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HAMILTONIAN RELATIVE EQUILIBRIA WITH CONTINUOUS ISOTROPY

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ABSTRACT. In symmetric Hamiltonian systems, relative equilibria usually arise in continuous families. The geometry of these families in the setting of free actions of the symmetry group is well-understood. Here we consider the question for non-free actions. Some results are already known in this direction, and we use the so-called bundle equations to provide a systematic treatment of this question which both consolidates the known results, extending the scope of the results to deal with non-compact symmetry groups, as well as producing new results. Specifically we address questions about the stability, persistence and bifurcations of these relative equilibria.

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1. INTRODUCTION

This article deals with a particular kind of integral curve of Hamiltonian flows on symplectic manifolds equivariant under the canonical action of a Lie group. They are usually known in the literature as relative equilibria, or steady state solutions, a terminology inherited from analytical mechanics. In this article we follow an approach to the qualitative study of some local properties of these solutions, based on the use of geometrically adapted tubular neighbourhoods. This approach is an evolution of methods previously proposed in the literature by several authors (see [29, 31, 34]).

Let (\mathcal{P}, ω) be a symplectic manifold, G a Lie group acting on \mathcal{P} by symplectomorphisms and h a smooth G -invariant function on \mathcal{P} . Hamilton's equations produce a vector field X_h on \mathcal{P} , and due to the invariance of h and the symplectic structure under the G -action, the flow of this vector field sends group orbits to group orbits. A relative equilibrium is an integral curve of X_h that belongs to a single group orbit.

The importance of relative equilibria is more apparent using orbit reduction. The flow of X_h naturally descends to a continuous flow on the quotient space \mathcal{P}/G , and relative equilibria project to fixed points of this reduced flow, therefore being (together with periodic orbits, which will not be addressed in this article) the primary object of qualitative local studies. If the action of G on \mathcal{P} is free and proper the quotient space \mathcal{P}/G is a manifold and the reduced flow can be shown to be smooth and Hamiltonian with respect to some reduced Hamiltonian function and Poisson structure on the quotient space. This means that relative equilibria can be studied via their projected fixed points using many of the standard available techniques from differential geometry and critical point theory on this quotient space.

For these reasons, this article deals with equivariant Hamiltonian flows for which the symmetry Lie group action has singularities. In this situation this geometric approach is no longer possible since, to start with, the quotient space \mathcal{P}/G is no longer smooth. We focus on the problems of nonlinear stability of relative equilibria, as well as on the organization of relative equilibria in parametrized branches and their bifurcations. The study of these topics in the presence of singularities has been going on for a number of decades now, and a variety of methods have been used to attack them, from the use of singular reduction and confinement arguments [21], to the use of suitable linearizations of the Hamiltonian field [11, 10, 4] passing by topological methods [15, 19, 27], to cite a few. Analogous questions for free actions are addressed in a number of papers, and in particular [25, 15, 26]

The motivation for this article is twofold. On the one hand, we wish to obtain a convenient framework adapted to the study of the local dynamics of equivariant Hamiltonian flows in the presence of singularities specifically adapted to the unique geometric features of the problem, and powerful enough to serve as a unifying approach to deal with all aspects related to this topic. For that, we have adopted the ‘‘bundle equations’’ proposed in [29] and [31] and have built on this formalism in order to obtain a self contained geometric machinery especially adapted to deal with Lie group actions with singularities. On the other hand, we show how these ideas are implemented by proving within this framework some standard results in the theory, with improved hypotheses, as well as obtaining several new relevant ones.

There are many systems to which these methods can be applied and we discuss two simple ones at the end of the paper. Other systems include recent work on magnetic confinement and the levitron [5, 7], the famous Riemann Ellipsoid problem (or affine rigid body) [30, 6, 32], the double (or triple etc) spherical pendulum [14], dynamics of molecules near collinear configurations [17], and recent work on the dynamics of the N-body problem in dimension greater than three [1, 3].

Organization. This article is organized as follows: Section 2 collects some standard generalities on symmetric Hamiltonian systems and relative equilibria. Then the bundle equations, the main theoretical framework of this article, are introduced. We present a modification of the bundle equations of [29] and [31] which explicitly incorporates the existence of continuous stabilizers for the group action, resulting in the bundle equations with isotropy (2.10), (2.11) and (2.12). These are then used, in Proposition 2.6, to obtain a system of differential equations locally characterizing relative equilibria for a given equivariant Hamiltonian flow. One of the main objectives of this article is to find solutions of these equations, and for that we prove two technical results (Lemmas 2.8 and 2.9) that will apply in the subsequent persistence and bifurcation results.

In Section 3 we use the bundle equations with isotropy in order to revisit the proof of well known stability results from [10, 21] and [18]. Our approach allows us to extend these to the statement in Theorem 3.6. Compared to [10, 21] we are able to avoid making use of the “orthogonality” choice of velocity, and compared to [18] we allow the possibility of non-compact momentum stabilizer G_μ . In Remark 3.7 we explicitly show with an example how our approach gives sharper stability results than those in [10] and [21].

In Section 4.1 we deal with the problem of persistence. That is, under which conditions a relative equilibrium persists to a continuous branch of relative equilibria. This differs with the problem of bifurcations, which considers as starting point a parametrized branch of relative equilibria and finds conditions for which at a particular value of the parameter, new relative equilibria exist arbitrarily close to the original branch. These new relative equilibria are typically organized in bifurcating branches.

We obtain four main results in this direction: Theorems 4.3, 4.4, 4.5 and 4.6, based on different non-degeneracy or degeneracy hypotheses. Some of these results, or particular cases of them, had been obtained in the past using quite different approaches. Here we show how they all follow in a straightforward way from the bundle equations with isotropy and are therefore unified under a common framework. Many of these results assume the group of symmetries to be compact, whereas we allow non-compact symmetry groups in all our results, thereby also providing minor extensions to the known results.

Finally in Section 5 we address the problem of finding bifurcations from parametrized branches of relative equilibria. Using the notion of branches of relative equilibria of the same symplectic type introduced in Definition 3.4, we are able to give, in Theorem 5.1, sufficient conditions under which one such branch exhibits bifurcations. This result is further elaborated in Theorem 5.2, where we show that branches of formally stable relative equilibria satisfying a dimensionality condition generically have a continuous range of bifurcation points. This property is specifically related to the existence of continuous stabilizers and shows that these bifurcations are purely singular phenomena,

since it is a well known result, originally due to Arnold [2], that branches of nondegenerate regular relative equilibria cannot bifurcate. In Section 6 the results of this article are illustrated in two study cases, the sleeping Lagrange top and the system of two point vortices on the sphere.

We use a mixture of notation for derivatives: $\mathbf{d}h$ denotes the differential of the function h , and if $h : N \times X \rightarrow \mathbb{R}$ is defined on a product of spaces, $\mathbf{D}_N h$ denotes the partial derivative(s) of h with respect to the N -variables. Occasionally, we write $\mathbf{d}_z^2 h$ to denote the Hessian matrix at the point z , but omit reference to the point z if it is clear.

2. HAMILTONIAN RELATIVE EQUILIBRIA

In this section we provide necessary and sufficient conditions for the existence of Hamiltonian relative equilibria. The study will be local, in a neighbourhood of a given point in a symplectic phase space of a symmetric Hamiltonian system. To this end, we will use the framework provided by the bundle equations obtained in [29] and generalized in [31] (see also Chapter 7 in [22]). In our approach we will additionally incorporate in a critical way the freedom in several choices that exists if the symmetry group action exhibits continuous isotropy groups.

To fix notation, we recall the standard definitions. A proper G -Hamiltonian system is a quintuple $(\mathcal{P}, \omega, G, \mathbf{J}, h)$, where (\mathcal{P}, ω) is a symplectic manifold, G is a Lie group acting properly and in a Hamiltonian fashion on \mathcal{P} . We write the action by concatenation: $(g, z) \mapsto g \cdot z$. The resulting momentum map $\mathbf{J} : \mathcal{P} \rightarrow \mathfrak{g}^*$ satisfies

$$\iota_{\xi_{\mathcal{P}}} \omega = \mathbf{d}\langle \mathbf{J}(\cdot), \xi \rangle, \quad \forall \xi \in \mathfrak{g}.$$

Here $\xi_{\mathcal{P}}(z) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \cdot z$ is the fundamental vector field associated to the element $\xi \in \mathfrak{g}$ (the Lie algebra of G) evaluated at $z \in \mathcal{P}$. We assume \mathbf{J} to be G -equivariant with respect to the coadjoint action on \mathfrak{g}^* :

$$\mathbf{J}(g \cdot z) = \text{Ad}_{g^{-1}}^*(\mathbf{J}(z)), \quad \forall g \in G, z \in \mathcal{P},$$

where $\text{Ad}^* : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the coadjoint representation of G . See also Remark 2.4 for the adaptation to coadjoint actions modified by a cocycle. Finally, $h \in C^\infty(\mathcal{P})$ is a G -invariant Hamiltonian function on \mathcal{P} . The symplectic manifold \mathcal{P} is the phase space for the Hamiltonian dynamical system given by the flow of the vector field X_h defined by Hamilton's equations

$$\iota_{X_h} \omega = \mathbf{d}h.$$

Since h and ω are G -invariant, the Hamiltonian vector field X_h is G -equivariant and so is its flow, therefore sending group orbits to group orbits for all times. A well known theorem by Noether states that \mathbf{J} is constant along integral curves of X_h . Our main object of study is defined next.

Definition 2.1. A relative equilibrium is a point in \mathcal{P} whose integral curve lies in a group orbit.

The following characterizations of relative equilibria are standard and can be found, for instance in [14].

Proposition 2.2. *Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a G -Hamiltonian system and $z \in \mathcal{P}$ with momentum $\mathbf{J}(z) = \mu$. The following are equivalent.*

- (i) z is a relative equilibrium.
- (ii) The group orbit $G \cdot z$ consists of relative equilibria. That is, if z is a relative equilibrium, then so is $g \cdot z$ for any g in G .
- (iii) There is an element $\xi \in \mathfrak{g}_\mu$ such that $X_h(z) = \xi_{\mathcal{P}}(z)$, where $\mathfrak{g}_\mu \leq \mathfrak{g}$ is the Lie algebra of the stabilizer of μ defined by $G_\mu = \{g \in G : \text{Ad}_{g^{-1}}^* \mu = \mu\}$.
- (iv) There is an element $\xi \in \mathfrak{g}_\mu$ such that $\bar{z}(t) = e^{t\xi} \cdot z$, where $\bar{z}(t)$ is the integral curve of X_h with $\bar{z}(0) = z$.
- (v) There is an element $\xi \in \mathfrak{g}_\mu$ such that z is a critical point of the augmented Hamiltonian

$$h_\xi := h - \mathbf{J}^\xi,$$

with $\mathbf{J}^\xi(z) := \langle \mathbf{J}(z), \xi \rangle$.

Here and throughout, we use \leq to denote the relation of being a Lie subalgebra or a closed subgroup (according to context), and \triangleleft to denote the relation of being a normal subgroup, or an ideal in a Lie algebra. We denote the normalizer of a subgroup H in G by $N_G(H)$.

Remark 2.3. The element ξ associated to z is called a velocity for the relative equilibrium. Notice that ξ is not uniquely defined if z has continuous isotropy, and this is a key observation that will follow a long way. The isotropy group, or stabilizer, of z , $G_z = \{g \in G : g \cdot z = z\}$ is a compact Lie subgroup of G by the properness hypothesis on the action. If G_z has positive dimension it has a non-trivial Lie algebra \mathfrak{g}_z whose elements η satisfy by definition $\eta_{\mathcal{P}}(z) = 0$.

Notice also that the equivariance property of \mathbf{J} implies in particular that $\mathfrak{g}_z \leq \mathfrak{g}_\mu$. Therefore, by (iii) in Proposition 2.2 if ξ is a velocity for z , so is $\xi + \eta$ for any $\eta \in \mathfrak{g}_z$. The converse is also true: any two admissible velocities for z must differ by an element of \mathfrak{g}_z . Also note that it follows again from (iii) and (iv) in the same proposition that if ξ is a velocity for z , then $\text{Ad}_g \xi$ is a velocity for the relative equilibrium $g \cdot z$.

Remark 2.4. It was stated earlier that we are assuming the momentum map to be equivariant with respect to the coadjoint action \mathfrak{g}^* . If G is compact, this can always be arranged by an averaging argument [15, 22]. However, for more general groups, \mathbf{J} may not be able to be chosen to be equivariant in this sense. In such cases, there is a cocycle $\theta : G \rightarrow \mathfrak{g}^*$ (the ‘Souriau cocycle’) for which the resulting modified action $\text{Coad}_g^\theta \mu := \text{Ad}_{g^{-1}}^* \mu + \theta(g)$ renders the momentum map equivariant: $\mathbf{J}(g \cdot z) = \text{Coad}_g^\theta \mathbf{J}(z)$. Moreover, as shown in [33] there is an analogous local model for the symplectic action of G and similar bundle equations. It follows that all the results of this paper stated for the coadjoint action also hold in this more general case of a modified coadjoint action, provided the momentum isotropy group G_μ is understood to be relative to this modified action.

We state a well-known algebraic lemma.

Lemma 2.5. *Let H be a compact subgroup of the Lie group G , and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be an $\text{Ad}(H)$ -invariant decomposition (as vector spaces).*

- (i) The subspace of fixed points \mathfrak{g}^H is a Lie subalgebra of \mathfrak{g} , and the Lie bracket on \mathfrak{g}^H descends to one on \mathfrak{m}^H .
- (ii) The Lie algebra of $N_G(H)$ satisfies $\mathfrak{Lie}(N_G(H)) = \mathfrak{h} \oplus \mathfrak{m}^H = \mathfrak{h} + \mathfrak{g}^H$. Consequently, if $L = N_G(H)/H$, and \mathfrak{l} is its Lie algebra, then $\mathfrak{l} \simeq \mathfrak{m}^H$ as Lie algebras.
- (iii) If H is a normal subgroup of G then $\mathfrak{m}^H = \mathfrak{m}$.

In fact $\mathfrak{m}^H = \mathfrak{m}$ is equivalent to the (weaker) statement that $N_G(H)$ contains the connected component G_0 of G .

Proof. The decomposition follows by using an $\text{Ad}(H)$ -invariant inner product, which exists because H is compact, and defining $\mathfrak{m} = \mathfrak{h}^\perp$.

(i) It is easy to check that \mathfrak{g}^H is a Lie subalgebra of \mathfrak{g} . Now, \mathfrak{h}^H is in the centre of \mathfrak{g}^H , and it follows that the Lie bracket descends to $\mathfrak{m}^H \simeq \mathfrak{g}^H/\mathfrak{h}^H$.

(ii) For any $\xi \in \mathfrak{g}$ write $\xi = \eta + \xi^\perp \in \mathfrak{h} \oplus \mathfrak{m}$. Now $g \in N_G(H)$ means $gHg^{-1} = H$, and putting $g = \exp(t\xi)$ and differentiating at $t = 0$ shows $\xi \in \mathfrak{n} := \mathfrak{Lie}(N_G(H))$ if and only if ξ satisfies the (linear) condition

$$\text{Ad}_h \xi - \xi \in \mathfrak{h}, \quad \forall h \in H.$$

Writing $\xi = \eta + \xi^\perp \in \mathfrak{h} \oplus \mathfrak{m}$ (an $\text{Ad}(H)$ -invariant decomposition), this condition becomes

$$\text{Ad}_h \xi^\perp - \xi^\perp = 0,$$

which is equivalent to $\xi^\perp \in \mathfrak{m}^H$, as required.

(iii) This follows immediately from the fact that $\mathfrak{n} = \mathfrak{h} + \mathfrak{m}^H$. □

2.1. The bundle equations with isotropy.

Since we are interested in the local properties of a Hamiltonian flow in a neighbourhood of a relative equilibrium, we will substitute the phase space \mathcal{P} by the symplectic tubular neighbourhood given by the Marle-Guillemin-Sternberg (MGS) model. Originally due to Marle [13] and Guillemin and Sternberg [8], it is now standard and details can be found for instance in [22]. We briefly recall its construction.

Let $z \in \mathcal{P}$ be an arbitrary point with momentum $\mathbf{J}(z) = \mu$. Let \mathfrak{g}_z and \mathfrak{g}_μ be the Lie algebras of the stabilizers of z and μ as before. We will assume through this paper that μ is a split element of \mathfrak{g}^* , meaning that there is a G_μ -coadjoint invariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$. Examples of split momentum values are the cases of compact or Abelian G_μ . For an example of non-split momentum see [31].

Since by the equivariance of \mathbf{J} we have $G_z \leq G_\mu$ (we use \leq to mean ‘is a closed subgroup of’), and since G_z is compact, we can find a G_z -invariant splitting

$$\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_z, \tag{2.1}$$

with associated dual invariant splitting $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_z^*$. Here \mathfrak{g}_z^* and \mathfrak{m}^* are identified with the annihilators of \mathfrak{m} and \mathfrak{g}_z in \mathfrak{g}_μ^* respectively. Finally, combining these splittings of the Lie algebras, we have a G_z -invariant splitting $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_z \oplus \mathfrak{q}$. For each of these splittings we will denote by $\mathbb{P}_\mathfrak{l}$ the projection onto the factor \mathfrak{l} .

Next, choose $N = N_z$ to be a complement to $\mathfrak{g}_\mu \cdot z$ in $\ker T_z \mathbf{J}$, G_z -invariant with respect to the induced linear representation of G_z on $T_z \mathcal{P}$. It follows from the general properties of Witt-Artin decompositions for Hamiltonian actions that such a choice is always possible. Moreover, (N, Ω) is a symplectic vector space, where Ω is the restriction

to N of $\omega(z)$, and the linear action of G_z on N is Hamiltonian, with associated equivariant momentum map $\mathbf{J}_N : N \rightarrow \mathfrak{g}_z^*$ defined by

$$\langle \mathbf{J}_N(v), \eta \rangle = \frac{1}{2} \Omega(\eta \cdot v, v), \quad \forall \eta \in \mathfrak{g}_z, v \in N, \quad (2.2)$$

where $\eta \cdot v = \left. \frac{d}{dt} \right|_{t=0} e^{t\eta} \cdot v$ is the infinitesimal generator at v corresponding to $\eta \in \mathfrak{g}_z$. The space N is usually called the symplectic normal space (or symplectic slice) at z for the G -action on \mathcal{P} .

The product space $G \times \mathfrak{m}^* \times N$ supports free actions of both G and G_z , given by

$$g' \cdot (g, \rho, v) = (g'g, \rho, v) \quad g' \in G \quad (2.3)$$

$$h \cdot (g, \rho, v) = (gh^{-1}, \text{Ad}_h^* \rho, h \cdot v) \quad h \in G_z \quad (2.4)$$

for all $g \in G$, $\rho \in \mathfrak{m}^*$ and $v \in N$. It is clear that these actions are free and commute. Consider the principal bundle associated to the G_z -action

$$\begin{aligned} \pi : G \times \mathfrak{m}^* \times N &\rightarrow G \times_{G_z} (\mathfrak{m}^* \times N) \\ (g, \rho, v) &\mapsto [g, \rho, v], \end{aligned}$$

where $[g, \rho, v]$ is the equivalence class of points consisting of the orbit $G_z \cdot (g, \rho, v) \subset G \times \mathfrak{m}^* \times N$. The G -action (2.3) on $G \times \mathfrak{m}^* \times N$ descends to a smooth action on $G \times_{G_z} (\mathfrak{m}^* \times N)$ given by

$$g' \cdot [g, \rho, v] = [g'g, \rho, v] \quad g' \in G. \quad (2.5)$$

It follows from the MGS construction that there are open neighbourhoods of the origin in \mathfrak{m}^* and N for which it is possible to define a local symplectic form on $G \times_{G_z} (\mathfrak{m}^* \times N)$, as well as a local G -equivariant symplectomorphism

$$\varphi : G \times_{G_z} (\mathfrak{m}^* \times N) \rightarrow \mathcal{P}$$

onto an open G -invariant neighbourhood of the orbit $G \cdot z$, and φ satisfies $\varphi([e, 0, 0]) = z$. We will denote by Y the domain of this diffeomorphism in $G \times_{G_z} (\mathfrak{m}^* \times N)$. The concrete expression of the mentioned symplectic form ω_Y can be given as follows: Every tangent vector to Y can be written as $\gamma_{\lambda, \dot{\rho}, \dot{v}}([g, \rho, v]) \in T_{[g, \rho, v]} Y$ with

$$\gamma_{\lambda, \dot{\rho}, \dot{v}}([g, \rho, v]) = T_{(g, \rho, v)} \pi(g \cdot \lambda, \dot{\rho}, \dot{v}),$$

where $\lambda \in \mathfrak{g}$, $\dot{\rho} \in \mathfrak{m}^*$ and $\dot{v} \in N$. Here, $g \cdot \lambda$ is the concatenation notation for $T_e L_g \lambda \in T_g G$, where $L : G \times G \rightarrow G$ is the left translation on G , given by $L_{g'} g = g'g$. We then have

$$\begin{aligned} \omega_Y(\gamma_{\lambda_1, \dot{\rho}_1, \dot{v}_1}([g, \rho, v]), \gamma_{\lambda_2, \dot{\rho}_2, \dot{v}_2}([g, \rho, v])) \\ = \langle \dot{\rho}_2 + T_v \mathbf{J}_N(\dot{v}_2), \lambda_1 \rangle - \langle \dot{\rho}_1 + T_v \mathbf{J}_N(\dot{v}_1), \lambda_2 \rangle \\ + \langle \mu + \rho + \mathbf{J}_N(v), [\lambda_1, \lambda_2] \rangle + \Omega(\dot{v}_1, \dot{v}_2). \end{aligned} \quad (2.6)$$

In addition, the induced G -action on (Y, ω_Y) given by (2.5) is Hamiltonian with a locally defined equivariant momentum map $\mathbf{J}_Y = \mathbf{J} \circ \varphi : Y \rightarrow \mathfrak{g}^*$. The explicit expression for \mathbf{J}_Y is very simple and provides in this local model a normal form for \mathbf{J} . It is given by

$$\mathbf{J}_Y([g, \rho, v]) = \text{Ad}_{g^{-1}}^* (\mu + \rho + \mathbf{J}_N(v)). \quad (2.7)$$

The idea exploited in [29] and [31] consists in taking the pullback by φ of the Hamiltonian vector field X_h to Y and then lifting it to $G \times \mathfrak{m}^* \times N$. Then the differential equations for the flow of a choice of lifted vector field are obtained, providing a general framework to study the local dynamics of X_h near $G \cdot z$.

In order to state these equations, note first that $h \circ \varphi$ is a G -invariant function on Y , so it lifts to a $G \times G_z$ -invariant function $h \circ \varphi \circ \pi$ on $G \times \mathfrak{m}^* \times N$. From equations (2.3) and (2.4) it follows that

$$\begin{aligned} (h \circ \varphi \circ \pi)(g, \rho, v) &= (\varphi \circ \pi)^* h(g \cdot (e, \rho, v)) = (\varphi \circ \pi)^* h(e, \rho, v) \\ (h \circ \pi \circ \varphi)(l \cdot (g, \rho, v)) &= (\varphi \circ \pi)^* h(gl^{-1}, \text{Ad}_{l^{-1}}^* \rho, l \cdot v) = (\varphi \circ \pi)^* h(e, \text{Ad}_{l^{-1}}^* \rho, l \cdot v), \end{aligned}$$

for all $g \in G$, $l \in G_z$, $\rho \in \mathfrak{m}^*$ and $v \in N$. Therefore we can identify $h \circ \varphi \circ \pi$ with a G_z -invariant function on $\mathfrak{m}^* \times N$ that we will denote by

$$\bar{h} \in C^\infty(\mathfrak{m}^* \times N)^{G_z}. \quad (2.8)$$

Since π is a locally trivial fibration, a π -projectable local vector field $X \in \mathfrak{X}(G \times \mathfrak{m}^* \times N)$ can be expressed as

$$X(g, \rho, v) = (g \cdot (X_{\mathfrak{g}}(g, \rho, v)), X_{\mathfrak{m}^*}(g, \rho, v), X_N(g, \rho, v)),$$

where $X_{\mathfrak{g}}(g, \rho, v) \in \mathfrak{g}$, $X_{\mathfrak{m}^*}(g, \rho, v) \in \mathfrak{m}^*$, and $X_N(g, \rho, v) \in N$. The bundle equations on $G \times \mathfrak{m}^* \times N$ define a vector field on $G \times \mathfrak{m}^* \times N$ whose components are given by:

$$\begin{aligned} X_{\mathfrak{g}}(g, \rho, v) &= D_{\mathfrak{m}^*} \bar{h}(\rho, v) \\ X_{\mathfrak{m}^*}(g, \rho, v) &= \mathbb{P}_{\mathfrak{m}^*} \left(\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v)}^* (\rho + \mathbf{J}_N(v)) \right) \\ X_N(g, \rho, v) &= \Omega^\sharp(D_N \bar{h}(\rho, v)). \end{aligned}$$

Here $D_{\mathfrak{m}^*} \bar{h}$ and $D_N \bar{h}$ denote the partial derivatives of $h \in C^\infty(\mathfrak{m}^* \times N)^{G_z}$ with respect to \mathfrak{m}^* and N respectively, and $\Omega^\sharp : N^* \rightarrow N$ is the linear G_z -equivariant isomorphism induced from Ω . As shown in [29, 31, 22], these are the differential equations for the flow of the unique π -projectable local vector field X on $G \times \mathfrak{m}^* \times N$ which is a lift of $\varphi^* X_h \in \mathfrak{X}(G \times_{G_z} (\mathfrak{m}^* \times N))$ (i.e. such that X and $\varphi^* X_h$ are π -related) and satisfies the additional condition

$$\mathbb{P}_{\mathfrak{g}_z}(X_{\mathfrak{g}}) = 0. \quad (2.9)$$

Notice that since $D_{\mathfrak{m}^*} \bar{h}(\rho, v) \in \mathfrak{m}$, the above equations imply that $\mathbb{P}_{\mathfrak{q}}(X_{\mathfrak{g}}) = 0$, which is a consequence of the condition on μ to be split, and it is not related to the condition (2.9). Notice also that since φ is a G -equivariant symplectomorphism, $\varphi^* X_h$ is actually $X_{h \circ \varphi}$, the Hamiltonian vector field corresponding to the local G -Hamiltonian system $(Y, \omega_Y, G, \mathbf{J}_Y, h \circ \varphi)$, which is a local model for $(\mathcal{P}, \omega, G, \mathbf{J}, h)$.

However, in order to obtain all the projectable vector fields on $G \times \mathfrak{m}^* \times N$ that are π -related to $\varphi^* X_h$ one needs to include all vertical vector fields tangent to the π -fibres. These are, in view of (2.4), of the form

$$X(g, \rho, v) = (g \cdot \eta, \text{ad}_{\eta}^* \rho, -\eta \cdot v),$$

where $\eta : G \times \mathfrak{m}^* \times N \rightarrow \mathfrak{g}_z$ is an arbitrary smooth map. In the following we will fix η to be an arbitrary constant in \mathfrak{g}_z , since the vector fields obtained in this way generate the module of vertical vector fields.

Using the facts that $\text{ad}_\eta^* \rho \in \mathfrak{m}^*$ (since \mathfrak{m}^* is G_z -invariant), $\text{ad}_\eta^*(\mathbf{J}_N(v)) \in \mathfrak{g}_z^*$ (since $\mathbf{J}_N(v) \in \mathfrak{g}_z^*$), and $\eta \cdot v = \Omega^\sharp(\mathbf{d}(\mathbf{J}_N(\cdot), \eta))$ (since the G_z -action on N is Hamiltonian), we can write the equations of the flow of the most general local vector field on $G \times \mathfrak{m}^* \times N$ that projects to $\varphi^* X_h$ as

$$\dot{g} = g \cdot (D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta) \quad (2.10)$$

$$\dot{\rho} = \mathbb{P}_{\mathfrak{m}^*} \left(\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) \right) \quad (2.11)$$

$$\dot{v} = \Omega^\sharp(D_N \bar{h}(\rho, v) - \mathbf{d}\mathbf{J}_N^\eta(v)). \quad (2.12)$$

where $\eta \in \mathfrak{g}_z$ is arbitrary and the function \mathbf{J}_N^η is defined by $\mathbf{J}_N^\eta(v) := \langle \mathbf{J}_N(v), \eta \rangle$ for all $\eta \in \mathfrak{g}_z$ and $v \in N$.

We also recall for future reference the following identity, which is satisfied for integral curves of the above flow (see equation (7.7.9) in [22]).

$$\mathbb{P}_{\mathfrak{g}_z^*} \left(\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) \right) = T_v \mathbf{J}_N(\dot{v}). \quad (2.13)$$

2.2. Characterization of relative equilibria.

We can now reduce the problem of finding relative equilibria of the G -Hamiltonian system $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ near $z \in \mathcal{P}$ to find solutions of (2.10), (2.11), (2.12) that project to relative equilibria of $X_{h \circ \varphi}$ near $[e, 0, 0]$ in Y . We then have the following characterization.

Proposition 2.6. *A point $[g, \rho, v]$ near $z = [e, 0, 0] \in Y$ is a relative equilibrium for $X_{h \circ \varphi}$ if and only if there exists an element $\eta \in \mathfrak{g}_z$ such that*

$$\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) = 0, \quad \text{and} \quad (2.14)$$

$$D_N \bar{h}_\eta(\rho, v) = 0, \quad (2.15)$$

or equivalently

$$\mathbb{P}_{\mathfrak{m}^*} \left(\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) \right) = 0, \quad \text{and} \quad (2.16)$$

$$D_N \bar{h}_\eta(\rho, v) = 0, \quad (2.17)$$

where

$$\bar{h}_\eta(\rho, v) := \bar{h}(\rho, v) - \mathbf{J}_N^\eta(v).$$

The element of \mathfrak{g} given by $\text{Ad}_g(D_{\mathfrak{m}^*} h(\rho, v) + \eta)$ is a velocity for the relative equilibrium $[g, \rho, v]$.

Proof. Let $[g, \rho, v]$ be in the domain Y of the diffeomorphism φ . First, by Remark 2.3, $[g, \rho, v]$ is a relative equilibrium if and only if $[e, \rho, v]$ also is, so we can restrict the study to points of this form. Let $[e, \rho, v](t)$ be the integral curve of $\varphi^* X_h$ with initial condition $[e, \rho, v](0) = [e, \rho, v]$. Therefore, we can find representative curves $g(t), \rho(t), v(t)$ with initial conditions $g(0) = e, \rho(0) = \rho, v(0) = v$, respectively, such that $[e, \rho, v](t) = [g(t), \rho(t), v(t)]$. According to Proposition 2.2, $[e, \rho, v]$ is a relative equilibrium for the local G -Hamiltonian system on $G \times_{G_z} (\mathfrak{m}^* \times N)$ corresponding to the Hamiltonian vector field $\varphi^* X_h = X_{h \circ \varphi}$, with velocity $\xi \in \mathfrak{g}$ if and only if

$$\left. \frac{d}{dt} \right|_{t=0} [e, \rho, v](t) = \xi_Y([e, \rho, v]).$$

This is equivalent, in view of (2.3) to

$$\begin{aligned}\dot{g}(0) &= \xi + \eta' \\ \dot{\rho}(0) &= \text{ad}_{\eta'}^* \rho \\ \dot{v}(0) &= -\eta' \cdot v.\end{aligned}$$

for some $\eta' \in \mathfrak{g}_z$. Since in the equations (2.10),(2.11),(2.12), η is arbitrary, we can absorb η' into η so that the relative equilibrium conditions are

$$\begin{aligned}\xi &= D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta \\ 0 &= \mathbb{P}_{\mathfrak{m}^*} \left(\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) \right) \\ 0 &= D_N \bar{h}_\eta(\rho, v)\end{aligned}\tag{2.18}$$

Note that using (2.10),(2.11),(2.12) with this choice of isotropy η , we have $\dot{v}(0) = 0$. Therefore, from (2.13) it follows that $\mathbb{P}_{\mathfrak{g}_z^*} \left(\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) \right) = 0$. This, together with (2.18) is equivalent to $\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta}^* (\rho + \mathbf{J}_N(v)) = 0$, since $D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta \in \mathfrak{g}_\mu$, $\rho + \mathbf{J}_N(v) \in \mathfrak{g}_\mu^*$ and $\mathfrak{g}_\mu^* = \mathfrak{g}_z^* \oplus \mathfrak{m}^*$. Therefore we have obtained that $[e, \rho, v]$ is a relative equilibrium on $G \times_{G_z} (\mathfrak{m}^* \times N)$ near $[e, 0, 0]$ with velocity ξ if and only if there exists $\eta \in \mathfrak{g}_z$ such that $\xi = D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta$ and $D_N \bar{h}_\eta(\rho, v) = 0$. And by Remark 2.3, this is equivalent to $[g, \rho, v]$ being a relative equilibrium with velocity $\text{Ad}_g(D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta)$. \square

Remark 2.7. Note that (2.11) implies that the \mathfrak{m}^* -components of the vector fields on $G \times \mathfrak{m}^* \times N$ that are π -related to $X_{h \circ \varphi}$ are always tangent to the coadjoint orbit of G_μ which contains $\rho + \mathbf{J}_N(v)$. This equation can be studied by topological methods, as in [15]. It can also be simplified by imposing conditions on G_μ . For instance, if G_μ is Abelian, (2.14) is automatically satisfied. Also, the study of some dynamical properties can be simplified if these coadjoint orbits are bounded, as we shall see in Theorem 3.6. On the other hand, equation (2.15) is a critical point equation for a family of functions on N parametrized by $(\eta, \rho) \in \mathfrak{g}_z \times \mathfrak{m}^*$. Therefore, (2.15) is naturally suited to be studied by methods based on the implicit function theorem or singularity theory. Note also that in general $\bar{h}_\eta \in C^\infty(\mathfrak{m}^* \times N)$ is no longer G_z -invariant, due to the presence of η , but only $(G_z)_\eta$ -invariant.

2.3. Two lemmas.

We will now focus on the solutions of equation (2.15) under the initial assumption that $0 \in N$ is a critical point of $\bar{h}_\eta(0, \cdot)$ for some $\eta \in \mathfrak{g}_z$. Notice that the dependence of \bar{h}_η on η is only through the term \mathbf{J}_N^η and this is a quadratic polynomial on N . It follows that if 0 is a critical point of $\bar{h}_\eta(\rho, \cdot)$ for some $\eta \in \mathfrak{g}_z$ then 0 is also a critical point of $\bar{h}_{\eta'}(\rho, \cdot)$ for any other element in $\eta' \in \mathfrak{g}_z$, since 0 is always a critical point of $\mathbf{J}_N^{\eta'}$. The problem consists then in finding triples $(\rho, \eta, v) \in \mathfrak{m}^* \times \mathfrak{g}_z \times N$ satisfying

$$D_N \bar{h}_\eta(\rho, v) = 0,$$

under the initial assumption

$$D_N \bar{h}_{\eta'}(0, 0) = 0 \quad \text{for every } \eta' \in \mathfrak{g}_z.$$

In order to attack this problem, we state two main lemmas, based respectively on non-degeneracy and degeneracy properties of suitable restrictions of a function on $\mathfrak{g}_z \times \mathfrak{m}^* \times N$. These two lemmas will be used in most of the results of this paper.

Recall that if G acts on a set X , and K is a subgroup of G then one writes

$$X^K = \{x \in X : k \cdot x = x \text{ for all } k \in K\}$$

for the fixed point set. Moreover, the action of G on X restricts to an action of the normalizer $N_G(K)$ on X^K , and hence defines an action on X^K of $N_G(K)/K$. Before stating the lemmas, we recall the Principle of Symmetric Criticality, due to Palais [23], which we will have recourse to a number of times.

Principle of Symmetric Criticality *Suppose a Lie group G acts properly on a manifold M , and let $f : M \rightarrow \mathbb{R}$ be a smooth G -invariant function. Let $x \in M^H$. Then $df(x) \in (T_x^*M)^H$, which as H is compact can be identified with $T_x^*(M^H)$. In particular, it follows that if x is a critical point of $f|_{M^H}$ then it is a critical point of f .*

We now have the first of the technical lemmas.

Lemma 2.8 (non-degeneracies). *Suppose that $H \leq G_z$ and let $f \in C^\infty(\mathfrak{g}_z \times \mathfrak{m}^* \times N)^H$ be a smooth H -invariant function. Suppose there is a subgroup $K \leq H$, satisfying*

$$\begin{aligned} D_N f(0, 0, 0) &= 0, \quad \text{and} \\ D_N^2 f(0, 0, 0)|_{N^K} &\text{ is non-degenerate.} \end{aligned}$$

*Then, there is a unique local (defined in a neighbourhood of $(0, 0)$) smooth $N_H(K)$ -equivariant map $v : \mathfrak{m}^{*K} \times \mathfrak{g}_z^K \rightarrow N^K$ satisfying $v(0, 0) = 0$ and*

$$D_N f(\eta', \rho, v(\rho, \eta')) = 0$$

*for every $(\rho, \eta') \in \mathfrak{m}^{*K} \times \mathfrak{g}_z^K$ near $(0, 0)$.*

*The stabilizer of $m = (\rho, v(\rho, \eta')) \in \mathfrak{m}^{*K} \times N^K$ is $(G_z)_m = (G_z)_\rho \cap (G_z)_{v(\rho, \eta')}$ and satisfies*

$$K \leq (G_z)_m \leq (G_z)_\rho.$$

Proof. Notice that \mathfrak{m}^{*K} , \mathfrak{g}_z^K and N^K support linear actions of $N_H(K)$, which is the maximal subgroup of H that leaves invariant these subspaces. By the pull-back property of Hessians, it follows that $D_N^2 f(0, 0, 0)|_{N^K} = D_{N^K}^2 (f|_{N^K})(0, 0, 0)$. If this bilinear form is non-degenerate, it follows from the Implicit Function Theorem that there is a unique map $v : \mathfrak{m}^{*K} \times \mathfrak{g}_z^K \rightarrow N^K$ defined in a neighbourhood of $(0, 0)$ such that

$$D_{N^K} (f|_{N^K})(\eta', \rho, v(\rho, \eta')) = 0$$

for every (ρ, η') in $\mathfrak{m}^{*K} \times \mathfrak{g}_z^K$, and these are all the points in $\mathfrak{m}^{*K} \times \mathfrak{g}_z^K \times N^K$ satisfying the equation $D_{N^K} (f|_{N^K})(\eta', \rho, v) = 0$ near $(0, 0, 0)$. By the invariance properties of $f|_{N^K}$, it follows that the map v is $N_H(K)$ -equivariant.

Since $f(\eta', \rho, \cdot) \in C^\infty(N)$ is in particular K -invariant for any $(\rho, \eta') \in \mathfrak{m}^{*K} \times \mathfrak{g}_z^K$, it follows from the Principle of Symmetric Criticality, and $v(\rho, \eta') \in N^K$, that

$$D_N f(\eta', \rho, v(\rho, \eta')) = 0.$$

The property about the stabilizer of $(\rho, v(\rho, \eta'))$ is obvious. \square

The second technical lemma involves degenerate critical points, and will be applied to finding bifurcating branches of relative equilibria. It is usual to apply methods of singularity theory (e.g., [28, 16]) to study the nature of critical points when a function with a degenerate critical point is deformed. However, to avoid assumptions on higher order terms, and discussions of finite determinacy and versal unfoldings, we consider here cases that can be reduced to one variable, where one can give a general existence result.

An action of a compact Lie group K on a manifold M is said to be of *cohomogeneity one* if the orbit space M/K is 1-dimensional. Note that any representation of cohomogeneity one is necessarily irreducible. Some simple examples are the representations of the trivial group on \mathbb{R} , $\mathrm{SO}(n)$ on \mathbb{R}^n and $\mathrm{SU}(n)$ on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.

Let $\sigma(u)$ be a continuous real-valued function on a topological space. We say σ *crosses 0* at u_0 if $\sigma(u_0) = 0$ and in any neighbourhood of u_0 , there exist u_1, u_2 such that $\sigma(u_1) > 0$ and $\sigma(u_2) < 0$.

Lemma 2.9 (degeneracies). *Let W and N be representations of a compact Lie group K , and let $f \in C^\infty(W \times N)^K$ be a smooth K -invariant function. Let $L \leq K$, and Λ a path connected open neighbourhood of the origin in W^L . Suppose that*

$$D_N f(\lambda, 0) = 0 \quad \text{for all } \lambda \in \Lambda.$$

Let $\mathcal{N} := \ker D_N^2 f(0, 0) \cap N^L$. Suppose $\mathcal{N} \neq 0$ and that the following two conditions are fulfilled:

- (i) *the representation of $N_K(L)$ on \mathcal{N} is of cohomogeneity one (and hence irreducible), and*
- (ii) *suppose that $\sigma(\lambda)$ crosses 0 at $\lambda = 0$, where $\sigma(\lambda)$ is the eigenvalue of $D_N^2 f(\lambda, 0)|_{\mathcal{N}}$ (thus $\sigma(0) = 0$).*

Let $N^L = \mathcal{N} \oplus S$ be an $N_K(L)$ -invariant decomposition. Then for each sufficiently small $v \in \mathcal{N}$ there is a $\lambda = \lambda_v \in \Lambda$ and $s = s_v \in S$ (not necessarily unique) such that $D_N f(\lambda_v, v, s_v) = 0$.

The stabilizer of the critical point $m = (\lambda_v, v, s_v)$ is $K_m = K_{\lambda_v} \cap K_v$ which satisfies $L \leq K_m \leq N_K(L)$.

Proof. Notice that by hypothesis, for any fixed $\lambda \in \Lambda$ the function $f(\lambda, \cdot) \in C^\infty(N)$ is L -invariant. Therefore, according to the Principle of Symmetric Criticality, a point $v \in N^L$ is a critical point of $f(\lambda, \cdot)$ if and only if the restriction of $f(\lambda, \cdot)$ to N^L , denoted by $f^L(\lambda, \cdot)$, has a critical point at v . This is equivalent to $D_N f(\lambda, v)|_{N^L} = 0$. Notice also that the pull-back property for Hessians implies

$$D_N^2 f(\lambda, v)|_{N^L} = D_{N^L}^2 f^L(\lambda, v).$$

Consequently, if (i) and (ii) are satisfied, then the (unique) eigenvalue $\sigma(\lambda)$ of $D_{N^L}^2 f^L(\lambda, 0)|_{\mathcal{N}}$ is the only eigenvalue of $D_{N^L}^2 f^L(\lambda, 0)$ that crosses 0 at $\lambda = 0$, the others being bounded away from zero. We now apply the Splitting Lemma from singularity theory, see for example [28, 16].

Let $N^L = \mathcal{N} \oplus S$ be the given splitting. Then $D_N^2 f(0, 0)|_S$ is non-degenerate. Writing $v = (y, s)$ with $y \in \mathcal{N}$, $s \in S$, there is an $N_K(L)$ -equivariant change of coordinates of

the form $(\lambda, y, \bar{s}) \mapsto (\lambda, y, s(\lambda, y, \bar{s}))$, such that

$$f(\lambda, y, s(\lambda, y, \bar{s})) = Q(\bar{s}) + g(\lambda, y).$$

Here Q is the $N_K(L)$ -invariant non-degenerate quadratic form $\frac{1}{2}D^2f|_{\mathcal{S}}$, and g is an $N_K(L)$ -invariant function for which $D_{\mathcal{N}}g(\lambda, 0) = 0$ and $D_{\mathcal{N}}^2g(0, 0) = 0$. It follows that $f(\lambda, \cdot)$ has a critical point at $v = (y, s) \in N^L$ if and only if $\bar{s}(\lambda, y, s) = 0$ and $g(\lambda, \cdot)$ has a critical point at y .

Since we are only considering critical points with respect to N^L , we can replace $f(\lambda, y, s)$ by $f(\lambda, y, s) - f(\lambda, 0, 0)$ and thus, without loss of generality, suppose that $f(\lambda, 0, 0) = 0$. And hence $g(\lambda, 0) = 0$.

For the remainder, we first treat the case $\dim \mathcal{N} = 1$, and then extend to more general representations of cohomogeneity 1. With $y \in \mathbb{R}$ and $g(\lambda, 0) = 0$ and $D_{\mathcal{N}}g(\lambda, 0) = 0$ we can write $g(\lambda, y) = y^2g_1(\lambda, y)$. It follows from (ii) that $\sigma(\lambda) = 2g_1(\lambda, 0)$. Therefore, by Taylor's theorem,

$$g(\lambda, y) = \frac{1}{2}y^2\sigma(\lambda) + y^3b(\lambda, y)$$

for some smooth function b . Differentiating with respect to y we have

$$g'(\lambda, y) = y\sigma(\lambda) + y^2c(\lambda, y),$$

for some new function c . Now use the fact that σ crosses 0: let $\lambda_1, \lambda_2 \in \Lambda$ be such that $\sigma(\lambda_1) > 0$ and $\sigma(\lambda_2) < 0$. There is then a neighbourhood V of 0 in \mathcal{N} such that $\sigma(\lambda_1) + yc(\lambda_1, y) > 0$ and $\sigma(\lambda_2) + yc(\lambda_2, y) < 0$ for all $y \in V$. Now fix $y \in V$. By the intermediate value theorem there is a $\lambda \in \Lambda$ (by taking any path in Λ joining λ_1 to λ_2) with $g'(\lambda, y) = 0$ as required.

Now consider the case where $\dim \mathcal{N} > 1$, and let $z(y)$ be the generator of the ring of invariants for the $N_K(L)$ -action on \mathcal{N} . Then we can write $g = g(\lambda, z)$, and the argument above can be repeated using z in place of y .

For the stabilizer, let $m = (\lambda, y, s)$ be a critical point. Then $s = s(\lambda, y, 0)$ (i.e., $\bar{s} = 0$) so the result follows. \square

3. STABILITY

In this paper we use equations (2.10), (2.11), (2.12) together with (2.14) and (2.15) to obtain results on the nonlinear stability, persistence and bifurcations of Hamiltonian relative equilibria. To do so, we take advantage of the available freedom in the isotropy, given by the indeterminacy in the Lie algebra element $\eta \in \mathfrak{g}_z$ in the bundle equations with isotropy, and clarify the appearance in the literature of ‘‘orthogonal velocities’’ and optimal stability conditions. In this section we will be concerned only about the stability problem of relative equilibria. We start by introducing some notions needed in the following. We will always assume that we are given a G -Hamiltonian system $(\mathcal{P}, \omega, G, \mathbf{J}, h)$, a point $z \in \mathcal{P}$ with split momentum $\mathbf{J}(z) = \mu$, and that we have chosen G_z -invariant splittings $\mathfrak{g}_\mu = \mathfrak{g}_z \oplus \mathfrak{m}$ and $\ker T_z\mathbf{J} = \mathfrak{g}_\mu \cdot z \oplus N$, and a G_μ -invariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$. The results in this section extend the main result of [18] to allow non-compact G_μ , although we still require \mathfrak{g} to admit a G_μ -invariant inner product.

Definition 3.1. Let $z \in \mathcal{P}$ be a relative equilibrium and $\xi \in \mathfrak{g}_\mu$ a velocity. Then the projection $\xi^\perp = \mathbb{P}_{\mathfrak{m}}(\xi) \in \mathfrak{m}$ is called the orthogonal velocity of z .

Note that the orthogonal velocity of a relative equilibrium is unique once a splitting (2.1) has been chosen, since all velocities of z differ by elements of \mathfrak{g}_z . On the other hand, in general there is no canonical orthogonal velocity for z , unless the splitting (2.1) is unique (see Remark 3.7 for a counterexample).

Definition 3.2. Let $z \in \mathcal{P}$ be a relative equilibrium with momentum $\mathbf{J}(z) = \mu$. We say z is nonlinearly stable (or stable modulo G_μ) if for every G_μ -invariant neighbourhood U of z there is an open neighbourhood O of z contained in U such that the integral curve of X_h through any point in O is contained in U for all time.

This definition of stability for symmetric Hamiltonian systems was introduced in [24] where it is shown to be a natural one in this setting, since it has been noticed that the existence of a conserved momentum map can allow a drift of the Hamiltonian dynamics along the G_μ -orbits which makes the obvious choice of orbital stability too restrictive in the symmetric Hamiltonian scenario. We will see below that stability modulo G_μ is related to the following notion of formal stability.

Definition 3.3. Let $z \in \mathcal{P}$ be a relative equilibrium with momentum $\mathbf{J}(z) = \mu$. We say that z is formally stable if it admits a velocity $\xi \in \mathfrak{g}_\mu$ such that $\mathbf{d}_z^2 h_\xi|_N$ is definite.

The lemma below will be of great importance in the exchange of information between a G -Hamiltonian system $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ and its local model $(Y, \omega_Y, G, \mathbf{J}_Y, h \circ \varphi)$. In particular, it shows that the concept of formal stability is independent of the choice of symplectic normal space N . First we fix some notation.

Definition 3.4. We say that an injective map $\bar{z} : W \rightarrow \mathcal{P}$, where W is a neighbourhood of 0 in a vector space, is a *parametrized branch of relative equilibria* if for every $w \in W$, the point $\bar{z}(w)$ is a relative equilibrium, and all the relative equilibria of the branch are inequivalent in the sense that they all belong to different G -orbits. Let $\bar{z}(0) = z$ with $\mathbf{J}(z) = \mu$. The points of the branch $\bar{z}(w)$ are said to be *of the same symplectic type* (or symplectic orbit type) if for every $w \in W$, there is some $g \in G$ such that $G_{\mathbf{J}(\bar{z}(w))} = gG_\mu g^{-1}$ and $G_{\bar{z}(w)} = gG_z g^{-1}$.

Notice that by construction of the Marle-Guillemin-Sternberg model, if $\bar{z}(w)$ is a family of constant symplectic type then the symplectic normal spaces $N_{g^{-1}\bar{z}(w)g}$ and N_z are isomorphic as G_z -modules. Since group orbits of relative equilibria consist of relative equilibria, it is customary to refer, by extension, to $G \cdot \bar{z}(w)$ as a parametrized branch of relative equilibria, although in this case we are not parametrizing points but G -orbits of relative equilibria.

For the following result, recall that the function \bar{h} is defined in (2.8).

Lemma 3.5.

(i) Let $z = [e, \rho, 0] \in Y$, with $\rho \in \mathfrak{m}^*$. Then for every $(\rho, \eta') \in \mathfrak{m}^* \times \mathfrak{g}_z$,

$$D_N \bar{h}_{\eta'}(\rho, \cdot) = 0 \quad \text{if and only if} \quad \mathbf{d}_{[e, \rho, 0]}(h \circ \varphi)_\xi = 0,$$

where

$$(h \circ \varphi)_\xi := (h \circ \varphi) - \mathbf{J}_Y^\xi(\cdot) = h_\xi \circ \varphi,$$

and $\xi = D_{\mathfrak{m}^*} \bar{h}(\rho, 0) + \eta'$. Furthermore, if $0 \in N$ is a critical point of $\bar{h}_{\eta'}(\rho', \cdot)$, then

$$D_N^2 \bar{h}_{\eta'}(\rho, 0) = \mathbf{d}_{[e, \rho, 0]}^2 (h \circ \varphi)_\xi \Big|_N,$$

where $D_N^2 \bar{h}_{\eta'}(\rho, 0)$ is the Hessian of the function $\bar{h}_{\eta'}(\rho, \cdot) \in C^\infty(N)$ at $0 \in N$ and $\mathbf{d}_{[e, \rho, 0]}^2 (h \circ \varphi)_\xi \Big|_N$ is the restriction to N of the Hessian of $(h \circ \varphi)_\xi$ at $[e, \rho, 0]$.

- (ii) Let $z \in \mathcal{P}$ be a critical point of h_ξ for some $\xi \in \mathfrak{g}$. Suppose that $\mathbf{d}_z^2 h_\xi \Big|_N$ is non-degenerate, degenerate, positive definite or negative definite. Then the corresponding property is satisfied for any other choice of symplectic normal space N' .
- (iii) The same conclusions hold if in (ii) we replace N by N^K and N' by $(N')^K$ respectively, for any compact subgroup $K \leq G_z$.

Proof. We will assume, without loss of generality, that $G_{\mathbf{J}_Y([e, \rho, 0])} = G_\mu$ for each point of the family. In particular, from (2.7), this assumption implies that $(G_\mu)_\rho = G_\mu$. Recall that from (v) in Proposition 2.2 the critical points of $(h \circ \varphi)_\xi$ correspond precisely to relative equilibria of the G -Hamiltonian system $(Y, \omega_Y, G, \mathbf{J}_Y, h \circ \varphi)$. Now using (2.7) and

$$(h \circ \varphi)([g, \rho, v]) = \bar{h}(\rho, v)$$

we have that

$$\begin{aligned} \langle \mathbf{d}(h \circ \varphi)_\xi([g, \rho, v]), \gamma_{\lambda, \dot{\rho}, \dot{v}}([g, \rho, v]) \rangle &= \left. \frac{d}{dt} \right|_{t=0} (\bar{h}(\rho + t\dot{\rho}, v + t\dot{v})) \\ &\quad - \left. \frac{d}{dt} \right|_{t=0} \langle \text{Ad}_{(ge^{t\lambda})}^* (\mu + \rho + t\dot{\rho} + \mathbf{J}_N(v + t\dot{v})), \xi \rangle \\ &= D_{\mathfrak{m}^*} \bar{h}(\rho, v) \cdot \dot{\rho} + D_N \bar{h}(\rho, v) \cdot \dot{v} \\ &\quad + \langle \text{Ad}_{g^{-1}}^* (\text{ad}_\lambda^* (\mu + \rho + \mathbf{J}_N(v))), \xi \rangle \\ &\quad - \langle \text{Ad}_{g^{-1}}^* (\dot{\rho} + D_N \mathbf{J}_N(v) \cdot \dot{v}), \xi \rangle \end{aligned}$$

It follows that if we make $g = e$ and $v = 0$ in the previous expression we obtain $\mathbf{d}(h \circ \varphi)_\xi([e, \rho, 0]) = 0$ if and only if

$$\begin{aligned} D_{\mathfrak{m}^*} \bar{h}(\rho, 0) - \mathbb{P}_{\mathfrak{m}}(\xi) &= 0 \\ D_N (\bar{h} - \langle \mathbf{J}_N, \mathbb{P}_{\mathfrak{g}_z}(\xi) \rangle) (\rho, 0) &= 0 \\ \text{ad}_\xi^* (\mu + \rho) &= 0 \end{aligned}$$

Therefore since $\mathbb{P}_{\mathfrak{g}_z}(\xi) = \eta'$ the first condition is automatically satisfied and the second is equivalent to

$$D_N \bar{h}_{\eta'}(\rho, 0) = 0.$$

Notice that since $\xi \in \mathfrak{g}_\mu$ and $(G_\mu)_\rho = G_\mu$ the third equation is automatically satisfied. This proves the first part of (i).

For the second part of (i), notice now that if $0 \in N$ is a critical point of $\bar{h}_{\eta'}(\rho, \cdot)$, then $D_N \bar{h}(\rho, 0) = 0$ and $D \mathbf{J}_N^{\eta'}(0) = 0$ for any $\rho \in \mathfrak{m}^*$. This follows from (2.2) since $\mathbf{J}_N^{\eta'}$ is a

quadratic polynomial on N . We can naturally identify N with the subspace of $T_{[e,\rho,0]}Y$ given by $\{\gamma_{0,0,\dot{v}}([e,\rho,0]) : \dot{v} \in N\}$. Then, we have

$$\begin{aligned} \mathbf{d}_{[e,\rho,0]}^2(h \circ \varphi)_\xi(\gamma_{0,0,\dot{v}_1}([e,\rho,0]), \gamma_{0,0,\dot{v}_2}([e,\rho,0])) \\ = D_N^2 \bar{h}(\rho, 0)(\dot{v}_1, \dot{v}_2) - D^2 \mathbf{J}'_N(0)(\dot{v}_1, \dot{v}_2) \\ = D_N^2 \bar{h}_{\eta'}(\rho, 0)(\dot{v}_1, \dot{v}_2) \end{aligned}$$

with $\xi = D_{\mathfrak{m}^*} \bar{h}(\rho, 0) + \eta'$.

For (ii), let N and N' be two symplectic normal spaces at z . That is, both are G_z -invariant complements to $\mathfrak{g}_\mu \cdot z$ in $\ker T_z \mathbf{J}$. Every element in N' can therefore be written as $v + \lambda \cdot z$ for some $v \in N$ and $\lambda \in \mathfrak{g}_\mu$. Since the Hamiltonian vector field is equivariant, the point $g \cdot z$ is a relative equilibrium with group velocity $\text{Ad}_g \xi$, and hence for all $g \in G$, the corresponding differential vanishes: $\mathbf{d}h_{\text{Ad}_g \xi}(g \cdot z) = 0$. Write $g = \exp(t\eta)$ for $\eta \in \mathfrak{g}_\mu$, and differentiate with respect to t at $t = 0$ to obtain

$$\mathbf{d}^2 h_\xi(\eta \cdot z, -) - \mathbf{d}\mathbf{J}_{[\eta, \xi]} = 0,$$

where the differentials are taken at z . It follows that for any $v \in \ker \mathbf{d}\mathbf{J}(z)$ and any $\eta \in \mathfrak{g}$, we have

$$\mathbf{d}^2 h_\xi(\eta \cdot z, v) = 0.$$

It then follows that, given any $\lambda \in \mathfrak{g}_\mu$ (so that $\lambda \cdot z \in \ker \mathbf{d}\mathbf{J}(p)$)

$$\mathbf{d}^2 h_\xi(v + \lambda \cdot z, v + \lambda \cdot z) = \mathbf{d}^2 h_\xi(v, v),$$

as required. Part (iii) is a straightforward consequence of the pull-back property of Hessians. \square

We can now extend a standard stability result, originally stated in [10] and [21] for the case of relative equilibria with continuous isotropy, extending the work of [24] on the nonlinear stability of regular relative equilibria. This also extends the theorem in [18], where it was assumed that G_μ is compact.

Theorem 3.6. *Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a G -Hamiltonian system and $z \in \mathcal{P}$ a relative equilibrium with momentum $\mathbf{J}(z) = \mu$. Suppose that \mathfrak{g} admits a G_μ -invariant inner product with respect to the adjoint representation. Suppose in addition that z admits a velocity $\xi \in \mathfrak{g}_\mu$ such that $\mathbf{d}_z^2 h_\xi|_N$ is definite, where N is any G_z -invariant complement to $\mathfrak{g}_\mu \cdot z$ in $\ker T_z \mathbf{J}$. Then z is nonlinearly stable, in the sense of Definition 3.2.*

Proof. Since $\varphi : Y = G \times_{G_z} (\mathfrak{m}^* \times N) \rightarrow \mathcal{P}$ is a local G -equivariant symplectomorphism, it is clear that $[e, 0, 0]$ is a relative equilibrium with momentum $\mathbf{J}_Y([e, 0, 0]) = \mu$ for the local G -Hamiltonian system $(Y, \omega_Y, G, \mathbf{J}_Y, h \circ \varphi)$ that models the original system in a neighbourhood of the group orbit $G \cdot z$. It is easy to verify that N , identified with $\{\gamma_{0,0,\dot{v}}([e, 0, 0]) : \dot{v} \in N\} \in T_{[e,0,0]}(G \times_{G_z} (\mathfrak{m}^* \times N))$ is a symplectic normal space at $[e, 0, 0]$. Let $\xi \in \mathfrak{g}_\mu$ be a velocity for $[e, 0, 0]$ and let $\xi = \xi^\perp + \eta \in \mathfrak{m} \oplus \mathfrak{g}_z$ according to the splitting (2.1). Then, it follows from (i) in Lemma 3.5 that $D_N \bar{h}_\eta(0, 0) = 0$ and $D_{\mathfrak{m}^*} \bar{h}(0, 0) = \xi^\perp$. Suppose now that $\mathbf{d}_{[e,0,0]}^2(h \circ \varphi)_\xi|_N$ is definite. Then, from (ii) also in Lemma 3.5 we have that $D_N^2 \bar{h}_\eta(0, 0)$ is definite. The existence of a G_μ -invariant inner product on \mathfrak{g} implies that μ is split, so (2.10–2.12) apply.

By the smoothness of the dependence of \bar{h} on \mathfrak{m}^* , it follows from the Morse lemma that $0 \in N$ is a Lyapunov stable point for (2.12). Also, the condition on the existence of the inner product guarantee that the coadjoint orbits of G_μ are contained in compact hypersurfaces, so they are bounded. This, using (2.11) implies that $0 \in \mathfrak{m}^*$ is Lyapunov stable for (2.11). Since $D_{\mathfrak{m}^*}h(\rho, v) + \eta \in \mathfrak{g}_\mu$ for all $(\rho, v) \in \mathfrak{m}^* \times N$, it follows from (2.10) that $g(t) \in g_0 G_\mu$ for any initial condition $g(0) = g_0$ and all t .

Let $O_G \subset G$ be an open neighbourhood of e . Any G_μ -invariant neighbourhood of $[e, 0, 0]$ in $G \times_{G_z} (\mathfrak{m}^* \times N)$ is given by $U = \{G_\mu \cdot [g, \rho, v] : g \in U_G, \rho \in U_{\mathfrak{m}^*}, v \in U_N\} \subset G \times_{G_z} (\mathfrak{m}^* \times N)$ where $U_G \subset G$, $U_{\mathfrak{m}^*} \subset \mathfrak{m}^*$ and $U_N \subset N$ are neighbourhoods of $e, 0$ and 0 respectively. It follows from the above discussion, and from the Lyapunov stability of $(0, 0)$ in $\mathfrak{m}^* \times N$ that we can find open neighbourhoods $O_{\mathfrak{m}^*} \subset U_{\mathfrak{m}^*}$ and $O_N \subset U_N$ of the origins such that $\rho(t) \in U_{\mathfrak{m}^*}$ and $v(t) \in U_N$ for all t if $\rho(0) \in O_{\mathfrak{m}^*}$ and if $v(0) \in O_N$. Therefore, calling $O = \{[g, \rho, v] : g \in U_G, \rho \in O_{\mathfrak{m}^*}, v \in O_N\} \subset G \times_{G_z} (\mathfrak{m}^* \times N)$, we have that the integral curves of $X_{h \circ \varphi}$ through points in O always lie inside U , proving the nonlinear stability of $[e, 0, 0]$, hence of z . It follows from Lemma 3.5(ii) that this is independent of the choice of symplectic normal space N . \square

Remark 3.7. Results analogous to Theorem 3.6 have been obtained first in [24] for the case when G_z is discrete and in [10, 21] under similar hypotheses as here. In [10] the result is proved using the symplectic slicing technique, and the proof of [21] is based on an extension of the methods of [24] to the case of continuous isotropy. Theorem 3.6 is, however, more general, since it could guarantee nonlinear stability in some cases in which the application of [10, 21] is inconclusive. The main difference with our result is that in those two references, the bilinear form on N used to test stability is $\mathbf{d}_z h_{\xi^\perp}$, where ξ^\perp is the orthogonal velocity associated to some choice of G_z -invariant splitting (2.1). The strongest stability results follow then by testing over all possible orthogonal velocities corresponding to all possible such invariant splittings. In our approach, we fix from the beginning one splitting, which is necessary only for the construction of the local model of the G -Hamiltonian system, and we test over all possible velocities admissible for the relative equilibrium under study. This is the same as testing stability for $\xi^\perp + \eta$ for all possible $\eta \in \mathfrak{g}_z$, where ξ^\perp is the orthogonal velocity corresponding to the fixed invariant splitting of \mathfrak{g}_μ . We provide now an example where the methods of [10, 21] don't predict stability, but Theorem 3.6 does. This is a different example to the one in [18], and here the relative equilibrium is not an equilibrium.

Let z be a relative equilibrium with $G_\mu = \text{SO}(3)$ and $G_z = S^1$. If we identify \mathfrak{g}_μ with \mathbb{R}^3 and \mathfrak{g}_z with $\text{span}\langle \mathbf{e}_3 \rangle$, the only G_z -invariant complement to \mathfrak{g}_z is given by $\mathfrak{m} = \text{span}\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$, which can be identified with \mathfrak{m}^* using the standard inner product on \mathbb{R}^3 .

Suppose that the symplectic normal space N at z is isomorphic to \mathbb{R}^4 , where in a Darboux basis the symplectic matrix Ω takes the form

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Suppose that in the linear coordinates $\{x_1, y_1, x_2, y_2\}$ associated to this basis we have

$$\bar{h}(\rho, v) = \frac{1}{2} (x_1^2 + y_1^2 - x_2^2 - y_2^2) + f(\rho),$$

where $v = (x_1, y_1, x_2, y_2)$, $\rho \in \mathfrak{m}^*$ and f is an arbitrary S^1 -invariant function on \mathfrak{m}^* . Suppose also that in this basis for N , the symplectic action of $G_z = S^1$ is given by

$$S^1 \ni \theta \mapsto \begin{pmatrix} R_\theta & 0 \\ 0 & R_\theta \end{pmatrix},$$

with R_θ the standard rotation through θ on \mathbb{R}^2 . The associated momentum map has the expression

$$\mathbf{J}_N(v) = \frac{1}{2} (x_1^2 + y_1^2 + x_2^2 + y_2^2).$$

It follows from Lemma 3.5 that testing definiteness of $\mathbf{d}_z h_{\xi^\perp}$, with ξ^\perp being the orthogonal velocity relative to the invariant splitting of \mathbb{R}^3 into \mathfrak{g}_z and \mathfrak{m} is equivalent to testing definiteness of

$$D_N^2 \bar{h}_0(0, 0) = \text{diag}(1, 1, -1, -1),$$

and the test is inconclusive. Since there are no more invariant splittings available, the results of [10, 21] can't predict the nonlinear stability of this relative equilibrium. In our approach, using Theorem 3.6 it follows from Lemma 3.5 that the relative equilibrium will be nonlinearly stable if there is some $\eta \in \mathbb{R} \simeq \mathfrak{g}_z$ for which $D_N^2 \bar{h}_\eta(0, 0)$ is definite. We have the general expression

$$\bar{h}_\eta(\rho, v) = \frac{1-\eta}{2} (x_1^2 + y_1^2) - \frac{1+\eta}{2} (x_2^2 + y_2^2) + f(\rho).$$

Then we obtain

$$D_N^2 \bar{h}_\eta(0, 0) = \text{diag}(1-\eta, 1-\eta, -(1+\eta), -(1+\eta)),$$

which is definite for $\eta \in (-\infty, -1) \cup (1, +\infty)$, and therefore the relative equilibrium is nonlinearly stable.

Finally, we remark that the first time that an approach similar to ours was used in the context of testing nonlinear stability of Hamiltonian relative equilibria is in [34]. In that reference, the authors only consider free Hamiltonian actions, which do not exhibit isotropy, however in that situation they investigate more general momentum values without requiring the existence of a G_μ -invariant inner product on \mathfrak{g} .

4. PERSISTENCE

The problem of persistence of relative equilibria consists in providing conditions under which a given relative equilibrium z belongs to a continuous family of relative equilibria, called the persisting branch. Additional properties about this family, such as the stability and stabilizers of its elements, or its geometric features, are usually also of interest.

Definition 4.1. A relative equilibrium z is said to persist, if for every G -invariant neighbourhood U containing z , the set $U \setminus G \cdot z$ contains a relative equilibrium.

In several references, it is also required in the definition of persistence that the persisting set of relative equilibria is the G -saturation of a parametrized branch of relative equilibria of the same symplectic type, as in Definition 3.4. We will not require this. In this section we will examine four different scenarios where a given relative equilibrium can persist, together with additional information about the persisting branch.

4.1. Persistence in the case of non-degeneracies.

The following theorem gives sufficient conditions for finding persisting branches of relative equilibria starting with a relative equilibrium satisfying a typical non-degeneracy hypothesis, as well as an estimate on the orbit types of the persisting relative equilibria. It extends Theorem 3.2 of [29] to non-compact groups, and uses a weaker hypothesis.

First recall that the orbit type of a point $z \in M$ is the conjugacy class in G of its stabilizer G_z and it is denoted by (G_z) . In particular, two distinct relative equilibria have the same orbit type if and only if they have conjugate stabilizers.

If M is a smooth manifold on which the Lie group G acts smoothly and properly, for each orbit type (H) , the set

$$\mathcal{P}_{(H)} = \{z \in M : G_z \text{ is conjugate to } H\},$$

called the manifold of orbit type (H) in \mathcal{P} , is a disjoint union of connected embedded submanifolds of \mathcal{P} . Our main persistence result in the case of non-degeneracies is the following.

Before stating the theorems, we discuss briefly an algebraic hypothesis we make below and in other theorems:

Lemma 4.2. *Let H be a compact subgroup of the Lie group G , with Lie algebras $\mathfrak{h} < \mathfrak{g}$ respectively. Suppose one can write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (as vector spaces), with $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$, the centre of \mathfrak{g} . Then*

- (i) *the decomposition is $\text{ad}(\mathfrak{h})$ -invariant;*
- (ii) *The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is an equality of Lie algebras, with \mathfrak{m} Abelian.*
- (iii) *$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$;*
- (iv) *any $\text{ad}(\mathfrak{h})$ -invariant decomposition (in particular if it is $\text{Ad}(H)$ -invariant) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ also satisfies $\mathfrak{m}' \subset \mathfrak{z}(\mathfrak{g})$;*

Furthermore, if G is compact then $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$ is equivalent to $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$.

When this condition that $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$ is satisfied, we say \mathfrak{h} is a *co-central* subalgebra of \mathfrak{g} . It holds for any subalgebra if \mathfrak{g} is Abelian, and a non-Abelian example is provided by $H = \text{SU}(n)$ as a subgroup of $G = \text{U}(n)$.

Proof. (i) For the decomposition to be $\text{ad}(\mathfrak{h})$ -invariant simply means $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, but since $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$, we have $[\mathfrak{h}, \mathfrak{m}] = 0 \subset \mathfrak{m}$.

(ii) This is clear, since $[\mathfrak{h}, \mathfrak{m}] = [\mathfrak{m}, \mathfrak{m}] = 0$.

(iii) This follows immediately from (ii).

(iv) Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}'$, the projection $\mathfrak{g} \rightarrow \mathfrak{m}$ with kernel \mathfrak{h} induces an isomorphism $f : \mathfrak{m}' \rightarrow \mathfrak{m}$. Moreover, since \mathfrak{m}' is $\text{ad}(\mathfrak{h})$ -invariant, the projection commutes with $\text{ad}(\mathfrak{h})$, so the two representations \mathfrak{m} and \mathfrak{m}' are equivalent. Thus $[\mathfrak{h}, \mathfrak{m}'] = 0$. Moreover, since $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$ it follows that $[\mathfrak{m}, \mathfrak{m}'] = 0$. Therefore $[\mathfrak{g}, \mathfrak{m}'] = 0$, whence $\mathfrak{m}' \subset \mathfrak{z}(\mathfrak{g})$, as required.

For the final part, suppose G is compact, and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$. It follows that $\mathfrak{h} \triangleleft \mathfrak{g}$ (an ideal) since $[\mathfrak{g}, \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, and hence (by compactness of G), there is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ that is $\text{ad}(\mathfrak{g})$ -invariant. In particular, this implies $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$. But since $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{h}$ by hypothesis, we conclude $[\mathfrak{g}, \mathfrak{m}] = 0$, whence $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$. \square

Note that since the adjoint action is by Lie algebra isomorphisms, fixed point sets in \mathfrak{g} are necessarily subalgebras (see Lemma 2.5). In particular if $K \leq G_z$ then $(\mathfrak{g}_\mu)^K$ is a subalgebra of \mathfrak{g}_μ , and moreover (since K is then compact), $(\mathfrak{g}_\mu^*)^K$ is naturally the dual of \mathfrak{g}_μ^K . The lemma above will be applied with \mathfrak{g}_z^K in place of \mathfrak{h} and \mathfrak{g}_μ^K in place of \mathfrak{g} .

Theorem 4.3. *Let $z \in \mathcal{P}$ be a relative equilibrium with momentum $\mathbf{J}(z) = \mu$, and let $\xi \in \mathfrak{g}_\mu$ be a velocity for z . Write $\xi = \xi^\perp + \eta \in \mathfrak{m} \oplus \mathfrak{g}_z$ according to the splitting (2.1). Suppose there exists a subgroup $K \leq (G_z)_\eta$ such that*

- (i) \mathfrak{g}_z^K is a co-central subalgebra of \mathfrak{g}_μ^K , and
- (ii) $\mathbf{d}_z^2 h_\xi|_{N^K}$ is non-degenerate.

Then, there is a smooth $N_{(G_z)_\eta}(K)$ -equivariant map

$$\begin{aligned} \bar{z} : \mathfrak{m}^{*K} \times \mathfrak{g}_z^K &\longrightarrow \mathcal{P}^K \\ (\rho, \eta') &\longmapsto \bar{z}(\rho, \eta') \end{aligned}$$

defined in a neighbourhood of $(0, 0)$ such that $\bar{z}(0, 0) = z$, and for each η' the map $\rho \mapsto \bar{z}(\rho, \eta')$ is an immersion, such that for each (ρ, η') in the domain, the point $\bar{z}(\rho, \eta')$ is a relative equilibrium with velocity of the form $\xi + \eta' \in \mathfrak{g}_\mu$ and stabilizer $G_{\bar{z}(\rho, \eta')}$ satisfying

$$K \leq G_{\bar{z}(\rho, \eta')} \leq (G_z)_\rho.$$

Moreover the branch $\bar{z}(\rho, \eta')$ consists of every relative equilibrium near z with stabilizer containing K and velocity ξ' satisfying $\xi' - \xi \in \mathfrak{g}_z^K$.

This theorem is only non-trivial if $\mathfrak{m}^{*K} \neq 0$. This is because, if $\mathfrak{m}^{*K} = 0$, then the theorem is satisfied by the constant map $\bar{z}(\eta') = z$, since z is by assumption a relative equilibrium with velocity $\xi + \eta'$ for every $\eta' \in \mathfrak{g}_z$.

Proof. Write $\xi = \xi^\perp + \eta \in \mathfrak{m} \oplus \mathfrak{g}_z$ according to the splitting (2.1), thus $\eta := \mathbb{P}_{\mathfrak{g}_z}(\xi) \in \mathfrak{g}_z$. Using the local model $(Y, \omega_Y, G, \mathbf{J}_Y, h \circ \varphi)$ around the group orbit $G \cdot z$, and putting $f(\eta', \rho, v) = \bar{h}_{\eta+\eta'}(\rho, v)$ the hypotheses imply that $D_N f(0, 0, 0) = 0$ and $D_N^2 f(0, 0, 0)|_{N^K}$ is non-degenerate (see (2.8) for definition of \bar{h}). Therefore, from Lemma 2.8 it follows that there exists a local smooth $N_{(G_z)_\eta}(K)$ -equivariant map $v : \mathfrak{m}^{*K} \times \mathfrak{g}_z^K \rightarrow N^K$ satisfying $D_N f(\eta', \rho, v(\rho, \eta')) = D_N \bar{h}_{\eta+\eta'}(\rho, v(\rho, \eta')) = 0$. Therefore (2.17) is satisfied for the isotropy Lie algebra element $\eta + \eta'$. In the local model, we therefore put $\bar{z}(\rho, \eta') = (\rho, v(\rho, \eta'))$, which for each η' is clearly an immersion. We will now show that (2.16) is also satisfied for the same choice of element. Using the G_z -invariant (and therefore K -invariant) splittings $\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_z$ and $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_z^*$, and noting that $\rho + \mathbf{J}_N(v(\rho, \eta')) \in \mathfrak{m}^* \oplus \mathfrak{g}_z^* = \mathfrak{g}_\mu^*$ we can identify $\rho + \mathbf{J}_N(v(\rho, \eta'))$ with an element $\lambda \in \mathfrak{g}_\mu^*$. Moreover, $\lambda \in \mathfrak{g}_\mu^{*K}$, since $\rho \in \mathfrak{m}^{*K}$ and by the equivariance of \mathbf{J}_N and the fact that $v(\rho, \eta') \in N^K$, $\mathbf{J}_N(v(\rho, \eta')) \in \mathfrak{g}_z^{*K}$. Therefore, (2.16) is equivalent to

$$\mathbb{P}_{\mathfrak{m}^*}(\text{ad}_\lambda^* \lambda) = 0, \tag{4.1}$$

with $\gamma = D_{\mathfrak{m}^*} \bar{h}(\rho, v(\rho, \eta')) + \eta + \eta'$. We claim that $\gamma \in \mathfrak{g}_\mu^K$. It then follows from hypothesis (i) that the equation above is satisfied. To see this, note that (4.1) is equivalent to

$$\langle \text{ad}_\gamma^* \lambda, \beta \rangle = 0, \quad \forall \beta \in \mathfrak{m}^K.$$

This in turn is equivalent to $\langle \lambda, [\gamma, \beta] \rangle = 0$ (for all $\beta \in \mathfrak{m}^K$). The hypothesis implies $\mathfrak{m}^K \subset \mathfrak{z}(\mathfrak{g}_\mu^K)$, whence $[\beta, \gamma] = 0$ for all $\beta \in \mathfrak{m}^K$, $\gamma \in \mathfrak{g}_\mu^K$ showing (4.1) to be satisfied, independently of the values of $\gamma \in \mathfrak{g}_\mu^K$ and $\lambda \in \mathfrak{g}_\mu^{*K}$.

To prove the claim that $\gamma \in \mathfrak{g}_\mu^K$, recall that $K \leq (G_z)_\eta$, and therefore $\eta + \eta' \in \mathfrak{g}_z^K$. It remains to show that $D_{\mathfrak{m}^*} \bar{h}(\rho, v(\rho, \eta')) \in \mathfrak{m}^K$. Now \bar{h} is G_z -invariant, and therefore for each $v \in N^K$, the function $\rho \mapsto \bar{h}(\rho, v)$ is K -invariant. It follows (from the Principle of Symmetric Criticality) that for $\rho \in \mathfrak{m}^{*K}$ we have $D_{\mathfrak{m}^*} \bar{h}(\rho, v) \in \mathfrak{m}^K$ as required.

Since we see that (2.16) and (2.17) are simultaneously satisfied, if we define $\bar{z}(\rho, \eta') \in \mathcal{P}$ by $\bar{z}(\rho, \eta') = \varphi([e, \rho, v(\rho, \eta')])$ then it follows by the equivariance of φ , Proposition 2.6 and the fact that φ is a local isomorphism of symmetric Hamiltonian systems that for each pair $(\rho, \eta') \in \mathfrak{m}^{*K} \times \mathfrak{g}_z^K$, the point $\bar{z}(\rho, \eta') \in \mathcal{P}$ is a relative equilibrium with velocity $\xi + \eta'$.

The stabilizer of $\bar{z}(\rho, \eta')$ is $G_{\bar{z}(\rho, \eta')}$. Since φ is G -equivariant this is equal to $G_{[e, \rho, v(\rho, \eta')]} = (G_z)_m$ where $(G_z)_m$ is the stabilizer of $m = (\rho, v(\rho, \eta')) \in \mathfrak{m}^* \times N$ with respect to the diagonal G_z -action. According to Lemma 2.8, $K \leq (G_z)_m$ (so $z \in \mathcal{P}^K$) and is contained in $(G_z)_\rho$. It also follows from Lemma 2.8 that the pairs $(\rho, v(\rho, \eta')) \in \mathfrak{m}^{*K} \times \mathfrak{g}_z^K$ give all the solutions to equation (2.15) with fixed ξ and η , from which the last statement follows. \square

4.2. Persistence of branches of constant orbit type.

As an application of Theorem 4.3, we now study in more detail under which conditions we can guarantee that a relative equilibrium belongs to a smooth branch of relative equilibria of the same orbit type (and possibly the same symplectic type), parametrized by momentum values. The following is an extension within our framework of the results of Patrick [25] in the free case and Lerman and Singer [10] in the singular case, each reference requiring slightly different hypotheses. It is also related to results of Patrick and Roberts [26], where G is assumed to be compact and acting freely, and they give, *inter alia*, necessary and sufficient conditions for the set of relative equilibria to form a symplectic submanifold. It would be interesting to adapt their approach to non-free actions.

Theorem 4.4. *Let $z \in \mathcal{P}$ be a relative equilibrium with momentum $\mu = \mathbf{J}(z)$ for which $\mathbf{d}_z^2 h_\xi|_{N_{G_z}}$ is non-degenerate for some admissible velocity $\xi \in \mathfrak{g}_\mu$. Suppose moreover that the Lie algebra \mathfrak{l} of $L := N_{G_\mu}(G_z)/G_z$ is Abelian, where $N_{G_\mu}(G_z)$ is the normalizer of G_z in G_μ .*

- (i) *There is a G -invariant neighbourhood U of z in $\mathcal{P}_{(G_z)}$ such that the set \mathcal{Z} of relative equilibria in U forms a smooth submanifold of dimension $\dim G/G_z + \dim(\mathfrak{m}^*)^{G_z}$. For each $\mu' \in \mathbf{J}(U)$ there is a relative equilibrium $z' \in \mathcal{Z}$ with $\mathbf{J}(z') = \mu'$.*
- (ii) *Suppose in addition that $G_z \triangleleft G_\mu$, or more generally $\Sigma \mathfrak{ic}(N_{G_\mu}(G_z)) = \mathfrak{g}_\mu$. Then the submanifold \mathcal{Z} is a symplectic submanifold of \mathcal{P} .*

- (iii) If G_μ is compact and Abelian then the momentum values of the relative equilibria of \mathcal{Z} have stabilizers conjugate to G_μ , and the family \mathcal{Z} is of constant symplectic type.
- (iv) Finally, still assuming the connected component of G_μ to be a torus, suppose z is formally stable. Then all the relative equilibria of this branch close enough to z are also formally stable and therefore nonlinearly stable.

Proof. (i) In the local model $Y = G \times_{G_z} (\mathfrak{m}^* \times N)$ of \mathcal{P} near $G \cdot z$, the point z corresponds to $[e, 0, 0]$. According to the property

$$G_{[g, \rho, v]} = g((G_z)_\rho \cap (G_z)_v)g^{-1},$$

the points of orbit type (G_z) are those of the form

$$[g, \rho, v], \quad \forall g \in G, \rho \in \mathfrak{m}^{*G_z}, v \in N^{G_z}.$$

We wish to find solutions of (2.14, 2.15) in $Y_{(G_z)}$. As usual, the velocity ξ can be decomposed as $\xi = \xi^\perp + \eta \in \mathfrak{m} \oplus \mathfrak{g}_z$, see (2.1). In the hypothesis of the theorem, we have that $D_N \bar{h}_\eta(0, 0) = 0$ and $D_N^2 \bar{h}_\eta(0, 0)|_{N^{G_z}}$ is non-degenerate. Using that \mathbf{J}_Y is a homogeneous quadratic polynomial on N and that $\eta \cdot v = 0$ for every $\eta \in \mathfrak{g}_z$ and $v \in N^{G_z}$, we have that $D_N \bar{h}(0, 0) = D_N \bar{h}_\eta(0, 0) = 0$ and that $D_N^2 \bar{h}(0, 0)|_{N^{G_z}} = D_N^2 \bar{h}_\eta(0, 0)|_{N^{G_z}}$ is non-degenerate.

By the Principle of Symmetric Criticality, and the pull-back property of Hessians, this means that the function $\bar{h}^{G_z}(\rho, \cdot) := \bar{h}(\rho, \cdot)|_{N^{G_z}}$, with $\rho \in \mathfrak{m}^{*G_z}$, has a non-degenerate critical point at $0 \in N^{G_z}$ for $\rho = 0$. Therefore, from the implicit function theorem there is a smooth function $v : \mathfrak{m}^{*G_z} \rightarrow N^{G_z}$ such that for any $g \in G$ and $\rho \in \mathfrak{m}^{*G_z}$, the points in the branch $[g, \rho, v(\rho)]$ are all the solutions of (2.15), with $\eta = 0$, with orbit type (G_z) for ρ sufficiently close to 0.

At points of this branch, condition (2.14) is equivalent to $\text{ad}_{D_{\mathfrak{m}^*} \bar{h}(\rho, v(\rho))}^* \rho = 0$, again because $\mathbf{J}_N(v) = 0$ for $v \in N^{G_z}$.

To see that this equality is satisfied for all ρ in a neighbourhood of 0, notice that since $\rho \in \mathfrak{m}^{*G_z}$ and $v(\rho) \in N^{G_z}$ then by the Principle of Symmetric Criticality, $D_{\mathfrak{m}^*} \bar{h}(\rho, v(\rho)) \in \mathfrak{m}^{G_z}$ (as in the proof of Theorem 4.3). Since \mathfrak{m}^{G_z} , together with the bracket induced from \mathfrak{g} is isomorphic to the Lie algebra \mathfrak{l} , and \mathfrak{m}^{*G_z} is isomorphic to $(\mathfrak{m}^{G_z})^* = \mathfrak{l}^*$, the hypothesis on the Abelian nature of \mathfrak{l} implies (2.14).

By (2.7), and since $\mathbf{J}_N(v) = 0$ for $v \in N^{G_z}$, the values of $\mathbf{J}|_{\mathcal{P}(G_z)}$ near μ form a neighbourhood of μ in the set

$$\{\text{Ad}_{g^{-1}}^*(\mu + \rho) : g \in G, \rho \in \mathfrak{m}^{*G_z}\}.$$

Let μ' be one of such values, close to the coadjoint orbit through μ . Then, there are $g \in G$ and $\rho \in \mathfrak{m}^{*G_z}$ close to 0, such that $\mu' = \text{Ad}_{g^{-1}}^*(\mu + \rho)$. It follows that the point $[g, \rho, v(\rho)]$ is a relative equilibrium of orbit type (G_z) close to the orbit $G \cdot z$.

Furthermore, if $\rho \in (\mathfrak{g}_\mu^*)^{G_\mu} \cap \mathfrak{m}^* \subset (\mathfrak{m}^*)^{G_z}$, then μ' has orbit type (G_μ) and the resulting relative equilibrium will have the same symplectic type as z .

(ii) The symplectic form on Y is given by ω_Y in (2.6). Pulling this back to the submanifold \mathcal{Z} of relative equilibria in (i) gives a closed form. To show it is non-degenerate

in a neighbourhood of $[e, 0, 0]$ it suffices to show it is non-degenerate at that point. Let $\gamma_{1,2}$ be arbitrary tangent vectors at this point:

$$\gamma_{1,2} = T\pi(g \cdot \lambda_{1,2}, \dot{\rho}_{1,2}, Dv(\rho) \cdot \dot{\rho}_{1,2}) \in T_{[g,\rho,v(\rho)]}Y.$$

Note that $\dot{\rho}_{1,2} \in (\mathfrak{m}^*)^{G_z}$, but by Lemma 2.5(iii) the hypothesis $G_z \triangleleft G_\mu$ implies $\mathfrak{m}^{G_z} = \mathfrak{m}$, and so

$$T_z\mathcal{Z} \simeq \mathfrak{g} \cdot z \times (\mathfrak{m}^*)^{G_z} = \mathfrak{g} \cdot z \times \mathfrak{m}^*.$$

It follows from (2.6) that at $[e, 0, 0]$,

$$\begin{aligned} \omega_Y(\gamma_1, \gamma_2) &= \langle \dot{\rho}_2, \lambda_1 \rangle - \langle \dot{\rho}_1, \lambda_2 \rangle + \langle \mu, [\lambda_1, \lambda_2] \rangle \\ &\quad + \Omega_N(Dv(\rho) \cdot \dot{\rho}_1, Dv(\rho) \cdot \dot{\rho}_2), \end{aligned}$$

since $\mathbf{J}_N|_{N^{G_z}} = T_{v(\rho)}\mathbf{J}_N|_{N^{G_z}} = 0$. We can decompose an element $\lambda \in \mathfrak{g}$ as

$$\lambda = \lambda^{\mathfrak{g}_z} + \lambda^{\mathfrak{m}} + \lambda^{\mathfrak{q}},$$

with respect to the fixed splitting $\mathfrak{g} = \mathfrak{g}_z \oplus \mathfrak{m} \oplus \mathfrak{q}$. Then

$$\begin{aligned} \omega_Y(\gamma_1, \gamma_2) &= \langle \dot{\rho}_2, \lambda_1^{\mathfrak{m}} \rangle - \langle \dot{\rho}_1, \lambda_2^{\mathfrak{m}} \rangle + \langle \mu, [\lambda_1^{\mathfrak{q}}, \lambda_2^{\mathfrak{q}}] \rangle \\ &\quad + \Omega_N(Dv(\rho) \cdot \dot{\rho}_1, Dv(\rho) \cdot \dot{\rho}_2). \end{aligned}$$

The first two terms are the standard symplectic form on $\mathfrak{m} \times \mathfrak{m}^*$, the last term can be seen as a ‘magnetic form’ on the same space, which does not alter the non-degeneracy, while the third term is the standard KKS symplectic form on $\mathfrak{g}_\mu^\circ = \mathfrak{q}^*$. It follows that ω_Y is indeed non-degenerate on $T_z\mathcal{Z} \simeq \mathfrak{g} \cdot z \times \mathfrak{m}^* = (\mathfrak{m} \times \mathfrak{q}) \times \mathfrak{m}^*$. (See also Lemma 4.2 of [10].)

(iii) If G_μ is compact then, using Palais’ tube theorem, one can choose a G_μ -invariant slice to the coadjoint orbit through μ , which can be identified with \mathfrak{g}_μ^* . Then the stabilizer of $\mathbf{J}_Y([e, \rho, v(\rho)]) = \mu + \rho$ is $(G_\mu)_\rho$, and since G_μ is Abelian, $(G_\mu)_\rho = G_\mu$. Thus all points of \mathcal{Z} have the same symplectic type.

(iv) Finally, suppose z is formally stable; that is, $\mathbf{d}_z^2 h_\xi|_N$ is definite for some admissible velocity $\xi \in \mathfrak{g}_\mu$. Since by (iii) the points of \mathcal{Z} are of the same symplectic type, they have isomorphic symplectic normal spaces $N' \simeq N$. By continuity, the appropriate nearby hessian $\mathbf{d}_{z'}^2 h_{\xi'}|_{N'}$ is also positive definite, and hence by Theorem 3.6 the relative equilibrium is nonlinearly stable. \square

4.3. Persistence in the case of degeneracies.

We now prove a persistence result specific for relative equilibria with continuous isotropy, which exploits the indeterminacy of the velocity in order to predict branches of persisting relative equilibria.

Theorem 4.5. *Let z be a relative equilibrium with momentum $\mathbf{J}(z) = \mu$ and suppose that z admits the velocity $\xi \in \mathfrak{g}_\mu$ with $\xi = \xi^\perp + \eta \in \mathfrak{m} \oplus \mathfrak{g}_z$. Let $K = (G_z)_\eta$ and suppose that there exists a subgroup $L \leq K$ for which $\mathcal{N} := \ker \mathbf{d}_z^2 h_\xi|_{N^L}$ is non-zero. Suppose moreover the following conditions are satisfied*

- (i) \mathfrak{g}_z^L is co-central in \mathfrak{g}_μ^L ,
- (ii) the action of $N_K(L)$ on \mathcal{N} is of cohomogeneity one, and
- (iii) the eigenvalue $\sigma(\eta')$ of $\mathbf{d}_z^2 h_{\xi+\eta'}|_{\mathcal{N}}$ for $\eta' \in \mathfrak{g}_z^K$, crosses 0 at $\eta' = 0$.

Then, for each $v \in \mathcal{N} \setminus \{0\}$ close to 0 there are elements $\eta'_v \in \mathfrak{g}_z^K$ close to 0 and $z_v \in \mathcal{P}$ close to z which are relative equilibria with velocity $\xi + \eta'_v$ and stabilizer K_v , which contains L .

In Section 7, we give an elementary illustration of this theorem in the system of pairs of point vortices on the sphere.

Proof. The proof is a direct application of Lemma 2.9, where we choose $W = \mathfrak{g}_z^K$, and $f(\eta', v) = \bar{h}_{\eta+\eta'}(0, v)$ (with $\rho = 0$). Notice that if z corresponds to $[e, 0, 0]$ in the local model, then Lemma 2.9, predicts that for each $v \in \mathcal{N}$ there is an η'_v such that v is a critical point of $\bar{h}_{\eta+\eta'_v}(0, \cdot)$. Therefore, $(0, v)$ satisfies (2.17) for $\eta + \eta'_v$. By an argument similar to the one used in Theorem 4.3, the Principle of Symmetric Criticality guarantees that $D_{\mathfrak{m}^*} \bar{h}(0, v) + \eta + \eta'_v \in \mathfrak{g}_\mu^L$, and then condition (i) implies that (2.16) is also satisfied. Therefore, in the local model the persisting relative equilibria predicted by this theorem are group orbits of points of the form $[e, 0, v]$ for each $v \in \mathcal{N}$. These are the points that correspond to z_v in \mathcal{P} .

The stabiliser is given by $(G_z)_v \cap (G_z)_{\eta'}$. But $\eta' \in \mathfrak{g}_z^K$, so $(G_z)_{\eta'} = K$, and $v \in N^L$ whence $L \leq (G_z)_v$. \square

4.4. Persistence from formally stable relative equilibria.

To end this section, we prove a more general result on momentum parametrized branches persisting from formally stable relative equilibria. It was originally obtained in [19] in a slightly different setup, but there it was required that G_μ be compact (see the remark below for more details). Here we present it as a product of the bundle equations formalism.

Theorem 4.6. *Let z be a formally stable relative equilibrium with $\mu = \mathbf{J}(z)$. Suppose that \mathfrak{g}_μ has a G_μ -invariant inner product (for example, G_μ is compact). Then there is a G -invariant neighbourhood U of z such that for every $\mu' \in \mathbf{J}(U)$ near μ there is a relative equilibrium z' near z with $\mathbf{J}(z') = \mu'$.*

Proof. Let U_0 be a neighbourhood of z for which the Marle-Guillemin-Sternberg normal form is valid. Recall that for a point $[g, \rho, v] \in Y := G \times_{G_z} (\mathfrak{m}^* \times N)$ we have

$$\mathbf{J}([g, \rho, v]) = \text{Ad}_{g^{-1}}^*(\mu + \rho + \mathbf{J}_N(v))$$

Therefore $\mathbf{J}(Y) = \{\text{Ad}_{g^{-1}}^*(\mu + \rho + \epsilon) : g \in G, \rho \in \mathfrak{m}^*, \epsilon \in \text{im } \mathbf{J}_N\}$. Let $Y_0 \subset Y$ be the subset corresponding to U_0 ; then $\mathbf{J}(Y_0) \subset \mathbf{J}(Y)$. If G_μ is compact then $\mathbf{J}(Y_0)$ is an open subset of $\mathbf{J}(Y)$, but it is not known in the more general setting here—see also Remark 4.7 below.

Step 1. Taking $\mu' = \mu + \bar{\rho} + \bar{\epsilon}$

Suppose $z' \in Y_0$ (or U_0) is a relative equilibrium with $\mathbf{J}(z') = \text{Ad}_{g^{-1}}^*(\mu + \bar{\rho} + \bar{\epsilon})$. This is equivalent to $g^{-1} \cdot z'$ being a relative equilibrium with momentum $\text{Ad}_g^*(\text{Ad}_{g^{-1}}^*(\mu + \bar{\rho} + \bar{\epsilon})) = \mu + \bar{\rho} + \bar{\epsilon}$. Therefore in the statement of the theorem we can choose, without loss of generality, $\mu' = \mu + \bar{\rho} + \bar{\epsilon}$.

Step 2. Using the bundle equations

According to Proposition 2.6, a point $[g, \rho, v]$ is a RE if and only if there exists $\eta \in \mathfrak{g}_z$

such that

$$\mathrm{ad}_{D_{\mathfrak{m}^*}^* \bar{h}(\rho, v) + \eta}(\rho + \mathbf{J}_N(v)) = 0 \quad (4.2)$$

$$D_N \bar{h}_\eta(\rho, v) = 0. \quad (4.3)$$

Equation (4.2) is an equation in \mathfrak{g}_μ^* and is equivalent to

$$(D_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta) \Big|_{T_{\rho + \mathbf{J}_N(v)} \mathcal{O}} = 0 \quad (4.4)$$

where $\mathcal{O} = \{\mathrm{Ad}_{l^{-1}}(\rho + \mathbf{J}_N(v)) : l \in G_\mu\} \subset \mathfrak{g}_\mu^*$ is the G_μ -orbit of $\rho + \mathbf{J}_N(v)$ for the coadjoint action.

Now we need to use the formal stability of $z = [e, 0, 0]$. By Lemma 3.5(i), this is equivalent to there being a $\gamma \in \mathfrak{g}_z$ (which we fix throughout) such that $D_N^2 \bar{h}_\gamma$ is definite (as a quadratic form on N). Let us assume it is positive definite, the negative definite case being similar.

Define $f \in C^\infty(\mathfrak{g}_\mu^* \times N)$ by

$$f(\alpha, v) := \bar{h}_\gamma(\alpha|_{\mathfrak{m}}, v) - \bar{h}_\gamma(0, 0)$$

and let $f_{\mathcal{O}}$ be its restriction to $\mathcal{O} \times N$. Define also $\phi : \mathfrak{g}_\mu^* \times N \rightarrow \mathfrak{g}_z^*$ by

$$\phi(\alpha, v) := \mathbf{J}_N(v) - \alpha|_{\mathfrak{g}_z},$$

and $\phi_{\mathcal{O}}$ its restriction to $\mathcal{O} \times N$.

Now (4.3) and (4.4) are together equivalent to the restriction of $f_{\mathcal{O}}$ to $\phi_{\mathcal{O}}^{-1}(0)$ having a critical point at the point $(\rho \oplus \mathbf{J}_N(v), v) \in \mathfrak{m}^* \oplus \mathfrak{g}_z^* \times N = \mathfrak{g}_\mu^* \times N$, where $\eta \in \mathfrak{g}_z$ is the Lagrange multiplier. Indeed, the latter can be written

$$\begin{aligned} D_N(f - \langle \phi, \eta \rangle) &= 0 \\ D_{\mathfrak{g}_\mu^*}(f - \langle \phi, \eta \rangle) \Big|_{T_{\alpha} \mathcal{O}} &= 0. \end{aligned}$$

where $\alpha = \rho + \mathbf{J}_N(v)$ (that is, $\phi(\alpha, v) = 0$). The first of these is equivalent to (4.3), while the second becomes (4.4), since $(f - \langle \phi, \eta \rangle)$ is independent of the \mathfrak{g}_z^* component of α .

Step 3. Existence of critical points

We claim that for sufficiently small \mathcal{O} , $f_{\mathcal{O}}$ is proper. To see this, we can apply the Splitting Lemma from singularity theory, see for example [28, 16]: since $D_N f(z) = 0$ and $D_N^2 f(z)$ is non-degenerate, there is a neighbourhood U of $(0, 0)$ in $\mathfrak{g}_\mu^* \times N$ and change of coordinates on U of the form $(\alpha, v) \mapsto (\alpha, V)$, where $V = V(\alpha, v)$, and a smooth function $g : \mathfrak{g}_\mu^* \rightarrow \mathbb{R}$ such that

$$f(\alpha, v) = g(\alpha) + f(0, 0) + D_N^2 f(V, V),$$

and similarly for $f_{\mathcal{O}}$. For simplicity, let us assume $f(0, 0) = 0$. Now let $I = [a, b] \subset \mathbb{R}$ be a compact interval. Then, since \mathcal{O} is compact,

$$f_{\mathcal{O}}(\alpha, v) \in I \Rightarrow D_N^2 f(V, V) \in [a - \max g(\mathcal{O}), b - \min g(\mathcal{O})].$$

Since $D_N^2 f(V, V)$ is positive definite, this implies V is contained in a compact set N_I . Then

$$f_{\mathcal{O}}^{-1}(I) \subset \mathcal{O} \times N_I$$

which is compact, and hence indeed $f_{\mathcal{O}}$ is proper. It is also clear that $f_{\mathcal{O}}$ is bounded below, by $\min g(\mathcal{O})$.

To finish we use a well-known variational argument: any continuous function $f : X \rightarrow \mathbb{R}$ (X a topological space) that is proper and bounded below attains its greatest lower bound, which is therefore a minimum and hence a critical point. To see this, let r_1 be the greatest lower bound of $f(X)$, and let $r_2 > r_1$. Then $K := f^{-1}([r_1, r_2])$ is non-empty and compact (as f is proper). Therefore $f(K)$ is a compact subset of \mathbb{R} so must be equal to $[r_1, r_2]$, and hence the greatest lower bound r_1 is attained by f , as claimed.

Applying this to the restriction of $f_{\mathcal{O}}$ to $\phi_{\mathcal{O}}^{-1}(0) \subset \mathcal{O} \times N$, for $\alpha = \bar{\rho} + \bar{\epsilon}$ sufficiently small that $\mathcal{O} \subset U$, shows that indeed this restriction of $f_{\mathcal{O}}$ has a critical point as required. \square

Remark 4.7. There are two differences with the persistence theorem in [19]. The first is that in [19] it is only assumed that the relative equilibrium is ‘extremal’, rather than formally stable. In practice, formal stability is the most straightforward way of testing for extremality, although it is of course a stronger condition. An example of an extremal relative equilibrium which is not formally stable is the famous Thompson heptagon in the planar point vortex problem, see the proof in [20] of its stability (Corollary 14.7 and the calculation in Sec. 14.3.5.2).

The second difference is more subtle. The conclusion in [19] is that relative equilibria exist for all μ' in a neighbourhood of $\mathbf{J}(z)$ in $\mathbf{J}(U_0)$ (where, as in the proof above, U_0 is the G -invariant neighbourhood on which the MGS normal form is valid). Here on the other hand, existence is only guaranteed for $\mu' \in \mathbf{J}(U)$, the image under \mathbf{J} in $\mathbf{J}(U_0)$ of some (small) G -invariant open set U . This image $\mathbf{J}(U)$ may not in general be a full neighbourhood of $\mu = \mathbf{J}(z)$ in $\mathbf{J}(U_0)$. On the other hand, if G_{μ} is compact, then in [19] it is proved that \mathbf{J} is G -open, meaning that the image of U will be open in $\mathbf{J}(U_0)$ and the results will coincide (in this respect).

5. BIFURCATIONS

We now consider the problem of bifurcations of branches of relative equilibria. Starting with a parametrized branch of relative equilibria of the same symplectic type, we will give sufficient conditions for the existence of bifurcating branches. At the end of this section we study the possibility of bifurcations from persisting branches consisting of formally stable relative equilibria. Recall that we assume G acts properly on the symplectic manifold \mathcal{P} (the phase space) with momentum map $\mathbf{J} : \mathcal{P} \rightarrow \mathfrak{g}^*$, and this is equivariant with respect to a (possibly modified) coadjoint action. Moreover, $h : \mathcal{P} \rightarrow \mathbb{R}$ is a G -invariant Hamiltonian function.

5.1. Bifurcations from branches of constant symplectic type.

The following result gives sufficient conditions for the occurrence of bifurcations from a parametrized branch of relative equilibria of the same symplectic type.

Theorem 5.1. *Let $z \in \mathcal{P}$ be a relative equilibrium with momentum $\mathbf{J}(z) = \mu$ and velocity $\xi \in \mathfrak{g}_{\mu}$ written $\xi = \xi^{\perp} + \eta \in \mathfrak{m} \oplus \mathfrak{g}_z$ according to (2.1). For a subgroup $K \leq (G_z)_{\eta}$ and a vector subspace $W \subset \mathfrak{m}^* \times \mathfrak{g}_z^K$ let $\bar{z} : W \rightarrow \mathcal{P}$, be a parametrized branch of relative equilibria of the same symplectic type, satisfying $\bar{z}(0, 0) = z$, $\mathbf{J}(\bar{z}(\rho, \eta')) = \mu + \rho$, $G_{\mathbf{J}(\bar{z}(\rho, \eta'))} = G_{\mu}$ and $G_{\bar{z}(\rho, \eta')} = G_z$. Let $\xi(\rho, \eta')$ be a family of velocities for points of this*

branch chosen such that $\mathbb{P}_{\mathfrak{g}_z}\xi(\rho, \eta') = \eta + \eta'$ (this is always possible). Now suppose that there exists a subgroup $L \leq K$ satisfying

- (i) \mathfrak{g}_z^L is co-central in \mathfrak{g}_μ^L (see Lemma 4.2),
- (ii) $N_{G_z}(L)$ acts on $\mathcal{N} := \ker \mathbf{d}_z^2 h_\xi|_{N^L}$ with cohomogeneity one, and
- (iii) the eigenvalue $\sigma(\rho, \eta')$, with $(\rho, \eta') \in W$, of $\mathbf{d}_{\bar{z}(\rho, \eta')}^2 h_{\xi(\rho, \eta')}|_{\mathcal{N}}$ crosses 0 at $0 \in W$.

Then, for every $v \in \mathcal{N}$ close enough to the origin, there is a relative equilibrium \tilde{z}_v near z with velocity $\xi_v \in \mathfrak{g}_\mu$ close to ξ and not in the G -orbit of any point of the original branch $\bar{z}(\rho, \eta')$. The stabilizer of \tilde{z}_v is

$$G_{\tilde{z}_v} = (G_z)_v \geq L.$$

Proof. In the local model, the given branch $\bar{z}(\rho, \eta')$ can be written as

$$\bar{z}(\rho, \eta) = [g(\rho, \eta'), \bar{\rho}(\rho, \eta'), v(\rho, \eta')],$$

with $[g(0, 0), \bar{\rho}(0, 0), v(0, 0)] = [e, 0, 0]$, for $(\rho, \eta') \in \mathfrak{m}^* \times \mathfrak{g}_z^K$. Since relative equilibria come in group orbits, we can choose in our analysis $g(\rho, \eta') = e$ without loss of generality.

Since points in the branch satisfy $\mathbf{J}(\bar{z}(\rho, \eta')) = \mu + \bar{\rho}(\rho, \eta') + \mathbf{J}_N(v(\rho, \eta')) = \mu + \rho$ it follows that

$$\bar{\rho}(\rho, \eta') = \rho \quad \text{and} \quad G_{\mu+\rho} = G_\mu$$

Since $\bar{z}(\rho, \eta')$ and z are of the same symplectic type, we can assume that $G_{\bar{z}(\rho, \eta')} = G_z$ and in particular

$$(\rho, v(\rho, \eta')) \in \mathfrak{m}^{*G_z} \times N^{G_z}.$$

So actually $W \subset \mathfrak{m}^{*G_z} \times \mathfrak{g}_z^K \subset \mathfrak{m}^{*(G_z)\eta} \times \mathfrak{g}_z^K$. Let $\varphi_{\rho, \eta'} : N \rightarrow N$ be the W -parametrized family of diffeomorphisms of N given by

$$\varphi_{\rho, \eta'}(v) = v + v(\rho, \eta').$$

These maps are obviously K -equivariant. Notice also that $\mathbf{J}_N(\varphi_{\rho, \eta'}(v)) = \mathbf{J}_N(v)$ since $v(\rho, \eta') \in N^{G_z}$.

Notice that $\mathbf{d}h_{\xi(\rho, \eta')}(\bar{z}(\rho, \eta')) = 0$ if and only if $\mathbf{d}\bar{h}_{\eta+\eta'}(\rho, v(\rho, \eta')) = 0$ which, putting $\bar{h}'_{\eta+\eta'}(\rho, v) = \bar{h}(\rho, \varphi_{\rho, \eta'}(v)) - \mathbf{J}_N^{\eta+\eta'}(\varphi_{\rho, \eta'}(v))$, is in turn equivalent to

$$D_N \bar{h}'_{\eta+\eta'}(\rho, 0) = 0.$$

Let us call $f(\eta', \rho, v) = \bar{h}'_{\eta+\eta'}(\rho, v)$. The function $f \in C^\infty(W \times N)$ is by construction $(G_z)_\eta$ -invariant. We have according to Lemma 3.5 (using \bar{h}' as the function \bar{h} in the statement of the lemma) that, for every $(\rho, \eta') \in W$,

$$D_N f(\eta', \rho, 0) = 0,$$

and that for any $L \leq K$, the eigenvalues (and their multiplicities) of

$$D_N^2 f(\eta', \rho, 0)|_{N^L}$$

and

$$\mathbf{d}_{\bar{z}(\rho, \eta')}^2 h_{\xi(\rho, \eta')}|_{N^L}$$

coincide. In particular, we also have

$$\ker D_N^2 f(0, 0, 0)|_{N^L} = \ker D_N^2 \bar{h}_\eta(0, 0)|_{N^L} = \ker \mathbf{d}_z^2 h_\xi|_{N^L}.$$

We have that f satisfies the hypotheses of Lemma 2.9 with $K = (G_z)_\eta$. Therefore we can conclude that for each $v \in \ker D_N^2 \bar{h}_\eta(0, 0)|_{N^L \setminus \{0\}}$ there is a pair $(\rho(v), \eta'(v)) \in W$ satisfying

$$D_N \bar{h}_{\eta+\eta'(v)}(\rho(v), v) = 0.$$

We have shown that the point $[e, \rho(v), v]$ satisfies (2.17). We now show that it also satisfies (2.16). The argument is again very similar to the one used in the proof of Theorem 4.3. Using the splitting $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_z$ we have that $\rho(v) + \mathbf{J}_N(v) \in \mathfrak{g}_\mu^*$. In fact, since $(\rho(v), v) \in \mathfrak{m}^{*G_z} \times N^L$, we can say that $\rho(v) + \mathbf{J}_N(v) \in \mathfrak{g}_\mu^{*L}$. Calling $\gamma = D_{\mathfrak{m}^*} \bar{h}(\rho(v), v) + \eta + \eta'$, if we show that $\gamma \in \mathfrak{g}_\mu^L$ then (2.17) will follow from the hypothesis (i). It is clear that $D_{\mathfrak{m}^*} \bar{h}(\rho(v), v) + \eta + \eta' \in \mathfrak{m} \oplus \mathfrak{g}_z = \mathfrak{g}_\mu$, so it only remains to show that this element is invariant by L . This is true since, on the one hand $\eta + \eta' \in \mathfrak{g}_z^L$ and on the other hand, using the Principle of Symmetric Criticality, an argument similar to the one used in Theorem 4.3 shows that $D_{\mathfrak{m}^*} \bar{h}(\rho(v), v) \in \mathfrak{m}^L$ since $\rho(v) \in \mathfrak{m}^{*G_z} \subset \mathfrak{m}^{*L}$.

Therefore for each $v \in \ker \mathcal{N} \setminus \{0\}$ close enough to the origin we have that $\tilde{z}(v) = \varphi([e, \rho(v), v])$ is a relative equilibrium with velocity $D_{\mathfrak{m}^*} \bar{h}(\rho(v), v) + \eta + \eta' \in \mathfrak{g}_\mu$. In addition, since $G_{\tilde{z}(v)} = (G_z)_{\rho(v)} \cap (G_z)_v$ and we found that $W \in \mathfrak{m}^{*G_z} \times \mathfrak{g}_z^L$ then $(G_z)_{\rho(v)} = G_z$ and therefore $G_{\tilde{z}(v)} = (G_z)_v$. Since $v \in N^L$ we also have $L \leq G_{\tilde{z}(v)}$. \square

5.2. Bifurcations from a branch of formally stable points of constant symplectic type.

As a consequence of Theorem 5.1 we now show that branches of formally stable relative equilibria of the same symplectic type with 2-dimensional fixed point subspaces and satisfying the usual group-theoretic conditions used in this article, typically have bifurcating solutions at each point of the branch. This is a phenomenon due exclusively to the existence of continuous isotropy and cannot happen for branches of formally stable relative equilibria with discrete stabilizers, for which the existing persisting branch of relative equilibria of the same symplectic type must persist without bifurcation.

Theorem 5.2. *Assume the hypotheses of Theorem 5.1 for a branch $\bar{z}(\rho, \eta')$ consisting of formally stable points, where conditions (ii), (iii) are replaced with*

- (ii) $\dim N^L = 2$, and
- (iii) $\mathbf{d}_z^2 h_\xi|_{N^L}$ is definite.

Then, generically, for every $v \in \ker \mathbf{d}_z^2 h_\xi|_{N^L} \setminus \{0\}$ close enough to the origin, there is a relative equilibrium

$$\tilde{z}_v$$

near z with velocity of the form $\xi_v \in \mathfrak{g}_\mu$ close to ξ and not in the G -orbit of any point of the original branch $\bar{z}(\rho, \eta')$.

Proof. Using the function $f \in C^\infty(W \times N)$ introduced in the proof of Theorem 5.1 we have that in the local model we can assume that $\bar{z}(\rho, \eta') = [e, \rho, 0]$ and that each of these points is a relative equilibrium with velocity $\xi^\perp + \eta + \eta'$. Furthermore, the signatures of $D_N^2 f(\eta', \rho, 0)|_{N^L}$ and $\mathbf{d}_{\bar{z}(\rho, \eta')}^2 h_{\xi(\rho, \eta')}|_{N^L}$ are the same. Since $\mathbf{d}_z^2 h_\xi|_{N^L}$ is definite then so is $D_N^2 f(\eta', \rho, 0)|_{N^L}$ in a neighbourhood of $(0, 0) \in W$. We therefore have a map $\psi : W \rightarrow \text{Sym}(2)$ from a neighbourhood of $(0, 0)$ in W to the space of symmetric 2×2 matrices and its image lies in the set of definite matrices.

If we give coordinates $\{x, y, z\}$ on $\text{Sym}(2)$, where x, y are the diagonal elements and z is off-diagonal, the set of definite matrices is given by the two connected components of the set $\text{Sym}(2, +)$ defined by $xy - z^2 > 0$. These connected components are the set of positive definite and negative definite matrices respectively. The set $\text{Sym}(2, -)$ of symmetric 2×2 matrices with index 1 is defined by $xy - z^2 < 0$. These two regions are separated by the set $\text{Sym}(2, 0)$ of matrices with at least one zero eigenvalue, given by $xy - z^2 = 0$. The origin in \mathbb{R}^3 represents the zero matrix. Therefore any point in $\text{Sym}(2, 0)$ other than the origin represents a matrix with only one zero eigenvalue. Suppose that there is an element $\bar{\eta} \in K$ such that $D^2 \mathbf{J}_N^{\bar{\eta}}(0)|_{N^L}$ is not proportional to $D_N^2 f(\eta', \rho, 0)|_{N^L}$. This implies that we can find some $t \in \mathbb{R}$ satisfying $D_N^2 f(0, 0)|_{N^L} + t D^2 \mathbf{J}_N^{\bar{\eta}}(0, 0)|_{N^L} \in \text{Sym}(2, 0) \setminus \{(0, 0, 0)\}$. Notice that this last matrix is equal to $D_N^2 \bar{f}(0, 0)|_{N^L}$ where $\bar{f}(\eta', \rho, v) = \bar{h}_{\eta+t\bar{\eta}+\eta'}(\rho, v)$. Notice also that we can rewrite our starting branch of relative equilibria as $\bar{z}(\rho, \eta') = \bar{z}(\rho, t\bar{\eta} + \eta')$ but now this produces the map $\bar{\psi} : W \rightarrow \text{Sym}(2)$ given by $\bar{\psi}(\rho, \eta') = D_N^2 \bar{f}(\eta', \rho, 0)|_{N^L}$ which satisfies $\bar{\psi}(0, 0) \in \text{Sym}(2, 0) \setminus \{(0, 0, 0)\}$ and generically will cross this surface. In other words, substituting the initial velocity of z , the element ξ , by the also admissible velocity $\xi^\perp + \eta + t\bar{\eta}$, we can force all the conditions of Theorem 5.1 to be satisfied generically, and the existence of the stated bifurcating branch follows. \square

6. EXAMPLE. THE SLEEPING LAGRANGE TOP

This classic example will allow us to illustrate our results on the persistence, stability and bifurcations of relative equilibria. We will briefly summarize the construction of this system, referring to [21] for more details, since we will be using the same notations and conventions.

The Lagrange top is a mechanical system consisting in a rigid body with an axis of symmetry and a fixed point in the presence of an homogeneous vertical gravitational field. Relative equilibria are steady rotations around the vertical axis and the axis of symmetry. Mathematically, this is the symmetric Hamiltonian system

$$(T^* \text{SO}(3), \omega_c, \mathbb{T}^2, \mathbf{J}, h),$$

where ω_c is the canonical symplectic form on the cotangent bundle $T^* \text{SO}(3)$ and $\mathbf{J} : T^* \text{SO}(3) \rightarrow \mathfrak{t}^2$ its canonical momentum map. We will use the right trivialization of $T^* \text{SO}(3)$, and will identify $\mathfrak{so}(3)$ with \mathbb{R}^3 and \mathfrak{t}^2 with \mathbb{R}^2 in the usual ways. The Euclidean structures on these two vector spaces will allow us to identify them also with the duals of their Lie algebras. In this setup, a point in $T^* \text{SO}(3)$ is represented by a pair (Λ, π) , where $\Lambda \in \text{SO}(3)$, viewed as a matrix group, and $\pi \in \mathbb{R}^3$. The several relevant elements for the analysis are:

- The action $\mathbb{T}^2 \times T^* \text{SO}(3) \rightarrow T^* \text{SO}(3)$, given by

$$((\theta_1, \theta_2), (\Lambda, \pi)) \mapsto (e^{\theta_1 \widehat{\mathbf{e}}_3} \Lambda e^{-\theta_2 \widehat{\mathbf{e}}_3}, e^{\theta_1 \widehat{\mathbf{e}}_3} \pi). \quad (6.1)$$

- The Hamiltonian

$$h(\Lambda, \pi) = mgl \mathbf{e}_3 \cdot \Lambda \mathbf{e}_3 + \frac{1}{2} \pi \cdot \mathbb{I}_\Lambda^{-1} \pi,$$

where $\mathbb{I} = \text{diag}(I_1, I_1, I_3)$ and $\mathbb{I}_\Lambda = \Lambda \mathbb{I} \Lambda^{-1}$. Here I_1, I_3 are the principal moments of inertia of the body, I_3 corresponding to the axis of symmetry.

- The momentum map

$$\mathbf{J}(\Lambda, \pi) = (\pi \cdot \mathbf{e}_3, -\pi \cdot \Lambda \mathbf{e}_3) \in \mathbb{R}^2 \simeq \mathfrak{t}^{2*}.$$

The sleeping Lagrange top is a family of relative equilibria of this Hamiltonian system, consisting in the top rotating around the vertical position, with the gravity and symmetry axes aligned. This corresponds to the points in phase space of the form $\mathbb{T}^2 \cdot z$, where $z = (\Lambda, \pi) = (I, \lambda I_3 \mathbf{e}_3)$, I is the identity 3×3 matrix and λ is any non-zero real number, corresponding to the angular velocity of the rotation. It is easy to see from (6.1) that the stabilizer of any sleeping Lagrange top is

$$G_z = \{(\theta_1, \theta_2) \in \mathbb{T}^2 : \theta_1 = \theta_2\} \simeq S^1.$$

The Lie algebra \mathfrak{g}_z can be identified with \mathbb{R} as

$$\mathfrak{g}_z = \{(\eta, \eta) \in \mathbb{R}^2 : \eta \in \mathbb{R}\}.$$

Therefore this relative equilibrium has continuous isotropy and its velocity is not uniquely determined. It is easy to see that a possible orthogonal velocity is given by $\xi^\perp = \frac{1}{2}(\lambda, -\lambda) \in \mathbb{R}^2$, corresponding to the choice

$$\mathfrak{m} = \{(\gamma, -\gamma) \in \mathbb{R}^2 : \gamma \in \mathbb{R}\}. \quad (6.2)$$

It follows that the set of all possible velocities for this relative equilibrium is $(\eta + \lambda/2, \eta - \lambda/2)$, for any real number η . Its momentum value is

$$\mu = (\lambda I_3, -\lambda I_3). \quad (6.3)$$

Since the symmetry group is Abelian, $G_\mu = G = \mathbb{T}^2$, and $\dim \mathfrak{m}^* = 1$.

It follows that with respect to the basis of N given by

$$v_1 = (\mathbf{e}_1, 0), v_2 = (\mathbf{e}_2, 0), v_3 = (0, \mathbf{e}_1), v_4 = (0, \mathbf{e}_2)$$

the symplectic normal space N at $z = (I, \lambda I_3 \mathbf{e}_3)$ can be identified with \mathbb{R}^4 , and

$$\mathbf{d}_z^2 h_{(\eta+\lambda/2, \eta-\lambda/2)} \Big|_N = \begin{pmatrix} A & 0 & 0 & B \\ 0 & A & -B & 0 \\ 0 & -B & C & 0 \\ B & 0 & 0 & C \end{pmatrix} \quad (6.4)$$

where

$$A = -mgl - \frac{\lambda^2 I_3}{2I_1} (2I_3 - I_1) - \eta \lambda I_3, \quad B = \frac{\lambda}{2I_1} (2I_3 - I_1) - \eta, \quad C = \frac{1}{I_1}.$$

Stability. We will fix $\lambda \neq 0$ and study the nonlinear stability of $z = (I, \lambda I_3)$. In order to apply Theorem 3.6 we need to find an admissible velocity ξ for z such that $\mathbf{d}_z^2 h_\xi \Big|_N$ is definite. This velocity will be of the form $\xi = (\eta + \lambda/2, \eta - \lambda/2)$.

The eigenvalues of $\mathbf{d}_z^2 h_{(\eta+\lambda/2, \eta-\lambda/2)} \Big|_N$ are given by

$$\sigma_\pm = \frac{1}{2} \left((A + C) \pm \sqrt{(A + C)^2 - 4(AC - B^2)} \right), \quad (6.5)$$

each with multiplicity 2. It follows that this matrix is definite provided $AC - B^2 > 0$. Since η is arbitrary and $\lambda \neq 0$, we will make $\eta = k\lambda$ for arbitrary $k \in \mathbb{R}$. Then we have

$$AC - B^2 = \frac{1}{4I_1} (\lambda^2 (I_3(4k+2) - I_1(4k^2 + 4k + 1)) - 4glm).$$

Then, the relative equilibrium will be stable if we can find some $k \in \mathbb{R}$ for which $\lambda^2 > \frac{4glm}{I_3(4k+2) - I_1(4k^2 + 4k + 1)}$. It is clear that this is always possible if λ^2 is sufficiently large, so we are interested in the element of the family of sleeping Lagrange tops with minimum $|\lambda|$ yet nonlinearly stable. For that, we find a minimum of the above function of k , which happens when $k = \frac{I_3 - I_1}{2I_1}$, and then, the relative equilibrium $z = (I, \lambda I_3 \mathbf{e}_3)$ is nonlinearly stable provided

$$\lambda^2 > \frac{4glmI_1}{I_3^2}$$

which is the well known classical stability condition (fast-top condition).

Persistence. It is straightforward to show that the linear action of $G_z \simeq S^1$ on N is given by

$$\theta \in S^1 \mapsto \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}. \quad (6.6)$$

It follows that $N^{G_z} = \{0\}$, whence $\mathbf{d}_z^2 h_\xi|_{N^{G_z}}$ is trivially non-degenerate. Note also that $G_\mu = \mathbb{T}^2$ is a torus. Recall that $(T^*\text{SO}(3))_{(S^1)}$ is the set of points of the form $(I, \lambda I_3 \mathbf{e}_3)$ with $\lambda \neq 0$, that is, the set of all Lagrange tops which are not at rest. Taking into account expression (6.3), it follows from Theorem 4.4 that near a sleeping Lagrange top with phase space point $(I, \lambda_0 I_3 \mathbf{e}_3)$ there is $\epsilon > 0$ such that for every $r \in (-\epsilon, \epsilon)$ there is a \mathbb{T}^2 -orbit of relative equilibria of orbit type (S^1) that forms a persisting branch of nonlinearly stable relative equilibria of the form $(I, (\lambda_0 + r)I_3 \mathbf{e}_3)$. Furthermore this branch is a local symplectic submanifold of $T^*\text{SO}(3)$.

Note that in this particular case, the persistence result already follows from Remark 4.3 in [10], although in that reference the conditions are stronger in general, requiring the relative equilibrium to be non-degenerate. By the uniqueness argument of Theorem 4.4, it follows that this persisting branch corresponds exactly to a neighbourhood of λ in the family of sleeping Lagrange tops, which exists for any non-zero λ , as can be shown by a direct computation. Note that the case $\lambda = 0$ is not contained in the persisting branch of sleeping Lagrange tops since in this case $G_\mu = \mathbb{T}^2$, and then the symplectic type of this point is different from the elements of the branch. The case $\lambda = 0$ corresponds to the unstable equilibrium of the top in the upright position.

Bifurcations. We now apply Theorem 5.1 to study the possible bifurcations of relative equilibria from the family of sleeping Lagrange tops. We will start by a relative equilibrium in the family of sleeping Lagrange tops of the form $z = (I, \lambda_0 I_3 \mathbf{e}_3)$ with momentum $\mu = (\lambda_0 I_3, -\lambda_0 I_3)$ and orthogonal velocity $\xi^\perp = (\lambda_0/2, -\lambda_0/2) \in \mathfrak{m}$ according

to the choice (6.2). Notice that since G_z is Abelian, $(G_z)_\eta = G_z \simeq S^1$. Let us choose $L = K = \{e\}$ and $W = \mathfrak{m}^* \times \{0\} \subset \mathfrak{m}^* \times \mathfrak{g}_z$. We can identify

$$W = \{(\rho, -\rho) : \rho \in \mathbb{R}\}$$

and define a local branch of relative equilibria $\bar{z}(\rho) = (I, (\lambda_0 + \rho/I_3)I_3\mathbf{e}_3)$. Notice that $\bar{z}(0) = z$ and $\mathbf{J}(\bar{z}(\rho)) = \mu + (\rho, -\rho)$. It is also clear that $G_{\mathbf{J}(\bar{z}(\rho))} = \mu + (\rho, -\rho) = G_\mu = G$ since G is Abelian, and also that $G_{\bar{z}(\rho)} = G_z \simeq S^1$. We choose the family of velocities $\xi(\rho) = ((\lambda_0 + r)/2, -(\lambda_0 + r)/2) + (\eta, \eta)$ for a fixed, although arbitrary, element $\eta \in \mathbb{R}$.

It is clear, again from the Abelian character of G , that condition (i) in Theorem 5.1 is trivially satisfied. In order to check condition (iii), notice that $N^L = N$. According to (6.5) we see that $\sigma_+(\rho) > 0$ for every value of ρ but

$$\text{sign}(\sigma_-(\rho)) = \text{sign}(AC - B^2).$$

Computing the value of this last expression along the branch $\bar{z}(\rho)$ we get

$$\begin{aligned} 4I_1I_3^2(AC - B^2) &= \rho^2(2I_3 - I_1) + \rho((2\eta + \lambda_0)I_3(I_1 - I_3) - 2I_3^2\lambda_0) \\ &\quad - ((2\eta + \lambda_0)^2I_1I_3^2 - 2(2\eta + \lambda_0)I_3^3\lambda_0 + 4mglI_3^2). \end{aligned}$$

It is straightforward to check that η can be chosen such that this expression vanishes at $\rho = 0$ if one can solve the independent term for η . This is possible precisely when

$$\lambda_0^2 > \frac{4mglI_1}{I_3^2},$$

that is, if the local branch $\bar{z}(\rho)$ is centred at a point in the formally stable range. In order to check that the eigenvalue $\sigma_-(\rho)$ actually changes sign at $\rho = 0$ we have to guarantee that the coefficient of the linear term is different from zero. But this is true since if both the coefficients of the linear and independent term vanish simultaneously we would have

$$\left(\frac{2I_3\lambda_0}{I_1 - I_3}\right)^2 I_1I_3^2 - 2\frac{2I_3\lambda_0}{I_1 - I_3}I_3^3\lambda_0 + 4mglI_3^2 = 0$$

and the above expression is always positive so $\sigma(\rho)$ changes sign at $\rho = 0$ and condition (iii) in Theorem 5.1 is satisfied.

Thus it remains to study condition (ii). For that, notice that since $L = \{e\}$ and $N_{S^1}(e) \simeq \text{SO}(2)$ we have to check if

$$\ker \mathbf{d}_z^2 h_{(\eta+\lambda/2, \eta-\lambda/2)} \Big|_N,$$

with η chosen in a way that the crossing condition (iii) is satisfied, is two-dimensional and invariant under the action of $G_z \simeq \text{SO}(2)$ given by (6.6). It is easy to see from (6.4) that for $\lambda = \lambda_0$,

$$\ker \mathbf{d}_z^2 h_{(\eta+\lambda/2, \eta-\lambda/2)} \Big|_N = \text{span} \left\langle \left(a, 0, 0, -\frac{B}{C}a \right), \left(0, b, \frac{B}{C}b, 0 \right) \right\rangle$$

with respect to the basis $\{v_1, v_2, v_3, v_4\}$ of N . It is clear from (6.6) that this space is a S^1 -module equivariantly isomorphic to \mathbb{R}^2 equipped with the standard action of S^1 . Therefore (ii) is also satisfied. It follows from Theorem 5.1 that for every $v \in \ker \mathbf{d}_z^2 h_{(\eta+\lambda/2, \eta-\lambda/2)} \Big|_N$ near the origin, there is a relative equilibrium for the Lagrange top system not contained in the branch of sleeping tops. Since the action of S^1 on

this space is free outside the origin, these relative equilibria have trivial isotropy and correspond to precessing tops, where the symmetry and gravity axes are not aligned.

7. EXAMPLE. 2 POINT VORTICES ON THE SPHERE

Consider the system of 2 point vortices on the sphere [9, 12], which is a simple system where every motion is a relative equilibrium. The phase space is $P = S^2 \times S^2 \setminus \Delta$, where Δ is the diagonal, with symplectic form $\omega = \Gamma_1 \omega_1 + \Gamma_2 \omega_2$, where ω_j is the natural $\text{SO}(3)$ -invariant symplectic form on the j^{th} copy of the sphere and $\Gamma_j \in \mathbb{R}$ are the vorticities of the two points (taken to be non-zero). The equation of motion is

$$\dot{\mathbf{x}}_1 = \Gamma_2 \frac{\mathbf{x}_2 \times \mathbf{x}_1}{1 - \mathbf{x}_1 \cdot \mathbf{x}_2}, \quad \dot{\mathbf{x}}_2 = \Gamma_1 \frac{\mathbf{x}_1 \times \mathbf{x}_2}{1 - \mathbf{x}_1 \cdot \mathbf{x}_2}.$$

This is a Hamiltonian system, with Hamiltonian

$$h(\mathbf{x}_1, \mathbf{x}_2) = -\Gamma_1 \Gamma_2 \log \|\mathbf{x}_1 - \mathbf{x}_2\|^2$$

where $\|\mathbf{x}_1 - \mathbf{x}_2\|$ is the Euclidean distance between the points (different versions of this differ by factors of 2π , but this can always be compensated by rescaling time). There are two possible G -Hamiltonian systems depending on whether Γ_1 or Γ_2 are different or equal. If $\Gamma_1 \neq \Gamma_2$ then the group $\text{SO}(3)$ acts on \mathcal{P} by the diagonal action. We think of each copy of S^2 as embedded as the unit sphere in \mathbb{R}^3 . This action is Hamiltonian with momentum map given by, after identifying $\mathfrak{so}(3)^*$ with \mathbb{R}^3 as usual,

$$\mathbf{J}(\mathbf{x}_1, \mathbf{x}_2) = \Gamma_1 \mathbf{x}_1 + \Gamma_2 \mathbf{x}_2. \quad (7.1)$$

If $\Gamma_1 = \Gamma_2$ then the system supports a Hamiltonian action of the direct product $\text{SO}(3) \times \mathbb{Z}_2^\tau$ where $\text{SO}(3)$ acts as in the previous case and \mathbb{Z}_2^τ is the reflection group generated by $\tau(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_2, \mathbf{x}_1)$. The momentum map for this action is also given by the expression (7.1).

The action of $\text{SO}(3)$ (respectively $\text{SO}(3) \times \mathbb{Z}_2^\tau$) on phase space is locally free except where $\mathbf{x}_1 = -\mathbf{x}_2$ (antipodal points), in which case the isotropy is $\text{SO}(2)$ (respectively $\widetilde{\text{O}}(2)$), which is generated by the group $\text{SO}(2)$ of rotations around \mathbf{x}_1 and an element $(R_{\mathbf{n}}, \tau) \in \text{SO}(3) \times \mathbb{Z}_2^\tau$. Here $R_{\mathbf{n}}$ denotes a rotation of angle π around a vector \mathbf{n} perpendicular to \mathbf{x}_1 . Since, for a given $\text{SO}(2)$, the subspace $\mathcal{P}^{\text{SO}(2)} = \mathcal{P}^{\widetilde{\text{O}}(2)}$ consists of just two points, they are necessarily equilibria for any invariant Hamiltonian. Non-antipodal pairs have stabilizers $\{e\}$ or $\mathbb{Z}_2(\mathbf{n})$ respectively, where $\mathbb{Z}_2(\mathbf{n})$ is the subgroup generated by the element $(R_{\mathbf{n}}, \tau) \in \text{SO}(3) \times \mathbb{Z}_2^\tau$.

With 2 point vortices, as already mentioned, every solution is a relative equilibrium, and indeed given any $\mathbf{x}_1, \mathbf{x}_2$ which are not antipodal, the angular velocity is

$$\xi = \frac{1}{2 \sin^2(\theta/2)} \mu$$

where $\mu = \mathbf{J}(\mathbf{x}_1, \mathbf{x}_2)$ and θ is the angle subtended by the two points. With respect to antipodal points, the situation changes depending on the two possibilities for the vorticities.

1. Case $\Gamma_1 \neq \Gamma_2$. In the limit of an antipodal pair, so as $\mathbf{x}_2 \rightarrow -\mathbf{x}_1$, one has $\xi \rightarrow \frac{1}{2} \mu = \frac{1}{2}(\Gamma_1 - \Gamma_2)\mathbf{x}_1 \neq 0$. This limiting ξ is of course an element of \mathfrak{g}_z for $z = (\mathbf{x}_1, -\mathbf{x}_1)$, in

order that z be an equilibrium point. In this case N is 2-dimensional. Specifically, take $z = (\mathbf{e}_3, -\mathbf{e}_3) \in S^2 \times S^2$, and using coordinates (x_1, x_2, y_1, y_2) the tangent space to the group orbit is spanned by $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$. Since $\dim \mathfrak{g}_z = 1$ the rank of $T_z \mathbf{J}$ is also 2, and its kernel is spanned by the tangent vectors

$$v_1 = (\Gamma_2, -\Gamma_1, 0, 0), \quad v_2 = (0, 0, \Gamma_2, -\Gamma_1). \quad (7.2)$$

This is therefore also the symplectic normal space N when $\Gamma_1 \neq \Gamma_2$. On this space, and with this basis,

$$\mathbf{d}_z^2 h|_N = \frac{1}{2} \Gamma_1 \Gamma_2 (\Gamma_1 - \Gamma_2)^2 I_N, \quad \mathbf{d}_z^2 \mathbf{J}|_N = \Gamma_1 \Gamma_2 (\Gamma_1 - \Gamma_2) I_N,$$

where I_N is the identity matrix on N . Thus, for $\eta \in \mathfrak{g}_z \simeq \mathbb{R}$,

$$\mathbf{d}_z^2 h_\eta|_N = \frac{1}{2} \Gamma_1 \Gamma_2 (\Gamma_1 - \Gamma_2) (\Gamma_1 - \Gamma_2 - 2\eta) I_N.$$

Thus, assuming $\Gamma_1 \neq \Gamma_2$, this is degenerate precisely at $\eta = \frac{1}{2}(\Gamma_1 - \Gamma_2)$; note that this is precisely the limiting velocity $\frac{1}{2}\mu$ mentioned above. In particular, note that $\mu = (\Gamma_1 - \Gamma_2)\mathbf{e}_3 \neq 0$ and then $G_\mu = G_z = \text{SO}(2)$ and $\mathfrak{m}^* = 0$.

With reference to Theorem 4.5, the relative equilibrium z is an equilibrium, so $\xi^\perp = 0$, and let $\eta = \frac{1}{2}(\Gamma_1 - \Gamma_2)$. We take $L = \{e\}$ and $K = G_z = \text{SO}(2)$. Therefore $N^L = N$ and $[\mathfrak{g}_\mu^L, \mathfrak{g}_\mu^L] = [\mathfrak{g}_\mu, \mathfrak{g}_\mu] = 0$ since G_μ is Abelian. We have

$$\mathbf{d}_z^2 h_{\eta+\eta'}|_N = \Gamma_1 \Gamma_2 (\Gamma_2 - \Gamma_1) \eta' I_N,$$

which has a unique eigenvalue $\lambda(\eta') = \Gamma_1 \Gamma_2 (\Gamma_2 - \Gamma_1) \eta'$ that changes sign at $\eta' = 0$, satisfying hypothesis (i). At $\eta' = 0$, $\ker \mathbf{d}_z^2 h_\eta|_N = N$ which is 2-dimensional, and from (7.2) is a $\text{SO}(2)$ -module isomorphic to \mathbb{R}^2 equipped with the standard circle action, satisfying hypothesis (ii). Since the action of G_z is free outside the origin, it follows from Theorem 4.5 that for every element of $N \setminus \{0\}$ there is a relative equilibrium near z with trivial isotropy. That is, a rotating configuration of non-antipodal pairs.

On the other hand the content of Theorem 4.3 is empty in this case, since $\mathfrak{m}^* = 0$ and therefore we only get the same equilibrium point, with different velocities $\eta' \in \mathfrak{g}_z$. On the third hand, Theorem 4.4 is not applicable here since $\mathcal{P}_{(\text{SO}(2))}$ consists of a single group orbit.

2. Case $\Gamma_1 = \Gamma_2$. In this case, noting that for $z = (\mathbf{x}_1, -\mathbf{x}_1)$ we have $\mu = \mathbf{J}(z) = 0$ and we find that $G_\mu = \text{SO}(3) \times \mathbb{Z}_2^T$ and $G_z = \widetilde{\text{O}(2)}$. Therefore the kernel of $T_z \mathbf{J}$ coincides with the tangent space to the group orbit at z , which implies that N is trivial. However $\mathfrak{m}^* = \mathbf{x}_1^\perp$ and is therefore 2-dimensional. Since $\mathbf{d}_z^2 h_{\eta+\eta'}|_N$ is trivially non-degenerate for any η' , Theorem 4.5 is not applicable. Also, since in this case $G_z = \widetilde{\text{O}(2)}$, then $\mathcal{P}_{\widetilde{\text{O}(2)}}$ consists of a single group orbit and Theorem 4.4 is not applicable either.

With respect to Theorem 4.3, let us choose $K = \mathbb{Z}_2(\mathbf{n})$. Then we have $\mathfrak{m}^{*K} = \mathbb{R}\mathbf{n}$ which is Abelian, fulfilling condition (i). We also have $\mathfrak{g}_z^K = 0$. Since N^K is trivial, condition (ii) is automatically satisfied too. It follows from Theorem 4.3 that there is a smooth branch of (G -orbits of) relative equilibria near z with stabilizers containing $\mathbb{Z}_2(\mathbf{n})$. Noticing that for $\Gamma_1 = \Gamma_2$ the stabilizer of non-antipodal pairs is precisely $\mathbb{Z}_2(\mathbf{n})$,

we find that the application of the theorem produces the branch of rotating non-antipodal relative equilibria near antipodal equilibria.

Since every motion is a relative equilibrium, the reduced dynamics for this system is trivial so stability holds trivially. Moreover, the theorems of Section 5 do not apply as there are no non-trivial branches of relative equilibria from which to bifurcate.

REFERENCES

- [1] Albouy, A, and Chenciner, A. [1998], Le problème des n corps et les distances mutuelles. *Invent. Math.* **131**, 151–184.
- [2] Arnold, V.I. [1989], *Mathematical methods of classical mechanics*. Graduate Texts in Mathematics **60**, Springer-Verlag, New York.
- [3] Chenciner A., [2013] The Lagrange reduction of the n -body problem, a survey. *Acta Math. Vietnam.* **38**, 165–186.
- [4] Chossat, P., Lewis, D., Ortega, J.-P. and Ratiu, T.S. [2003], Bifurcation of relative equilibria in mechanical systems with symmetry. *Adv. in Appl. Math.* **31**, 10–45.
- [5] Dullin, H. and Easton, R. W. [1999], Stability of levitrons, *Physica D* **126**, 1–17.
- [6] Fassò F. and Lewis, D. [2001], Stability properties of the Riemann ellipsoids, *Archive for Rational Mechanics and Analysis* **158**, 259–292.
- [7] Grigoryeva, L., Ortega, J.-P. and Zub, S. [2014], Stability of Hamiltonian relative equilibria in symmetric magnetically confined rigid bodies. *J. Geom. Mech.* **6**, 373–415.
- [8] Guillemin, V. and Sternberg, S. [1984], A normal form for the moment map. In *Differential Geometric Methods in Mathematical Physics*, S. Sternberg ed. Mathematical Physics Studies, **6**, D. Reidel.
- [9] Kidambi, R. and Newton, P. [1998], Motion of three point vortices on a sphere. *Physica D* **116**, 143–175.
- [10] Lerman, E. and Singer, S.F. [1998], Stability and persistence of relative equilibria at singular values of the moment map, *Nonlinearity* **11**, 1637–1649.
- [11] Lewis, D. [1992], Lagrangian block-diagonalization. *Journal of Dynamics and Differential Equations* **4**, 1–41.
- [12] Lim, C., Montaldi, J. and Roberts, M. [2001], Relative equilibria of point vortices on the sphere, *Physica D* **148**, 97–135.
- [13] Marle, C.-M. [1985], Modèle d’action hamiltonienne d’un groupe de Lie sur une variété symplectique. *Rend. Sem. Mat. Univ. Politec. Torino*, **43** no. 2, 227–251.
- [14] Marsden, J.E. [1992], *Lectures on Mechanics*, Lecture Note Series **174**, LMS, Cambridge University Press.
- [15] Montaldi, J. [1997], Persistence and stability of relative equilibria, *Nonlinearity*, **10**, 449–466.
- [16] Montaldi, J., *Singularities, Bifurcations and Catastrophes*. (In preparation)
- [17] Montaldi, J. and Roberts, M. [1999], Relative equilibria of molecules. *J. Nonlinear Science* **9**, 53–88.
- [18] Montaldi, J. and Rodríguez-Olmos, M. [2011], On the stability of Hamiltonian relative equilibria with non-trivial isotropy, *Nonlinearity* **24**, 2777–2783
- [19] Montaldi, J. and Tokieda, T. [2003], Openness of momentum maps and persistence of extremal relative equilibria. *Topology* **42**, 833–844.
- [20] Montaldi, J. and Tokieda, T. [2013], Deformation of geometry and bifurcations of vortex rings. Springer Proceedings in Mathematics & Statistics, vol. 35. pp. 335–370.
- [21] Ortega, J.-P. and Ratiu, T.S. [1999], Stability of Hamiltonian relative equilibria, *Nonlinearity*, **12**, 693–720.
- [22] Ortega, J.-P. and Ratiu, T.S. [2004], *Momentum maps and Hamiltonian reduction*, Progress in Mathematics **222**, Birkhauser-Verlag.
- [23] Palais, R.S. [1979], The principle of symmetric criticality. *Comm. Math. Phys.* **69**, 1, 19–30.
- [24] Patrick, G.W. [1992], Relative equilibria in Hamiltonian systems: the dynamics interpretation of nonlinear stability on the reduced phase space. *J. Geom. Phys.* **9**, 111–119.

- [25] Patrick, G.W. [1995], Relative equilibria of Hamiltonian systems with symmetry: linearization, smoothness, and drift. *J. Nonlinear Science* **5** 373–418.
- [26] Patrick, G.W. and Roberts, M. [2000], The transversal relative equilibria of a Hamiltonian system with symmetry. *Nonlinearity* **13**, 2089–2105.
- [27] Patrick, G.W., Roberts, M. and Wulff, C. [2004], Stability of Poisson and Hamiltonian relative equilibria by energy methods. *Arch. Rational Mech. Anal.* **174**, 301–344.
- [28] Poston, T. & Stewart, I. [1978] *Catastrophe Theory and Applications*. Dover, 1996. (Original edition Prentice Hall).
- [29] Roberts, M. and Sousa-Dias, M.E. [1997], Bifurcations from relative equilibria of Hamiltonian systems. *Nonlinearity* **10**, 1719–1738.
- [30] Roberts, M. and Sousa-Dias, M.E. [1999], Symmetries of Riemann ellipsoids. *Resenhas IME-USP* **4**, 183–221.
- [31] Roberts, M., Wulff, C. and J.S. Lamb [2002], Hamiltonian systems near relative equilibria, *J. Differential Equations*, **179**, 562–604.
- [32] Rodríguez-Olmos M. and Sousa-Dias, M.E. [2009] Nonlinear stability of Riemann ellipsoids with symmetric configurations. *J. Nonlinear Science* **19**, 179–219.
- [33] Scheerer, U. & Wulff, C. [2001], Reduced dynamics for momentum maps with cocycles. *C.R. Acad. Sci. Paris*, **333**, Série I, 999–1004.
- [34] Wulff, C. Patrick, G. and Roberts, M. [2001], Stability of Hamiltonian relative equilibria by energy methods. *Symmetry and perturbation theory (Cala Gonone, 2001)*, World Sci. Publ., 214–221.

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