An Equivalent of Language Invariance

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Abstract

We give a simpler equivalent of the Principle of Language Invariance within the framework of Pure Inductive Logic which is more evidently rational.

Key words: Language Invariance, Pure Inductive Logic, Rationality.

Introduction

The purpose of this short note is to give an equivalent version of the Principle of Language Invariance, which is simpler, easier to verify, and arguably more evidently rational. This reformulation of Language Invariance has actually been known for several years, and even been used implicitly, but without any printed referent to it existing. The present note aims at correcting this omission.

We shall assume the reader is familiar with the context and notation of Polyadic Pure Inductive Logic, see for example [2], [3].

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The Main Theorem

Let w be a probability function on a possibly polyadic language L with, as usual, constants a_i, $i \in \mathbb{N}^+$, and relation symbols[1] $R_1, R_2, \ldots, R_q$ of arities $r_1, r_2, \ldots, r_q$ respectively, satisfying Constant Exchangeability (Ex) and Predicate Exchangeability (Px).

According to the 'standard definition':

Language Invariance I

The probability function w satisfies Language Invariance if there is a family of probability functions $w^{\mathcal{L}}$, one on each language $\mathcal{L}$, satisfying Ex+Px such that $w^L = w$ and whenever $\mathcal{L} \subseteq \mathcal{L}'$, $w^{\mathcal{L}} = w^{\mathcal{L}'} \restriction S\mathcal{L}$ (i.e. $w^{\mathcal{L}'}$ restricted to $S\mathcal{L}$).

We shall refer to this version as Li1.

Let L_∞ be a language extending L with countably infinite numbers of relation symbols of each arity (and as standard still the constants a_i, $i \in \mathbb{N}^+$). Then Li1 is well known to be equivalent to Li2:

Language Invariance II

The probability function w satisfies Language Invariance if there is a probability function w_∞ on L_∞ extending w and satisfying Ex+Px.

To see this equivalence, if w satisfies Li1 then by taking the union of the $w^{\mathcal{L}}$ from the language invariant family with $\mathcal{L}$ a finite sub-language of L_∞ we obtain the required w_∞.

Conversely given w_∞ and $\mathcal{L}$ a (finite) language pick a sub-language $\mathcal{L}'$ of L_∞ isomorphic to $\mathcal{L}$, extend this isomorphism to one, σ say, between $S\mathcal{L}$ and $S\mathcal{L}'$ and define for $\theta \in S\mathcal{L}$,

$$w^{\mathcal{L}}(\theta) = w_\infty(\sigma\theta).$$

Because L_∞ satisfies Px the particular choice of $\mathcal{L}'$ does not matter and these $w^{\mathcal{L}}$ give the required language invariant family for Li1.

In this note we are interested in the following apparent weakening, Li3, of Li1:

Language Invariance III

The probability function w satisfies Language Invariance if for each language $\mathcal{L}$ extending L there is a probability function $w^{\mathcal{L}}$ on $S\mathcal{L}$ satisfying Ex+Px and extending w.

[1] As usual all the languages considered will have constants a_i for $i \in \mathbb{N}^+$ and only finitely many relation symbols unless stated otherwise.

We shall refer to this as Li3. The main result of this paper is:

Theorem 1. *Li1 and Li3 are equivalent.*

Proof. Clearly Li1 $\Rightarrow$ Li3 so, in view of the observation that Li2 $\Rightarrow$ Li1, it is enough to show Li3 $\Rightarrow$ Li2.

Suppose that w satisfies Li3. Let

$$L = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_n \subset \ldots$$

be (finite) languages with union L_∞. As given by Li3 let w_n be a probability function on SL_n satisfying Ex+Px and extending w. Let θ_n for $n \in \mathbb{N}$ enumerate $QFSL_\infty$ and such that $\theta_n \in SL_n$.

We now repeatedly apply a standard sequential compactness construction. Starting at stage 0 with the sequence of all natural numbers suppose that by stage n we have produced a subsequence

$$m_0^n < m_1^n < m_2^n < m_3^n < \ldots \tag{1}$$

such that for each $i < n$ and $r \leq j, k$,

$$|w_{m_j^n}(\theta_i) - w_{m_k^n}(\theta_i)| \leq 2^{-r}. \tag{2}$$

Now consider the values of $w_{m_j^n}(\theta_n)$ for $j \in \mathbb{N}$. These will have an accumulation point so we can pick a subsequence[2]

$$m_0^{n+1} < m_1^{n+1} < m_2^{n+1} < m_3^{n+1} < \ldots$$

of (1) such that for $r \leq j, k$,

$$|w_{m_j^{n+1}}(\theta_n) - w_{m_k^{n+1}}(\theta_n)| \leq 2^{-r}. \tag{3}$$

Clearly then for this subsequence (2) will hold with $n+1$ in place of n.

Noticing that the sequence

$$m_0^0 < m_1^1 < m_2^2 < m_3^3 < \ldots$$

will be a subsequence of each of the sequences (1) eventually we can by (2) properly define (i.e. the limit will exist) w_∞ on $QFSL_\infty$ by

$$w_\infty(\theta) = \lim_{r \to \infty} w_{m_r^r}(\theta).$$

[2]We can even *define* it here so there is no need to invoke the Axiom of Choice.

Since for $\theta, \phi \in QFSL_\infty$ and r sufficiently large the standard conditions to be a probability function on quantifier free sentences, i.e.

(P1) If $\models \phi$ then $w_{m_r^r}(\phi) = 1$;

(P2) If $\theta \models \neg\phi$ the $w_{m_r^r}(\theta \vee \phi) = w_{m_r^r}(\theta) + w_{m_r^r}(\phi)$;

hold for the probability function $w_{m_r^r}$ by taking limits we see that they must also hold for w_∞. Furthermore since the $w_{m_r^r}$ satisfy Ex+Px and extend w so will w_∞ on $QFSL_\infty$.

By Gaifman's Theorem, see [1] (or in the notation of this paper [3, Theorem 7.1]) we can uniquely extend w_∞ to a probability function[3] on SL_∞ satisfying Ex+Px.

To show that on SL w_∞ extends w notice that since all the w_m extend w we must have that w_∞ agrees with w on $QFSL$. Since w and $w_\infty \restriction SL$ are both extensions of w on $QFSL$ to SL and any such extension is unique by Gaifman's Theorem, they must agree too on SL. So we have as required Li2. $\square$

Recall from [3] that we say that w satisfies *Language Invariance with* $\mathcal{P}$, where $\mathcal{P}$ is some property, if the members $w^{\mathcal{L}}$ of the Li1 family also all satisfy the property $\mathcal{P}$. From the proof of Theorem 1 it follows that this too is equivalent to the corresponding enhancement of Li3 for properties preserved under limits, for example Strong Negation, Spectrum Exchangeability, Permutation Invariance.

Conclusion

We would argue that noting the equivalence of Li3 and Language Invariant (Li1) is worthwhile for two reasons. Firstly it is ostensibly weaker than Li1 and hence in practice may be more easily derived. Secondly it seems more obviously rational in that it simply requires of an agent's chosen, supposedly rational, probability function on SL that for each extension of L it has *some* extension which is also rational in the sense of satisfying Ex+Px without additionally requiring that these extensions put together form a consistent family.

References

[1] Gaifman, H., Concerning Measures on First Order Calculi, *Israel Journal of Mathematics*, 1964, **2**:1-18.

[3] So satisfying (P1), (P2) for general $\theta, \phi \in SL$ and in addition for $\exists x\, \psi(x) \in SL_\infty$,

(P3) $w_\infty(\exists x\, \psi(x)) = \lim_{n\to\infty} w_\infty(\psi(a_1) \vee \psi(a_2) \vee \ldots \vee \psi(a_n))$.

[2] Paris, J.B., Pure Inductive Logic, in *The Continuum Companion to Philosophical Logic*, Eds. L.Horsten & R.Pettigrew, Continuum International Publishing Group, London, 2011, pp428-449.

[3] Paris, J.B. & Vencovská, A., *Pure Inductive Logic*, in the Association of Symbolic Logic Perspectives in Mathematical Logic Series, Cambridge University Press, April 2015.