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INTRINSIC VOLUMES OF POLYHEDRAL CONES: A COMBINATORIAL PERSPECTIVE

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ABSTRACT. These notes provide a self-contained account of the combinatorial theory of intrinsic volumes for polyhedral cones. Streamlined derivations of the General Steiner formula, the conic analogues of the Brianchon-Gram-Euler and the Gauss-Bonnet relations, and the Principal Kinematic Formula are given. In addition, a connection between the characteristic polynomial of a hyperplane arrangement and the intrinsic volumes of the regions of the arrangement, due to Klivans and Swartz, is generalized and some applications presented.

1. INTRODUCTION

The theory of conic intrinsic volumes (or solid/internal/external/Grassmann angles) has a rich and varied history, with origins dating back almost 100 years [Som27]. This theory has recently found renewed interest, owing to newly found connections with measure concentration and resulting applications in compressive sensing and related fields [DT09, ALMT14, MT14, GNP14]. Despite this recent surge in interest, the theory still remains somewhat inaccessible; this is, in part, due to the fact that many of the results are spread out in the literature with uneven terminology (cf. Section 2.3), or are available as special cases of a more sophisticated theory of integral geometry [SW08] that treats the subject in a level of generality (involving curvature/support measures or relying on differential geometry) that may appear daunting to the uninitiated. In addition, some results, such as the relation to the theory of hyperplane arrangements, have so far not been connected to the existing body of research.

The main aim of these notes is to provide the practitioner with a self-contained account of the basic theory of intrinsic volumes of polyhedral cones, that requires little more background than some elementary polyhedral geometry and properties of the Gaussian distribution. The focus of this text is thus put on simplicity rather than generality, on finding the most natural relations between different results that may be derived in different orders from each other, and on highlighting parallels between different results. Despite this, the text does contain some generalizations of known results, provided these can be derived with little additional effort. In the interest of brevity, this article does not discuss the probabilistic properties of intrinsic volumes, such as their moments and concentration properties, nor does it go into the geometric problems that gave rise to the ideas presented here in the first place.

Section 2 is devoted to some preliminaries from the theory of polyhedral cones including a discussion of conic intrinsic volumes, a section devoted to clarifying the connections between different notation and terminology used in the literature, and a section introducing some concepts and techniques from the theory of partially ordered sets. In Section 3 we present a modern interpretation of the conic Steiner formula that underlies the recent developments in [ALMT14, MT14, GNP14], and in Section 4 we derive and discuss the Gauss-Bonnet relation for intrinsic volumes. Section 5 contains a crisp proof of the Principal Kinematic Formula for polyhedral cones, and Section 6 is devoted to a generalization of a result by Klivans and Swartz [KS11] and some applications thereof.

1.1. **Notation and conventions.**

Throughout, we use boldface letters for vectors and linear transformations. To lighten the notation we denote the set consisting solely of the zero vector by $\mathbf{0}$. We use calligraphic letters for families of sets.

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2. PRELIMINARIES

General references for basic facts about convex cones that are stated here are, for example, [Bar02, Zie95, Roc70]. More precise references will be given when necessary. A convex cone $C \subseteq \mathbb{R}^d$ is a convex set such that $\lambda C = C$ for all $\lambda > 0$. A convex cone is polyhedral if it is a finite intersection of closed half-spaces. In particular, linear subspaces are polyhedral, and polyhedral cones are closed. In what follows, unless otherwise stated, all cones are assumed to be polyhedral. A supporting hyperplane is a hyperplane H such that C lies entirely in one of the closed half-spaces induced by H. A face of C is a set of the form $F = C \cap H$, where H is a supporting hyperplane. The linear span $\operatorname{lin}(C)$ of a cone C is the smallest linear subspace containing C and is given by $\operatorname{lin}(C) = C + (-C)$. The dimension of a face F is $\dim F := \dim \operatorname{lin}(F)$, and the relative interior $\operatorname{relint}(F)$ is the interior of F in $\operatorname{lin}(F)$. A cone is pointed if the origin $\mathbf{0}$ is a zero-dimensional face, or equivalently, if it does not contain a linear subspace of dimension greater than zero. If C is not pointed, then it contains a nontrivial linear subspace of maximal dimension $k > 0$, given by $L = C \cap (-C)$, and L is contained in every supporting hyperplane (and thus, in every face) of C. Denoting by C/L the orthogonal projection of C on the orthogonal complement of L, the projection C/L is pointed, and $C = L + C/L$ is an orthogonal decomposition of C; we call this the canonical decomposition of C.

We denote by $\mathcal{F}(C)$ the set of faces, $\mathcal{F}_k(C)$ the set of k-dimensional faces, and let $f_k(C) = |\mathcal{F}_k(C)|$ denote the number of k-faces of C. The tuple $\boldsymbol{f}(C) = (f_0(C), \dots, f_d(C))$ is called the *f-vector* of C. Note that if $C = L + C/L$ is the canonical decomposition, then $\boldsymbol{f}(C)$ is a shifted version of $\boldsymbol{f}(C/L)$. The most fundamental property of the f-vector is the *Euler relation*.

Theorem 2.1 (Euler). *Let $C \subseteq \mathbb{R}^d$ be a polyhedral cone. Then*

$$\sum_{i=0}^{d} (-1)^i f_i(C) = \begin{cases} (-1)^{\dim L} & \text{if } C = L \text{ is a linear subspace,} \\ 0 & \text{else.} \end{cases} \tag{2.1}$$

This relation is usually stated and proved in terms of polytopes [Zie95, Ch. 8], but intersecting a pointed cone with a suitable affine hyperplane yields a polytope with an equivalent face structure as the cone; the general case can be reduced to the pointed case through the canonical decomposition. A short proof of the Euler relation along with remarks on the history of this result can be found in [Law97].

2.1. Duality.

The *polar* cone of a cone $C \subseteq \mathbb{R}^d$ is defined as

$$C^\circ = \{\boldsymbol{x} \in \mathbb{R}^d : \forall \boldsymbol{y} \in C, \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 0\}.$$

If $C = L$ is a linear subspace, then $C^\circ = L^\perp$ is just the orthogonal complement. To any face $F \in \mathcal{F}_k(C)$ we can associate the *normal face* $N_F C \in \mathcal{F}_{d-k}(C^\circ)$ defined as $N_F C = C^\circ \cap F^\perp$. The resulting map $\mathcal{F}_k(C) \to \mathcal{F}_{d-k}(C^\circ)$ is a bijection, which satisfies $N_{F^\diamond}(C^\circ) = F$, if $F^\diamond = N_F(C)$.

Central to convex geometry and optimization are a variety of theorems of the alternative, the most prominent of which is known as Farkas' Lemma (among the countless references, see for example [Zie95, Chapter 2]). All versions of Farkas' Lemma follow from a special case of the Hahn-Banach theorem, the separating hyperplane theorem. We refer to [Roc70, Theorem 11.3] for a proof of the following fundamental result.

Theorem 2.2 (Separating hyperplane). *Let $K_1, K_2 \subset \mathbb{R}^d$ be non-empty convex sets. Then* $\operatorname{relint}(K_1) \cap \operatorname{relint}(K_2) = \emptyset$ *if and only if there exists a hyperplane H, not containing both K_1 and K_2, such that $K_1 \subset H_1$ and $K_2 \subset H_2$, where H_1, H_2 denote the closed half-spaces defined by H.*

The following variation of Farkas' Lemma for convex cones, which is slightly more general than the usual one, is taken from [AL14].

Lemma 2.3 (Farkas). *Let C, D be closed convex cones. Then*

$$\operatorname{relint}(C) \cap D = \emptyset \iff C^\circ \cap -D^\circ \neq \mathbf{0}.$$

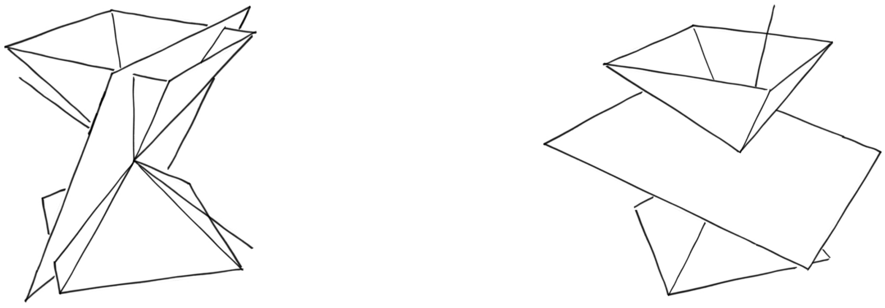

FIGURE 1. Either a subspace intersects C, or its complement intersects C°.

In particular, if $D = L$ is a linear subspace, then

$$\operatorname{relint}(C) \cap L = \emptyset \iff C^\circ \cap L^\perp \neq \mathbf{0}. \tag{2.2}$$

The situation in which $D = L$ is a hyperplane is best visualised as in Figure 1.

Proof. If $\operatorname{relint}(C) \cap D = \emptyset$, then by Theorem 2.2 there exists a separating hyperplane $H = \boldsymbol{h}^\perp$, $\boldsymbol{h} \neq \mathbf{0}$, such that $\langle \boldsymbol{h}, \boldsymbol{x} \rangle \leq 0$ for all $\boldsymbol{x} \in C$ and $\langle \boldsymbol{h}, \boldsymbol{y} \rangle \geq 0$ for all $\boldsymbol{y} \in D$. But this means $\boldsymbol{h} \in C^\circ \cap (-D^\circ)$. On the other hand, if $\boldsymbol{x} \in \operatorname{relint}(C) \cap D$ then only in the case $C = \mathbb{R}^d$, for which the claim is trivial, can $\boldsymbol{x} = \mathbf{0}$ hold. If $\boldsymbol{x} \neq \mathbf{0}$, then $C^\circ \setminus \mathbf{0}$ lies in the open half-space $\{\boldsymbol{h} : \langle \boldsymbol{h}, \boldsymbol{x} \rangle < 0\}$ and $-D^\circ$ lies in the closed half-space $\{\boldsymbol{h} : \langle \boldsymbol{h}, \boldsymbol{x} \rangle \geq 0\}$, and thus $C^\circ \cap (-D^\circ) = \mathbf{0}$. The case $D = L$ follows immediately. □

In view of some of the later developments, it is important to understand the behaviour of duality under intersections. The following is a conic variant of [Roc70, Corollary 23.8.1] (see also [Zie95, Chapter 7] for a similar theme).

Proposition 2.4. *The polar operation of intersection is the Minkowski sum,*

$$(C \cap D)^\circ = C^\circ + D^\circ.$$

Moreover, every face of $C \cap D$ is of the form $F \cap G$ for some $F \in \mathcal{F}(C), G \in \mathcal{F}(D)$, and the polar face is given by

$$N_{F \cap G}(C \cap D) = N_F C + N_G D.$$

Proof. For the first claim, note that

$$\begin{aligned}(C \cap D)^\circ &= \{\boldsymbol{z} \in \mathbb{R}^d : \forall (\boldsymbol{x}, \boldsymbol{y}) \in C \times D, \langle \boldsymbol{z}, \boldsymbol{x} \rangle \leq 0, \langle \boldsymbol{z}, \boldsymbol{y} \rangle \leq 0\} \\ &= \{\boldsymbol{z} \in \mathbb{R}^d : \forall (\boldsymbol{x}, \boldsymbol{y}) \in C \times D, \langle \boldsymbol{z}, \boldsymbol{x} + \boldsymbol{y} \rangle \leq 0\} = C^\circ + D^\circ.\end{aligned}$$

Clearly, a face of $C \cap D$, being the intersection of $C \cap D$ with supporting hyperplanes of C and D, is of the form $F \cap G$. The claim about the polar face is easily verified. □

Two faces $F \in \mathcal{F}(C)$ and $G \in \mathcal{F}(D)$ are said to intersect transversely, written $F \pitchfork G$, if their interiors have a non-empty intersection, $\operatorname{relint}(F) \cap \operatorname{relint}(G) \neq \emptyset$, and $\dim F \cap G = \dim F + \dim G - d$.

Corollary 2.5. *Let C, D be cones and $F \in \mathcal{F}(C)$, $G \in \mathcal{F}(D)$ be faces that intersect transversely. Then $N_F C + N_G D$ is a face of $C^\circ + D^\circ$ of dimension $(d - \dim F) + (d - \dim G)$.*

For a polyhedral cone $C \subseteq \mathbb{R}^d$, denote by $\boldsymbol{\Pi}_C$ the Euclidean projection,

$$\boldsymbol{\Pi}_C(\boldsymbol{x}) = \arg\min\{\|\boldsymbol{x} - \boldsymbol{y}\|^2 : \boldsymbol{y} \in C\}.$$

The Moreau decomposition of a point $\boldsymbol{x} \in \mathbb{R}^d$ is the sum representation

$$\boldsymbol{x} = \boldsymbol{\Pi}_C(\boldsymbol{x}) + \boldsymbol{\Pi}_{C^\circ}(\boldsymbol{x}), \tag{2.3}$$

where $\mathbf{\Pi}_C(\boldsymbol{x})$ and $\mathbf{\Pi}_{C^\circ}(\boldsymbol{x})$ are orthogonal. A direct consequence is the disjoint decomposition

$$\mathbb{R}^d = \bigcup_{F\in\mathcal{F}(C)} (\mathrm{relint}(F) + N_F C), \tag{2.4}$$

see also [McM75, Lemma 3].

2.2. **Intrinsic volumes.** For $C \subseteq \mathbb{R}^d$ polyhedral cone and for a face $F \in \mathcal{F}(C)$, define

$$v_F(C) = \mathbb{P}\{\mathbf{\Pi}_C(\boldsymbol{g}) \in \mathrm{relint}\, F\} = \mathbb{P}\{\boldsymbol{g} \in F + N_F C\},$$

where $\boldsymbol{g} \sim \mathcal{N}(\mathbb{R}^d)$ is a standard Gaussian vector in $\mathbb{R}^d$; the second equality follows from (2.4). Define the k-th intrinsic volumes of C, $0 \leq k \leq d$, to be

$$v_k(C) = \sum_{F\in\mathcal{F}_k(C)} v_F(C).$$

The intrinsic volumes form a probability distribution on $\{0, 1, \dots, d\}$, and intiuitively, $v_k(C)$ measures the extent to which a cone C is k-dimensional. Note that if $F \in \mathcal{F}_k(C)$ then

$$v_F(C) = v_k(F)\, v_{d-k}(N_F C).$$

For later reference, we note that in combination with Corollary 2.5, we get for cones C, D and faces $F \in \mathcal{F}_k(C)$, $G \in \mathcal{F}_\ell(D)$ that intersect transversely, with $j = k + \ell - d$,

$$v_{F\cap G}(C \cap D) = v_j(F \cap G)\, v_{d-j}(N_F C + N_G D). \tag{2.5}$$

This follows from Corollary 2.5 in conjunction with (2.1).

Example 2.6. Let $C = L \subseteq V$ be a linear subspace of dimension i. Then

$$v_k(C) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

Example 2.7. Let $C = \mathbb{R}^d_{\geq 0}$ be the non-negative orthant, i.e., the cone consisting of points with non-negative coordinates. A vector $\boldsymbol{x}$ projects orthogonally to a k-dimensional face of C if and only if exactly k coordinates are non-positive. By symmetry considerations and the invariance of the Gaussian distribution under permutations of the coordinates, it follows that

$$v_k(\mathbb{R}^d_{\geq 0}) = \binom{d}{k} 2^{-d}.$$

The following important properties of the intrinsic volumes, which are easily verified in the setting of polyhedral cones, will be used frequently:

(a) **Orthogonal invariance.** For an orthogonal transformation $\boldsymbol{Q} \in O(d)$,

$$v_k(\boldsymbol{Q}C) = v_k(C);$$

(b) **Polarity.**

$$v_k(C) = v_{d-k}(C^\circ);$$

(c) **Product rule.**

$$v_k(C \times D) = \sum_{i+j=k} v_i(C) v_j(D). \tag{2.6}$$

Note that the product rule implies $v_i(C \times L) = v_{i-k}(C)$ if $i \geq k$ and L is a subspace of dimension k. We will sometimes be working with the intrinsic volume generating polynomial,

$$v_C(t) = \sum_{k=0}^{d} v_k(C) t^k.$$

The product rule then states that the generating polynomial is multiplicative with respect to direct products. A direct consequence of the orthogonal invariance and the polarity rule is that the intrinsic volume sequence is symmetric for self-dual cones (i.e., cones such that $C = -C^\circ$). An important summary parameter is the expected value of the distribution associated to the intrinsic volumes, the *statistical dimension*, which coincides with the expected squared norm of the projection of a Gaussian vector on the cone,

$$\delta(C) = \sum_{k=0}^{d} k v_k(C) = \mathbb{E}\left[\|\boldsymbol{\Pi}_C(\boldsymbol{g})\|^2\right].$$

The statistical dimension reduces to the usual dimension for linear subspaces. The coincidence of the two expected values is a special case of the generalized Steiner formula 3.1, and is crucial in applications of the statistical dimension. More on the statistical dimension and its properties and applications can be found in [ALMT14, MT14, GNP14].

2.3. **Angles.** In the classical works on polyhedral cones, intrinsic volumes were viewed as polytope angles, see [FK09] for some perspective. Polyhedral cones arise as tangent or normal cones of polyhedra $K \subseteq \mathbb{R}^d$. Given such a polyhedron K and a face $F \subset K$, with $\boldsymbol{x}_0 \in \operatorname{relint}(K)$, the *tangent cone* $T_F K$ is defined as

$$T_F K = \bigcup_{\tau>0} \{\boldsymbol{v} \in \mathbb{R}^d : \boldsymbol{x}_0 + \tau \boldsymbol{v} \in K\}.$$

The *normal cone* to K at F is the polar of the tangent cone. To clarify the relations to the terminology used in this paper and to facilitate a translation of the results of some of the referenced papers, we provide the following list.

2.3.1. *Solid angle.* When speaking about the solid angle of a cone $C \subseteq \mathbb{R}^d$, sometimes denoted $\alpha(C)$, one usually assumes that C has nonempty interior, and one defines $\alpha(C)$ as the (Gaussian) volume of C; we extend this definition to also cover lower-dimensional cones, and define

$$\alpha(C) := v_C(C) = v_0(C^\circ).$$

The second expression is convenient because it avoids the fact that one has to take the volume of C relative to the linear hull of C.

2.3.2. *Internal/external angle.* The internal and external angle of a d-dimensional polyhedral set $K \subseteq \mathbb{R}^d$ at a face F are defined as the solid angle of the tangent and normal cone of K at F, respectively,

$$\beta(F,K) = \alpha(T_F K) = v_0(N_F K), \qquad \gamma(F,K) = \alpha(N_F K) = v_0(T_F K).$$

Note that we have $v_F(C) = \beta(\mathbf{0}, F)\gamma(F, C)$. Furthermore, conic polarity swaps between internal and external angles:

$$\beta(F,C) = \gamma(F^\diamond, C^\circ), \qquad \gamma(F,C) = \beta(F^\diamond, C^\circ),$$

where we use the notation $F^\diamond := N_F C$ for the face of C°, which is polar to the face F of C. This shows that any formula involving the internal and external angles of a cone C has a polar version in terms of the internal and external angles of C° where the roles of internal and external have been exchanged. (Some of the formulas in [McM75] are stated in this polar version.)

Remark 2.8. The Brianchon-Gram-Euler relation [PS67, Thm. (1)] of a convex polytope K translates in the above notation as

$$\sum_{F\in\mathcal{F}(K)} (-1)^{\dim F} \beta(F,K) = \sum_{F\in\mathcal{F}(K)} (-1)^{\dim F} v_0(N_F K) = 0.$$

Replacing the bounded polytope by an unbounded cone makes this relation invalid. However, there exists a closely related conic version, which is called Sommerville's Theorem [PS67, Thm. (37)]. This in turn underlies the Gauss-Bonnet relation, cf. Section 4.

2.3.3. *Grassmann angle.* The Grassmann angles of a cone C, which have been introduced and mostly been analyzed by Grünbaum [Grü68], are defined through the probability that a uniformly random linear subspace of a specific (co)dimension intersects the cone nontrivially. The kinematic/Crofton formulae express this probability in terms of the intrinsic volumes, cf. Section 5. More precisely, we have

$$\mathbb{P}\{C \cap L_k \neq \mathbf{0}\} = 2 \sum_{i \geq 1 \text{ odd}} v_{k+i}(C) =: 2h_{k+1}(C),$$

where $L_k \subseteq \mathbb{R}^d$ denotes a uniformly random linear subspace of codimension k. Notice that when considering the intrinsic volumes and the Grassmann angles as vectors, $(v_0, v_1, \dots, v_d)$ and $(h_0, h_1, \dots, h_d)$, then these are related through a nonsingular linear transformation. Hence, any formula in the intrinsic volumes of a cone has an equivalent form in terms of Grassmann angles and vice versa; in this paper we prefer the intrinsic volume versions.

2.4. **Some poset techniques.** In this section we recall some notions from the theory of partially ordered sets (posets) that we will need in Section 6. We only recall those properties that we will directly use, see [Sta12, Ch. 3] for more details and context.

A lattice is a poset with the property that any two elements have both a least upper bound and a greatest lower bound. We will only consider finite lattices; in particular, for these lattices the greatest and the least elements $\hat{1}, \hat{0}$ both exist. More precisely, we will consider the following two (types of) finite lattices.

Example 2.9 (Face lattice)**.** Let $C \subseteq \mathbb{R}^d$ be a polyhedral cone. Then the set of faces $\mathcal{F}(C)$ with partial order given by inclusion is a finite lattice. The elements $\hat{1}, \hat{0}$ are given by $\hat{1} = C$ and $\hat{0} = C \cap (-C)$.

Example 2.10 (Intersection lattice of hyperplane arrangement)**.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a set of (linear) hyperplanes $H_i \subset \mathbb{R}^d$, $i = 1, \dots, n$. The set of all intersections $\mathcal{L}(\mathcal{A}) = \big\{ \bigcap_{i \in I} H_i : I \subseteq \{1, \dots, n\} \big\}$, endowed with the partial order given by reverse inclusion, is called the intersection lattice of the hyperplane arrangement $\mathcal{A}$. This lattice has a disjoint decomposition into $\mathcal{L}_0(\mathcal{A}), \dots, \mathcal{L}_d(\mathcal{A})$, where $\mathcal{L}_j(\mathcal{A}) = \{L \in \mathcal{L}(\mathcal{A}) : \dim L = j\}$. The minimal and maximal elements are given by $\hat{0} = \mathbb{R}^d$ and $\hat{1} = \bigcap_{i=1}^n H_i$.

One can define the (real) incidence algebra of a (locally) finite poset $(P, \preceq)$ as the set of all functions $f \colon P \times P \to \mathbb{R}$, which besides having the usual vector space structure also possesses the multiplication

$$fg \colon P \times P \to \mathbb{R}, \qquad fg(x, y) = \sum_{x \preceq z \preceq y} f(x, z)\, g(z, y).$$

The identity element in this algebra is the Kronecker delta, $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ else. Another important element is the characteristic function of the partial order, $\zeta(x, y) = 1$ if $x \preceq y$ and $\zeta(x, y) = 0$ else. This function is invertible, and its inverse μ, called *Möbius function* on P, can be recursively defined by $\mu(x, y) = 0$ if $x \not\preceq y$, and

$$\mu(x, x) = 1, \qquad \mu(x, y) = -\sum_{x \preceq z \prec y} \mu(x, z) \quad \text{if } x \prec y. \tag{2.7}$$

The so-called Möbius inversion is the simple fact that for two functions f, g in the incidence algebra one has $f\zeta = g$ iff $f = g\mu$. Explicitly, this equivalence can be written out as follows:

$$\forall y \in P : g(y) = \sum_{x \preceq y} f(x) \quad \iff \quad \forall y \in P : f(y) = \sum_{x \preceq y} g(x)\mu(x, y). \tag{2.8}$$

The Möbius function of the face lattice from Example 2.9 is given by $\mu(F, G) = (-1)^{\dim G - \dim F}$. For a whole range of techniques for computing Möbius functions we refer to [Sta12, Ard14].

2.4.1. *Some elementary facts about hyperplane arrangements.* The last concept we need to introduce is that of a characteristic polynomial, which can be defined for any finite graded lattice; we only introduce the characteristic polynomial for hyperplane arrangements, as we will only use it in this context. We use the notation from Example 2.10. The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is defined as [Sta12, Sec. 3.11.2]

$$\chi_{\mathcal{A}}(t) = \sum_{L\in\mathcal{L}(\mathcal{A})} \mu(\mathbb{R}^d, L)t^{\dim L}.$$

More generally, we introduce the jth-level characteristic polynomial of $\mathcal{A}$ as follows,

$$\chi_{\mathcal{A},j}(t) = \sum_{\tilde{L}\in\mathcal{L}_j(\mathcal{A})} \sum_{L\in L(\mathcal{A})} \mu(\tilde{L}, L)t^{\dim L}, \tag{2.9}$$

so that $\chi_{\mathcal{A}} = \chi_{\mathcal{A},d}$, and we also introduce the bivariate polynomial[1]

$$X_{\mathcal{A}}(s,t) := \sum_{j=0}^{d} s^j \chi_{\mathcal{A},j}(t) = \sum_{\tilde{L},L\in\mathcal{L}(\mathcal{A})} \mu(\tilde{L}, L)s^{\dim \tilde{L}}t^{\dim L}. \tag{2.10}$$

The jth level characteristic polynomial can be written in terms of characteristic functions by considering restrictions of $\mathcal{A}$: If $L \subseteq \mathbb{R}^d$ is a linear subspace, then the arrangement $\mathcal{A}^L = \{H\cap L : H\in\mathcal{A}, L\not\subseteq H\}$ is a hyperplane arrangement relative to L. It is easily seen that we obtain

$$\chi_{\mathcal{A},j}(t) = \sum_{L\in\mathcal{L}_j(\mathcal{A})} \chi_{\mathcal{A}^L}(t). \tag{2.11}$$

The Möbius function of the intersection lattice alternates in sign [Sta12, Prop.3.10.1], and so do the coefficients of the (jth-level) characteristic polynomial. Note that $\chi_{\mathcal{A},j}(t)$ (is either zero or) has degree j and the leading coefficient is given by $|\mathcal{L}_j(\mathcal{A})| =: \ell_j(\mathcal{A})$. For future reference we also note that in the cases $j = 0, 1$ we have

$$\chi_{\mathcal{A},0}(t) = \ell_0(\mathcal{A}), \qquad \chi_{\mathcal{A},1}(t) = \ell_1(\mathcal{A})(t - \ell_0(\mathcal{A})). \tag{2.12}$$

The complement of the hyperplanes of an arrangement $\mathcal{A}$, $\mathbb{R}^d\setminus\bigcup_{H\in\mathcal{A}} H$, decomposes into open convex cones. We denote by $\mathcal{R}(\mathcal{A})$ the set of polyhedral cones given by the closures of these regions, and we denote $r(\mathcal{A}) := |\mathcal{R}(\mathcal{A})|$. More generally, we define

$$\mathcal{R}_j(\mathcal{A}) = \bigcup_{C\in\mathcal{R}(\mathcal{A})} \mathcal{F}_j(C), \qquad r_j(\mathcal{A}) = |\mathcal{R}_j(\mathcal{A})|, \tag{2.13}$$

so that $\mathcal{R}(\mathcal{A}) = \mathcal{R}_d(\mathcal{A})$ and $r(\mathcal{A}) = r_d(\mathcal{A})$. The following theorem by Zaslavsky [Zas75] lies at the heart of the result by Klivans and Swartz [KS11] that we will present in Section 6.

Theorem 2.11 (Zaslavsky)**.** *Let $\mathcal{A}$ be an arrangement of linear hyperplanes in $\mathbb{R}^d$. Then*

$$r_j(\mathcal{A}) = (-1)^j\,\chi_{\mathcal{A},j}(-1).$$

Note that since the coefficients of the characteristic polynomial alternate in sign, the number of j-dimensional regions, $r_j(\mathcal{A})$, is given by the sum of the absolute values of the coefficients of the jth-level characteristic polynomial.

[1]This bivariate polynomial (or simple transformations thereof) is also known as Möbius polynomial [Zas75] or Whitney polynomial [Ath96a, Ath96b]; it should not be confused with the coboundary/Tutte polynomial [Jur12].

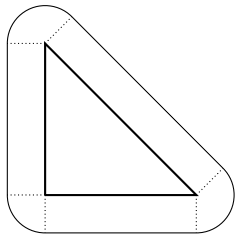

FIGURE 2. $\mathrm{vol}(K + \varepsilon B^2) =$ area $+$ circumference $\cdot\, \varepsilon + \pi \cdot \varepsilon^2$

3. PROBABILISTIC TUBES

A classic result in integral geometry is the Steiner Formula: the d-dimensional measure of the ε-neighbourhood of a convex body $K \subset \mathbb{R}^d$ (compact convex) can be expressed as a polynomial in ε of degree at most d, with the *intrinsic volumes* as coefficients:

$$\mathrm{vol}(K + \varepsilon B^d) = \sum_{i=0}^{d} V_i(C)\omega_{d-i}\varepsilon^{d-i}, \tag{3.1}$$

where B^d denotes the unit ball, $\omega_{d-i} = \mathrm{vol}(B^{d-i}) = \frac{2\pi^{(d-i)/2}}{\Gamma((d-i)/2+1)}$, and the $V_i(K)$ are the Euclidean intrinsic volumes (see, e.g., [KR97, Theorem 9.2.3]). For example, in the two-dimensional case, we have the situation of Figure 2.

In order to state an analogous result for convex cones or spherically convex sets, we have to agree on a notion of distance. A natural choice here is the angular distance. Define the (capped) angle between a nonzero vector $\boldsymbol{x}$ and C as

$$\sphericalangle(C, \boldsymbol{x}) = \arccos\left(\frac{\langle \boldsymbol{x}, \boldsymbol{\Pi}_C(\boldsymbol{x})\rangle}{\|\boldsymbol{x}\|^2}\right).$$

Note that with this definition, $\sphericalangle(C, \boldsymbol{x}) \leq \pi/2$ and is equal to $\pi/2$ iff $\boldsymbol{x} \in C^\circ$. Note also that for $\boldsymbol{x}$ with $\|\boldsymbol{x}\| = 1$, we have $\sphericalangle(C, \boldsymbol{x}) \leq \alpha$ if and only if $\|\Pi_C(\boldsymbol{x})\|^2 \geq \cos^2(\alpha)$. Using this notion of distance one obtains a formula similar to the Euclidean Steiner formula (3.1), which is usually called *spherical/conic Steiner formula* (see below for an explicit formula).

It turns out that, when working with cones rather than spherically convex sets, it is convenient to work with the squared length of the projection on C rather than with the angle. Moreover, it turns out quite useful to also consider the squared length of the projection on the polar cone C°. The following general Steiner formula in the conic setting is due to McCoy and Tropp [MT14, Theorem 3.1]; its formulation in probabilistic terms, as suggested by Goldstein, Nourdin and Peccati [GNP14], is remarkably elegant. We sketch their proof (in the polyhedral case) below.

Theorem 3.1. *Let $C \subseteq \mathbb{R}^d$ convex polyhedral cone, let $\boldsymbol{g} \in \mathbb{R}^d$ Gaussian, and let the discrete random variable V on $\{0, 1, \dots, d\}$ be given by $\mathbb{P}\{V = k\} = v_k(C)$. Then*

$$\big(\|\Pi_C(\boldsymbol{g})\|^2, \|\Pi_{C^\circ}(\boldsymbol{g})\|^2\big) \stackrel{d}{=} \big(X_V, Y_{d-V}\big) \tag{3.2}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and X_k, Y_k are independent χ^2-distributed random variables with k degrees of freedom.

A geometric interpretation of this form of the conic Steiner formula is readily obtained by considering *moments* of the two sides in (3.2). Indeed, the expectation of $f\big(\|\Pi_C(\boldsymbol{g})\|^2, \|\Pi_{C^\circ}(\boldsymbol{g})\|^2\big)$ equals the Gaussian volume of the angular tube around C of radius $\alpha \leq \frac{\pi}{2}$, if one sets $f(x, y) = 1$ if $x/(x + y) \geq \cos\alpha$, and $f(x, y) = 0$ otherwise. For this choice of f the expectation of $f\big(X_V, Y_{d-V}\big)$ becomes a finite sum $\sum_{k=0}^{d} v_k(C)\mathbb{P}\{\boldsymbol{g} \in T_\alpha(L_k)\}$, where $T_\alpha(L_k)$ denotes the angular tube of radius α around a

k-dimensional linear subspace. So by taking a suitable moment of (3.2) we obtain the usual conic Steiner formula.

Proof sketch of Theorem 3.1. In order to show the claimed equality in distribution (3.2) it suffices to show that the moments coincide. Let $f\colon \mathbb{R}_+^2 \to \mathbb{R}$ be a bounded, continuous function. In view of the decomposition (2.1) we can express the expectation of $f\big(\|\Pi_C(\boldsymbol{g})\|^2, \|\Pi_{C^\circ}(\boldsymbol{g})\|^2\big)$ as

$$\mathbb{E}\left[f\big(\|\Pi_C(\boldsymbol{g})\|^2, \|\Pi_{C^\circ}(\boldsymbol{g})\|^2\big)\right] = \sum_{k=0}^{d} \sum_{F\in\mathcal{F}_k(C)} \mathbb{E}[f(\|\boldsymbol{\Pi}_C(\boldsymbol{g})\|, \|\boldsymbol{\Pi}_{C^\circ}(\boldsymbol{g})\|)\, 1_{\{\boldsymbol{\Pi}_C(\boldsymbol{g})\in\mathrm{relint}(F)\}}].$$

Notice now that for $g \in (\mathrm{relint}\, F) + N_F C$ we have $\boldsymbol{\Pi}_C(\boldsymbol{g}) = \boldsymbol{\Pi}_{\mathrm{lin}(F)}(\boldsymbol{g})$ and $\boldsymbol{\Pi}_{C^\circ}(\boldsymbol{g}) = \boldsymbol{\Pi}_{\mathrm{lin}(N_F C)}(g)$. This implies

$$\begin{aligned}&\mathbb{E}\left[f\big(\|\boldsymbol{\Pi}_C(\boldsymbol{g})\|, \|\boldsymbol{\Pi}_{C^\circ}(\boldsymbol{g})\|\big)\, 1_{\{\boldsymbol{\Pi}_C(\boldsymbol{g})\in\mathrm{relint}(F)\}}\right]\\ &= \mathbb{E}\left[f\big(\|\boldsymbol{\Pi}_{\mathrm{lin}(F)}(\boldsymbol{g})\|, \|\boldsymbol{\Pi}_{\mathrm{lin}(N_F C)}(\boldsymbol{g})\|\big)\, 1_{\{\boldsymbol{\Pi}(\boldsymbol{g})\in\mathrm{relint}(F)\}}\right]\end{aligned}$$

Using spherical coordinates and the orthogonal invariance of Gaussian vectors, one can deduce that the above expectation equals

$$\begin{aligned}&\mathbb{E}\left[f\big(\|\boldsymbol{\Pi}_{\mathrm{lin}(F)}(\boldsymbol{g})\|, \|\boldsymbol{\Pi}_{\mathrm{lin}(N_F C)}(\boldsymbol{g})\|\big)\, 1_{\{\boldsymbol{\Pi}_C(\boldsymbol{g})\in\mathrm{relint}(F)\}}\right]\\ &= \mathbb{E}\left[f\big(\|\boldsymbol{\Pi}_{L_k}(\boldsymbol{g})\|, \|\boldsymbol{\Pi}_{L_k^\perp}(\boldsymbol{g})\|\big)\right] \mathbb{P}\{\boldsymbol{\Pi}_C(\boldsymbol{g}) \in \mathrm{relint}(F)\} = \mathbb{E}[f(X_k, Y_{d-k})]\, v_F(C),\end{aligned}$$

where L_k denotes an arbitrary k-dimensional linear subspace. Summing up the terms gives rise to the claimed coincidence of moments, which shows equality of the distributions. □

A useful consequence of the general Steiner formula is that the moment generating functions of the discrete random variable V from Theorem 3.1 and the continuous random variable $\|\Pi_C(\boldsymbol{g})\|^2$ coincide up to reparametrization:

$$\mathbb{E}[e^{tV}] = \mathbb{E}[e^{s\|\Pi_C(\boldsymbol{g})\|^2}], \qquad s = \tfrac{1-e^{-2t}}{2}$$

which directly follows from (3.2) by the well-known formula for the moment generating function of χ^2-distributed random variables, $\mathbb{E}[e^{sX_k}] = (1-2s)^{-k/2}$. This result is from [MT14], where it is used to derive a concentration result for the random variable V, and also underlies the argumentation in [GNP14], where a central limit theorem for V is derived.

4. Gauss-Bonnet and the face lattice

The Gauss-Bonnet Theorem is a celebrated result in differential geometry connecting curvature with the Euler characteristic. In the setting of polyhedral cones, this theorem asserts that the alternating sum of the intrinsic volumes equals the alternating sum of the f-vector,

$$\sum_{k=0}^{d} (-1)^k v_k(C) = \sum_{k=0}^{d} (-1)^k f_k(C).$$

The main goal of this section is to show the connections between the Gauss-Bonnet relation, a result by Sommerville [Som27], which can be seen as a conic version of the Brianchon-Euler-Gram relation for polytopes [Grü03, 14.1], and a result by Grünbaum [Grü68, Thm. 2.8]. More precisely, we will provide an elementary proof of the result by Sommerville, which is basically an application of Farkas' Lemma, and show how the other relations are easily deduced by this. Historically, Grünbaum's result, which was formulated in terms of Grassmann angles, cp. Section 2.3, relied on Gauss-Bonnet, but we will see that it is more natural to argue the other way round.

Theorem 4.1 (Sommerville)**.** *For any polyhedral cone $C \subseteq \mathbb{R}^d$,*

$$v_0(C) = \sum_{F\in\mathcal{F}(C)} (-1)^{\dim F} v_0(F). \tag{4.1}$$

Proof. Both sides in (4.1) are zero if C contains a nonzero linear subspace. So we assume in the following that C is pointed, $C \cap (-C) = \mathbf{0}$. Let $\boldsymbol{g}$ be a random Gaussian vector and $H = \boldsymbol{g}^\perp$ the orthogonal complement, which is almost surely a hyperplane. By Farkas' Lemma 2.3,

$$\mathbb{P}\{C \cap H = \mathbf{0}\} = \mathbb{P}\{\boldsymbol{g} \in C^\circ \cup -C^\circ\} = 2\mathbb{P}\{\boldsymbol{g} \in C^\circ\} = 2v_0(C). \tag{4.2}$$

Note that with probability 1, the intersection $C \cap H$ is either $\mathbf{0}$ or has dimension $\dim C - 1$. Setting $\overline{\chi} = \sum_{i=0}^{d-1} (-1)^i f_i(C \cap H)$, the Euler relation (2.1) implies $\overline{\chi} = 0$ if $C \cap H \neq \mathbf{0}$ and $\overline{\chi} = 1$ if $C \cap H = \mathbf{0}$. Using (4.2) we get the expected value

$$\mathbb{E}\left[\overline{\chi}\right] = \mathbb{E}\left[\overline{\chi} \mid C \cap H \neq \mathbf{0}\right](1 - 2v_0(C)) + \mathbb{E}\left[\overline{\chi} \mid C \cap H = \mathbf{0}\right] 2v_0(C) = 2v_0(C). \tag{4.3}$$

On the other hand, for $0 < i < d$ and using (4.2),

$$\mathbb{E}\left[f_i(H \cap C)\right] = \sum_{F \in \mathcal{F}_{i+1}(C)} \mathbb{P}\{F \cap H \neq \mathbf{0}\} = f_{i+1}(C) - 2 \sum_{F \in \mathcal{F}_{i+1}(C)} v_0(F),$$

where in the first step we used the fact that almost surely every i-dimensional face of $C \cap H$ is of the form $F \cap H$, with $F \in \mathcal{F}_{i+1}(C)$, and for every $F \in \mathcal{F}_{i+1}(C)$ the intersection $F \cap H$ is either an i-dimensional face of $C \cap H$ or $\mathbf{0}$. Alternating the sum and using linearity of expectation,

$$\begin{aligned}
\mathbb{E}[\overline{\chi}] &= 1 + \sum_{i=1}^{d-1} (-1)^i \,\mathbb{E}[f_i(C \cap H)] = 1 + \sum_{i=1}^{d-1} (-1)^i \Bigg(f_{i+1}(C) - 2 \sum_{F \in \mathcal{F}_{i+1}(C)} v_0(F) \Bigg) \\
&= 1 - \sum_{i=2}^{d} (-1)^i f_i(C) + 2 \sum_{i=2}^{d} \sum_{F \in \mathcal{F}_i(C)} (-1)^{\dim F} v_0(F) \\
&= 1 + f_0 - f_1 - \sum_{i=0}^{d} (-1)^i f_i(C) + 2 \Bigg(- v_0(\mathbf{0}) + \sum_{F \in \mathcal{F}_1(C)} v_0(F) + \sum_{F \in \mathcal{F}(C)} (-1)^{\dim F} v_0(F) \Bigg) \\
&= 2 \sum_{F \in \mathcal{F}(C)} v_0(F)(-1)^{\dim F},
\end{aligned}$$

where in the final step we used the Euler relation (2.1), the fact that $f_1(C) = 2 \sum_{\dim F = 1} v_0(F)$ (because each F° is a halfspace), and $f_0 = v_0(\mathbf{0}) = 1$. Combining this with (4.3) yields the claim. □

The following theorem is a simple generalization of Sommerville's Theorem.

Theorem 4.2. *Let $C \subseteq \mathbb{R}^d$ be a polyhedral cone. Then for any face $G \subseteq C$,*

$$(-1)^{\dim G} v_G(C) = \sum_{F \in \mathcal{F}(C)} (-1)^{\dim F} v_G(F). \tag{4.4}$$

Proof. If $G = \mathbf{0}$ then we obtain Sommerville's Theorem 4.1. Let $G \neq \mathbf{0}$ and let C/G denote the orthogonal projection of C onto the orthogonal complement of the linear span of G. It follows from the Gaussian distribution that $v_G(C) = v_G(G)\, v_0(C/G)$, which can be expressed as

$$v_G(G)\, v_0(C/G) = v_G(G) \sum_{F/G \in \mathcal{F}(C/G)} (-1)^{\dim F/G} v_0(F/G) = \sum_{F \in \mathcal{F}(C)} (-1)^{\dim F - \dim G} v_G(F),$$

where in the first step we used Sommerville's Theorem, and in the second step we used that $v_G(F) = 0$ if G is not a face of F, and $\dim F/G = \dim F - \dim G$. This shows the claim. □

The following corollary is [Grü68, Thm. 2.8], cf. Section 2.3.3.

Corollary 4.3. *Let $C \subseteq \mathbb{R}^d$ be a closed convex cone. Then*

$$(-1)^k v_k(C) = \sum_{F \in \mathcal{F}(C)} (-1)^{\dim F} v_k(F). \tag{4.5}$$

Proof. Follows by summing in (4.4) over all k-dimensional faces and noting that for every face F of C we have $\mathcal{F}_k(F) \subseteq \mathcal{F}_k(C)$. $\square$

Corollary 4.4 (Gauss-Bonnet). *For a polyhedral cone C,*

$$\sum_{i=0}^{d}(-1)^i v_i(C) = \sum_{i=0}^{d}(-1)^i f_i(C) = \begin{cases} (-1)^{\dim C} & \text{if } C \text{ is a linear subspace,} \\ 0 & \text{else.} \end{cases} \tag{4.6}$$

Proof. Summing the terms in (4.5) over k and using $\sum_{k=0}^{d} v_k(C) = 1$ yields

$$\sum_{k=0}^{d}(-1)^k v_k(C) = \sum_{k=0}^{d}\sum_{F\in\mathcal{F}(C)}(-1)^{\dim F} v_k(F) = \sum_{F\in\mathcal{F}(C)}(-1)^{\dim F}\sum_{k=0}^{d} v_k(F) = \sum_{k=0}^{d}(-1)^k f_k(C).$$

The rest follows from the Euler relation (2.1). $\square$

If C is not a linear subspace, then the Gauss-Bonnet relation can be interpreted as saying that the random variable V on $\{0,1,\dots,d\}$ given by $\mathbb{P}\{V=k\} = v_k(C)$, actually decomposes into two random variables V^0, V^1 on $\{0,2,4,\dots,2\lfloor\frac{d}{2}\rfloor\}$ and $\{1,3,5,\dots,2\lfloor\frac{d-1}{2}\rfloor+1\}$, respectively, such that

$$\mathbb{P}\{V^0 = k\} = 2v_k(C) \quad \text{if } k \text{ even}, \qquad \mathbb{P}\{V^1 = k\} = 2v_k(C) \quad \text{if } k \text{ odd}.$$

In fact, the same argument that gives the general Steiner formula (3.2) also shows that

$$\big(\|\Pi_C(\boldsymbol{g}^0)\|^2, \|\Pi_{C^\circ}(\boldsymbol{g}^0)\|^2\big) \stackrel{d}{=} \big(X_{V^0}, Y_{d-V^0}\big), \qquad \big(\|\Pi_C(\boldsymbol{g}^1)\|^2, \|\Pi_{C^\circ}(\boldsymbol{g}^1)\|^2\big) \stackrel{d}{=} \big(X_{V^1}, Y_{d-V^1}\big),$$

where $\boldsymbol{g}^0$ and $\boldsymbol{g}^1$ denote Gaussian vectors conditioned on their projection on C falling in an even- or odd-dimensional face, respectively, and X_k, Y_k are independent χ^2-distributed random variables with k degrees of freedom. We can paraphrase (4.5) in terms of the moments of these random variables.

Corollary 4.5. *Let $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ be a bounded, continuous function, and for $C \subseteq \mathbb{R}^d$ a polyhedral cone, which is not a linear subspace, let $\varphi_f(C), \varphi_f^0(C), \varphi_f^1(C)$ denote the moments*

$$\varphi_f(C) = \mathbb{E}\left[f\big(\|\Pi_C(\boldsymbol{g})\|^2, \|\Pi_{C^\circ}(\boldsymbol{g})\|^2\big)\right], \qquad \varphi_f^{0/1}(C) = \mathbb{E}\left[f\big(\|\Pi_C(\boldsymbol{g}^{0/1})\|^2, \|\Pi_{C^\circ}(\boldsymbol{g}^{0/1})\|^2\big)\right].$$

Then we have

$$\frac{\varphi_f^0(C) - \varphi_f^1(C)}{2} = \sum_{F\in\mathcal{F}(C)}(-1)^{\dim F}\varphi_f(F), \qquad \varphi_f(C) = \sum_{F\in\mathcal{F}(C)}(-1)^{\dim F}\frac{\varphi_f^0(F) - \varphi_f^1(F)}{2}.$$

Proof. The first equation is obtained by invoking the general Steiner formula and applying (4.5):

$$\begin{aligned}\frac{\varphi_f^0(C) - \varphi_f^1(C)}{2} &= \sum_{k=0}^{d}(-1)^k\,\mathbb{E}\left[f\big(X_k, Y_{d-k}\big)\right] v_k(C) \\ &= \sum_{k=0}^{d}\mathbb{E}\left[f\big(X_k, Y_{d-k}\big)\right]\sum_{F\in\mathcal{F}(C)}(-1)^{\dim F} v_k(F) = \sum_{F\in\mathcal{F}(C)}(-1)^{\dim F}\varphi_f(F).\end{aligned}$$

The second equation is obtained by using Möbius inversion (2.8) and noting that the Möbius function of the face lattice is $\mu(F,C) = (-1)^{\dim C - \dim F}$. $\square$

We list a few more corollaries, the usefulness of which may yet need to be established. The proofs are variations of the proof of Corollary 4.4.

Corollary 4.6. *For the statistical dimension $\delta(C)$ we obtain*

$$\sum_{k=0}^{d}(-1)^k k\cdot v_k(C) = \sum_{F\in\mathcal{F}(C)}(-1)^{\dim F}\delta(F).$$

In particular, if $\dim C$ *is even, then*

$$2\sum_{k\ \mathit{even}} k\, v_k(C) = \sum_{F\subset C} (-1)^{\dim F}\delta(F),$$

and if $\dim C$ *is odd, then*

$$2\sum_{k\ \mathit{odd}} k\, v_k(C) = -\sum_{F\subset C} (-1)^{\dim F}\delta(F).$$

Corollary 4.7. *Let* V_C *be the random variable on* $\{0,1,\dots,d\}$ *defined by* $\mathbb{P}\{V_C = k\} = v_k(C)$*. The alternating sum of the exponential generating function satisfies*

$$\mathbb{E}\left[(-1)^{V_C} e^{tV_C}\right] = \sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}\,\mathbb{E}\left[e^{tV_F}\right].$$

Remark 4.8. The Gauss-Bonnet relation can also be written out as $\sum_{F\in\mathcal{F}(C)}(-1)^{\dim F} v_F(C) = 0$, if C is not a linear subspace. If $G \in \mathcal{F}(C)$ is a proper face, i.e., $G \neq C$, then one can apply Gauss-Bonnet to the projected cone C/G, as in the deduction of Theorem 4.2 from Sommerville's Theorem 4.1, to obtain

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F} v_{F/G}(C/G) = 0.$$

Rewriting this formula in terms of internal/external angles, and extending this to include also the case $G = C$, one obtains

$$\sum_{G\leq F\leq C} (-1)^{\dim F-\dim G}\beta(G,F)\,\gamma(F,C) = \begin{cases} 1 & \text{if } F = G\\ 0 & \text{else.}\end{cases}$$

In [McM75] McMullen observed that this relation means that the internal and external angle functions (one of them multiplied by the Möbius function) are mutual inverses in the incidence algebra of the face lattice, cf. Section 2.4. More precisely, the Gauss-Bonnet relation only shows that one of them is the left-inverse of the other (and of course the other is a right-inverse of the first), but since left-inverse, right-inverse, or two-sided inverse are equivalent in the incidence algebra [Sta12, Prop. 3.6.3] one obtains the following additional relation "for free":

$$\sum_{G\leq F\leq C} (-1)^{\dim C-\dim F}\gamma(G,F)\,\beta(F,C) = \begin{cases} 1 & \text{if } F = G\\ 0 & \text{else.}\end{cases}$$

This is [McM75, Thm. 3], but while being an interesting observation, this additional relation has not found any real use yet.

The relation (4.2) used in the proof of Sommerville's Theorem 4.1 is a special case of the principal kinematic formula, to be derived in more detail next.

5. Elementary kinematics for polyhedral cones

The principal kinematic formulae of integral geometry relate the intrinsic volumes, or certain measures that localizes these quantities, of the intersection of two or more randomly moved geometric objects to those of the individual objects. This section presents a self-contained derivation of the principal kinematic formula in the setting of two polyhedral cones. The focus is on simplicity rather than generality. In what follows, by the uniform distribution on $O(d)$ we mean the unique normalized Haar measure.

Theorem 5.1 (Kinematic Formula)**.** *Let* $C, D \subseteq \mathbb{R}^d$ *be polyhedral cones and let* $O(d)$ *denote the orthogonal group. Then, for* $\boldsymbol{Q} \in O(d)$ *uniformly at random,* $k > 0$*,*

$$\mathbb{E}[v_k(C\cap \boldsymbol{Q}D)] = v_{k+d}(C\times D), \qquad \mathbb{E}[v_0(C\cap \boldsymbol{Q}D)] = \sum_{j=0}^{d} v_j(C\times D). \tag{5.1}$$

If $D = L$ is a linear subspace of dimension $d - m$, then

$$\mathbb{E}[v_k(C \cap \boldsymbol{Q}L)] = v_{k+m}(C), \qquad \mathbb{E}[v_0(C \cap \boldsymbol{Q}L)] = \sum_{j=0}^{m} v_j(C). \tag{5.2}$$

Recall that the intrinsic volumes of $C \times D$ are obtained by convoluting the intrinsic volumes of C and D, cf. Section 2.2. The second equation in (5.1) follows from the first from the equation $\sum_k v_k(C) = 1$, and statement (5.2) follows easily from (5.1) by applying the product rule (2.6). Note also that using polarity on both sides of (5.1) we obtain the polar kinematic formulas

$$\mathbb{E}[v_{d-k}(C + \boldsymbol{Q}D)] = v_{d-k}(C \times D), \qquad \mathbb{E}[v_d(C + \boldsymbol{Q}D)] = \sum_{j=0}^{d} v_{d+j}(C \times D), \tag{5.3}$$

and similarly for (5.2). Combining Theorem 5.1 with the Gauss-Bonnet relation (4.6) yields the so-called *Crofton formulas*, which we formulate in the following corollary.

Corollary 5.2. *Let $C, D \subseteq \mathbb{R}^d$ be polyhedral cones such that not both of C and D are linear subspaces. Then, for $\boldsymbol{Q} \in O(d)$ uniformly at random,*

$$\mathbb{P}\{C \cap \boldsymbol{Q}D \neq \boldsymbol{0}\} = 2 \sum_{i \geq 1 \text{ odd}} v_{d+i}(C \times D).$$

In particular, if $D = L$ is a linear subspace of dimension $d - m$,

$$\mathbb{P}\{C \cap \boldsymbol{Q}L \neq \boldsymbol{0}\} = 2 \sum_{i \geq 1 \text{ odd}} v_{m+i}(C).$$

For the derivation of this corollary we need the following simple genericity lemma, for which we also provide a proof for completeness.

Lemma 5.3. *Let $C, D \subseteq \mathbb{R}^d$ polyhedral cones such that not both of C and D are linear subspaces. Then the intersection $C \cap \boldsymbol{Q}D$, where $\boldsymbol{Q} \in O(d)$ uniformly at random, is almost surely either* **0** *or not a linear subspace.*

Proof. Let $L_1 := C \cap (-C)$ and $L_2 := D \cap (-D)$ denote the lineality spaces of C and D and, assuming that C is not a linear subspace, let $\boldsymbol{x}_0 \in C \setminus L_1$ and define $\tilde{L}_1 := L_1 + \mathbb{R}\boldsymbol{x}_0$. The lineality space of the intersection $C \cap \boldsymbol{Q}D$ is given by $L_1 \cap \boldsymbol{Q}L_2 =: L_0$, and it is easily seen, for example by using Gaussian matrices, that the dimension of L_0 is almost surely $\max\{\dim L_1 + \dim L_2 - d, 0\}$. So assuming that L_0 is nonzero, and denoting $\tilde{L}_0 := \tilde{L}_1 \cap \boldsymbol{Q}L_2$, we have $\dim \tilde{L}_0 = 1 + \dim L_0$ (almost surely). Since $C \cap \tilde{L}_1 = L_1 + \mathbb{R}_+\boldsymbol{x}_0$, and $\tilde{L}_0$ does not lie entirely in L_1, the intersection $C \cap \tilde{L}_0$ contains a point that is not in $L_1 \cap L_2 = L_0$. This shows $C \cap \boldsymbol{Q}D \neq L_0$, and thus $C \cap \boldsymbol{Q}D$ is not a linear subspace. ☐

Proof of Corollary 5.2. Denoting $\chi(C) := \sum_{i=0}^{d} (-1)^i v_i(C)$, the Gauss-Bonnet relation (4.6) says that $\chi(C) = 0$ if C is not a linear subspace, and $\chi(\boldsymbol{0}) = 1$. By Lemma 5.3 we see that χ is almost surely the indicator function for the event that C and D only intersect at the origin. We can therefore conclude,

$$\begin{aligned}\mathbb{P}\{C \cap QD = \boldsymbol{0}\} &= \mathbb{E}[\chi(C \cap QD)] = \mathbb{E}[\underbrace{v_0(C \cap QD)}_{=1-\sum_{i=1}^{d} v_i(C \cap \boldsymbol{Q}D)}] + \sum_{i=1}^{d} (-1)^i\, \mathbb{E}[v_i(C \cap QD)] \\ &= 1 - 2 \sum_{i \geq 1 \text{ odd}} \mathbb{E}[v_i(C \cap QD)] \overset{(5.1)}{=} 1 - 2 \sum_{i \geq 1 \text{ odd}} v_{d+i}(C \times D).\end{aligned}$$

The second claim follows by replacing D with L. ☐

Remark 5.4. There are essentially three different strategies to derive kinematic formulas (in the conic setting, but also in other settings):

(1) Use a characterisation theorem for the intrinsic volumes (or a suitable localization thereof) that shows that certain types of functions in a cone must be linear combinations of the intrinsic volumes. To apply this strategy, show that the expectation of interest, considered as function in the cones, satisfies these properties, and then compute the constants by considering specific cones. This approach is common in integral geometry [SW08, KR97].
(2) Assume that the boundary of the cone (without the origin) is a smooth hypersurface; then argue over the curvature of the intersection of the boundaries. For a general version of this approach, with references to related work, see [How93].
(3) Assume that the cones are polyhedral; then argue over the face structure of the cones.

Each of these approaches has its benefits and drawbacks: The first strategy requires, especially in the conic setting, where one needs to argue over localisations of the intrinsic volumes, substantial technical preparations, and the proofs, though conceptually simple, are indirect and maybe not too informative. The second strategy easily allows to go beyond the convex case, but the proofs require a fair amount of differential geometric background. The third strategy, which is what we will employ here, might be considered the weakest proof strategy. But at the same time one may argue that it is also the most illuminating strategy, as the technical requirements are minimal. The main idea is a "double counting" argument; to illustrate this, we first consider an analogous situation with finite sets.

Proposition 5.5. *Let Ω be a finite set and G be a finite group acting transitively on Ω. Let $M, N \subseteq \Omega$ be subsets. Then for uniformly random $\gamma \in G$,*

$$\mathbb{E}_{\gamma\in G}\,|M \cap \gamma N| = \frac{|M||N|}{|\Omega|}. \tag{5.4}$$

Proof. Taking $\xi \in \Omega$ uniformly at random, we obtain the cardinality of M as $|\Omega|\cdot\mathbb{P}\{\xi \in M\}$. Introduce the indicator function $1_M(\xi)$ for the event $\xi \in M$ and note that $1_{\gamma N}(x) = 1_N(\gamma^{-1}x)$ and $\mathbb{E}_{\gamma\in G}[1_N(\gamma^{-1}x)] = |N|/|\Omega|$ for any $x \in \Omega$. It follows that the random variables $1_M(\xi), 1_{\gamma N}(\xi)$ are uncorrelated:

$$\begin{aligned}\mathbb{E}_{\gamma\in G}\,|M\cap\gamma N| &= |\Omega|\cdot\mathbb{E}_{\gamma\in G}[\mathbb{E}_{\xi\in\Omega}[1_M(\xi)\,1_{\gamma N}(\xi)]] = |\Omega|\cdot\mathbb{E}_{\xi\in\Omega}[1_M(\xi)\;\mathbb{E}_{\gamma\in G}[1_N(\gamma^{-1}x)]]\\ &= \mathbb{E}_{\xi\in\Omega}[1_M(\xi)]\cdot|N| = \frac{|M||N|}{|\Omega|}.\end{aligned}$$ □

The proof of the kinematic formula in the case $k = d$ is almost a carbon copy of the above proof, which is why we include it here although this case is also covered below by the proof for the case $\dim C + \dim D = k + d$.

Proof of Theorem 5.1 for $k = d$. We need to show that

$$\mathbb{E}_{\boldsymbol{Q}\in O(d)}[v_d(C \cap \boldsymbol{Q}D)] = v_d(C)\,v_d(D).$$

Note that we can write $v_d(C)$ as the probability $\mathbb{P}\{\boldsymbol{g} \in C\}$, where $\boldsymbol{g} \sim \mathcal{N}(\mathbb{R}^d)$ denotes a Gaussian vector. We introduce the indicator function $1_C(\boldsymbol{x})$ for the event $\boldsymbol{x} \in C$ and note that $1_{\boldsymbol{Q}D}(\boldsymbol{x}) = 1_D(\boldsymbol{Q}^T\boldsymbol{x})$, as well as $\mathbb{E}_{\boldsymbol{Q}\in O(d)}[1_D(\boldsymbol{Q}^T\boldsymbol{x})] = v_d(D)$ for any $\boldsymbol{x} \in \mathbb{R}^d \setminus \boldsymbol{0}$. We thus obtain

$$\begin{aligned}\mathbb{E}_{\boldsymbol{Q}\in O(d)}[v_d(C\cap\boldsymbol{Q}D)] &= \mathbb{E}_{\boldsymbol{Q}\in O(d)}[\mathbb{E}_{\boldsymbol{g}\sim\mathcal{N}(\mathbb{R}^d)}[1_C(\boldsymbol{g})\,1_{\boldsymbol{Q}D}(\boldsymbol{g})]]\\ &= \mathbb{E}_{\boldsymbol{g}\sim\mathcal{N}(\mathbb{R}^d)}[1_C(\boldsymbol{g})\;\mathbb{E}_{\boldsymbol{Q}\in O(d)}[1_D(\boldsymbol{Q}^T\boldsymbol{g})]]\\ &= \mathbb{E}_{\boldsymbol{g}\sim\mathcal{N}(\mathbb{R}^d)}[1_C(\boldsymbol{g})]\,v_d(D) = v_d(C)\,v_d(D).\end{aligned}$$ □

Next, we prove Theorem 5.1 in the case $\dim C + \dim D = k + d$, which is the main step in the overall proof. The argumentation is similar to the case $k = d$, but with some important (and subtle) changes.

Proof of Theorem 5.1 for $\dim C + \dim D = k + d$. Let $j := \dim C$ and $\ell := \dim D = k + d - j$. Then $v_i(C) = 0$ for $i > j$ and similarly for D, so that

$$v_{k+d}(C \times D) = \sum_i v_i(C)\,v_{k+d-i}(D) = v_j(C)\,v_\ell(D).$$

We thus need to show that

$$\mathbb{E}_{\boldsymbol{Q}\in O(d)}[v_k(C\cap \boldsymbol{Q}D)] = v_j(C)\, v_\ell(D).$$

Introduce the notation $L_C := \operatorname{lin}(C)$, $L_D := \operatorname{lin}(D)$, and define for linear subspaces L_0, L with $\dim L_0 \geq \dim L = k$,

$$O(L_0, L) := \{\varphi\colon L' \to L \text{ isometry} : L' \in \operatorname{Gr}_k(L_0)\},$$

where $\operatorname{Gr}_k(L_0)$ denotes the set of all k-dimensional linear subspaces of L_0. Then the random cone $C \cap \boldsymbol{Q}D$ with $\boldsymbol{Q} \in O(d)$ uniformly at random has the same distribution as the random cone $C \cap \boldsymbol{Q}_L D$ where $L \in \operatorname{Gr}_k(L_C)$ and $\boldsymbol{Q} \in O(L_D, L)$ both uniformly at random. We thus obtain, similar to the case $k = d$,

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{Q}\in O(d)}[v_k(C\cap \boldsymbol{Q}D)] &= \mathbb{E}_{L\in\operatorname{Gr}_k(L_C)}[\mathbb{E}_{\boldsymbol{Q}_L\in O(L_D,L)}[v_k(C\cap \boldsymbol{Q}_L D)]] \\
&= \mathbb{E}_{L\in\operatorname{Gr}_k(L_C)}\left[\mathbb{E}_{\boldsymbol{Q}_L\in O(L_D,L)}\left[\mathbb{E}_{\boldsymbol{g}\in\mathcal{N}(L)}[1_C(\boldsymbol{g})\, 1_{\boldsymbol{Q}_L D}(\boldsymbol{g})]\right]\right] \\
&= \mathbb{E}_{L\in\operatorname{Gr}_k(L_C)}\left[\mathbb{E}_{\boldsymbol{g}\in\mathcal{N}(L)}\left[1_C(\boldsymbol{g})\; \mathbb{E}_{\boldsymbol{Q}_L\in O(L_D,L)}[1_D(\boldsymbol{Q}_L^{-1}\boldsymbol{g})]\right]\right] \\
&= \mathbb{E}_{L\in\operatorname{Gr}_k(L_C)}\left[\mathbb{E}_{\boldsymbol{g}\in\mathcal{N}(L)}[1_C(\boldsymbol{g})]\right] v_\ell(D) \\
&= v_j(C)\, v_\ell(D). \qquad \square
\end{aligned}$$

As in Section 4, we reduce the proof of the kinematic formula in the general case $k < d$ to the boundary case $k = d$ through the facial structure of the cones.

Proof of Theorem 5.1 for $k < d$. We proceed by induction on d, where the basis step $d = 0$ is already covered. Recall the notation $F \pitchfork G$ for transverse intersection. It is enough to show that if $F \in \mathcal{F}_j(C)$, $G \in \mathcal{F}_\ell(D)$, with $j + \ell > d$, then

$$\mathbb{E}[v_{F\cap\boldsymbol{Q}G}(C\cap\boldsymbol{Q}D)\cdot 1_{\{F\pitchfork\boldsymbol{Q}G\}}] = v_F(C)\, v_G(D). \tag{5.5}$$

The kinematic formula then follows by noting that $v_F(C)v_G(D) = v_{F\times G}(C\times D)$ and

$$\begin{aligned}
\mathbb{E}[v_k(C\cap\boldsymbol{Q}D)] &= \sum_{\substack{(F,G)\in\mathcal{F}(C)\times\mathcal{F}(D)\\ \dim F+\dim G = k+d}} \mathbb{E}[v_{F\cap\boldsymbol{Q}G}(C\cap\boldsymbol{Q}D)\cdot 1_{\{F\pitchfork\boldsymbol{Q}G\}}] \\
&= \sum_{\substack{(F,G)\in\mathcal{F}(C)\times\mathcal{F}(D)\\ \dim F+\dim G = k+d}} v_{F\times G}(C\times D) = v_{k+d}(C\times D).
\end{aligned}$$

Now, recall that $v_F(C) = v_d(F + N_F C)$, so that, with $k = \dim F + \dim G - d$, and assuming $F \pitchfork G$, by (2.5)

$$v_{F\cap\boldsymbol{Q}G}(C\cap\boldsymbol{Q}D) = v_k(F\cap\boldsymbol{Q}G)\, v_{d-k}(N_F C + \boldsymbol{Q}N_G D).$$

Note that if $L_{\boldsymbol{Q}} = \operatorname{lin}(F\cap\boldsymbol{Q}G)$, then $\operatorname{lin}(N_F C + \boldsymbol{Q}N_G D) = L_{\boldsymbol{Q}}^\perp$. For given $\boldsymbol{Q}$ let $\tilde{\boldsymbol{Q}}$ be uniformly at random in the subgroup of $O(d) = O(\mathbb{R}^d)$ that keeps $L_{\boldsymbol{Q}}$ fixed. In other words, $\tilde{\boldsymbol{Q}}$ is uniformly at random in $O(L_{\boldsymbol{Q}}^\perp)$. Note that the product of these random matrices, $\tilde{\boldsymbol{Q}}\boldsymbol{Q}$, is a uniformly random element in $O(d)$, and note that $F\cap\tilde{\boldsymbol{Q}}\boldsymbol{Q}G = F\cap\boldsymbol{Q}G$ and $F\pitchfork\tilde{\boldsymbol{Q}}\boldsymbol{Q}G$ iff $F\pitchfork\boldsymbol{Q}G$. We thus obtain

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{Q}}[v_{F\cap\boldsymbol{Q}G}(C\cap\boldsymbol{Q}D)\cdot 1_{\{F\pitchfork\boldsymbol{Q}G\}}] &= \mathbb{E}_{\boldsymbol{Q}}[v_k(F\cap\boldsymbol{Q}G)\, v_{d-k}(N_F C + \boldsymbol{Q}N_G D)\cdot 1_{\{F\pitchfork\boldsymbol{Q}G\}}] \\
&= \mathbb{E}_{\boldsymbol{Q}}\left[v_k(F\cap\boldsymbol{Q}G)\; \mathbb{E}_{\tilde{\boldsymbol{Q}}}[v_{d-k}(N_F C + \tilde{\boldsymbol{Q}}\boldsymbol{Q}N_G D)]\cdot 1_{\{F\pitchfork\boldsymbol{Q}G\}}\right].
\end{aligned}$$

For the inner expectation we may apply the polar kinematic formula (5.3), which holds by induction hypothesis as $\dim L_{\boldsymbol{Q}}^\perp = d - k < d$. So we obtain

$$\mathbb{E}_{\tilde{\boldsymbol{Q}}}[v_{d-k}(N_F C + \tilde{\boldsymbol{Q}}\boldsymbol{Q}N_G D)] = \sum_{i=0}^{d-k} v_{d-k+i}(N_F C\times\boldsymbol{Q}N_G D) = v_{d-j}(N_F C)\, v_{d-\ell}(N_G D),$$

where the second equation follows from a dimension count. Continuing the above computation, we thus obtain

$$\begin{aligned}\mathbb{E}_{\boldsymbol{Q}}[v_{F\cap \boldsymbol{Q}G}(C\cap \boldsymbol{Q}D)\cdot 1_{\{F\pitchfork \boldsymbol{Q}G\}}] &= \mathbb{E}_{\boldsymbol{Q}}[v_k(F\cap \boldsymbol{Q}G)]\, v_{d-j}(N_FC)\, v_{d-\ell}(N_GD) \\ &\overset{(*)}{=} v_j(F)\, v_\ell(G)\, v_{d-j}(N_FC)\, v_{d-\ell}(N_GD) \\ &= (v_j(F)\, v_{d-j}(N_FC))\, (v_\ell(G)\, v_{d-\ell}(N_GD)) \\ &= v_F(C)\, v_G(D),\end{aligned}$$

where equation $(*)$ follows from the case $\dim C + \dim D = k + d$. This proves (5.5) and thus finishes the proof. ☐

6. The Klivans-Swartz relation for hyperplane arrangements

While the most natural lattice structure associated to a polyhedral cone is arguably its face lattice, there is also the intersection lattice generated by the hyperplanes that are spanned by the facets of the cone (assuming that the cone has nonempty interior; otherwise one can argue within the linear span of the cone). In this section we present a deep and useful relation between this intersection lattice and the intrinsic volumes of the regions of the hyperplane arrangement, which is due to Klivans and Swartz [KS11], and which we will generalize to also include the faces of the regions. We finish this section with some applications of this result.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$. Recall from (2.13) the notation $\mathcal{R}_j(\mathcal{A})$ and $r_j(\mathcal{A})$ for the set of j-dimensional regions of the arrangement and for their cardinality, respectively. Also recall Zaslavsky's Theorem 2.11, which is the briefly stated identity $r_j(\mathcal{A}) = (-1)^j\, \chi_{\mathcal{A},j}(-1)$, where $\chi_{\mathcal{A},j}$ denotes the jth-level characteristic polynomial of the arrangement. Expressing this polynomial in the form

$$\chi_{\mathcal{A},j}(t) = \sum_{k=0}^{j} a_{jk} t^k,$$

and using the identity $\sum_k v_k(C) = 1$, we can rewrite Zaslavsky's result in the form

$$\sum_{k=0}^{j} \sum_{F\in\mathcal{R}_j(\mathcal{A})} v_k(F) = \sum_{k=0}^{j} (-1)^{j-k} a_{jk}.$$

Klivans and Swartz [KS11] have proved that in the case $j = d$ this equality of sums is in fact an equality of the summands. We will extend this and show that for all j the summands are equal. In particular, taking the sum of intrinsic volumes of all regions of a certain dimension j in a hyperplane arrangement yields a quantity that is solely expressible in the lattice structure of the hyperplane arrangement. So while the intrinsic volumes of a single region are certainly not necessarily invariant under any nonsingular linear transformations, the sum of intrinsic volumes over all regions of a fixed dimension is indeed invariant under any nonsingular linear transformations.

Theorem 6.1. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$. Then for $0 \le j \le d$,*

$$\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_F(t) = (-1)^j \chi_{\mathcal{A},j}(-t),$$

where $v_F(t) = \sum_k v_k(F) t^k$. In terms of the intrinsic volumes, for $0 \le k \le j$,

$$\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_k(F) = (-1)^{j-k} a_{jk}, \tag{6.1}$$

where a_{jk} is the coefficient of t^k in $\chi_{\mathcal{A},j}(t)$. In particular, $\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_j(F) = \ell_j(\mathcal{A})$.

We derive a concise proof of Theorem 6.1 by combining Zaslavsky's Theorem with the kinematic formula. A similar, though slightly different, proof strategy using the kinematic formula was recently employed in [KVZ15] to derive Klivans and Swartz's result.

The cases $j = 0, 1$ will be shown directly; in the case $j \geq 2$ we prove (6.1) by induction on k. This proof by induction naturally consists of two steps:

(1) For the case $k = 0$ let H be a hyperplane in general position relative to $\mathcal{A}$, that is, H intersects all subspaces in $\mathcal{L}(\mathcal{A})$ transversely. In H consider the restriction $\mathcal{A}^H = \{H' \cap H : H' \in \mathcal{A}\}$. The number of $(j-1)$-dimensional regions in $\mathcal{A}^H$ is given by the number of j-dimensional regions in $\mathcal{A}$, which are hit by the hyperplane H. With the simplest case of the Crofton formula (4.2), we obtain for a uniformly random hyperplane H,

$$\mathbb{E}\left[r_{j-1}(\mathcal{A}^H)\right] = \sum_{F\in\mathcal{R}_j(\mathcal{A})} \mathbb{P}\{F\cap H \neq \mathbf{0}\} = \sum_{F\in\mathcal{R}_j(\mathcal{A})} (1-2v_0(F)) = r_j(\mathcal{A}) - 2\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F). \tag{6.2}$$

We will see below that $r_{j-1}(\mathcal{A}^H)$ is almost surely constant, which eliminates the expectation on the left-hand side, and is in fact expressible in terms of $\chi_{\mathcal{A},j}$. This will give the basis step in a proof by induction on k of (6.1).

(2) For the induction step we use the kinematic formula (5.2) with $m = 1$, that gives for a uniformly random hyperplane H,

$$\begin{aligned} \sum_{F\in\mathcal{R}_j(\mathcal{A})} v_1(F) &= \sum_{F\in\mathcal{R}_j(\mathcal{A})} \big(\mathbb{E}[v_0(F\cap H)] - v_0(F)\big) \\ &= \mathbb{E}\Big[\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F\cap H)\Big] - \sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F), \end{aligned} \tag{6.3}$$

$$\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_k(F) = \sum_{F\in\mathcal{R}_j(\mathcal{A})} \mathbb{E}[v_{k-1}(F\cap H)] = \mathbb{E}\Big[\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_{k-1}(F\cap H)\Big], \quad \text{if } k\geq 2. \tag{6.4}$$

Notice that if the summation would be over the regions in $\mathcal{A}^H$, then we could (and in fact can if $k \geq 2$) apply the induction hypothesis and express $\sum v_k(C\cap H)$ in terms of the characteristic function of $\mathcal{A}^H$, which, as we will see below, is constant for generic H and expressible in the characteristic function of $\mathcal{A}$. Since the summation is over the regions of $\mathcal{A}$ we need to be a bit careful in the case $k = 1$.

To implement this idea we need to understand how the characteristic polynomial of a hyperplane arrangement changes when adding a hyperplane in general position.

Lemma 6.2. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$, and let $j \geq 2$. If $H \subset \mathbb{R}^d$ is a linear hyperplane in general position relative to $\mathcal{A}$, then the $(j-1)$th-level characteristic function of the reduced arrangement $\mathcal{A}^H$ and the number of $(j-1)$-dimensional regions of $\mathcal{A}^H$ are given by*

$$\chi_{\mathcal{A}^H,j-1}(t) = \chi_{\mathcal{A},j}(0) + \frac{\chi_{\mathcal{A},j}(t)-\chi_{\mathcal{A},j}(0)}{t}, \qquad r_{j-1}(\mathcal{A}^H) = r_j(\mathcal{A}) - (-1)^j 2\chi_{\mathcal{A},j}(0).$$

In terms of coefficients, if $\chi_{\mathcal{A},j}(t) = \sum_k a_{jk}\, t^k$, then

$$\chi_{\mathcal{A}^H,j-1}(t) = a_{j0} + \sum_{k=1}^{j} a_{jk}\, t^{k-1}, \qquad r_{j-1}(\mathcal{A}^H) = r_j(\mathcal{A}) - (-1)^j 2a_{j0}. \tag{6.5}$$

Proof. Note first that the assumption that H is in general position relative to $\mathcal{A}$ implies that if $\tilde{L}, L \in \mathcal{L}(\mathcal{A})$, with $\dim\tilde{L}, \dim L \geq 2$, then $\tilde{L} \supseteq L$ iff $\tilde{L}\cap H \supseteq L\cap H$. Indeed, if $\tilde{L}\cap H \supseteq L\cap H$, then

$$\dim(\tilde{L}\cap L) \geq \dim(\tilde{L}\cap L\cap H) = \dim(L\cap H) = \dim L - 1 \geq 1,$$

and thus

$$\dim L - 1 = \dim(L\cap H) = \dim(\tilde{L}\cap L\cap H) = \dim(\tilde{L}\cap L) - 1,$$

by the assumption that H intersects all subspaces in $\mathcal{L}(\mathcal{A})$ transversely. Hence, $\dim L = \dim(\tilde{L} \cap L)$, and $\tilde{L} \supseteq L$. In other words, the map $L \mapsto L \cap H$ is a bijection between $\mathcal{L}_j(\mathcal{A})$ and $\mathcal{L}_{j-1}(\mathcal{A}^H)$ for all $j \geq 2$ that is compatible with the partial orders on $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A}^H)$. Of course, all elements in $\mathcal{L}_0(\mathcal{A}) \cup \mathcal{L}_1(\mathcal{A})$ are mapped to $\mathbf{0}$.

Now, recall the form of the jth-level characteristic polynomial (2.9)

$$\chi_{\mathcal{A},j}(t) = \sum_{k=0}^{j} a_{jk} t^k, \qquad a_{jk} = \sum_{\tilde{L} \in \mathcal{L}_j(\mathcal{A})} \sum_{L \in \mathcal{L}_k(\mathcal{A})} \mu(\tilde{L}, L),$$

and also recall the recursive definition of the Möbius function (2.7), $\mu(\tilde{L}, L) = 0$ if $\tilde{L} \not\supseteq L$, and

$$\mu(L, L) = 1, \qquad \mu(\tilde{L}, L) = -\sum_{\tilde{L} \supseteq M \supset L} \mu(\tilde{L}, M) \quad \text{if } \tilde{L} \supset L.$$

From the above observation about the sets $\mathcal{L}_j(\mathcal{A})$ and $\mathcal{L}_{j-1}(\mathcal{A}^H)$ for $j \geq 2$ we obtain

$$\forall \tilde{L}, L \in \mathcal{L}(\mathcal{A}),\ \dim \tilde{L}, \dim L \geq 2 : \mu(\tilde{L}, L) = \bar{\mu}(\tilde{L} \cap H, L \cap H),$$

where $\bar{\mu}$ shall denote the Möbius function on $\mathcal{L}(\mathcal{A}^H)$. This shows the claimed formula for the nonconstant coefficients of $\chi_{\mathcal{A}^H,j-1}$. We obtain the claim for the constant coefficient by noting that for $L \in \mathcal{L}(\mathcal{A})$, $\dim L \geq 2$, and $\bar{L} := L \cap H$,

$$\bar{\mu}(\bar{L}, \mathbf{0}) = -\sum_{\bar{L} \supseteq \bar{M} \supset \mathbf{0}} \bar{\mu}(\bar{L}, \bar{M}) = -\sum_{\substack{L \supseteq M \\ \dim M \geq 2}} \bar{\mu}(L \cap H, M \cap H) = -\sum_{\substack{L \supseteq M \\ \dim M \geq 2}} \mu(L, M),$$

so that the constant coefficient of $\chi_{\mathcal{A}^H,j-1}$ is given by

$$\bar{a}_{j-1,0} = -\sum_{L \in \mathcal{L}_k(\mathcal{A})} \sum_{\substack{L \supseteq M \\ \dim M \geq 2}} \mu(L, M).$$

The constant and linear coefficients of $\chi_{\mathcal{A},j}$ are given by

$$\begin{aligned} a_{j0} &= \sum_{L \in \mathcal{L}_j(\mathcal{A})} \mu(L, \mathbf{0}) = -\sum_{L \in \mathcal{L}_j(\mathcal{A})} \sum_{\substack{L \supseteq M \\ \dim M \geq 1}} \mu(L, M) \\ &= -\sum_{L \in \mathcal{L}_j(\mathcal{A})} \sum_{\substack{L \supseteq M \\ \dim M \geq 2}} \mu(L, M) - \sum_{L \in L_j(\mathcal{A})} \sum_{M \in \mathcal{L}_1(\mathcal{A}), L \supseteq M} \mu(L, M), \\ a_{j1} &= \sum_{L \in \mathcal{L}_j(\mathcal{A})} \sum_{M \in \mathcal{L}_1(\mathcal{A}), L \supseteq M} \mu(L, M), \end{aligned}$$

which shows that indeed $\bar{a}_{j-1,0} = a_{j0} + a_{j1}$. As for the claimed formula for $r_{j-1}(\mathcal{A}^H)$ we use Zaslavsky's Theorem 2.11 to obtain

$$r_{j-1}(\mathcal{A}^H) = (-1)^{j-1} \chi_{\mathcal{A}^H,j-1}(-1) = (-1)^{j-1} \big(2\chi_{\mathcal{A},j}(0) - \chi_{\mathcal{A},j}(-1)\big) = r_j(\mathcal{A}) - (-1)^j 2\chi_{\mathcal{A},j}(0),$$

which finishes the proof. □

Proof of Theorem 6.1. We first verify the cases $j = 0, 1$ directly. Recall from (2.12) that $\chi_{\mathcal{A},0}(t) = \ell_0(\mathcal{A})$ and $\chi_{\mathcal{A},1}(t) = \ell_1(\mathcal{A})(t - \ell_0(\mathcal{A}))$, where $\ell_j(\mathcal{A}) = |\mathcal{L}_j(\mathcal{A})|$. In a linear hyperplane arrangement we have at most one 0-dimensional region, and $\mathcal{R}_0(\mathcal{A}) = \mathcal{L}_0(\mathcal{A})$ (possibly both empty). Therefore,

$$\sum_{F \in \mathcal{R}_0(\mathcal{A})} v_F(t) = r_0(\mathcal{A}) = \ell_0(\mathcal{A}) = \chi_{\mathcal{A},0}(-t).$$

As for the case $j = 1$, note first that if $r_0(\mathcal{A}) = 0$, then $\mathcal{R}_1(\mathcal{A}) = \mathcal{L}_1(\mathcal{A})$ and the claim follows as in the case $j = 0$. If on the other hand $r_0(\mathcal{A}) = 1$, then every line $L \in \mathcal{L}_1(\mathcal{A})$ corresponds to two rays $F_+, F_- \in \mathcal{R}_1(\mathcal{A})$, that is, $r_1(\mathcal{A}) = 2\ell_1(\mathcal{A})$. Since $v_1(F_\pm) = v_0(F_\pm) = \frac{1}{2}$, and $\ell_0(\mathcal{A}) = 1$, we obtain

$$\sum_{F\in\mathcal{R}_1(\mathcal{A})} v_F(t) = \frac{r_1(\mathcal{A})}{2}(t+1) = \ell_1(\mathcal{A})(t+\ell_0(\mathcal{A})) = -\chi_{\mathcal{A},1}(-t).$$

We now assume $j \geq 2$ and proceed by induction on k starting with $k = 0$. In (6.2) we have seen that

$$\mathbb{E}\left[r_{j-1}(\mathcal{A}^H)\right] = r_j(\mathcal{A}) - 2\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F).$$

From Lemma 6.2 we obtain that $r_{j-1}(\mathcal{A}^H)$ is almost surely constant and given by $r_j(\mathcal{A}) - (-1)^j 2\chi_{\mathcal{A},j}(0)$. Therefore,

$$\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F) = \tfrac{1}{2}\Big(r_j(\mathcal{A}) - \big(r_j(\mathcal{A}) - (-1)^j 2\chi_{\mathcal{A},j}(0)\big)\Big) = (-1)^j\chi_{\mathcal{A},j}(0) = (-1)^j a_{j0}.$$

This settles the case $k = 0$. For $k > 0$ we need to distinguish between $k = 1$ and $k \geq 2$. From (6.3), we obtain, using the case $k = 0$ and Lemma 6.2,

$$\begin{aligned}
\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_1(F) &= \mathbb{E}\Big[\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F\cap H)\Big] - \sum_{F\in\mathcal{R}_j(\mathcal{A})} v_0(F) \\
&= \mathbb{E}\Big[\sum_{\bar{F}\in\mathcal{R}_{j-1}(\mathcal{A}^H)} v_0(\bar{F}) + |\{F\in\mathcal{R}_j(\mathcal{A}) : F\cap H = \mathbf{0}\}|\Big] - (-1)^j a_{j0} \\
&= (-1)^{j-1}(a_{j0}+a_{j1}) + \sum_{F\in\mathcal{R}_j(\mathcal{A})} \underbrace{\mathbb{P}\{F\cap H = \mathbf{0}\}}_{=2v_0(F)} - (-1)^j a_{j0} \\
&= (-1)^{j-1}(a_0+a_1) + 2(-1)^j a_{j0} - (-1)^j a_{j0} = (-1)^{j-1}a_1.
\end{aligned}$$

This settles the case $k = 1$. Finally, in the case $k \geq 2$ we argue similarly, using that $v_i(\mathbf{0}) = 0$ if $i > 0$,

$$\begin{aligned}
\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_k(C) &\overset{(6.4)}{=} \mathbb{E}\Big[\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_{k-1}(F\cap H)\Big] = \mathbb{E}\Big[\sum_{\bar{F}\in\mathcal{R}_{j-1}(\mathcal{A}^H)} v_{k-1}(\bar{F})\Big] \\
&= (-1)^{j-1-(k-1)}\bar{a}_{jk} = (-1)^{j-k}a_{jk}.
\end{aligned}$$

□

6.1. **Applications.** In this section we compute some examples and applications of Theorem 6.1.

6.1.1. *Product arrangements.* Let $\mathcal{A}, \mathcal{B}$ be two hyperplane arrangements in $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively. The product arrangement in $\mathbb{R}^{d+e}$ is defined as

$$\mathcal{A}\times\mathcal{B} = \{H\times\mathbb{R}^e : H\in\mathcal{A}\} \cup \{\mathbb{R}^d\times H : H\in\mathcal{B}\}.$$

The characteristic polynomial is multiplicative, $\chi_{\mathcal{A}\times\mathcal{B}}(t) = \chi_{\mathcal{A}}(t)\chi_{\mathcal{B}}(t)$, and so is the bivariate polynomial (2.10), $X_{\mathcal{A}\times\mathcal{B}}(s,t) = X_{\mathcal{A}}(s,t)X_{\mathcal{B}}(s,t)$. This can either be shown directly [OT92, Ch. 2], or deduced from Theorem 6.1, as the intrinsic volumes polynomial satisfies $v_{C\times D}(t) = v_C(t)v_D(t)$.

6.1.2. *Generic arrangements.* A hyperplane arrangement $\mathcal{A}$ is said to be in general position if the corresponding normal vectors are linearly independent.[2] Combinatorial properties of such arrangements have been studied by Cover and Efron [CE67], who generalize results of Schläfli [Sch50] and Wendel [Wen62] to get expressions for, among other things, the average number of j-dimensional faces of a region in the arrangement. We set out to compute the characteristic polynomial of an arrangement of hyperplanes in general position, and in the process recover the formulas of Cover and Efron and a formula of Hug and Schneider [HS15] for the expected intrinsic volumes of the regions.

[2] We only discuss linear hyperplane arrangements; for generic affine hyperplane arrangements see for example [Ard14].

Lemma 6.3. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a generic hyperplane arrangement in $\mathbb{R}^d$ with $n \geq d$. Then for $0 < j \leq d$,*

$$(-1)^j \chi_{\mathcal{A},j}(-t) = \binom{n}{d-j}\left(\binom{n-d+j-1}{j-1} + \sum_{k=1}^{j}\binom{n-d+j}{j-k}t^k\right). \tag{6.6}$$

Proof. Assume first that $j = d$. The proof in this case relies on Whitney's theorem [Sta12, Prop. 3.11.3]

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}\subseteq\mathcal{A}} (-1)^{|\mathcal{B}|} t^{d-\rho(\mathcal{B})},$$

where ρ denotes the rank of the arrangement $\mathcal{B}$. We can subdivide the sum into two parts:

$$\sum_{|\mathcal{B}|<d} (-1)^{|\mathcal{B}|} t^{d-\rho(\mathcal{B})} + \sum_{|\mathcal{B}|\geq d} (-1)^{|\mathcal{B}|} t^{d-\rho(\mathcal{B})}.$$

Since $\mathcal{A}$ is in general position, $\rho(\mathcal{B}) = |\mathcal{B}|$ if $|\mathcal{B}| \leq d$, and $\rho(\mathcal{B}) = d$ if $|\mathcal{B}| \geq d$. Collecting terms with equal rank, we obtain

$$\chi_{\mathcal{A}}(t) = \sum_{k=0}^{d-1}\binom{n}{k}(-1)^k t^{d-k} + \sum_{k=d}^{n}\binom{n}{k}(-1)^k.$$

An easy induction proof shows that $\sum_{k=d}^{n}\binom{n}{k}(-1)^k = \binom{n-1}{d-1}(-1)^d$, which settles the case $j = d$.

For the case $0 < j < d$ note that if $L \in \mathcal{L}_j(\mathcal{A})$, then L is the intersection of $d - j$ uniquely determined hyperplanes, and the restriction $\mathcal{A}^L$ is a generic hyperplane arrangement in L consisting of $n - d + j$ hyperplanes. Furthermore, there are exactly $\binom{n}{d-j}$ linear subspaces of dimension j in $\mathcal{L}(\mathcal{A})$. Therefore, using the characterisation (2.11) of the jth-level characteristic polynomial, we obtain

$$(-1)^j \chi_{\mathcal{A},j}(-t) = \sum_{L\in\mathcal{L}_j(\mathcal{A})} (-1)^j \chi_{\mathcal{A}^L}(-t) = \binom{n}{d-j}\left(\binom{n-d+j-1}{j-1} + \sum_{k=1}^{j}\binom{n-d+j}{j-k}t^k\right),$$

where the second equality follows from the case $j = d$. □

From Zaslavsky's Theorem 2.11 we obtain from (6.6) the number of j-dimensional regions in a generic hyperplane arrangement, $r_j(\mathcal{A})$, by setting $t = 1$. Using the simplification

$$\binom{n-d+j-1}{j-1} + \sum_{k=1}^{j}\binom{n-d+j}{j-k} = 2\sum_{k=1}^{j}\binom{n-d+j-1}{j-k}$$

we recognize the right-hand side as Schläfli's formula [CE67, (1.1)] for the number of regions of a generic arrangement of $n - d + j$ hyperplanes in j-dimensional space. The resulting formula for $r_j(\mathcal{A})$ allows us to recover the formula of Cover and Efron [CE67, Theorem 1] for the sum of the $f_j(C)$ over all regions.

If one takes one of these j-dimensional regions uniformly at random, then one also recovers the expression for the average number of j-dimensional faces from [CE67, Theorem 3']. Moreover, then (6.6) and Theorem 6.1 together yield a closed formula for the expected intrinsic volumes of the regions. In particular, the d-dimensional regions have expected intrinsic volumes of

$$\mathbb{E}_{C\in\mathcal{R}_d(\mathcal{A})}[v_0(C)] = \frac{1}{r_d(\mathcal{A})}\binom{n-1}{d-1}, \qquad \mathbb{E}_{C\in\mathcal{R}_d(\mathcal{A})}[v_k(C)] = \frac{1}{r_d(\mathcal{A})}\binom{n}{d-k}, \quad \text{if } k > 0.$$

This is [HS15, Theorem 4.1].

6.1.3. *Braid and Coxeter arrangements.* Finally, we compute the jth-level characteristic polynomial for the three families of arrangements

$$\begin{aligned}\mathcal{A}_A &:= \big\{\{\boldsymbol{x}\in\mathbb{R}^d : x_i = x_j\} : 1\le i<j\le d\big\},\\ \mathcal{A}_{BC} &:= \big\{\{\boldsymbol{x}\in\mathbb{R}^d : x_i = \pm x_j\} : 1\le i<j\le d\big\}\cup\big\{\{\boldsymbol{x}\in\mathbb{R}^d : x_i = 0\} : 1\le i\le d\big\},\\ \mathcal{A}_D &:= \big\{\{\boldsymbol{x}\in\mathbb{R}^d : x_i = \pm x_j\} : 1\le i<j\le d\big\}.\end{aligned}$$

These arrangements are particularly nice to work with as the d-dimensional regions are all isometric; these chambers are indeed given by

$$\begin{aligned}\mathcal{A}_A:&\ \{\boldsymbol{x}\in\mathbb{R}^d : x_{\pi(1)}\le\cdots\le x_{\pi(d)}\}, && \pi\in S_d,\\ \mathcal{A}_{BC}:&\ \{\boldsymbol{x}\in\mathbb{R}^d : 0\le s_1x_{\pi(1)}\le\cdots\le s_dx_{\pi(d)}\}, && s_1,\dots,s_d\in\{\pm1\},\pi\in S_d,\\ \mathcal{A}_D:&\ \{\boldsymbol{x}\in\mathbb{R}^d : -s_1x_{\pi(1)}\le s_1x_{\pi(1)}\le\cdots\le s_dx_{\pi(d)}\}, && s_1,\dots,s_d\in\{\pm1\},\pi\in S_d.\end{aligned}$$

The characteristic polynomials of these arrangements are well known, see for example [Ard14, Sec. 6.4],

$$\chi_{\mathcal{A}_A}(t)=\prod_{i=0}^{d-1}(t-i),\qquad \chi_{\mathcal{A}_{BC}}(t)=\prod_{i=0}^{d-1}(t-2i-1),\tag{6.7}$$
$$\chi_{\mathcal{A}_D}(t)=(t-d+1)\prod_{i=0}^{d-2}(t-2i-1)=\chi_{\mathcal{A}_{BC}}(t)+d\prod_{i=0}^{d-2}(t-2i-1).$$

The bivariate polynomial $X_{\mathcal{A}_A}(s,t)$ (along with affine generalizations) has been computed in [Ath96a, Thm. 8.3.1]. We derive this again, along with polynomials for the other two arrangements, from the known characteristic polynomials.

Lemma 6.4. *The jth-level characteristic polynomials for the above defined hyperplane arrangements are given by*

$$\chi_{\mathcal{A}_A,j}(t)=\left\{{d\atop j}\right\}\prod_{i=0}^{j-1}(t-i),\qquad \chi_{\mathcal{A}_{BC},j}(t)=\left\{{d+1\atop j+1}\right\}\prod_{i=0}^{j-1}(t-2i-1),$$
$$\chi_{\mathcal{A}_D,j}(t)=\chi_{\mathcal{A}_{BC},j}(t)+j\left\{{d\atop j}\right\}\prod_{i=0}^{j-2}(t-2i-1),$$

where $\left\{{d\atop j}\right\}$ denote the Stirling numbers of the second kind.

Proof. We first discuss the case $\mathcal{A}=\mathcal{A}_A$. From the formula for the chambers of $\mathcal{A}$ it is seen that an element in $\mathcal{L}(\mathcal{A})$ is of the form

$$L=\{\boldsymbol{x}\in\mathbb{R}^d : x_{\pi(k_1)}=\cdots=x_{\pi(\ell_1)},\ x_{\pi(k_2)}=\cdots=x_{\pi(\ell_2)},\ \dots\},$$

where $k_1\le\ell_1<k_2\le\ell_2<\dots$. More precisely, for $L\in\mathcal{L}_j(\mathcal{A})$ there exists a unique partition $I_1,\dots,I_j$, each nonempty, of $\{1,\dots,d\}$ such that $L=\{\boldsymbol{x}\in\mathbb{R}^d : \forall i=1,\dots,j,\forall a,b\in I_i, x_a=x_b\}$. The corresponding reduction $\mathcal{A}^L$ is easily seen to be a nonsingular linear transformation of the j-dimensional braid arrangement, so that $\chi_{\mathcal{A}^L}(t)=\prod_{i=0}^{j-1}(t-i)$. Since the number of partitions of $\{1,\dots,d\}$ into j nonempty sets is given by $\left\{{d\atop j}\right\}$, cf. [Sta12], and by the characterisation (2.11) of $\chi_{\mathcal{A},j}(t)$, we obtain the claim in the case $\mathcal{A}=\mathcal{A}_A$.

In the case $\mathcal{A}=\mathcal{A}_{BC}$ we can argue similarly, but we need to keep in mind the extra role of the origin. For every element $L\in\mathcal{L}(\mathcal{A})$ there exists a subset I of $\{1,\dots,d\}$ of cardinality $|I|\ge j$, and a partition $I_1,\dots,I_j$ of I such that $L=\{\boldsymbol{x}\in\mathbb{R}^d : \forall a\notin I, x_a=0 \text{ and } \forall i=1,\dots,j,\forall a,b\in I_i, x_a=x_b\}$. The same argument as in the case $\mathcal{A}=\mathcal{A}_A$, along with the identity $\sum_{i=j}^d\binom{d}{i}\left\{{i\atop j}\right\}=\left\{{d+1\atop j+1}\right\}$, then settles the case $\mathcal{A}=\mathcal{A}_{BC}$.

In the case $\mathcal{A} = \mathcal{A}_D$ we have two types of linear subspaces:

$$L_1 = \{\boldsymbol{x} \in \mathbb{R}^d : x_{\pi(k_1)} = \dots = x_{\pi(\ell_1)},\ x_{\pi(k_2)} = \dots = x_{\pi(\ell_2)},\ \dots\},$$
$$L_2 = \{\boldsymbol{x} \in \mathbb{R}^d : 0 = x_{\pi(k_1)} = \dots = x_{\pi(\ell_1)},\ x_{\pi(k_2)} = \dots = x_{\pi(\ell_2)},\ \dots\}.$$

For the first type of linear subspace we obtain a reduction $\mathcal{A}^{L_1}$ that is isomorphic to the arrangement $\mathcal{A}_D$, while for the second type we obtain a reduction $\mathcal{A}^{L_2}$ that is isomorphic to the arrangement $\mathcal{A}_{BC}$ (each, of course, of the corresponding dimension). The number of subspaces of type L_1 is given by $\left\{ {d \atop j} \right\}$ (as in the case $\mathcal{A} = \mathcal{A}_A$), while the number of subspaces of type L_2 is given by $\left\{ {d+1 \atop j+1} \right\} - \left\{ {d \atop j} \right\}$ (as in the case $\mathcal{A} = \mathcal{A}_{BC}$, but noting that $|I| = d$ does not give a BC-type reduction). The same argument as before now yields the formula

$$\begin{aligned}\chi_{\mathcal{A}_D,j}(t) &= \left\{ {d \atop j} \right\}(t-j+1)\prod_{i=0}^{j-2}(t-2i-1) + \left(\left\{ {d+1 \atop j+1} \right\} - \left\{ {d \atop j} \right\} \right) \prod_{i=0}^{j-1}(t-2i-1) \\ &= \left\{ {d+1 \atop j+1} \right\} \prod_{i=0}^{j-1}(t-2i-1) + j\left\{ {d \atop j} \right\} \prod_{i=0}^{j-1}(t-2i-1),\end{aligned}$$

which settles the case $\mathcal{A} = \mathcal{A}_D$. □

As before in the case of generic hyperplanes in Section 6.1.2, we finish by considering resulting formulas for uniformly random j-dimensional regions of the arrangement. We restrict to the arrangements $\mathcal{A}_A$ and $\mathcal{A}_{BC}$, and we restrict the formulas to the statistical dimensions. These statistical dimensions are particularly interesting for applications as seen in [ALMT14], where only the d-dimensional regions were considered. (Here, of course, the expectation vanishes since all d-chambers of these arrangements are isometric; for the lower-dimensional regions this is no longer true.)

Recall that the statistical dimension is given by $\delta(C) = v_C'(1)$. Using again $r_j(\mathcal{A}) = (-1)^j \chi_{\mathcal{A},j}(-1)$, we obtain

$$\frac{1}{r_j(\mathcal{A})} \sum_{F \in \mathcal{R}_j(\mathcal{A})} \delta(F) = \frac{1}{(-1)^j \chi_{\mathcal{A},j}(-1)} \sum_{F \in \mathcal{R}_j(\mathcal{A})} v_F'(1) = -\frac{\chi_{\mathcal{A},j}'(-1)}{\chi_{\mathcal{A},j}(-1)}.$$

We thus obtain:

$$\chi_{\mathcal{A}_A,j}'(t) = \chi_{\mathcal{A}_A,j}(t) \sum_{i=0}^{j-1} \frac{1}{t-i}, \qquad \chi_{\mathcal{A}_{BC},j}'(t) = \chi_{\mathcal{A}_{BC},j}(t) \sum_{i=0}^{j-1} \frac{1}{t-2i-1},$$

$$-\frac{\chi_{\mathcal{A}_A,j}'(-1)}{\chi_{\mathcal{A}_A,j}(-1)} = \sum_{i=0}^{j-1} \frac{1}{1+i} = H_j, \qquad -\frac{\chi_{\mathcal{A}_{BC},j}'(-1)}{\chi_{\mathcal{A}_{BC},j}(-1)} = \sum_{i=0}^{j-1} \frac{1}{1+2i+1} = \tfrac{1}{2} H_j,$$

where H_j denotes the jth harmonic number.

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