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THE MAXIMAL 2-LOCAL GEOMETRY FOR J_4, II

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INTRODUCTION

This paper is the second in a three part series which is devoted to a study of the maximal 2-local geometry for the sporadic simple group J_4. We continue the section numbering of [RW1], and direct the reader to Section 1 for notation and statements of the main results of this enterprise. The present state of play is that we know a great deal about $\Delta_1(a) \cup \Delta_2(a)$ (a being a fixed point of the point-line collinearity graph $\mathcal{G}$). So, for example, it has been shown that $\Delta_2(a)$ is the union of three G_a-orbits $\Delta_2^1(a)$, $\Delta_2^2(a)$ and $\Delta_2^3(a)$. Of these three G_a-orbits, our information about $\Delta_2^1(a)$ is the most complete - indeed in Lemma 6.4 we are able to finish our analysis of $\Delta_2^1(a)$. Section 7 sees us refining our earlier knowledge gained in Theorems 4.7 and 4.8, about G_{ax} for $x \in \Delta_2^2(a) \cup \Delta_2^3(a)$. In Theorems 7.1 and 7.2 we show that $G_{ax}^{*x} \cong 2^6 : (S_3 \times S_4)$ (when $x \in \Delta_2^2(a)$) and $G_{ax}^{*x} \cong 2^6 : S_5$ (when $x \in \Delta_2^3(a)$).Building on this information, in the balance of Section 7, we determine the G_{ax}-orbits upon $\Gamma_1(x)$ for $x \in \Delta_2^2(a) \cup \Delta_2^3(a)$. Section 8 is concerned with the point distribution of lines in $\Gamma_1(x)$ for $x \in \Delta_2^2(a)$.

The investigation of $\Delta_3(a)$, the third and (as we learn) final disc of a, occupies [RW2]. However we do dip our toe into $\Delta_3(a)$- in Section 6 we introduce and examine the set $\Delta_3^1(a)$, showing that it is a G_a-orbit contained in $\Delta_3(a)$.

We end this paper with two appendices which contain information needed for some of our calculations.

6. THE G_a-ORBIT $\Delta_3^1(a)$

As mentioned in the introduction, we now begin delving into the third disc of $\mathcal{G}$. Ultimately we shall show that $\Delta_3(a)$ is the union of two G_a-orbits, one of which is $\Delta_3^1(a)$, defined below. In this section we prove some elementary properties of $\Delta_3^1(a)$, and complete the proof of Theorem C. We shall return, in Section 10 of [RW2], to $\Delta_3^1(a)$ and subject it to an in-depth investigation.

Definition 6.1

$$\Delta_3^1(a) := \{d \in \Gamma_0 | \text{ there exists } c \in \Delta_2^1(a) \cap \Delta_1(d) \text{ such that } c + d \in \beta_1(c, X(c,a))\}$$

Lemma 6.2 $\Delta_3^1(a) \subseteq \Delta_3(a)$.

Proof Let $d \in \Delta_3^1(a)$, and let $c \in \Delta_2^1(a) \cap \Delta_1(d)$ be such that $c + d \in \beta_1(c, X(c,a))$. From Theorem 4.3(ii) $d \notin \Delta_1(a)$ and, by Theorem 5.2, $d \notin \Delta_2^3(a)$, also. Now suppose $d \in \Delta_2^1(a)$. Then Lemma 5.1 implies that $\{a,c\}^\perp \cap \{a,d\}^\perp \neq \emptyset$ whence, by Lemma 5.4, $c + d \notin \beta_1(c, X(c,a))$, a contradiction. Thus $d \notin \Delta_2^1(a)$. Furthermore $d \notin \Delta_2^2(a)$ because of Theorem 5.8(i). So, in view of Lemma 4.2, we conclude that $d \notin \Delta_1(a) \cup \Delta_2(a)$. Hence $d \in \Delta_3(a)$, so giving the lemma.

Lemma 6.3 $\Delta_3^1(a)$ is a G_a-orbit.

Proof Let $d \in \Delta_3^1(a)$. Then there exists $c \in \Delta_2^1(a) \cap \Delta_1(d)$ such that $l = c + d \in \beta_1(c, X(c,a))$, by definition of $\Delta_3^1(a)$. As a result of (2.5)(i) and Theorem 4.3(v), we only need prove that G_{ac} is transitive on $\Gamma_0(l)\backslash\{c\}$. Put $X = X(c,a)$. Again by (2.5)(i) and Theorem 4.3(v) we may assume, without loss of generality, that in Ω_c, X is the standard sextet and l is $\begin{array}{|cc|cc|cc|}\hline \circ & + & + & + & + & + \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ + & \circ & - & - & - & - \\ \hline\end{array}$.

There exists a trio lying in $\beta_2(c,X)\cap\alpha_2(c,l)$, namely $k_1 = \begin{array}{|cc|cc|cc|}\hline + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline\end{array}$. Let Y be the unique sextet in $\Gamma_2(c,l,k_1)$ and choose $k_2 \in \Gamma_1(c,Y)$ to be $\begin{array}{|cc|cc|cc|}\hline \circ & + & + & + & + & + \\ + & \circ & - & - & - & - \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ \hline\end{array}$. By inspection we have

(6.3.1) $k_1 \in \alpha_3(c,k_2)$;

(6.3.2) $k_1,k_2 \in \alpha_2(c,l)$.

In Ω_c, let Z be the sextet $\begin{array}{|cc|cc|cc|}\hline \square & \cdot & \ast & \cdot & \ast & \square \\ \square & \circ & \ast & \circ & \times & + \\ + & \circ & \times & \cdot & \times & \square \\ + & \cdot & \times & \circ & \ast & + \\ \hline\end{array}$. Then we note that

(6.3.3) (i) $l \in \beta_1(c,X)\cap\beta_1(c,Z)$; and
(ii) $k_1 \in \beta_2(c,X), k_2 \in \beta_3(c,Z)$.

By Lemma 3.7(ii) and (6.3.1) $k_1 \in \delta_1(Y,c,k_2)$ and as a consequence k_1 and k_2 determine a different pairing of the points in $\Gamma_0(l)\backslash\{c\}$ by (2.13)(iv). For $i = 1,2$ we fix $d_i \in \Gamma_0(k_i)\backslash\{c\}$. Now we can label the points in $\Gamma_0(l)\backslash\{c\}$ a_1,a_2,a_3,a_4 in such a way that $\{a_1,a_2\} = \Delta_1(d_1)\cap(\Gamma_0(l)\backslash\{c\})$ and $\{a_1,a_3\} = \Delta_1(d_2)\cap(\Gamma_0(l)\backslash\{c\})$. From Lemma 3.3 and (6.3.3)(i) $\tau(X)$ and $\tau(Z)$ act regularly upon $\Gamma_0(l)\backslash\{c\}$. Also $\tau(X)$ fixes d_1 and $\tau(Z)$ fixes d_2 by (6.3.3)(ii) and Lemma 3.3. Therefore $\tau(X)$ must interchange a_1 with a_2 and a_3 with a_4 while $\tau(Z)$ interchanges a_1 with a_3 and a_2 with a_4. Hence $<\tau(X),\tau(Z)>$ is transitive on $\Gamma_0(l)\backslash\{c\}$. Since $Z \in \gamma_3(c,X)$, $\tau(Z) \in Q(X)$ by Lemma 3.4. Thus $<\tau(X),\tau(Z)>\leq G_a$ and so the lemma is proved.

Lemma 6.4 Let $c \in \Delta_2^1(a)$, $l \in \Gamma_1(c)$ with $l \in \beta_i(c,X(c,a))$. Then the following table describes the point distribution of $\Gamma_0(l)$.

i	point distribution
0	$3\Delta_1 2\Delta_2^1$
1	$1\Delta_2^1 4\Delta_3^1$
2	$3\Delta_2^1 2\Delta_2^2$
3	$1\Delta_2^1 4\Delta_2^2$

Proof If $l \in \beta_0(c, X(c,a))$, then $l \in \Gamma_1(X(c,a))$. Applying (2.16) to $\Lambda = \Gamma_{X(c,a)}$ yields that $|\Gamma_0(l) \cap \Delta_1(a)| = 3$ with the remaining two points of $\Gamma_0(l)$ lying in $\Delta_2(a) \cap \Gamma_0(X(c,a))$. Hence, these two points lie in $\Delta_2^1(a)$, by the definition of $\Delta_2^1(a)$. Theorem 5.8(ii), (iii) gives the point distribution for the cases $i = 2, 3$ and finally $i = 1$ follows from the definition of $\Delta_3^1(a)$.

Combining Lemma 6.4, Theorem 4.3(v) and (2.5)(i) we see that we have verified Theorem C.

Lemma 6.5 If $d \in \Delta_3^1(a)$, then $a \in \Delta_3^1(d)$.

Proof Let $c \in \Delta_2^1(a) \cap \Delta_1(d)$ be such that $c + d \in \beta_1(c, X)$ where $X = X(c,a)$. By Theorem 4.3(v) and (2.5)(i), without loss of generality, we may assume that X is the standard sextet in Ω_c and $c + d$ is $\begin{array}{|cc|cc|cc|} \hline \circ & + & + & + & + & + \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ + & \circ & - & - & - & - \\ \hline \end{array}$. Let $l \in \Gamma_1(c)$ be the trio $\begin{array}{|cc|cc|cc|} \hline + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline \end{array}$. Then $l \in \beta_2(c, X) \cap \alpha_2(c, c+d)$, whence Lemma 6.4 implies that $|\Gamma_0(l) \cap \Delta_2^2(a)| = 2$ and $|\Gamma_0(l) \cap \Delta_2^1(a)| = 3$. By Lemma 3.8(i) d is collinear with exactly two points of $\Gamma_0(l) \backslash \{c\}$. Since $\Delta_3^1(a)$ is a G_a-orbit by Lemma 6.3 and $\Gamma_0(c+d) \backslash \{c\} \subseteq \Delta_3^1(a)$ from Lemma 6.4, we know that d is collinear with a point, c' say, in $\Gamma_0(l) \cap \Delta_2^2(a)$. When viewed in Ω_c, $l = c' + c$ is the unique trio incident with every sextet in $\mathcal{S}(c', a)$ by Lemma 5.6 and thus $l \in \alpha_2(c', c'+b)$ for all $b \in \{c', a\}^{\perp}$ by Theorem 4.7. Since $c' + d \in \alpha_2(c', l)$, in $\Gamma_{c'}$ the trios $c' + d$ and l have a common octad O. Using Theorem 4.7 again there exists $b \in \{c', a\}^{\perp}$ such that the trio $c' + b$ contains O. Therefore $d \in \Delta_2^1(b)$ and the result follows because $d(a,d) = 3$ and $\Delta_1(b) \cap \Delta_3(d) \subseteq \Delta_3^1(d)$ by Lemma 6.2.

7. STABILIZERS AND LINE ORBITS FOR POINTS IN $\Delta_2^2(a)$ AND $\Delta_2^3(a)$

In this section we investigate the G_{ac}-orbits of lines in $\Gamma_1(c)$, where c is first a point of $\Delta_2^2(a)$ and then a point of $\Delta_2^3(a)$. Before commencing this study we need to determine the structure of G_{ac} in both cases as well as showing that $\Delta_2^2(a)$ and $\Delta_2^3(a)$ are G_a-orbits.

Starting with $c \in \Delta_2^2(a)$, by Theorem 4.7 there exists a unique sextet line $\mathcal{S}(c,a)$ in Γ_c fixed by G_{ac}. We let l_c denote the unique line in $\Gamma_1(c)$ incident with each sextet in $\mathcal{S}(c,a)$.

Theorem 7.1 Let $c \in \Delta_2^2(a)$. Then,

(i) $\Delta_2^2(a)$ is a G_a-orbit;

(ii) $|\Delta_2^2(a)| = 2^7.3.5.7.11.23$;

(iii) if g is a 3-element of G_{ac} which acts non-trivially on the octads of l_c, then g acts non-trivially on $\Gamma_0(l_c) \cap \Delta_2^1(a)$;

(iv) if $\mathcal{S}(c,a) = \{X_c, Y_c, Z_c\}$, then $\tau(X_c)\tau(Y_c) = \tau(Z_c)$; and

(v) G_{ac}^{*c} is the stabilizer of the sextet line $\mathcal{S}(c,a)$ and the trio l_c incident with each sextet of $\mathcal{S}(c,a)$ in Ω_c. Moreover $G_{ac}^{*c} \cong 2^6 : (S_3 \times S_4)$ with $Q(c)_a \cong 2^4$.

Proof Combining Theorem 3.6(ii) and (2.3)(i) it suffices to prove the following result to show part (i) is true.

(7.1.1) For $b \in \{a,c\}^{\perp}$, $Q(b)_{ac} \cong 2^7$.

Since $Q(b) \cong 2^9$ by Theorem 3.6(ii) we need to show that $Q(b)_{ac}$ is transitive on $\Gamma_0(b+c)\backslash\{b\}$. The strategy we employ is the same as that used in Lemma 6.3. Without loss of generality we may take, in Γ_b, $b+c$ to be the standard trio and $b+a = \left|\begin{array}{cc|cc|cc} - & + & + & + & + & + \\ + & - & - & - & - & - \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \end{array}\right|$.

Consider the following lines and planes in Γ_b:-

$$k_1 = \left|\begin{array}{cc|cc|cc} + & + & - & \circ & - & \circ \\ + & + & - & \circ & - & \circ \\ + & + & - & \circ & - & \circ \\ + & + & - & \circ & - & \circ \end{array}\right| \qquad k_2 = \left|\begin{array}{cc|cc|cc} - & \circ & + & + & - & \circ \\ - & \circ & + & + & - & \circ \\ - & \circ & + & + & - & \circ \\ - & \circ & + & + & - & \circ \end{array}\right|$$

$$Y_1 = \left|\begin{array}{cc|cc|cc} \times & \square & + & + & \cdot & \cdot \\ \ast & \circ & + & \cdot & \cdot & + \\ \ast & \times & \square & \ast & \square & \ast \\ \square & \circ & \times & \circ & \times & \circ \end{array}\right| \qquad Y_2 = \left|\begin{array}{cc|cc|cc} + & + & \times & \square & \cdot & \cdot \\ + & \cdot & \ast & \circ & \cdot & + \\ \square & \ast & \ast & \times & \square & \ast \\ \times & \circ & \square & \circ & \times & \circ \end{array}\right|.$$

Since $k_1, k_2, b+c$ are incident with the standard sextet and $k_1, k_2 \in \alpha_2(b, b+c)$, $k_1 \in \alpha_3(b, k_2)$, appealing to (2.13)(iv) gives that k_1 and k_2 determine a different pairing of the points in $\Gamma_0(b+c)\backslash\{b\}$. Moreover we observe that $b+c \in \beta_1(b,Y_1) \cap \beta_1(b,Y_2)$ and $b+a \in \beta_3(b,Y_1) \cap \beta_3(b,Y_2)$. Also $k_1 \in \beta_3(b,Y_1)$ and $k_2 \in \beta_3(b,Y_2)$. Using Lemma 3.3 gives $\tau(Y_1), \tau(Y_2) \in Q(b)_a$ with $\tau(Y_1)$ fixing $\Gamma_0(k_1)$ pointwise and $\tau(Y_2)$ fixing $\Gamma_0(k_2)$ pointwise. Hence, using Lemma 3.3 again we see that $< \tau(Y_1), \tau(Y_2) >$ acts transitively on $\Gamma_0(b+c)\backslash\{b\}$, which completes the proof of (7.1.1).

Part (ii) now follows from (i) together with the fact that $|\Delta_1(a)| = 2^2.3.5.11.23$, $|\{a,c\}^{\perp}| = 18$ and $|\alpha_1(b, b+a)| = 2^4.3^2.7$ for any $b \in \Delta_1(a)$ by (2.3)(ii) and Theorems 3.6(i) and 4.7(v).

For part (iii) it is enough to prove

(7.1.2) Let $d \in \Gamma_0(l_c) \cap \Delta_2^1(a)$. Then $|\{a,d\}^{\perp} \cap \{a,c\}^{\perp}| = 6$ and their exists an octad of the trio l_c in Ω_c which lies in the trio $c+b$ for all $b \in \{a,d\}^{\perp} \cap \{a,c\}^{\perp}$.

For each $b \in \{a,c\}^{\perp}$, $c+b \in \alpha_2(c, l_c)$ by Theorem 4.7. So b is incident with some point in $\Gamma_0(l_c) \cap \Delta_2^1(a)$ because $|\Gamma_0(l_c) \cap \Delta_2^1(a)| = 3$ by Theorem 5.8(ii). Let $b_1 \in \{a,d\}^{\perp} \cap \{a,c\}^{\perp}$. In Γ_{b_1} we may suppose that $b_1 + a$ is the standard trio, $b_1 + c = \left|\begin{array}{cc|cc|cc} - & + & + & + & + & + \\ + & - & - & - & - & - \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \end{array}\right|$ and $b_1 + d =$

$\begin{array}{|cc|cc|cc|}\hline + & + & - & - & - & - \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}$. If $c' \in \Delta_1(b_1) \cap \Gamma_0(l_c) \backslash \{c, d\}$, then $b_1 + c' = \begin{array}{|cc|cc|cc|}\hline - & + & - & - & - & - \\ + & - & + & + & + & + \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \\ \hline \end{array} \in$ $\alpha_1(b_1, b_1 + a)$. So $c' \in \Delta_2^2(a)$ and b is incident with a unique point in $\Gamma_0(l_c) \cap \Delta_2^1(a)$. Since G_{ac} is transitive on $\Gamma_0(l_c) \cap \Delta_2^1(a)$ and $|\{a, c\}^\perp| = 18$ we have that $|\{a, d\}^\perp \cap \{a, c\}^\perp| = 6$. Let $\{a, d\}^\perp \cap \{a, c\}^\perp = \{b_1, b_2, b_3, b_4, b_5, b_6\}$. Arguing as in Theorem 4.7 we have that $b_i \in \Delta_1(b_1)$ for $i = 2, \ldots, 6$ and we may suppose that $b_1 + b_2 = b_1 + d$, $b_1 + b_3 = b_1 + b_4 = \begin{array}{|cc|cc|cc|}\hline - & - & + & + & - & - \\ - & - & + & + & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}$ $(\in \alpha_2(b_1, b_1 + c))$ and $b_1 + b_5 = b_1 + b_6 = \begin{array}{|cc|cc|cc|}\hline + & + & + & + & - & - \\ + & + & + & + & - & - \\ - & - & \circ & \circ & \circ & \circ \\ - & - & \circ & \circ & \circ & \circ \\ \hline \end{array}$ $(\in \alpha_2(b_1, b_1 + c))$. So we have $c + b_i \in \alpha_2(c, c + b_j)$ for all $i \neq j \in \{1, \ldots, 6\}$. By examining the possibilities in the table following Theorem 4.7 we see that each trio $c + b_i (i = 1, \ldots, 6)$ contains a fixed octad of the trio l_c. This proves (7.1.2) and so part (iii).

Turning to part (iv), since $Z_1(c) = 1$ by Lemma 3.2(iv) we need to show that $\tau(X_c)\tau(Y_c)$ has the same action as $\tau(Z_c)$ on $\Gamma_0(l)$ for every $l \in \Gamma_1(c)$. By Theorem 3.6(iii) $\tau(X_c)\tau(Y_c) = \tau(X)$ for some $X \in \Gamma_2(l_c)$. In Γ_c we identify $\mathcal{S}(c, a)$ with the concrete description presented after Theorem 4.7. So l_c is the standard trio. Suppose that $X = \begin{array}{|cc|cc|cc|}\hline + & + & * & * & \times & \times \\ \cdot & \cdot & \square & \square & \circ & \circ \\ + & + & * & * & \times & \times \\ \cdot & \cdot & \square & \square & \circ & \circ \\ \hline \end{array}$. Let $l = \begin{array}{|cc|cc|cc|}\hline + & \circ & + & \circ & \circ & \circ \\ + & \circ & + & \circ & - & - \\ + & - & + & - & - & \circ \\ + & - & + & - & \circ & - \\ \hline \end{array}$. Then $l \in \beta_2(c, X_c) \cap \beta_3(c, Y_c) \cap \beta_1(c, X)$. Now Lemma 3.3 yields a contradiction because $\tau(X_c)\tau(Y_c)$ fixes $\Gamma_0(l)$ pointwise but $\tau(X)$ does not. Similar considerations rule out the other three sextets in $\Gamma_2(l_c) \backslash \mathcal{S}(c, a)$. Hence $X = Z_c$ as required.

We now prove (v). Let b be some fixed point of $\{a, c\}^\perp$. From Lemma 4.5(ii) there are exactly three $X \in \Gamma_2(b, c)$ such that $\tau(X)$ fixes a. One of these three lies in $\mathcal{S}(c, a)$ and the other two are incident with exactly three $x \in \{a, c\}^\perp$ by Theorem 4.7. So we get 15 distinct $X \in \Gamma_2(c)$ incident with some $x \in \{a, c\}^\perp$ such that $\tau(X)$ fixes a. Hence $|Q(c)| \geq 2^4$. By Theorem 4.7 G_{ac} is contained in $Stab_{G_c}\mathcal{S}$. We recall that $L := (Stab_{G_c}\mathcal{S})^{*c} \cong 2^6 : (S_3 \times S_4)$. We have $G_{ac}^{*c} \leq G_{acl_c}^{*c}$. Set $M = O_2(G_{acl_c}^{*c}) \cong 2^6$; note that $M \leq L$ and $L/M \cong S_3 \times S_4$. From part (i) and Theorem 4.7(vi), $|G_{ac}| = 2^{14}.3^2$. By Theorem 5.8(ii), $|\Gamma_0(l_c) \cap \Delta_2^2(a)| = 2$ and $|\Gamma_0(l_c) \cap \Delta_2^1(a)| = 3$. Choose $d \in \Gamma_0(l_c) \cap \Delta_2^1(a)$. Put $X = X(d, a)$. From Theorem 4.3(v) $G_{ad}^{*d} \cong 2^6 3 S_6$ with $Q(d)_a \cong 2^7$; as a $GF(2)3S_6$-module $Q(d)_a$ has composition series $1 \backslash 6$ where $< \tau(X) >$ is the trivial submodule and $Q(d)_a / < \tau(X) >$ is a 6-dimensional irreducible which is dual to $O_2(G_{ad}^{*d})$ (see [Theorem

3;MS]). Viewing things from d, we have by Lemma 5.6 that $d+c \in \beta_2(d,X)$, a line orbit of size 180. Using Appendix 2 we deduce that $G^{*d}_{adl_c} \cong 2^4 : (S_4 \times 2)$. We claim that $Q(d)_{ac} \cong 2^6$. Suppose X is the standard sextet, $d+c = \begin{array}{|cc|cc|cc|} \hline + & + & - & - & - & - \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}$ and let $Y = \begin{array}{|cc|cc|cc|} \hline \cdot & \square & \ast & \ast & \cdot & \square \\ + & \times & \ast & \circ & + & \times \\ \cdot & \square & \circ & \circ & \cdot & \square \\ + & \times & \circ & \ast & + & \times \\ \hline \end{array}$.
Then $Y \in \gamma_3(d,X)$ and $d+c \in \beta_1(d,Y)$. So $Y \in Q(X) \leq G_a$ by Lemma 3.4 and $\tau(Y) \notin G_c$ by Lemma 3.3. Therefore $Q(d)_{ac} \cong 2^6$. Hence $G_{acd} \cong 2^6 2^4 : (S_4 \times 2)$ and so $[G_{ac} : G_{acd}] = 3$. In Γ_c let $X_1 = \begin{array}{|cc|cc|cc|} \hline + & + & \cdot & \times & \cdot & \times \\ \circ & \circ & \cdot & \times & \cdot & \times \\ \circ & + & \square & \ast & \square & \ast \\ + & \circ & \square & \ast & \square & \ast \\ \hline \end{array}$, $X_2 = \begin{array}{|cc|cc|cc|} \hline \cdot & \times & + & + & \cdot & \times \\ \cdot & \times & \circ & \circ & \cdot & \times \\ \square & \ast & \circ & + & \square & \ast \\ \square & \ast & + & \circ & \square & \ast \\ \hline \end{array}$, $X_3 = \begin{array}{|cc|cc|cc|} \hline \cdot & \times & \cdot & \times & + & + \\ \cdot & \times & \cdot & \times & \circ & \circ \\ \square & \ast & \square & \ast & \circ & + \\ \square & \ast & \square & \ast & + & \circ \\ \hline \end{array}$. Since part (iv) applies to any sextet line at c it is easily verified that $< \tau(X_1), \tau(X_2), \tau(X_3), \tau(X_c), \tau(Y_c) >$ is a group of order 2^4. Put

$$N_1 = < \tau(X_c), \tau(Y_c) > (\cong 2^2) \leq Q(c) \cap Q(d) \cap G_a \text{ and}$$
$$N_2 = < \tau(X_1), \tau(X_2), \tau(X_3), \tau(X_c), \tau(Y_c) > .$$

Observe that N_1 and N_2 are both normal subgroups of G_{ac} contained in $Q(c)$.

(7.1.3) $N_1 = Q(c) \cap Q(d) \cap G_a \lneqq < T(c+d) >$.

First we note that $N_1 \leq Q(c) \cap Q(d) \cap G_a$. Of the seven sextets in $\Gamma_2(d+c)$, four lie in $\gamma_1(d,X)$ and three lie in $\gamma_3(d,X)$. By Lemma 3.4 $\{\tau(Y) | Y \in \Gamma_2(d+c) \cap \gamma_3(d,X)\} \subseteq Q(X) (\leq G_a)$. Also if $Y \in \Gamma_2(d+c) \cap \gamma_1(d,X)$, then there exists $k \in \Gamma_1(X)$ with $k \in \beta_1(d,Y)$. So $\tau(Y)$ acts regularly on $\Gamma_0(k) \setminus \{b\}$ which consists of three points in $\Delta_1(a)$ and one point in $\Delta_2^1(a)$. Hence $\tau(Y) \notin G_a$. So $|Q(c) \cap Q(d) \cap G_a| = 2^2$ using Theorem 3.6 (iii).

We now assume that $Q(c)_a \gneqq N_2$ and seek a contradiction. Let $P \in Syl_3(G_{ac})$ and let $R \leq P$ be a subgroup of order 3 which bodily permutes the octads of l_c. Then $C_{Q(c)}(R) = < T(c+d) >$.

(7.1.4) $|Q(c)_a| = 2^6$.

If $C_{Q(c)_a/N_2}(P) \neq 1$, then, as R centralizes N_1, $|C_{Q(c)_a}(R)| \geq 2^3$, whence $< T(c+d) > = C_{Q(c)_a}(R)$. But then $< T(c+d) > \leq Q(c) \cap Q(d) \cap G_a$ contrary to (7.1.3). Therefore $|Q(c)_a/N_2| \neq 2$ or 2^3. If $|Q(c)_a| \geq 2^8$, then by (7.1.3) $Q(c)^{*d}_a = O_2(G^{*d}_{d,d+c})$, which is not the case. Since $Q(c)_a \leq Q(c)_{ad}$ and $Q(c) \cong 2^9$ (as $[G_{ac} : G_{acd}] = 3$) the only remaining possibility is $|Q(c)_a| = 2^6$.

Since $C_M(P) = 1$, (7.1.4) yields

(7.1.5) One of the following holds:-

(i) $|M \cap G_{ac}^{*c}| = 2^4$ and G_{ac}^{*c} covers L/M; or

(ii) $M \leq G_{ac}^{*c}$ and $G_{ac}^{*c}/M \cong S_3 \times A_4$, $S_3 \times S_3$ or $(3 \times A_4)2$.

(7.1.6) If g is an element of order 3 in G_{acd}, then g leaves invariant the three octads of $l_c = c + d$.

Since g fixes d, part (iii) implies (7.1.6).

Let g be an element of order 3 in G_{acd}, and without loss of generality suppose $g \in P$. Suppose (7.1.5)(i) holds. Since $Q(d)_{ac} \cong 2^6$ and $Q(c) \cap Q(d) \cap G_a \cong 2^2$, $Q(d)_{ac}^{*c} \cong 2^4$ and hence $M \cap G_{ac}^{*c} = Q(d)_{ac}^{*c}$. Noting that
$N_1 < \tau(X) > / < \tau(X) > \cong 2^2$ we infer that $|C_{M \cap G_{ac}^{*c}}(g)| = 2^2$ (see [Lemma 3.5;R]). Then as G_{ac}^{*c} covers $L/M \cong S_3 \times S_4$ and $C_M(P) = 1$ using (7.1.6) we deduce that g centralizes $M/M \cap G_{ac}^{*c}$ and consequently $|C_M(g)| \geq 2^4$, whereas $|C_M(g)| = 2^2$. So (7.1.5)(i) cannot hold and so (7.1.5)(ii) must pertain. Looking in G_{acd} we see that g is inverted by an involution and so $G_{ac}^{*c}/M \cong S_3 \times S_4$ or $(3 \times A_4)2$. Examining the action upon $Q(c)_a/N_2$ and N_2/N_1 and using the fact that R is fixed-point free on these sections yields that g centralizes both $Q(c)_a/N_2$ and N_2/N_1. So $|C_{Q(c)_a}(g)| \geq 2^4$. Since $Q(c)_a \leq Q(c)_{ad} \leq Q(c)_d$ this gives $|C_{Q(c)_d}(g)| \geq 2^4$. This is impossible as $|C_{Q(c)_a}(g)| = 2^3$ and this gives us our final contradiction. Thus we must have $Q(c)_a = N_2 \cong 2^4$ from which (iv) follows.

Theorem 7.2 Let $c \in \Delta_2^3(a)$. Then

(i) $\Delta_2^3(a)$ is a G_a-orbit;

(ii) $|\Delta_2^3(a)| = 2^{11}.3^2.7.11.23$;

(iii) $G_{ac}^{*c} \cong 2^6.S_5$ with $Q(c)_a \cong 2$;

(iv) G_{ac}^{*c} stabilizes a unique sextet $X(c,a)$ in Γ_c and $Q(c)_a =< \tau(X(c,a)) >$ (and a similar statement holds with a and c interchanged). Further if $t = \tau(X(a,c))^{*c}$, then G_{ac}^{*c} is the centralizer in G_c^{*c} of t; and

(v) the set of lines $\{c + x | x \in \{a,c\}^{\perp}\}$ form a non-sparse triangle in $\Gamma_1(X(c,a))$ and $\{a,c\}^{\perp} = \Gamma_0(l)$ for some line l.

Proof For any $b \in \Delta_1(a)$, G_{ab} is transitive on $\alpha_0(b, b+a)$ by (2.3)(ii) and Theorem 3.6(ii). Hence to prove part (i) it is enough to show that G_{ab} is transitive on $\Gamma_0(b+c) \backslash \{b\}$ for any $b \in \{a,c\}^{\perp}$. Without loss of generality we may choose b and c such that in Γ_b, $b+a$ is the standard trio and $b + c = \begin{array}{|cc|cc|cc|} + & \circ & + & + & + & + \\ - & + & \circ & - & \circ & - \\ - & + & \circ & - & - & \circ \\ \circ & + & \circ & \circ & - & - \end{array}$. Consider the sextets $Y_1 =$ standard sextet and $Y_2 = \begin{array}{|cc|cc|cc|} + & * & \times & \circ & \cdot & \square \\ + & * & \times & \circ & \cdot & \square \\ * & + & \circ & \times & \square & \cdot \\ * & + & \circ & \times & \square & \cdot \end{array}$ and the trios $l_1 = \begin{array}{|cc|cc|cc|} + & \circ & \circ & + & + & + \\ - & \circ & \circ & - & + & - \\ - & \circ & \circ & - & - & + \\ + & \circ & \circ & + & - & - \end{array}$ and $l_2 = \begin{array}{|cc|cc|cc|} + & \circ & - & + & + & + \\ + & - & \circ & + & \circ & - \\ - & - & \circ & - & + & \circ \\ \circ & - & \circ & \circ & - & + \end{array}$ in Γ_b. Then $b + c \in \beta_1(b, Y_i)$ and $b + c \in \alpha_2(b, l_i)$ for $i = 1,2$. Since $l_1 \in \beta_2(b, Y_1)$ and $l_2 \in \beta_3(b, Y_2)$, $\tau(Y_i)$ fixes $\Gamma_0(l_i)$ for each $i = 1,2$ by Lemma 3.3. Furthermore $l_1 \in \alpha_3(b, l_2)$ and so (2.13)(iv) and Lemma 3.7(i) imply that l_1 and l_2 determine a different pairing of the points in $\Gamma_0(b+c)\setminus\{b\}$. However $\tau(Y_1)$ and $\tau(Y_2)$ move the points in $\Gamma_0(b+c)\setminus\{b\}$ and so $<\tau(Y_1),\tau(Y_2)>$ is transitive on $\Gamma_0(b+c)\setminus\{b\}$. Since $\tau(Y_1)$ and $\tau(Y_2)$ lie in G_{ab}, part (i) is proved.

Part (ii) is just Theorem 4.8(ii).

We next establish parts (iii) and (iv). By Theorem 4.8(i) the lines in $\{c+x|x\in\{a,c\}^{\perp}\}$ form a non-sparse triangle at c and so are incident with a unique sextet, say $X(c,a)$, in $\Gamma_2(c)$. Likewise $\{a+x|x\in\{a,c\}^{\perp}\}$ is contained in a unique sextet $X(a,c)$ in $\Gamma_2(a)$. Therefore $G_{ac}\leq \mathrm{Stab}_{G_a}(X(a,c))\cap \mathrm{Stab}_{G_c}(X(c,a))$. By Lemma 3.3 we have $\tau(X(a,c)),\tau(X(c,a))\in G_{ac}$, with $<\tau(X(a,c)),\tau(X(c,a))>\leq Z(G_{ac})$. Let $b\in\{a,c\}^{\perp}$ and , without loss of generality in Γ_b, take $b+c$ to be the standard trio and $b+a=\begin{array}{|cc|cc|cc|}\hline - & + & + & + & + & + \\ + & - & \circ & \circ & - & - \\ + & \circ & \circ & - & - & \circ \\ + & \circ & - & \circ & - & \circ \\ \hline\end{array}$. Then $\{a,c\}^{\perp}=\Gamma_0(l)$ where $l=\begin{array}{|cc|cc|cc|}\hline + & + & - & - & + & + \\ + & + & - & - & + & + \\ - & - & \circ & \circ & \circ & \circ \\ - & - & \circ & \circ & \circ & \circ \\ \hline\end{array}$ $(\in\Gamma_1(b))$. Set $X_a=\begin{array}{|cc|cc|cc|}\hline \circ & \ast & \cdot & \cdot & \ast & \ast \\ \ast & \circ & \square & \square & \circ & \circ \\ \cdot & \square & + & \times & \times & + \\ \cdot & \square & \times & + & \times & + \\ \hline\end{array}$ and $X_c=\begin{array}{|cc|cc|cc|}\hline + & + & \ast & \ast & \square & \square \\ + & + & \ast & \ast & \square & \square \\ \cdot & \cdot & \circ & \circ & \times & \times \\ \cdot & \cdot & \circ & \circ & \times & \times \\ \hline\end{array}$ (regarded as sextets in Γ_b). Then $X(a,c)=X_a$ and $X(c,a)=X_c$ (regarded as planes in Γ). Put $\tau=\tau(X_a)$. Set $k=\begin{array}{|cc|cc|cc|}\hline + & - & + & - & \circ & \circ \\ + & - & + & - & \circ & \circ \\ + & - & + & - & \circ & \circ \\ + & - & + & - & \circ & \circ \\ \hline\end{array}\in\Gamma_1(b)$. By Lemma 3.3 τ acts regularly on $\Gamma_0(k)\setminus\{b\}$ as $k\in\beta_1(b,X_a)$. Since $k\in\alpha_2(b,b+c)$ we deduce that $\tau\notin Q(c)$. Because $\tau\in Z(G_{ac})$ we must have $\tau^{*c}\in Z(G_{ac}^{*c})$. Now parts (i) and (ii) yield that $|G_{ac}|=2^{10}.3.5$ and consequently, as $G_{ac}^{*c}\leq Stab_{G_c}(X(c,a))=L\cong 2^6 3S_6$, we must have $G_{ac}^{*c}\leq C_L(t^{*c})=K\cong 2^6S_5$. Furthermore we must have that $G_{ac}^{*c}O_2(K)/O_2(K)\cong A_5$ or S_5. We now show that $|G_{ac}^{*c}\cap O_2(K)|\geq 2^4$. Using Theorem 3.6(ii) we see that $2^7\leq|Q(b)_{ac}|$. Recall, from Theorem 3.6, that

$$Q(b)\cap Q(c)=<T(b+c)>=\{\tau(X)|X\in\Gamma_2(b+c)\}\cup\{1\}$$

with $Q(b)_c^{*c}\cong 2^6$. We observe that $b+a\in\beta_3(b,X)$ for one X in $\Gamma_2(b+c)$ while $b+a\in\beta_1(b,Y)$ for the other six Y in $\Gamma_2(b+c)$. Consequently, by Lemma 3.3, $(Q(b)\cap Q(c))_a$ has order 2. Therefore $Q(b)_{ac}\cap Q(c)$ has order 2 and hence $|Q(b)_{ac}^{*c}|\geqslant 2^6$. Since $Q(b)_{ac}^{*c}$ is elementary abelian and $G_{ac}^{*c}O_2(K)/O_2(K)\cong A_5$ or S_5, we infer that $|G_{ac}^{*c}\cap O_2(K)|\geq 2^4$. As a $GF(2)S_5$-module, $O_2(K)$ is indecomposable and has a composition series $1\backslash 1\backslash 4$. This then forces $O_2(K)\leq G_{ac}^{*c}$ so we have $G_{ac}^{*c}\cong 2^6A_5$ or 2^6S_5. If $G_{ac}^{*c}\cong 2^6A_5$, then $|Q(c)_a|=2^2$. Referring to [MS] and using their notation, as a $GF(2)L$-module $Q(c)$ has a decomposition series $1\backslash\overline{6}\backslash 4$ where the $\overline{6}$ is dual to $O_2(K)$. So $\overline{6}$ is an indecomposable $GF(2)S_5$-module with composition series $4\backslash 1\backslash 1$ and so cannot contain a 1-dimensional subspace invariant under A_5. Hence $|Q(c)_a|\neq 2^2$ and so $G_{ac}^{*c}\cong 2^6S_5$ with $Q(c)_a\cong 2$. This establishes parts (iii) and (iv).

For part (v) see Theorem 4.8(i).

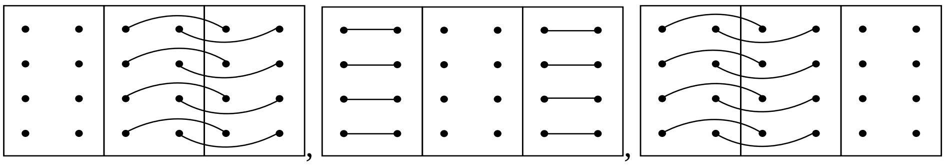
Remark In Theorem 7.2(iv), because of the involution structure of M_{24}, t must be an involution of cycle type 2^{12}.

We now fix a point $c \in \Delta_2^2(a)$ until the end of Lemma 7.8. Recall from Theorem 4.7 that associated with c there is a unique sextet line $\mathcal{S}(c,a) = \{X_c, Y_c, Z_c\}$ in Ω_c. Let l_c be the unique trio incident with each $S \in \mathcal{S}(c,a)$. We can now classify the lines in $\Gamma_1(c)$ by viewing them as trios in Ω_c and looking at their intersections with the sextets in $\mathcal{S}(c,a)$ and l_c.

Definition 7.3 Let $(\beta_r\beta_s\beta_t, \alpha_u)$ be the set of lines $l \in \Gamma_1(c)$ with $l \in \beta_r(c,X_c) \cap \beta_s(c,Y_c) \cap \beta_t(c,Z_c)$ for $\{X_c, Y_c, Z_c\} = \mathcal{S}(c,a)$ and $l \in \alpha_u(c,l_c)$.

Lemma 7.4 $(\beta_0\beta_2\beta_2, \alpha_2)$ and $(\beta_0\beta_3\beta_3, \alpha_3)$ are G_{ac}-orbits with $|(\beta_0\beta_2\beta_2, \alpha_2)| = 18$ and $|(\beta_0\beta_3\beta_3, \alpha_3)| = 24$.

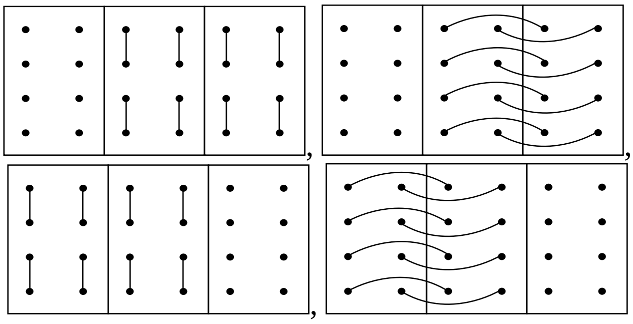
Proof Of the 14 lines in $\Gamma_1(c,X_c)\backslash\{l_c\}$, 8 lie in $S_1 := \beta_3(c,Y_c) \cap \beta_3(c,Z_c) \cap \alpha_3(c,l_c)$ and 6 lie in $S_2 := \beta_2(c,Y_c) \cap \beta_2(c,Z_c) \cap \alpha_2(c,l_c)$. Since G_c is transitive on sextet lines, by Theorem 7.1(i) we may assume that, in Ω_c, $\mathcal{S}(c,a)$ is the standard sextet line, that is $X_c = \left|\begin{array}{cc|cc|cc} \ast & \circ & \times & \cdot & + & \square \\ \ast & \circ & \times & \cdot & + & \square \\ \ast & \circ & \times & \cdot & + & \square \\ \ast & \circ & \times & \cdot & + & \square \end{array}\right|$, $Y_c = \left|\begin{array}{cc|cc|cc} \ast & \ast & \times & \times & + & + \\ \ast & \ast & \times & \times & + & + \\ \circ & \circ & \cdot & \cdot & \square & \square \\ \circ & \circ & \cdot & \cdot & \square & \square \end{array}\right|$ and $Z_c = \left|\begin{array}{cc|cc|cc} \ast & \circ & \times & \cdot & + & \square \\ \ast & \circ & \times & \cdot & + & \square \\ \circ & \ast & \cdot & \times & \square & + \\ \circ & \ast & \cdot & \times & \square & + \end{array}\right|$ with l_c the standard trio. Let $H = G_{acX_c}^{*c}$. Then



are elements of H by Theorem 7.1(v) and [Cu2]. Hence S_1 and S_2 are H-orbits and now the result follows because G_{ac} is transitive on $\mathcal{S}(c,a)$.

Lemma 7.5 $(\beta_2\beta_2\beta_2, \alpha_2)$ and $(\beta_3\beta_3\beta_3, \alpha_3)$ are G_{ac} orbits with $|(\beta_2\beta_2\beta_2, \alpha_2)| = 24$ and $|(\beta_3\beta_3\beta_3, \alpha_3)| = 32$.

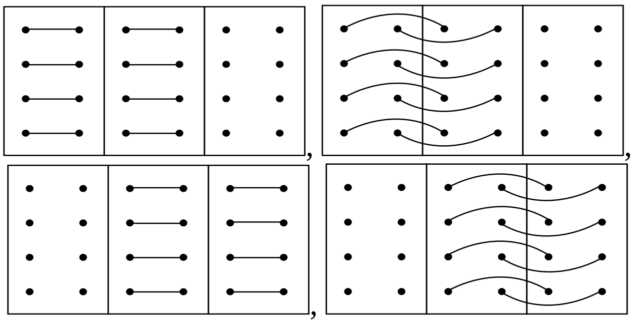
Proof Any line in $(\beta_2\beta_2\beta_2, \alpha_2) \cup (\beta_3\beta_3\beta_3, \alpha_3)$ must lie in a plane incident with l_c. Let S be one of the four sextets in $\Gamma_2(c,l_c)\backslash\mathcal{S}(c,a)$. Of the 14 lines in $\Gamma_1(c,S)\backslash\{l_c\}$, 8 lie in $(\beta_3\beta_3\beta_3, \alpha_3)$ and 6 lie in $(\beta_2\beta_2\beta_2, \alpha_2)$. Since G_a is transitive on sextet lines and we may assume that $\mathcal{S}(c,a)$ is the standard sextet line. Let $S = \left|\begin{array}{cc|cc|cc} \ast & \ast & \times & \times & + & + \\ \circ & \circ & \cdot & \cdot & \square & \square \\ \circ & \circ & \cdot & \cdot & \square & \square \\ \ast & \ast & \times & \times & + & + \end{array}\right|$ and $H = G_{acS}^{*c}$. Then

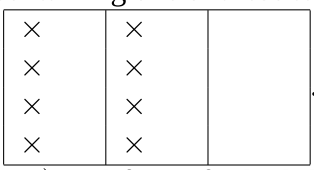


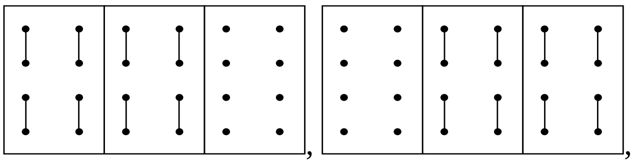
are elements of H by Theorem 7.1(v) and [Cu2]. Hence $(\beta_3\beta_3\beta_3,\alpha_3)\cap\Gamma_1(c,S)$ and $(\beta_2\beta_2\beta_2,\alpha_2)\cap\Gamma_1(c,S)$ are H-orbits and now the result follows because G_{ac} is transitive on $\Gamma_2(c,l_c)\backslash\mathcal{S}(c,a)$ by Theorem 7.1(iii).

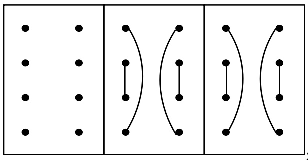
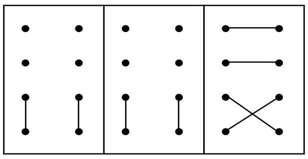
Lemma 7.6 $(\beta_1\beta_1\beta_2,\alpha_1)$ and $(\beta_2\beta_3\beta_3,\alpha_1)$ are G_{ac}-orbits with $|(\beta_1\beta_1\beta_2,\alpha_1)| = 288$ and $|(\beta_2\beta_3\beta_3,\alpha_1)| = 144$.

Proof Since G_{ac} is transitive on the three sextets in $\mathcal{S}(c,a)$ we need only show that G_{ac} is transitive on lines in $(\beta_1\beta_1\beta_2,\alpha_1)$ which lie in $\beta_2(c,X_c)$ and lines in $(\beta_2\beta_3\beta_3,\alpha_1)$ which lie in $\beta_2(c,X_c)$. We may assume, without loss of generality, that $\mathcal{S}(c,a)$ is the standard sextet line and X_c is the standard sextet. Since



lie in G_{acX_c} by [Cu2], we see that G_{acX_c} is transitive on the twelve octads incident with X_c but not lying in the standard trio l_c. Notice that any line l in $\beta_2(c,X_c)\cap((\beta_1\beta_1\beta_2,\alpha_1)\cup(\beta_2\beta_3\beta_3,\alpha_1))$ must be a trio containing one of these twelve octads. By the above we may assume that l contains the octad $O =$ . Then there are twelve possibilities for l, eight of which lie in $(\beta_1\beta_1\beta_2,\alpha_1)$ and four of which lie in $(\beta_2\beta_3\beta_3,\alpha_1)$. We can find subsets of G_{acX_c} which stabilize O and act transitively on these two sets of trios. For instance



,


generate a suitable subgroup of G_{acX_c} by [Cu2]. We have $|(\beta_1\beta_1\beta_2,\alpha_1)| = 3.12.8 = 288$ and $|(\beta_2\beta_3\beta_3,\alpha_1)| = 3.12.4 = 144$ which completes the proof of the lemma.

Lemma 7.7 $(\beta_1\beta_1\beta_3,\alpha_0)$ and $(\beta_1\beta_1\beta_1,\alpha_0)$ are G_{ac}-orbits with $|(\beta_1\beta_1\beta_3,\alpha_0)| = 1152$ and $|(\beta_1\beta_1\beta_1,\alpha_0)| = 1536$.

Proof Since G_a is transitive on sextet lines we may assume, without loss, that $\mathcal{S}(c,a)$ is the standard sextet line and l_c is the standard trio. Let

$l =$

○	+	+	+	+	+
+	○	−	−	○	○
+	−	○	−	−	○
+	−	−	○	−	○

. Then $l \in (\beta_1\beta_1\beta_3,\alpha_0)$. Set $H = G_c^{*c}$ and $K = G_{ac}^{*c}$. First we observe that $K_l \leq H_{l_c}$ and so $K_l = H_{l_cl} \cap K$. Since $l \in \alpha_0(c,l_c)$, (2.3)(i),(ii) implies that $|H_{l_cl}| = 2^3.3$. Put $g_1 = (0,1)(2,17,16,11)(3,5,15,6)$
$(4,10,13,7)(8,19)(9,14)(12,20,21,18)(22,\infty)$, $g_2 = (0,9)(1,14)(2,11)(3,5)(4,10)$
$(6,15)(7,13)(8,22)(12,18)(16,17)(19,\infty)(20,21)$ and $g_3 = (0,6,4)(1,7,3)$
$(2,20,19)(5,13,14)(8,12,17)(9,10,15)(11,\infty,21)(16,18,22)$ where we use the labelling of the MOG given in Section 2. It is straightforward to check that $< g_1,g_2 >$ is a dihedral group of order 8 and that $< g_1,g_2,g_3 >$ has order $2^3.3$. Further we check that $< g_1,g_2,g_3 >$ stabilizes both l_c and l. Using [Cu2] we see that $< g_1,g_2,g_3 >$ is a subgroup of M_{24} and hence $H_{l_cl} =< g_1,g_2,g_3 >$. By inspection we see that $< g_1,g_2 >$ stabilizes the sextet line $\mathcal{S}(c,a)$ whereas g_3 does not. Thus $H_{l_cl} \cap K =< g_1,g_2 >$ and therefore $|K_l| = 2^3$. So the K-orbit of l contains $\frac{2^{10}.3^2}{2^3} = 1152$ lines.

Let $k =$

+	−	+	+	+	+
○	+	−	○	−	○
○	+	○	−	−	○
−	+	○	○	−	−

. Then $k \in (\beta_1\beta_1\beta_1,\alpha_0)$. By an argument similar to the above we have $K_k = H_{l_ck} \cap K$ with $|H_{l_ck}| = 2^3.3$. Set
$h_1 = (0,22,7)(1,2,\infty)(3,19,4)(5,11,14)(6,13,8)(9,17,15)(10,18,12)(16,20,21)$,
$h_2 = (0,5)(1,8)(2,13)(3,9)(4,17)(6,\infty)(7,11)(10,16)(12,20)(14,22)(15,19)$
$(18,21)$, $h_3 = (0,3)(2,10)(6,21)(8,20)(11,17)(14,15)(18,\infty)(19,22)$
and $h_4 = (1,12)(4,7)(5,9)(6,21)(13,16)(14,15)(18,\infty)(19,22)$.
Then it is readily checked that $< h_1,h_2 >\cong S_3$ with $< h_3,h_4 >$ a fours group normalized by $< h_1,h_2 >$. Hence $| < h_1,h_2,h_3,h_4 > | = 2^3.3$. By inspection we see that $< h_1,h_2,h_3,h_4 >$ stabilizes both l_c and k and, employing [Cu2] again, we deduce that $< h_1,h_2,h_3,h_4 >= H_{l_ck}$. Further scrutiny reveals that $< h_1,h_2 >$ stabilizes the sextet line $\mathcal{S}(c,a)$ but that no non-trivial element in $< h_3,h_4 >$ does. Consequently $H_{l_ck} \cap K =< h_1,h_2 >$ and so $|K_k| = 2.3$. Hence the K-orbit of k consists of $\frac{2^{10}.3^2}{2.3} = 1536$ lines. Since $|\alpha_0(c,l_c)| = 2688$ by (2.3)(ii) the lemma is proven.

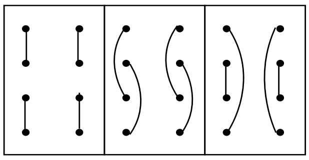
Lemma 7.8 $(\beta_1\beta_1\beta_3, \alpha_1)$ is a G_{ac}-orbit of size 576.

Proof Let $H = G_{ac}^{*c} \cong 2^6(S_3 \times S_4)$ and $Q = O_2(H)$. Since we know all the other G_{ac}-orbits on $\Gamma_1(c)$, to prove the lemma it is enough to show that $|H_l| = 2^4$ where $l = \begin{array}{|cc|cc|cc|} \hline \circ & + & + & + & + & + \\ + & \circ & - & - & - & - \\ + & \circ & - & - & - & - \\ + & \circ & \circ & \circ & \circ & \circ \\ \hline \end{array}$ $\in (\beta_1\beta_1\beta_3, \alpha_1)$. Any 3-element in H either permutes the three blocks of Ω_c or fixes each block and exactly two elements of each block. Therefore 3 does not divide $|H_l|$. By using Appendix 2 we have $|H_l \cap Q| = 2$. Suppose, for a contradiction, that $|H_l| = 2^5$. Then $H_l/H_l \cap Q$ is isomorphic to a Sylow 2-subgroup of H/Q of shape $2 \times D_8$. Also there exists a subgroup isomorphic to D_8 which is transitive on the elements in any of the three blocks. We now have a contradiction because of the way in which the three octads of l intersect the three blocks. So $|H_l| = 2^4$ as required.

Lemma 7.9 The G_{ac}-orbits on $\Gamma_1(c)$ are as listed in Theorem D.

Proof This is a consequence of Lemmas 7.3(iii) 7.4, 7.5, 7.6 and 7.7.

For our last result of this section, c is a point in $\Delta_2^3(a)$. In the light of Lemma 7.2(iv) and the accompanying remark, we may suppose $X(c,a)$ is the standard sextet and $G_{ac}^{*c} = C_{G_c^{*c}}(t)$ where

$t =$  .

Lemma 7.10 For $c \in \Delta_2^3(a)$ the G_{ac}-orbits of $\Gamma_1(c)$, together with their sizes, are as follows.

ORBIT	SIZE	REPRESENTATIVE
$(\beta_0, *)$	5	standard trio
$(\beta_0, **)$	10	$T_1 = \begin{array}{\|cc\|cc\|cc\|}\hline + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ \hline\end{array}$
$(\beta_1; 21^6; 21^6; 1^8)$	960	$T_2 = \begin{array}{\|cc\|cc\|cc\|}\hline \circ & + & + & + & + & + \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ + & \circ & - & - & - & - \\ \hline\end{array}$
$(\beta_1; 21^6; 21^6; 2^2 1^4)$	1920	$T_3 = \begin{array}{\|cc\|cc\|cc\|}\hline \circ & + & + & + & + & + \\ + & \circ & - & - & - & - \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ \hline\end{array}$
$(\beta_2; 2^4; 1^8; 1^8)$	60	$T_4 = \begin{array}{\|cc\|cc\|cc\|}\hline + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline\end{array}$
$(\beta_2; 2^4; 2^2 1^4; 2^2 1^4)$	120	$T_5 = \begin{array}{\|cc\|cc\|cc\|}\hline + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ \hline\end{array}$
$(\beta_3; 1^8; 1^8; 1^8)$	160	$T_6 = \begin{array}{\|cc\|cc\|cc\|}\hline + & \circ & + & \circ & - & - \\ \circ & + & + & \circ & - & - \\ \circ & \circ & - & - & + & \circ \\ + & + & - & - & + & \circ \\ \hline\end{array}$
$(\beta_3; 2^2 1^4; 2^2 1^4; 1^8)$	240	$T_7 = \begin{array}{\|cc\|cc\|cc\|}\hline + & \circ & + & - & - & \circ \\ + & \circ & + & - & - & \circ \\ \circ & + & - & + & \circ & - \\ \circ & + & - & + & \circ & - \\ \hline\end{array}$
$(\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$	320	$T_8 = \begin{array}{\|cc\|cc\|cc\|}\hline + & + & + & - & + & - \\ \circ & + & - & \circ & - & \circ \\ + & \circ & - & \circ & - & \circ \\ \circ & \circ & + & - & + & - \\ \hline\end{array}$

Moreover $(\beta_0,*) = \{$

+	+	○	○	−	−
+	+	○	○	−	−
+	+	○	○	−	−
+	+	○	○	−	−

,

+	○	+	−	○	−
+	○	+	−	○	−
+	○	+	−	○	−
+	○	+	−	○	−

,

+	○	−	+	−	○
+	○	−	+	−	○
+	○	−	+	−	○
+	○	−	+	−	○

,

+	○	−	○	+	−
+	○	−	○	+	−
+	○	−	○	+	−
+	○	−	○	+	−

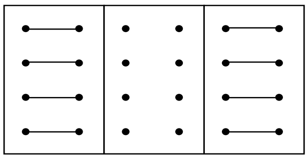
,

+	○	○	−	−	+
+	○	○	−	−	+
+	○	○	−	−	+
+	○	○	−	−	+

$\}$

$$= \{c + x | x \in \{a,c\}^{\perp}\}.$$

Proof First we note that the G_{ac}-orbits will partition the sets $\beta_i(c, X(c,a))$, $i = 0,1,2,3$. Put $H = G^{*c}_{ac}$. We begin by looking at $\beta_0(c, X(c,a))$ which, by Theorem 7.2(v), must contain $\{c + x | x \in \{a,c\}^{\perp}\}$. Observe that

$g_1 =$  $\in H$.

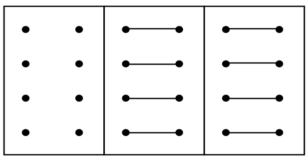
Now g_1 fixes the standard trio and interchanges

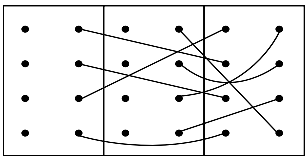
+	+	−	○	−	○
+	+	−	○	−	○
+	+	−	○	−	○
+	+	−	○	−	○

and

+	+	−	○	○	−
+	+	−	○	○	−
+	+	−	○	○	−
+	+	−	○	○	−

. Hence Theorem 7.2(v) implies that the standard trio must be in $\{c + x | x \in \{a,c\}^{\perp}\}$. Let g_2 and g_3 be





respectively. Then $g_2, g_3 \in C_{G^{*c}_c}(t) = H$ and $g_2 g_3$ has order 5. Applying $g_2 g_3$ to the standard trio yields $\{c + x | x \in \{a,c\}^{\perp}\}$ (= $(\beta_0,*)$) as stated. Using $g_2 g_3$ and g_1 it is straight forward to verify that $\beta_0(c, X(c,a)) \setminus (\beta_0,*)$ (= $(\beta_0,**)$) is a G_{ac}-orbit of length 10.

Since an element of H of order 5 must fix (pointwise) one column of the standard sextet and cycle the remaining 5 columns (in some order), on examining the above listed orbit representatives we see that the stabilizer in H of each representative is a $\{2,3\}$-group.

We next consider $\beta_1(c, X(c,a))$. An element of order 3 in G^{*c}_{ac} must be of cycle type 3^2 on the columns of the MOG and hence no element of order 3 can stabilize T_2 or T_3. Scanning Appendix 2 we readily see that $Stab_{O_2(H)}(T_i) = 1$ ($i = 2,3$). Therefore $|Stab_H(T_2)| \leq 2^3$. Let

$g \in Stab_H(T_3)$. Then g must either fix columns one and two of the standard sextet or interchange them. Because g must also centralize t we observe that g can only fix or interchange the third and fourth columns and fix or interchange the fifth and sixth columns. Hence $Stab_H(T_3)$ contains no elements of order 4 and so $|Stab_H(T_3)| \leq 2^2$. From

$$|T_2^H| + |T_3^H| \geq 960 + 1920 = |\beta_1(c, X(c,a))|$$

we then deduce that $(\beta_1; 21^6; 21^6; 1^8)$ and $(\beta_1; 21^6; 21^6; 2^2 1^4)$ are H-orbits of sizes 960 and 1920 respectively.

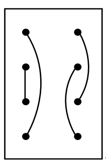
Looking at $\beta_2(c, X(c,a))$, we again observe that no element of H of order 3 can stabilize T_4 or T_5. While Appendix 2 reveals that $|Stab_{O_2(H)}(T_i)| = 2^4$ for $i = 4, 5$. Let $g \in Stab_H(T_5) \backslash O_2(H)$ and suppose g has order 4. Then g must cycle columns three to six in some order. But then g cannot simultaneously stabilize T_5 and centralize t. Therefore $|Stab_H(T_4)| \leq 2^7$ and $|Stab_H(T_5)| \leq 2^6$ from which we infer that $(\beta_2; 2^4; 1^8; 1^8) \cup (\beta_2; 2^4; 2^2 1^4; 2^2 1^4) = \beta_2(c, X(c,a))$ with $|(\beta_2; 2^4; 1^8; 1^8)| = 60$ and $|(\beta_2; 2^4; 2^2 1^4; 2^2 1^4)| = 120$.

To complete the proof of the lemma we must finally examine $\beta_3(c, X(c,a))$. Using Appendix 2 again we see that $|Stab_{O_2(H)}(T_i)| = 2^2$ for $i = 6, 7, 8$. By inspection, a 3-element of H cannot leave T_7 invariant and therefore $|Stab_H(T_7)| \leq 2^5$. Turning to T_8 we observe that there are exactly three columns of the MOG for which the 2|2 partition of the column induced by t is compatible with the partition induced by the octads of T_8. This together with the cycle structure of elements of S_5 in its degree 6 permutation representation implies that $Stab(T_8)$ induces a group of order at most 6 upon the columns of the MOG. Consequently $|Stab_H(T_8)| \leq 2^3.3$.

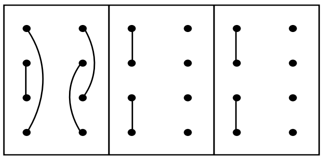
We next examine $Stab_H(T_6)$. So far we know that $|Stab_H(T_6)| \leq 2^2.2^3.3 = 2^5.3$. If 3 does not divide $|Stab_H(T_6)|$, then

$$|T_6^H| + |T_7^H| + |T_8^H| \geq 240 + 240 + 320 = 800 > |\beta_3(c, X(c,a))|,$$

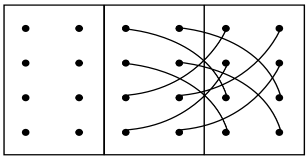
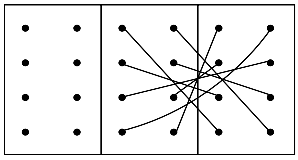
a contradiction. So $3 \,|\, |Stab_H(T_6)|$. Supposing $|Stab_H(T_6)| = 2^5.3$ we derive a contradiction. Then $Stab_H(T_6)$ must contain an element g of order 4 which cycles four of the columns of the MOG. If, say, g leaves columns one and three invariant, then g must leave the + octad of T_6 invariant. But by considering the action of g on the other columns we see that g cannot leave the + octad of T_6 invariant. Thus g cannot leave columns one and three invariant; similar considerations show that g must leave invariant either columns one and two or columns three and four or columns five and six. Looking at $Stab_{O_2(H)}(T_6)$ we infer that $Stab_H(T_6)$ must contain an element of order 3 cycling O_1, O_2 and O_3 in some order. Therefore we may suppose that g leaves columns one and two of the MOG invariant. The action of g on columns three to six means that g cannot leave the + octad of T_6 invariant. Hence in columns one and two g interchanges the + and o entries. Therefore g^2 must leave all octads of T_6 invariant and g^2 either fixes columns one and two pointwise or acts on O_1 as

.

Now

$h =$ 

is an element of $Stab_{O_2(H)}(T_6)$. So multiplying by h if necessary we may suppose that g^2 fixes O_1 pointwise. Since g^2 must send the subset $\{17,4\}$ of the MOG to $\{12,21\}$, by [Cu1] there are two possibilities for g^2, namely

 .

However neither of these elements stabilize T_6 which gives us our desired contradiction. So $|Stab_H(T_6)| \neq 2^5.3$. Consequently $|Stab_H(T_6)| \leq 2^4.3$. We therefore conclude that $(\beta_3;1^8;1^8;1^8)$, $(\beta_3;2^21^4;2^21^4;1^8)$ and $(\beta_3;2^21^4;2^21^4;2^21^4)$ are G_{ac}-orbits of sizes 160, 240 and 320 (respectively), since

$$|\beta_3(c,X(c,a))| \geq |T_6^H| + |T_7^H| + |T_8^H| \geq 160 + 240 + 320 = |\beta_3(c,X(c,a)|.$$

This completes the proof of Lemma 7.10.

8. LINES INCIDENT WITH $\Delta_2^2(a)$ POINTS

Throughout this section c is a fixed point in $\Delta_2^2(a)$ and $l \in \Gamma_1(c)$.

Lemma 8.1 If l is in $(\beta_0\beta_2\beta_2,\alpha_2)$, then $|\Gamma_0(l) \cap \Delta_1(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_2^2(a)| = 4$.

Proof Let $S = \{c+b | b \in \{a,c\}^\perp\}$. If there exists $b, b' \in \{a,c\}^\perp$ such that $b \neq b'$ and $c+b = c+b'$, then Lemma 3.11 implies that $\Gamma_2(a,c) \neq \emptyset$. So $c \in \Delta_2^1(a)$ contrary to Theorems 4.3 and 4.7. Hence $|S| = 18$ by Theorem 4.7(v). Since S is a union of G_{ac}-orbits on $\Gamma_1(c)$, Lemma 7.9 gives that S is the G_{ac}-orbit $(\beta_0\beta_2\beta_2,\alpha_2)$. Let $l \in S$. We know that $|\Gamma_0(l) \cap \Delta_1(a)| = 1$ from the above. For some $b \in \{a,c\}^\perp$, $b+c \in \alpha_1(b,b+a)$ by the definition of $c \in \Delta_2^2(a)$ and so for any $c' \in \Gamma_0(b+c)\backslash\{b\}$ we have $c' \in \Delta_2^2(a)$. The result now follows by Theorem 7.1(i) and because S is a G_{ac}-orbit on $\Gamma_1(c)$.

Lemma 8.2 If l is the trio incident with all sextets in $\mathcal{S}(c,a)$, then $|\Gamma_0(l) \cap \Delta_2^1(a)| = 3$ and $|\Gamma_0(l) \cap \Delta_2^2(a)| = 2$. Moreover if $x \in \Gamma_0(l) \cap \Delta_2^1(a)$, then $l \in \beta_2(x,X(x,a))$.

Proof Let

$$S = \{m \in \Gamma_0(c) |\text{ there exists } x \in \Delta_2^1(a) \cap \Gamma_0(m) \text{ with } m \in \beta_2(x,X(x,a))\}$$

($S \neq \emptyset$ by Lemma 6.4 because $\Delta_2^1(a)$ and $\Delta_2^2(a)$ are G_a-orbits). Then Lemma 6.4 implies that $|\Gamma_0(m) \cap \Delta_2^1(a)| = 3$ and $|\Gamma_0(m) \cap \Delta_2^2(a)| = 2$ for every $m \in S$. Using Theorem 4.3(iv), Theorem 7.1(ii) and (2.5)(ii) we get that $|S| = 1$ and therefore, by Lemma 7.9, the lemma is proven.

Lemma 8.3 If l is in $(\beta_0\beta_3\beta_3, \alpha_3)$, then $|\Gamma_0(l) \cap \Delta_2^1(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_2^2(a)| = 4$. Moreover if $x \in \Gamma_0(l) \cap \Delta_2^1(a)$, then $l \in \beta_3(x, X(x,a))$.

Proof We first show that $\Gamma_0(l) \cap \Delta_2^1(a) \neq \emptyset$. Let $x \in \Delta_2^1(a)$ and $y \in \Delta_2^2(a) \cap \Delta_1(x)$ with $x+y \in \beta_3(x, X(x,a))$ (such a point y exists by Lemma 6.4). Then

(8.3.1) $y+x$ lies in the G_{ay}-orbit $(\beta_0\beta_3\beta_3, \alpha_3)$.

By (2.11)(iv) there exists $k \in \Gamma_1(X(x,a))$ with $x+y \in \alpha_3(x,k)$. Let $b \in \Gamma_0(k) \cap \{a,x\}^\perp$. Then $b \in \{a,y\}^\perp$ by Lemma 3.8(ii). Since $b+x \in \alpha_3(b, b+y)$ by Lemma 3.9 and $b+c \in \alpha_2(b, b+a)$, we must have $b+x \in \Gamma_1(X)$ for some $X \in \mathcal{S}(y,a)$ using Theorem 4.7. Thus $y+x \in \beta_0(y,X)$. Appealing to Lemma 6.4, Lemma 7.9, Theorem 4.3(iv) and Theorem 7.1(ii) concludes the proof of (8.3.1).

Since $\Delta_2^1(a)$ and $\Delta_2^2(a)$ are G_a-orbits we must have some $z \in \Gamma_0(l) \cap \Delta_2^1(a)$ with $l \in \beta_3(z, X(z,a))$. Lemma 8.3 now follows from Lemma 6.4.

The following result covers all G_{ac}-orbits of $\Gamma_1(c)$ which contain a line incident with a point in $\Delta_2^3(a)$.

Lemma 8.4 Let $x \in \Delta_2^3(a) \cap \Delta_1(c)$ and $m \in \Gamma_1(c)$ be the trio in Ω_c incident with every sextet in $\mathcal{S}(c,a)$. Then $\{a,x\}^\perp \cap \{a,c\}^\perp \neq \emptyset$ and we have the following three possibilities:

(i) $c+x$ is in $(\beta_1\beta_1\beta_2, \alpha_1)$ with $|\Gamma_0(c+x) \cap \Delta_2^2(a)| = 1$, $|\Gamma_0(c+x) \cap \Delta_2^3(a)| = 2$ and $|\Gamma_0(c+x) \cap \Delta_3^1(a)| = 2$; or

(ii)$c+x$ is in $(\beta_2\beta_3\beta_3, \alpha_1)$ with $|\Gamma_0(c+x) \cap \Delta_2^2(a)| = 3$ and $|\Gamma_0(c+x) \cap \Delta_2^3(a)| = 2$; or

(iii) $c+x$ is in $(\beta_1\beta_1\beta_3, \alpha_1)$ with $|\Gamma_0(c+x) \cap \Delta_2^2(a)| = 1$ and $|\Gamma_0(c+x) \cap \Delta_2^3(a)| = 4$.

Proof Assume there exists $b \in \{a,c\}^\perp \cap \{a,x\}^\perp$ and let $X \in \mathcal{S}(c,a)$ with $b \in \Gamma_0(X)$. Then $c+x \in \alpha_i(c, c+b)$ for $i = 2$ or 3 by Lemma 3.11 and (2.3). First assume that $c+x \in \alpha_2(c, c+b)$ and let O be the octad in Ω_c incident with both the sextets $c+x$ and $c+b$. Suppose, for a contradiction, that O is an octad of the trio m and choose $b' \in \{a,c\}^\perp \backslash \{b\}$ such that $c+b'$ is incident with O. We must have $b' \in \{a,x\}^\perp$ for otherwise $x \in \Delta_2^1(b')$ which contradicts Theorem 5.2 because $x \in \Delta_2^3(a)$. However Theorem 4.7 implies that there are six points $y \in \{a,c\}^\perp$ such that $c+y$ is incident with O, whence $|\{a,x\}^\perp| \geqslant 6$, contrary to Theorem 4.8(i). Therefore we conclude that $c+x$ has no octad in common with m in Ω_c. Since $c+x \notin \beta_0(X)$ by Lemmas 8.1, 8.2, 8.3 and 7.9, we must have $c+x \in \beta_2(c,X)$. Then $c+x \in \alpha_1(c,m)$ by (2.11)(iii) because $c+x$ has no octad in common with m. Therefore Lemma 7.9 implies that $c+x$ lies in one of the G_{ac}-orbits $(\beta_1\beta_1\beta_2, \alpha_1)$ or $(\beta_2\beta_3\beta_3, \alpha_1)$. Let $x' \in \Gamma_0(c+x) \backslash \{c,x\}$ be such that $x' \in \Delta_1(b)$. By Theorem 4.7 we may assume that, in Ω_b, $b+c$ is the standard trio and $b+a$ is

○	+	+	+	+	+
+	○	○	○	○	○
+	○	–	–	–	–
+	○	–	–	–	–

. Since

$b+x \in \alpha_2(b,b+c) \cap \alpha_0(b,b+a)$ we see (using Appendix 1) that there are 24 possibilities for the trio $b+x$. In Ω_b the trio $b+x'$ contains the octad common to both $b+c$ and $b+x$ and is incident with the sextet in $\Gamma_2(b+c,b+x)$. By checking for each of the 24 possibilities we see that $b+x'$ must lie in $\alpha_0(b,b+a)$ and thus $x' \in \Delta_2^3(a)$. For any $y \in \Gamma(c+x)\backslash\{c,x,x'\}$, $y \in \Delta_2^1(b)$, whence $y \notin \Delta_2^3(a)$ by Theorem 5.2. Therefore $|\Gamma_0(c+x) \cap \Delta_2^3(a)| = 2$ as required.

(8.4.1) b is the unique point in $\{a,c\}^\perp \cap \{a,x\}^\perp$ such that $c+x \in \alpha_2(c,c+b)$.

Since $c+x$ has no common octad with m, Theorem 4.7 implies that b is the unique point in $\Gamma_0(X) \cap \{c,a\}^\perp$ such that $c+b \in \alpha_2(c,c+x)$. Let $Y \in \mathcal{S}(c,a)\backslash\{X\}$. If $c+x \in \beta_1(c,Y)$, then x is not collinear with any point of $\Gamma_0(Y)$ by (2.11)(ii). If $c+x \in \beta_3(c,Y)$, then (2.11)(iv) implies that $\Gamma_1(Y) \cap \alpha_2(c,c+x) = \emptyset$. Hence (8.4.1) is proven.

Let

$$S = \{y \in \Delta_2^3(a) \cap \Delta_1(c) | \text{ there exists } z \in \{a,c\}^\perp \cap \{a,y\}^\perp \text{ with } c+y \in \alpha_2(c,z)\}.$$

Since $|\Gamma_0(b+x) \cap \Delta_2^3(a)| = 4$ and $|\Delta_1(c) \cap (\Gamma_0(b+x)\backslash\{b\})| = 2$ we have, using Theorem 4.7(v) and Appendix 1,

$$|S| = 2.24.18 = 864.$$

However, $|\Gamma_0(c+x) \cap \Delta_2^3(a)| = 2$ and since $2|(\beta_1\beta_1\beta_2,\alpha_1)| + |\beta_2\beta_3\beta_3,\alpha_1)| = 864$ by Lemma 7.9 we must have $\{c+y|y \in S\} = (\beta_1\beta_1\beta_2,\alpha_1) \cup (\beta_2\beta_3\beta_3,\alpha_1)$. Therefore we have shown that if $l \in (\beta_1\beta_1\beta_2,\alpha_1) \cup (\beta_2\beta_3\beta_3,\alpha_1)$, then there exists $x \in \Gamma_0(l) \cap \Delta_2^3(a)$ and $\{a,x\}^\perp \cap \{a,c\}^\perp \neq \emptyset$. Let Z be the unique plane in $\gamma_2(c+b,c+x)$ and $Y \in \mathcal{S}(c,a)\backslash\{X\}$.

(8.4.2) If $c+x$ is in $\beta_1(Y)$, then $b+a \in \beta_1(b,Z)$.

By (2.8) there are twelve lines in $\Gamma_1(Y) \cap \beta_1(c,Z)$ and so from Theorem 4.7(v) and Lemma 3.3 there exists $b' \in \Gamma_0(Y) \cap \{a,c\}^\perp$ such that $b'^{\tau(Z)} \neq b'$. However $b' \in \Gamma_0(c+b')$ and so we must have that $a^{\tau(Z)} \neq a$ because $\Gamma_2(a,c) = \emptyset$. Using Lemma 3.3 again gives (8.4.2)

If $y \in \Gamma_0(c+x)\backslash\{c,x,x'\}$, then $b \in \Delta_2^1(a)$. Hence by (8.4.2) and Lemma 6.4, $a \in \Delta_3^1(y)$. Hence $y \in \Delta_3^1(a)$ by Lemma 6.5. This completes the proof of (i).

Now assume $c+x \in \alpha_3(c,c+b)$. By Lemmas 7.9, 8.1, 8.2 and 8.3 $c+x \notin \beta_0(Y)$ for each $Y \in \mathcal{S}(c,a)$ because $x \in \Delta_2^3(a)$. Thus $c+x$ is incident with a plane in $\Gamma_2(c,c+b)\backslash\{X\}$. Then (2.8)(iii) and (2.11)(iii) imply that $c+x \in \beta_3(c,X)$. Furthermore $c+x \in \alpha_1(c,m)$ by (2.11)(iv) because $c+b$ and m have a common octad. Therefore Lemma 7.9 implies that $c+x$ lies in $(\beta_1\beta_1\beta_3,\alpha_1)$ or $(\beta_2\beta_3\beta_3,\alpha_1)$. Assume $c+x$ is in$(\beta_1\beta_1\beta_3,\alpha_1)$. By (2.11)(ii),(iv), b is the unique point in $\{a,c\}^\perp \cap \{a,x\}^\perp$. Let Z be the unique plane in $\Gamma_2(c+x,c+b)$. Then $b+a \in \beta_1(b,Z)$ by (8.4.2). If $y \in \Gamma_0(c+x)\backslash\{x\}$, then (2.11)(ii) gives that $b+a \in \alpha_0(b,b+y)$ and thus $y \in \Delta_2^3(a)$ by Definition 4.1. Suppose $c+x$ is in $(\beta_2\beta_3\beta_3,\alpha_1)$. Then $b+a \in \alpha_0(b,b+x) \cap \alpha_1(b,b+c)$ and so (2.11) implies that $b+a \in \beta_i(b,Z)$ for $i=1$ or 3. If $b+a \in \beta_1(b,Z)$, then $y \in \Delta_2^3(a)$ for every $y \in \Gamma_0(c+x)\backslash\{c\}$ as above. However we already know that $|\Gamma_0(c+x) \cap \Delta_2^3(a)| = 2$ and so we must have $b+a \in \beta_3(b,Z)$. Using (2.11)(iv) and the fact that $b+y \in \alpha_3(b,b+c)$

for all $y \in \Gamma_0(c+x)\backslash\{c\}$ we conclude that $b+a \in \alpha_0(b,b+y)$ for three $y \in \Gamma_0(c+x)$. Thus $|\Gamma_0(c+x)\cap\Delta_2^2(a)| = 3$ as required for part (ii). This completes the proof of the lemma.

For the rest of this section we let $l_c \in \Gamma_1(c)$ be the unique line in $\beta_0(c,S)$ for all $S \in \mathcal{S}(c,a)$.

Lemma 8.5 If l is in $(\beta_2\beta_2\beta_2,\alpha_2)$, then $|\Gamma_0(l)\cap\Delta_2^2(a)| = 3$ and $|\Gamma_0(l)\cap\Delta_3^1(a)| = 2$.

Proof Since $l \in \alpha_2(c,l_c)$ we have that $l \in \alpha_2(c,c+b)$ for six points $b \in \{a,c\}^{\perp}$, using Theorem 4.7, with each trio $c+b$ containing a common octad. Fix $b \in \{a,c\}^{\perp}$ with $l \in \alpha_2(c,c+b)$ and let X' be the unique plane in $\Gamma_2(b,l)$. By Lemma 3.8(i) there are exactly two points in $(\Gamma_0(l)\backslash\{c\})\cap\Delta_1(b)$. Let $\{x,x'\} = (\Gamma_0(l)\backslash\{c\})\cap\Delta_1(b)$. Then $x,x' \notin \Delta_2^1(a)\cup\Delta_2^3(a)$ by Lemmas 7.9, 8.1, 8.2, 8.3 and 8.4. Furthermore $x,x' \notin \Delta_1(a)$ because $l \in \beta_2(X)$ for all $X \in \mathcal{S}(c,a)$, and so $x,x' \in \Delta_2^2(a)$. Let $y \in \Gamma_0(l)\backslash\Delta_1(b)$. Then $b \in \Delta_2^1(y)$ and so to complete the proof of the lemma it is enough to show that $b+a \in \beta_1(b,X')$ by Lemmas 6.4 and 6.5. Using (2.5)(i), without loss of generality we may assume that, in Ω_b, $b+c$ is the standard trio and $b+a$ is $\begin{array}{|cc|cc|cc|} \circ & + & + & + & + & + \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ + & \circ & - & - & - & - \end{array}$. The trios $b+x$ and $b+x'$ are incident with the same sextet in $\Gamma_2(b,b+c)$ and both lie in $\alpha_2(b,b+c)$. Thus they are two of the trios labelled B in Appendix 1. Then X' is one of the four sextets in $\Gamma_2(b,b+c)$ such that $b+a \in \beta_1(b,X')$, as required.

Lemma 8.6 If l is in $(\beta_3\beta_3\beta_3,\alpha_3)$, then $\Gamma_0(l)\backslash\{c\} \subseteq \Delta_3^1(a)$.

Proof Since $l \in \alpha_3(c,l_c)$ there exists $x \in \Gamma_0(l_c)\cap\Delta_2^1(a)$ by Lemma 8.2, whence Lemma 3.8(ii) implies that x is collinear with all points $y \in \Gamma_0(l)$. Fix $y \in \Gamma_0(l)\backslash\{c\}$. Using Lemma 8.2 again we have $l_c \in \beta_2(x,X(x,a))$ and so, of the seven sextets in $\Gamma_2(c,l)$, four lie in $\gamma_1(x,X(x,a))$ and three lie in $\gamma_3(x,X(x,a))$.

Let $X \in \mathcal{S}(c,a)$. Then $x \in \Gamma_0(X)$. If $X \notin \gamma_3(x,X(x,a))$, then there exists $k \in \Gamma_1(x,X(x,a))\cap\beta_1(x,X)$. Since $\tau(X)$ moves every point in $\Gamma_0(k)\backslash\{x\}$ by Lemma 3.3, $\tau(X)$ moves a because a is collinear with exactly three points of $\Gamma_0(k)\backslash\{x\}$. However $\tau(X) \in G_a$ by Theorem 4.7 and so we deduce that $\mathcal{S}(c,a) \subseteq \gamma_3(x,X(x,a))$. Let X' be the unique plane in $\Gamma_2(c,x,y)$. Then $X' \in \gamma_1(x,X(x,a))$ and so, in Ω_x, the sextets $X(x,a)$ and X' are incident with a common octad O. Since $l \in \beta_2(x,X(x,a))\cap\beta_0(x,X')$, O must be an octad of l. Using (2.8)(ii) we have that there are exactly three trios in $\beta_0(x,X')\backslash\beta_1(x,X(x,a))$ and each one contains the octad O. However $x+y \in \alpha_3(x,l_c)$ and so $x+y$ cannot contain O. Therefore $x+y \in \beta_1(x,X(x,a))$. Appealing to Lemma 6.4 completes the proof of the lemma.

Lemma 8.7 If l is in $(\beta_1\beta_1\beta_3,\alpha_0)$, then $\Gamma_0(l)\backslash\{c\} \subseteq \Delta_3^1(a)$.

Proof Let $X \in \mathcal{S}(c,a)$ be such that $l \in \beta_3(c,X)$. Then there exists $k \in \Gamma_1(c,X)\cap\alpha_3(c,l)$. Since $l \in \alpha_0(c,l_c)$, k has no octad in common with l_c and so Lemma 7.9 implies that k lies in $(\beta_0\beta_3\beta_3,\alpha_3)$. By Lemmas 8.3 and 3.8(ii) there exists $x \in \Gamma_0(k)\cap\Delta_2^1(a)$ and x is collinear with every point in $\Gamma_0(l)$. Fix $y \in \Gamma_0(l)\backslash\{c\}$. Using Lemmas 6.4 and 6.5, to prove the lemma it is enough to show that $x+y \in \beta_1(x, X(x,a))$. In Ω_x, $x+c \in \beta_3(x, X(x,a))$ by Lemma 8.3. Therefore, of the seven sextets in $\Gamma_2(x, x+c)$, six lie in $\gamma_0(x, X(x,a))$ and one lies in $\gamma_3(x, X(x,a))$. If $X \in \gamma_0(x, X(x,a))$, then there exists $k' \in \Gamma_1(x, X(x,a))$ such that $\tau(X)$ moves all points in $\Gamma_0(k') \backslash \{x\}$ by (2.8)(i) and Lemma 3.3. However this means that $\tau(X)$ moves a which contradicts Theorem 4.7. Hence $X \in \gamma_3(x, X(x,a))$. If X' is the unique plane in $\Gamma_2(c,x,y)$, then $X' \in \gamma_0(x, X(x,a))$. By (2.8)(i) there are precisely three trios in $\Gamma_1(x, X') \backslash \beta_1(x, X(x,a))$ and each one contains a common octad O. Since l_c is one of these trios and $x+y \in \alpha_3(x,m)$, $x+y$ cannot contain O. Therefore $x+y \in \beta_1(x, X(x,a))$ as required.

At this point we have determined the point distribution for all lines in $\Gamma_1(c)$ except those in $(\beta_1\beta_1\beta_1, \alpha_0)$. We define the set

$$\Delta_3^2(a) = \{x \in \Gamma_0 | \text{ there exists } c \in \Delta_1(x) \cap \Delta_2^2(a) \text{ with } c+x \in (\beta_1\beta_1\beta_1, \alpha_0)\}.$$

APPENDIX 1

In this Appendix we list all 105 trios which are incident with one of the 7 sextets which are themselves incident with the standard trio. This list is used extensively in Section 8.

Let $b \in \Gamma_0$ and put $\Lambda = \Gamma_b$. Let l denote the standard trio in Λ and set

$$k_1 = \begin{array}{|cc|cc|cc|} \hline \circ & + & + & + & + & + \\ + & \circ & \circ & \circ & \circ & \circ \\ + & \circ & - & - & - & - \\ + & \circ & - & - & - & - \\ \hline \end{array} \text{ and}$$

$$k_2 = \begin{array}{|cc|cc|cc|} \hline - & + & + & + & + & + \\ + & - & \circ & \circ & - & - \\ + & \circ & \circ & - & - & \circ \\ + & \circ & - & \circ & - & \circ \\ \hline \end{array}.$$

We annotate the trios k below in the following way:-

A means that $k \in \alpha_0(b, k_1)$;
B means that $k \in \alpha_1(b, k_1)$;
C means that $k \in \alpha_0(b, k_2)$.

Further we use $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ to mean that (respectively) A, B and C hold and that $k \in \alpha_2(b, l)$. So, for example, $\mathbb{A}$ means that $k \in \alpha_0(b, k_1)$ and k contains exactly one of the octads of the standard trio l.

1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6

+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

12|34|56

+	+	○	—	○	—
+	+	○	—	○	—
+	+	○	—	○	—
+	+	○	—	○	—

12|35|46

$\mathbb{B}\ \mathbb{C}$

+	+	○	—	—	○
+	+	○	—	—	○
+	+	○	—	—	○
+	+	○	—	—	○

12|36|45

$\mathbb{B}\ \mathbb{C}$

+	○	+	○	—	—
+	○	+	○	—	—
+	○	+	○	—	—
+	○	+	○	—	—

13|24|56

$\mathbb{A}\ \mathbb{C}$

+	○	+	—	○	—
+	○	+	—	○	—
+	○	+	—	○	—
+	○	+	—	○	—

13|25|46

$A\ C$

+	○	+	—	—	○
+	○	+	—	—	○
+	○	+	—	—	○
+	○	+	—	—	○

13|26|45

$A\ C$

+	○	○	+	—	—
+	○	○	+	—	—
+	○	○	+	—	—
+	○	○	+	—	—

14|23|56

$\mathbb{A}\ \mathbb{C}$

+	○	—	+	○	—
+	○	—	+	○	—
+	○	—	+	○	—
+	○	—	+	○	—

14|25|36

$A\ C$

+	○	—	+	—	○
+	○	—	+	—	○
+	○	—	+	—	○
+	○	—	+	—	○

14|26|35

$A\ C$

+	○	○	—	+	—
+	○	○	—	+	—
+	○	○	—	+	—
+	○	○	—	+	—

15|23|46

A

+	○	—	○	+	—
+	○	—	○	+	—
+	○	—	○	+	—
+	○	—	○	+	—

15|24|36

A

+	○	—	—	+	○
+	○	—	—	+	○
+	○	—	—	+	○
+	○	—	—	+	○

15|26|34

$\mathbb{A}$

+	○	○	—	—	+
+	○	○	—	—	+
+	○	○	—	—	+
+	○	○	—	—	+

16|23|45

$A\ C$

+	○	—	○	—	+
+	○	—	○	—	+
+	○	—	○	—	+
+	○	—	○	—	+

16|24|35

$A\ C$

+	○	—	—	○	+
+	○	—	—	○	+
+	○	—	—	○	+
+	○	—	—	○	+

16|25|34

$\mathbb{A}\ \mathbb{C}$

1	2	3	4	5	6
1	2	3	4	5	6
2	1	4	3	6	5
2	1	4	3	6	5

+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

$12|34|56$

+	+	○	—	○	—
+	+	○	—	○	—
+	+	—	○	—	○
+	+	—	○	—	○

$12|35|46$

$\mathbb{B}\ \mathbb{C}$

+	+	○	—	—	○
+	+	○	—	—	○
+	+	—	○	○	—
+	+	—	○	○	—

$12|36|45$

$\mathbb{B}\ \mathbb{C}$

+	○	+	○	—	—
+	○	+	○	—	—
○	+	○	+	—	—
○	+	○	+	—	—

$13|24|56$

$\mathbb{A}\ \mathbb{C}$

+	○	+	—	○	—
+	○	+	—	○	—
○	+	—	+	—	○
○	+	—	+	—	○

$13|25|46$

$A\ C$

+	○	+	—	—	○
+	○	+	—	—	○
○	+	—	+	○	—
○	+	—	+	○	—

$13|26|45$

A

+	○	○	+	—	—
+	○	○	+	—	—
○	+	+	○	—	—
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$14|23|56$

$\mathbb{A}\ \mathbb{C}$

+	○	—	+	○	—
+	○	—	+	○	—
○	+	+	—	—	○
○	+	+	—	—	○

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$A\ C$

+	○	—	+	—	○
+	○	—	+	—	○
○	+	+	—	○	—
○	+	+	—	○	—

$14|26|35$

A

+	○	○	—	+	—
+	○	○	—	+	—
○	+	—	○	—	+
○	+	—	○	—	+

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$A\ C$

+	○	—	○	+	—
+	○	—	○	+	—
○	+	○	—	—	+
○	+	○	—	—	+

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$A\ C$

+	○	—	—	+	○
+	○	—	—	+	○
○	+	—	—	○	+
○	+	—	—	○	+

$15|26|34$

$\mathbb{A}$

+	○	○	—	—	+
+	○	○	—	—	+
○	+	—	○	+	—
○	+	—	○	+	—

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$A\ C$

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○	+	○	—	+	—
○	+	○	—	+	—

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$A\ C$

+	○	—	—	○	+
+	○	—	—	○	+
○	+	—	—	+	○
○	+	—	—	+	○

$16|25|34$

$\mathbb{A}\ \mathbb{C}$

1	2	3	4	5	6
2	1	4	3	6	5
1	2	3	4	5	6
2	1	4	3	6	5

+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

12|34|56

+	+	○	—	○	—
+	+	—	○	—	○
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+	+	—	○	—	○

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$\mathbb{B}\ \mathbb{C}$

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+	+	○	—	—	○
+	+	—	○	○	—

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$\mathbb{A}\ \mathbb{C}$

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○	+	—	+	—	○

13|25|46

$A\ C$

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+	○	+	—	—	○
○	+	—	+	○	—

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$A\ C$

+	○	○	+	—	—
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+	○	○	+	—	—
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14|23|56

$\mathbb{A}$

+	○	—	+	○	—
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+	○	—	+	○	—
○	+	+	—	—	○

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$A\ C$

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+	○	—	+	—	○
○	+	+	—	○	—

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$A\ C$

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○	+	—	○	—	+
+	○	○	—	+	—
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A

+	○	—	○	+	—
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$A\ C$

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$\mathbb{A}\ \mathbb{C}$

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+	○	○	—	—	+
○	+	—	○	+	—

16|23|45

A

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○	+	○	—	+	—

16|24|35

$A\ C$

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○	+	—	—	+	○
+	○	—	—	○	+
○	+	—	—	+	○

16|25|34

$\mathbb{A}\ \mathbb{C}$

1	2	3	4	5	6
2	1	4	3	6	5
2	1	4	3	6	5
1	2	3	4	5	6

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+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

12|34|56

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+	+	—	○	—	○
+	+	—	○	—	○
+	+	○	—	○	—

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$\mathbb{B}\ \mathbb{C}$

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+	+	—	○	○	—
+	+	○	—	—	○

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$\mathbb{B}\ \mathbb{C}$

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+	○	+	○	—	—

13|24|56
$\mathbb{A}$

+	○	+	—	○	—
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+	○	+	—	○	—

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$A\ C$

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$A\ C$

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+	○	○	+	—	—

14|23|56
$\mathbb{A}\ \mathbb{C}$

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+	○	—	+	○	—

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$A\ C$

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○	+	+	—	○	—
+	○	—	+	—	○

14|26|35
$A\ C$

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+	○	○	—	+	—

15|23|46
$A\ C$

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○	+	○	—	—	+
+	○	—	○	+	—

15|24|36
A

+	○	—	—	+	○
○	+	—	—	○	+
○	+	—	—	○	+
+	○	—	—	+	○

15|26|34
$\mathbb{A}\ \mathbb{C}$

+	○	○	—	—	+
○	+	—	○	+	—
○	+	—	○	+	—
+	○	○	—	—	+

16|23|45
$A\ C$

+	○	—	○	—	+
○	+	○	—	+	—
○	+	○	—	+	—
+	○	—	○	—	+

16|24|35
A

+	○	—	—	○	+
○	+	—	—	+	○
○	+	—	—	+	○
+	○	—	—	○	+

16|25|34
$\mathbb{A}\ \mathbb{C}$

1	1	3	3	5	5
1	1	3	3	5	5
2	2	4	4	6	6
2	2	4	4	6	6

+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

12|34|56

+	+	○	○	○	○
+	+	○	○	○	○
+	+	—	—	—	—
+	+	—	—	—	—

12|35|46

+	+	○	○	—	—
+	+	○	○	—	—
+	+	—	—	○	○
+	+	—	—	○	○

12|36|45
$\mathbb{B}$ $\mathbb{C}$

+	+	+	+	—	—
+	+	+	+	—	—
○	○	○	○	—	—
○	○	○	○	—	—

13|24|56
$\mathbb{B}$ $\mathbb{C}$

+	+	+	+	○	○
+	+	+	+	○	○
○	○	—	—	—	—
○	○	—	—	—	—

13|25|46

+	+	+	+	—	—
+	+	+	+	—	—
○	○	—	—	○	○
○	○	—	—	○	○

13|26|45
B C

+	+	○	○	—	—
+	+	○	○	—	—
○	○	+	+	—	—
○	○	+	+	—	—

14|23|56
B

+	+	—	—	○	○
+	+	—	—	○	○
○	○	+	+	—	—
○	○	+	+	—	—

14|25|36
B C

+	+	—	—	—	—
+	+	—	—	—	—
○	○	+	+	○	○
○	○	+	+	○	○

14|26|35
B C

+	+	○	○	+	+
+	+	○	○	+	+
○	○	—	—	—	—
○	○	—	—	—	—

15|23|46

+	+	—	—	+	+
+	+	—	—	+	+
○	○	○	○	—	—
○	○	○	○	—	—

15|24|36
B

+	+	—	—	+	+
+	+	—	—	+	+
○	○	—	—	○	○
○	○	—	—	○	○

15|26|34
$\mathbb{B}$

+	+	○	○	—	—
+	+	○	○	—	—
○	○	—	—	+	+
○	○	—	—	+	+

16|23|45
B

+	+	—	—	—	—
+	+	—	—	—	—
○	○	○	○	+	+
○	○	○	○	+	+

16|24|35
B C

+	+	—	—	○	○
+	+	—	—	○	○
○	○	—	—	+	+
○	○	—	—	+	+

16|25|34
$\mathbb{B}$ $\mathbb{C}$

1	1	3	3	5	5
2	2	4	4	6	6
1	1	3	3	5	5
2	2	4	4	6	6

+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

12|34|56

+	+	○	○	○	○
+	+	—	—	—	—
+	+	○	○	○	○
+	+	—	—	—	—

12|35|46

+	+	○	○	—	—
+	+	—	—	○	○
+	+	○	○	—	—
+	+	—	—	○	○

12|36|45

$\mathbb{B}\ \mathbb{C}$

+	+	+	+	—	—
○	○	○	○	—	—
+	+	+	+	—	—
○	○	○	○	—	—

13|24|56

$\mathbb{A}\ \mathbb{C}$

+	+	+	+	○	○
○	○	—	—	—	—
+	+	+	+	○	○
○	○	—	—	—	—

13|25|46

B

+	+	+	+	—	—
○	○	—	—	○	○
+	+	+	+	—	—
○	○	—	—	○	○

13|26|45

$A\ C$

+	+	○	○	—	—
○	○	+	+	—	—
+	+	○	○	—	—
○	○	+	+	—	—

14|23|56

$\mathbb{A}\ \mathbb{C}$

+	+	—	—	○	○
○	○	+	+	—	—
+	+	—	—	○	○
○	○	+	+	—	—

14|25|36

$A\ C$

+	+	—	—	—	—
○	○	+	+	○	○
+	+	—	—	—	—
○	○	+	+	○	○

14|26|35

$B\ C$

+	+	○	○	+	+
○	○	—	—	—	—
+	+	○	○	+	+
○	○	—	—	—	—

15|23|46

B

+	+	—	—	+	+
○	○	○	○	—	—
+	+	—	—	+	+
○	○	○	○	—	—

15|24|36

$A\ C$

+	+	—	—	+	+
○	○	—	—	○	○
+	+	—	—	+	+
○	○	—	—	○	○

15|26|34

$\mathbb{A}\ \mathbb{C}$

+	+	○	○	—	—
○	○	—	—	+	+
+	+	○	○	—	—
○	○	—	—	+	+

16|23|45

$A\ C$

+	+	—	—	—	—
○	○	○	○	+	+
+	+	—	—	—	—
○	○	○	○	+	+

16|24|35

$B\ C$

+	+	—	—	○	○
○	○	—	—	+	+
+	+	—	—	○	○
○	○	—	—	+	+

16|25|34

$\mathbb{A}\ \mathbb{C}$

1	1	3	3	5	5
2	2	4	4	6	6
2	2	4	4	6	6
1	1	3	3	5	5

+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—
+	+	○	○	—	—

12|34|56

+	+	○	○	○	○
+	+	—	—	—	—
+	+	—	—	—	—
+	+	○	○	○	○

12|35|46

+	+	○	○	—	—
+	+	—	—	○	○
+	+	—	—	○	○
+	+	○	○	—	—

12|36|45

$\mathbb{B}\ \mathbb{C}$

+	+	+	+	—	—
○	○	○	○	—	—
○	○	○	○	—	—
+	+	+	+	—	—

13|24|56

$\mathbb{A}\ \mathbb{C}$

+	+	+	+	○	○
○	○	—	—	—	—
○	○	—	—	—	—
+	+	+	+	○	○

13|25|46

B

+	+	+	+	—	—
○	○	—	—	○	○
○	○	—	—	○	○
+	+	+	+	—	—

13|26|45

$A\ C$

+	+	○	○	—	—
○	○	+	+	—	—
○	○	+	+	—	—
+	+	○	○	—	—

14|23|56

$\mathbb{A}\ \mathbb{C}$

+	+	—	—	○	○
○	○	+	+	—	—
○	○	+	+	—	—
+	+	—	—	○	○

14|25|36

$A\ C$

+	+	—	—	—	—
○	○	+	+	○	○
○	○	+	+	○	○
+	+	—	—	—	—

14|26|35

$B\ C$

+	+	○	○	+	+
○	○	—	—	—	—
○	○	—	—	—	—
+	+	○	○	+	+

15|23|46

B

+	+	—	—	+	+
○	○	○	○	—	—
○	○	○	○	—	—
+	+	—	—	+	+

15|24|36

$A\ C$

+	+	—	—	+	+
○	○	—	—	○	○
○	○	—	—	○	○
+	+	—	—	+	+

15|26|34

$\mathbb{A}\ \mathbb{C}$

+	+	○	○	—	—
○	○	—	—	+	+
○	○	—	—	+	+
+	+	○	○	—	—

16|23|45

$A\ C$

+	+	—	—	—	—
○	○	○	○	+	+
○	○	○	○	+	+
+	+	—	—	—	—

16|24|35

$B\ C$

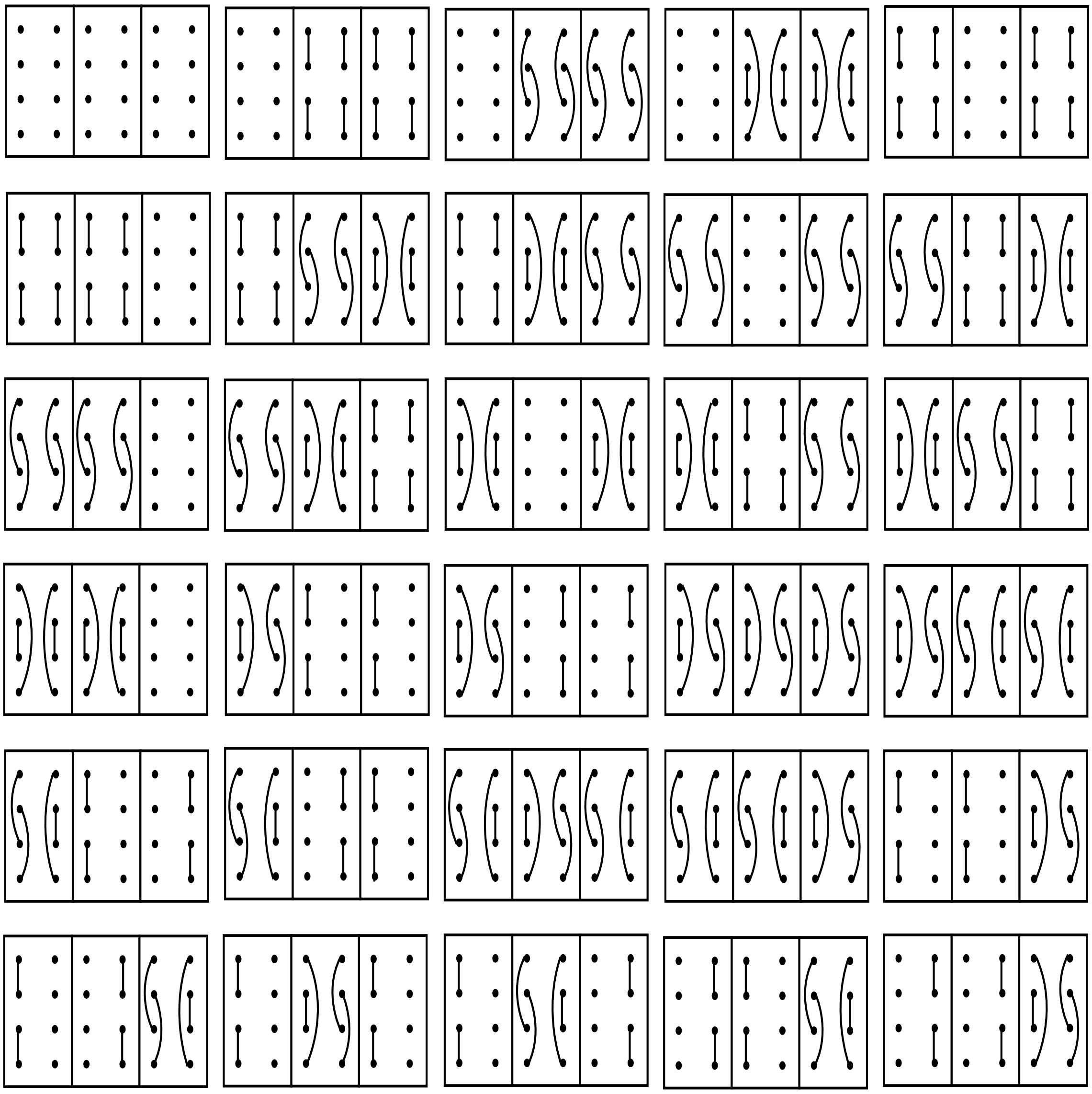
+	+	—	—	○	○
○	○	—	—	+	+
○	○	—	—	+	+
+	+	—	—	○	○

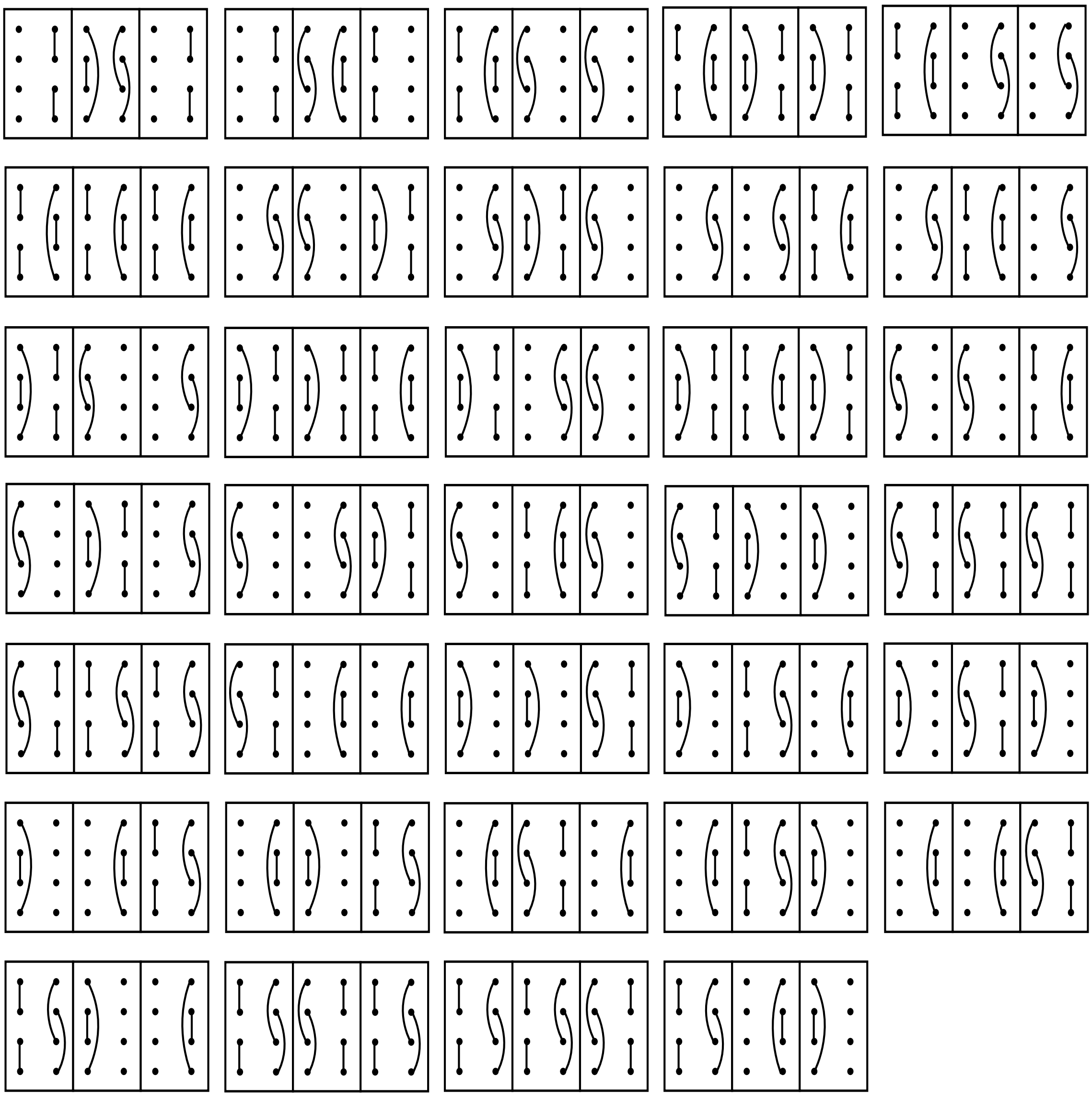
16|25|34

$\mathbb{A}\ \mathbb{C}$

APPENDIX 2

Let H be the stabilizer in M_{24} of the standard sextet. Then $H \cong 2^6 3 S_6$. Here we list all 64 elements of $O_2(H)$; see also the sextet stabilizer section in [Cu1].





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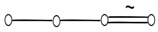
[R] P.J. Rowley, *Minimal parabolic systems with diagram*  , J. of Algebra, 141 (1991), 204-251.
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