V\) if and only if \((u v) \cdot (u v) = I\). Then define the action \(\phi : V \times \text{Sp}(2) \to V\) by the matrix product

\[
\phi((u v), A) = (u v)A
\]

for all \((u v) \in V\) and \(A \in \text{Sp}(2)\). This action is well defined, because

\[
((u v)A) (u v)A = A(u v)(u v)A = A^2 = I.
\]

The same formulae clearly restrict to the case \(V_{n+1}\).

If \(A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\) and \((u v)A = \left(\begin{array}{c} ua + vc \\ ub + vd \end{array}\right) = (u v)\), then \(A\) is the identity matrix; that is, the action of \(\text{Sp}(2)\) on \(V\) is free. Also, \(\text{Pin}(4)\) and \(\text{Sp}(1) \times \text{Sp}(1)\) act freely on the contractible space \(V\) since they are subgroups of \(\text{Sp}(2)\). So we may state

**Proposition 3.6.** The spaces \(V/(\text{Sp}(1) \times \text{Sp}(1))\), \(V/\text{Pin}(4)\), \(V/\text{Sp}(2)\) are models for the classifying spaces \(B(\text{Sp}(1) \times \text{Sp}(1))\), \(B\text{Pin}(4)\), \(B\text{Sp}(2)\), respectively.

We should recall from (3.4) that \(V/\text{Sp}(2)\) is identified with \(G_{r \mathbb{H}}(2, \infty)\).

If \(G\) is a semidirect product of \(N\) by \(H\), then there is a short exact sequence (Appendix B.2), \(1 \to N \to G \to H \to 1\). Thus for the case \(G = \text{Pin}(4)\), \(N = \text{Spin}(4)\), \(H = \mathbb{Z}/2\), we have the exact sequence

\[
1 \to \text{Spin}(4) \to \text{Pin}(4) \to \mathbb{Z}/2 \to 1,
\]

which gives rise to the fibration \(p\) of classifying spaces

\[
B\text{Spin}(4) \to B\text{Pin}(4) \xrightarrow{p} B\mathbb{Z}/2
\]

by [43, Theorem 11.4].

Let \(\text{Pin}(4)\) act on \(S^\infty\) by projecting onto \(\mathbb{Z}/2\), and applying the antipodal map. Then we notice that \(S^\infty \times_{\text{Pin}(4)} V\) is homotopy equivalent to \(V/\text{Pin}(4)\), since there is a fibration \(S^\infty \to S^\infty \times_{\text{Pin}(4)} V \to V/\text{Pin}(4)\). Also there is a \(\mathbb{Z}/2\)-equivariant homotopy equivalence

\[
V/(\text{Sp}(1) \times \text{Sp}(1)) \xrightarrow{\sim} (S^{4\infty+3} \times S^{4\infty+3})/(\text{Sp}(1) \times \text{Sp}(1)) \cong \mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty,
\]

induced by including pairs of orthonormal vectors into all pairs of unit vectors. This yields an equivalence \(S^\infty \times_{\text{Pin}(4)} V \xrightarrow{\sim} B = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)\). Assembling these
we have a commutative diagram

\[
\begin{array}{ccc}
B(\text{Sp}(1) \times \text{Sp}(1)) & \rightarrow & V/\text{Pin}(4) \rightarrow \mathbb{BZ}/2 \\
\approx & & \approx \\
V/(\text{Sp}(1) \times \text{Sp}(1)) & \rightarrow & S^\infty \times_{\text{Pin}(4)} V \rightarrow S^\infty /\mathbb{Z}/2 \\
\approx & & \approx \\
\mathbb{H}P^\infty \times \mathbb{H}P^\infty & \rightarrow & B \rightarrow S^\infty /\mathbb{Z}/2
\end{array}
\]

which shows that the Borel space $B$ is an alternative model for $B\text{Pin}(4)$. It also shows that we may assume the homotopy equivalence between them identifies $\mathbb{H}P^\infty \times \mathbb{H}P^\infty$ with our model for $B(\text{Sp}(1) \times \text{Sp}(1))$, and identifies the natural projection $B \rightarrow \mathbb{RP}^\infty$ with the map $B\text{Pin}(4) \rightarrow \mathbb{BZ}/2$ obtained by projecting $\text{Pin}(4)$ onto $\mathbb{Z}/2$.

**Remark 3.8.** We may think of the space $S^\infty \times_{\text{Pin}(4)} V$ in diagram (3.7) as $\hat{\Gamma}$, because it is homeomorphic to $S^\infty \times_{\mathbb{Z}/2} V/(\text{Sp}(1) \times \text{Sp}(1))$. Then our two models for $B\text{Pin}(4)$ are related by

\[
B \leftarrow \hat{\Gamma} \xrightarrow{\pi} \Gamma := V/\text{Pin}(4),
\]

where the map $\hat{\Gamma} \rightarrow \Gamma$ is the restriction map $\pi|\hat{\Gamma}$ of $\pi$ (2.15), and $\hat{\Gamma} \rightarrow B$ lies above the inclusion of the subspace $\Gamma \subset \text{SP}^2$.

### 3.5 On Vector Bundles over $B\text{Pin}(4)$

This section concerns vector bundles over $B\text{Pin}(4)$ that are closely related to certain $\text{Spin}(m)$ bundles. Some useful properties of the universal $\text{Spin}(m)$ bundles when $m = 3, 4, 5$ were described in [5] whilst the characteristics of $\text{Pin}^\pm$ structures on vector bundles were studied in [30].

First we shall describe the vector bundle $\theta$ that is associated to the action of $\text{Pin}(4)$ on $\mathbb{R}^4$ as defined in Section 3.2. Recall that $V$ stands for $V_{\mathbb{H}}(2, \infty)$ as in Section 3.3, and that

\[
\Gamma = V/\text{Pin}(4) = V/ \sim
\]

is a model for $B\text{Pin}(4)$, where $(u, v) \sim (ua, vb) \sim (v, u)$ for $a, b \in S^3$.

**Definition 3.10.** The real 4-plane vector bundle

\[
\mathbb{R}^4 \rightarrow \mathbb{R}(\theta) \rightarrow B\text{Pin}(4)
\]
has total space $\mathbb{R}(\theta)$ given by

$$\mathbb{R}(\theta) = V \times_{\text{Pin}(4)} \mathbb{R}^4 = V \times \mathbb{R}^4 / \sim,$$

where $((u, v), h) \sim ((v, u), \bar{h}) \sim ((ua, vb), \bar{ahb})$ for $h \in \mathbb{H} \cong \mathbb{R}^4$, and the projection map is given by

$$V \times_{\text{Pin}(4)} \mathbb{R}^4 \to V/\text{Pin}(4), \quad [(u, v), h] \mapsto [u, v].$$

We shall confirm below (and again in Section 6.2.2) that the vector bundle $\theta$ is the universal $\text{Pin}^-(4)$ bundle. Since $\text{Pin}^+(4)$ and $\text{Pin}^-(4)$ are isomorphic Lie groups as mentioned in Section 3.1, $\Gamma$ is also a model for $B\text{Pin}^+(4)$. The following question then arises naturally: “What is the $\text{Pin}^+(4)$ action on $\mathbb{R}^4$, that gives rise to the universal $\text{Pin}^+(4)$ bundle over $\Gamma$?”

**Proposition 3.12.** The $\text{Pin}^+(4)$ action on $\mathbb{R}^4 \cong \mathbb{H}$ can be given by

- $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ acts on $h \in \mathbb{H}$ by $h \mapsto ah\overline{b}$
- $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ acts on $h \in \mathbb{H}$ by $h \mapsto -a\overline{h}\overline{b}$,

and in particular, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts on $h$ by $h \mapsto -\bar{h}$.

**Proof.** Let $\rho_+: \text{Pin}(4) \to O(4)$ be the stated representation, and $r \in O(4)$ be reflection across the hyperplane $e_4^\perp$. Then the preimage $\rho_+^{-1}(\{I, r\})$ consists of the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$ 

This subgroup is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, so it follows from [30, page 180] that $\rho_+$ is the standard representation of $\text{Pin}^+(4)$, (and not $\text{Pin}^-(4)$). \qed

**Remark 3.13.** Let $\rho_-: \text{Pin}(4) \to O(4)$ be the representation of $\text{Pin}(4)$ in Section 3.2. Then applying the same procedure gives

$$\rho_-^{-1}(\{I, r\}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

which is isomorphic to $\mathbb{Z}/4$. So by [30, page 180], $\rho_-$ gives the $\text{Pin}^-(4)$ action, and $\theta$ of Definition 3.10 must be the universal $\text{Pin}^-(4)$ bundle.
We may also check the action of $Pin^+(4)$ of Proposition 3.12 against Kirby and Taylor’s characterisation of $Pin^\pm$ structures on a given vector bundle. To do this, we write $Pin^-(\xi)$, $Pin^+(\xi)$ and $Spin(\xi)$ respectively for the sets of all possible $Pin^-$, $Pin^+$ and $Spin$ structures on an $n$-plane vector bundle $\xi$. By definition, each of these structures consists of a homotopy class of lifts of its classifying map from $BO(n)$ to $BPin^\pm(n)$ or $BSpin(n)$.

**Lemma 3.14** ([30], Lemma 1.7). Let $\xi$ be a vector bundle $\mathbb{R}^n \to E \to X$. Then there exist natural bijections

\[
Pin^-(\xi) \leftrightarrow Spin(\xi \oplus \det \xi) \quad (3.15)
\]
\[
Pin^+(\xi) \leftrightarrow Spin(\xi \oplus 3 \det \xi) \quad (3.16)
\]

where $\det \xi$ is the determinant line bundle associated to $\xi$.

Before considering the case $Pin^+$ (3.16), it is helpful to interpret (3.15) first, in terms of our Definition 3.10. Our computations in Section 3.2 show that the pull back of the universal $Spin(5)$ bundle along $BPin(4) \to BSpin(5)$ is a Whitney sum. It is isomorphic to $\theta \oplus \lambda$, where $\lambda$ is the line bundle coming from the action of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{R}$ by $r \mapsto -r$. This action defines $\det \theta$. In other words, our $Pin^-(4)$ structure on $\theta$ corresponds to the pull back $Spin(5)$ structure on $\theta \oplus \det \theta$ under (3.15).

We shall take an analogous approach to the case $Pin^+$. Let $q$ be a pure quaternion (so $\bar{q} = -q$), and let $h$ be any quaternion $h \in \mathbb{H} \cong \mathbb{R}^4$. Then, we find below that the $Pin^+(4)$ action as in Proposition 3.12 agrees with the action of $Pin(4)$ on $\mathbb{R}^4 \cong \mathbb{H}$ deduced from conjugation on quaternionic Skew-Hermitian matrices

\[
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \bar{q} & h \\ -h & q \end{pmatrix} \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \end{pmatrix} = \begin{pmatrix} \bar{q} & ahb \\ -b\bar{a} & q \end{pmatrix}
\]
\[
\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} \bar{q} & h \\ -h & q \end{pmatrix} \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} = \begin{pmatrix} q & -ahb \\ b\bar{a} & \bar{q} \end{pmatrix}
\]

for $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{R}^4 \cong Pure(\mathbb{H}) \oplus \mathbb{H}$. This illustrates (3.16), by showing how the pull back of the universal $Spin(7)$ bundle along $BPin(4) \to BSpin(7)$ splits as the universal $Pin^+(4)$ bundle plus 3 copies of its determinant line bundle.
Remark 3.17. The action of \( \text{Pin}^- (4) \) of Section 3.2 has occasionally occurred in the literature, for example in Dupont [16, chapter 6, page 48]. On the other hand, we have not been able to find any reference to the action of \( \text{Pin}^+ (4) \) of Proposition 3.12.
Chapter 4

The Cohomology Ring of $BPin(4)$

In this chapter we describe the cohomology ring of $BPin(4)$ via two models $B$ and $\Gamma$. Fred W. Roush [52] computed the integral cohomology of the Borel space $B$, however, his result has not been published. We shall extend his work and describe the integral cohomology of $B_n$. To compute the cohomology of $\Gamma$ with $\mathbb{Z}$ and $\mathbb{Z}/2$ coefficients, we consider the fibre bundle $\mathbb{RP}^4 \rightarrow \Gamma \rightarrow Gr_{\mathbb{H}}(2,\infty)$. We also describe the integral cohomology of $\Gamma_n$, and therefore the integral cohomology ring of the configuration space of unordered pairs of distinct points in $\mathbb{HP}^n$. Our calculation uses a quaternionic version of a symmetric orthogonalisation procedure that originally appeared in the literature [35] of chemical physics.

4.1 The Cohomology of $\Gamma$ and $\Gamma_n$

In this section we shall compute the cohomology rings of $\Gamma$ and $\Gamma_n$.

From [5,49] we have that $Spin(5)/Spin(4) \cong S^4$ and $Sp(2)/Pin(4) \cong \mathbb{RP}^4$. Then analogous to the complex case in [57] there is a commutative diagram

$$
\begin{array}{ccc}
Sp(1) \times Sp(1) & \longrightarrow & Sp(2) \longrightarrow Sp(2)/Sp(1) \times Sp(1) \cong S^4 \\
\downarrow & & \downarrow \\
Pin(4) & \longrightarrow & Sp(2) \longrightarrow Sp(2)/Pin(4) \cong \mathbb{RP}^4
\end{array}
$$

(4.1)

of fiber bundles, which in turn induces two commutative diagrams

$$
\begin{array}{ccc}
S^4 & \longrightarrow & V_{n+1}/Sp(1) \times Sp(1) \longrightarrow Gr_{\mathbb{H}}(2, n + 1) \\
\downarrow & & \downarrow c_{n} \\
\mathbb{RP}^4 & \longrightarrow & \Gamma_n \longrightarrow Gr_{\mathbb{H}}(2, n + 1)
\end{array}
$$

(4.2)
CHAPTER 4. THE COHOMOLOGY RING OF $BP\text{IN}(4)$

\[ S^4 \longrightarrow V/Sp(1) \times Sp(1) \longrightarrow Gr_{\mathbb{H}}(2, \infty) \tag{4.3} \]
\[ \mathbb{CP}^4 \longrightarrow \Gamma \longrightarrow Gr_{\mathbb{H}}(2, \infty) \]

\[ \Gamma = \bigcup_n \Gamma_n \text{ and } Gr_{\mathbb{H}}(2, \infty) = \bigcup_n Gr_{\mathbb{H}}(2, n+1) \]

of fibre bundles, where $c, c', c_n$ are double coverings. So for any diagram

\[ \mathbb{CP}^4 \longrightarrow \Gamma_n \longrightarrow Gr_{\mathbb{H}}(2, n+1) \tag{4.4} \]

mapping into the colimit, the upper bundle is the restriction, and therefore the pull back, of the lower bundle.

We should recall from Section 3.4 that the space $\Gamma_n$ is identified with $V_{n+1}/\text{Pin}(4) \sim V_{n+1}$, where $(u, v) \sim (v, u) \sim (ua, vb)$ for $a, b \in S^3$. For the infinite case we take colimits and identify $\Gamma$ with $V/\text{Pin}(4) = \bigcup_n V_{n+1}/\text{Pin}(4)$. Similarly, the space $V_{n+1}/Sp(1) \times Sp(1)$ is $V_{n+1} \sim (ua, vb)$ for $a, b \in S^3$. For the infinite case we take the colimit $V/Sp(1) \times Sp(1) = \bigcup_n V_{n+1}/Sp(1) \times Sp(1)$.

We shall use the fibre bundles from (4.4) to compute the cohomology rings of $\Gamma$ and $\Gamma_n$. As input we need the following standard facts.

From [22, page 212–214],

\[ H^*(\mathbb{RP}_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[a], \quad H^*(\mathbb{RP}_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/(a^{n+1}) \tag{4.5} \]

where $|a| = 1$, and

\[ H^*(\mathbb{RP}^\infty_+; \mathbb{Z}) \cong \mathbb{Z}[w]/(2w), \quad |w| = 2 \tag{4.6} \]
\[ H^*(\mathbb{RP}^{2k}_+; \mathbb{Z}) \cong \mathbb{Z}[w]/(2w, w^{k+1}), \quad |w| = 2 \tag{4.7} \]
\[ H^*(\mathbb{RP}^{2k+1}_+; \mathbb{Z}) \cong \mathbb{Z}[w, v]/(2w, w^{k+1}, v^2, vw), \quad |w| = 2, \quad |v| = 2k + 1. \tag{4.8} \]

Reduction mod 2 is the injective ring homomorphism

\[ \rho: H^*(\mathbb{RP}_+; \mathbb{Z}) \rightarrow H^*(\mathbb{RP}_+; \mathbb{Z}/2) \text{ given by } w \mapsto a^2, \tag{4.9} \]

for $*>0$. The unique non-zero element of $H^1(\mathbb{RP}_+; \mathbb{Z}/2)$ is called the 1st Stiefel-Whitney class of the tautological line bundle $\lambda$ over $\mathbb{RP}_+$, written $a = w_1(\lambda)$. 

\[ a \mapsto a = a^2. \tag{4.9} \]
CHAPTER 4. THE COHOMOLOGY RING OF $BPIN(4)$

For any $n \geq 1$, a real $n$-plane bundle $\zeta$ over the base space $X$, has Stiefel-Whitney classes $w_i(\zeta) \in H^i(X_+; \mathbb{Z}/2)$, $i \geq 0$, which satisfy the following [36, 40, 42].

1. $w_0(\zeta) = 1$, and $w_i(\zeta) = 0$ when $i > n$  \hspace{1cm} (4.10)
2. $w_i(\zeta \oplus \mathbb{R}^1) = w_i(\zeta)$ where $\mathbb{R}^1$ is the trivial line bundle over $X$  \hspace{1cm} (4.11)
3. $w_i(\zeta \oplus \eta) = \sum_{j=0}^{i} w_j(\zeta) \cdot w_{i-j}(\eta)$,  \hspace{1cm} (4.12)

for any other real vector bundle $\eta$ over $X$. The total Stiefel-Whitney class $w(\zeta)$ is defined to be $\sum_{i \geq 0} w_i(\zeta) \in H^*(X_+; \mathbb{Z}/2)$. So (4.12) implies the formula

$$w(\zeta \oplus \eta) = w(\zeta) \cdot w(\eta).$$  \hspace{1cm} (4.13)

Similarly, a quaternionic vector bundle $\kappa$ over $X$ has Symplectic Pontryagin classes $p_i(\kappa) \in H^{4i}(X_+)$, $i \geq 0$, with analogous properties [40]. It is important to note that $\rho(p_i(\kappa)) = w_{4i}(\kappa)$ in $H^{4i}(X_+; \mathbb{Z}/2)$.

Other essential input includes the cohomology rings for $Gr_{\mathbb{H}}(2, n + 1)$, and the model $Gr_{\mathbb{H}}(2, \infty)$ for the classifying space $BSp(2)$.

Assume $R = \mathbb{Z}$ or $\mathbb{Z}/2$. From [40], we have

$$H^*(Gr_{\mathbb{H}}(2, \infty)_+; R) \cong R[p_1, p_2]$$

where $|p_i| = 4i$ for $i = 1, 2$, and $p_i = p_i(\xi)$ for the tautological quaternionic 2-plane bundle $\xi$ over $Gr_{\mathbb{H}}(2, \infty)$. Also

$$H^*(Gr_{\mathbb{H}}(2, n + 1)_+; R) \cong R[p_1, p_2, \bar{p}_1, \ldots, \bar{p}_{n-1}]/\mathcal{I}$$

where $\bar{p}_j \in H^{4j}(Gr_{\mathbb{H}}(2, n + 1); R)$ is $p_j(\xi^\perp)$ and the ideal $\mathcal{I}$ is given by the relation

$$(1 + p_1 + p_2)(1 + \bar{p}_1 + \bar{p}_2 + \cdots + \bar{p}_{n-1}) = 1.$$

This equation arises from $\xi \oplus \xi^\perp \cong \mathbb{H}^{n+1}$, where $\mathbb{H}^{n+1}$ is the trivial $n + 1$-plane quaternionic bundle over the Grassmannian. Equivalently, from [48]

$$H^*(Gr_{\mathbb{H}}(2, n + 1)_+; R) \cong R[p_1, p_2]/(q_n, q_{n+1})$$  \hspace{1cm} (4.14)

where $q_j$ is the $j$-th complete symmetric polynomial in $z_1, z_2$ for $p_1 = z_1 + z_2$, $p_2 = z_1 z_2$. 
4.1.1 The Integral Cohomology

To compute the cohomology of $\Gamma$ and $\Gamma_n$ it is helpful to recall from the literature some results on the Serre spectral sequence (Appendix B.3) for a fibration $F \xrightarrow{i} E \xrightarrow{p} M$ when the fundamental group of the base $M$ is trivial. We assume below that $R$ is a commutative ring with unity, and that the spectral sequence is denoted by $E_2^{p,q} \Rightarrow H^*(E+; R)$.

**Lemma 4.15** ([40], Lemma 2.15). Suppose the local coefficient ring is trivial and the base space $M$ and the fibre $F$ of the fibration are path connected, and either $H^*(M+; R)$ or $H^*(F+; R)$ is a finitely generated free $R$-module in each dimension. Then the product $E_2^{p,0} \otimes E_2^{0,q} \rightarrow E_2^{p,q}$ induces an isomorphism $E_2^{*,*} \cong H^*(M+; R) \otimes_R H^*(F+; R)$ for any $p, q \geq 0$.

**Theorem 4.16** ([40], Theorem 4.2). Let the base space $M$ and the fibre $F$ of the fibration $F \xrightarrow{i} E \xrightarrow{p} M$ be path connected. Suppose that the local coefficient ring is trivial and all differentials are zero, $d_k = 0, k \geq 2$, that is $E_2 = E_\infty$. Then $p^*: H^*(M+; R) \rightarrow H^*(E+; R)$ is a monomorphism, and $i^*: H^*(E+; R) \rightarrow H^*(F+; R)$ is an epimorphism.

Lemma 4.15 and Theorem 4.16 are applicable to the bundle $\mathbb{R}P^4 \xrightarrow{i} \Gamma \xrightarrow{\ell} Gr_{\mathbb{H}}(2, \infty)$ with $R = \mathbb{Z}$, since the base $Gr_{\mathbb{H}}(2, \infty)$ is simply connected, its integral cohomology is the polynomial ring $\mathbb{Z}[p_1, p_2]$, and $H^*(\mathbb{R}P^4) = 0$ in odd dimensions by (4.7).

The homotopy exact sequence of the bundle shows that $i_\ast: \pi_1(\mathbb{R}P^4) \cong \mathbb{Z}/2 \rightarrow \pi_1(\Gamma)$ is an isomorphism.

**Proposition 4.17.** The integral cohomology ring of $\Gamma$ is

$$H^*(\Gamma_+) \cong \mathbb{Z}[\bar{x}, \bar{y}, w]/(2w, \bar{x}w + w^3), \quad |\bar{x}| = 4, \ |\bar{y}| = 8, \ |w| = 2.$$ 

**Proof.** We shall consider the SSS for the bundle $\mathbb{R}P^4 \xrightarrow{i} \Gamma \xrightarrow{\ell} Gr_{\mathbb{H}}(2, \infty)$, where $E_2^{p,q} \cong H^p(Gr_{\mathbb{H}}(2, \infty)_+; H^q(\mathbb{R}P^4_+))$. Then $E_2^{*,*} \cong H^*(Gr_{\mathbb{H}}(2, \infty)_+) \otimes \mathbb{Z} H^*(\mathbb{R}P^4_+)$.
by Lemma 4.15, and $E_2 = E_\infty$ for dimensional reasons. Then $i^*$ is an epimorphism and $\ell^*$ is a monomorphism by Theorem 4.16.

The group $Pin(4)$ is the semidirect product of $Spin(4)$ by $\mathbb{Z}/2$, so there are maps $\mathbb{Z}/2 \hookrightarrow Pin(4) \to \mathbb{Z}/2$, which yields maps $B\mathbb{Z}/2 \to BPin(4) \to B\mathbb{Z}/2$ of classifying spaces, whose composition is the identity. Let $f$ be the map $\Gamma = BPin(4) \to B\mathbb{Z}/2 = \mathbb{R}P^\infty$. Then we define $w := f^*(w)$ in $H^2(\Gamma_+)$, where $2w = 0$, $|w| = 2$, and note that all powers $w^k$ are non-zero, because they pull back non-zero to $H^{2k}(\mathbb{R}P^\infty)$. Also, $f_\ast : \pi_1(\Gamma) \to \pi_1(\mathbb{R}P^\infty)$ must be non-zero, and therefore an isomorphism. It follows that $(f \circ i)_\ast : \pi_1(\mathbb{R}P^4) \to \pi_1(\mathbb{R}P^\infty)$ is an isomorphism as well, and therefore $i^*(w^k) = w^k$ in $H^{2k}(\mathbb{R}P^4')$ for $k = 1, 2$.

We can now say that $w$ and $w^2$ represent the classes $1 \otimes w$ and $1 \otimes w^2$ in $E^0_{\infty} \otimes \mathbb{Z}$ and $E^0_{\infty} \otimes \mathbb{Z}$, respectively, and have order 2. Similarly, $\ell_\ast(p_1)$ and $\ell_\ast(p_2)$ represent the classes $p_1 \otimes 1$ and $p_2 \otimes 1$ in $E^{4,0}_{\infty} \otimes \mathbb{Z}$ and $E^{8,0}_{\infty} \otimes \mathbb{Z}$, respectively. Therefore $\ell_\ast(p_1)^i \ell_\ast(p_2)^j w^k$ represents $p_1^i p_2^j \otimes w^k$ in $E^{*}_{\infty} \otimes \mathbb{Z}$, and $E^*_{\infty} \cong H^*(\Gamma_+)$ additively. Let us consider the product structure, remembering that $w^3 \neq 0$, which in turn suggests that $w^3 = \ell_\ast(p_1) w$ because $H^6(\Gamma_+) \cong E^{3,2}_{\infty} \cong \mathbb{Z}/2$ on generator $p_1 \otimes w$. We now complete the proof by setting $\tilde{x} := \ell_\ast(p_1)$ and $\tilde{y} := \ell_\ast(p_2)$.

\[\square\]

**Remark 4.18.** We should note that the generator $\tilde{x}$ in Proposition 4.17 can be replaced by $x := \tilde{x} + w^2$; in this case, the ideal $(2w, \tilde{x}w + w^3)$ is given by $(2w, xw)$.

Also, it follows from [5] that the bundle $\mathbb{R}P^4 \to \Gamma \to Gr_{\mathbb{H}}(2, \infty)$ is the projectivisation of the universal $Spin(5)$ vector bundle $\chi_5$ over $BSp(2)$. So $\Gamma \to \mathbb{R}P^\infty$ classifies the tautological line bundle $\lambda$ over $\mathbb{R}P(\chi_5) = \Gamma$.

We shall now express the cohomology of $\Gamma_n$ in terms of $H^*(\Gamma_+)$.  

**Proposition 4.19.** For any $n \geq 1$,

$$H^*(\Gamma_n)_+ \cong H^*(\Gamma_+)/\langle h_n, h_{n+1} \rangle$$

where $|h_i| = 4i$.

**Proof.** Apply $H^*(-)$ to the commutative diagram (4.4) of fibre bundles

\[\begin{array}{ccc}
\mathbb{R}P^4 & \xrightarrow{i_n} & \Gamma_n \\
\downarrow & & \downarrow \gamma \\
\mathbb{R}P^4 & \xrightarrow{i} & \Gamma \\
\end{array}\]

\[\begin{array}{ccc}
\Gamma_n & \xrightarrow{\ell_n} & Gr_{\mathbb{H}}(2, n + 1) \\
\downarrow & & \downarrow r \\
\Gamma & \xrightarrow{\ell} & Gr_{\mathbb{H}}(2, \infty). \\
\end{array}\]
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The homomorphism $r^*$ is surjective where the kernel is given by the ideal $(q_n, q_{n+1})$ by (4.14). The base space $Gr_{2n}(2, n + 1)$ is simply connected and its cohomology is a finitely generated free $\mathbb{Z}$-module in each dimension. Thus both $i^*$ and $i^*_n$ are epimorphisms whilst $\ell^*$ and $\ell^*_n$ are monomorphisms. The homomorphism $\ell^*$ is given by $\ell^*(p_1) =: \tilde{x}$, $\ell^*(p_2) =: \tilde{y}$, and so $\gamma^*(\ell^*(q_n)) = 0 = \gamma^*(\ell^*(q_{n+1}))$. The SSS for $\Gamma_n$ is analogous to that for $\Gamma$, therefore

$$H^*((\Gamma_n)_+) \cong H^*(\Gamma_+)/((\ell^*(q_n), \ell^*(q_{n+1}))$$

where $h_n := \ell^*(q_n)$, $h_{n+1} := \ell^*(q_{n+1})$ as required. \hfill $\square$

**Example 4.20.** For $n = 2$,

$$H^*((\Gamma_2)_+) \cong H^*(\Gamma_+)/((\tilde{x}^2 - \tilde{y}, \tilde{x}^3 - 2\tilde{x}\tilde{y})].$$

### 4.1.2 The Mod 2 Cohomology

The mod 2 cohomology $H^*(\Gamma_+; \mathbb{Z}/2)$ plays an important role in Section 6.2.2 and 7.2. Compared to the integral cohomology ring, the $\mathbb{Z}/2$-cohomology is relatively easy to describe by using the Leray-Hirch theorem and a well-known result on product structure for the cohomology of projective bundles [57, Theorem 2.3]. The additive structure of $H^*(\Gamma_+; \mathbb{Z}/2)$ can also be read off from Proposition 4.17 by applying the Universal Coefficient Theorem.

**Theorem 4.21** (Leray-Hirsch [22]). Let $F \rightarrow E \rightarrow X$ be a fibre bundle such that, for some commutative ring $R$

(i) $H^m(F_+; R)$ is a finitely generated free $R$-module for each $m$

(ii) there exist elements $c_j \in H^*(E_+; R)$ such that the $i^*(c_j)$ form a basis for $H^*(F_+)$. Then the map

$$\varphi: H^*(X_+; R) \otimes_R H^*(F_+; R) \rightarrow H^*(E_+; R), \quad \sum_{k,j} x_k \otimes i^*(c_j) \mapsto \sum_{k,j} p^*(x_k) c_j,$$

is an isomorphism of $R$-modules. In other words, $H^*(E_+; R)$ is a free $H^*(X_+; R)$-module, with basis the $c_j$. 
Lemma 4.22. The ring $H^*(\Gamma_+; \mathbb{Z}/2)$ is given by

$$H^*(\Gamma_+; \mathbb{Z}/2) \cong H^*(\text{Gr}_H(2, \infty)_+; \mathbb{Z}/2)[a]/(a^5 - p_1 a),$$

where $|a| = 1$, $H^*(\text{Gr}_H(2, \infty)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2]$, $|p_1| = 4i$.

Proof. We shall use the Leray-Hirsch theorem for projective bundles of vector bundles and consider $\mathbb{RP}^4 \xrightarrow{f} \Gamma \xrightarrow{\ell} \text{Gr}_H(2, \infty)$. We define $a := f^*(a)$ in $H^1(\Gamma_+; \mathbb{Z}/2)$ by using the map $f : \Gamma \to \mathbb{RP}^\infty$ from the proof of Proposition 4.17. It follows that $i^*(a) = a$, so $H^*(\mathbb{RP}^4_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/(a^5)$ has basis $i^*(a)$, $i^*(a^2)$, $i^*(a^3)$, $i^*(a^4)$ as required. Note from Remark 4.18 that $a$ is $w_1(\lambda)$, by definition.

In order to find the multiplicative structure, we can apply the reduction mod 2 to Proposition 4.17, or use the fact $\chi_5$ has only one non-zero Stiefel-Whitney class $w_4(\chi_5) = p_1$ in $H^4(\text{Gr}_H(2, \infty)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2$. So applying [57, Theorem 2.3] gives $a^5 = \sum_{j=0}^4 w_{5-j}(\chi_5) a^j = p_1 a$, as required.

Remark 4.23. Analogous to $H^*(\Gamma_+; \mathbb{Z}/2)$, we also have

$$H^*((\Gamma_n)_+; \mathbb{Z}/2) \cong H^*(\text{Gr}_H(2, n+1)_+; \mathbb{Z}/2)[a]/(a^5 - p_1 a),$$

where $|a| = 1$, $p_1 \in H^4(\text{Gr}_H(2, n+1)_+; \mathbb{Z}/2)$.

Proof. Again, this follows from the Leray-Hirsch theorem and [57, Theorem 2.3].

4.2 Configuration Space

In this section we shall show that the second configuration space $C_2(\mathbb{HP}^n)$ and the space $\Gamma_n$ are homotopy equivalent. To establish the homotopy equivalence we shall use a quaternionic version of the symmetric orthogonalisation process.

4.2.1 The Configuration Space $C_2(\mathbb{HP}^n)$

The $n$-th unordered configuration space$^1$ $C_n(X)$ of a topological space $X$ is the quotient $C_n(X) = \tilde{C}_n(X)/\Sigma_n$ by the natural action of the $n$-th symmetric group $\Sigma_n$, where $\tilde{C}_n(X) = \{(x_1, x_2, \cdots, x_n) \mid x_i \neq x_j \text{ for } i \neq j\}$ is the space of ordered $n$-tuples of distinct points of $X$.

$^1$The space $C_n(X)$ is sometimes called the $n$-th braid space [12,27] of $X$. 

During the late 60’s - 70’s, Samuel Feder [17] worked on the mod 2 cohomology of $C_2(\mathbb{R}P^n)$ and $C_2(\mathbb{C}P^n)$; the integral and the mod 2 cohomology rings of $C_2(\mathbb{C}P^n)$ are also described by Tsutomu Yasui in [57]. More recently Jesús González and Peter Landweber computed $H^*(C_2(\mathbb{R}P^n);\mathbb{Z})$ as groups [18], and also as rings [15] together with Carlos Domínguez. By analogy with the complex case and the real case in [17,57], we have a fibration $\xi$, $$C_2(\mathbb{H}P^1) \to C_2(\mathbb{H}P^n) \to Gr_{\mathbb{H}}(2,n+1),$$ and there is a homotopy equivalence $C_2(\mathbb{H}P^1) \simeq \mathbb{R}P^4$, given by the fact that $C_2(\mathbb{H}P^1)$ is an open 4-disk bundle over $\mathbb{R}P^4$ [27, Lemma 2.1]. We may also notice that there is a subspace of $C_2(\mathbb{H}P^n)$, which consists of all pairs of lines $(m,l)$ through the origin in $\mathbb{H}^{n+1}$ with the inner product $\langle m,l \rangle = 0$; this is the space $\Gamma_n$ of Definition 2.1.

### 4.2.2 The Symmetric Orthogonalisation

The idea of the (Löwdin) symmetric orthogonalisation originates in a paper [35] by Per-Olov Löwdin. The process was used in the literature of quantum chemistry [38], and still seems less known to mathematicians than the non-symmetrical Gram-Schmidt orthogonalisation. Given any $n \times k$ complex matrix $X$ of linearly independent unit column vectors $x_1, \ldots, x_k$, let $M$ be the positive definite Hermitian matrix $X^*X$. Then $M$ has a unique positive definite Hermitian square root, and $V = XM^{-1/2}$ is Löwdin’s $n \times k$ matrix of orthogonalised columns. The process does not appear in the literature over the quaternions, but is just as valid, by applying [58, Corollary 6.2]. We make it explicit for the case $k = 2$ below, by introducing the $2 \times 2$ matrix $M^{-1/2}$ over $\mathbb{H}$.

**Definition 4.25.** For any linearly independent unit vectors $x, y \in \mathbb{H}^{n+1}$, let $M$ be the Hermitian matrix $M = \begin{pmatrix} x^*x & x^*y \\ y^*x & y^*y \end{pmatrix} = \begin{pmatrix} 1 & x^*y \\ y^*x & 1 \end{pmatrix}$, and define the associated Hermitian matrix $N$ to be

$$N = N_{x,y} = \begin{pmatrix} C+D \frac{-h}{(C+D)CD} & \frac{-\bar{h}}{C+D} \\ \frac{-\bar{h}}{C+D} & \frac{C+D}{2CD} \end{pmatrix},$$

where $h = x^*y \in \mathbb{H}$, $\bar{h} = y^*x \in \mathbb{H}$, $|h| = \sqrt{hh} = \sqrt{\bar{h}h} \in \mathbb{R}$, $|h| < 1$,

$$C = (1 + |h|)^{1/2}, \quad D = (1 - |h|)^{1/2}.$$

If $x, y$ are orthogonal, then $M$ is the identity matrix, and so is $N$. 
Simple computations show that $N^2 = M^{-1}$, and that $N$ transforms $x, y$ to an orthonormal pair $u, v$ by $(u \; v) = (x \; y)N$. The most important property is that the process is symmetrical; if we start with $(y \; x)$ then we get $(v \; u)$. Note also that $N$ is well defined, because $|h| < 1$, and depends continuously on $x$ and $y$.

**Proposition 4.26.** For any $n \geq 1$, there is a homotopy equivalence $\Gamma_n \simeq C_2(\mathbb{H}P^n)$, and therefore an isomorphism $H^*((\Gamma_n)_+) \cong H^*(C_2(\mathbb{H}P^n)_+)$. 

**Proof.** Let unit column vectors $x, y \in \mathbb{H}^{n+1}$, with $x \neq y$, represent elements $[x], [y] \in \mathbb{H}P^n$ respectively. Then an element of $C_2(\mathbb{H}P^n)$ can be expressed as an $(n+1) \times 2$ matrix $(x \; y)$, up to the action of $Pin(4)$. So consider the maps $s : \Gamma_n \to C_2(\mathbb{H}P^n)$, $o : C_2(\mathbb{H}P^n) \to \Gamma_n$ given by $s(u \; v) = (u \; v)$ and $o(x \; y) = (x \; y)N_{x,y}$, where we factor out by $Pin(4)$ on both sides. If $(x' \; y') := (x \; y)N$, then for any $a, b \in S^3$, we have $|x^*y| = |(xa)^*yb| = |y^*x|$, $(xa yb)N_{xa,yb} = (x'a \; y'b)$ and $(y \; x)N_{y,x} = (y' \; x')$. Clearly $o \circ s = 1_{\Gamma_n}$, the identity map on $\Gamma_n$. We need to prove that $s \circ o$ is homotopic to $1_{C_2(\mathbb{H}P^n)}$.

For $0 \leq t \leq 1$, and any unit vectors $x, y \in \mathbb{H}^{n+1}$, define the $2 \times 2$ matrix

$$N_t = N_{x,y,t} = \begin{pmatrix} \frac{C+D}{2CD} - 1 & t + 1 \\ \frac{-h}{(C+D)CD} & \frac{C+D}{2CD} - 1 \end{pmatrix}$$

where

$$h = x^*y \in \mathbb{H}, \quad \bar{h} = y^*x \in \mathbb{H}, \quad |h| \in \mathbb{R},$$

$$C = (1 + |h|)^{1/2}, \quad D = (1 - |h|)^{1/2}$$

which varies continuously with $t$. Then $M_0$ is the $2 \times 2$ identity matrix, and $M_1$ is the matrix $N$ of Definition 4.25. So the map $(x \; y) \mapsto (x \; y)M_t$ induces a homotopy $1_{C_2(\mathbb{H}P^n)} \simeq s \circ o$. □

We will use this process later in Proposition 6.4.

### 4.3 The Cohomology of the Borel Space

In this section we shall describe the integral cohomology ring $H^*((B_n)_+)$, by extending the calculation of $H^*(B_+)$ in [52]. There is, of course, an isomorphism $H^*(B_+) \cong$
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$H^*(\Gamma_+)$ because $B \simeq \Gamma$ are both models for $BP\text{in}(4)$, by Remark 3.8. To compute the cohomology, we first recall some useful results regarding group cohomology.

Let $H^i(G, A)$ be the $i$-th cohomology of a finite group $G$ with coefficients in a $G$-module $A$, where we can think of a $G$-module $A$ as an abelian group together with an action of $G$. From the topological point of view, this is the same as $H^i(BG; A)$, where $\pi_1(BG) = G$ acts on $A$. We shall consider only the case when $G$ is the cyclic group $C_k$ of order $k$ with generator $\tau$, and use [24, Proposition 7.1] to define the cohomology groups $H^*(G, A)$ as follows.

**Definition 4.27.** Let $\alpha, \beta: A \to A$ be the $C_k$-homomorphisms given by

$$\alpha(a) = (\tau - 1)a, \quad \beta(a) = (\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1)a, \quad a \in A.$$  

Then $H^0(C_k, A) = \ker \alpha$, and for $n \geq 1$

$$H^{2n-1}(C_k, A) = \ker \beta / \im \alpha, \quad H^{2n}(C_k, A) = \ker \alpha / \im \beta. \quad (4.28)$$

The integral cohomology ring of the Borel space $B = S^\infty \times \mathbb{Z}/2 (\mathbb{H}P^\infty \times \mathbb{H}P^\infty)$ can be computed by using the Serre spectral sequence (Appendix B.3), and by applying (4.28) with $k = 2$, $A = H^*((\mathbb{H}P^\infty \times \mathbb{H}P^\infty)_+)$, where $\mathbb{Z}/2$ acts on $H^*((\mathbb{H}P^\infty \times \mathbb{H}P^\infty)_+) \cong \mathbb{Z}[z_1, z_2]$ by swapping $z_1$ and $z_2$. Before starting the computation, we recall that $H^*((\mathbb{H}P^\infty \times \mathbb{H}P^\infty)_+) \cong \mathbb{Z}[z_1, z_2]$ where $|z_i| = 4$, and $z_1, z_2$ are the pull-backs of $z \in H^4(\mathbb{H}P^\infty)$ under the two projections.

**Theorem 4.29 ( [52] ).** The Serre spectral sequence of the fibre bundle

$$\mathbb{H}P^\infty \times \mathbb{H}P^\infty \to S^\infty \times \mathbb{Z}/2 (\mathbb{H}P^\infty \times \mathbb{H}P^\infty) \xrightarrow{p} S^\infty / \mathbb{Z}/2 = \mathbb{R}P^\infty \quad (4.30)$$

in integral cohomology can be described as follows.

$$E_2^{0, q} = \text{invariant classes of } H^*(\mathbb{H}P^\infty \times \mathbb{H}P^\infty)_+ \text{ under interchange of the factors}$$

$$E_2^{2i, 0} = \mathbb{Z}/2 \quad \text{for } i > 0 \text{ with the generator } w^i(z_1 z_2)^q$$

and all other $E_2^{p, q}$ are zero. The relation $w(z_1 + z_2) = 0$ holds in $E_2^{2, 4}$, Moreover $E_2 = E_\infty$ and $E_\infty^{*, *} \cong H^*(B_+)$ as graded rings.
Therefore, Theorem 4.29 can be summarised as
\[ H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx), \quad |x| = 4, \quad |y| = 8, \quad |w| = 2 \]  \tag{4.31}
where \( x, y \in H^*(B_+) \) represent \( z_1 + z_2, \ z_1z_2 \in E_{\infty}^{0,*} \) respectively, and \( w \in H^2(B_+) \) represents \( w \in E_{\infty}^{2,0} \). We should note that the generator \( x \) in (4.31) could be replaced by \( x + w^2 \) by amending the ideal of relations. Indeed, setting \( \tilde{x} := x + w^2 \) and \( \tilde{y} := y \) yields
\[ H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx) \cong \mathbb{Z}[\tilde{x}, \tilde{y}, w]/(2w, \tilde{x}w + w^3) \cong H^*(\Gamma_+) \],
which confirms Roush’s result (4.31) and agrees with Proposition 4.17, our computation of \( H^*(BPin(4)_+) \) using the other model \( \Gamma \) for \( BPin(4) \).

The proof of Theorem 4.29 above proceeds by applying (4.28) with \( A = \mathcal{H}^*((\mathbb{HP}^\infty \times \mathbb{HP}^\infty)_+) \). In other words, we have the \( E_2 \) page of the spectral sequence
\[ E_2^{p,q} = H^p((\mathbb{RP}^\infty)_+, \mathcal{H}^q((\mathbb{HP}^\infty \times \mathbb{HP}^\infty)_+)). \]

For example, \( E_2^{0,2} = H^0(\mathbb{RP}^\infty_+, \mathcal{H}^2((\mathbb{HP}^\infty \times \mathbb{HP}^\infty)_+)) = \ker\alpha \), which means that \( E_2^{0,*} \) is the invariant classes of \( H^*((\mathbb{HP}^\infty \times \mathbb{HP}^\infty)_+) \); that is, the polynomial ring \( \mathbb{Z}[z_1 + z_2, \ z_1z_2] \).

Then, we also have that \( E_2^{2,4} = 0 \) because
\[ H^2(\mathbb{RP}^\infty_+, \mathcal{H}^4((\mathbb{HP}^\infty \times \mathbb{HP}^\infty))) = \ker\alpha/\im\beta, \]
and \( \im\beta = \ker\alpha = \mathbb{Z}(z_1 + z_2) \) for \( (p,q) = (2,4) \). So \( w(z_1 + z_2) = 0 \) holds in the product structure on \( E_2^{*,*} \).

All differentials are zero because all non-zero classes are in even dimensions, hence \( E_2 = E_\infty \). By diagram (3.7) and Remark 3.8, we may take \( w := p^*(w) \) in \( H^2(B) \) to be a representative of order 2 for \( w \) in \( E_{\infty}^{2,0} \cong H^2(\mathbb{RP}^\infty) \) where \( p \) is the projection \( B \to \mathbb{RP}^\infty \).

Powers of \( w \) are therefore representatives of order 2 for all of \( E_{\infty}^{*,0} \cong H^*(\mathbb{RP}^\infty) \).

Applying diagram (3.7) twice more, the elements \( \tilde{x} \) and \( \tilde{y} \) of Proposition 4.17 may be pulled back to \( H^4(B) \) and \( H^8(B) \) respectively, and restrict to \( z_1 + z_2 \) and \( z_1z_2 \) in \( H^*(\mathbb{HP}^\infty \times \mathbb{HP}^\infty) \) because they are given by \( \ell^*(p_1) \) and \( \ell^*(p_2) \) under \( \ell : \Gamma \to BS\Pi(2) \).

So they represent \( z_1 + z_2 \) and \( z_1z_2 \) in \( E_{\infty}^{0,*} \). Monomials in \( w, \tilde{x} \) and \( \tilde{y} \) therefore give order 2 representatives for all generators in \( E_{\infty}^{p,0} \) with \( p > 0 \). This means that all the extension problems are trivial, thus \( E_{\infty} \cong H^*(B) \) additively.

To show \( E_{\infty} \cong H^*(B_+) \) multiplicatively, it is enough to identify a representative \( x \) for \( z_1 + z_2 \in E_{\infty}^{0,4} \) such that \( w(z_1 + z_2) = 0 \), where \( H^4(B) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \) and \( H^6(B) \cong \mathbb{Z}/2 \) on \( w^3 \). This we shall now do.
The following facts about the subgroups of $Sp(2)$ are useful, and we shall also use them in Section 5.3.

For $a \in S^3$, the composition of the subgroup inclusions

$$S^3 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2)$$

is given by $a \mapsto \left( \begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right) \in Sp(2)$ and therefore agrees with

$$S^3 \hookrightarrow S^3 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2)$$

where the first subgroup is the diagonal.

Let $\tau$ be the generator of $\mathbb{Z}/2$ and consider the composition of inclusions

$$\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2),$$

which is given by $\tau \mapsto \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$.

**Remark 4.34.** The map (4.33), which is given by $\tau \mapsto \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ can also be rewritten as $\tau \mapsto \left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ in terms of the basis of eigenvectors $(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}), (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$, which agrees with the composition $\mathbb{Z}/2 = O(1) \rightarrow U(1) \rightarrow Sp(1) \rightarrow Sp(2)$. So both homomorphisms induce the same map on the cohomology of classifying spaces.

The inclusion map $\mathbb{R}P^\infty \xrightarrow{i} B$ induces the homomorphism $i^*$, given by $i^*(w^k) = w^k$, $k \geq 1$. Also, the maps $\mathbb{R}P^\infty \xrightarrow{i} B \xrightarrow{\ell} BSp(2)$ induce the homomorphisms $i^*, \ell^*$ such that $(i^* \circ \ell^*)(p_1) = w^2$, where $p_1 \in H^4(BSp(2))$; this follows from Remark 4.34.

Then, $i^*(w \ell^*(p_1)) = w^3$ and $i^*(w (\ell^*(p_1) + w^2)) = w^3 + w^3 = 0$. Because $H^6(B) \cong \mathbb{Z}/2$ on $w^3$, we may now identify $\ell^*(p_1) + w^2$ as the required representative for $z_1 + z_2$.

The integral cohomology ring of $B_n = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}P^n \times \mathbb{H}P^n)$ can be computed analogously, by combining Theorem 4.29 and (4.28) for $k = 2$, $A = H^*((\mathbb{H}P^n \times \mathbb{H}P^n)_+)$, together with the cohomology $H^*((\mathbb{H}P^n \times \mathbb{H}P^n)_+) \cong \mathbb{Z}[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$. This means that we have the spectral sequence for the bundle

$$\mathbb{H}P^n \times \mathbb{H}P^n \rightarrow S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}P^n \times \mathbb{H}P^n) \rightarrow \mathbb{R}P^\infty$$

where

$$E^{p, q}_2 = H^p(\mathbb{R}P^\infty, \mathcal{H}^q((\mathbb{H}P^n \times \mathbb{H}P^n)_+))$$

$$E^{0, *}_2 = \text{invariant classes of } H^*(\mathbb{H}P^n \times \mathbb{H}P^n)_+ \text{ under interchange of the factors}$$

$$= \mathbb{Z}[z_1 + z_2, z_1 z_2]/\mathcal{I}_n \text{ where the ideal } \mathcal{I}_n \text{ is described below}$$

$$E^{2i, 8q}_2 = \mathbb{Z}/2 \text{ for } i > 0, q \leq n \text{ with the generator } w^i(z_1 z_2)^q$$
and other \( E_{n}^{p,q} \) are zero. The relation \( E_{2}^{2,4} = w(z_{1} + z_{2}) = 0 \) holds for the same reason as the the case \( B \). Also, again all differentials are zero for dimensional reason, and \( E_{2} = E_{\infty} \cong H^{*}((B_{n})_{+}) \) as rings.

**Theorem 4.35.** For any \( n \geq 1 \), there exist elements \( f_{n,1} \) and \( f_{n,2} \) in the cohomology ring \( H^{*}(B_{+}) \cong \mathbb{Z}[x, y, w]/(2w, wx) \) such that

\[
H^{*}((B_{n})_{+}) \cong H^{*}(B_{+})/(f_{n,1}, f_{n,2}, y^{n+1}),
\]

(4.36)

where \( |f_{n,1}| = 4(n + 1), \ |f_{n,2}| = 4(n + 2) \); also \( f_{n,1}, f_{n,2} \) are given by

\[
\begin{align*}
 f_{1,1} &= x^{2} - 2y & \text{for } n = 1, & f_{n,1} &= x f_{n-1,1} - f_{n-1,2} & \text{for } n \geq 2, \\
 f_{1,2} &= xy & f_{n,2} &= y f_{n-1,1} \\
\end{align*}
\]

(4.37)

**Proof.** It is left to show that the ideal \( I_{n} \) is given by \( (f_{n,1}, f_{n,2}, y^{n+1}) \) in \( H^{*}(B) \). Any invariant polynomial in \( \mathbb{Z}[z_{1}, z_{2}]/(z_{1}^{n+1}, z_{2}^{n+1}) \) may be lifted to a unique polynomial in \( \mathbb{Z}[z_{1}, z_{2}] \) that has no monomial containing \( z_{1}^{n+1} \) or \( z_{2}^{n+1} \), or any higher powers of \( z_{1} \) or \( z_{2} \). The same is true after interchanging \( z_{1} \) and \( z_{2} \), so the lifts of the invariants are actually invariant in \( \mathbb{Z}[z_{1}, z_{2}] \), and must lie in \( \mathbb{Z}[z_{1} + z_{2}, z_{1} z_{2}] \). So the ideal \( I_{n} \) above is the kernel of the composition

\[
\mathbb{Z}[z_{1} + z_{2}, z_{1} z_{2}] \xrightarrow{\phi} \mathbb{Z}[z_{1}, z_{2}] \rightarrow \mathbb{Z}[z_{1}, z_{2}]/(z_{1}^{n+1}, z_{2}^{n+1}),
\]

and contains \( r_{n,1} := z_{1}^{n+1} + z_{2}^{n+1} \) and \( r_{n,2} := z_{1} z_{2} (z_{1}^{n} + z_{2}^{n}) \). So

\[
I_{n} \supseteq (r_{n,1}, r_{n,2}, (z_{1} z_{2})^{n+1}) := R_{n},
\]

and we need to prove the opposite inclusion.

For this we note that

\[
r_{n+1,1} = (z_{1} + z_{2}) r_{n,1} - r_{n,2} \quad \text{and} \quad r_{n+1,2} = z_{1} z_{2} r_{n,1}
\]

(4.38)

hold in \( \mathbb{Z}[z_{1} + z_{2}, z_{1} z_{2}] \), so \( r_{n+1,1} \) and \( r_{n+1,2} \) lie in \( R_{n} \). Repeating the process shows that \( r_{n+k,1} \ r_{n+k,2} \) also lie in \( R_{n} \) for any \( k \geq 0 \). Now let \( g(z_{1}, z_{2}) \in I_{n} \), which means that \( g(z_{1}, z_{2}) = g(z_{2}, z_{1}) \) and every monomial in \( g \) is divisible by \( z_{1}^{n+1} \) or \( z_{2}^{n+1} \). The monomials must therefore

(i) either occur in pairs such as \( z_{1}^{n+1+k} z_{2}^{i} + z_{1}^{n+1+k} z_{2}^{i} \)
(ii) or be of the form \((z_1z_2)^{n+1+k}\),

for some \(i, k \geq 0\). These can be written as \((z_1^i + z_2^i)r_{n+k,1} - r_{n+k+i,1}\) and \((z_1 z_2)^k(z_1 z_2)^{n+1}\) respectively. Because they are both contained in \(R_n\), so is \(g(z_1, z_2)\). It follows that \(I_n \subset R_n\), and they equal as required.

Finally, we remember from the justification for (4.31) that \(z_1 + z_2\) and \(z_1 z_2\), when considered as elements in \(E^{0,*}_\infty\), have unique representatives \(x\) and \(y\) in \(H^{\ast}(BSp(2))\). So \(r_{n,1}\) and \(r_{n,2}\) in \(\mathbb{Z}[z_1 + z_2, z_1 z_2]\) have corresponding representatives \(f_{n,1}(x, y)\) and \(f_{n,2}(x, y)\) in \(H^{\ast}(B)\), and \((z_1 z_2)^{n+1}\) has representative \(y^{n+1}\). Then by (4.38) the expressions \(f_{n,1} = x f_{n-1,1} - f_{n-1,2}\) and \(f_{n,2} = y f_{n-1,1}\) hold in \(H^{\ast}(B)\) for \(n \geq 2\), where \(f_{1,1} = x^2 - 2y\) and \(f_{1,2} = xy\).

\[\square\]

**Example 4.39.** For \(n = 3\),

\[H^\ast((B_3)_+) \cong H^\ast(B_+)/(x^4 - 4x^2y + 2y^2, x^3y - 3xy^2, y^4).\]

We shall observe the following Remarks.

**Remark 4.40.** For any \(n \geq 1\), there is an expression for \(f_{n,i}\), \(i = 1, 2\)

\[f_{n,i} = \lambda_k y^k + \sum_{\ell + 2m = n+i} \lambda_{\ell,m} x^\ell y^m\]

where \(\lambda_k = \pm 2\), \(\lambda_{\ell,m} \in \mathbb{Z}\), \(2k = n + i\). This expression comes from (4.37).

**Remark 4.41.** From Theorem 4.35, for any \(n \geq 1\) we have that

\[H^r((B_n)_+) \cong H^r(B_+) \text{ if } r \leq 4n + 3\]

\[H^r(B_n) \cong (\mathbb{Z}/2)^{n+1} \text{ if } r = 8n + 2\ell, \ell = 1, 2, \ldots\]

\[H^r(B_n) = 0 \text{ if } r \text{ is odd.}\]

Also, for \(x^{n+1} \in H^{4(n+1)}(B_n)\) there is an expression

\[x^{n+1} = \sum_{a,b} \lambda_{a,b} x^a y^b, \quad \lambda_{a,b} \in \mathbb{Z}\] (4.42)

where \(b \geq 1\), \(a + 2b = n + 1\).

For future use we include the following Lemmas.
Lemma 4.43. For any $k \geq 2$, there is an expression
\[
2y^k = \sum_m \mu_m f_{m,2} \in H^{sk}(B)
\] (4.44)
where $m \geq k - 1$, $\mu_m \in \mathbb{Z}\langle x^iy^j \rangle$, $j \geq 0$, $i \geq 1$.

Proof. We use the relation $y f_{n,2} = xy f_{n,1} - y f_{n+1,1} = x f_{n+1,2} - f_{n+2,2}$, which comes from (4.37).

For $k = 2$, it is clear that $2y^2 = x f_{1,2} - y f_{1,1} = x f_{1,2} - f_{2,2}$.

For $k \geq 3$, we have that
\[
2y^k = y^{k-2}(x f_{1,2} - f_{2,2})
\]
\[
= xy^{k-3}y f_{1,2} - y^{k-3}y f_{2,2}
\]
\[
= xy^{k-3}(x f_{2,2} - f_{3,2}) - y^{k-3}(x f_{3,2} - f_{4,2})
\]
\[
\vdots
\]
\[
= \sum_{m=k-1}^{2k-2} \mu_m f_{m,2}, \text{ where } \mu_m \in \mathbb{Z}\langle x^iy^j \rangle, \; j \geq 0, \; i \geq 1.
\]

Lemma 4.45. For $n \geq 1$, $f_{n,1} = f_{n,2} = 0$ implies that $f_{m,2} = 0$ for all $m \geq n$.

Proof. Suppose $f_{n,1} = f_{n,2} = 0$, then
\[
f_{n+1,2} = y f_{n,1} = 0
\]
\[
f_{n+2,2} = y f_{n+1,1} = xy f_{n,1} - y f_{n,2} = 0
\]
\[
f_{n+3,2} = y f_{n+2,1} = xy f_{n+1,1} - y f_{n+1,2} = 0
\]
\[
f_{n+4,2} = y f_{n+3,1} = xy f_{n+2,1} - y f_{n+2,2} = 0
\]
\[
\vdots
\]
\[
f_{n+k,2} = y f_{n+k-1,1} = xy f_{n+k-2,2} - y f_{n+k-2,2} = 0
\]
for any $k \geq 1$, by using the relation $f_{n+1,1} = x f_{n,1} - f_{n,2}$. 
\qed
Chapter 5

The Classifying Space of $\mathbb{Z}/2 \times S^3$

In Section 2.1 we viewed the subspace $A_n \subset SP^2(\mathbb{H}\mathbb{P}^n)$

$$A_n \overset{\text{def}}{=} \{([u], [v]) \mid ([u], [v]) \sim ([v], [u]), \quad d = 1/2, \quad \|u\| = \|v\| = 1\}$$

as the total space of the (real) projectivisation of the tangent bundle of $\mathbb{H}\mathbb{P}^n$. In this chapter we shall identify $A_n$ with the quotient of the quaternionic Stiefel manifold $V_{n+1}$ by the action of the subgroup $\mathbb{Z}/2 \times S^3$ of $Pin(4)$. The main aim of the Chapter is to compute the homomorphism induced by the map $B(\mathbb{Z}/2 \times S^3) \rightarrow BPin(4)$ of classifying spaces in integral cohomology.

We will usually write $G_A$ for the group $\mathbb{Z}/2 \times S^3$.

5.1 The Quotient $V_{n+1}/G_A$

Theorem 5.1. The quotient space $V_{n+1}/G_A$ is homeomorphic to the total space $A_n$ of the (real) projectivisation $\mathbb{RP}(\tau_{\mathbb{H}\mathbb{P}^n})$ of the tangent bundle of $\mathbb{H}\mathbb{P}^n$

$$\mathbb{RP}^{4n-1} \rightarrow A_n \rightarrow \mathbb{H}\mathbb{P}^n$$

where $V_{n+1}$ is the quaternionic Stiefel manifold, of ordered pairs of orthonormal vectors in $\mathbb{H}^{n+1}$.

Proof. We define the action of $\mathbb{Z}/2$ on $V_{n+1}$ by swapping the pair of vectors, $(u, v) \mapsto (v, u)$ and the action of $S^3$ on $V_{n+1}$ by $(v, u) \mapsto (va, ua)$. Then the quotient of $V_{n+1}$ by the action of $G_A := \mathbb{Z}/2 \times S^3$ is given by

$$V_{n+1}/G_A = V_{n+1}/\sim$$
where \((u, v) \sim (v, u) \sim (va, ua)\) for \(a \in S^3\). By [42], the tangent space \(T[h]\) at any point \([h] \in \mathbb{H}P^n\) may be thought of as the space of quaternionic linear transformations \(\alpha: [h] \to [h]^\perp\). This works by identifying a transformation \(\alpha\) with the line through \((h, \alpha(h))\); this line is generated over \(\mathbb{H}\) by \(h + \alpha(h)\) in \(\mathbb{H}^{n+1}\). In other words, \(\alpha\) is identified with its own graph, and \(T[h]\) is a \(4n\)-dimensional real vector space (but is not quaternionic [49, page 68]). We write the total space of the tangent bundle as \(\mathbb{R}(\tau_{\mathbb{H}P^n}) = \bigcup_n T[h]\), and its unit sphere bundle as \(S(\tau_{\mathbb{H}P^n}) = \bigcup_n S[h]\), where \(S[h]\) denotes the transformations that have \(\|\alpha(h)\| = 1\).

We first define a homeomorphism \(V_{n+1}/S^3\) to \(S(\tau_{\mathbb{H}P^n})\) by \((u, v) \mapsto \alpha_{u,v}\), where

\[
\alpha_{u,v} : [(u + v)/\sqrt{2}] \mapsto [(u + v)/\sqrt{2}]^\perp
\]

is given by

\[
\alpha_{u,v}((u + v)q/\sqrt{2}) = \alpha_{u,v}((u + v)/\sqrt{2})q = (u - v)q/\sqrt{2}
\]

for any \(q \in \mathbb{H}\). The map is well defined because \((ua, va) \mapsto \alpha_{ua,va}\), and we have \(\alpha_{ua,va}((u + v)a/\sqrt{2}) = (u - v)a/\sqrt{2}\) for any \(a \in S^3\). So \(\alpha_{ua,va}\) is the same function as \(\alpha_{u,v}\). We have used the fact that \((u + v)^*(u - v) = 0\), and that \(\alpha_{u,v}\) corresponds to a unit tangent vector because it maps unit vectors to unit vectors.

The map is homeomorphism because we can describe its inverse.

Suppose \([h] \in \mathbb{H}P^n\), then any unit tangent vector in \(S[h]\) corresponds to a linear transformation \(\alpha: [h] \to [h]^\perp\), which is given by \(\alpha(h) = k\) for some \(k \in [h]^\perp \subset \mathbb{H}^{n+1}\) with \(\|k\| = 1\). Then define the inverse function from \(S(\tau_{\mathbb{H}P^n})\) to \(V_{n+1}/S^3\) by

\[
\alpha \mapsto ((h + k)/\sqrt{2}, (h - k)/\sqrt{2}).
\]

It is well defined because \(\alpha(hq) = kq\) for any \(q \in S^3\).

Both of these maps are continuous.

Finally, we factor out by the action of \(\mathbb{Z}/2\), which acts by \((u, v) \sim (v, u)\) and \(\alpha \sim -\alpha\). The previous homeomorphism and its inverse are equivariant, so they pass to orbit spaces. 

**Corollary 5.2.** We have a homeomorphism \(A \cong V/G_A\) by defining \(A := \bigcup_n A_n\) and \(V/G_A := \bigcup_n V_{n+1}/G_A\).

**Proof.** We can check that the above homeomorphisms are compatible when we include \(\mathbb{H}^{n+1}\) in \(\mathbb{H}^{n+2}\). So we get a homeomorphism of colimits. 

\[\square\]
5.2 The Cohomology of $A_2 \subset SP^2(\mathbb{HP}^2)$

Corollary 5.2 implies that the colimit $A = \bigcup_n A_n$ is a model for the classifying space $B(\mathbb{Z}/2 \times S^3) = BG_A$. So the quotient $V/G_A \to V/Pin(4)$ is a model for the map $f: BG_A \to BPin(4)$ induced by the inclusion of the subgroup $G_A \to Pin(4)$, and we have a commutative diagram

$$
\begin{array}{ccc}
A_n & \xrightarrow{f} & V_{n+1}/Pin(4) = \Gamma_n \\
V_{n+1}/\mathbb{Z}/2 \times S^3 & \xrightarrow{\iota} & V_{n+1}/Pin(4) \\
B(\mathbb{Z}/2 \times S^3) & \xrightarrow{\iota} & BPin(4)
\end{array}
$$

The map $f$ induces the homomorphism $f^*$, which is important for our cohomological computation of $SP^2(\mathbb{HP}^n)$. In this section we will describe the integral cohomology ring of $A_n$ where $n = 2$, by considering the Serre spectral sequence for the fibre bundle

$$\mathbb{RP}^7 \xrightarrow{\iota} A_2 \to \mathbb{HP}^2.$$ 

The homotopy exact sequence of this bundle shows that $\iota_*$ is an isomorphism on $\pi_1(-)$ so that $\pi_1(A_2) = \mathbb{Z}/2$.

Since $A_2$ is a projectivisation by Theorem 5.1, it has a canonical line bundle over it, which restricts to the tautological line bundle $\lambda$ on each fibre. So, in mod 2 cohomology, its 1st Stiefel-Whitney class $a \in H^1(A_2; \mathbb{Z}/2)$ pulls back to $i^*(a) = a \in H^1(\mathbb{RP}^7; \mathbb{Z}/2)$. This abuse of notation is not necessarily confusing, because $a$ is the only non-zero element of $H^1(A_2; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

It helps to note that $a$ may also be defined in $H^1(A_2; \mathbb{Z}/2)$ by considering the map $A_2 \xrightarrow{g} \mathbb{RP}^\infty$, given by the composition of the map $A_2 \to A = BG_A$ of diagram (5.3) and the map $r_1: BG_A \to B(\mathbb{Z}/2) \cong \mathbb{RP}^\infty$ of classifying spaces induced by the projection $G_A \to \mathbb{Z}/2$. So we have $a := r_1^*(a)$ in $H^1(A; \mathbb{Z}/2) = H^1(BG_A; \mathbb{Z}/2)$, which restricts to the element $g^*(a)$ in $H^1(A_2; \mathbb{Z}/2)$. It must agree with $a$ as first defined, because $g_*$ is non-zero on $\pi_1(-)$, so $g$ represents the line bundle $\lambda$.

**Remark 5.4.** We can describe an integral element $w \in H^2(A) = H^2(BG_A)$ in exactly the same way, by pulling back $w \in H^2(\mathbb{RP}^\infty)$ along $r_1^*$. This element restricts to $w := g^*(w)$ in $H^2(A_2)$. Then reduction mod 2 satisfies $\rho(w) = a^2$ in both $H^2(A_2; \mathbb{Z}/2)$ and $H^2(A; \mathbb{Z}/2)$, because the same equation is true in $H^2(\mathbb{RP}^\infty; \mathbb{Z}/2)$.
We can now describe the mod 2 cohomology ring $H^*(A_2; \mathbb{Z}/2)$, which helps to compute the integral cohomology. We apply the Leray-Hirsch Theorem 4.21, which tells us that the mod 2 cohomology looks additively like that of a product bundle if certain conditions are satisfied. In this case the conditions hold because the elements $a, a^2, \ldots, a^7$ in $H^*(A_2; \mathbb{Z}/2)$ are enough to map onto $H^*(\mathbb{R}P^7; \mathbb{Z}/2)$. So the Theorem gives that the mod 2 cohomology of $A_2$ is a free $H^*(\mathbb{H}P^2_+; \mathbb{Z}/2)$-module

$$H^*((A_2)_+; \mathbb{Z}/2) \cong H^*((\mathbb{H}P^2)_+; \mathbb{Z}/2) \langle 1, a, a^2, \ldots, a^7 \rangle.$$  \hfill (5.5)

Let $\pi: A_2 \to \mathbb{H}P^2$ be the bundle projection. Then the isomorphism (5.5) includes the fact that $\pi^*: H^*(\mathbb{H}P^2; \mathbb{Z}/2) \to H^*(A_2; \mathbb{Z}/2)$ is a monomorphism, its image is a direct summand, and it induces the module structure. So we set $z := \pi^*(z)$ in $H^4(A_2; \mathbb{Z}/2)$, where $H^*((\mathbb{H}P^2)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[z]/(z^3)$, and $z$ is the mod 2 reduction of the first symplectic Pontryagin class of the tautological line bundle over $\mathbb{H}P^2$.

Since $A_2$ is the projectivisation of $\tau_{\mathbb{H}P^2}$, multiplication in $H^*(A_2; \mathbb{Z}/2)$ involves its Stiefel-Whitney classes [57, Theorem 2.3]. According to Szczarba [55, Corollary 2.3], the total Stiefel-Whitney class is

$$w(\tau_{\mathbb{H}P^2}) = (1 + z)^3 = 1 + z + z^2 \quad \text{in} \quad H^*(\mathbb{H}P^2; \mathbb{Z}/2)$$

so we have that

$$a^8 = \sum_{j=1}^{8} w_j(\tau_{\mathbb{H}P^2})a^{8-j} = za^4 + z^2$$

in $H^8(A_2; \mathbb{Z}/2)$. Note that $a^{11} = za^7 + z^2a^3 \neq 0$, and $a^{12} = 0$. To summarise:

**Proposition 5.6.** There is an isomorphism of graded rings

$$H^*((A_2)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[z, a]/(z^3, a^8 + za^4 + z^2)$$

where $|a| = 1$ and $|z| = 4$.

The integral cohomology of $A_2$ is more delicate compared to the $\mathbb{Z}/2$-cohomology. However, considering the Serre spectral sequence (SSS) for the associated tangent sphere bundle will lead to Proposition 5.15.

We shall write $\tau$ for $\tau_{\mathbb{H}P^2}$.

Let $S(\tau) := A_2'$ be the total space of the tangent sphere bundle and let $D(\tau)$ be the total space of the corresponding disk bundle $D^8 \to D(\tau) \xrightarrow{p} \mathbb{H}P^2$. Then the
CHAPTER 5. THE CLASSIFYING SPACE OF $\mathbb{Z}/2 \times S^3$

combinative diagram of (homotopy) cofibre sequences

$$
\begin{array}{cccc}
S(\tau) & \longrightarrow & D(\tau) & \longrightarrow D(\tau)/S(\tau) \\
\parallel & \parallel & p \doteq & \doteq \\
S(\tau) & \longrightarrow & \mathbb{H}P^2 & \longrightarrow \text{Th}(\tau)
\end{array}
$$

induces the long exact sequence in cohomology

$$
\cdots \leftrightarrow H^*(S(\tau)_+) \leftrightarrow H^*(\mathbb{H}P^2_+) \leftrightarrow \cdots 
\leftrightarrow H^*(\text{Th}(\tau)) \leftrightarrow \cdots.
\tag{5.7}
$$

This is a version of the Gysin sequence [22] for $S(\tau)$.

Since $\tau$ is orientable, there is a Thom class $t \in H^8(\text{Th}(\tau); \mathbb{Z})$, which, by definition (see Section 6.1 for the definition of a Thom space), pulls back along the zero section to the Euler class $e(\tau) \in H^8(\mathbb{H}P^2_+)$; that is

$$
t^*(t) = e(\tau) = \chi z^2
\tag{5.8}
$$

where $\chi = 3$ is the Euler characteristic of $\mathbb{H}P^2$.

For the tangent sphere bundle $S^7 \to A_2 \to \mathbb{H}P^2$, the $E_2$ page of the SSS is

$$
E_2^{p,q} \cong H^p(\mathbb{H}P^2_+; H^q(S_+^7; \mathbb{Z})) = \begin{cases} 
\mathbb{Z} & \text{for } p = 0, 4, 8 \text{ and } q = 0, 7 \\
0 & \text{otherwise}
\end{cases}
$$

because the local coefficients are trivial, and it converges to $H^*(A_2^+)$. We call this spectral sequence $SS_1$.

Similarly, for $\mathbb{R}P^7 \stackrel{i}{\to} A_2 \to \mathbb{H}P^2$, the $E_2$ page is

$$
E_2^{p,q} \cong H^p(\mathbb{H}P^2_+; H^q(\mathbb{R}P^7_; \mathbb{Z})) = \begin{cases} 
\mathbb{Z} & \text{for } p = 0, 4, 8 \text{ and } q = 0, 7 \\
\mathbb{Z}/2 & \text{for } p = 0, 4, 8 \text{ and } q = 2, 4, 6 \\
0 & \text{otherwise},
\end{cases}
$$

and it converges to $H^*((A_2)_+)$. We call this spectral sequence $SS_2$.

In order to apply the Leray-Hirsch Theorem above, we used that $i^*$ is an epimorphism in mod 2 cohomology. This forces all differentials to be 0 in the spectral sequence $E_2^{p,q} \cong H^p(\mathbb{H}P^2_+; H^q(\mathbb{R}P^7_; \mathbb{Z}/2))$, which converges to $H^*((A_2)_+; \mathbb{Z}/2)$ (and is the reason why (5.5) is true). We call this spectral sequence $SS_3$. 
In all three spectral sequences, the only possible non-zero differentials are \(d_4\) and \(d_8\). The fact that all differentials \(d_4: \mathbb{Z}/2 \to \mathbb{Z}/2\) are trivial in \(SS_3\) therefore implies that the corresponding differentials \(d_4\) in \(SS_2\) are zero, by naturality of reduction mod 2. Also for dimensional reasons all the other \(d_4\)s in \(SS_2\) are trivial. So it remains to look at the only other possible non-trivial differential \(d_8: E_{0,7} \to E_{8,0}\) in \(SS_2\), which may be written as \(d_8: H^7(\mathbb{R}P^7; \mathbb{Z}) \to H^8(\mathbb{H}P^2_+; \mathbb{Z})\), and therefore as \(d_8: \mathbb{Z} \to \mathbb{Z}\).

Let \(v, g\) and \(z^2\) be generators
\[
v \in H^7(\mathbb{R}P^7_+; \mathbb{Z}) \cong \mathbb{Z}, \quad g \in H^7(S^7_+; \mathbb{Z}) \cong \mathbb{Z}, \quad z^2 \in H^8(\mathbb{H}P^2_+; \mathbb{Z}) \cong \mathbb{Z}.
\]

Factoring out the action of \(\mathbb{Z}/2\) on the fibres of \(S(\tau)\) gives a bundle map from the sphere bundle of \(\tau\) to its projectivisation. On each fibre it is projection \(\sigma: S^7 \to \mathbb{R}P^7\). The bundle map induces a map of spectral sequences from \(SS_2\) to \(SS_1\), which works on the \(E_2\) pages by applying \(\sigma^*: H^7(\mathbb{R}P^7_+) \to H^7(S^7_+)\). This is actually given by \(v \mapsto 2g\), which can be justified by considering the long exact sequence in cohomology induced by the cofibre sequence
\[
S^7 \xrightarrow{\sigma} \mathbb{R}P^7 \to \mathbb{R}P^8,
\]
or by the geometric fact that \(\sigma\) has degree 2. Note \(\sigma^*\) is the identity in dimension 0.

One of the properties of the Gysin sequence [39, Example 5.C, page 143] tells us that \(d'_8\) is given by (5.8), in the form \(d'_8(1 \otimes g) = 3(z^2 \otimes 1)\). So we can compute
\[
d_8(1 \otimes v) = \sigma^*(d_8(1 \otimes v)) \quad (5.9)
= d_8(\sigma^*(1 \otimes v)) \quad (5.10)
= d_8(1 \otimes 2g) \quad (5.11)
= 2d'_8(1 \otimes g) \quad (5.12)
= 2 \cdot 3(z^2 \otimes 1) \quad (5.13)
= 6(z^2 \otimes 1). \quad (5.14)
\]

Also, \(E_9 = E_\infty\) for \(SS_2\).

Assembling all the above together we have the following.

**Proposition 5.15.** The integral cohomology ring of \(A_2\) is described by Table 5.1.
Table 5.1: $H^* := H^*(A_2)$

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^*$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$w$</td>
<td>$\tilde{z}$, $w^2$</td>
<td>$w\tilde{z}$, $w^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>*</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^*$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/6$</td>
<td>0</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$w^2\tilde{z}$, $\tilde{z}^2$</td>
<td>$w^3\tilde{z}$, $w\tilde{z}^2$</td>
<td>$m$</td>
<td>$w^2\tilde{z}^2$</td>
<td>$w^3\tilde{z}^2$</td>
<td>$m\tilde{z}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $w^4 = w^2\tilde{z} + 3\tilde{z}^2 \neq 0$, and $w^6 = 0$. \hfill (5.16)

Proof. In the spectral sequence $SS_2$, (5.14) shows that the $E_\infty$ term differs from the $E_2$ term only because $E_2^{0,7}$ has become 0, and $E_2^{8,0}$ has become $\mathbb{Z}/6$ on generator $\tilde{z}^2$. So we have to check the extension problems and the multiplicative structure.

From Remark 5.4, we have the element $w$ in $H^2(A_2)$, which represents the generator of $E_\infty^{0,2} \cong \mathbb{Z}/2$. Also, we have $\tilde{z} := \pi^*(z) \in H^4(A_2)$, which represents the generator of $E_\infty^{4,0} \cong \mathbb{Z}$. Therefore every element $w^i\tilde{z}^j$ of $H^{2i+4j}(A_2)$ represents the corresponding generator of $E_\infty^{4i,2j}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 2$, and has order 2. So every extension problem in which they arise is trivial. The rest of the non-zero groups in $E_\infty$ are $E_\infty^{4,7} \cong E_\infty^{8,7} \cong \mathbb{Z}$, in dimensions 11 and 15, where there are no extension problems.

To find the relation (5.16), we first reduce mod 2 and look back at Proposition 5.6. Since $\rho(w^4) = a^8 = za^4 + z^2$, we must have $w^4 = w^2\tilde{z} + k\tilde{z}^2$ for some odd integer $k$. Hence $w^6 = w^4\tilde{z} + kw^2\tilde{z}^2 = 0$. Also, $w^4$ has order 2 and $\tilde{z}^2$ has order 6, so $k \equiv 3 \mod 6$.

The appearance of an element of order 6 is interesting, and suggests that further work should be done on $H^*(A_n)$ for larger $n$. In particular, since $\chi(\mathbb{H}P^n) = n + 1$, the corresponding SSS with coefficient in $\mathbb{Z}[1/2]$ shows that

$$H^*(A_n; \mathbb{Z}[1/2]) \cong H^*(\mathbb{H}P^n; \mathbb{Z}[1/2])/(n + 1)p_n^\ast).$$

So for any odd prime $p$, each of $H^{4n}(A_n; \mathbb{Z}[1/2])$ and $H^{4n}(A_n)$ contains a summand isomorphic to $\mathbb{Z}/p^n$, where $p^n$ is the highest power of $p$ dividing $(n + 1)$.  \hfill $\Box$
5.3 The Map $B(\mathbb{Z}/2 \times S^3) \to BPin(4)$

For this section, we should remember our notation $p_i \in H^4(BSp(2))$ for the symplectic Pontryagin class, where $i = 1$ and 2, and $H^*(BSp(2)_+) \cong \mathbb{Z}[p_1, p_2]$. Restriction to $BS^3 = BSp(1) = \mathbb{HP}^\infty$ acts by mapping $p_1$ to $z$ and $p_2$ to 0, and restriction to $BS^3 \times BS^3 = \mathbb{HP}^\infty \times \mathbb{HP}^\infty$ by mapping $p_1$ to $z_1 + z_2$ and $p_2$ to $z_1 z_2$, where $z_1 = z \otimes 1$ and $z_2 = 1 \otimes z$. Then $H^*(B(S^3 \times S^3)_+) \cong H^*((\mathbb{HP}^\infty \times \mathbb{HP}^\infty)_+) \cong \mathbb{Z}[z_1, z_2]$.

For $BG_A$ we need to define generators more carefully. Of course we know that $H^*(BG_A) \cong \mathbb{Z}[w, z]/(2w)$, and we have already defined $w$ in Remark 5.4. So we must describe $z \in H^4(BG_A)$ in terms of the model $A = V/G_A$. We shall use the map $\pi_\infty: V/G_A \to \mathbb{HP}^\infty$, defined as in Theorem 5.1 by

$$\pi_\infty([u, v]) = [(u + v)/\sqrt{2}]$$

where $(u, v)$ lies in $V$.

**Lemma 5.17.** There is a homotopy commutative triangle

$$
\begin{array}{ccc}
A = V/G_A & \xrightarrow{\pi_\infty} & \mathbb{HP}^\infty \\
\epsilon \downarrow & & \downarrow r_2 \\
\mathbb{RP}^\infty \times \mathbb{HP}^\infty & \xrightarrow{r_2} & \mathbb{HP}^\infty
\end{array}
$$

where $\epsilon$ is a homotopy equivalence and $r_2$ is projection onto the second factor.

**Proof.** We define $\epsilon$ by first considering the maps

$$V \leftarrow S^\infty \times V \rightarrow S^\infty \times S^{4\infty + 3}.$$ 

Each space is contractible, and is acted on freely by $G_A$. The element $\tau$ of order 2 acts antipodally on $S^\infty$, and diagonally on $S^\infty \times V$, and $S^3$ acts only on the second factors, in the usual way ( $S^3$ acts trivially on $S^\infty$). The left map is projection onto the second factor, and the right map is defined by $(s, (u, v)) \mapsto (s, (u + v)/\sqrt{2})$. So both maps are homotopy equivalences and equivariant. The induced maps of orbit spaces are also homotopy equivalences. They combine to give the triangle. \qed

The Lemma states that $\pi_\infty$ is a model for $r_2$, and allows us to use $e^{-1}i_1: \mathbb{RP}^\infty \to A$ and $e^{-1}i_2: \mathbb{HP}^\infty \to A$ as models for the maps induced by the subgroup inclusions $\mathbb{Z}/2 \to G_A$ and $S^3 \to G_A$. 

Corollary 5.18. The element \( \tilde{z} \in H^4(A_2) \) arises by pulling back \( z \) from \( H^4(BG_A) \).

Proof. This follows from the commutative diagram

\[
\begin{array}{ccc}
A_2 & \xrightarrow{\pi} & H^4 \Phi^2 \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi_{\infty}} & H^4 \Phi^\infty
\end{array}
\]

by applying \( H^4(-) \).

Note that the map \( g \) of Remark 5.4 can also be constructed from the homotopy equivalence of Lemma 5.17 by applying the projection \( r_1 \) instead of \( r_2 \).

Recall (4.32), (4.33), and Remark 4.34. For \( a \in S^3 \), the composition of the subgroup inclusions \( S^3 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2) \) is given by \( a \mapsto \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \in Sp(2) \) and therefore agrees with \( S^3 \hookrightarrow S^3 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2) \) where the first subgroup is the diagonal. Let \( \tau \) be the generator of \( \mathbb{Z}/2 \) and consider the composition of inclusions \( \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2) \), which is given by \( \tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in Sp(2) \). This may be rewritten as \( \tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) in terms of the basis of eigenvectors \( \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), which agrees with the composition \( \mathbb{Z}/2 = O(1) \rightarrow U(1) \rightarrow Sp(1) \rightarrow Sp(2) \).

Proposition 5.19. Let \( \tilde{x}, \tilde{y} \) be the generators for \( H^*(BPin(4)_+) \) given in Proposition 4.17. Then, the homomorphism \( f^*: H^*(BPin(4)) \rightarrow H^*(BG_A) \) induced by the inclusion \( G_A \rightarrow Pin(4) \) satisfies

\[
\begin{align*}
f^*(\tilde{x}) &= 2z + w^2 \\
f^*(\tilde{y}) &= z^2 + w^2z,
\end{align*}
\]

where \( z \) and \( w \) are defined as in Remark 5.4 and Lemma 5.17.

Proof. Given \( \ell: BPin(4) \rightarrow BSp(2) \), then by definition \( \tilde{x} = \ell^*(p_1) \) and \( \tilde{y} = \ell^*(p_2) \).

We first show (5.20). Since there are no cross terms in \( H^4(BG_A) \),

\[
H^4(BSp(2)) \xrightarrow{\ell^*} H^4(BPin(4)) \xrightarrow{f^*} H^4(BG_A)
\]

acts by \( p_1 \mapsto \tilde{x} \mapsto Az + Bw^2 \),

where \( A \in \mathbb{Z} \) and \( B = 0 \) or \( 1 \).

To find \( A \), we consider

\[
BSp(2) \xleftarrow{\ell} BPin(4) \xleftarrow{B(S^3 \times S^3)} BS^3
\]
and apply $H^4(-)$. The induced homomorphisms are given by $p_1 \mapsto x \mapsto z_1 + z_2 \mapsto 2z$, so $A = 2$ by comparison with (4.32).

To find $B$, we consider

$$BSp(2) \leftarrow BSp(1) \leftarrow BU(1) \leftarrow B\mathbb{Z}/2,$$

where $BSp(1) = HP^\infty, BU(1) = CP^\infty$ and $B\mathbb{Z}/2 = RP^\infty$, and apply $H^4(-)$. The induced homomorphisms are given by $p_1 \mapsto x \mapsto z_1 + z_2 \mapsto 2z$, so $A = 2$ by comparison with (4.32).

We now show (5.21). Suppose $f^*(\tilde{y}) = Cz^2 + Dw^2z + Ew^4$, where $C \in \mathbb{Z}$ and $D, E = 0$ or 1, and let us consider

$$BSp(2) \leftarrow BPin(4) \leftarrow B(S^3 \times S^3) \leftarrow BS^3.$$

Then the induced homomorphisms in $H^8(-)$ are $p_2 \mapsto \tilde{y} \mapsto z_1z_2 \mapsto z^2$, so $C = 1$ by comparison with (4.32).

To find $D$ and $E$, we consider the commutative diagram

$$\begin{array}{ccc}
V_{n+1}/G_A & \longrightarrow & V_{n+1}/Pin(4) \\
\downarrow f_2 & & \downarrow f_2 \\
BG_A & \longrightarrow & BPin(4) \\
\downarrow f & & \downarrow f \\
BSp(2) & \longrightarrow & BS^3
\end{array}$$

where $V_{n+1}/G_A \cong A_n$, and $V_{n+1}/Pin(4)$ is identified with $\Gamma_n$. For $n = 2$, we get

$$\begin{array}{ccc}
A_2 & \longrightarrow & \Gamma_2 \\
\downarrow f_2 & & \downarrow f_2 \\
BG_A & \longrightarrow & BPin(4) \\
\downarrow f & & \downarrow f \\
BSp(2) & \longrightarrow & BS^3
\end{array}$$

to which we may apply $H^8(-)$ and the results of Section 5.2.

We recall that $H^8((A_2)_+) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/6$ from Proposition 5.15 with corresponding generators are $w^2\tilde{z}$ and $\tilde{z}^2$, which pull back from $w^2z$ and $z^2$ in $H^8(BG_A)$ by Remark 5.4 and Corollary 5.18. Also, $w^4 = w^2\tilde{z} + 3\tilde{z}^2$ from (5.1). We saw that $\tilde{x}^2 = \tilde{y} \in H^6(\Gamma_2)$ in Example 4.20, and $f^*(\tilde{x}) = 2z + w^2$ implies that $f^*(\tilde{x}^2) = f^*(\tilde{x})^2 = (2z + w^2)^2 = 4z^2 + w^4$. Hence by the commutativity of the diagram, we deduce that

$$\tilde{z}^2 + Dw^2\tilde{z} + Ew^4 = 4\tilde{z}^2 + w^4.$$
CHAPTER 5. THE CLASSIFYING SPACE OF $\mathbb{Z}/2 \times S^3$

in $H^8(A_2)$. So $(1 + 3E)\hat{z}^2 + (D + E)w^2\hat{z} = 7\hat{z}^2 + w^2\hat{z}$, and

\[1 + 3E \equiv 1 \mod 6 \quad \text{and} \quad D + E \equiv 1 \mod 2.\]

Therefore, we have that $D = 1$ and $E = 0$, as we wanted. \qed

**Corollary 5.22.** Given $H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx)$, then

\[f^*(x^m) = (2z)^m, \quad f^*(y^m) = (z^2 + w^2z)^m = \sum_{k=0}^{m} \binom{m}{k} (z^2)^{m-k}(w^2z)^k.\]

in $H^{4m}(B)$ and $H^{8m}(B)$ for any $m \geq 1$. \[\]
Chapter 6

The Thom Space $\text{Th}(\theta)$

In this Chapter, we will show that the quotient $L/A$ is a Thom space $\text{Th}(\theta)$, where $\theta$ is the universal vector bundle over $BPin(4)$, which is sometimes denoted by $MPin(4)$. The homotopy equivalences $L/A \simeq SP^2/\Delta \simeq B/\hat{\Delta}$ of Chapter 2 induce isomorphisms in cohomology, which are important in our calculation of $H^*(SP^2)$. In earlier chapters we have shown the interconnections between the Borel space and the symmetric square.

Recall from Section 2.2 the commutative diagram

$$
\begin{array}{ccc}
\Delta_n & \overset{i}{\longrightarrow} & SP^2_n \\
\downarrow{\pi} & & \downarrow{\pi} \\
\hat{\Delta}_n & \overset{f}{\longrightarrow} & B_n \\
\end{array}
$$

$$
\simeq
\begin{array}{ccc}
\Delta_n & \overset{i}{\longrightarrow} & SP^2_n \\
\downarrow{\pi} & & \downarrow{\pi} \\
\hat{\Delta}_n & \overset{f}{\longrightarrow} & B_n \\
\end{array}
$$

where $i$ and $f$ are cofibrations and $\pi: B_n \to SP^2_n$ is the canonical projection. Also, from the decomposition $SP^2_n = L_n \cup_{A_n} N_n$ in Section 2.1, and the natural homeomorphism $SP^2_n/N_n \cong L_n/A_n$, we have a commutative diagram

$$
\begin{array}{ccc}
N_n & \overset{i}{\longrightarrow} & SP^2_n \\
\downarrow{\pi} & & \downarrow{\pi} \\
A_n & \overset{j}{\longrightarrow} & L_n \\
\end{array}
$$

$$
\cong
\begin{array}{ccc}
N_n & \overset{i}{\longrightarrow} & SP^2_n \\
\downarrow{\pi} & & \downarrow{\pi} \\
A_n & \overset{j}{\longrightarrow} & L_n \\
\end{array}
$$

where $i$ and $j$ are also cofibrations.

Applying $H^*(-)$ to either diagram yields a ladder of long exact sequences (LES) in cohomology. For the infinite dimensional cases, we consider the spaces $X = \bigcup_n X_n$ as colimits.
6.1 \( MPin(4) \)

A real \( n \)-plane vector bundle \( \xi \) over \( X \) has the structure group \( O(n) \) when a continuous choice of inner product is given each fibre. After making this choice, the Thom space \( Th(\xi) := D(\xi)/S(\xi) \) is constructed by taking the unit disk bundle \( D(\xi) \) and collapsing its boundary, the unit sphere bundle \( S(\xi) \), to a single point \( \infty \). Equivalently, \( Th(\xi) \) is homeomorphic to the mapping cone \( C(p) = X \cup_p CS(\xi) \) of the projection, \( S(\xi) \xrightarrow{p} X \), where \( \infty \) corresponds to the cone point.

Now remember that for the model \( \Gamma = V/\text{Pin}(4) \) for the classifying space \( B\text{Pin}(4) \), the corresponding action of \( \text{Pin}(4) \) on \( \mathbb{R}^4 \sim \mathbb{H} \) is given in Section 3.2.

Remark 6.2. The universal 4-disk and 3-sphere bundles associated to \( \theta \) are given by

\[
D^4 \to D(\theta) \to \Gamma \quad \text{for} \quad |h| \leq 1 \quad \text{and} \quad S^3 \to S(\theta) \to \Gamma \quad \text{for} \quad |h| = 1,
\]

where \( D(\theta) = V \times_{\text{Pin}(4)} D^4 \) and \( S(\theta) = V \times_{\text{Pin}(4)} S^3 \).

Lemma 6.3. There exists a homeomorphism \( g \) between the total space of the sphere bundle \( S(\theta) \) and the quotient space \( V/G_A \), where \( G_A = \mathbb{Z}/2 \times S^3 \).

Proof. For \( a, b, h \in S^3 \) and \( (u, v) \in V \), we have that

\[
S(\theta) = \{((u, v), h) \mid ((u, v), h) \sim ((ua, vb), \bar{a}hb) \sim ((v, u), \bar{h})\}
\]

and

\[
V/G_A = \{(u, v) \mid (u, v) \sim (ub, vb) \sim (v, u)\}.
\]

So we define \( g: S(\theta) = V \times_{\text{Pin}(4)} S^3 \to V/G_A \) on equivalence classes by sending \([u, v], h] \) to \([uh, v] \). For \( g^{-1}: V/G_A \to V \times_{\text{Pin}(4)} S^3 \), we send \([u, v] \) to \([u, v], 1] \).

These are well-defined, continuous, and inverse to each other. \( \square \)

In the subsequent parts, so long as the meaning is clear we use the notation \( S(\theta) \) and \( D(\theta) \) for \( S^3 \to S(\theta) \to \Gamma_n \) and \( D^4 \to D(\theta) \to \Gamma_n \) respectively, as well as infinite dimensional cases.

Proposition 6.4. For any \( n \geq 1 \), there exist homeomorphisms

\[
S(\theta) \cong A_n \quad \text{and} \quad D(\theta) \cong L_n,
\]

that are consistent with the inclusions \( S(\theta) \hookrightarrow D(\theta) \) and \( A_n \hookrightarrow L_n \).
\textbf{CHAPTER 6. THE THOM SPACE }\textbf{Th(\theta)}

\textit{Proof.} For any \( n \geq 1 \), consider the open disk bundle \( \hat{D}^4 \to \hat{D}(\theta) \to \Gamma_n \), where \( \hat{D} = \{ h \in \mathbb{H} \mid |h| < 1 \} \) and \( \hat{D}(\theta) = V_{n+1} \times_{\text{Pin}(4)} \hat{D}^4 \). We first show a certain map

\[ \epsilon: \hat{D}(\theta) \to SP^2_n \setminus \Delta_n \]

is a homeomorphism. If \( h \in \mathbb{H} \) has \( |h| < 1 \), then \( \epsilon \) is given by \([u, v], h] \mapsto (u, v)N^{-1}_h \) where \( N^{-1}_h = \left( \begin{smallmatrix} (C+D)/2 & h/(C+D) \\ h/(C+D) & (C+D)/2 \end{smallmatrix} \right) \) and \( C = (1 + |h|)^{1/2}, \quad D = (1 - |h|)^{1/2} \) (the matrix \( N \) is the symmetric orthogonalisation operator for \((x, y)\) as in Definition 4.25). The map \( \epsilon \) is well-defined because \([[(u, v), h] \sim [(v, u), \tilde{h}]] \sim [(ua, vb), \bar{ahb}] \) and \( N^{-1}_h(\frac{0}{1} \frac{1}{0}) = (\frac{0}{1} \frac{1}{0})N^{-1}_h, \ N^{-1}_h(\frac{a}{0} \frac{0}{b}) = (\frac{a}{0} \frac{0}{b})N^{-1}_{ahb}, \) so \( (u, v)N^{-1}_h \sim (v, u)N^{-1}_h \sim (ua, vb)N^{-1}_{ahb}. \)

For any pair of unit column vectors \( x, y \in \mathbb{H}^{n+1} \), \( d([x], [y]) \neq 0 \) (chordal distance \( d \) as in Definition 2.1) representing an element of \( SP^2_n \setminus \Delta_n \), the inverse of \( \epsilon \),

\[ \epsilon^{-1}: SP^2_n \setminus \Delta_n \to \hat{D}(\theta) \]

is given by \([x, y] \mapsto [(\hat{x}, \hat{y}), h] \) where \( (\hat{x}, \hat{y}) = (x, y)N \) and \( h = x^*y \in \mathbb{H} \), so \( \epsilon^{-1} \) is also well-defined because \( |\hat{h}| = |h| = |x^*y| = |(xa)^*(yb)|. \)

Recalling Section 4.2.2, \( \epsilon \) maps \( D(\theta)_{\sqrt{3}/2} \) homeomorphically onto \( A_n \) and \( D(\theta)_{\leq \sqrt{3}/2} \) homeomorphically onto \( L_n \) where

\[ D(\theta)_{\sqrt{3}/2} := \{ [(u, v), h] \mid |h| = \sqrt{3}/2 \} \]
\[ D(\theta)_{\leq \sqrt{3}/2} := \{ [(u, v), h] \mid |h| \leq \sqrt{3}/2 \}. \]

Clearly \( D(\theta)_{\sqrt{3}/2} \cong S(\theta) \) and \( D(\theta)_{\leq \sqrt{3}/2} \cong D(\theta) \).

\textbf{Corollary 6.5.} For any \( n \geq 1 \), the space \( L_n/A_n \cong SP^2_n/N_n \) is homeomorphic to the Thom space \( \text{Th}(\theta) \) over \( \Gamma_n \).

\textit{Proof.} This follows from the homeomorphism of pairs given by Proposition 6.4. \( \square \)

\textbf{Proposition 6.6.} The space \( L/A \cong SP^2/N \) is homeomorphic to the Thom space \( \text{MPin}(4) \) over \( \Gamma \).

\textit{Proof.} As \( n \) increases, the homeomorphism \( \alpha \) and their restrictions to \( D(\theta)_{\leq \sqrt{3}/2} \) and \( D(\theta)_{\sqrt{3}/2} \) are compatible, so we get homeomorphisms \( S(\theta) \cong A \) and \( D(\theta) \cong L \). \( \square \)
CHAPTER 6. THE THOM SPACE

6.2 The Cohomology of $MPin(4)$

The main aim of this section is to describe the cohomology ring

$$H^*(MPin(4)) \cong H^*(B/\widehat{\triangle}) \cong H^*(SP^2/\triangle)$$

with $\mathbb{Z}$ and $\mathbb{Z}/2$ coefficients. For example, we shall show that

$$H^*(B/\widehat{\triangle};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \text{if } \ast = 4k > 4 \\ \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 & \text{if } \ast = 4k + 1, 4k + 3, k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

We use the ladder of long exact sequences

$$
\begin{array}{cccccccc}
\cdots & H^{*+1}(SP^2/\triangle) & \xleftarrow{\delta} & H^*(\triangle) & \xleftarrow{i^*} & H^*(SP^2) & \xleftarrow{\pi^*} & H^*(SP^2/\triangle) & \cdots \\
\cong & & \uparrow & & \downarrow & & \uparrow & & \\
\cdots & H^{*+1}(B/\widehat{\triangle}) & \xleftarrow{\delta} & H^*(\widehat{\triangle}) & \xleftarrow{f^*} & H^*(B) & \xleftarrow{\beta^*} & H^*(B/\widehat{\triangle}) & \cdots
\end{array}
$$

(6.7)

that comes from applying $H^*(-) := H^*(-;\mathbb{Z})$ to (2.29). As input we have that $H^*(B_+;\mathbb{Z}) \cong \mathbb{Z}[x,y,w]/(xw,2w), |x| = 4, |y| = 8, |w| = 2$ from (4.31), and the cohomology of the diagonals $H^*(\triangle_+;\mathbb{Z}) \cong H^*(HP^\infty_+;\mathbb{Z}) \cong \mathbb{Z}[z], |z| = 4$ and $H^*(\widehat{\triangle}_+;\mathbb{Z}) \cong H^*([RP^\infty \times HP^\infty]_+;\mathbb{Z}) \cong \mathbb{Z}[w,z]/(2w), |z| = 4, |w| = 2$, which come from Section 2.3. Also, from Section 5.3

$$f^*(x) = 2z, \quad f^*(y) = z^2 + w^2z$$

(6.8)

in $H^*(\widehat{\triangle})$, where $x = \bar{x} + w^2$ and $y = \bar{y}$. It follows from the right hand side of diagram (3.7), and the description of $w \in H^2(B)$ in (4.31), that we may also use the same symbol $w$ in both $H^2(B)$ and $H^2(\widehat{\triangle})$, as a way of simplifying the notation.

6.2.1 The Cohomology of $B/\widehat{\triangle}$

We shall start with the case $\ast = 4k$.

Lemma 6.9. If $k = 1$, then $H^{4k}(B/\widehat{\triangle}) := H^{4k}(B/\widehat{\triangle};\mathbb{Z}) = 0$.

Proof. We have the exact sequence

$$
\cdots \xrightarrow{f^*} H^4(\widehat{\triangle}) \xrightarrow{\beta^*} H^4(B/\widehat{\triangle}) \xrightarrow{\delta} H^3(\widehat{\triangle}) = 0
$$
where \( H^4(B) \cong \mathbb{Z}/2(w^2) \oplus \mathbb{Z}(x), \) \( H^4(\hat{\Delta}) \cong \mathbb{Z}/2(w^2) \oplus \mathbb{Z}(z). \) We also have the images \( f^*(w^2) = w^2, \) \( f^*(x) = 2z, \) which yield that \( \ker f^* = 0, \) and therefore \( H^4(B/\hat{\Delta}) \cong \ker f^* = 0. \)

**Lemma 6.10.** If \( k \geq 2, \) then the cohomology group \( H^{4k}(B/\hat{\Delta}) := H^{4k}(B/\hat{\Delta}; \mathbb{Z}) \) is torsion free, that is, \( H^{4k}(B/\hat{\Delta}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}. \)

**Proof.** From the exact sequence

\[
0 \longrightarrow H^{4k+1}(B/\hat{\Delta}) \xrightarrow{\delta} H^{4k}(\hat{\Delta}) \xrightarrow{f^*} H^{4k}(B) \xrightarrow{\beta^*} H^{4k}(B/\hat{\Delta}) \longrightarrow 0
\]

we have \( H^{4k}(B/\hat{\Delta}) \cong \ker f^*, \) so it is sufficient to show that \( \ker f^* \) is torsion free. Suppose, for contradiction, that \( \ker f^* \subset H^{4k}(B) \) has a nonzero torsion element \( \sigma; \) that is, \( 2\sigma = 0 \) and \( f^*(\sigma) = 0. \) Then

\[
f^*(\sigma) = f^* \left( \sum_{a,b} \lambda_{a,b} w^a y^b \right), \quad \lambda_{a,b} \in \{0, 1\}, \ a \geq 1, \ b \geq 0, \ a + 4b = 2k
\]

\[
= f^* \left( w^{a_1} y^{b_1} + w^{a_2} y^{b_2} + \cdots + w^{a_j} y^{b_j} \right)
\]

\[
[1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j, \ \beta_1 > \beta_2 > \cdots > \beta_j \geq 0]
\]

\[
= w^{\alpha_1} z^{2\beta_1} + \text{other terms} \neq 0.
\]

This is a contradiction. Thus, for \( k \geq 2, \) \( \ker f^* \cong H^{4k}(B/\hat{\Delta}) \) is torsion free. \[\square\]

**Lemma 6.11.** For any \( k \geq 0, \) \( H^{4k+2}(B/\hat{\Delta}; \mathbb{Z}) = 0. \)

**Proof.** From the exact sequence

\[
\xrightarrow{\delta} H^{4k+2}(\hat{\Delta}) \xrightarrow{f^*} H^{4k+2}(B) \xrightarrow{\beta^*} H^{4k+2}(B/\hat{\Delta}) \xrightarrow{\delta} H^{4k+1}(\hat{\Delta}) = 0
\]

we have \( H^{4k+2}(B/\hat{\Delta}) \cong \ker f^*, \) so it is sufficient to show that \( \ker f^* = 0. \) From (4.31), \( H^{4k+2}(B; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \) with basis elements \( w^i y^j, \) where \( i + 4j = 2k + 1. \) By (5.22), the images \( f^*(w^i y^j) \) are given by \( w^i (z^2 + w^2 z^j). \) Suppose, for contradiction, that \( \ker f^* \neq 0 \) and let \( \sigma \) be a nonzero element of \( \ker f^* \subset H^{4k+2}(B). \) Then

\[
f^*(\sigma) = f^* \left( \sum_{a,b} \lambda_{a,b} w^a y^b \right), \quad \lambda_{a,b} \in \{0, 1\}, \ a \geq 1, \ b \geq 0, \ a + 4b = 2k + 1
\]

\[
= f^* \left( w^{\alpha_1} y^{\beta_1} + w^{\alpha_2} y^{\beta_2} + \cdots + w^{\alpha_j} y^{\beta_j} \right)
\]

\[
[1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j, \ \beta_1 > \beta_2 > \cdots > \beta_j \geq 0]
\]

\[
= w^{\alpha_1} z^{2\beta_1} + \text{other terms} \neq 0.
\]

This is a contradiction. Thus, \( H^{4k+2}(B/\hat{\Delta}) \cong \ker f^* = 0. \) \[\square\]
Lemma 6.12. If \( k \geq 1 \), then
\[
H^{4k+1}(B/\hat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2.
\]

Proof. Case \( k = 1 \). We have the exact sequence
\[
0 \to H^5(B/\hat{\Delta}) \xrightarrow{\delta} H^4(\hat{\Delta}) \xrightarrow{f^*} H^4(B) \xrightarrow{\beta^*} H^4(B/\hat{\Delta}) = 0
\]
where \( H^4(B) \cong \mathbb{Z}/2 \langle w^2 \rangle \oplus \mathbb{Z}(x) \) and \( H^4(\hat{\Delta}) \cong \mathbb{Z}/2 \langle w^2 \rangle \oplus \mathbb{Z}(z) \) where \( |w| = 2 \), \( |x| = |z| = 4 \). Also, \( (\delta \circ f^*)(x) = \delta(f^*(x)) = \delta(2z) = 0 \), and \( (\delta \circ f^*)(w^2) = \delta(w^2) = 0 \). Clearly \( \delta \) is surjective and \( \delta(z) \neq 0 \). Thus \( H^5(B/\hat{\Delta}) \cong \mathbb{Z}/2 = \langle \delta(z) \rangle \).

Case \( k \geq 2 \). We have the exact sequence
\[
0 \to H^{4k+1}(B/\hat{\Delta}) \xrightarrow{\delta} H^{4k}(\hat{\Delta}) \xrightarrow{f^*} H^{4k}(B) \xrightarrow{\beta^*} H^{4k}(B/\hat{\Delta}) = 0
\]
where \( H^{4k}(\hat{\Delta}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \) with minimal generating set \( w^az^b, z^k \) for \( a+2b = 2k \). Clearly, \( \delta \) is surjective from the exact sequence. We shall show that \( \delta(z^k) \neq 0 \) and \( 2\delta(z^k) = 0 \).

If \( k \geq 2 \) is even, then a minimal generating set for \( H^{4k}(B) \) is \( y^{k/2}, w^iy^j, x^ay^b, x^k \) where \( i + 4j = 2k, i \geq 4, j \geq 0, \alpha + 2\beta = k \). The images of these elements under the homomorphism \( f^* \) are given by \( f^*(y^{k/2}) = (z^2 + w^2 z)^{k/2} = z^k + w^k z^{k/2} + \) other terms, \( f^*(w^iy^j) = w^i(z^2 + w^2 z)^j, f^*(x^ay^b) = (2z)^\alpha z^{2\beta}, f^*(x^k) = (2z)^k \), where \( k/2 > j \). Then, the preimage \( (f^*)^{-1}(z^k) \) is empty, due to the fact \( k/2 > j \); thus \( \delta(z^k) \neq 0 \). Furthermore, \( (\delta \circ f^*)(y^{k/2}) = \delta((z^2 + w^2 z)^{k/2}) = \delta(z^k + \sum_{a,b} \lambda_{a,b} w^a z^b) = 0, \lambda_{a,b} \in \{0,1\} \), implies that \( \delta(z^k) = \delta \left( \sum_{a,b} \lambda_{a,b} w^a z^b \right) \), from which we deduce that \( 2\delta(z^k) = 0 \).

An analogous argument is applied to the case \( k \) is odd, and the only major difference is that there is no element \( y^{k/2} \) for \( H^{4k}(B) \). If \( k \geq 3 \) is odd, then a minimal generating set for \( H^{4k}(B) \) is \( w^iy^j, x^ay^b, x^k \) where \( i + 4j = 2k, i \geq 2, j \geq 0, \alpha + 2\beta = k, \alpha \neq 0 \). Then from the images of these elements under \( f^* \), it is clear that the preimage \( (f^*)^{-1}(z^k) \) is empty; so again \( \delta(z^k) \neq 0 \). Furthermore, the image of \( xy^m \in H^{4k}(B), m = \frac{k-1}{2} \) under the composition \( \delta \circ f^* \) is given by \( (\delta \circ f^*)(xy^m) = \delta(2z(z^2 + w^2 z)^m) = \delta(2z^{2m+1}) = 2\delta(z^{2m+1}) = 0 \).

We shall now proceed to the \( (4k+3) \)-th cohomology.

Lemma 6.13. If \( k = 0 \), then \( H^{4k+3}(B/\hat{\Delta}; \mathbb{Z}) = 0 \).
Proof. It follows from the exact sequence

\[ 0 \rightarrow H^3(B/\hat{\Delta}) \xrightarrow{\delta} H^2(\hat{\Delta}) \xrightarrow{f^*} H^2(B) \xrightarrow{\beta^*} H^2(B/\hat{\Delta}) \rightarrow 0 \]

where \( H^2(\hat{\Delta}) \cong \mathbb{Z}/2(w), \) \( H^2(B) \cong \mathbb{Z}/2(w) \) and \( f^*(w) = w. \) Clearly \( \delta \) is surjective and the composition \( (\delta \circ f^*)(w) = \delta(w) = 0. \)

\[ \square \]

Lemma 6.14. If \( k \geq 1, \) then

\[ H^{4k+3}(B/\hat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2. \]

Proof. We have the exact sequence

\[ H^{4k+3}(B) = 0 \xrightarrow{\beta^*} H^{4k+3}(B/\hat{\Delta}) \xrightarrow{\delta} H^{4k+2}(\hat{\Delta}) \xrightarrow{f^*} H^{4k+2}(B) \xrightarrow{0} \]

where \( H^{4k+2}(\hat{\Delta}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \) with basis \( w^i z^j, \) \( i + 2j = 2k + 1, i \geq 1, j \geq 0 \) and \( H^{4k+2}(B) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \) with basis \( w^\alpha y^\beta, \) \( \alpha + 4\beta = 2k + 1, \alpha \geq 1, \beta \geq 0. \) Clearly \( \delta \) is surjective, and its images is the quotient \( \mathbb{Z}/2 \) vector space, as required. Note that \( (\delta \circ f^*)(w^\alpha y^\beta) = \delta(w^\alpha(z^2 + zw^2)^\beta) = 0 \) in \( H^{4k+3}(B/\hat{\Delta}; \mathbb{Z}). \)

\[ \square \]

Example 6.15. Case \( k = 1. \) We have

\[ 0 \rightarrow H^7(B/\hat{\Delta}) \xrightarrow{\delta} H^6(\hat{\Delta}) \xrightarrow{f^*} H^6(B) \xrightarrow{0} \]

where \( H^6(B) \cong \mathbb{Z}/2(w^3), \) \( H^6(\hat{\Delta}) \cong \mathbb{Z}/2(w^3) \oplus \mathbb{Z}/2(wz) \) and \( f^*(w^3) = w^3. \) Then \( (\delta \circ f^*)(w^3) = \delta(w^3) = 0. \) Clearly \( \delta \) is surjective, and \( H^7(B/\hat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \) is generated by \( \delta(wz). \)

6.2.2 Thom Isomorphism

There is a useful reference book [53] for this section; we shall list some basics related to the Thom isomorphism below.

Given the Thom space \( \text{Th}(\xi) \) of an \( n \)-plane bundle \( \mathbb{R}^n \rightarrow \mathbb{R}(\xi) \rightarrow X, \)

1. a unique Thom isomorphism \( H^*(X_+; \mathbb{Z}/2) \cong H^{*+n}(\text{Th}(\xi); \mathbb{Z}/2) \) always exists over \( \mathbb{Z}/2 \) coefficients, and is given by relative cup product with the \( \mathbb{Z}/2 \) Thom class in \( H^n(\text{Th}(\xi); \mathbb{Z}/2) \)

\[ \]
CHAPTER 6. THE THOM SPACE \( \text{Th}(\theta) \)

(2) the existence of a Thom isomorphism \( H^*(X_+; \mathbb{Z}) \cong H^{*+n}(\text{Th}(\xi); \mathbb{Z}) \) is equivalent to the vanishing of the first Stiefel-Whitney class \( w_1(\xi) \) of \( \xi \) in \( H^1(X; \mathbb{Z}/2) \), and to the existence of a Thom class \( t \) (or \( -t \)) in \( H^n(\text{Th}(\xi)) \)

(3) the reduced cohomology groups \( H^*(\text{Th}(\xi); R) \) are trivial for \( * < n \) and any coefficient ring \( R \)

(4) if \( \xi \) is orientable, the Euler class \( e = e(\xi) \in H^n(X; \mathbb{Z}) \) may be defined \([23, \text{page 88}]\) to be the restriction of the Thom class \( t \in H^n(\text{Th}(\xi); \mathbb{Z}) \) to the zero section; it can also be defined as the class \( et \) corresponding to \( t^2 \) under the Thom isomorphism.

By a Theorem of René Thom \([42, \text{page 91}]\), for \( i \leq n = |t| \) we have that

\[
\text{Sq}^i t = w_i(\xi) t
\]

in \( H^{n+i}(\text{Th}(\xi); \mathbb{Z}/2) \), where \( w_i(\xi) \) is the \( i \)-th Stiefel-Whitney class. This implies that the mod 2 Euler class \( e = w_n(\xi) \) since the Steenrod square \( \text{Sq}^k \) with \( k = |t| \) acts by \( \text{Sq}^k t = t^2 = w_k(\xi) t \).

We shall apply (1) – (4) to our case, \( \text{Th}(\theta) = M\text{Pin}(4) \) and \( X = B\text{Pin}(4) \), where \( \theta \) is the universal \( \text{Pin}(4) \) vector bundle, \( \mathbb{R}^4 \to \mathbb{R}(\theta) \to B\text{Pin}(4) \) as in Definition 3.10, and \( B\text{Pin}(4) = \Gamma \).

(1) There exists the Thom isomorphism \( \tau \)

\[
\tau: H^*(B\text{Pin}(4)_+; \mathbb{Z}/2) \xrightarrow{\cong} H^{*+4}(M\text{Pin}(4); \mathbb{Z}/2)
\]

defined by \( \tau(x) = xt \) where \( t \) is the unique Thom class

\[
t \in H^4(M\text{Pin}(4); \mathbb{Z}/2).
\]

(2) There is no such isomorphism (6.17) over \( \mathbb{Z} \), because \( w_1(\theta) \) is non-zero (6.21) (in other words, the bundle \( \theta \) is not orientable).

(3) The bundle \( \theta \) is a 4-plane bundle, so

\[
H^*(M\text{Pin}(4); \mathbb{Z}/2) \cong H^*(B/\tilde{\Delta}; \mathbb{Z}/2) = 0 \quad \text{if} \quad * < 4.
\]
(4) The Euler class $e = e(\theta) \in H^4(BPin(4); \mathbb{Z}/2)$ is
\[ e \overset{\text{def}}{=} j^*(t), \quad j^*: H^*(MPin(4); \mathbb{Z}/2) \to H^*(BPin(4); \mathbb{Z}/2) \]
where $j^*$ is induced by the inclusion of the zero section $BPin(4) \to MPin(4)$.

So $e = w_n(\theta)$.

By Lemma 4.22 we have
\[ H^*(BPin(4)_+; \mathbb{Z}/2) \cong H^*(BSp(2)_+; \mathbb{Z}/2)[a]/(a^5 - p_1 a), \]
where $H^*(BSp(2)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2], |p_i| = 4i, |a| = 1$.

**Lemma 6.20.** Let $\tau$ be the Thom isomorphism (6.17)
\[ \tau: H^*(BPin(4)_+; \mathbb{Z}/2) \to H^{*+4}(MPin(4); \mathbb{Z}/2). \]
Then the relation
\[ t^2 = (p_1 + a^4)t \]
holds in $H^8(MPin(4); \mathbb{Z}/2)$.

**Proof.** Consider the map $BPin(4) \overset{j}{\to} BSp(2) \cong BSpin(5)$ and the universal $Spin(5)$ vector bundle $\chi_5$ over $BSpin(5)$. It follows from Section 3.2 that $\ell^*(\chi_5) = \theta \oplus \lambda$ where $\theta$ is a 4-plane bundle and $\lambda$ is a non-trivial line bundle. This agrees with results in [30] and [5], although $Pin(4)$ was not introduced in [5]; instead the wreath product of $S^3$ and $\mathbb{Z}/2$, and the normaliser of $Spin(4)$ in $Spin(5)$ were used.

Let $w(-)$ be the total Stiefel-Whitney class and $p_1 := \ell^*(p_1)$ in $H^4(BPin(4); \mathbb{Z}/2)$ be the mod 2 symplectic Pontryagin class. Because $w(\chi_5) = 1 + p_1$ in $H^*(BSp(2))$, we have that $w(\ell^*(\chi_5)) = w(\theta \oplus \lambda) = w(\theta) w(\lambda) \Rightarrow 1 + p_1 = w(\theta)(1 + a)$, so $w(\theta)$ is given by
\[ w(\theta) = 1 + w_1(\theta) + w_2(\theta) + w_3(\theta) + w_4(\theta) \]
\[ = 1 + a + a^2 + a^3 + (a^4 + p_1) + a^5 + p_1 a + \cdots. \]
Equating terms with the same dimensions yields
\[ w_1(\theta) = a \] (6.21)
\[ w_2(\theta) = a^2 \] (6.22)
\[ w_3(\theta) = a^3 \] (6.23)
\[ w_4(\theta) = a^4 + p_1 \] (6.24)
\[ 0 = a^5 + p_1 a \] (6.25)
and hence that \( t^2 = et = w_4(\theta)t = (a^4 + p_1)t \).

From [30], the obstructions to putting \( Pin^\pm \) and \( Spin \) structures on an \( n \)-plane bundle \( \xi, \mathbb{R}^n \to E \to X \) are

- \( w_2(\xi) \) for a \( Pin^+ \) structure
- \( w_1^2(\xi) + w_2(\xi) \) for a \( Pin^- \) structure
- \( w_1(\xi) \) and \( w_2(\xi) \) for a \( Spin \) structure,

where \( w_i(\xi) \in H^i(X; \mathbb{Z}/2) \) is the \( i \)-th Stiefel-Whitney class. These are consistent with our computations, because from (6.21) and (6.22), we have that \( (w_1(\theta))^2 = a^2 = w_2(\theta) \).

This confirms that \( \theta \) is indeed the universal \( Pin(4) \) bundle.

We also recall a form of the Universal Coefficient Theorem (UCT) that is very useful for our computations.

**Theorem 6.26** ([54], page 66). For any abelian group \( G \) and any \( m \geq 1 \), there exists an isomorphism

\[
H^m(X, G) \cong H^m(X, \mathbb{Z}) \otimes G \oplus \text{Tor}(H^{m+1}(X, \mathbb{Z}), G). \tag{6.27}
\]

For any finitely generated abelian groups \( G \) and \( G' \), the torsion product \( \text{Tor}(G, G') \) may be built up from the properties [54]

\[
\text{Tor}(G, G') \cong \text{Tor}(G', G), \quad \text{Tor}(G, \mathbb{Z}) \cong \text{Tor}(\mathbb{Z}, G) = 0,
\]

\[
\text{Tor}(\mathbb{Z}/p, \mathbb{Z}/q) \cong \mathbb{Z}/\gcd(p, q) \text{ for any } p, q \geq 2,
\]

if \( G = \bigoplus_i G_i, \; G' = \bigoplus_i G'_i \), then \( \text{Tor}(G, G') \cong \bigoplus_{i,j} \text{Tor}(G_i, G'_j) \).

**Lemma 6.28.** If \( * = 4k + 1, \; 4k + 3, \; k \geq 1 \), then the reduction mod 2 homomorphism

\[
\rho: H^*(B/\hat{\Delta}; \mathbb{Z}) \to H^*(B/\hat{\Delta}; \mathbb{Z}/2)
\]

is an isomorphism.

**Proof.** Apply formula (6.27) for \( G = \mathbb{Z}/2, \; X = B/\hat{\Delta}, \; m = 4k + 3, \; 4k + 1 \).

Case \( * = 4k + 3 \). From Section 6.2, \( H^{4(k+1)}(B/\hat{\Delta}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \) is torsion free, and \( H^{4k+3}(B/\hat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \) by Lemma 6.14. By applying (6.27)
we have an isomorphism $H^{4k+3}(B/\Delta; \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H^{4k+3}(B/\Delta; \mathbb{Z}/2)$, and therefore
\[ \rho: H^{4k+3}(B/\Delta; \mathbb{Z}) \cong H^{4k+3}(B/\Delta; \mathbb{Z}/2). \]

Case $* = 4k + 1$. The cohomology $H^{4k+2}(B/\Delta; \mathbb{Z}) = 0$ by Lemma 6.11, and $H^{4k+1}(B/\Delta; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ by Lemma 6.12. By applying (6.27) we have an isomorphism $H^{4k+1}(B/\Delta; \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H^{4k+1}(B/\Delta; \mathbb{Z}/2)$, and therefore deduce $\rho: H^*(B/\Delta; \mathbb{Z}) \cong H^*(B/\Delta; \mathbb{Z}/2)$. \hfill \Box

We shall further describe $H^*(B/\Delta; \mathbb{Z}/2)$ for $* = 4k + 1, 4k + 3$.

**Proposition 6.29.** If $* = 4k + 1, 4k + 3$, then

\[ H^*(B/\Delta; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 =: (\mathbb{Z}/2)^{\ell+1} \]

where $\ell = \frac{k-1}{2}$ when $k$ is odd, and $\ell = \frac{k}{2} - 1$ when $k$ is even.

**Proof.** We use the Thom isomorphism

\[ H^*(\Gamma_+; \mathbb{Z}/2) \cong H^{*+4}(B/\Delta; \mathbb{Z}/2). \quad (6.30) \]

First observe that $H^{4m}(\Gamma; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 = (\mathbb{Z}/2)^{m+1}$, because a basis is given by the $4m$ dimensional classes

\[ p_1^m, p_1^{m-1} a^4, p_1^{m-2} p_2, p_1^{m-3} p_2 a^4, p_1^{m-4} p_2^2, \cdots, p_1 p_2^{\ell-1} a^4, p_2^\ell, \text{ if } m = 2\ell \]

\[ p_1^m, p_1^{m-1} a^4, p_1^{m-2} p_2, p_1^{m-3} p_2 a^4, \cdots, p_1 p_2^\ell, p_2^\ell a^4, \text{ if } m = 2\ell + 1, \]

in $H^*(Gr_{\mathbb{Z}/2}(2, \infty); \mathbb{Z}/2)(1, a, a^2, a^3, a^4)$, where $|a| = 1, |p_i| = 4i, a^5 = p_1 a$.

Then, multiplying each basis element of $H^{4m}(\Gamma; \mathbb{Z}/2)$ by $a$ with the relation $a^5 = p_1 a$, yields

\[ H^{4m+1}(\Gamma; \mathbb{Z}/2) = \begin{cases} (\mathbb{Z}/2)^{\ell+1} & \text{if } m = 2\ell \\ (\mathbb{Z}/2)^{\frac{1}{2}(m+1)} & \text{if } m = 2\ell + 1 \end{cases} \]

with basis elements of the form $p_1^\lambda p_2^\mu a$. Multiplying by $a^2$ then gives basis elements $p_1^\lambda p_2^\mu a^3$ for $H^{4m+3}(\Gamma; \mathbb{Z}/2)$.

We may now write for both $* = 4m + 1$ and $* = 4m + 3$

\[ H^*(\Gamma; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\ell+1} \]

where $\ell = \frac{m}{2}$ if $m$ is even, and $\ell = \frac{1}{2}(m - 1)$ if $m$ is odd.
For \( * = 4k + 1, 4k + 3 \) with \( k = m + 1 \), we deduce, by the Thom isomorphism (6.30), that there is an isomorphism

\[
H^*(B/\widehat{\triangle}; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\ell+1},
\]

where \( \ell = \frac{k-1}{2} \) if \( k \) is odd, and \( \ell = \frac{k}{2} - 1 \) if \( k \) is even.

We shall say more about \( H^*(B/\widehat{\triangle}; \mathbb{Z}/2) \) later in Section 7.2.

**Corollary 6.31.** By Lemma 6.28, if \( * = 4k + 1, 4k + 3 \), then there is an isomorphism

\[
H^*(B/\widehat{\triangle}; \mathbb{Z}) \cong (\mathbb{Z}/2)^{\ell+1}
\]

of abelian groups, where \( \ell = \frac{k-1}{2} \) if \( k \) is odd, and \( \ell = \frac{k}{2} - 1 \) if \( k \) is even.

**Example 6.32.** Corollary 6.31 may be written in an alternative way as follows.

- \( H^*(B/\widehat{\triangle}; \mathbb{Z}) \cong \mathbb{Z}/2 \) for \( * = 5, 7, 9, 11 \)
- \( H^*(B/\widehat{\triangle}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) for \( * = 13, 15, 17, 19 \)
- \( H^*(B/\widehat{\triangle}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) for \( * = 21, 23, 25, 27 \)
- \( \vdots \)
- \( H^*(B/\widehat{\triangle}; \mathbb{Z}) \cong (\mathbb{Z}/2)^a \), \( a = 1, 2, 3, \ldots \) for \( * = 8a - 3, 8a - 1, 8a + 1, 8a + 3 \).

Now recall from (6.7) the LES

\[
\cdots \leftarrow H^{*+1}(B/\widehat{\triangle}) \overset{\delta}{\leftarrow} H^*(\widehat{\triangle}) \overset{f^*}{\leftarrow} H^*(B) \leftarrow H^*(B/\widehat{\triangle}) \leftarrow \cdots
\]

**Lemma 6.33.** For any \( m \geq 1 \), there exists an expression

\[
\delta(w^{2m}z^m) = \sum_{0 \leq j < 2i} \lambda_{j,i} \delta(w^jz^i), \quad \lambda_{j,i} \in \{0, 1\}, \ 2i + j = 4m \quad (6.34)
\]

in \( H^{8m+1}(B/\widehat{\triangle}; \mathbb{Z}) \).

**Proof.** This follows directly from the fact that \( (\delta \circ f^*)(y^m) = \delta((z^2 + w^2z)^m) = 0 \) in the 4-term exact sequence

\[
0 \longleftarrow H^{8m+1}(B/\widehat{\triangle}) \overset{\delta}{\longleftarrow} H^{8m}(\widehat{\triangle}) \overset{f^*}{\longleftarrow} H^{8m}(B) \longleftarrow H^{8m}(B/\widehat{\triangle}) \longleftarrow 0.
\]

For example, \( \delta(w^6z^3) = \delta(z^6) + \delta(w^2z^5) + \delta(w^4z^4) \) when \( m = 3 \).
Corollary 6.35. Any element \( \delta(w^b z^m) \in H^*(B/\~\triangle) \), \( b > 2m \), \( * = 2b + 4m + 1 \) can be expressed as

\[
\delta(w^b z^m) = \sum_{0 \leq j < 2i} \lambda_{j,i} \delta(w^j z^i), \quad \lambda_{j,i} \in \{0, 1\}.
\]

Proof. Let \((2m, m)\) denote \( \delta(w^{2m} z^m) \) in \( H^{8m+1}(B/\~\triangle) \). For any \( m \geq 1 \), we have

\[
(2m, m) = (b_1, a_1) + (b_2, a_2) + \cdots + (b_r, a_r)
\]

where \( b_i < 2a_i \) by Lemma 6.33. So

\[
(2m + \beta, m) = (b_1 + \beta, a_1) + (b_2 + \beta, a_2) + \cdots + (b_r + \beta, a_r);
\]

and if \( b_i + \beta \geq 2a_i \), then the process may be iterated until it terminates. \( \square \)

Example 6.36. Given the expressions \((12, 6) = (0, 12) + (4, 10) + (8, 8), (10, 5) = (0, 10) + (2, 9) + (8, 6), (8, 4) = (0, 8), (6, 3) = (0, 6) + (2, 5) + (4, 4), (4, 2) = (0, 4), (2, 1) = (0, 2) \). Then, \((b, m) = (22, 3)\) can be written as

\[
(22, 3) = (6 + 16, 3)
\]

\[
= (16, 6) + (18, 5) + (20, 4)
\]

\[
= (12 + 4, 6) + (10 + 8, 5) + (8 + 12, 4)
\]

\[
= (4, 12) + (8, 10) + (12, 8) + (8, 10) + (10, 9) + (16, 6) + (12, 8)
\]

\[
= (4, 12) + (10, 9) + (16, 6)
\]

\[
= (4, 12) + (10, 9) + (12 + 4, 6)
\]

\[
= (4, 12) + (10, 9) + (4, 12) + (8, 10) + (12, 8)
\]

\[
= (10, 9) + (8, 10) + (12, 8),
\]

and the process has terminated.

Proposition 6.37. If \( * = 4k + 3, 4k + 1, k \geq 1 \), then a basis for the \( \mathbb{Z}/2 \) vector space \( H^*(B/\~\triangle; \mathbb{Z}) \) is given by \( \delta(w^j z^i) \) where \( i \geq 1, 0 \leq j < 2i \).

Proof. If \( * = 4k + 3, 4k + 1, k \geq 1 \) then we have the LES

\[
0 = H^*(B) \leftarrow H^*(B/\~\triangle) \overset{\delta}{\leftarrow} H^{*-1}(\~\triangle) \leftarrow H^{*-1}(B)
\]
with the obvious surjection $\delta$. By Corollary 6.35 for any $\delta(w^b z^a) \in H^*(B/\triangle)$ with $b \geq 2a$, there exists an expression $\delta(w^b z^a) = \sum_{0 \leq j < 2i} \lambda_{j,i} \delta(w^j z^i)$, $\lambda_{j,i} \in \{0,1\}$. Also, from Corollary 6.31 and Example 6.32, for the positive integers $m' = 1, 2, 3, 4, \cdots$

$$H^{*+1}(B/\triangle; \mathbb{Z}) \cong (\mathbb{Z}/2)^m'$$

where $* = 8m' - 4$, $8m' - 2$, $8m'$, $8m' + 2$. It is easy to check that the number of images $\delta(w^j z^i)$ with $1 \leq i$, $0 \leq j < 2i$ is equal to $m'$, by considering generators $w^j z^i$ for $H^*(\triangle)$ with $1 \leq i$, $0 \leq j < 2i$ for each dimension $* = 8m' - 4$, $8m' - 2$, $8m'$, $8m' + 2$.

They are:

- for $* = 8m' - 4$: $w^{2m'-2}z^{m'}$, $w^{2m'-4}z^{m'+1}$, $\cdots$, $w^2z^{2m'-2}$, $z^{2m'-1}$
- for $* = 8m' - 2$: $w^{2m'-1}z^{m'}$, $w^{2m'-3}z^{m'+1}$, $\cdots$, $w^3z^{2m'-2}$, $wz^{2m'-1}$
- for $* = 8m'$: $w^{2m'-2}z^{m'+1}$, $\cdots$, $w^2z^{2m'-1}$, $z^{2m'}$
- for $* = 8m' + 2$: $w^{2m'-1}z^{m'+1}$, $\cdots$, $w^3z^{2m'-1}$, $wz^{2m'}$

There are indeed $m'$ of these in each case, and the other generators all have $j \geq 2i$. So $\delta$ is a surjection from the $\mathbb{Z}/2$ vector space they generate (after tensoring with $\mathbb{Z}/2$ in cases $8m' - 4$ and $8m'$) onto the $\mathbb{Z}/2$ vector space $H^*(B/\triangle; \mathbb{Z})$. Since both vector space have dimension $m'$, the restriction of $\delta$ is an isomorphism, as we wanted. \qed
Chapter 7

The Cohomology Ring I

The main aim of this chapter is to justify Theorem 1.7. As we mentioned earlier in Chapter 2, for a quotient orbifold such as the symmetric product $SP^n(M)$, there exists a rational isomorphism [1, page 38] between the orbit space $X = M^n/\Sigma_n$ and the associated Borel space $\mathcal{B}$ in cohomology

$$H^*(\mathcal{B}; \mathbb{Q}) \cong H^*(X; \mathbb{Q}). \quad (7.1)$$

This well known fact is quite useful for our computation, from which we can immediately deduce that the numbers of $\mathbb{Z}$-summands in the integral cohomology of the symmetric square and the associated Borel construction are the same.

Another relevant result comes from Nakaoka’s work [46], through which we have an isomorphism $H^*(SP^2(X); \mathbb{Z}) \cong H^*(X; \mathbb{Z})$ for $* \leq r + 1$ where $X$ is a connected CW complex space such that its homology is $H_*(X; \mathbb{Z}) = 0$ for $* < r$. Applying this to our case yields

$$H^*(SP^2; \mathbb{Z}) = 0 \quad \text{if} \quad * = 1, 2, 3, 5, \quad \text{and} \quad H^4(SP^2; \mathbb{Z}) \cong \mathbb{Z}. \quad (7.2)$$

We shall reprove these results in passing; the integral cohomology ring $H^*(SP^2)$ when $* > 5$ is our primary interest.
7.1 \( H^*(SP^2(\mathbb{HP}^\infty); \mathbb{Z}) \)

Our main focus in this section is to describe \( H^*(SP^2) := H^*(SP^2(\mathbb{HP}^\infty); \mathbb{Z}) \). For example, we shall show that

\[
H^*(SP^2) \cong \begin{cases} 
  \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \text{if } * = 4k \\
  \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 & \text{if } * = 4k + 1 \geq 13, \\
  0 & \text{otherwise.}
\end{cases}
\]

We use the ladder of long exact sequences

\[
\begin{array}{cccccc}
  \\
  \leftarrow H^{*+1}(SP^2/\triangle) \xrightarrow{\delta} H^*(\triangle) \xrightarrow{i^*} H^*(SP^2) \xrightarrow{s^*} H^*(SP^2/\triangle) \leftarrow \\
  \cong \downarrow \pi_\triangle \downarrow \pi^* \cong \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \\
  \leftarrow H^{*+1}(B/\hat{\triangle}) \xrightarrow{\delta} H^*(\hat{\triangle}) \xrightarrow{f^*} H^*(B) \xrightarrow{\beta^*} H^*(B/\hat{\triangle}) \leftarrow
\end{array}
\]

(7.3)

that comes from applying \( H^*(-) := H^*(-; \mathbb{Z}) \) to (2.29), where \( \pi_\triangle^* \) is induced by \( \pi|_\triangle \).

Remark 7.4. It is occasionally helpful to write the upper and lower coboundary homomorphisms in this ladder as \( \delta_u \) and \( \delta_l \) respectively.

As input we have

\[
H^*(B_+; \mathbb{Z}) \cong \mathbb{Z}[x, y, w]/(xw, 2w) \text{ with } |x| = 4, |y| = 8, |w| = 2
\]

from (4.31), and the cohomology of the diagonals

\[
H^*(\triangle_+; \mathbb{Z}) \cong H^*(\mathbb{HP}^\infty_+; \mathbb{Z}) \cong \mathbb{Z}[z] \text{ with } |z| = 4
\]

and

\[
H^*(\hat{\triangle}_+; \mathbb{Z}) \cong H^*((\mathbb{RP}^\infty \times \mathbb{HP}^\infty)_+; \mathbb{Z}) \cong \mathbb{Z}[w, z]/(2w) \text{ with } |z| = 4, |w| = 2,
\]

which come from Section 2.3. The homomorphism \( f^* \) from (6.8) given by \( f^*(x) = 2z, f^*(y) = z^2 + w^2z, f^*(w) = w, \) and the restricted homomorphism \( \pi_\triangle^*(z^k) = z^k \), which comes from the projection map \( \mathbb{RP}^\infty \times \mathbb{HP}^\infty \to \mathbb{HP}^\infty \), are important and used throughout this chapter and the next. We should also recall Proposition 6.37 which states that if \( * = 4k + 3, 4k + 1, k \geq 1 \) then \( H^*(B/\hat{\triangle}; \mathbb{Z}) = \mathbb{Z}/2 \langle \delta(w^jz^i) \rangle \) where \( i \geq 1, 0 \leq j < 2i \) and \( |\delta(w^jz^i)| = 4i + 2j + 1. \)
7.1.1 $H^{4k}(SP^2(\mathbb{H}\mathbb{P}^\infty))$

We shall begin with the $4k$-th cohomology $H^{4k}(SP^2) := H^{4k}(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$.

For $k = 1$, the cohomology group $H^4(SP^2) \cong \mathbb{Z}$ from (7.2) is expected by Nakaoka’s work [46]. We shall confirm his result by using the commutative ladder

$$
\begin{array}{cccccc}
\rightarrow & H^5(SP^2/\Delta) & \rightarrow & H^4(\Delta) & \rightarrow & H^4(SP^2) \leftarrow 0 \\
\cong & \downarrow & & \downarrow & \cong & \\
0 & \rightarrow & H^5(B/\hat{\Delta}) & \rightarrow & H^4(\hat{\Delta}) & \rightarrow H^4(B) \rightarrow H^4(B/\hat{\Delta}) \rightarrow 0
\end{array}
$$

induced by (2.29). Here, $H^4(\Delta) \cong \mathbb{Z}$, and by Lemma 6.9 we have that $H^4(B/\hat{\Delta}) = 0$. The rational isomorphism $H^*(SP^2; \mathbb{Q}) \cong H^*(B; \mathbb{Q})$ from (7.1) and $H^4(B) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ from (4.31), suggest that the cohomology group $H^4(SP^2) \cong \mathbb{Z} \oplus \text{torsion}$. However, the LES

$$
\cdots \leftarrow \mathbb{Z} \cong H^4(\Delta) \leftarrow i^* \rightarrow H^4(SP^2) \leftarrow s^* \rightarrow H^4(SP^2/\Delta) = 0
$$

(of the upper row) in the ladder (7.5), implies that $\ker i^* \cong \text{im} s^* = 0$. Then, the 4-th cohomology group $H^4(SP^2)$ must be torsion free, which confirms the result (7.2).

In general, we have the following.

**Lemma 7.6.** For any $k \geq 1$, the integral cohomology $H^{4k}(SP^2)$ is torsion free, that is, $H^{4k}(SP^2) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$.

**Proof.** We have the ladder

$$
\begin{array}{cccccc}
\rightarrow & H^{4k+1}(SP^2/\Delta) & \rightarrow & H^{4k}(\Delta) & \rightarrow & H^{4k}(SP^2) \leftarrow 0 \\
\cong & \downarrow & & \downarrow & \cong & \\
\rightarrow & H^{4k+1}(B/\hat{\Delta}) & \rightarrow & H^{4k}(\hat{\Delta}) & \rightarrow H^{4k}(B) \rightarrow H^{4k}(B/\hat{\Delta}) \rightarrow 0
\end{array}
$$

where $H^{4k}(\Delta) \cong \mathbb{Z}$, and the cohomology $H^{4k}(B/\hat{\Delta})$ is torsion free by Lemma 6.10. Then, the canonical isomorphism $H^*(SP^2/\Delta) \cong H^*(B/\hat{\Delta})$ yields

$$
\cdots \leftarrow \mathbb{Z} \leftarrow i^* \rightarrow H^{4k}(SP^2) \leftarrow s^* \rightarrow \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \leftarrow 0
$$

which shows that $H^{4k}(SP^2)$ must be torsion free. \qed

Let us describe the cohomology ladder in more detail, this time paying attention to generators and the maps between them. We shall use various inputs mentioned at the beginning of this chapter.
CHAPTER 7. THE COHOMOLOGY RING I

Case $k = 1$. Let us describe the ladder of exact sequences

$$
\cdots \longrightarrow H^5(SP^2/\Delta) \xrightarrow{\delta} H^4(\Delta) \xrightarrow{\iota^*} H^4(SP^2) \cong \mathbb{Z} \longrightarrow 0 \quad (7.8)
$$

and

$$
0 \longrightarrow H^5(B/\widehat{\Delta}) \xrightarrow{\delta} H^4(\widehat{\Delta}) \xrightarrow{f^*} H^4(B) \longrightarrow 0
$$

Here, $H^4(\Delta) \cong \mathbb{Z}\langle z \rangle$, and by Proposition 6.37 we have $H^5(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(z) \rangle$. The homomorphism $\pi^*$ sends $z \mapsto z$, and the isomorphism $H^5(SP^2/\Delta) \cong H^5(B/\widehat{\Delta})$ yields that $\text{im} \ i^* \cong \ker \delta = \mathbb{Z}\langle 2z \rangle$. Furthermore, $f^*$ is an injection, $\pi^*$ is an injection because it is a rational isomorphism, and $f^*(x) = 2z$. These facts lead to the following.

**Definition 7.9.** Given the homomorphism $\pi^*: H^4(SP^2) \rightarrow H^4(B)$, we define the generator

$$g \in H^4(SP^2) \cong \mathbb{Z}$$

as the unique preimage of $x \in H^4(B)$ under $\pi^*$. So $i^*(g) = 2z$ in $H^4(\Delta)$.

Case $k = 2$. Let us describe the cohomology ladder

$$
\cdots \longrightarrow H^9(SP^2/\Delta) \xrightarrow{\delta} H^8(\Delta) \xrightarrow{\iota^*} H^8(SP^2) \xrightarrow{\iota^*} H^8(\Delta) \xrightarrow{\pi^*} H^8(SP^2/\Delta) \xrightarrow{\delta} H^9(SP^2/\Delta) \longrightarrow 0
$$

and

$$
0 \longrightarrow H^9(B/\widehat{\Delta}) \xrightarrow{\delta} H^8(\widehat{\Delta}) \xrightarrow{f^*} H^8(B) \longrightarrow 0
$$

This time, we have $H^8(B) \cong \mathbb{Z}\langle x^2 \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}/2\langle w^4 \rangle$, and $H^8(\Delta) \cong \mathbb{Z}\langle z^2 \rangle \oplus \mathbb{Z}/2\langle w^2z \rangle \oplus \mathbb{Z}/2\langle w^4 \rangle$. From the ladder, $H^8(B/\widehat{\Delta}) \cong \ker f^*$, and $\ker f^* \cong \mathbb{Z}$, generated by $x^2 - 4y$, which follows from (6.8). Then, the canonical isomorphism $H^8(SP^2/\Delta) \cong H^8(B/\widehat{\Delta})$ implies that $\ker i^* \cong H^8(SP^2/\Delta) \cong \mathbb{Z}$. So we may choose a generator $\sigma$ for ker $i^*$ such that $\pi^*(\sigma) = x^2 - 4y \in H^8(B)$, and a generator $\tilde{\sigma}$ for $H^8(SP^2/\Delta)$ such that $s^*(\tilde{\sigma}) = \sigma$. Input from Proposition 6.37 gives an isomorphism $H^5(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(z^2) \rangle$. By an analogous argument to the 4-th cohomology case, we then have that $\text{im} i^* \cong \ker \delta = \mathbb{Z}\langle 2z \rangle$, where $H^8(\Delta) \cong \mathbb{Z}\langle z^2 \rangle$. So there is a short exact sequence

$$
0 \leftarrow \mathbb{Z}\langle 2z \rangle \xrightarrow{\iota^*} H^8(SP^2) \xrightarrow{\iota^*} \mathbb{Z}\langle \tilde{\sigma} \rangle \leftarrow 0,
$$

which must be split. This proves that $H^8(SP^2) \cong \mathbb{Z}\langle \alpha \rangle \oplus \mathbb{Z}\langle \sigma \rangle$, where $i^*(\alpha) = 2z^2$, so $f^*(\pi^*(\alpha)) = 2z^2$. But also $f^*(2y) = 2f^*(y) = 2(z^2 + w^2z) = 2z^2$, and therefore $f^*(\pi^*(\alpha) - 2y) = 0$, which means that $\pi^*(\alpha) - 2y = m(x^2 - 4y) = m\pi^*(\sigma)$ for some integer $m$. Thus $\pi^*(\alpha - m\sigma) = 2y$, and $\pi^*$ is injective because it is a rational isomorphism. This finally leads to the following.
Definition 7.11. Given the homomorphism $\pi^*: H^8(\text{SP}^2) \to H^8(B)$, we define the generator

$$h \in H^8(\text{SP}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

as the unique preimage of $2y \in H^8(B)$ under $\pi^*$. So $i^*(h) = 2z^2$ in $H^8(\triangle)$.

Definitions 7.9 and Definition 7.11 can be generalised as we shall see below, which leads to basis elements

$$(1/2)^{s-1}h^s, (1/2)^uh^u \text{ where } s, t \geq 1, u \geq 0$$

for $H^{4k}(\text{SP}^2)$.

Case $k = 2m \geq 2$ for $H^{4k}(\text{SP}^2)$.

From (4.31), we have $H^{4(2m)}(B) \cong (\mathbb{Z}/2z)^m \oplus (\mathbb{Z})^{m+1} = \mathbb{Z}/2(w^n y^k) \oplus \mathbb{Z}(x^c y^d)$ where $4 \leq a \leq 4m$, $0 \leq b \leq m - 1$, $0 \leq c \leq 2m$, $0 \leq d \leq m$; explicitly, the generators are

$$w^{4m}, w^{4m-4}y, w^{4m-8}y^2, \ldots, w^4y^{m-1}, y^m, x^2y^{m-1}, \ldots, x^{2m-2}y, x^{2m}.$$ 

Then, we observe that $\ker f^*$ contains the subgroup $(\mathbb{Z})^m = \mathbb{Z}\langle k_0, k_1, \ldots, k_{m-1} \rangle$ where

$$k_0 = x^{2m} - 2^{2m}y^m, \quad k_1 = x^{2m-2}y - 2^{2m-2}y^m, \quad k_2 = x^{2m-4}y^2 - 2^{2m-4}y^m$$

$$\ldots k_j = x^{2(m-j)}y^j - 2^{2(m-j)}y^m, \quad \ldots k_{m-1} = x^2y^{m-1} - 2^2y^m$$

for $0 \leq j \leq m - 1$. So $k_j = (x^2 - 4y) p_j(x, y)$ for some polynomial $p_j(x, y)$.

Actually, this subgroup is the whole kernel. For let $g(x, y) + h(w, y)$ be

$$\sum_{0 \leq i \leq m} g_i x^{2(m-i)}y^i + \sum_{0 \leq j \leq m-1} h_j w^{4(m-j)}y^j, \quad g_i, h_j \in \mathbb{Z}$$

in $\ker f^* \subseteq H^{8m}(B)$. Then

$$g(2z, z^2 + w^2z) + h(w, z^2 + w^2z) = 0$$

(7.12)

in $\mathbb{Z}[z, w]/(2w)$. Equating coefficients of $w^{4m}$, $w^{4m-2}z$, $w^{4m-4}z^2$, $\ldots$, $w^{2m+2}z^{m-1}$ shows that $h_0, h_1, \ldots, h_{m-1} \equiv 0 \mod 2$, so $h(w, y) = 0$. Equating coefficients of $z^{2m}$ shows that $g := \sum_{i=0}^m 2^{2(m-i)}g_i = 0$, so $g(x, y)|_{x^2 = 4y} = g y^m = 0$. Therefore $x^2 - 4y$ divides $g(x, y)$, and $g(x, y) + h(w, y)$ lies in the subgroup we have given.
By analogy with the case $m = 1$, the canonical isomorphism $H^8m(SP^2/\triangle) \cong H^8m(B/\widehat{\triangle})$ implies that $\ker i^* \cong H^8m(SP^2/\triangle) \cong \mathbb{Z}^m$, and we may choose basis elements $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ for $\ker i^*$ with $\pi^*(\sigma_i) = k_i$ for $i = 0, 1, \ldots, m - 1$. Similarly there are basis elements $k'_0, k'_1, \ldots, k'_{m-1}$ for $H^8m(SP^2/\triangle)$ such that $s^*(k'_i) = \sigma_i$.

Proposition 6.37 gives an isomorphism $H^8m+1(B/\widehat{\triangle}) \cong \mathbb{Z}/2(\delta(2z^{2m}))$. So $\ker \delta = \mathbb{Z}(2z^{2m})$, where $H^8m(\triangle) \cong \mathbb{Z}(z^{2m})$. We then get a short exact sequence

$$0 \leftarrow \mathbb{Z}(2z^{2m}) \xrightarrow{i^*} H^8m(SP^2) \xrightarrow{s^*} \mathbb{Z}(k'_0, k'_1, \ldots, k'_{m-1}) \leftarrow 0 \quad (7.13)$$

which must be split. This proves that $H^8m(SP^2) \cong \mathbb{Z}(\alpha_m) \oplus \mathbb{Z}(\sigma_0, \sigma_1, \ldots, \sigma_{m-1})$ where $i^*(\alpha_m) = 2z^{2m}$, so $f^* \pi^*(\alpha_m) = 2z^{2m}$. But also $f^*(2y^m) = 2z^{2m}$, and therefore $\pi^*(\alpha_m) - 2y^m := \kappa$ lies in $\ker f^*$, and $\kappa = \pi^*(\sigma)$ for some $\sigma$ in $\im s^*$. Finally, $\pi^*(\alpha_m - \sigma) = 2y^m$, and $\pi^*$ is injective because it is a rational isomorphism. This leads to the following.

**Definition 7.14.** For any $m \geq 1$, we define the generator

$$(1/2)^{m-1}h^m \in H^8m(SP^2)$$

as the unique preimage of $2y^m$ under $\pi^*$. So $i^*((1/2)^{m-1}h^m) = 2z^{2m}$.

**Remark 7.15.** The notation we have used for this generator is important for the multiplicative structure later; it refers to the fact that $2^{m-1} \cdot (1/2)^{m-1}h^m = h^m$. This is true because both map to $2^my^m$ under the monomorphism $\pi^*$.

Having defined the generator $(1/2)^{m-1}h^m$ as in definition 7.14, it has now become more natural and convenient for us to choose the alternative basis $2y^m, x^{2(m-j)}y^j$, where $0 \leq j \leq m - 1$, for $\im \pi^*$.

**Definition 7.16.** For $0 \leq j \leq m - 1$, we define the generator

$$(1/2)^j g^{2(m-j)}h^j \in H^8m(SP^2)$$

as the unique preimage of $x^{2(m-j)}y^j$ under $\pi^*$. So $i^*((1/2)^j g^{2(m-j)}h^j) = 2^{2(m-j)}z^{2m}$.

The notation indicates that $2^j \cdot (1/2)^j g^{2(m-j)}h^j = g^{2(m-j)}h^j$ in $H^8m(SP^2)$.

We shall repeat a similar process for $H^{4k}(SP^2)$, $k = \text{odd}$.

Case $k = 2m + 1 \geq 3$. 
We have $H^{4(2m+1)}(B) \cong (\mathbb{Z}/2)^{m+1} \oplus \mathbb{Z}^{m+1} = \mathbb{Z}/2^a y^b \oplus \mathbb{Z}^c y^d$, where $2 \leq a \leq 4m+2$, $0 \leq b \leq m$, $1 \leq c \leq 2m+1$, $0 \leq d \leq m$, explicitly; the generators are

$$w^{4m+2}, w^{4m-2}y, w^{4m-6}y^2, \ldots, w^2y^m, \ xym, x^3y^{m-1}, \ldots, x^{2m-1}y, x^{2m+1}.$$ 

Then, we observe that $\ker f^*$ contains the subgroup $\mathbb{Z}^m = \mathbb{Z}\langle k_0, k_1, \ldots, k_{m-1} \rangle$ where

$$k_0 = x^{2m+1} - 2^{2m}xy^m, \ k_1 = x^{2m-1}y - 2^{2m-2}xy^m, \ k_2 = x^{2m-3}y^2 - 2^{2m-4}xy^m \ldots \ k_j = x^{2(m-j)+1}y^j - 2^{2(m-j)}xy^m, \ldots \ k_{m-1} = x^3y^{m-1} - 2^2xy^m$$

for $0 \leq j \leq m-1$, and $k_j = (x^2 - 4y)p_j(x,y)$ for some polynomial $p_j(x,y)$.

As in the case $k = 2m$, we may prove that this subgroup is actually the whole of $\ker f^*$, and that there is a short exact sequence

$$0 \to \mathbb{Z}\langle 2z^{2m+1} \rangle \to^s H^{8m+4}(SP^2) \to^s \mathbb{Z}\langle k_0, k_1', \ldots, k_{m-1}' \rangle \to 0 \quad (7.17)$$

which must be split. This proves that $H^{8m+4}(SP^2) \cong \mathbb{Z}\langle \alpha_m \rangle \oplus \mathbb{Z}^m$ where $i^*(\alpha_m) = 2z^{2m+1}$, so $f^*\pi^*(\alpha_m) = 2z^{2m+1}$. By an analogous argument to the case $k = 2m$, we may change $\alpha_m$ so that its image under $\pi^*$ is $xy^m$. Also, $\pi^*$ is injective because it is a rational isomorphism. This leads to the following.

**Definition 7.18.** For any $m \geq 1$, we define the generator

$$(1/2)^m gh^m \in H^{8m+4}(SP^2)$$

as the unique preimage of $xy^m \in H^{8m+4}(B)$ under $\pi^*$. So $i^*((1/2)^m gh^m) = 2z^{2m+1}$.

The notation indicates that $2^m \cdot (1/2)^m gh^m = gh^m$ in $H^{8m+4}(SP^2)$.

As before, it is now become more natural and convenient for us to choose the alternative basis $x^{2(m-j)+1}y^j$ instead of $x^{2(m-j)+1}y^j - 2^{2(m-j)}xy^m$ for $\im \pi^*$.

**Definition 7.19.** We define the generator

$$(1/2)^j g^{2(m-j)+1}h^j \in H^{8m+4}(SP^2), \quad 0 \leq j \leq m - 1$$

as the unique preimage of $x^{2(m-j)+1}y^j$ under $\pi^*$. So $i^*((1/2)^j g^{2(m-j)+1}h^j) = 2^{2(m-j)+1}z^{2m+1}$.

Finally, summarising the above results we arrive at the following.
We shall now describe Lemma 7.22. If \( \ell, s \geq 1 \) from Nakaoka [46]. We shall confirm his results for \( \pi^* \): \( H^*(SP^2) \to H^*(B) \).

### 7.1.2 \( H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^\infty)) \)

We now proceed to \( H^{4k+2}(SP^2) := H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z}) \).

**Proposition 7.21.** For any \( k \geq 0 \), \( H^{4k+2}(SP^2) = 0 \).

**Proof.** We have the commutative ladder

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^{4k+2}(\Delta) & \overset{\pi^*}{\longrightarrow} & H^{4k+2}(SP^2) & \overset{s^*}{\longrightarrow} & H^{4k+2}(SP^2/\Delta) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& \longrightarrow & H^{4k+2}(\hat{\Delta}) & \overset{f^*}{\longrightarrow} & H^{4k+2}(B) & \overset{\beta^*}{\longrightarrow} & H^{4k+2}(B/\hat{\Delta}) & \longrightarrow & 0 \\
\end{array}
\]

where \( H^{4k+2}(B) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \) with basis \( w^i y^j \) where \( i + 4j = 2k + 1 \), \( j \geq 0 \), \( i \geq 1 \) from (4.31). The image \( f^*(w^i y^j) = w^i(z^2 + w^2 z)^j \) confirms that \( \ker f^* = 0 \). Then, by the isomorphisms \( \ker f^* \cong H^{4k+2}(B/\hat{\Delta}) \cong H^*(SP^2/\Delta) \cong H^{4k+2}(SP^2) \), we must have \( H^{4k+2}(SP^2) = 0 \). \( \square \)

### 7.1.3 \( H^{4k+1}(SP^2(\mathbb{H}\mathbb{P}^\infty)) \) and \( H^{4k+3}(SP^2(\mathbb{H}\mathbb{P}^\infty)) \)

We shall now describe \( H^*(SP^2) := H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z}) \) for \( * = 4k + 1, 4k + 3 \).

For \( * = 1, 3, 5 \), the cohomology group \( H^*(SP^2) = 0 \) is given by (7.2), which comes from Nakaoka [46]. We shall confirm his results for \( * = 3, 5 \) by using our cohomology ladder. This works as below.

**Lemma 7.22.** If \( * = 1, 3, 5, 9 \), then \( H^*(SP^2(\mathbb{H}\mathbb{P}^\infty)) = 0 \).

**Proof.** Case \( * = 3 \). The ladder

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^3(SP^2) & \overset{s^*}{\longrightarrow} & H^3(SP^2/\Delta) & \overset{\delta}{\longrightarrow} & H^2(\Delta) = 0 \\
& & \downarrow \cong & & \downarrow & & \\
& 0 = H^3(B) & \overset{\beta^*}{\longrightarrow} & H^3(B/\hat{\Delta}) & \overset{\delta}{\longrightarrow} & H^2(\hat{\Delta}) & \longrightarrow \\
\end{array}
\]
shows that $H^3(SP^2/\triangle) \xrightarrow{s^*} H^3(SP^2)$ is an isomorphism, and also, $H^3(B/\hat{\triangle}) = 0$ by Lemma 6.13. Then, using the canonical isomorphism $H^3(B/\hat{\triangle}) \cong H^3(SP^2/\triangle)$, we confirm that $H^3(SP^2) = 0$.

Case $* = 5$. We have the ladder

$$
0 \xrightarrow{\delta} H^5(SP^2) \xrightarrow{s^*} H^5(SP^2/\triangle) \xrightarrow{\delta} H^4(\triangle) \xrightarrow{\delta} H^4(SP^2)
$$

where $H^4(\triangle) \cong \mathbb{Z}\langle z \rangle$, and $H^5(B/\hat{\triangle}) \cong \mathbb{Z}/2\langle \delta(z) \rangle$ by Proposition 6.37. Then, since $s^*(\delta(z)) = 0$ and $s^*$ is a surjection, we confirm that $H^5(SP^2) = 0$.

Case $* = 9$. We have the ladder

$$
0 \xrightarrow{\delta} H^9(SP^2) \xrightarrow{s^*} H^9(SP^2/\triangle) \xrightarrow{\delta} H^8(\triangle) \xrightarrow{\delta} H^8(SP^2)
$$

where $H^8(\triangle) \cong \mathbb{Z}\langle z^2 \rangle$, and $H^9(B/\hat{\triangle}) \cong \mathbb{Z}/2\langle \delta(z^2) \rangle$ by Proposition 6.37.

So, $H^9(SP^2) = 0$ follows from the fact that $s^*(\delta(z^2)) = 0$, and that $s^*$ is a surjection.

In general, $H^{4k+1}(SP^2)$ has the following property.

**Lemma 7.23.** For any $k \geq 1$, the LES

$$
H^{4k+1}(\triangle) \xleftarrow{\delta^*} H^{4k+1}(SP^2) \xleftarrow{s^*} H^{4k+1}(SP^2/\triangle) \xleftarrow{\delta} H^{4k}(\triangle) \xleftarrow{\cdots}
$$

is given by

$$
0 \xleftarrow{\delta^*} (\mathbb{Z}/2)^m \xleftarrow{s^*} (\mathbb{Z}/2)^m \oplus \mathbb{Z}/2 \xleftarrow{\delta} \mathbb{Z} \xleftarrow{\cdots}
$$

for some $m \geq 0$, and $\ker s^* \cong \mathbb{Z}/2\langle \delta(z^k) \rangle$.

**Proof.** We use the notation of Remark 7.4. It is clear that $H^{4k}(\triangle) \cong \mathbb{Z}\langle z^k \rangle$ and $H^{4k+1}(\triangle) = 0$. By Proposition 6.37, we have that $H^{4k+1}(B/\hat{\triangle}) \cong (\mathbb{Z}/2)^{m+1} = \mathbb{Z}/2\langle \delta_i(w^jz^i) \rangle$, where $i \geq 1, 0 \leq j < 2i$. So $H^{4k+1}(SP^2/\triangle) \cong (\mathbb{Z}/2)^{m+1}$. If $j \geq 1$, then $\delta_i(w^jz^i)$ cannot lie in the image of $\delta_u$, so the only non-zero element in its image is $\delta_u(z^k)$, which must generate $\ker s^*$. \qed
Corollary 7.24. For any \( k \geq 3 \), we have \( H^{4k+1}(SP^2) \cong \mathbb{Z}/2\langle \delta(w^j z^i) \rangle \), where \( i \geq 2, 2 \leq j < 2i \).

On the other hand, \( H^{4k+3}(SP^2) \) has the following property.

Lemma 7.25. For any \( k \geq 0 \), there exists an isomorphism

\[
H^{4k+3}(SP^2) \cong H^{4k+3}(SP^2/\triangle).
\] (7.26)

Proof. It follows from the exact sequence

\[
0 = H^{4k+3}(\triangle) \xrightarrow{s^*} H^{4k+3}(SP^2) \xrightarrow{s^*} H^{4k+3}(SP^2/\triangle) \xrightarrow{\delta} H^{4k+2}(\triangle) = 0
\]

that the homomorphism \( s^* \) is an isomorphism for any \( k \geq 0 \).

Corollary 7.27. For any \( k \geq 0 \), there exists an isomorphism

\[
H^{4k+3}(SP^2) \cong H^{4k+3}(B/\hat{\triangle}).
\]

Assembling the properties of the \((4k+1)st\) and \((4k+3)rd\) cohomology groups described above, we may now introduce the following.

Definition 7.28. For \( * = 4k + 3 \geq 7, 4k + 1 \geq 13 \), we define a basis for \( H^*(SP^2) \) by

\[
t_{ij} := u^* (\delta(w^j z^i)), \quad 1 \leq j < 2i, 1 \leq i, |t_{ij}| = 4i + 2j + 1
\]

where \( \delta(w^j z^i) \in H^*(B/\hat{\triangle}) \), and \( u^* := s^* \circ (\pi_1^*)^{-1} \) in the commutative ladder

\[
\begin{array}{c}
\xrightarrow{s^*} H^*(SP^2) \xrightarrow{s^*} H^*(SP^2/\triangle) \xrightarrow{\delta} H^{*+1}(\triangle) \\
\xrightarrow{\pi^*} \cong \xrightarrow{\pi_1^*} \xrightarrow{s^*} \xrightarrow{\delta} H^*(B) \xrightarrow{\delta} H^*(B/\hat{\triangle}) \xrightarrow{s^*} H^{*+1}(\hat{\triangle})
\end{array}
\]

7.2 Product Structures

In this section we will describe the multiplicative structures for \( H^*(SP^2) \).

If \( * = 4k \), then the product structures

- \( (1/2)^{s-1} h^s \cdot (1/2)^{s'-1} h^{s'} = 2 \cdot (1/2)^{s+s'-1} h^{s+s'} \)

- \( (1/2)^m g^l h^m \cdot (1/2)^{s-1} h^s = 2 \cdot (1/2)^{m+s} g^l h^{m+s} \)
\( (1/2)^m g^t h^m \cdot (1/2)^{m'} g'^{t'} h'^{m'} = (1/2)^{m+m'} g^{t+t'} h^{m+m'} \).

For \( (1/2)^m g^t h^m, (1/2)^{s-1} h^s \), \( s, t \geq 1, m \geq 0, |g| = 4, |h| = 8 \), come naturally from Proposition 7.20.

The product structure between the torsion elements \( t_{i,j} \) can be deduced relatively easy, too.

**Proposition 7.29.** The relation
\[
t_{i,j} t_{k,\ell} = 0
\]
holds for any of the torsion elements \( t_{i,j} \) of Definition 7.28.

**Proof.** The dimension of any product \( t_{i,j} t_{k,\ell} \) is given by
\[
|t_{i,j} t_{k,\ell}| = (4i + 2j + 1) + (4k + 2\ell + 1) = 2(2i + 2k + j + \ell + 1).
\]
By Lemma 7.6 and Proposition 7.21, when \( * = \text{even} \), the cohomology group is either \( H^*(SP^2) \cong (\mathbb{Z})^r \) or 0. Thus, any product \( t_{i,j} t_{k,\ell} \) of torsion elements must be zero, since it is also a torsion element.

The remaining products \( t_{i,j} (1/2)^m g^t h^m, t_{i,j} (1/2)^{s-1} h^s \) are more delicate. Our strategy is to use the mod 2 reduction homomorphism,
\[
\rho: H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}/2).
\]
From Section 7.1, if \( * \neq 4k \) then the cohomology group \( H^*(SP^2) \) is either 0 or \( (\mathbb{Z}/2)^r \), so \( \rho \) is injective for \( * \neq 4k \). The dimensions of \( t_{i,j} (1/2)^m g^t h^m \) and \( t_{i,j} (1/2)^{s-1} h^s \) are not equal to \( 4k \), so
\[
\rho(t_{i,j} \cdot (1/2)^m g^t h^m) = 0 \quad \text{implies} \quad t_{i,j} \cdot (1/2)^m g^t h^m = 0,
\]
\[
\rho(t_{i,j} \cdot (1/2)^{s-1} h^s) = 0 \quad \text{implies} \quad t_{i,j} \cdot (1/2)^{s-1} h^s = 0.
\]
We shall use the cohomology ladders
\[
\begin{array}{cccccc}
H^*(\triangle) & \overset{i^*}{\longrightarrow} & H^*(SP^2) & \overset{s^*}{\longrightarrow} & H^*(SP^2/\triangle) & \overset{\delta}{\longrightarrow} & H^{*+1}(\triangle) \\
\rho \downarrow & & \rho \downarrow & & \rho \downarrow & & \rho \downarrow \\
H^*(\triangle; \mathbb{Z}/2) & \overset{i^*_\mathbb{Z}}{\longrightarrow} & H^*(SP^2; \mathbb{Z}/2) & \overset{s^*_\mathbb{Z}}{\longrightarrow} & H^*(SP^2/\triangle; \mathbb{Z}/2) & \overset{\delta}{\longrightarrow} & H^{*+1}(\triangle; \mathbb{Z}/2)
\end{array}
\]
and

\[ H^*\hat{\Delta} \xleftarrow{f_*^*} H^*(B) \xrightarrow{\delta} H^{*-1}(\hat{\Delta}) \]  \hspace{1cm} (7.31)

\[ \rho \]

\[ H^*(\hat{\Delta}; \mathbb{Z}/2) \xleftarrow{f_*^*} H^*(B; \mathbb{Z}/2) \xrightarrow{\delta} H^{*-1}(\hat{\Delta}; \mathbb{Z}/2) \]

which arise because \( \rho \) is a natural transformation, and also the ladder

\[ H^*(\Delta; \mathbb{Z}/2) \xleftarrow{i_*^*} H^*(SP^2; \mathbb{Z}/2) \xrightarrow{s_*^*} H^*(SP^2/\Delta; \mathbb{Z}/2) \xrightarrow{\delta} H^{*-1}(\Delta; \mathbb{Z}/2) \]  \hspace{1cm} (7.32)

\[ \pi_\Delta \]

\[ \pi^* \]

that comes from applying \( H^*(-; \mathbb{Z}/2) \) to (2.29).

Using (6.27), we have

\[ H^{4k}(B; \mathbb{Z}/2) \cong H^{4k}(B; \mathbb{Z}) \otimes \mathbb{Z}/2 \oplus \text{Tor}(H^{4k+1}(B; \mathbb{Z}), \mathbb{Z}/2) = H^{4k}(B; \mathbb{Z}) \otimes \mathbb{Z}/2 \]

since \( H^{4k+1}(B; \mathbb{Z}) = 0 \). Also \( H^*(\Delta_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, z] \), where \( |a| = 1, |z| = 4 \), and \( \rho(w) = a^2 \). So \( f^*_2 \) is given by

\[ f^*_2(x) = 0, \quad f^*_2(y) = z^2 + a^4z. \]  \hspace{1cm} (7.33)

**Lemma 7.34.** If \( * = 4k \), then \( \text{im} i_2^* = 0 \).

**Proof.** Since \( H^{4k}(\Delta; \mathbb{Z}/2) \cong \mathbb{Z}/2(z^k) \), it is enough to prove that \( z^k \) does not lie in \( \text{im} i_2^* \).

\[ H^{4k}(\Delta; \mathbb{Z}/2) \xleftarrow{i_*^*} H^{4k}(SP^2; \mathbb{Z}/2) \]

\[ H^*(\hat{\Delta}; \mathbb{Z}/2) \xleftarrow{f_*^*} H^{4k}(B; \mathbb{Z}/2) \]

This follows at once from (7.33).

**Lemma 7.36.** For \( k \geq 1 \), there is a split short exact sequence

\[ 0 \xleftarrow{} H^{4k+1}(SP^2; \mathbb{Z}/2) \xleftarrow{s_*^*} H^{4k+1}(SP^2/\Delta; \mathbb{Z}/2) \xrightarrow{\delta} H^{4k}(\Delta; \mathbb{Z}/2) \xrightarrow{} 0 \]  \hspace{1cm} (7.37)

given by

\[ 0 \xleftarrow{i_2^*} (\mathbb{Z}/2)^m \xleftarrow{s_*^*} (\mathbb{Z}/2)^m \oplus \mathbb{Z}/2 \xrightarrow{\delta} \mathbb{Z}/2 \xrightarrow{} 0 \]

for some \( m \geq 0 \) where \( \ker s_*^* \cong \mathbb{Z}/2(\delta(z^k)) \).
Proof. It follows by applying mod 2 reduction to the LES described in Lemma 7.23, together with Lemma 7.34.\qed

Lemma 7.38. For \( * = 4k, 4k + 2, 4k + 3, k \geq 1 \), there exists an isomorphism

\[
H^*(SP^2; \mathbb{Z}/2) \cong H^*(SP^2/\triangle; \mathbb{Z}/2).
\]

Proof. Case \( * = 4k + 2, 4k + 3 \). We have the exact sequence

\[
H^*(\triangle; \mathbb{Z}/2) \xrightarrow{i_*^2} H^*(SP^2; \mathbb{Z}/2) \xrightarrow{s_*^2} H^*(SP^2/\triangle; \mathbb{Z}/2) \xrightarrow{\delta} H^{*-1}(\triangle; \mathbb{Z}/2)
\]

where \( H^{4k+3}(\triangle; \mathbb{Z}/2) = H^{4k+2}(\triangle; \mathbb{Z}/2) = H^{4k+1}(\triangle; \mathbb{Z}/2) = 0 \), so clearly, \( s_*^2 \) is an isomorphism.

Case \( * = 4k \). From the LES

\[
H^{4k}(\triangle; \mathbb{Z}/2) \xrightarrow{i_*^2} H^{4k}(SP^2; \mathbb{Z}/2) \xrightarrow{s_*^2} H^{4k}(SP^2/\triangle; \mathbb{Z}/2) \xrightarrow{0}
\]

\( s_*^2 \) is injective. On the other hand, by Lemma 7.34, \( \text{im} i_*^2 = 0 \), so \( s_*^2 \) is surjective, too. \qed

The \( \mathbb{Z}/2 \) cohomology of symmetric squares was studied by Nakaoka in 1950’s, and Lemma 7.34 and 7.38 agree with Nakaoka [47, Lemma 11.3].

Proposition 7.39. The relations \( t_{i,j} (1/2)^m g^l h^m = 0 \) and \( t_{i,j} (1/2)^{s-1} h^s = 0 \) hold in \( H^*(SP^2) \).

Proof. By Lemma 4.22, \( H^*(\Gamma_+; \mathbb{Z}/2) \cong H^*(Gr_H(2, \infty); \mathbb{Z}/2)[a]/(a^5 - p_1 a) \). Let \( \tau \) be the Thom isomorphism \( \tau: H^*(\Gamma_+; \mathbb{Z}/2) \rightarrow H^{*+4}(\text{Th}(\theta); \mathbb{Z}/2) \), and \( t \) the Thom class \( t \in H^4(\text{Th}(\theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \). Then for \( * \geq 4 \), any generator of \( H^*(\text{Th}(\theta); \mathbb{Z}/2) \) can be expressed as \( t p_{\ell}^m a^s \) where \( \ell, m, s \geq 0 \). In particular, we have an expression \( t p_{\ell}^m a^s \) where \( \ell, m \geq 0 \) and \( s \geq 1 \) for generators with odd dimensions. Suppose that

\[
H^{4k}(\text{Th}(\theta); \mathbb{Z}/2) = \mathbb{Z}/2 \langle \nu_i \rangle, \quad H^*(\text{Th}(\theta); \mathbb{Z}/2) = \mathbb{Z}/2 \langle \zeta_s \rangle, \quad * = 4k + 1, 4k + 3, k \geq 1,
\]
then, for any basis elements $\nu_i$ and $\zeta_\kappa$

$$\nu_i \zeta_\kappa = (tp_1^b p_2^c a^\gamma)(tp_1^d p_2^e a^\delta), \quad \gamma \geq 1$$

$$= t^2 p_1^{\ell + i} p_2^{m + j} a^{s + u}$$

$$= (p_1 + a^4) t p_1^{\ell + i} p_2^{m + j} a^{s + u}$$

$$= (p_1^{1+\ell+i} p_2^{m+j} a^{s+u} + p_1^{\ell+i} p_2^{m+j} a^{4+s+u}) t$$

$$= (p_1^{1+\ell+i} p_2^{m+j} a^{s+u} + p_1^{\ell+i} p_2^{m+j} a^{s+u-1}) t$$

$$= (p_1^{1+\ell+i} p_2^{m+j} a^{s+u} + p_1^{\ell+i} p_2^{m+j} a^{s+u}) t$$

$$= 0,$$

where $t^2 = (p_1 + a^4) t$, by Lemma 6.20.

Suppose that $\alpha \in \{(1/2)^m g^h m, (1/2)^{s-1} h^s\}$, and consider the mod 2 reductions $\rho(\alpha), \rho(t_{i,j}) \in H^*(SP^2; \mathbb{Z}/2)$. Then, there are expressions

$$\rho(\alpha) = \sum_\iota s_2^\iota(\lambda_\iota \nu_\iota) \quad \text{and} \quad \rho(t_{i,j}) = \sum_\kappa s_2^\kappa(\mu_\kappa \zeta_\kappa)$$

where $\lambda_\iota, \mu_\kappa \in \{0, 1\}$, by Lemmas 7.38 and 7.36. The product $\nu_\iota \zeta_\kappa = 0$ for any $\nu_\iota$ and $\zeta_\kappa$, hence

$$\sum_\iota \lambda_\iota \nu_\iota \sum_\kappa \mu_\kappa \zeta_\kappa = 0,$$

which then implies that $\rho(\alpha)\rho(t_{i,j}) = \rho(\alpha t_{i,j}) = 0$. \qed

**Remark 7.40.** Any product

$$(tp_1^\lambda p_2^\mu a^\nu)(tp_1^\gamma p_2^\delta a^\gamma), \quad \gamma \geq 1$$

in $H^*(Th(\theta); \mathbb{Z}/2)$ vanishes; the only non-zero products are of the form

$$(tp_1^\lambda p_2^\mu)(tp_1^\rho p_2^\delta).$$

**7.3 $H^*(SP^2(H\mathbb{P}^\infty); \mathbb{Z}/2)$**

We finish this chapter by bringing together certain properties of $H^*(SP^2; \mathbb{Z}/2)$ and $H^*(Th(\theta); \mathbb{Z}/2)$ from above, and using them to start improving our understanding of the role of the Thom isomorphism.
CHAPTER 7. THE COHOMOLOGY RING I

By Lemma 7.36, there is a split short exact sequence

$$0 \leftarrow H^{4k+1}(SP^2; \mathbb{Z}/2) \xrightarrow{s_2^*} H^{4k+1}(SP^2/\triangle; \mathbb{Z}/2) \xrightarrow{\delta} H^{4k}(\triangle; \mathbb{Z}/2) \leftarrow 0$$

where \( \text{ker } s_2^* \cong \mathbb{Z}/2\langle \delta(z^k) \rangle \), and by Lemma 7.38, if \( * \neq 4k + 1 \) then

$$s_2^*: H^*(SP^2/\triangle; \mathbb{Z}/2) \rightarrow H^*(SP^2; \mathbb{Z}/2)$$

is an isomorphism. Then we may write

$$H^*(SP^2; \mathbb{Z}/2) \cong H^*(SP^2/\triangle; \mathbb{Z}/2)/\langle \delta(z^k) \rangle$$

where the only non-vanishing products in \( H^* (\text{Th}(\theta); \mathbb{Z}/2) \cong H^*(SP^2/\triangle; \mathbb{Z}/2) \) are of the form \((tp_1^*p_2^*)(tp_1^*p_2^\beta)\), by Corollary 7.40.

For \( H^*(\text{Th}(\theta); \mathbb{Z}/2) \) as described in the proof of Proposition 7.39, we write

$$H^*(\Gamma_+; \mathbb{Z}/2) t := H^*(\text{Th}(\theta); \mathbb{Z}/2)$$

where \( t \) is the unique Thom class in \( H^4(\text{Th}(\theta); \mathbb{Z}/2) \) and

$$H^*(\Gamma_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2](1, a, a^2, a^3, a^4)$$

with the relations \( a^5 = p_1 a, \ t^2 = (a^4 + p_1) t \).

Recall from (4.14) that

$$H^*(Gr_{H}(2,n+1)_+; \mathbb{Z}/2) \cong H^*(Gr_{H}(2,\infty)_+; \mathbb{Z}/2)/(q_n, q_{n+1}) \quad (7.41)$$

where \( H^*(Gr_{H}(2,\infty)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2], \ |p_i| = 4i \), and from [48] that \( q_n \) is the \( n \)-th complete symmetric polynomial in \( z_1, z_2 \), by setting \( p_1 = z_1 + z_2 \), \( p_2 = z_1z_2 \). Also,

$$H^*((\Gamma_+)_+; \mathbb{Z}/2) \cong H^*(Gr_{H}(2,n+1)_+; \mathbb{Z}/2)(1, a, a^2, a^3, a^4)$$

from (4.24), and in particular, the homomorphisms

$$H^*(Gr_{H}(2,n+1)_+; \mathbb{Z}/2) \rightarrow H^*((\Gamma_+)_+; \mathbb{Z}/2), \ H^*(Gr_{H}(2,\infty)_+; \mathbb{Z}/2) \rightarrow H^*(\Gamma_+; \mathbb{Z}/2)$$

induced by the projections are injective. Combining with the Thom isomorphism \( \tau \), these may be summarised as a diagram

$$\begin{array}{ccc}
H^*(Gr_{H}(2,n+1)_+; \mathbb{Z}/2) & \xrightarrow{\eta_1} & H^*((\Gamma_+)_+; \mathbb{Z}/2) & \xrightarrow{\tau} & H^{*+4}(\text{Th}(\theta_n); \mathbb{Z}/2) \\
H^*(Gr_{H}(2,\infty)_+; \mathbb{Z}/2) & \xrightarrow{\eta_2} & H^*(\Gamma_+; \mathbb{Z}/2) & \xrightarrow{\tau} & H^{*+4}(\text{Th}(\theta); \mathbb{Z}/2)
\end{array} \quad (7.42)$$

where \( \eta^*_i \), \( i = 1, 2, 3 \) are surjective, and induced by the inclusions.

In particular, observe that in \( H^{*(4n+1)_+4}(\text{Th}(\theta); \mathbb{Z}/2) \), we have \( \text{ker } \eta^*_3 \cong \mathbb{Z}/2(q_n a t) \).


**Remark 7.43.** We have the identifications

\[(1) \delta(z) = at, \quad (2) \delta(z^{n+1}) = q_n at,\]

where \(q_n, |q_n| = 4n,\) is the \(n\)-th complete symmetric polynomial as in (7.41).

(1) follows from the fact \(H^5(\text{Th}(\theta); \mathbb{Z}/2) \cong \mathbb{Z}/2.\)

(2) follows from (7.42) together with the commutative ladder

\[
\begin{array}{cccc}
H^{*+1}(SP^2_n/\Delta_n; \mathbb{Z}/2) & \xrightarrow{\delta} & H^*(\Delta_n; \mathbb{Z}/2) & \xrightarrow{i^*_2} H^*(SP^2_n; \mathbb{Z}/2) \\
\eta^* & & & \zeta^* \\
H^{*+1}(SP^2/\Delta; \mathbb{Z}/2) & \xrightarrow{\delta} & H^*(\Delta; \mathbb{Z}/2) & \xrightarrow{i^*_2} H^*(SP^2; \mathbb{Z}/2)
\end{array}
\]

where \(H^*(SP^2/\Delta; \mathbb{Z}/2) \cong H^*(\text{Th}(\theta); \mathbb{Z}/2),\)

\(H^*(SP^2_n/\Delta_n; \mathbb{Z}/2) \cong H^*(\text{Th}(\theta_n); \mathbb{Z}/2),\)

and the vertical maps are induced by inclusions. Note that for all \(r,\) there is an isomorphism \(H^r(X; \mathbb{Z}/2) \cong \lim_{\longleftarrow} H^r(X_n; \mathbb{Z}/2),\) which comes from [22, Proposition 3F.5] and Section 2.3, where \(X_n = SP^2_n, \Delta_n, SP^2_n/\Delta_n\) and \(X = \bigcup_n X_n.\) The kernel of \(\zeta^*\) is generated by \(z^{n+1}\) because \(H^*(\Delta_n; \mathbb{Z}/2) \cong \mathbb{Z}/2[z]/(z^{n+1}).\) Also, from Lemma 7.34, if \(* = 4k\) then \(0 = \text{im} i^*_2 = \ker \delta.\) Then, observe that for \(H^{4(n+1)+1}(SP^2/\Delta; \mathbb{Z}/2)\) we have \(\ker \eta^* \cong \mathbb{Z}/2(\delta(z^{n+1})).\) Therefore, it must be that \(\delta(z^{n+1}) = q_n at.\)
Chapter 8

The Cohomology Ring II

In this chapter, we mainly focus on the computation of the integral cohomology ring $H^\ast(SP^2(\mathbb{H}P^n))$. We shall also check our results with M. Nakaoka’s work [44, 47], in which he studied the mod $p$ cohomology of $p$-fold cyclic products $CP_p(X)$.

8.1 $H^\ast(SP^2(\mathbb{H}P^n); \mathbb{Z})$

We shall prove Theorem 1.8 in this Section. Nakaoka’s work included a computation of $H^\ast(CP_p(S^n))$ as an abelian group for any $n$-sphere $S^n$. For $p = 2$, we may compare our results on $H^\ast(SP^2(\mathbb{H}P^1))$ with his computations of $H^\ast(CP_2(S^4))$, as follows.

Theorem 8.1 ([47]).

$$H^\ast(SP^2(\mathbb{H}P^1)_+) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 4, 8 \\ \mathbb{Z}/2 & \text{if } * = 7 \\ 0 & \text{otherwise.} \end{cases}$$

Nakaoka also proved [46] that if $X$ is a connected CW complex such that its integral homology is $H_\ast(X_+) = 0$ for $0 < * < r$, then

$$H^\ast(SP^m(X)_+) \cong H^\ast(X_+) \quad \text{for} \quad * \leq r + 1$$

holds for any $m \geq 1$. Taking $X = \mathbb{H}P^n$ and $r = 4$ gives an isomorphism

$$H^\ast(SP^m(\mathbb{H}P^n)) \cong H^\ast(\mathbb{H}P^n) \quad \text{for} \quad * \leq 5, \quad (8.2)$$

Nakaoka used the notation $\mathbb{Z}_p(X)$
with any \( m \geq 1 \).

We build on his work, and on our results for \( H^*(SP^2) \) from Chapter 7, and describe the cohomology ring \( H^*(SP^2_n) := H^*(SP^2(\mathbb{H}P^n); \mathbb{Z}) \).

We shall use the commutative ladder of LES

\[
\begin{array}{cccccc}
H^*(\Delta_n) & \xleftarrow{\iota^*} & H^*(SP^2_n) & \xleftarrow{s^*} & H^*(SP^2_n/\Delta_n) & \xleftarrow{\delta} H^{*-1}(\Delta_n) \\
\downarrow \pi^* & & \downarrow \cong & & \downarrow \cong & \\
H^*(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^*(B_n) & \xleftarrow{\beta^*} & H^*(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} H^{*-1}(\widehat{\Delta}_n) \\
\end{array}
\]

that comes from applying \( H^*(-) \) to (2.22).

As input we have that for any \( n \geq 1 \)

\[
H^*((B_n)_+) \cong H^*(B_+)/f_{n,1}, f_{n,2}, y^{n+1})
\]

where \( H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx) \) from (4.36). We also have the integral cohomology of the diagonal spaces

\[
H^*((\Delta_n)_+) \cong H^*(\mathbb{H}P^n_+) \cong \mathbb{Z}[z]/(z^{n+1}), |z| = 4
\]

from Section 2.1, and

\[
H^*((\widehat{\Delta}_n)_+) \cong H^*((\mathbb{R}P^\infty \times \mathbb{H}P^n)_+) \cong \mathbb{Z}[w, z]/(2w, z^{n+1}), |z| = 4, |w| = 2,
\]

which comes from Remark 2.18.

### 8.1.1 \( H^*(SP^2(\mathbb{H}P^n)) \) for \( * > 8n \)

As a simple check on our computations we first confirm that the cohomology group \( H^*(SP^2_n) \) is zero for \( * > 8n \), as required by the cell structure [47].

Assume \( * > 8n \), and recall from Remark 4.41 that the cohomology group of the Borel space is

\[
H^*(B_n) \cong (\mathbb{Z}/2)^{n+1}, n \geq 1
\]

for \( * = 8n + 2\ell, \ell = 1, 2, 3, \ldots \), and \( H^*(B_n) = 0 \) when \( * \) = odd. There are four cases to consider, \( * = 4k, 4k + 1, 4k + 2 \) and \( 4k + 3 \).

If \( * = 4k > 8n \), then we have the ladder

\[
\begin{array}{cccccc}
0 & \xleftarrow{i^*} & H^{4k}(SP^2_n) & \xleftarrow{s^*} & H^{4k}(SP^2_n/\Delta_n) & \xleftarrow{\delta} H^{4k-1}(\Delta_n) = 0 \\
\downarrow \pi^* & & \downarrow \cong & & \downarrow \cong & \\
H^{4k}(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^{4k}(B_n) & \xleftarrow{\beta^*} & H^{4k}(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} H^{4k-1}(\widehat{\Delta}_n) = 0
\end{array}
\]
where $H^{4k}(B_n) \cong (\mathbb{Z}/2)^{n+1} = \mathbb{Z}/2(w^ay^b)$ for $2k - 4n \leq a \leq 2k$, $0 \leq b \leq n$, $a + 4b = 2k$ and $H^{4k}(\widehat{\Delta}_n) \cong (\mathbb{Z}/2)^{n+1} = \mathbb{Z}/2(w^cz^d)$ for $2k - 2n \leq c \leq 2k$, $0 \leq d \leq n$, $2c + 4d = 4k$.

We have $H^{4k}(B_n/\widehat{\Delta}_n) \cong \ker f^* = 0$, because $f^*(\sum_\lambda \lambda_\alpha w^ay^b) = 0$ implies that

$$\sum_{\alpha \leq 2k} \lambda_\alpha w^a(z^2 + w^az)^b = 0,$$

so equating coefficients of $w^{2k}$, $w^{2k-2}z$, ... in succession gives $\lambda_{2k} = \lambda_{2k-4} = \cdots = 0$. But $H^{4k}(SP_n^2/\Delta_n)$ is isomorphic to the kernel of $f^*$, so

$$H^{4k}(SP_n^2/\Delta_n) \cong H^{4k}(SP_n^2) = 0, \quad k > 2n.$$

The remaining three cases are similar. If $* = 4k + 2$, then $\ker f^* = 0$ again, and

$$\ker f^* \cong H^{4k+2}(B_n/\widehat{\Delta}_n) \cong H^{4k+2}(SP_n^2/\Delta_n) \cong H^{4k+2}(SP_n^2).$$

If $* = 4k + 1$, $4k + 3$, then we have the exact sequence

$$H^*(B_n) = 0 \xrightarrow{f^*} H^*(B_n/\widehat{\Delta}_n) \xrightarrow{\delta^{-1}} H^*(\widehat{\Delta}_n) \xrightarrow{f^*} H^*(B_n) = 0$$

where $H^*(B_n) \cong H^*(\widehat{\Delta}_n) \cong (\mathbb{Z}/2)^{n+1}$, so $f^*$ is an isomorphism.

Therefore, $0 = H^*(B_n/\widehat{\Delta}_n) \cong H^*(SP_n^2/\Delta_n) \cong H^*(SP_n^2)$.

### 8.1.2 $H^{4k}(SP^2(\mathbb{H}^n))$

In the previous section we confirmed that $H^*(SP_n^2) = 0$ if $* > 8n$. In this section we shall describe the integral cohomology ring $H^{4*}(SP_n^2)$.

We always assume $n \geq 1$.

**Lemma 8.5.** For any $k \leq 2n$, the cohomology group $H^{4k}(B_n/\widehat{\Delta}_n)$ is torsion free; that is, $H^{4k}(B_n/\widehat{\Delta}_n) \cong \mathbb{Z}^m$ for some $m$.

**Proof.** From the LES

$$L \xrightarrow{f^*} H^{4k}(\widehat{\Delta}_n) \xleftarrow{f^*} H^{4k}(B_n) \xleftarrow{\beta^*} H^{4k}(B_n/\widehat{\Delta}_n) \xleftarrow{\delta} H^{4k-1}(\widehat{\Delta}_n) = 0$$

we see that $H^{4k}(B_n/\widehat{\Delta}_n)$ is isomorphic to $\ker f^*$. So we need to show that $\ker f^*$ is torsion free.

Case $k \leq n$. 


If \( k \leq n \), then the cohomology ring \( H^*(B_n) \) is isomorphic to \( H^*(B) \) by Remark 4.41, and clearly \( H^{4k}(\hat{\Delta}_n) \cong H^{4k}(\hat{\Delta}) \). So, computing \( \ker f^* \) is the same as finding the kernel of \( f_\infty^* \) in the diagram

\[
\begin{array}{cccccc}
H^{4k}(\hat{\Delta}_n) & \xleftarrow{f^*} & H^{4k}(B_n) & \xleftarrow{\delta} & H^{4k-1}(\hat{\Delta}_n) & = 0 \\
\cong & & \cong & & \downarrow & \\
H^{4k}(\hat{\Delta}) & \xleftarrow{f_\infty^*} & H^{4k}(B) & \xleftarrow{\delta} & H^{4k-1}(\hat{\Delta}) & = 0
\end{array}
\]

which we found to be torsion free in Section 7.1.1.

Case \( k \geq n + 1 \).

From (4.36), we have \( H^*(B_n) \cong H^*(B_+)/(f_{n,1}, f_{n,2}, y^n+1) \), and from (4.42), any \( x^k \in H^{4k}(B_n) \), \( k \geq n + 1 \) can be expressed as

\[
x^k = \sum_{\ell,m} \lambda_{\ell,m} x^\ell y^m \quad \text{where} \quad \lambda_{\ell,m} \in \mathbb{Z}, \ \ell \geq 0, \ 0 < m \leq n;
\]

in other words, any element in \( H^{4k}(B_n) \) can be written as

\[
\sum_{a,b} \lambda_{a,b} w^a y^b + \sum_{\ell,m} \lambda_{\ell,m} x^\ell y^m
\]

where \( \lambda_{a,b} \in \{0,1\}, \ \lambda_{\ell,m} \in \mathbb{Z}, \ b, \ell \geq 0, \ a, m > 0 \). So every torsion element has the form \( u := \sum_j \lambda_j w^{a_j} y^{b_j} \), where \( \lambda_j \in \{0,1\}, \ 2k = a_1 > a_2 > \cdots, 0 = b_1 < b_2 < \cdots \), and \( a_j + 4b_j = 2k \).

We know that \( f^*(w^{a_j} y^{b_j}) = w^a(z^2 + w^2 z)^b \) in \( H^{4k}(\hat{\Delta}_n) \), so

\[
f^*(u) = \sum_j \lambda_j w^{a_j} (z^2 + w^2 z)^{b_j} = 0
\]

implies that \( \lambda_1 = \lambda_2 = \cdots = 0 \), by equating coefficients of \( w^{2k}, w^{2k-2}z, \ldots \) in succession. Therefore \( u = 0 \), and there are no non-zero torsion elements in \( \ker f^* \).

For future reference, note that

\[
\begin{align*}
f^*(2y^{k/2}) &= 2(z^2 + w^2 z)^{k/2} = 2z^k = 0, \quad \text{(when k is even)} \quad \text{(8.6)} \\
f^*(x^\ell y^m) &= (2z)^\ell (z^2 + w^2 z)^m = 2^\ell z^{\ell+2m} = 0, \quad \text{when} \ \ell > 0. \quad \text{(8.7)}
\end{align*}
\]

by (8.4), Remark 4.18 and Proposition 5.19. So, \( 2y^{k/2}, x^\ell y^m \in \ker f^* \).

Since \( B_n/\hat{\Delta}_n \) is connected, we also have \( H^0((B_n/\hat{\Delta}_n)_+) \cong \mathbb{Z} \).
Proposition 8.8. Given $f^* : H^{4k}(B_n) \to H^{4k}(\hat{\triangle}_n)$ as above; if $n + 1 \leq k \leq 2n$, then any element of $\ker f^*$ can be written as

$$\lambda_k 2y^{k/2} + \sum_{a,b} \mu_{a,b} x^a y^b, \quad \lambda_k, \mu_{a,b} \in \mathbb{Z}, \quad a + 2b = k.$$

Proof. Suppose $\kappa = Ay^{k/2} + \sum_{a,b} B_{a,b} x^a y^b + \sum_{i,j} C_{i,j} w^i y^j$ is a non-zero element of $\ker f^*$, then

$$f^*(\kappa) = f^* \left( Ay^{k/2} + \sum_{a,b} B_{a,b} x^a y^b + \sum_{i,j} C_{i,j} w^i y^j \right) = A(z^2 + w^2 z)^{k/2} + \sum_{i,j} C_{i,j} w^i (z^2 + w^2 z)^j = 0$$

where $a + 2b = k, i + 4j = 2k, i > 0, j > 0$ for some $A, B_{a,b}, C_{i,j} \in \mathbb{Z}$. If $A = 0$ then $f^*(\kappa) = \sum_{i,j} C_{i,j} w^i (z^2 + w^2 z)^j = 0$ implies that $C_{2k,0} = C_{2k-4,1} = \cdots = 0$ by equating coefficients of $w^{2k}, w^{2k-2}, \cdots$ in succession. If $A \neq 0$ then $C_{i,j} = 0$ by equating coefficients of $w^{k+2} z^{2-1}, w^{k+4} z^{2-2}, \cdots$ in succession. This, in turn, yields that $A$ must be even. \qed

Lemma 8.9. For any $n \geq 1$, the cohomology group $H^{4k}(SP^2_n)$ is torsion free.

Proof. Case $k \leq n$.

From the LES

$$\begin{array}{ccccc}
\longrightarrow & H^{4k}(\triangle_n) & \overset{s^*}{\longrightarrow} & H^{4k}(SP^2_n) & \overset{s^*}{\longrightarrow} & H^{4k}(SP^2_n/\triangle_n) & \overset{\delta}{\longrightarrow} & H^{4k-1}(\triangle_n) = 0
\end{array} \quad (8.10)$$

$s^*$ is injection and $H^{4k}(SP^2_n/\triangle_n) \cong \ker s^*$, and also, by Lemma 8.5, the cohomology group $H^{4k}(B_n/\hat{\triangle}_n) \cong H^{4k}(SP^2_n/\triangle_n)$ is torsion free. Then (8.10) is given by

$$\begin{array}{cccc}
\longrightarrow & \mathbb{Z} & \overset{s^*}{\longrightarrow} & H^{4k}(SP^2_n) & \overset{s^*}{\longrightarrow} & \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \longrightarrow 0
\end{array}$$

which shows that $H^{4k}(SP^2_n)$ must be torsion free.

Case $k \geq n + 1$.

From the exact sequence

$$\begin{array}{ccc}
0 & \overset{i^*}{\longrightarrow} & H^{4k}(SP^2_n) & \overset{s^*}{\longrightarrow} & H^{4k}(SP^2_n/\triangle_n) & \overset{\delta}{\longrightarrow} & 0
\end{array}$$

we have that $s^* : H^{4k}(SP^2_n/\triangle_n) \cong H^{4k}(SP^2_n)$ is an isomorphism, and Lemma 8.5 shows that $H^{4k}(B_n/\hat{\triangle}_n) \cong H^{4k}(SP^2_n/\triangle_n)$ is torsion free. \qed
Now let us consider the commutative ladder

\[
\begin{array}{cccc}
\cdots & H^{4k}(\triangle) & \overset{f^*}{\leftarrow} & H^{4k}(B) & \overset{\beta^*}{\leftarrow} & H^{4k}(B/\triangle) & \overset{q^*}{\leftarrow} & H^{4k-1}(\triangle) = 0 \\
\downarrow d^* & \downarrow b^* & \downarrow & \downarrow & \downarrow & = \\
\cdots & H^{4k}(\triangle_n) & \overset{f^*}{\leftarrow} & H^{4k}(B_n) & \overset{\beta^*}{\leftarrow} & H^{4k}(B_n/\triangle_n) & \overset{q^*}{\leftarrow} & H^{4k-1}(\triangle_n) = 0
\end{array}
\]  

(8.11)

If \( k \leq n \), then \( b^* \) is an isomorphism by Remark 4.41, and clearly \( H^{4k}(\triangle_n) \cong H^{4k}(\triangle) \). So \( q^* \) is an isomorphism, by applying the Five Lemma [22, page 129] to the ladder (8.11). This, in turn, implies that \( \phi^*: H^{4k}(SP^2/\triangle) \to H^{4k}(SP^2_n/\triangle_n) \) is an isomorphism in the commutative ladder

\[
\begin{array}{cccc}
\cdots & H^{4k}(\triangle) & \overset{s^*}{\leftarrow} & H^{4k}(SP^2) & \overset{s^*}{\leftarrow} & H^{4k}(SP^2/\triangle) & \overset{\phi^*}{\leftarrow} & H^{4k-1}(\triangle) = 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & = \\
\cdots & H^{4k}(\triangle_n) & \overset{s^*}{\leftarrow} & H^{4k}(SP^2_n) & \overset{s^*}{\leftarrow} & H^{4k}(SP^2_n/\triangle_n) & \overset{\phi^*_n}{\leftarrow} & H^{4k-1}(\triangle_n) = 0
\end{array}
\]  

(8.12)

by appealing to the diagram (2.27). Now (8.12) and the Five Lemma show that

**Remark 8.13.** For \( k \leq n \), \( \phi^*: H^{4k}(SP^2) \to H^{4k}(SP^2_n) \) is an isomorphism.

**Lemma 8.14.** For \( n + 1 \leq k \leq 2n \), additive generators for \( H^{4k}(SP^2_n) \) may be chosen to be of the form

\[(\frac{1}{2})^m g^l h^m \quad \text{and} \quad (\frac{1}{2})^k \frac{k}{2} h^{k/2} \quad \text{(if } k \text{ is even)} \]

(8.15)

where \( |h| = 8 \), \( |g| = 4 \), \( \ell \geq 1 \), \( m \geq 0 \), \( \ell + 2m = k \).

**Proof.** If \( n + 1 \leq k \leq 2n \) then we have the commutative ladder

\[
\begin{array}{cccc}
0 & \overset{\pi^*_n}{\leftarrow} & H^{4k}(SP^2_n) & \overset{s^*_n}{\leftarrow} & H^{4k}(SP^2_n/\triangle_n) & \overset{\pi^*_n}{\leftarrow} & 0 \\
\downarrow & & \downarrow \pi^*_n & & \downarrow & = & \\
H^{4k}(\triangle_n) & \overset{f^*_n}{\leftarrow} & H^{4k}(B_n) & \overset{\beta^*_n}{\leftarrow} & H^{4k}(B_n/\triangle_n) & \overset{q^*_n}{\leftarrow} & 0
\end{array}
\]

where \( H^{4k}(B_n/\triangle_n) \cong \ker f^* \) and \( H^{4k}(SP^2_n) \cong H^{4k}(SP^2_n/\triangle_n) \). For \( n + 1 \leq k \leq 2n \), any element of \( \ker f^* \) can be written as

\[
\lambda_m 2y^m + \sum_{a,b} \mu_{a,b} x^a y^b, \quad \lambda_m, \mu_{a,b} \in \mathbb{Z}, \quad 2m = a + 2b = k \geq 2
\]

by Proposition 8.8. Using Lemma 8.9 and (7.1), we define \((\frac{1}{2})^{m-1} h^m\), \((\frac{1}{2})^k g^l h^m\) as preimages \((\pi^*_n)^{-1}(2y^m)\) and \((\pi^*_n)^{-1}(x^a y^b)\) corresponding to the additive generators for \( \ker f^* \), respectively. \(\square\)
Furthermore, combining Proposition 7.20 with Lemma 8.14 yields the following statement.

**Corollary 8.16.** For $4(n + 1) \leq 4\ast$, the homomorphism

$$ \phi^* : H^{4\ast}(SP^2) \to H^{4\ast}(SP^2_n) $$

induced by the inclusion is a surjection.

Alternatively, it is possible to prove Corollary 8.16 by showing that the restriction homomorphism $H^\ast(SP^2_{n+1}) \to H^\ast(SP^2_n)$ is surjective for any $n \geq 1$, and referring to [22, page 313–314 and Theorem 3F.8] for the properties of inverse limit.

To describe $\ker \phi^*$, we use the commutative diagram

$$
\begin{array}{ccc}
H^{4\ast}((B_n)_+) & \xrightarrow{\pi^*_n} & H^{4\ast}((SP^2_n)_+)
\\
\downarrow{b^*} & & \downarrow{\phi^*}
\\
H^{4\ast}(B_+) & \xleftarrow{\pi^*} & H^{4\ast}(SP^2_+)
\end{array}
$$

where $b^* : H^{4\ast}(B_+) \to H^{4\ast}((B_n)_+)$ is a surjection by Remark 4.41. The homomorphisms $\pi^*$ and $\pi^*_n$ are injective, because there exists the rational isomorphism in cohomology between the Borel space and the symmetric square (7.1) and $H^{4\ast}(SP^2_n)$ and $H^{4\ast}(SP^2)$ are torsion free by Lemma 8.9 and 7.6.

**Definition 8.19.** We define $\theta_{n,1}$ in $H^{4n+4}(SP^2)$ and $\theta_{n,2}$ in $H^{4n+8}(SP^2)$ by the unique preimages

$$(\pi^*)^{-1}(f_{n,1}) = : \theta_{n,1} \text{ and } (\pi^*)^{-1}(f_{n,2}) = : \theta_{n,2}.$$  

**Example 8.20.** Case $n = 2$.

From Proposition 7.20, $H^{4\ast}((B_2)_+) \cong H^{4\ast}(B_+)/(f_{2,1}, f_{2,2}, g^3)$, where $f_{2,1} = x^3 - 3xy$ and $f_{2,2} = x^2y - 2y^2$. Also, $f_{2,1} = \pi^*(g^3 - 3 \cdot \frac{1}{2} gh)$ and $f_{2,2} = \pi^*(\frac{1}{2} g^2 h - \frac{1}{2} h^2)$ by Proposition 7.20, so $\theta_{2,1} := g^3 - 3 \cdot \frac{1}{2} gh$ and $\theta_{2,2} := \frac{1}{2} g^2 h - \frac{1}{2} h^2$.

Combining Theorem 4.35 and Definition 8.19 yields the following.

**Remark 8.21.** The polynomials $\theta_{n,1}$ and $\theta_{n,2}$ are given by

$$
\begin{align*}
\theta_{1,1} &= g^2 - h \\
\theta_{1,2} &= \frac{1}{2} gh \\
\end{align*}
$$

for $n = 1$, and

$$
\begin{align*}
\theta_{n,1} &= g \theta_{n-1,1} - \theta_{n-1,2} \\
2 \theta_{n,2} &= h \theta_{n-1,1} \\
\end{align*}
$$

for $n \geq 2$.

For further low dimensional examples, see Appendix C.2.
Remark 8.22. For some fixed \( n \geq 1 \), \( \theta_{m,2m \geq n} = 0 \) implies that \((\frac{1}{2})^{k-1}h^k = 0\) for all \( k \geq n+1 \). Because \( \pi^*((\frac{1}{2})^{k-1}h^k) = 2y^k \) by Proposition 7.20, and for any \( k \geq 2 \), there is an expression \( 2y^k = \sum_m \mu_m f_{m,2} \), where \( \mu_m \in \mathbb{Z}(x^i y^j), \ j \geq 0, \ i \geq 1, \) and \( k - 1 \leq m \leq 2(k - 1) \) by Lemma 4.43. Therefore, if \( \theta_{m,2m \geq n} = 0 \) then \((\frac{1}{2})^{k-1}h^k = \sum_m \mu'_m \theta_{m,2} = 0\) where \( \mu'_m \in \mathbb{Z}((\frac{1}{2})^i g^j h^j), \ j \geq 0, \ i \geq 1\).

Lemma 8.23. Suppose \( \kappa \) is any element of \( \ker \phi^* \subset H^4^*(SP^2_\ell) \). Then the image \( \pi^*(\kappa) \) can be expressed as

\[
\pi^*(\kappa) = \lambda_1 f_{n,1} + \lambda_2 f_{n,2} + \lambda_3 y^{n+1}
\]

where \( \lambda_3 \in \mathbb{Z}(2y^n, x^m y^\ell) \), and \( \lambda_1, \lambda_2 \in \mathbb{Z}(y^n, x^m y^\ell), \ s, m \geq 1, \ell \geq 0 \).

Proof. Recall \( H^*(B_n)_+ \cong H^*(B_+)/(f_{n,1}, f_{n,2}, y^{n+1}) \) from Theorem 4.35. So any element \( \kappa' \in \ker b^* \) where \( b : B_n \hookrightarrow B \) can be written as \( \kappa' = \lambda_1' f_{n,1} + \lambda_2' f_{n,2} + \lambda_3' y^{n+1} \) where \( \lambda_1', \lambda_2', \lambda_3' \in H^*(B_+) \). Also, \( \ker \phi^* \subset \ker b^* \) from the diagram (8.18), so an expression of the form (8.24) is valid.

If \( \lambda_1, \lambda_2 \in \mathbb{Z}/2\langle w^i y^j \rangle, \ i \geq 1, \ j \geq 0 \), then \( \lambda_1 f_{n,1} = \lambda_2 f_{n,2} = 0 \) because \( wx = 0 \) and \( 2w = 0 \). We may therefore assume that no monomials of \( \lambda_1 \) and \( \lambda_2 \) involve \( w \). So the same is true for \( \lambda_3 \), because the image of \( \pi^* \) does not involve \( w \) either. In fact, \( \lambda_3 \) lies in \( \mathbb{Z}(2y^n, x^m y^\ell) \), because the coefficient of \( y^k \) in \( f_{n,1}, f_{n,2} \) is always 2 or 0. \( \square \)

Example 8.25. \( n = 2 \).

\[
\pi^*(\frac{1}{2}g^3h) = x^3 y = 3x f_{2,2} - 2y f_{2,1}
\]

\[
\pi^*(\frac{1}{2}g^2h^2) = x^2 y^2 = y f_{2,2} + 2y^3
\]

Lemma 8.26. There exist expressions

\[
y^i f_{n,1} = \sum_{m_j} \mu_{m,J} f_{m,2}
\]

where \( \mu_m = \pm xy^k, \ 0 \leq k \leq i - 2, \ m = n + 2, n + 4, \cdots, n' - 1, \) and \( n' = n + 2i - 1 \), and

\[
y^j f_{n,2} = \sum_{m_j} \mu_{m,J} f_{m',2}
\]

where \( \mu_m = \pm xy^k, \ 0 \leq k \leq j - 1, \ m = n + 1, n + 3, \cdots, n'' - 1, \) and \( n'' = n + 2j \).
CHAPTER 8. THE COHOMOLOGY RING II

Proof. We use the relations $f_{n-1,2} = x f_{n-1,1} - f_{n,1}, f_{n,2} = y f_{n-1,1}$ from Theorem 4.35. For any $n \geq 1$ and $i \geq 1$ we then have that

$$y^i f_{n,1} = y^{i-1} f_{n+1,2} = xy^{i-1} f_{n+1,1} - y^{i-2} f_{n+3,2} = xy^{i-1} f_{n+1,1} - (xy^{i-2} f_{n+3,1} - y^{i-3} f_{n+5,2})$$

$$\vdots$$

$$= xy^{i-1} f_{n+1,1} - xy^{i-2} f_{n+3,1} + \cdots + (-1)^i x y f_{n-1,1} + (-1)^{i-1} f_{n',2}$$

$$= xy^{i-2} f_{n+2,2} - xy^{i-3} f_{n+4,2} + \cdots + (-1)^i x f_{n-1,2} + (-1)^{i-1} f_{n',2}$$

$$= \sum_m \mu_m f_{m,2} + (-1)^i f_{n',2}$$

where $\mu_m = \pm x y^k$, $0 \leq k \leq i - 2$, $m = n + 2$, $n + 4, \cdots, n' - 1$, and $n' = n + 2i - 1$.

Similarly, for any $n \geq 1$ and $j \geq 1$ we have that

$$y^j f_{n,2} = y^j (x f_{n,1} - f_{n+1,1}) = xy^j f_{n,1} - y^{j-1} f_{n+2,2} = xy^j f_{n,1} - y^{j-1} (x f_{n+1,1} - f_{n+3,1})$$

$$\vdots$$

$$= xy^j f_{n,1} - xy^{j-1} f_{n+1,2} + \cdots + (-1)^j x y f_{n-2,1} + (-1)^j f_{n',2}$$

$$= xy^{j-1} f_{n+1,2} - xy^{j-2} f_{n+3,2} + \cdots + (-1)^j x f_{n-2,2} + (-1)^j f_{n',2}$$

$$= \sum_m \mu_m f_{m,2} + (-1)^j f_{n',2}$$

where $\mu_m = \pm x y^k$, $0 \leq k \leq j - 1$, $m = n + 1$, $n + 3, \cdots, n'' - 1$, and $n'' = n + 2j$.

Using the relation $f_{n,2} = x f_{n,1} - f_{n+1,1}$ also yields the following.

Remark 8.27. For $k \geq 1$, $m \geq 1$, there is an expression

$$x^m y^k = x^{m-1} y^{k-1} f_{1,2} = \sum_{m_k} \mu_{m_k} f_{m_k,2}$$

where $m_k \geq k$, $\mu_{m_k} \in \mathbb{Z}[x]$, $k \leq i \leq m + 2k - 2$. 

\end{proof}
**Example 8.28.** For \( m \geq 1 \), we have that

\[
x^m y = x^{m-1} f_{1,2} \\
x^m y^2 = x^{m-1} y f_{1,2} \\
= x^{m-1} (xy f_{1,1} - y f_{2,1}) \\
= x^{m-1} (x f_{2,2} - f_{3,2}).
\]

We can now describe \( \ker \phi^* \) in dimension 4*.

**Proposition 8.29.** For any \( n \geq 1 \), there is an isomorphism

\[
H^{4*}((SP^2_n)^+) \cong H^{4*}(SP^2_+) / I
\]

where \( I = (\theta_{n,1}, \theta_{m,2;m \geq n}) \).

**Proof.** First we note that \((f_{n,1}, f_{m,2;m \geq n}) \subset \ker b^*\); by Lemma 4.45; it then follows that \( I \subseteq \ker \phi^*\), by the commutative diagram (8.18).

Any element \( \kappa \in \ker \phi^* \subset H^{4*}(SP^2_2) \) can be described by

\[
\pi^*(\kappa) = \lambda f_{n,1} + \lambda_2 f_{n,2} + \lambda_3 y^{n+1} \quad \text{by Lemma 8.23}
\]

\[
= \lambda' f_{n,1} + \sum \lambda'_m f_{m,2;m \geq n} \quad \text{by Lemmas 8.26, 4.43, and Remark 8.27}
\]

where \( \lambda', \lambda'_m \in \mathbb{Z}(2y^s, x^m y^f) \). So,

\[
\kappa = (\pi^*)^{-1} \left( \lambda' f_{n,1} + \sum \lambda'_m f_{m,2;m \geq n} \right)
\]

\[
= \gamma \theta_{n,1} + \sum \gamma_m \theta_{m,2;m \geq n}
\]

where \( \gamma, \gamma_m \in H^{4*}(SP^2_2) \). Thus \( \kappa \in I \) and \( \ker \phi^* \subseteq I \).

**Example 8.30.** Case \( n = 2 \).

\[
H^{4*}((SP^2_2)^+) \cong H^{4*}(SP^2_+) / (\theta_{1,1}, \theta_{m,2;m \geq 2}),
\]

and the elements \( g^4 - 3 \frac{1}{2} g^2 h, \frac{1}{2} g^3 h, \) and \( \frac{1}{4} g^2 h^2 \) in \( \ker \phi^* \), for example, can be expressed as

\[
g^4 - 3 \frac{1}{2} g^2 h = g \theta_{2,1}, \quad \frac{1}{2} g^3 h = 3g \theta_{2,2} - h \theta_{2,1}, \quad \frac{1}{4} g^2 h^2 = g^2 \theta_{2,2} - g \theta_{3,2}.
\]
8.1.3 $H^{4k+2}(SP^2(\mathbb{H}P^n))$

Lemma 8.31. For any $n \geq 1$, $H^{4k+2}(SP^2_n) = 0$.

Proof. The proof is similar to the infinite case (Proposition 7.21). We have the commutative ladder

$$
0 = H^{4k+2}(\Delta_n) \xrightarrow{\delta} H^{4k+2}(SP^2_n) \xrightarrow{\delta^*} H^{4k+2}(SP^2_n/\Delta_n) \xrightarrow{\delta^*} 0
$$

$$
\xrightarrow{\beta^*} H^{4k+2}(B_n/\widehat{\Delta}_n) \xrightarrow{\beta^*} H^{4k+2}(B_n) \xrightarrow{\beta^*} 0
$$

where $H^{4k+2}(B_n) = \mathbb{Z}/2(w^iy^j)$ for $0 \leq j \leq n$, $i \geq 1$, $i + 4j = 2k + 1$.

If $f^* \left( \sum_{i,j} \lambda_{i,j} w^i y^j \right) = \sum_{i,j} \lambda_{i,j} w^i (z^2 + w^2 z)^j = 0$, then $\lambda_{i,j} = 0$ by equating the coefficients of $w^{2k+1}, w^{2k-3}z, \ldots$, which implies that $\ker f^* = 0$. Therefore

$$
0 = \ker f^* \cong H^{4k+2}(B_n/\widehat{\Delta}_n) \cong H^{4k+2}(SP^2_n/\Delta_n) \cong H^{4k+2}(SP^2_n),
$$

as required. \qed

8.1.4 $H^{4k+1}(SP^2(\mathbb{H}P^n))$ and $H^{4k+3}(SP^2(\mathbb{H}P^n))$

We shall describe $H^*(SP^2_n)$ for $* = 4k + 1, 4k + 3$.

Lemma 8.32. For $* = 4k + 1, 4k + 3$,

$$
q^*: H^*(B/\Delta) \to H^*(B_n/\widehat{\Delta}_n)
$$

is an isomorphism if $k \leq n$, and a surjection if $k \geq n + 1$.

Proof. If $* \leq 4n + 3$, then by Remark 4.41, the cohomology ring $H^*(B)$ is isomorphic to $H^*(B_n; \mathbb{Z})$, and we know that $H^*(\Delta) \cong H^*(\widehat{\Delta}_n)$.

Then, for $* = 4k + 1, 4k + 3$, $k \leq n$, the commutative ladder

$$
0 = H^*(B_n) \xrightarrow{\delta} H^*(B_n/\widehat{\Delta}_n) \xrightarrow{\delta} H^{*-1}(\widehat{\Delta}_n) \xrightarrow{\delta} H^{*-1}(B_n) \xrightarrow{\delta} 0
$$

$$
\xrightarrow{\delta} H^*(B) \xrightarrow{\delta} H^*(B/\Delta) \xrightarrow{\delta} H^{*-1}(\Delta) \xrightarrow{\delta} H^{*-1}(B) \xrightarrow{\delta} 0
$$

shows that $q^*$ is an isomorphism.
If \( * = 4k + 1, 4k + 3, \ k \geq n + 1 \), then we have the ladder

\[
\begin{array}{ccccc}
H^*(B_n) & = & 0 & \hookrightarrow & H^*(B_n/\triangle_n) \hookrightarrow H^{*-1}(\triangle_n) \cong \mathbb{Z}[w, z]/(2w, z^{n+1}) \\
\downarrow q^* & & & & \downarrow d^* \\
H^*(B) & = & 0 & \hookrightarrow & H^*(B/\triangle) \hookrightarrow H^{*-1}(\triangle) \cong \mathbb{Z}[w, z]/(2w)
\end{array}
\]

where \( d^* \) and both homomorphisms \( \delta \) are surjective. Therefore, \( q^* \) is surjective.

**Remark 8.33.** Both of the above two proofs are actually special cases of the Five Lemma, as applied in (8.11). So we may continue as in (8.12), and replace \( B/\triangle \) with \( SP^2/\triangle \) and \( B_n/\triangle_n \) with \( SP^2_n/\triangle_n \), then use a version of the Five Lemma again.

**Proposition 8.34.** For \( * = 4k + 1, 4k + 3 \), the homomorphism

\[
\phi^*: H^*(SP^2) \to H^*(SP^2_n)
\]

is an isomorphism if \( k \leq n \), and surjective if \( k \geq n + 1 \).

**Proof.** We consider four cases.

Case 1. If \( * = 4k + 3 \) with \( k \leq n \), then we have

\[
\begin{array}{ccccc}
H^{4k+3}(\triangle_n) & = & 0 & \hookrightarrow & H^{4k+3}(SP^2_n) \cong H^{4k+3}(SP^2_n/\triangle_n) \hookrightarrow H^{4k+2}(\triangle_n) = 0 \\
\downarrow \phi^* & & & & \downarrow \cong \\
H^{4k+3}(\triangle) & = & 0 & \hookrightarrow & H^{4k+3}(SP^2) \cong H^{4k+3}(SP^2/\triangle) \hookrightarrow H^{4k+2}(\triangle) = 0
\end{array}
\]

by Lemma 8.32 and Remark 8.33. Thus \( \phi^* \) is an isomorphism.

Case 2. If \( * = 4k + 3, \ k \geq n + 1 \), then we have

\[
\begin{array}{ccccc}
0 & \hookrightarrow & H^{4k+3}(SP^2_n) \cong H^{4k+3}(SP^2_n/\triangle_n) & \hookrightarrow & 0 \\
\downarrow \phi^* & & & & \downarrow \cong \\
0 & \hookrightarrow & H^{4k+3}(SP^2) \cong H^{4k+3}(SP^2/\triangle) & \hookrightarrow & 0
\end{array}
\]

where \( \phi^*_1 \) is a surjection by Remark 8.33. Thus \( \phi^* \) is a surjection.

Case 3. If \( * = 4k + 1, \ k \leq n \), then we have

\[
\begin{array}{ccccc}
H^{4k+1}(\triangle_n) & = & 0 & \hookrightarrow & H^{4k+1}(SP^2_n) \hookrightarrow H^{4k+1}(SP^2_n/\triangle_n) \hookrightarrow H^{4k}(\triangle_n) \\
\downarrow \phi^* & & & & \downarrow \cong \\
H^{4k+1}(\triangle) & = & 0 & \hookrightarrow & H^{4k+1}(SP^2) \hookrightarrow H^{4k+1}(SP^2/\triangle) \hookrightarrow H^{4k}(\triangle)
\end{array}
\]

by Lemma 8.32 and Remark 8.33. Thus \( \phi^* \) is an isomorphism.
Case 4. If $\ast = 4k + 1$, $k \geq n + 1$, then we have

$$0 \leftarrow H^{4k+1}(\mathbb{P}^2_n) \overset{\Phi}{\leftarrow} H^{4k+1}(\mathbb{P}^2_n/\Delta_n) \overset{\Phi^*}{\leftarrow} H^{4k}(\Delta_n) = 0$$

where $\phi^*_1$ is surjective by Lemma 8.32 and Remark 8.33. Thus $\phi^*$ is a surjection.

**Example 8.35.** The groups $H^5(\mathbb{P}^2_n; \mathbb{Z}) \cong H^9(\mathbb{P}^2_n; \mathbb{Z}) = 0$, $\forall n \geq 1$, are two trivial cases (see Appendix C.1).

Proposition 8.34 and Definition 7.28 for $t_{i,j}$, and the fact $z^{n+1} = 0$ for $H^*(\mathbb{P}^n)$, yields the following.

**Corollary 8.36.** For $\ast = 4k + 1, 4k + 3$, $\ker \phi^* = \mathbb{Z}/2\langle t_{i,j} : i \geq n + 1 \rangle$.

We can now complete our description of $\ker \phi^*$ by combining Proposition 8.29 and Corollary 8.36 with the product structure for $H^*(\mathbb{P}^2)$ obtained in Section 7.2. the outcome is that we may write

$$H^*((\mathbb{P}^2_n)_+) \cong H^*(\mathbb{P}^2_n)/\langle \theta_{n,1}, \theta_{m,2m \geq n}, t_{i,j ; i \geq n+1} \rangle$$

for any $n \geq 1$, and therefore complete the proof of Theorem 1.8.

### 8.2 Relation to Nakaoka’s Work

In this section, we shall compare our computations with Nakaoka’s work [44, 47] on the cohomology of the $p$-fold cyclic products $CP_p(X)$. The results of ([47], Chapter II) were stated for a prime number $p$ and a connected finite simplicial complex $X$.

We shall apply them to the particular case $p = 2$ and $X = \mathbb{P}^n$, because there is a homeomorphism $CP_2(X) \cong SP^2(X)$. It is convenient to work with $H^*(\mathbb{P}^2_n/\Delta_n; \mathbb{Z}/2)$ rather than $H^*(\mathbb{P}^2_n, \Delta_n; \mathbb{Z}/2)$ for $\ast \geq 1$, but they are isomorphic [36, page 143].

Nakaoka introduced two types of elements, $\Phi^*_0(x_1 \times x_2)$ and $E_s(x)$ in [44, 47], which were defined via additive homomorphisms

$$\Phi^*_0 : H^*(X \times X; \mathbb{Z}/2) \to H^*(\mathbb{P}^2(X)/\Delta_n; \mathbb{Z}/2)$$

$$E_s : H^q(X; \mathbb{Z}/2) \to H^{q+s}(\mathbb{P}^2(X)/\Delta_n; \mathbb{Z}/2), \; s \geq 1$$
for $x, x_1, x_2 \in H^*(X; \mathbb{Z}/2)$, where $\Phi_0^*(x \times y) = \Phi_0^*(y \times x)$.

The multiplicative properties of these elements ([47], Theorem 11.8) are:

\[ \Phi_0^*(x_1 \times x_2) \Phi_0^*(y_1 \times y_2) = \Phi_0^*(x_1 y_1 \times x_2 y_2 + x_1 y_2 \times x_2 y_1) \quad (8.39) \]

\[ E_s(x) E_r(y) = 0 \quad \text{for} \quad s, r \geq 1 \quad (8.40) \]

\[ E_s(x) \Phi_0^*(x_1 \times x_2) = 0 \quad \text{for} \quad s \geq 1 \quad (8.41) \]

**Theorem 8.42.** ([47], Theorem 9.9) The homomorphism

\[ E_s: H^q(X; \mathbb{Z}/2) \rightarrow H^{q+s}(SP^2(X)/\triangle_n; \mathbb{Z}/2) \]

is injective for $1 \leq s \leq q$.

By applying Nakaoka’s theorems and lemmas to our case, we have the following reformulations of his results.

**Theorem 8.43.** ([47], Theorem 10.8)

For $z^k \in H^{4k}(\mathbb{H}P^n; \mathbb{Z}/2)$, $1 \leq k \leq n$,

\[ \Phi_0^*(z^k \times z^k) = E_{4k}(z^k) \]

holds.

**Theorem 8.44.** ([47], Theorem 11.2)

For any $m > 0$, a basis of $H^m(SP^2_n/\triangle_n; \mathbb{Z}/2)$ is given by

\[ E_r(z^d), \ 1 \leq r \leq 4d, \quad \Phi_0^*(z^p \times z^q), \ p \neq q \]

for $z^d \in H^{4d}(\mathbb{H}P^n; \mathbb{Z}/2)$, $r + 4d = m$ and $z^p, z^q \in H^*(\mathbb{H}P^n; \mathbb{Z}/2)$, $4(p + q) = m$.

**Lemma 8.45.** ([47], Lemma 11.3)

The sequence

\[ 0 \rightarrow H^{m-1}(\triangle_n; \mathbb{Z}/2) \rightarrow H^m(SP^2_n/\triangle_n; \mathbb{Z}/2) \xrightarrow{s^*} H^m(SP^2_n; \mathbb{Z}/2) \rightarrow 0 \]

is exact for $m > 1$.

**Theorem 8.46.** ([47], Theorem 11.4)

A basis of $H^m(SP^2_n; \mathbb{Z}/2)$ is given by

\[ s^* E_r(z^d), \ 2 \leq r \leq 4d, \quad \Phi_0^*(z^p \times z^q), \ p \neq q \]

for $z^d \in H^{4d}(\mathbb{H}P^n; \mathbb{Z}/2)$, $r + 4d = m$ and $z^p, z^q \in H^*(\mathbb{H}P^n; \mathbb{Z}/2)$, $4(p + q) = m$. 
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Note that $H^*(\mathbb{HP}^n \times \mathbb{HP}^n; \mathbb{Z}/2) \to H^*(SP_n^2; \mathbb{Z}/2)$ is the composition $i^* \circ \Phi_0$

$$H^*(\mathbb{HP}^n \times \mathbb{HP}^n; \mathbb{Z}/2) \xrightarrow{\Phi_0} H^*(SP_n^2/\triangle_n; \mathbb{Z}/2) \xrightarrow{i^*} H^*(SP_n^2; \mathbb{Z}/2)$$

where $i^*$ is the inclusion homomorphism ([47], Lemma 2.14).

We may now complete our cohomological calculations by checking that they agree with Nakaoka’s work. This means we must first use Theorem 6.26 (UCT) to translate our integral calculations into $H^*(SP_n^2; \mathbb{Z}/2)$. From (6.27), for any abelian group $G$ and any $m \geq 1$, there is an isomorphism

$$H^m(Y; G) \cong H^m(Y; \mathbb{Z}) \otimes G \bigoplus \text{Tor}(H^{m+1}(Y; \mathbb{Z}), G).$$

We take $G = \mathbb{Z}/2$ and $Y = SP_n^2$.

Case $* = 1, 2, 3, 5, 9 : H^*(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{*+1}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2) = 0$. (8.47)

Case $* = 4k + 3 \geq 7, 4k + 1 \geq 13 : H^*(SP_n^2; \mathbb{Z}/2) \cong H^*(SP_n^2; \mathbb{Z}) \otimes \mathbb{Z}/2$. (8.48)

Case $* = 4k + 2, k \geq 1 : H^{4k+2}(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{4k+3}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2)$. (8.49)

Case $* = 4k : H^{4k}(SP_n^2; \mathbb{Z}/2) \cong H^{4k}(SP_n^2; \mathbb{Z}) \otimes \mathbb{Z}/2 \oplus H^{4k+1}(SP_n^2; \mathbb{Z}/2)$. (8.50)

We shall explain (8.48) (8.49) (8.50), and introduce our notations for the mod 2 generators.

Let $\rho$ be the mod 2 reduction homomorphism, $H^*(SP_n^2; \mathbb{Z}) \xrightarrow{\rho} H^*(SP_n^2; \mathbb{Z}/2)$.

From Sections 8.1.2 and 8.1.3, we have that $H^{4k+4}(SP_n^2; \mathbb{Z})$ is torsion free and $H^{4k+2}(SP_n^2; \mathbb{Z}) = 0$, which yield the isomorphism (8.48). So, if $* = 4k + 1, 4k + 3$, then $\rho$ is an isomorphism on $H^*(SP_n^2; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 = \mathbb{Z}/2\langle t_{i,j} \rangle$ by Section 8.1.4. We write $b_{i,j}$ for $\rho(t_{i,j}), 1 \leq i, 1 \leq j < 2i$.

The facts that $H^{4k}(SP_n^2; \mathbb{Z})$ is torsion free and $H^{4k+1}(SP_n^2; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ as above, yield the isomorphism (8.50). We may write the generators corresponding to the first summand of (8.50) as $\rho\left((\frac{1}{2})^{s-1}h^s\right), \rho\left((\frac{1}{2})^m g^m h^m\right)$ where $s, \ell \geq 1, m \geq 0$.

Also, we have $H^{4k+1}(SP_n^2; \mathbb{Z})$ as a $\mathbb{Z}/2$ vector space on the elements $t_{i,j}$, so there exists an isomorphism $H^{4k+1}(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{4k+1}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2)$. We may now define a monomorphism

$$\sigma : H^{4k+1}(SP_n^2; \mathbb{Z}/2) \to H^{4k}(SP_n^2; \mathbb{Z}/2)$$

(8.51)
of \(\mathbb{Z}/2\) vector spaces, by composing with inclusion of the second summand \((8.50)\).
Then, we write the generators corresponding to the second summand of \((8.50)\) as
\[ b_{i,2j-1} \text{ for } \sigma(b_{i,2j}) \] where \(i, j \geq 1, 2i + j = 2k\).

The vanishing cohomology \(H^{4k+2}(SP^2_n; \mathbb{Z}) = 0\) from Lemma 8.31 yields \((8.49)\). We have \(H^{4k+3}(SP^2_n; \mathbb{Z}/2)\) as a \(\mathbb{Z}/2\) vector space on the \(b_{i,2j}\), so there exists an isomorphism
\[
\sigma': H^{4k+3}(SP^2_n; \mathbb{Z}/2) \cong \text{Tor}(H^{4k+3}(SP^2_n; \mathbb{Z}), \mathbb{Z}/2) \xrightarrow{\cong} H^{4k+2}(SP^2_n; \mathbb{Z}/2) \tag{8.52}
\]
and we write \(b_{i,2j-1}\) for the corresponding generators \(\sigma'(b_{i,2j})\) where \(i, j \geq 1, 2i + j = 2k + 1\).

Our generators and Nakaoka’s both involve the groups \(H^*(SP^2_n/\Delta_n)\), so it is very important to consider them at this point. From Section 8.1.4, if \(* = 4k+1, 4k+3, \) then \(s^*/H^*(SP^2_n/\Delta_n) \rightarrow H^*(SP^2_n)\) splits, or it is an isomorphism. Also, we have used the canonical isomorphism \(\pi^*|_{SP^2_2/\Delta_n} : H^*(SP^2_n/\Delta_n) \rightarrow H^*(B_n/\Delta_n)\) to describe elements in \(H^*(SP^2_n)\). In the case of \(\text{mod } 2\), the same construction gives new elements \((\pi^*)^{-1}\delta'(a^iz^i)\) in \(H^{4i+j+1}(SP^2_n/\Delta_n; \mathbb{Z}/2)\), where \(\rho(w) = a^2 \in H^2(\mathbb{R}P^\infty; \mathbb{Z}/2)\).

Amongst these elements we are especially interested in the ones that arise from \(\delta'(a^iz^i)\), which are the mod 2 reductions \(\rho(\delta(w^jz^j))\). They determine the integral basis elements \(t_{i,j} := u^*(\delta(w^jz^j))\) as in Proposition 8.34 and Definition 7.28.

The products \(\rho(t_{i,j})\rho(t_{k,l}) = 0, \rho(t_{i,j})\rho((1/2)^mg^\ell h^m) = 0, \rho(t_{i,j})\rho((1/2)^s h^s) = 0,\) and Nakaoka’s results \((8.40) (8.41)\), together with the fact \(z^i = 0, i > n, \) yield a bijection
\[
s^*E_{2j+1}(z^i) \leftrightarrow b_{i,2j} := \rho(t_{i,j}), \quad 1 \leq j < 2i, 1 \leq i \leq n, \tag{8.53}
\]
between Nakaoka’s basis elements and our own in \(H^{4i+2j+1}(SP^2_n; \mathbb{Z}/2)\), which is also consistent with Corollary 7.40.

Case \(j = 0, 1 \leq i \leq n, \) Nakaoka’s result \(s^*E_1(z^i) = 0\) is equivalent to (the mod 2 reduction of) our \(s^*(\delta(z^i)) = 0\) under the bijection \(E_1(z^i) \leftrightarrow \delta(z^i)\).

For \(4 \leq * \leq 7, \) we have \(H^*(SP^2_n; \mathbb{Z}/2) \cong \mathbb{Z}/2, n \geq 1, \) so it is clear that
\[
\rho(g) = \Phi_0^*(z \times 1), \quad b_{1,1} = s^*E_2(z), \quad b_{1,2} = s^*E_3(z)
\]
hold; then \(\rho(g) = \Phi_0^*(z \times 1)\) and \((8.39)\) yield
\[
\rho(g^i) = (\Phi_0^*(z \times 1))^i = \Phi_0^*(z^i \times 1) + \text{other terms} \tag{8.54}
\]
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where \( \Phi_i^*(z^i \times 1) \neq 0 \) if \( i \leq n \). Then we have a bijection

\[
\rho(g^i) \leftrightarrow \Phi_0^*(z^i \times 1)
\]

by choosing the term \( \Phi_0^*(z^i \times 1) \) as a representative.

We will show \( \rho(h) = s^*E_4(z) \). For \( n \geq 2 \), \( H^8(SP_n^2; \mathbb{Z}/2) \cong \mathbb{Z}/2(\rho(h)) \oplus \mathbb{Z}/2(\rho(g^2)) \), and \( \rho(g^2) = (\Phi_0^*(z \times 1))^2 = \Phi_0^*(z^2 \times 1) + s^*E_4(z) \) by (8.54) (8.39). This suggests that either \( \rho(h) = s^*E_4(z) \) or \( \rho(h) = \Phi_0^*(z^2 \times 1) \). If \( \rho(h) = \Phi_0^*(z^2 \times 1) \) then \( \rho(g)\rho(h) = \Phi_0^*(z^2 \times 1)\Phi_0^*(z \times 1) \neq 0 \), but \( \rho(g)\rho(h) = \rho(gh) = \rho(2 \cdot \frac{1}{2}gh) = 0 \), so we have a contradiction. Therefore \( \rho(h) = s^*E_4(z) \).

Combining \( \sigma, \sigma' : H^*(SP_n^2; \mathbb{Z}/2) \to H^*-1(SP_n^2; \mathbb{Z}/2), * = 4k + 1, 4k + 3 \) from (8.51)(8.52) and \( s^*E_{2j+1}(z^i) \leftrightarrow b_{i,2j} \) from (8.53) implies a bijection

\[
s^*E_{2j}(z^i) \leftrightarrow b_{i,2j-1},
\]

which agrees with Nakaoka’s (8.40), (8.41) and Corollary 7.40.

For \( n \geq 3 \), we have \( H^{12}(SP_n^2; \mathbb{Z}/2) \cong \mathbb{Z}/2(\rho(g^2)) \oplus \mathbb{Z}/2(\rho(\frac{1}{2}gh)) \oplus \mathbb{Z}/2(b_{2,3}) \) where \( \rho(g^3) \leftrightarrow \Phi_0^*(z^3 \times 1) \) and \( b_{2,3} \leftrightarrow s^*E_4(z^2) \). So we obtain a bijection by defining

\[
\rho \left( \frac{1}{2}gh \right) \leftrightarrow \Phi_0^*(z^2 \times z).
\]

Then, the products \( \rho \left( \left( \frac{1}{2} \right)^{s-1}h^s \right) \rho \left( \left( \frac{1}{2} \right)^{s'}^{-1}h^{s'} \right) = 0, \rho \left( \left( \frac{1}{2} \right)^m g^t h^m \right) \rho \left( \left( \frac{1}{2} \right)^{s-1}h^s \right) = 0 \) and (8.40)(8.41) suggest that \( \rho \left( \left( \frac{1}{2} \right)^{s-1}h^s \right) = \sum_{n,t} \lambda_{n,t} s^*E_n(z^t), \lambda_{n,t} \in \{0,1\} \); so we have a bijection

\[
\rho \left( \left( \frac{1}{2} \right)^{i-1}h^i \right) \leftrightarrow s^*E_{4i}(z^i)
\]

by choosing \( s^*E_{4i}(z^i) \) as a representative.

On the other hand, the product \( \rho \left( \left( \frac{1}{2} \right)^m g^t h^m \right) \rho \left( \left( \frac{1}{2} \right)^m' g'^t h'^m' \right) \neq 0 \) and Nakaoka’s results (8.39) (8.40) (8.41) suggest that \( \rho \left( \left( \frac{1}{2} \right)^m g^t h^m \right) = \Phi_0^*(z^{\ell+m} \times z^m) + \text{other terms} \); we now have a bijection

\[
\rho \left( \left( \frac{1}{2} \right)^m g^t h^m \right) \leftrightarrow \Phi_0^*(z^{\ell+m} \times z^m)
\]

by choosing \( \Phi_0^*(z^{\ell+m} \times z^m) \) as a representative.

It is possible to compute \( H^*(SP_n^2) \) from \( H^*(SP_N^2) \) with an arbitrarily large \( N \). In fact, truncation has been our own approach to compute \( H^*(SP_n^2) \) throughout this
thesis. In the $\mathbb{Z}/2$ cohomology case, it is clear that putting $z^{n+1} = 0$ into Nakaoka’s basis elements for $H^*(SP^2_n; \mathbb{Z}/2)$ yields $H^*(SP^2_n; \mathbb{Z}/2)$. We include Table 8.2 and 8.3, to compare our basis elements with Nakaoka’s for $n = 3$ and $n = N \geq 6$ in low dimensions.

In this section we have described a basis for $H^*(SP^2_n; \mathbb{Z}/2)$ and $H^*(SP^2_n/\triangle_n; \mathbb{Z}/2)$ using Theorem 6.26 (UCT) in order to check our results. On the other hand, in Chapter 6, we showed the isomorphism $H^*(SP^2_n/\triangle_n; \mathbb{Z}/2) \cong H^*(\text{Th}(\theta_n); \mathbb{Z}/2)$, which comes from $\text{Th}(\theta_n) \cong SP^2_n/N_n \simeq SP^2_n/\triangle_n$. It is interesting to compare Nakaoka’s basis elements and our basis elements for

$$H^*(\text{Th}(\theta_n); \mathbb{Z}/2) \cong H^*(\Gamma_n; \mathbb{Z}/2) t$$

where $t$ is the Thom class in $H^4(\text{Th}(\theta_n); \mathbb{Z}/2)$.

We compare them for $n = 2$ as in Table 8.1.

### 8.3 Finale

It is a general belief that the integral cohomology ring is harder to compute than $\mathbb{Z}/2$-cohomology ring. In Section 8.2, $\mathbb{Z}/2$-cohomology was used to check that our integral calculations were consistent with Nakaoka’s work. Nakaoka compared [47] his elements with the action of the Steenrod square operation $\text{Sq}^1$; we have started to work on the connections between $\text{Sq}^1$ and our generators, and think it should be possible to find formulae for their action. As we write this thesis, we are also considering the relationship between our cohomological computation and the cohomology of the infinite symmetric product $SP^\infty(X) = \bigcup_n SP^n(X)$. The most recent research related to the cohomology of the symmetric products $SP^n(X)$ includes a 2015 paper [21] by Dmitry Gugnin of Moscow State University in 2015. For $n \geq 2$, the structure of the integral cohomology ring $H^*(SP^n(X))$ remains an open problem except for a few basic cases.

Many of our methods have complex versions that apply to the case $SP^2(\mathbb{C}P^n)$, and will be discussed in forthcoming work [7]. Currently, we are also working on the integral cohomology rings of the symmetric squares of the octonionic projective line and plane; we aim to complete our computations by spring 2016.
Table 8.1: $H^* := H^*(SP^2_2/\triangle_2; \mathbb{Z}/2)$

<table>
<thead>
<tr>
<th>$*$</th>
<th>$H^*$</th>
<th>our basis</th>
<th>Nakaoka’s basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$\mathbb{Z}/2$</td>
<td>$t$</td>
<td>$\Phi_0^*(z \times 1)$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/2$</td>
<td>$at$</td>
<td>$E_1(z)$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}/2$</td>
<td>$a^2t$</td>
<td>$E_2(z)$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}/2$</td>
<td>$a^3t$</td>
<td>$E_3(z)$</td>
</tr>
<tr>
<td>8</td>
<td>$\mathbb{Z}/2$</td>
<td>$a^4t$</td>
<td>$E_4(z)$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1t$</td>
<td>$\Phi_0^*(z^2 \times 1)$</td>
</tr>
<tr>
<td>9</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1at$</td>
<td>$E_1(z^2)$</td>
</tr>
<tr>
<td>10</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1a^2t$</td>
<td>$E_2(z^2)$</td>
</tr>
<tr>
<td>11</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1a^3t$</td>
<td>$E_3(z^2)$</td>
</tr>
<tr>
<td>12</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1^2t$</td>
<td>$\Phi_0^*(z^2 \times z)$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1a^4t$</td>
<td>$E_4(z^2)$</td>
</tr>
<tr>
<td>13</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1a^4t(=p_1a^5t)$</td>
<td>$E_5(z^2)$</td>
</tr>
<tr>
<td>14</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1^2a^2t(=p_1a^5t)$</td>
<td>$E_6(z^2)$</td>
</tr>
<tr>
<td>15</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1^2a^3t(=p_1a^5t)$</td>
<td>$E_7(z^2)$</td>
</tr>
<tr>
<td>16</td>
<td>$\mathbb{Z}/2$</td>
<td>$p_1^2a^4t(=p_1a^5t)$</td>
<td>$E_8(z^2)$</td>
</tr>
</tbody>
</table>

$p_1^2 - p_2 = 0, \quad p_1^3 = 0$

$a^5 = p_1a, \quad t^2 = (a^4 + p_1)t.$
Table 8.2: $H^* := H^*(SP^2_3; \mathbb{Z}/2)$

<table>
<thead>
<tr>
<th>*</th>
<th>$H^*$</th>
<th>our generators</th>
<th>Nakaoka’s basis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$b_{i,j} := \rho(t_{i,j})$</td>
<td>$\Phi_0^<em>(x \times y) = \Phi_0^</em>(y \times x)$</td>
<td>$s^<em>E_4(z^2) = \Phi_0^</em>(z^2 \times z^2)$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(g)$</td>
<td>$\Phi_0^*(z \times 1)$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/2$</td>
<td>$0$</td>
<td>$-$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{1,1}$</td>
<td>$s^*E_2(z)$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{1,2}$</td>
<td>$s^*E_3(z)$</td>
</tr>
<tr>
<td>8</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(h)$</td>
<td>$s^*E_4(z)$</td>
</tr>
<tr>
<td>9</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(g^2)$</td>
<td>$\Phi_0^*(z^2 \times 1)$</td>
</tr>
<tr>
<td>10</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{2,1}$</td>
<td>$s^*E_2(z^2)$</td>
</tr>
<tr>
<td>11</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{2,2}$</td>
<td>$s^*E_3(z^2)$</td>
</tr>
<tr>
<td>12</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(g^3)$</td>
<td>$\Phi_0^*(z^3 \times 1)$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(\frac{1}{2}gh)$</td>
<td>$\Phi_0^*(z^2 \times z)$</td>
</tr>
<tr>
<td>13</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{2,3}$</td>
<td>$s^*E_4(z^2)$</td>
</tr>
<tr>
<td>14</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{2,4}$</td>
<td>$s^*E_5(z^2)$</td>
</tr>
<tr>
<td>15</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,1}$</td>
<td>$s^*E_2(z^3)$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{2,5}$</td>
<td>$s^*E_5(z^2)$</td>
</tr>
<tr>
<td>16</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,2}$</td>
<td>$s^*E_3(z^3)$</td>
</tr>
<tr>
<td>17</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{2,6}$</td>
<td>$s^*E_7(z^2)$</td>
</tr>
<tr>
<td>18</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(\frac{1}{2}h^2)$</td>
<td>$s^*E_8(z^2)$</td>
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<tr>
<td>19</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,3}$</td>
<td>$s^*E_3(z^3)$</td>
</tr>
<tr>
<td>20</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(\frac{1}{2}gh^2)$</td>
<td>$\Phi_0^*(z^3 \times z^2)$</td>
</tr>
<tr>
<td>21</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,7}$</td>
<td>$s^*E_5(z^3)$</td>
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<tr>
<td>22</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,8}$</td>
<td>$s^*E_9(z^3)$</td>
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<td>23</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,9}$</td>
<td>$s^*E_{10}(z^3)$</td>
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<tr>
<td>24</td>
<td>$\mathbb{Z}/2$</td>
<td>$b_{3,10}$</td>
<td>$s^*E_{11}(z^3)$</td>
</tr>
<tr>
<td>25</td>
<td>$\mathbb{Z}/2$</td>
<td>$\rho(\frac{1}{4}h^4)$</td>
<td>$s^*E_{12}(z^3)$</td>
</tr>
</tbody>
</table>

$\rho(g^4) = \rho(\frac{1}{2}h^2)$, $\rho(g^5) = 0$, $\rho(\frac{1}{2}g^3h) = \rho(\frac{1}{2}gh^2)$

$\rho(\frac{1}{2}g^4h) = \rho(\frac{1}{4}g^2h^2) = \rho(\frac{1}{4}h^3)$. 
Analogous to the integral cohomology (Section 8.1),

\[ H^*(SP^2_n; \mathbb{Z}/2) \cong H^*(SP^2; \mathbb{Z}/2), \quad \text{if } * < 4(n + 1). \]

<table>
<thead>
<tr>
<th>* ≤ 15</th>
<th>( H^<em>(SP^2_N) \cong H^</em>(SP^2_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 ( \mathbb{Z}/2 )</td>
<td>( \rho(g^1) ) ( \Phi_0^*(z^4 \times 1) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( \rho(\frac{1}{2}g^2h) ) ( \Phi_0^*(z^3 \times z) )</td>
</tr>
<tr>
<td>( \mathbb{Z}/2 )</td>
<td>( \rho(\frac{1}{2}h^2) ) ( s^*E_8(z^2) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( b_{3,3} ) ( s^*E_4(z^3) )</td>
</tr>
<tr>
<td>17 ( \mathbb{Z}/2 )</td>
<td>( b_{3,4} ) ( s^*E_5(z^3) )</td>
</tr>
<tr>
<td>18 ( (\mathbb{Z}/2)^2 )</td>
<td>( b_{3,5}, b_{1,1} ) ( s^*E_6(z^3), s^*E_2(z^4) )</td>
</tr>
<tr>
<td>19 ( (\mathbb{Z}/2)^2 )</td>
<td>( b_{3,6}, b_{1,2} ) ( s^*E_7(z^3), s^*E_3(z^4) )</td>
</tr>
<tr>
<td>20 ( \mathbb{Z}/2 )</td>
<td>( \rho(g^5) ) ( \Phi_0^*(z^5 \times 1) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( \rho(\frac{1}{2}g^3h) ) ( \Phi_0^*(z^4 \times z) )</td>
</tr>
<tr>
<td>( \mathbb{Z}/2 )</td>
<td>( \rho(\frac{1}{2}gh^2) ) ( \Phi_0^*(z^3 \times z^2) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( b_{4,3} ) ( s^*E_4(z^4) )</td>
</tr>
<tr>
<td>( \mathbb{Z}/2 )</td>
<td>( b_{3,7} ) ( s^*E_8(z^3) )</td>
</tr>
<tr>
<td>21 ( (\mathbb{Z}/2)^2 )</td>
<td>( b_{3,8}, b_{1,4} ) ( s^*E_6(z^3), s^*E_5(z^3) )</td>
</tr>
<tr>
<td>22 ( (\mathbb{Z}/2)^3 )</td>
<td>( b_{3,9}, b_{1,5}, b_{5,1} ) ( s^*E_{10}(z^3), s^*E_6(z^5), s^*E_2(z^5) )</td>
</tr>
<tr>
<td>23 ( (\mathbb{Z}/2)^3 )</td>
<td>( b_{3,10}, b_{1,6}, b_{5,2} ) ( s^*E_{11}(z^3), s^*E_7(z^3), s^*E_3(z^5) )</td>
</tr>
<tr>
<td>24 ( \mathbb{Z}/2 )</td>
<td>( \rho(g^6) ) ( \Phi_0^*(z^6 \times 1) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( \rho(\frac{1}{2}g^4h) ) ( \Phi_0^*(z^5 \times z) )</td>
</tr>
<tr>
<td>( \mathbb{Z}/2 )</td>
<td>( \rho(\frac{1}{2}g^2h^2) ) ( \Phi_0^*(z^4 \times z^2) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( \rho(\frac{1}{2}h^3) ) ( s^*E_{12}(z^3) )</td>
</tr>
<tr>
<td>( \mathbb{Z}/2 )</td>
<td>( b_{4,7} ) ( s^*E_8(z^4) )</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>( b_{5,3} ) ( s^*E_4(z^5) )</td>
</tr>
</tbody>
</table>
Bibliography


[33] L. Gaunce Lewis, JR. *When is the Natural map X → ΩΣX a Cofibration?*, Transactions of the American Mathematical Society, Volume 273, Number1, 1982.


Appendix A

Lemmas

A.1 Gluing Lemma

The following is known as the gluing lemma.

**Lemma A.1** ([9, 10, 37]). Assume given a commutative diagram of topological spaces

\[
\begin{array}{ccc}
X & \xrightarrow{h} & A \\
\beta & & \downarrow \gamma \\
X' & \xrightarrow{h'} & A' \\
\end{array}
\]

where \(h\) and \(h'\) are cofibrations. If \(\alpha, \beta, \gamma\) are homotopy equivalences, then so is their pushout

\[X \cup_A Y \to X' \cup_{A'} Y'.\]

A.2 Ladder Lemma

The following is called the ladder lemma.

**Lemma A.2** ([37], Lemma 2.1.10). Assume given a commutative diagram

\[
\begin{array}{cccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_i} & X_{i+1} & \xrightarrow{f_{i+1}} & \cdots \\
\downarrow{a_0} & & \downarrow{a_1} & & \cdots & \downarrow{a_i} & & \downarrow{a_{i+1}} \\
X'_0 & \xrightarrow{f'_0} & X'_1 & \xrightarrow{f'_1} & \cdots & \xrightarrow{f'_i} & X'_{i+1} & \xrightarrow{f'_{i+1}} & \cdots
\end{array}
\]

in which \(f_i\) and \(f'_i\) are cofibrations. If the maps \(a_i\) are homotopy equivalences, then so is their colimit

\[\text{colim}X_i \to \text{colim}X'_i.\]
Appendix B

Background Materials

B.1 NDR

**Definition B.1** ([14], page 114). Let $A$ be a subspace $A \subset X$. Then, we call the pair $(X, A)$ a neighbourhood deformation retract (NDR), if there exists a homotopy $\psi: X \times I \to X$, and a function $v: X \to I$ such that:

1. $A = v^{-1}(0)$
2. $\psi(x, 0) = x$ for $x \in X$
3. $\psi(a, t) = a$ for $(a, t) \in A \times I$
4. $\psi(x, 1) \in A$ for $1 > v(x)$.

B.2 Semidirect Products

We recall below some basics of semidirect products. The references include [25,51,56].

**Definition B.2** ([25]). A group $G$ is a semidirect product of a subgroup $N$ by a subgroup $H$ if the following conditions are satisfied:

- $G = NH$
- $N$ is a normal subgroup of $G$
- $H \cap N = \{1\}$. 
Remark B.3 ([25]). If $G$ is the semidirect product of $N$ by $H$, then

$$G/N = NH/N \cong H/(H \cap N) = H/\{1\} \cong H,$$

and we say that $G$ is an extension of $N$ by $H$. (Although in general when we say $G$ is an extension of $A$ by $B$, $B$ is not required to be a subgroup of $G$).

Example B.4. The direct product of groups $N$, $H$ is a semidirect product where both $H$ and $N$ are normal subgroups of $G$. The dihedral group $D(4)$ is a semidirect product, which is not a direct product.

**Notation:** A common notation for the semidirect product $N$ by $H$ is $N \rtimes H$ or $H \ltimes N$, however some authors use the symbol pointing in the opposite direction. To avoid ambiguity, it is better to state which subgroup is a normal subgroup when necessary.

**Proposition B.5 ([25]).** Let $G$ be a semidirect product of $N$ by $H$. For each element $h$ of $H$, the map $\theta_h: N \to N$ defined by $\theta_h(n) = hnh^{-1}$ is an automorphism of $N$. The map $\theta: H \to \text{Aut}(N)$ defined by $\theta(h) = \theta_h$ is a homomorphism.

An element of a semidirect product $N$ by $H$ can be uniquely written as an ordered pair $(n, h)$, $n \in N$, $h \in H$, or more simply $nh$. Then the group multiplication is given by either $(n_1, h_1)(n_2, h_2) = (n_1\theta_{h_1}(n_2), h_1h_2)$, or $n_1h_1n_2h_2 = n_1\theta_{h_1}(n_2)h_1h_2$.

The semidirect product $G$ can be put into a short exact sequence

$$1 \to N \to G \to H \to 1,$$

which is split [51, page 313] where maps are given by $n \mapsto (n, 1)$, $(n, h) \mapsto h$, and the splitting by $h \mapsto (1, h)$.

### B.3 Serre Spectral Sequence

There are many references for spectral sequences, which include [39], a very good reference where readers can find full details on the subject.

We list the main properties related to the cohomology Serre or Leray-Serre spectral sequence.
Theorem B.6. ([39, Theorem 5.2, page 135]) Let $R$ be a commutative ring with unit. Suppose $F \to X \xrightarrow{\pi} M$ is a fibration, where $M$ is path-connected and $F$ is connected. Then there is a first quadrant spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, converging to $H^*(X_+; R)$ as an algebra, with

$$E_2^{p,q} \cong H^p(M_+; \mathcal{H}^q(F_+; R)),$$

the cohomology of the space $M$ with local coefficients in the cohomology of the fibre of $\pi$. This spectral sequence is natural with respect to fibre-preserving maps of fibrations. Furthermore, the cup product on cohomology with local coefficients and the product $\cdot_2$ on $E_2^{*,*}$ are related by $u \cdot_2 v = (-1)^{p'q} uv$ when $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$.

Note that the differential $d_r$ is given by

$$d_r: E_r^{p,q} \to E_r^{p+r,q-r+1} \quad \text{and} \quad E_r^{p,q+1} = \ker d_r / \text{Im } d_r \quad \text{at } E_r^{p,q}$$

where

$$\ker d_r: E_r^{p,q} \to E_r^{p+r,q-r+1} \quad \text{and} \quad \text{Im } d_r: E_r^{p-r,q+r-1} \to E_r^{p,q}.$$

The spectral sequence becomes more manageable in some cases.

Proposition B.7. ([40, Lemma 2.15, page 111]) Given a fibration $F \to X \to M$. Suppose that the system of local coefficient rings $\mathcal{H}^*(F_+; R)$ is trivial, $M$ and $F$ are path-connected, and either $H^*(M_+; R)$ or $H^*(F_+; R)$ is a finitely generated free $R$-module in each dimension. Then the product $E_2^{p,0} \otimes E_2^{0,q} \to E_2^{p,q}$ induces an isomorphism

$$H^p(M_+; R) \otimes_R H^q(F_+; R) \cong E_2^{p,q}.$$

The above condition on $\mathcal{H}^*(F_+; R)$ means that the action of $\pi_1(M)$ is trivial. For example,

$$E_2^{p,0} \cong H^p(M_+; H^0(F_+; R)) \cong H^p(M_+; R),$$

$$E_2^{0,q} \cong H^0(M_+; H^q(F_+; R)) \cong H^q(F_+; R).$$
Appendix C

Examples

C.1 $H^5(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$ and $H^9(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$

The following is a simple check that $H^5(SP_n^2) = H^9(SP_n^2) = 0$ for any $n \geq 1$ using the commutative ladder

\[
\begin{array}{c}
\downarrow H^*(\triangle_n) \overset{i^*}{\leftarrow} H^*(SP_n^2) \overset{s^*}{\leftarrow} H^*(SP_n^2/\triangle_n) \overset{\delta}{\leftarrow} H^*\!-\!1(\triangle_n) \leftarrow \\
\downarrow \pi^* \quad \downarrow \pi^* \quad \downarrow \pi^* \quad \downarrow \pi^* \\
\downarrow H^*(\hat{\triangle}_n) \overset{f^*}{\leftarrow} H^*(B_n) \overset{\beta^*}{\leftarrow} H^*(B_n/\hat{\triangle}_n) \overset{\delta}{\leftarrow} H^*\!-\!1(\hat{\triangle}_n) \leftarrow \\
\end{array}
\]

where $H^*(-) := H^*(-; \mathbb{Z})$.

Case $*=9$. We have the ladder

\[
\begin{array}{c}
0 \leftarrow H^9(SP_n^2) \overset{s^*}{\leftarrow} H^9(SP_n^2/\triangle_n) \overset{\delta}{\leftarrow} H^8(\triangle_n) \overset{i^*}{\leftarrow} H^8(SP_n^2) \leftarrow \\
\downarrow \pi^* \quad \downarrow \pi^* \quad \downarrow \pi^* \quad \downarrow \pi^* \\
0 = H^9(B_n) \overset{\beta^*}{\leftarrow} H^9(B_n/\hat{\triangle}_n) \overset{\delta}{\leftarrow} H^8(\hat{\triangle}_n) \overset{f^*}{\leftarrow} H^8(B_n) \leftarrow \\
\end{array}
\]

If $n \geq 2$, then $H^9(SP_n^2) \cong H^9(SP^2) = 0$, since

\[
H^8(SP_n^2) = \mathbb{Z}(g^2) \oplus \mathbb{Z}(h), \quad H^8(B_n) = \mathbb{Z}(x^2) \oplus \mathbb{Z}(y) \oplus \mathbb{Z}/2 \langle w^4 \rangle
\]

\[
H^8(\triangle_n) = \mathbb{Z}(z^2), \quad H^8(\hat{\triangle}_n) = \mathbb{Z}/2 \langle w^4 \rangle \oplus \mathbb{Z}/2 \langle w^2 z \rangle
\]

and the homomorphisms are given by

\[
\begin{align*}
f^*(w^4) &= w^4, & f^*(x^2) &= (2z)^2, & f^*(y) &= z^2 + w^2 z \\
i^*(g^2) &= (2z)^2, & i^*(h) &= 2z^2, & (\delta \circ f^*)(y) &= \delta(z^2 + w^2 z) = 0
\end{align*}
\]
so \( H^g(B_n/\widetilde{\Delta}_n) = \mathbb{Z}/2\langle \delta(z^2) \rangle \) where \( \delta(z^2) = \delta(w^2z) \). All the above are the same as for \( H^g(SP^2) \).

Case \( n = 1 \). Again, \( H^g(SP^2_1) = 0 \), which follows from
\[
\begin{array}{c}
0 \rightarrow H^g(SP^2_1) \xrightarrow{\pi^*} H^g(SP^2_1/\Delta_1) \xrightarrow{\cong} 0 \\
\| \rightarrow 0 = H^g(B_1) \xrightarrow{\beta^*} H^g(B_1/\widetilde{\Delta}_1) \xrightarrow{\cong} 0
\end{array}
\]

Case \( * = 5 \). This is similar to \( H^g(SP^2_n) \), and \( H^5(SP^2) \cong H^5(SP^2_n) \)
\[
\begin{array}{c}
0 \rightarrow H^5(SP^2_n) \xrightarrow{\pi^*} H^5(SP^2_n/\Delta_n) \xrightarrow{\delta} H^4(\Delta_n) \xrightarrow{i^*} H^4(SP^2_n) \rightarrow 0 \\
\| \rightarrow 0 = H^5(B_n) \xrightarrow{\beta^*} H^5(B_n/\widetilde{\Delta}_n) \xrightarrow{\delta} H^4(\Delta_n) \xrightarrow{f^*} H^4(B_n) \rightarrow 0
\end{array}
\]

where \( H^4(SP^2_n) = \mathbb{Z}\langle g \rangle \), \( H^4(B_n) = \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}/2\langle w^2 \rangle \), \( H^4(\Delta_n) = \mathbb{Z}\langle z \rangle \) and \( H^4(\widetilde{\Delta}_n) = \mathbb{Z}/2\langle w^2 \rangle \oplus \mathbb{Z}\langle z \rangle \). The homomorphisms are given by \( f^*(x) = 2z \), \( f^*(w^2) = w^2 \), \( i^*(g) = 2z \), \( (\delta \circ f^*)(x) = \delta(2z) = 0 \). Thus \( H^5(B_n/\widetilde{\Delta}_n) = \mathbb{Z}/2\langle \delta(z) \rangle \), and \( (s^* \circ \delta)(z) = 0 \). Therefore, \( H^5(SP^2_n) = 0 \), which confirms Nakaoka (8.2).

### C.2 \( \theta_{m,1} \) and \( \theta_{m,2} \)

For the reader’s convenience and future use, we describe the polynomials \( \theta_{m,1} \) and \( \theta_{m,2} \) for \( 1 \leq m \leq 4 \).

For \( m = 1 \), we have
\[
\theta_{1,1} = g^2 - h, \quad \theta_{1,2} = \frac{1}{2} gh,
\]
and for any \( m \geq 2 \) we have expressions
\[
\theta_{m,1} = g \theta_{m-1,1} - \theta_{m-1,2}, \quad 2 \theta_{m,2} = h \theta_{m-1,1} ;
\]
for example,
\[
\begin{align*}
\theta_{2,1} &= g^3 - 3 \cdot \frac{1}{2} gh \\
\theta_{2,2} &= \frac{1}{2} g^2 h - \frac{1}{2} h^2 \\
\theta_{3,1} &= g^4 - 4 \cdot \frac{1}{2} g^2 h + \frac{1}{2} h^2 \\
\theta_{3,2} &= \frac{1}{2} g^3 h - 3 \cdot \frac{1}{2} gh^2 \\
\theta_{4,1} &= g^5 - 5 \cdot \frac{1}{2} g^3 h + 5 \cdot \frac{1}{4} gh^2 \\
\theta_{4,2} &= \frac{1}{2} g^4 h - 4 \cdot \frac{1}{4} g^2 h^2 + \frac{1}{2} h^3.
\end{align*}
\]