MANCHESTER
1824

The University of Manchester

# *Essential partial differential equations*

Griffiths, David F. and Dold, John W. and Silvester, David J.

2015

MIMS EPrint: **2016.6**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

# Contents


**1 Setting the Scene** . . . . . 1
1.1 Some Classical PDEs … . . . . . 2
1.2 … and Some Classical Solutions. . . . . 4

**2 Boundary and Initial Data** . . . . . 11
2.1 Operator Notation . . . . . 14
2.2 Classification of Boundary Value Problems . . . . . 16
2.2.1 Linear Problems . . . . . 16
2.2.2 Nonlinear Problems. . . . . 20
2.2.3 Well-Posed Problems. . . . . 21

**3 The Origin of PDEs**. . . . . 27
3.1 Newton's Laws. . . . . 27
3.1.1 The Wave Equation for a String . . . . . 27
3.2 Conservation Laws . . . . . 29
3.2.1 The Heat Equation . . . . . 29
3.2.2 Laplace's Equation and the Poisson Equation . . . . . 31
3.2.3 The Wave Equation in Water . . . . . 32
3.2.4 Burgers' Equation . . . . . 34

**4 Classification of PDEs** . . . . . 37
4.1 Characteristics of First-Order PDEs . . . . . 38
4.2 Characteristics of Second-Order PDES. . . . . 44
4.2.1 Hyperbolic Equations . . . . . 45
4.2.2 Parabolic Equations. . . . . 46
4.2.3 Elliptic Equations . . . . . 49
4.3 Characteristics of Higher-Order PDEs . . . . . 52
4.4 Postscript. . . . . 53

**5 Boundary Value Problems in $\mathbb{R}^1$** . . . . . 59
5.1 Qualitative Behaviour of Solutions . . . . . 61
5.2 Comparison Principles and Well-Posedness . . . . . 64

5.3 Inner Products and Orthogonality . . . . . 68
5.3.1 Self-adjoint Operators . . . . . 71
5.3.2 A Clever Change of Variables . . . . . 72
5.4 Some Important Eigenvalue Problems . . . . . 74

**6 Finite Difference Methods in $\mathbb{R}^1$** . . . . . 85
6.1 The Approximation of Derivatives. . . . . 86
6.2 Approximation of Boundary Value Problems . . . . . 90
6.3 Convergence Theory . . . . . 95
6.4 Advanced Topics and Extensions . . . . . 103
6.4.1 Boundary Conditions with Derivatives. . . . . 103
6.4.2 A Fourth-Order Finite Difference Method. . . . . 109

**7 Maximum Principles and Energy Methods** . . . . . 119
7.1 Maximum Principles . . . . . 119
7.2 Energy Methods . . . . . 124

**8 Separation of Variables** . . . . . 129
8.1 The Heat Equation Revisited . . . . . 129
8.1.1 Extension to Other Parabolic Problems .... . . . . 136
8.1.2 ... and to Inhomogeneous Data . . . . . 140
8.2 The Wave Equation Revisited. . . . . 143
8.3 Laplace's Equation . . . . . 148

**9 The Method of Characteristics** . . . . . 161
9.1 First-Order Systems of PDEs . . . . . 161
9.2 Second-Order Hyperbolic PDEs . . . . . 166
9.3 First-Order Nonlinear PDES . . . . . 171
9.3.1 Characteristics of Burgers' Equation . . . . . 174
9.3.2 Shock Waves . . . . . 179
9.3.3 Riemann Problems and Expansion Fans. . . . . 184

**10 Finite Difference Methods for Elliptic PDEs** . . . . . 195
10.1 A Dirichlet Problem in a Square Domain . . . . . 196
10.1.1 Linear Algebra Aspects . . . . . 198
10.1.2 Convergence Theory . . . . . 202
10.1.3 Improving the Solution Accuracy . . . . . 204
10.2 Advanced Topics and Extensions . . . . . 207
10.2.1 Neumann and Robin Boundary Conditions. . . . . 207
10.2.2 Non-rectangular Domains. . . . . 211
10.2.3 Polar Coordinates . . . . . 217
10.2.4 Regularity of Solutions . . . . . 221
10.2.5 Anisotropic Diffusion. . . . . 223
10.2.6 Advection–Diffusion . . . . . 226

**11 Finite Difference Methods for Parabolic PDEs** . . . . . 237
11.1 Time Stepping Algorithms . . . . . 238
11.1.1 An Explicit Method (FTCS) . . . . . 238
11.1.2 An Implicit Method (BTCS) . . . . . 247
11.1.3 The θ-Method . . . . . 251
11.2 Von Neumann Stability . . . . . 257
11.3 Advanced Topics and Extensions . . . . . 263
11.3.1 Neumann and Robin Boundary Conditions . . . . . 263
11.3.2 Multiple Space Dimensions . . . . . 264
11.3.3 The Method of Lines . . . . . 268

**12 Finite Difference Methods for Hyperbolic PDEs** . . . . . 275
12.1 Explicit Methods . . . . . 277
12.1.1 Order Conditions . . . . . 280
12.1.2 Stability Conditions . . . . . 282
12.1.3 First-Order Schemes . . . . . 284
12.1.4 Second-Order Schemes . . . . . 287
12.1.5 A Third-Order Scheme . . . . . 291
12.1.6 Quasi-implicit Schemes . . . . . 292
12.2 The Two-Way Wave Equation . . . . . 295
12.3 Convergence Theory . . . . . 296
12.4 Advanced Topics and Extensions . . . . . 299
12.4.1 Dissipation and Dispersion . . . . . 299
12.4.2 Nonperiodic Boundary Conditions . . . . . 304
12.4.3 Nonlinear Approximation Schemes . . . . . 308

**13 Projects** . . . . . 319

**Appendix A: Glossary and Notation** . . . . . 333

**Appendix B: Some Linear Algebra** . . . . . 335

**Appendix C: Integral Theorems** . . . . . 347

**Appendix D: Bessel Functions** . . . . . 351

**Appendix E: Fourier Series** . . . . . 355

**References** . . . . . 361

**Index** . . . . . 363

http://www.springer.com/978-3-319-22568-5

Essential Partial Differential Equations
Analytical and Computational Aspects
Griffiths, D.F.; Dold, J.W.; Silvester, D.J.
2015, XI, 368 p. 106 illus., 1 illus. in color., Softcover
ISBN: 978-3-319-22568-5