Coadjoint orbits of the special Euclidean group

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COADJOINT ORBITS OF THE SPECIAL EUCLIDEAN GROUP

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Contents

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In this report we classify the coadjoint orbits for compact semisimple Lie groups by establishing a correspondence between orbits and subsets of Dynkin diagrams. Particular attention is given to the special unitary and orthogonal groups for which the orbits are complex flags and real Hermitian flags respectively. The orbits for non-compact and non-semisimple affine groups are discussed and in the example of the special Euclidean group a geometric bijection between adjoint and coadjoint orbits is found.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

The coadjoint orbits of a Lie group are examples of symplectic homogeneous spaces, indeed it is the case that any symplectic homogeneous G space for G semisimple is up to symplectic coverings a coadjoint orbit of G ([7] Chapter 2.25). For compact and semisimple groups the adjoint and coadjoint representations are equivalent and so it suffices to deal with the adjoint orbits. For matrix Lie groups the adjoint orbits are relatively easy to describe since the adjoint action is given by matrix multiplication, $\mathrm{Ad}_g X = gXg^{-1}$, $g \in G$, $X \in \mathfrak{g}$.

For $SU(n)$ the adjoint orbits are shown to be flags in $\mathbb{C}^n$. These include the complex Grassmannians which are known to be symplectic unlike the case for real Grassmannians and more generally real flag manifolds. It turns out however that the adjoint orbits of $SO(n)$ are real flag manifolds equipped with additional structure, namely a choice of complex structure on certain subspaces of the flag. We call such a flag a Hermitian flag and show that the adjoint orbits of $SO(n)$ are Hermitian flag manifolds.

Generalizing the methods used to find the orbits for $SU(n)$ and $SO(n)$ we establish a correspondence between adjoint orbits of compact semisimple groups with subsets of its Dynkin diagram. Though known, this correspondence appears to the author to be missing from the standard literature. Owing to the symmetry of particular Dynkin diagrams and low dimmensional accidental isomorphisms between Lie algebras we find corresponding accidental diffeomorphisms between flag manifolds.

The more general problem of classifying adjoint orbits for any classical semisimple group is solved in [2]. Here they decompose Lie algebra elements into their commuting semisimple and nilpotent parts to determine what they refer to as a type decomposition which classifies orbit types along with their dimension and modulus. The Dynkin diagram method may be thought of as a specific example of their approach.

For non-compact and non-semisimple Lie groups the adjoint and coadjoint actions are no longer necessarily equivalent and so the coadjoint orbits must be found through other means. In [6] the coadjoint orbits for a semidirect product are discussed with a view to quantization. Here a notion of 'little subgroup' is employed to describe the orbits and a bijection between coadjoint orbits and bundles over little subgroups is found. Continuing their work in [2] on type decompositions Cushman and van der Kallen extend there method to classify orbits for affine orthogonal groups in [3]. In this paper they find a 'curious bijection' between adjoint and coadjoint orbits for such groups including the Poincaré group. In this report we exhibit geometrically this bijection for the example of the special Euclidean group. We show that along with the modulus of the orbits the bijection additionally preserves homotopy type.

Chapter 2

Definitions and motivation

2.1 Co/Adjoint actions

Let G be a Lie group. We can probe the non-abelian nature of G using the conjugation operation $C_g : G \to G$ given by,

$$C_g(h) = ghg^{-1}. \tag{2.1}$$

This diffeomorphism fixes the identity and so we may consider the tangent space map,

$$(C_g|_e)_* : T_eG \longrightarrow T_eG.$$

Identifying T_eG with the Lie algebra $\mathfrak{g}$ of G we denote by $\mathrm{Ad}_g \in GL(\mathfrak{g})$ the above isomorphism $(C_g|_e)_*$. By using the fact that $C_{gh} = C_g \circ C_h$ we can define the following homorphism and hence representation of G,

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g}); g \longmapsto (\mathrm{Ad}_g). \tag{2.2}$$

This is called the *Adjoint representation of G*. For when G is given as a matrix Lie group $G \subset \mathrm{Mat}(\mathbb{F}^n)$ this is just the usual conjugation $\mathrm{Ad}_g(X) = gXg^{-1}$.

The *adjoint representation* of the Lie algebra $\mathfrak{g}$ is the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$,

$$(\mathrm{ad}_X)\, H = \frac{d}{dt}\Big|_{t=0} \big(\mathrm{Ad}_{\exp(tX)}\, H\big) = [X, H], \quad X, H \in \mathfrak{g} \tag{2.3}$$

Remark 2.1.1. When discussing the Adjoint representation of connected Lie groups there is in fact nothing to be lost from exclusively dealing with matrix Lie groups. Given a connected Lie group G with Lie algebra $\mathfrak{g}$ we have by Ado's theorem an isomorphic copy of $\mathfrak{g}$ inside $\mathfrak{gl}(V)$ for some vector space V. Exponentiating this matrix Lie algebra gives us a matrix group with the same Lie algebra as G. Groups with the same Lie algebra we will call *isogeneous*. Two isogeneous connected Lie groups are related by a covering space homomorphism $\widetilde{G} \to G$ with kernel in the center $Z(\widetilde{G})$. The following proposition justifies our claim.

Proposition 2.1.1. Let $\widetilde{G}$ and G be isogeneous connected Lie groups with covering space map $\pi : \widetilde{G} \to G$. Then $\operatorname{Ad}_{\tilde{g}} X = \operatorname{Ad}_{\pi(\tilde{g})} X$ for all $X \in \mathfrak{g}$ and $\tilde{g} \in \widetilde{G}$.

Proof. By the definition of covering space we may take a neighbourhood U of e in G such that $\pi^{-1}(U)$ is a disjoint collection of sets,

$$\pi^{-1}(U) = \coprod U_\alpha$$

where each U_α projects diffeomorphically onto U. Let U_α denote the set in $\widetilde{G}$ containing the identity. Let $X \in \mathfrak{g}$ be arbitrary and choose $t \in \mathbb{R}$ small enough so that $\exp(tX) \in U$. Since the Lie algebras of $\widetilde{G}$ and G are the same we may consider $\exp(tX)$ as being π-related elements in both U_α and U. Now take any $\tilde{g} \in U_\alpha$ and shrink t if necessary so that $C_{\tilde{g}}(\exp(tX))$ is in U_α. Since π is a homomorphism we have,

$$C_{\tilde{g}}(\exp(tX)) = C_{\pi(g)}(\exp(tX)).$$

It suffices to show that for any $\tilde{h}$ in $\widetilde{G}$ satisfying $\pi(\tilde{h}) = \pi(\tilde{g})$ that $C_{\tilde{g}}(\exp(tX)) = C_{\tilde{h}}(\exp(tX))$. Now we use the fact that $\operatorname{Ker}(\pi) \subset Z(\widetilde{G})$ to see that $\tilde{h}\tilde{g}^{-1}$ and $\tilde{g}\tilde{h}^{-1}$ are central and so,

$$C_{\tilde{h}}(\exp(tX)) = \tilde{h}\exp(tX)\tilde{h}^{-1} = (\tilde{h}\tilde{g}^{-1})g\exp(tX)\tilde{g}^{-1}(\tilde{g}\tilde{h}^{-1}) = C_{\tilde{g}}(\exp(tX)).$$

Differentiating this and using U to generate all of G gives us our result. □

To every representation there is a dual (or contragredient) representation. For the Adjoint representation we call this the *Coadjoint representation*. It is defined by,

$$\langle \operatorname{Coad}_g \eta, Y \rangle = \langle \eta, \operatorname{Ad}_{g^{-1}} Y \rangle, \tag{2.4}$$

for all $\eta \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$ and $g \in G$. Here $\langle , \rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Similarly we may define the *coadjoint representation*, $\operatorname{coad}_X \eta$ dual to the adjoint representation. It is easily checked that this definition is equivalent to defining coad by,

$$(\operatorname{coad}_X)\,\eta = \frac{d}{dt}\Big|_{t=0} \left(\operatorname{Coad}_{\exp(tX)} \eta\right). \tag{2.5}$$

It is not usually the case that the Adjoint and Coadjoint representations are equivalent. We do however have the following result.

Proposition 2.1.2. Let $\rho : G \to GL(V)$ be a representation of a group G and $\rho^* : G \to GL(V^*)$ its dual. If there exists a G-invariant non-degenerate bilinear form B on V then the two representations are equivalent.

Proof. The definition of dual representation is that for all $g \in G$, $\eta \in V^*$ and $X \in V$ that,

$$\langle \rho^*(g)\eta, X\rangle = \langle \eta, \rho(g^{-1})X\rangle.$$

We may identify V and V^* via the isomorphism $X \mapsto B(X, \cdot)$ (an isomorphism made possible since B is non-degenerate). Identifying η above with an element in V allows us to write,

$$\langle \eta, \rho(g^{-1})X\rangle = B(\eta, \rho(g)^{-1}X) = B(\rho(g)\eta, X).$$

Above we have used that ρ is a homomorphism and that B is $\rho(G)$-invariant. It follows then that under the identification $V \equiv V^*$ that $\rho^*(g)\eta \equiv \rho(g)\eta$. Hence the representations are equivalent. □

Remark 2.1.2. It follows from the above proposition that if G is a compact Lie group any representation is equivalent to its dual. This is because we may create a G-invariant inner product on V by the 'averaging method'. For a semisimple Lie group the Killing form is a G-invariant non-degenerate inner product on $\mathfrak{g}$. It also then follows from the above proposition that the Adjoint and Coadjoint representations of semisimple Lie groups are equivalent.

In view of this remark it is worthwhile to recall the classification of compact connected Lie groups.

Theorem 1. (Classification of compact Lie groups) Let G be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Then $\mathfrak{g}$ is of the form $\mathfrak{s} \oplus \mathfrak{a}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian. It follows that G is isogeneous to a group of the form,

$$S_1 \times \cdots \times S_r \times \mathbb{T}^n$$

where $S_1, ..., S_n$ are compact simple Lie groups and $\mathbb{T}^n$ is an n-torus.

From Remark 2.1.1 we see then that the Adjoint orbits of any compact connected Lie group (and so by remark 2.1.2 the Coadjoint orbits) are just products of the Adjoint orbits for simple compact Lie algebras. The classification of such orbits will be the main goal of this discussion.

There are now two good questions; why should we care about the Coadjoint representation and why should we care that it may be equivalent to the Adjoint representation? In answer to the first question it so happens that Coadjoint orbits in $\mathfrak{g}^*$ are naturally symplectic homogeneous manifolds.

Given $\eta \in \mathfrak{g}^*$ denote the Coadjoint orbit through η by,

$$\mathcal{O}_{\mathrm{Coad}}(\eta) = \{\mathrm{Coad}_g\, \eta \mid \forall g \in G\}.$$

Often we will just write this as $\mathcal{O}(\eta)$ when there is no confusion between Co- and Adjoint orbits. This is a homogeneous G-space. The tangent space may be written with the aid of 2.5 as;

$$T_\eta \mathcal{O}(\eta) = \{\operatorname{coad}_X \eta \mid \forall X \in \mathfrak{g}\}.$$

There is a natural G-invariant symplectic structure ω on $\mathcal{O}(\eta)$ called the *Kirillov-Kostant-Souriau (KKS) form* given by,

$$\omega(\operatorname{coad}_X \eta, \operatorname{coad}_Y \eta) = \langle \eta, [X, Y] \rangle. \tag{2.6}$$

In answer to the second question, it is the case that computing Adjoint orbits is relatively easy since by Remark 2.1.1 this boils down to a matrix calculation. For Coadjoint orbits however it is not so straightforward to write the action in terms of matrices (an example for $SE(2)$ is given later). It is therefore helpful when the two representations are equivalent and hence the orbits are the same. For non-semisimple and non-compact Lie groups where the representations are usually inequivalent the Coadjoint orbits are harder to describe. This difficulty will become more apparent in Section 5 when we carry out the description for affine groups.

2.2 Matrix methods and examples

Let $G \subset GL(V)$ be a matrix Lie group and $\mathfrak{g} \subset \mathfrak{gl}(V)$ its Lie algebra. The Adjoint action is simply $\operatorname{Ad}_g X = gXg^{-1}$. We may identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the non-degenerate inner product $\langle\ ,\ \rangle$ on $\mathfrak{gl}(V)$ given by,

$$\langle A, B \rangle = \operatorname{Tr}(A^T B), \quad \forall A, B \in \mathfrak{gl}(V).$$

Therefore,

$$\langle \operatorname{Coad}_g \eta, X \rangle = \langle \eta, \operatorname{Ad}_{g^{-1}} X \rangle = \operatorname{Tr}(\eta^T g^{-1} X g) = \operatorname{Tr}(g\eta^T g^{-1} X) = \langle g^{-T} \eta g^T, X \rangle.$$

From this it is tempting to conclude that we may write $\operatorname{Coad}_g \eta \stackrel{?}{=} g^{-T}\eta g^T$ however this operation may not necessarily preserve the subspace $\mathfrak{g} \subset \mathfrak{gl}(V)$. To correct this we write $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{g}^\circ$ where $\mathfrak{g}^\circ$ is the annihilator of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ with respect to $\langle , \rangle$. Then the correct Coadjoint action is given by,

$$\operatorname{Coad}_g \eta = \operatorname{Pr}_1\left(g^{-T} \eta g^T\right), \tag{2.7}$$

where Pr_1 is the projection onto the first factor of $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{g}^\circ$.

Example 1. (Co/Adjoint orbits of $SO(3)$) We may identify the Lie algebra $\mathfrak{so}(3)$ of anti-symmetric real 3×3 matrices with $\mathbb{R}^3$ as follows,

$$A_v = \begin{pmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} := v$$

It is easily checked that for $g \in SO(3)$, $\mathrm{Ad}_g A_v = A_{gv}$. It follows that the Adjoint orbits are points and spheres. Though we have already established that the Coadjoint and Adjoint orbits will be equivalent for $SO(3)$ we can see it more directly using the fact that $gg^T = I$; from (2.7) we have that,

$$\mathrm{Coad}_g \eta = \mathrm{Pr}_1 \left(g^{-T} \eta g^T\right) = \mathrm{Pr}_1 \left(g \eta g^{-1}\right) = \mathrm{Ad}_g \eta.$$

Example 2. (Co/Adjoint orbits of $SL(2;\mathbb{R})$) The group $SL(2;\mathbb{R})$ is an example of a simple yet non-compact group. Despite being non-compact the Adjoint and Coadjoint actions coincide from simplicity of the group. The Lie algebra is,

$$\mathfrak{sl}(2;\mathbb{R}) = \left\{ \begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

The eigenvalues of $A \in \mathfrak{sl}(2;\mathbb{R})$ are $\lambda = \pm\sqrt{x^2 + y^2 - z^2}$. Provided $SL(2;\mathbb{R})$ acts transitively on the isospectral sets in $\mathfrak{sl}(2;\mathbb{R})$, the orbits are then given by the quadratic surfaces $-\lambda^2 = \det(A) = x^2 + y^2 - z^2$. Thus we have four types of orbits: a point orbit through $A = 0$; a cone (minus vertex) for A non-zero, nilpotent/parabolic elements, $\det(A) = 0$; a one-sheeted hyperboloid for elliptic A, $\det(A) > 0$; and finally a two-sheeted hyperboloid for hyperbolic elements $\det(A) < 0$.

Example 3. (Co/Adjoint orbits of $SE(2)$) The special Euclidean group is the semidirect product of the special orthogonal group with translations, $SE(n) := SO(n) \ltimes \mathbb{R}^n$. It is a non-semisimple and non-compact group. There is therefore no reason why the Coadjoint and Adjoint representations should be equivalent and indeed in this example we shall see just that. There is a faithful matrix representation of $SE(2)$;

$$SE(2) \hookrightarrow GL(3);\ (r_\theta, x) \mapsto \begin{pmatrix} r_\theta & x \\ 0 & 1 \end{pmatrix} \text{ where, } r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } x \in \mathbb{R}^2.$$

In this matrix representation the Lie algebra is given by,

$$\mathfrak{se}(2) = \left\{ \begin{pmatrix} \rho J & X \\ 0 & 0 \end{pmatrix} \mid \rho \in \mathbb{R},\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ X \in \mathbb{R}^2 \right\}$$

This allows us to compute the Adjoint action,

$$Ad_{(r_\theta,x)}\begin{pmatrix}\rho J & X\\ 0 & 0\end{pmatrix}=\begin{pmatrix}r_\theta & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho J & X\\ 0 & 0\end{pmatrix}\begin{pmatrix}r_{-\theta} & -r_{-\theta}x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\rho J & r_\theta X-\rho Jx\\ 0 & 0\end{pmatrix}.$$

And so acting by (r_θ, x) on (ρ, X) sends it to $(\rho, r_\theta X-\rho Jx)$. It is then easy to see that the Adjoint orbit through (ρ, X) for $\rho\neq 0$ is the plane $\left\{(\rho,X)|\, X\in\mathbb{R}^2\right\}$ and for $\rho=0$ the orbits are circles or points $\{(0,Y)\ :\ |Y|=|X|\}$.

For the Coadjoint action it is straightforward to compute $g^{-T}\eta g^T$,

$$g^{-T}\eta g^T=\begin{pmatrix}r_\theta & 0\\ -x^Tr_\theta & 1\end{pmatrix}\begin{pmatrix}\rho J & X\\ 0 & 0\end{pmatrix}\begin{pmatrix}r_{-\theta} & 0\\ x^T & 1\end{pmatrix}=\begin{pmatrix}\rho J+r_\theta Xx^T & r_\theta X\\ -\rho x^TJ-x^Tr_\theta Xx^T & -x^Tr_\theta X\end{pmatrix}$$

This looks quite messy and is clearly not inside $\mathfrak{se}(2)$. However we now note that $\mathfrak{g}^\circ$ is given by,

$$\mathfrak{g}^\circ=\left\{\begin{pmatrix}\lambda I & 0\\ a & b\end{pmatrix}\mid \lambda,a,b\in\mathbb{R}\right\}.$$

We may therefore project away the terms in the above expression to show that the Coadjoint action of (r_θ,x) on $(\rho J, X)\in\mathfrak{se}(2)^*$ sends it to $(\rho J+r_\theta Xx^T, r_\theta X)$. The orbits are therefore the points $(\rho,0)$ and cylinders $\mathbb{R}\times\{r_\theta X\}$ for $X\neq 0$.

Chapter 3

Co/Adjoint orbits of the special orthogonal and unitary groups

3.1 Flag manifolds

A *flag* in a vector space W, (for us W will be $\mathbb{R}^n$ or $\mathbb{C}^n$) is defined to be a strictly ascending sequence of subspaces,

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = W. \tag{3.1}$$

If we equip W with an inner product then using a Gram-Schmidt procedure we may reinterpret the flag as being an ordered sequence of mutually orthogonal subspaces $V_1, \ldots, V_k$ where $V_1 = E_1$ and $E_{i+1} = E_i \oplus V_{i+1}$,

$$0 \subset V_1 \subset V_1 \oplus V_2 \subset \cdots \subset \bigoplus_{i=1}^{k} V_i = W. \tag{3.2}$$

Let d_i denote the dimension of each V_i. A flag is said to have *signature* $(d_1, \ldots, d_k)$, where of course $\Sigma_i d_i = \dim(W) = n$. If each d_i is equal to one (and hence $k = n$) we call the flag a *full flag*. Otherwise the flag is called a *partial flag*. The manifold of all flags of signature $(d_1, \ldots, d_k)$ will be denoted $\mathcal{F}(d_1, \ldots, d_k)$. If the field of the vector space is ambiguous it will be indicated in subscript. For example the real and complex projective spaces and Grassmannians are partial flags; $\mathbb{R}P^n = \mathcal{F}_{\mathbb{R}}(1, n-1)$, $\mathbb{C}P^n = \mathcal{F}_{\mathbb{C}}(1, n-1)$, $\mathrm{Gr}_{\mathbb{C}}(k; n-k) = \mathcal{F}_{\mathbb{C}}(k, n-k)$. The following proposition shows how we may identify the tangent space to a flag manifold with a series of linear maps.

Proposition 3.1.1. Given a flag $F = 0 \subset V_1 \subset V_1 \oplus V_2 \subset \cdots \subset \bigoplus_{i=1}^{k} V_i = W$ in $\mathcal{F}(d_1, \ldots, d_k)$ we can identify the tangent space with a series of linear maps,

$$T_F \mathcal{F} = \bigoplus_{i=1}^{k} \mathcal{L}(V_i, E_i^{\perp}), \tag{3.3}$$

where $\mathcal{L}(V,U)$ is the set of linear maps $V \to U$ between vector spaces V, U.

Proof. Let $F(t)$ be a curve in $\mathcal{F}$ so that $F(0) = F$ and $V_i(t)$ the corresponding curve of subspaces. Let $A_i(t)$ be a curve in $\mathcal{L}(W, W/V_i)$ satisfying $\operatorname{Ker} A_i(t) = V_i(t)$ for all t. Let $\gamma_1(t)$ be an arbitrary curve in W such that $\gamma_1(t) \in V_1(t)$ for all t. Differentiating $A_1(t)\gamma_1(t) = 0$ at $t = 0$ gives,

$$A_1(0)\gamma_1'(0) + A_1'(0)\gamma_1(0) = 0.$$

The tangent vector is determined by $A_1'(0)$ and $\gamma_1(0)$ up to $\operatorname{Ker} A_1(0) = V_1$ and hence defines a class $\gamma_1'(0) + V_1 \in W/V_1$ which we may identify with a unique $\alpha_1 \in V_1^\perp$. The map sending $\gamma_1(0)$ to α_1 is linear. Hence we have a linear map in $\mathcal{L}(V_1, V_1^\perp) = \mathcal{L}(V_1, E_1^\perp)$ determined uniquely by $V_1'(0)$. Now consider arbitrary curves $\gamma_1(t), \ldots, \gamma_i(t)$ in W each satisfying $\gamma_j(t) \in V_j(t)$ for all t and $j \leq i$. As before we can show that $\gamma_i'(0)$ may be represented by an element in $V_i^\perp$. However since the $V_j(t)$ are mutually orthogonal we additionally require that $\langle \gamma_i(t), \gamma_j(t) \rangle = 0$ for all $j < i$. Differentiating this at $t = 0$ gives,

$$\langle \gamma_i'(0), \gamma_j(0) \rangle + \langle \gamma_i(0), \gamma_j'(0) \rangle = 0$$

This condition along with the fact that the γ_js were arbitrary imply that the projections of $\gamma_i'(0)$ onto each V_j, $(j < i)$ are predetermined by $\gamma_i(0)$ and $\gamma_j'(0)$. Therefore the vector $\gamma_i'(0)$ defines a class α_i in $V_i^\perp \cap_{j<i} V_j^\perp = E_i^\perp$. The map $\gamma_i(0) \to \alpha_i$ is linear and therefore $V_i'(0)$ determines a unique map in $\mathcal{L}(V_i, E_i^\perp)$. □

Let $\mathcal{F} = \mathcal{F}_\mathbb{C}$ be a manifold of complex flags in $\mathbb{C}^n$. Write a tangent vector to $\mathcal{F}$ at F as $\mathbf{A}$ where,

$$\mathbf{A} = A_1 \oplus \cdots \oplus A_k \in \bigoplus_{j=1}^{k} \mathcal{L}(V_j, E_j^\perp).$$

If $\mathbf{B}$ is another such vector we can define a natural symplectic structure on $\mathcal{F}$,

$$\omega_F(\mathbf{A}, \mathbf{B}) := \sum_{j=1}^{k} \operatorname{Tr}\left(i[A_j, B_j]\right). \tag{3.4}$$

This is non-degenerate since $\operatorname{Tr}\left(i[A, A^\dagger]\right) = 2\operatorname{Tr}\left(AA^\dagger\right)$, (here Tr means the real part of the trace). To show that ω_F is closed we use the formula,

$$\begin{aligned} d\omega_F(A,B,C) &= A\left(\omega_F(B,C)\right) - B\left(\omega_F(C,A)\right) + C\left(\omega_F(A,B)\right) \\ &\quad + \omega_F([A,B],C) + \omega_F([C,A],B) + \omega_F([B,C],A). \end{aligned}$$

By the Jacobi identity the final three terms vanish. For the first three terms note that the action of $SU(n)$ on flags is transitive. From this we can extend the vectors A, B, C to vector fields on $\mathcal{F}$ generated by the infinitesimal action of $SU(n)$. The action of $a \in SU(n)$ on the vector fields A, B, C is that of conjugation which leaves the trace invariant. Hence the first three terms vanish. Thus we see that complex flag manifolds are symplectic, indeed we will see later that they are precisely the Coadjoint orbits of $SU(n)$.

For real flag manifolds it is no longer true that they are symplectic (for example $\mathbb{R}P^3 \cong SO(3)$ has dimension 3 so is certainly not symplectic). There exist symplectic real flag manifolds but they are endowed with more structure than the flags we have defined so far possess. The following definitions are applicable to *real* flag manifolds.

An *oriented flag* in $\mathbb{R}^n$ is one where each E_i is given an orientation. Observe how this will also define an orientation on each V_i and $\mathbb{R}^n$. We denote the manifold of oriented flags with signature $(d_1, ..., d_k)$ by $\widetilde{\mathcal{F}}(d_1, ...d_k)$. We can also define a mixed flag to be one where only specific subspaces V_i receive an orientation. We denote such a flag manifold as $\mathcal{F}(\tilde{d_1}, ..., \tilde{d_k})$, where the tilde above a given d_i indicates that V_i receives an orientation.

An *(oriented) Hermitian flag* is a flag where each V_i is given a complex structure compatible with the metric on $\mathbb{R}^n$ as well as with a choice of orientation (note therefore that each d_i must be even). This is equivalent to each V_i having a complex structure, $J_i : V_i \to V_i$ satisfying $J_i^2 = -I$ with J_i orthogonal and such that their exists an oriented basis for each V_i where J_i takes the form,

$$\begin{pmatrix} 0 & -I \\ +I & 0 \end{pmatrix}. \tag{3.5}$$

We will denote the manifold of Hermitian flags by $\mathcal{HF}(d_1, ..d_k)$. We will also need to consider mixed flags whereupon certain subspaces V_i are given a complex structure, an orientation or nothing at all. We will write such a mixed flag manifold as $\mathcal{F}(d_1, \tilde{d_2}, ..., {d_k}^{\mathbb{C}})$, where ${d_i}^{\mathbb{C}}$ indicates that E_i has a complex structure and the tilde an orientation as before.

The group $SO(n)$ acts naturally on such flags by sending each subspace V_i to $a \cdot V_i$ for $a \in SO(n)$. If the flag is Hermitian then each complex structure J_i defined on V_i is sent to $a \circ J_i \circ a^{-1}$ on $a \cdot V_i$. We can therefore write the manifold of flags as a homogeneous $SO(n)$-space;

$$\mathcal{F}(d_1, ..., \tilde{d_2}, ..., d_3^{\mathbb{C}}) = \frac{SO(n)}{S\left(O(d_1) \times \cdots \times SO(d_2) \times \cdots \times U(\frac{d_3}{2})\right)} \tag{3.6}$$

The isotropy group of the flag requires some explanation. Clearly for a to fix a flag it must leave invariant each subspace V_i. The restriction of a to each subspace is orthogonal. Therefore a must belong to the subgroup $S\left(O(d_1) \times \cdots \times O(d_k)\right)$. If V_i is oriented then the corresponding action of a on V_i must restrict to an element of $SO(d_i)$. Finally should V_i

possess a complex structure given by $J_i \in \mathrm{Aut}(V_i)$ then the action of a restricted to V_i must satisfy $aJ_ia^{-1} = J_i$. If we identity V_i with $\mathbb{C}^{d_i/2}$ (which we may using the complex structure J_i) then this condition is precisely that which says that a acting on $\mathbb{C}^{d_i/2}$ should commute with multiplication by i. This is equivalent to saying that a is a complex linear map on $\mathbb{C}^{d_i/2}$. Moreover the map is unitary since a is orthogonal. Note that since $U(1) \cong SO(2)$, a complex structure on a plane is equivalent to a choice of orientation.

The description of complex flags as homogeneous spaces is simpler,

$$\mathcal{F}(d_1, \ldots, d_k) = \frac{SU(n)}{S\left(U(d_1) \times \cdots \times U(d_k)\right)} \tag{3.7}$$

We can also define an affine real flag manifold. These will turn out to include the Coadjoint orbits for the special Euclidean group. Given a flag F we can displace each of its subspaces E_i by a fixed vector $x \in \mathbb{R}^n$ to get an *affine flag* $F + x$. The bottom subspace $V_1 + x$ has added significance since the flag $F + x$ is invariant under translations belonging to V_1. Given such an affine flag we refer to the space $V_1 + x$ as the *flag pole*.

Given a flag manifold $\mathcal{F}$ we can define a tautological vector bundle $\mathrm{Taut}\mathcal{F}$ by defining the fibre over each flag F to be $\bigoplus_{i=2}^k V_i$, i.e. the sum of all subspaces but the first. The construction of this bundle is analogous to the tautological bundle over a projective space or grassmannian. Now consider the manifold of affine flags $\mathrm{Aff}\mathcal{F}(d_1; d_2, ..., d_k)$. We can define a bijection between this manifold and $\mathrm{Taut}\mathcal{F}(d_1; d_2, ..., d_k)$ as follows; the affine flag $F + x$ is determined uniquely by F and the flag pole $V_1 + x$. However the flag pole $V_1 + x$ is determined uniquely by its intersection with the orthogonal complement $V_1^\perp = \bigoplus_{i=2}^k V_i$. We may identify the flag $F + x$ uniquely with a particular F and a point in $\bigoplus_{i=2}^k V_i$. This defines a unique point in $\mathrm{Taut}\mathcal{F}(d_1; d_2, ..., d_k)$. This bijection between the two spaces is clearly smooth. We have thus proved the following proposition;

Proposition 3.1.2. The tautological bundle over a flag manifold, $\mathrm{Taut}\mathcal{F}(d_1; d_2, ..., d_k)$ is diffeomorphic to the affine flag manifold $\mathrm{Aff}\mathcal{F}(d_1; d_2, ..., d_k)$.

Example 4. The space of oriented lines in $\mathbb{R}^n$, $\mathrm{Aff}\widetilde{\mathcal{F}}(1; n-1)$ is then the tautological $n-1$ hyperplane bundle over the Grassmannian $\widetilde{\mathrm{Gr}}(1; n-1)$ of $n-1$ hyperplanes in $\mathbb{R}^n$, which itself is diffeomorphic to the sphere $S^{n-1} \subset \mathbb{R}^n$. The tautological fibre over each point in the sphere is then the hyperplane orthogonal to that point and hence tangent to the sphere. Thus we see that $\mathrm{Aff}\widetilde{\mathcal{F}}(1; n-1) \cong T^*S^{n-1}$. Observe that this tells us that $\mathrm{Aff}\widetilde{\mathcal{F}}(1; n-1)$ is symplectic, (a fact which will also follow later when we show that it is a Coadjoint orbit of $SE(n)$).

There is a transitive action of $SE(n)$ on $\mathrm{Aff}\mathcal{F}(d_1; d_2, ..., d_k)$ defined by sending $F + x$

to $a \cdot F + (ax + v)$ where $(a, v) \in SO(n) \ltimes \mathbb{R}^n = SE(n)$. Affine flag manifolds are then homogeneous $SE(n)$-spaces with isotropy subgroup isomorphic to $H_{\mathcal{F}} \ltimes \mathbb{R}^{d_1}$ where $H_{\mathcal{F}}$ is the isotropy subgroup of $SO(n)$ fixing a flag in $\mathcal{F}(d_1, d_2, ..., d_k)$ and d_1 is the dimension of the flag pole;

$$\mathrm{Aff}\mathcal{F}(d_1; d_2, ..., d_k) = \frac{SE(n)}{H_{\mathcal{F}} \ltimes \mathbb{R}^{d_1}}. \tag{3.8}$$

3.2 $SU(n)$ orbits

The Lie algebra of $SU(n)$ is,

$$\mathfrak{su}(n) = \left\{ X \in \mathrm{Mat}(\mathbb{C}^n) |\ X + X^\dagger = 0,\ \ \mathrm{Tr}(X) = 0 \right\}$$

Fix some $X \in \mathfrak{su}(n)$. Since X is skew-Hermitian it may be put into diagonal form with purely imaginary diagonal entries by conjugating with a unitary matrix. That is, there exists $a \in SU(n)$ such that,

$$\mathrm{Ad}_a X = i \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\Sigma_k \lambda_k = 0$. It is easy to see that by conjugating by a suitable permutation matrix in $SU(n)$ we may wlog group the eigenvalues that are equal together so that we may write $\mathrm{Ad}_a X$ in block form as,

$$H := \mathrm{Ad}_a X = i \begin{pmatrix} \lambda_1 I_{d_1} & & \\ & \ddots & \\ & & \lambda_r I_{d_r} \end{pmatrix}. \tag{3.9}$$

Here each I_{d_k} is a $d_k \times d_k$ identity matrix where d_k is the multiplicity of the eigenvalue λ_k. Define the subspaces V_k to be the eigenspaces for each respective eigenvalue λ_k i.e. the spaces on which each block in (3.9) acts. The decomposition of H into this form is referred to as a *type decomposition.* Borrowing notation from [2] we write this decomposition as

$$\Delta_H = d_1 \Delta_{\lambda_1} + \ldots d_r \Delta_{\lambda_r}.$$

It is this type decomposition which determines the orbit manifold.

Proposition 3.2.1. The Adjoint orbit $\mathcal{O}(H)$ through H where H is of the form in (3.9) is diffeomorphic to the complex flag manifold $\mathcal{F}(d_1, \ldots, d_r)$ via an $SU(n)$-equivariant diffeomorphism.

Proof. For any $a \in SU(n)$ define the following flag $\Psi(\mathrm{Ad}_a H) \in \mathcal{F}(d_1, \ldots, d_r)$,

$$\Psi(\mathrm{Ad}_a H) = 0 \subset a \cdot V_1 \subset \cdots \subset \bigoplus_{k=1}^{r} a \cdot V_k = \mathbb{C}^n$$

The matrix $\mathrm{Ad}_a H$ has eigenvalues $\lambda_1, \ldots \lambda_r$ and corresponding eigenspaces $V_1, \ldots, V_r$. Since a matrix is determined uniquely by its eigenvalues and eigenspaces it follows that Ψ is a well defined injective map $\Psi : \mathcal{O}(H) \to \mathcal{F}(d_1, \ldots, d_r)$. Clearly Ψ is equivariant. Since $SU(n)$ acts transitively on oriented unitary frames in $\mathbb{C}^n$ it also acts transitively on flags, therefore Ψ is a smooth $SU(n)$-equivariant bijection. □

Example 5. For $SU(2)$ the possible forms for H in (3.9) are $H = 0$ or $H = \mathrm{diag}(i\lambda, -i\lambda)$. For $H = 0$ the orbit is clearly the point but for the other case the proposition above shows that the orbit is the complex flag $\mathcal{F}(1,1) \cong \mathbb{C}P^1$ which is diffeomorphic to the 2-sphere.

We will now try to find the KKS form ω on a Coadjoint orbit of $SU(2)$. Wlog we may suppose that the point on our orbit is $H = \mathrm{diag}(i\lambda, -i\lambda)$ (for $\lambda \neq 0$ so as to avoid the trivial orbit). We will write typical elements $X, Y \in \mathfrak{su}(2)$ as,

$$X = \begin{pmatrix} ix & z \\ -z^* & -ix \end{pmatrix}, \; Y = \begin{pmatrix} iy & w \\ -w^* & -iy \end{pmatrix}, \quad \text{where } x, y \in \mathbb{R} \text{ and } z, w \in \mathbb{C}.$$

The tangent vectors $\hat{X} = \mathrm{ad}_H X$ and $\hat{Y} = \mathrm{ad}_H Y$ in $T_H\mathcal{O}$ are then,

$$\hat{X} = 2i\lambda \begin{pmatrix} 0 & z^* \\ z^* & 0 \end{pmatrix}, \; \hat{Y} = 2i\lambda \begin{pmatrix} 0 & w^* \\ w^* & 0 \end{pmatrix}.$$

From equation 2.6 we have $\omega_H(\hat{X}, \hat{Y}) = \langle H, [X, Y] \rangle$, which if we use $\langle A, B \rangle = \mathrm{Tr}(A^\dagger B)$ to identify $\mathfrak{su}(2)$ with $\mathfrak{su}(2)^*$ we then calculate,

$$\omega_H(\hat{X}, \hat{Y}) = 2\lambda i \left(zw^* - wz^* \right). \tag{3.10}$$

Since $\lambda \neq 0$ we have shown in the example above that the orbit is the complex flag $\mathcal{F}(1,1)$. The diffeomorphism in Proposition 3.2.1 sends the matrix H to the flag $F = \langle e_1 \rangle \subset \langle e_1 \rangle \oplus \langle e_2 \rangle = \mathbb{C}^2$. From Proposition 3.1.1 we see that a tangent vector to such a flag may be identified with a linear map from $\langle e_1 \rangle$ to $\langle e_2 \rangle$ i.e. a linear map $\mathbb{C} \longrightarrow \mathbb{C}$. For our tangent vectors $\hat{X}$,$\hat{Y}$ we may regard the multiplication by the complex numbers z and w to represent such tangent space maps. In this interpretation the symplectic form given above in (3.10) coincides with the natural symplectic form on flags given in (3.4) modulo the factor 2λ. Writing the symplectic form explicity in terms of $\hat{X}$ and $\hat{Y}$ we have,

$$\omega_H(\hat{X}, \hat{Y}) = Tr \left(\frac{1}{2\lambda} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \hat{X}\hat{Y} \right). \tag{3.11}$$

There is another nice interpretation for this result. The tangent vectors $\hat{X}$ and $\hat{Y}$ were defined by the complex numbers $z, w \in \mathbb{C}$. If we write $z = a + ib$ and $w = c + id$ as vectors $\tilde{z}$,$\tilde{w}$ in $\mathbb{R}^2$ the symplectic form $\omega_H(\hat{X}, \hat{Y}) = 2\lambda i\,(zw^* - wz^*)$ is equal to the area form $4\lambda(\tilde{v} \times \tilde{v})$ on $\mathbb{R}^2$, i.e. the area form for vectors tangent to a sphere where $\tilde{z}, \tilde{w} \in T_pS^2 \cong \mathbb{R}^2$.

Example 6. For $SU(3)$ we list below the possible type decompositions corresponding to the possible forms of H in (3.9), the corresponding isotropy subgroups and orbit flag manifold.

Δ_H	$Stab_{\mathrm{Ad}}(H)$	$\mathcal{O}_{\mathrm{Ad}}(H)$	$Notes$
$3\Delta_0$	$SU(3)$	Point	.
$2\Delta_\lambda + \Delta_{-2\lambda}$	$S\,(U(2) \times U(1))$	$\mathcal{F}(2,1)$	$\cong \mathbb{C}P^2$
$\Delta_\lambda + \Delta_\mu + \Delta_{-(\lambda+\mu)}$	$S\,(U(1) \times U(1) \times U(1))$	$\mathcal{F}(1,1,1)$	= full flag $\lambda \neq \mu$

Question: Given tangent vectors $\hat{X} := \mathrm{ad}_X H$, $\hat{Y} := \mathrm{ad}_Y H$ can we explicitly write out the KKS form $\omega_H(\hat{X}, \hat{Y})$ for any $SU(n)$ as we did in (3.11)?

3.3 $SO(n)$ orbits

The technique used to describe the Adjoint orbits for $SU(n)$ can be almost exactly translated to work for $SO(n)$ with slight changes. There is therefore a certain sense of déjà vu. The Lie algebra is,

$$\mathfrak{so}(n) = \left\{A \in \mathrm{Mat}(\mathbb{R}^n) |\; A + A^T = 0\right\}.$$

Fix an $A \in \mathfrak{so}(n)$. A is skew-symmetric and so iA is Hermitian. This implies that A has purely imaginary eigenvalues $i\rho_1, \dots, i\rho_n$. In fact the non-zero eigenvalues occur in pairs $(i\rho_k, -i\rho_k)$.

Let v belong to the eigenspace $E_{i\rho} \subset \mathbb{C}^n$. It is easy to show that $v^* \in E_{-i\rho}$. Define $x = \frac{1}{2}\,(v - v^*)$, and $y := \frac{1}{2i}\,(v - v^*)$. It can then be shown that the plane spanned by x, y is invariant under A. In particular the action of A on x, y is,

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\rho \\ +\rho & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ +x \end{pmatrix} \tag{3.12}$$

For $i\rho_k$ with multiplicity d_k take a unitary basis $v_1, \dots, v_{d_k}$ of $E_{i\rho_k}$ and using the above procedure produce d_k orthogonal planes where the action of A is given by (3.12). With respect to a basis in this decomposition we may write A as,

$$\left(\begin{array}{c|ccc} 0 & & & \\ \hline & \rho_1 J_{d_1} & & \\ & & \ddots & \\ & & & \rho_r J_{d_r} \end{array}\right) \tag{3.13}$$

where 0 is a $k \times k$ zero matrix, $k = \dim(\operatorname{Ker} A)$ and,

$$J_{d_s} = \underbrace{J \oplus \cdots \oplus J}_{d_s \text{ times}} \text{ for } J = J_{\pm} \quad \text{where } J_+ = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \; J_- = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}.$$

Here we have grouped together $\rho_i = \pm \rho_j$ and permuted the vectors x, y as necessary in (3.13) so that they form an oriented basis for V_s where V_s is the d_s dimensional subspace upon which the block $\rho_s J_{d_s}$ acts. Note that J_{d_s} is an orthogonal complex structure on V_s. Since the basis in which A takes the form in (3.13) is orthogonal it follows that there exists an $a \in SO(n)$ such that $H := \operatorname{Ad}_a A$ is of this form. Wlog we therefore consider the Adjoint orbits through H of this form.

Proposition 3.3.1. The Adjoint orbit $\mathcal{O}(H)$ through $H \in \mathfrak{so}(n)$ where H is of the form given in (3.13) is diffeomorphic to the mixed Hermitian flag manifold $\mathcal{F}\left(k, d_1^{\mathbb{C}}, \ldots, d_r^{\mathbb{C}}\right)$ via an $SO(n)$-equivariant diffeomorphism.

Proof. Given any $a \in SO(n)$ we define the following flag $\Psi(\operatorname{Ad}_a H) \in \mathcal{F}\left(k, d_1^{\mathbb{C}}, \ldots, d_r^{\mathbb{C}}\right)$,

$$0 \subset (a \cdot \operatorname{Ker} A) \subset (a \cdot \operatorname{Ker} A) \oplus (a \cdot V_1) \subset \cdots \subset (a \cdot \operatorname{Ker} A) \oplus \bigoplus_{s=1}^{r} (a \cdot V_s) = \mathbb{R}^n,$$

and where the complex structure on each $a \cdot V_s$ is $a \circ J_{d_s} \circ a^{-1}$. It is clear from (3.13) that each $A \in \mathfrak{so}(n)$ is determined uniquely by its kernel, invariant even dimensional subspaces and their corresponding eigenvalues and complex structures. It follows that Ψ is a well defined injection $\Psi : \mathcal{O}(H) \to \mathcal{F}\left(k, d_1^{\mathbb{C}}, \ldots, d_r^{\mathbb{C}}\right)$. Equivariance of this map is obvious. Surjectivity comes from the fact that $SO(n)$ acts transitively on oriented orthogonal frames and therefore on flags. To show transitivity on oriented complex structures on each V_s we firstly define the orientation on V_s to be that for which J_{d_s} is oriented - that is there is an oriented basis for V_s for which J_{d_s} takes the form given in (3.5). The action of $SO(2m)$ on oriented complex structures on $V \cong \mathbb{R}^{2m}$ is transtive. Hence Ψ is an $SO(n)$-equivariant bijection. $\square$

Example 7. For $SO(3)$ the only possible forms for A in (3.13) are $A = 0$ and for $\rho \neq 0$,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\rho \\ 0 & +\rho & 0 \end{pmatrix}.$$

Therefore the orbits are the point and the flag manifold $\mathcal{F}(1, 2^{\mathbb{C}})$. Recall that a choice of complex structure on a plane is equivalent to a choice of orientation. Therefore this orbit is the manifold of oriented planes in $\mathbb{R}^3$ which is diffeomorphic to the 2-sphere.

Remark 3.3.1. Observe that the orbits were the same for $SO(3)$ and $SU(2)$. This however is expected from Proposition 2.1.1 since $SU(2)$ is a 2-1 covering of $SO(3)$.

Example 8. For $SO(4)$ we list below the possible type decompositions for $A \in \mathfrak{so}(4)$ along with the corresponding isotropy subgroup and adjoint orbit through A.

Δ_A	$Stab_{\mathrm{Ad}}(A)$	$\mathcal{O}_{\mathrm{Ad}}(A)$	$Notes$
$4\Delta_0$	$SO(4)$	Point	.
$\Delta_\rho + \Delta_\kappa$	$S\left(U(1) \times U(1)\right)$	$\mathcal{HF}(2,2)$	$= \widetilde{\mathrm{Gr}}(2;2), \rho \neq \kappa$
$2\Delta_\rho$	$U(2)$	$\mathcal{HF}(4)$	$=$ (Complex structures on $\mathbb{R}^4$) $\cong S^2$.
$2\Delta_0 + \Delta_\rho$	$S\left(O(2) \times U(1)\right)$	$\mathcal{F}(2, 2^{\mathbb{C}})$	$= \widetilde{\mathrm{Gr}}(2;2)$.

Whenever we write $\rho, \kappa, \epsilon, \lambda$ and μ when parametrizing Lie algebra elements, we mean them to be distinct and non-zero real numbers.

Chapter 4

Co/Adjoint orbits of compact connected simple Lie groups

4.1 A generalization of the diagonalization of symmetric matrices.

Having described the Adjoint orbits of $SU(n)$ and $SO(n)$ we have completed just short of three quarters of the work required to classify the Adjoint orbits of all compact simple Lie groups which we list below.

$\mathfrak{su}(n+1),\ \mathfrak{so}(2n+1),\ \mathfrak{sp}(n),\ \mathfrak{so}\left(2(n+1)\right)$, for $n \geq 1$ and the exceptionals $\mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$.

Of the classical Lie algebras the only one that we have not touched upon is the compact symplectic group. Our methods used for $SU(n)$ and $SO(n)$ involved diagonalizing a Lie algebra element and putting it into a more manageable form. The compact symplectic group and its Lie algebra are defined as,

$$Sp(n) = \left\{ a \in GL(\mathbb{H}^n) |\ aa^\dagger = I \right\},\ \mathfrak{sp}(n) = \left\{ A \in \mathrm{Mat}(\mathbb{H}^n) |\ A + A^\dagger = 0 \right\}$$

The group is sometimes called the hyper unitary group since it is the group of isomorphisms of a quaternionic (left/right)-module preserving a Hermitian inner product. The Lie algebra consists of skew-'hyper' Hermitian matrices and so one might hope to apply the same method before and diagonalize elements. However over the quaternions there are issues; since the algebra is not commutative there are left and right eigenvalues and eigenvectors. Furthermore some elements may have infinitely many eigenvalues. For example,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sp}(2)$$

has characteristic polynomial $\chi(t) = t^2 + 1$. Over $\mathbb{H}$ the solution space is homeomorphic to S^3. It therefore does not seem straightforward how one might diagonalize elements.

Question: The Adjoint orbits of $SU(n)$ were complex flags in $\mathbb{C}^n$. A complex flag may be identified with a Hermitian flag in $\mathbb{R}^{2n}$. The converse however is not true. The fact that the Adjoint orbits of $SU(n)$ may be viewed as submanifolds of Adjoint orbits of $SO(2n)$ can be seen from the identiy,

$$SU(n) \cong SO(2n) \cap \mathrm{Symp}(2n; \mathbb{R}).$$

Here the role of $\mathrm{Symp}(2n; \mathbb{R})$ may be thought of as ensuring that a real even dimensional subspace in $\mathbb{R}^{2n}$ corresponds to a complex subspace in $\mathbb{C}^n$ and that the complex structures on the flag coincide with multiplication by i. Similarly for the compact symplectic group we have,

$$Sp(n) \cong SU(2n) \cap \mathrm{Symp}(2n; \mathbb{C}).$$

Can we in some way analogously identify Adjoint orbits of $SU(n)$, that is complex flags in $\mathbb{C}^{2n}$ as submanifolds of orbits of $Sp(n)$? Here however we cannot obviously identify a complex line with a quaternionic subspace. Could the Adjoint orbits be quaternionic flags?

For the exceptional Lie algebras the situation is even worse. There are no 'nice' faithful matrix representations of the compact exceptional Lie groups. Since there are only five such groups one could compute by hand the Adjoint orbits using one of these representations. This however is certainly an almost impossibly tedious and unenlightening task (the smallest representation of $\mathfrak{e}_8$ is of dimension 248). Fortunately we can nicely generalize the diagonalisation method used so far.

Recall that for a semisimple Lie algebra $\mathfrak{g}$ there exist Cartan subalgebras $\mathfrak{h}$. These may be defined as maximal abelian subalgebras where every $H \in \mathfrak{h}$ is semisimple (that is ad_X is a semisimple operator in $\mathfrak{gl}(\mathfrak{g})$). Since we are dealing with compact Lie groups this final condition is redundant as every element is semisimple. An element $H \in \mathfrak{g}$ is called regular if,

$$\mathfrak{g}_0(H) = \{X \in \mathfrak{g} |\ [X, H] = 0\}$$

is a Cartan subalgebra. It is a fact that every Cartan subalgebra may be written in this way for some H. We now prove a theorem which generalizes the diagonalisation of Hermitian matrices ([1][p. 73]).

Theorem 2. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of a compact semisimple Lie algebra $\mathfrak{g}$. Let $\mathcal{O}(Y)$ denote the Adjoint orbit through $Y \in \mathfrak{g}$. Then for any $Y \in \mathfrak{g}$ the intersection $\mathcal{O}(Y) \cap \mathfrak{h}$ is non-empty, moreover the intersection is clean.

Proof. Fix some regular element H satisfying $\mathfrak{g}_0(H) = \mathfrak{h}$. Define the function f on $\mathfrak{g}$ to be the squared distance function from H, that is $f(X) = \langle H - X, H - X \rangle$ where we are using the Killing form on $\mathfrak{g}$. Since $\mathfrak{g}$ is compact the Killing form is negative definite and so f is a function $f : \mathfrak{g} \to \mathbb{R}$ with $f(X) \leq 0$ for all $X \in \mathfrak{g}$ and $f(X) = 0 \Leftrightarrow X = H$. Restrict f to $\mathcal{O}(Y)$. If $f(X) = 0$ for some $X \in \mathcal{O}(Y)$ we are done since then $X = H \in \mathfrak{h}$. Suppose then that $f(X) \neq 0$ for all $X \in \mathcal{O}(Y)$. Then f is a negative real function bounded above by zero. It follows that $f|_{\mathcal{O}(Y)}$ must have a critical point, that is a point $Y' \in \mathcal{O}(Y)$ such that $df|_{Y'} = 0$. For $\mathrm{ad}_X Y' \in T_{Y'}\mathcal{O}(Y)$ we have,

$$df|_{Y'} \left(\mathrm{ad}_X Y'\right) = \frac{d}{dt}\Big|_{t=0} f\left(\mathrm{Ad}_{\exp tX} Y'\right) = -2\langle \mathrm{ad}_X Y', H \rangle.$$

So if $df|_{Y'} = 0$ then $\langle [X, Y'], H \rangle = 0$ for all $X \in \mathfrak{g}$. This is equivalent to $\langle X, [Y', H] \rangle = 0$, so by non-degeneracy $[Y', H] = 0$ and so $Y' \in \mathfrak{g}_0(H) = \mathfrak{h}, \implies \mathcal{O}(Y) \cap \mathfrak{h} \neq \emptyset$.

$T_{Y'}\mathcal{O}(Y) = \{\mathrm{ad}_X Y' | X \in \mathfrak{g}\}$. So for $H' \in \mathfrak{h}$ arbitrary $\langle \mathrm{ad}_X Y', H' \rangle = \langle X, [Y', H'] \rangle = 0$. Therefore the intersection is orthogonal and hence clean. This also implies that the set $\mathcal{O}(Y) \cap \mathfrak{h}$ is discrete. □

This theorem in fact holds for any compact group G; from Theorem 1 we see that the Adjoint action is just that of the Adjoint action on the semisimple factor of the group. Alternatively the definition of Cartan subalgebra does not require the group to be semisimple - defining it as a maximal abelian subalgebra will do. Note that it was crucial to the proof that the Killing form was negative definite not just non-degenerate. Take for example the simple yet non-compact group $SL(2; \mathbb{R})$. The Lie algebra $\mathfrak{sl}(2; \mathbb{R})$ consists of traceless real 2×2 matrices and a Cartan subalgebra is,

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}.$$

For any element $H \in \mathfrak{h}$, $\mathrm{Ad}_a H$ will always have negative determinant. Clearly this is not true for all members in $\mathfrak{sl}(2; \mathbb{R})$ and so the above result does not hold in general for non-compact semisimple Lie groups.

Proposition 4.1.1. Any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ are related by the Adjoint action.

Proof. Let H_1 and H_2 be regular elements such that $\mathfrak{h}_1 = \mathfrak{g}_0(H_1)$ and $\mathfrak{h}_2 = \mathfrak{g}_0(H_2)$. By the above theorem there exists $g \in G$ such that $\mathrm{Ad}_g H_1 = H_2$. Consider $\mathrm{Ad}_g \overline{H}_1$ for any $\overline{H}_1 \in \mathfrak{g}_1$. Then $[\mathrm{Ad}_g \overline{H}_1, H_2] = \mathrm{Ad}_g[\overline{H_1}, H_1] = 0$ which implies $\mathrm{Ad}_g \overline{H}_1 \in \mathfrak{g}_0(H_2)$ and so $\mathrm{Ad}_g \mathfrak{h}_1 \subseteq \mathfrak{h}_2$. By the maximal abelian property of Cartan subalgebras we must have $\mathrm{Ad}_g \mathfrak{h}_1 = \mathfrak{h}_2$. □

Now recall that one definition of the *Weyl group* is the group of Adjoint actions which leave $\mathfrak{h}$ invariant modulo those which fix $\mathfrak{h}$. That is,

$$W(H) := \frac{N(H)}{Z(H)}.$$

Here H is the maximal torus in G with Lie algebra $\mathfrak{h}$ and $N(H), Z(H)$ are the normalizer and centralizer of H respectively. It is readily checked that $Z(H)$ is a normal subgroup of $N(H)$. It is a remarkable fact that this defintion of the Weyl group coincides with the definition of $W(H)$ as generated by the finite group of reflections in the orthogonal complements of roots in $\mathfrak{h}$.

Proposition 4.1.2. For G compact and semisimple the set $\mathrm{Crit}(f) = \mathcal{O}(Y) \cap \mathfrak{h}$ is an orbit of the Weyl group.

Proof. Let X, Y belong to $\mathcal{O}(Y) \cap \mathfrak{h}$. Then we have shown that there exists a g such that $\mathrm{Ad}_g Y = X$. We first claim that $\mathfrak{g}_0(X) = \mathrm{Ad}_g \mathfrak{g}_0(Y)$; let $H \in \mathfrak{g}_0(Y)$ so that $[H, Y] = 0$. Then $[\mathrm{Ad}_g H, X] = \mathrm{Ad}_g[H, Y] = 0$ and so $\mathrm{Ad}_g \mathfrak{g}_0(Y) \subseteq \mathfrak{g}_0(X)$. Similarly $\mathrm{Ad}_{g^{-1}} \mathfrak{g}_0(X) \subseteq \mathfrak{g}(Y)$ and so $\mathfrak{g}_0(X) = \mathrm{Ad}_g \mathfrak{g}_0(Y)$. Since $\mathfrak{h} \subseteq \mathfrak{g}_0(Y)$ we therefore have $\mathrm{Ad}_g \mathfrak{h} \subseteq \mathfrak{g}_0(X)$. Now let G_X be the stabilizer of X and $\mathfrak{g}_X$ its Lie algebra. Note that $\mathfrak{g}_X = \mathfrak{g}_0(X)$. $G_X \subseteq G$ is compact and so the above proposition applies to G_X, namely that any two Cartan subalgebras are related by the Adjoint action. Now we observe that both $\mathfrak{h}$ and $\mathrm{Ad}_g \mathfrak{h}$ are both Cartan subalgebras in $\mathfrak{g}_0(X)$ and therefore there exists a $Z \in \mathfrak{g}_0(X)$ such that $\mathrm{Ad}_{\exp Z} \mathrm{Ad}_g \mathfrak{h} = \mathfrak{h}$. Also $\mathrm{Ad}_{\exp Z} X = X$ and so $\mathrm{Ad}_{\exp Z} \mathrm{Ad}_g Y = X$. Therefore $\mathrm{Ad}_{\exp Z \cdot g}$ preserves $\mathfrak{h}$ and sends Y to X and so is a member of the Weyl group. □

Question: For H regular, the function $f(X) = \langle H - X, H - X \rangle$ used in the proof of the Theorem is a Morse function with $\mathrm{Crit} f$ an orbit of the Weyl group in $\mathfrak{h}$. What can we say about the indices of these critical points and thus the homology of the orbits?

4.2 Adjoint orbit correspondence with subsets of Dynkin diagrams

Let $\mathfrak{u}$ be a compact real form of a simple complex Lie algebra $\mathfrak{g}$. It is a fact that all compact forms of $\mathfrak{g}$ are isomorphic. We therefore will use a convenient choice of compact real form given by a Weyl-Chevalley basis of $\mathfrak{g}$.

Theorem 3. (Weyl-Chevalley basis) Let $\mathfrak{g}$ be a simple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$ and consider the root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \text{ where, } \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \operatorname{ad}_H X = \alpha(H)X, \ \forall H \in \mathfrak{h}\} \tag{4.1}$$

where the sum is over the non-zero roots $R \subset \mathfrak{h}^*$. There exists a basis $\{H_\alpha, X_\alpha, Y_\alpha\}_{\alpha \in R^+}$ satisfying the following properties;

- R^+ is the set of positive roots given a notion of *height* on R
- $\mathfrak{h} = \text{Span}\,\{H_\alpha\}_{\alpha \in R^+}$
- $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$
- $\alpha\,(H_\beta) \in \mathbb{R}$ for all $\alpha, \beta \in R$
- Under the isomorphism $\mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}$ induced by the Killing form, $\alpha \mapsto H_\alpha$ for every $\alpha \in R^+$, (this is equivalent to $H_\alpha = [X_\alpha, Y_\alpha]$ where $\langle X_\alpha, Y_\alpha \rangle = 1$).

Proof. See [5] Chapter III, Section 5. □

A corollary to this theorem is that the following real subalgebra is in fact a compact real form of $\mathfrak{g}$ ([5] p. 181),

$$\mathfrak{r} = \text{Span}_{\mathbb{R}}\{\underbrace{iH_\alpha}_{\widetilde{H}_\alpha}, \underbrace{i\,(X_\alpha + Y_\alpha)}_{\widetilde{X}_\alpha}, \underbrace{X_\alpha - Y_\alpha}_{\widetilde{Y}_\alpha}\}. \tag{4.2}$$

The subalgebra $\mathfrak{h} = \text{Span}_{\mathbb{R}}\{\widetilde{H}_\alpha\}_{\alpha \in R}$ is a Cartan subalgebra for $\mathfrak{r}$. We can show that the action of $\operatorname{ad}_{\mathfrak{h}}$ is invariant on the planes $\langle \widetilde{X}_\alpha, \widetilde{Y}_\alpha \rangle$,

$$\begin{aligned} \operatorname{ad}_H \widetilde{X}_\alpha &= -1\,(\alpha(H)(X_\alpha - Y\alpha)) = -\alpha(H)\widetilde{Y}_\alpha \\ \operatorname{ad}_H \widetilde{Y}_\alpha &= \quad i\alpha(H)(X_\alpha + Y_\alpha) \quad\ = +\alpha(H)\widetilde{X}_\alpha \end{aligned} \tag{4.3}$$

Let $S \in R^+$ be the set of simple roots. The Weyl group is generated by the reflections in the orthogonal complements of the vectors $\widetilde{H}_\alpha$ for $\alpha \in S$. Denote such a hyperplane in $\mathfrak{h}$ by Π_α. A *Weyl chamber* is defined to be the closure of a connected component of the set,

$$\mathfrak{h} \setminus \bigcup_{\alpha \in S} \Pi_\alpha.$$

Of the Weyl chambers we select one in particular which we shall call the *principal Weyl chamber* satisfying,

$$\Delta = \{H \in \mathfrak{h} \,|\, \langle H, H_\alpha \rangle > 0, \quad \forall \alpha \in S\}\,. \tag{4.4}$$

The action of the Weyl group on the Weyl chambers is simple and transitive. Therefore from Proposition 4.1.2 each Adjoint orbit intersects each Weyl chamber in precisely one point. It follows that the classification of the Adjoint orbits reduces to the classification of Adjoint orbits $\mathcal{O}(H)$ where $H \in \Delta$.

Proposition 4.2.1. Let H belong to the Weyl chamber $\Delta \subset \mathfrak{h}$ of a simple compact real form $\mathfrak{r}$ as given in (4.2). Denote the stabilizer of H under the Adjoint action by $G_H \subseteq G$ where G is a connected Lie group with Lie algebra $\mathfrak{r}$. Then the Lie algebra $\mathfrak{g}_H$ of G_H is,

$$\mathfrak{g}_H = \mathfrak{r}_0(H) = \mathfrak{h} \oplus \sum_{\forall \alpha \in R | H \in \Pi_\alpha} \langle \widetilde{X}_\alpha, \widetilde{Y}_\alpha \rangle. \tag{4.5}$$

Proof. Recall the definition of $\mathfrak{r}_0(H)$,

$$\mathfrak{r}_0(H) = \{X \in \mathfrak{r} \mid [X, H] = 0\}.$$

Let X belong to $\mathfrak{g}_H$. Then,

$$\mathrm{ad}_X H = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp tX} H = 0,$$

and so $\mathfrak{g}_H \subseteq \mathfrak{r}_0(H)$. Conversely let $Y \in \mathfrak{r}_0(H)$. Consider the curve $\gamma(t) := \mathrm{Ad}_{\exp(tY)} H$. Then,

$$\frac{d}{dt}\bigg|_{t=s} \gamma(t) = \frac{d}{dt}\bigg|_{t=s} \mathrm{Ad}_{\exp(tY)} H = \mathrm{Ad}_{\exp(sY)} \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tY)} H = \mathrm{Ad}_{\exp(sY)}(\mathrm{ad}_Y H) = 0.$$

Since $\gamma(0) = H$ we therefore have that $\mathrm{Ad}_{\exp(tY)} H = H$ for all t. Hence $Y \in \mathfrak{g}_H$ and so $\mathfrak{g}_H = \mathfrak{r}_0(H)$. Now clearly $\mathfrak{h}$ is contained in $\mathfrak{r}_0(H)$ by the abelian property of Cartan subalgebras. From (4.3) we see that for any $X \in \mathfrak{r}$ not in $\mathfrak{h}$ satisfying $[H, X] = 0$ that X must belong to a plane $\langle \widetilde{X}_\alpha, \widetilde{Y}_\alpha \rangle$ where $\alpha(H) = 0$, that is $\langle H_\alpha, H \rangle = 0$, equivalently $H \in \Pi_\alpha$. It follows then that all of $\langle \widetilde{X}_\alpha, \widetilde{Y}_\alpha \rangle$ is contained in $\mathfrak{r}_0(H)$. Conversely if $\langle H_\alpha, H \rangle = 0$ for some root α then since $\alpha(H) = 0$ we see again from (4.3) that $\mathfrak{r}_0(H)$ contains $\langle \widetilde{X}_\alpha, \widetilde{Y}_\alpha \rangle$ giving us (4.5). □

Remark 4.2.1. Since the Adjoint orbits are determined by a point in Δ, the above proposition shows that the isotropy subgroup is determined by the geometry of the Weyl chamber. If $H \in \Delta$ is regular then it is in the interior of the Weyl chamber and so $\mathfrak{g}_H = \mathfrak{h}$. The resulting Adjoint orbit is the homogeneous space G/H where as before H is the maximal torus in G with Lie algebra $\mathfrak{h}$. The remaining orbit types are determined by which faces Π_α (for α a simple root) of the principal Weyl chamber H is inside. Let d denote the number of such hyperplanes that the point H lies inside. We see then from (4.5) that,

$$\dim(\mathfrak{g}_H) \geq k + 2d, \tag{4.6}$$

where k is the dimension of $\mathfrak{h}$ - the *rank* of $\mathfrak{r}$. The inequality is not always achieved as we will see from later examples. This is because being orthogonal to a given set of simple roots does not mean that they cannot also be orthogonal to other non-simple roots. We can also establish the *modulus* of a particular orbit from the geometry. The modulus is the degree of freedom that a point in Δ may move through inside Δ and still have the same orbit up to diffeomorphism. For a regular element in the interior of Δ the modulus is k. For an H belonging to the intersection of d faces of Δ, the modulus is $k-d$. This is due to the fact that the codimension of the intersection of d linearly independent hyperplanes is d.

Example 9. We will go through a detailed example for when $\mathfrak{r} = \mathfrak{su}(3)$. In this case the complexification $\mathfrak{su}(3)_{\mathbb{C}}$ is is the 8 dimensional complex simple Lie algebra $\mathfrak{sl}(3;\mathbb{C})$. We choose the following Cartan subalgebra,

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda-\mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{C} \right\}. \tag{4.7}$$

The action of $\mathfrak{h}$ on $\mathfrak{sl}(3;\mathbb{C})$ has 6 roots in $\mathfrak{h}^*$; these are $\alpha_{ij} \in \mathfrak{h}^*$ for $i,j = 1,2,3$, $i \neq j$ where,

$$\alpha_{ij}(H) = (H)_{ii} - (H)_{jj}, \quad \forall H \in \mathfrak{h}. \tag{4.8}$$

The eigenvector for each root α_{ij} is the matrix E_{ij} whose only non zero entry is a 1 in the (i,j) position. Temporarily let H_{ij} denote $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. Let T_{ij} denote the element in $\mathfrak{h}$ which is the image of the root α_{ij} under the isomorphism $\mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}$ induced by the Killing form $\langle\ ,\ \rangle$. It can be shown that,

$$T_{ij} = \alpha_{ij}(H_{ij}) \frac{H_{ij}}{\langle H_{ij}, H_{ji} \rangle} \tag{4.9}$$

From (4.8) we get $\alpha_{ij}(H_{ij}) = +2$. We can also calculate $\langle H_{ij}, H_{ji} \rangle$ (ref. [4] Chapter 14),

$$\langle H_{ij}, H_{ji} \rangle = \sum_{\alpha \in R} \alpha(H_{ij})^2 = 2\left(2^2 + 1^2 + 1^2\right) = 12.$$

We then have,

$$T_{ij} = \frac{1}{6} H_{ij}.$$

We have therefore found a basis $\{T_{ij}, E_{ij}, E_{ji} \mid i < j\}$ satisfying the conditions in Theorem 3. We thus have a real form given by (4.2), with Cartan subalgebra $\mathfrak{h} = \mathrm{Span}_{\mathbb{R}}\{iT_{ij} \mid i < j\}$. Indeed since $H_{ij} = E_{ii} - E_{jj}$ this Cartan subalgebra is the same as (4.7) except where λ and μ are purely imaginary. So for $\mathfrak{su}(3)$ we have Cartan subalgebra,

$$\mathfrak{h} = \left\{ i \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda-\mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}. \tag{4.10}$$

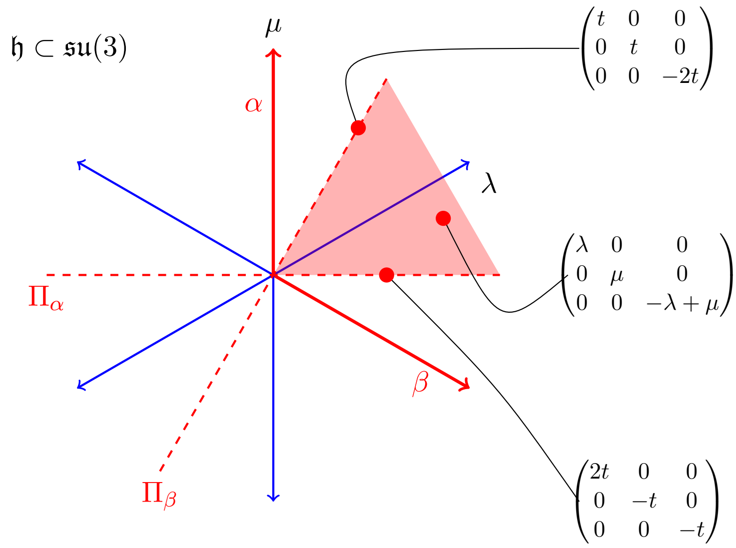


Figure 4.1: The root diagram of $\mathfrak{h} \subset \mathfrak{su}(3)$. The simple roots α,β are for an arbitrary choice of height on $\mathfrak{h}$ are indicated in red. The remaining roots are indicated by blue arrows. The parameters λ and μ correspond to those in (4.10). We have shown on the principal Weyl chamber (shaded in pink) the positions of typical elements with distinct orbit types.

The root vectors are given by $iT_{ij} \in \mathfrak{h}$. Since the Weyl chambers are determined by the Euclidean geometry of $\mathfrak{h}$ we should also calculate the Killing form restricted to $\mathfrak{h}$. This is easy since the vectors iT_{ij} were defined so that $\langle T_{ij}, H\rangle = \alpha_{ij}(H)$ for all $H \in \mathfrak{h}$. We therefore have,

$$\langle iT_{12}, iT_{ij}\rangle = -\alpha_{12}(T_{ij}) = -\left(\delta_{1i} + \delta_{2j}\right)$$

Using this we can calculate that the mutual angles between root vectors in $\mathfrak{h}$ are one of $0, \pi/3, 2\pi/3$. We are now in a position to draw the Cartan subalgebra and highlight a Weyl chamber corresponding to an arbitrary choice of simple roots, this is given in Figure 4.1.

With reference to Example 6 and Figure 4.1 we can see from the Weyl chamber how the points correspond to orbit type (see Table 4.1).

We will now provide further examples of root diagrams and Adjoint orbits for the groups $SO(n)$ for $n = 4, 5, 6, 7, 8$. We will not carry out a derivation of the root system as we did in the above example as these details may be found in the literature ([4]).

For $\mathfrak{so}(4)$ the Cartan subalgebra we use is,

$$\mathfrak{h} = \left\{ \begin{pmatrix} \rho & 0 \\ 0 & \kappa \end{pmatrix} \;\middle|\; \rho, \kappa \in \mathbb{R} \right\} \quad \text{where we are using the abbreviation } \rho = \begin{pmatrix} 0 & -\rho \\ +\rho & 0 \end{pmatrix}. \tag{4.11}$$

Remark 4.2.2. From Figure 4.2 we see that the root system is reducible to that of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. From Remark 2.1.1 we have that the Adjoint orbits of $SO(4)$ must be the same as

Typical element H	Type decomposition Δ_H	Stabilizer G_H	$\dim(G_H)$	Orbit
(0)	$3\Delta_0$	$SU(3)$	8	$\{0\}$
$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -2t \end{pmatrix}$	$2\Delta_t + \Delta_{-2t}$	$S\left(U(2) \times U(1)\right)$	4	$\mathcal{F}(2,1)$
$\begin{pmatrix} 2t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & -t \end{pmatrix}$	$\Delta_{2t} + 2\Delta_{-t}$	$S\left(U(1) \times U(2)\right)$	4	$\mathcal{F}(1,2)$
$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda-\mu \end{pmatrix}$	$\Delta_\lambda + \Delta_\mu + \Delta_{-\lambda-\mu}$	$S\left(U(1) \times U(1) \times U(1)\right)$	2	$\mathcal{F}(1,1,1)$

Table 4.1: Orbit breakdown for $SU(3)$. The typical elements for H correspond to those in the interior of the Weyl chambers or in the orthogonal planes to the simple roots α, β (see Figure 4.1). Here λ and μ are distinct and not all zero.

Typical element H	Type decomposition Δ_H	Stabilizer G_H	$\dim(G_H)$	Orbit
(0)	$2\Delta_0$	$SO(4)$	6	$\{0\}$
$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$	$2\Delta_\rho$	$U(2)$	4	$\mathcal{HF}(4) \cong S^2$
$\begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix}$	$\Delta_\rho + \Delta_{-\rho}$	$U(2)$	4	$\mathcal{HF}(4) \cong S^2$
$\begin{pmatrix} \rho & 0 \\ 0 & \kappa \end{pmatrix}$	$\Delta_\rho + \Delta_\kappa$	$SO(2) \times SO(2)$	2	$\widetilde{\mathcal{F}}(2,2) \cong \mathcal{HF}(2,2)$

Table 4.2: Orbit characterization for $\mathfrak{so}(4)$, ($\rho \neq \kappa$, $\rho, \kappa \neq 0$).

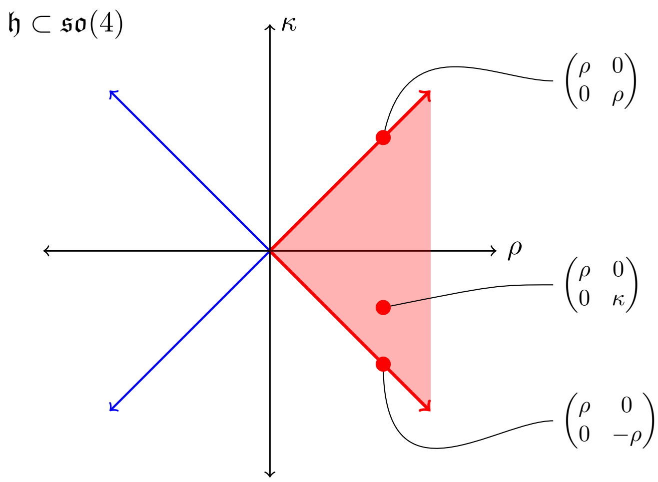


Figure 4.2: Root diagram $\mathfrak{h} \subset \mathfrak{so}(4)$. Simple roots are the red arrows and the remaining two are in blue. The parametrization $\rho, \kappa \in \mathbb{R}$ is that in (4.11). Notice how this root system is reducible to that of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

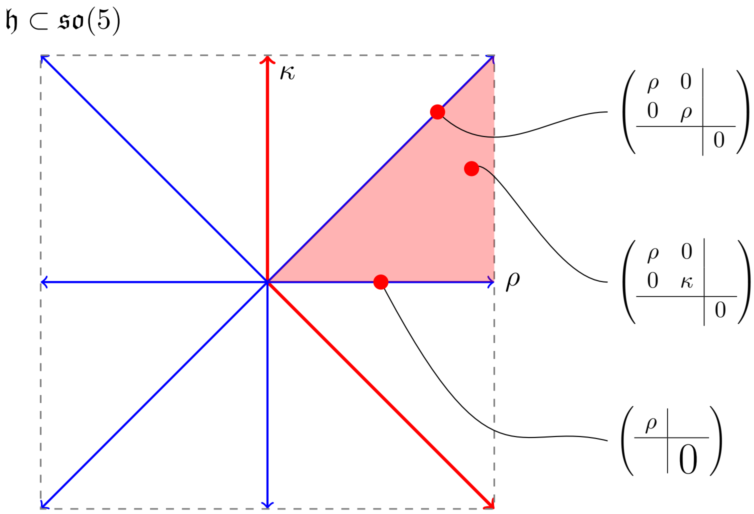


Figure 4.3: Root diagram for $\mathfrak{h} \subset \mathfrak{so}(5)$. Simple roots in red, others in blue. We remark that there is an accidental isomorphism $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ and so the orbits in this picture will be the same for $Sp(2)$.

Typical element H	Type decomposition Δ_H	Stabilizer G_H	$\dim(G_H)$	Orbit
(0)	$2\Delta_0$	$SO(5)$	10	$\{0\}$
$\left(\begin{array}{cc\|c} \rho & 0 & \\ 0 & \rho & \\ \hline & & 0 \end{array}\right)$	$2\Delta_\rho$	$U(2)$	4	$\mathcal{F}(\tilde{1}, 4^{\mathbb{C}})$
$\left(\begin{array}{c\|c} \rho & \\ \hline & 0 \end{array}\right)$	$\Delta_\rho + \Delta_0$	$SO(2) \times SO(3)$	4	$\widetilde{\mathcal{F}}(2,3) \cong \widetilde{Gr}(2;3)$
$\left(\begin{array}{cc\|c} \rho & 0 & \\ 0 & \kappa & \\ \hline & & 0 \end{array}\right)$	$\Delta_\rho + \Delta_\kappa$	$SO(2) \times SO(2)$	2	$\widetilde{\mathcal{F}}(1,2,2)$

Table 4.3: Orbit characterization for $\mathfrak{so}(5)$.

those of $SO(3) \times SO(3)$. Thus the orbits are products of two points and/or spheres (the Adjoint orbits of $SO(3)$). We therefore see that the generic orbit $\widetilde{\mathcal{F}}(2,2) = \widetilde{Gr}(2;2)$ is homeomorphic to $S^2 \times S^2$.

For $SO(5)$ we have essentially the same Cartan subalgebra as in $SO(4)$,

$$\mathfrak{h} = \left\{ \left(\begin{array}{cc|c} \rho & 0 & \\ 0 & \kappa & \\ \hline & & 0 \end{array}\right) \,\middle|\, \rho, \kappa \in \mathbb{R} \right\}.$$

Here as before, ρ and κ represent 2×2 matrix blocks and the bottom right zero is a single entry. Note that in $\mathfrak{so}(5)$ the following two elements are conjugate (which we can see since they are sent into each other by a reflection in the line perpendicular to the ρ-axis which is a root),

$$\left(\begin{array}{cc|c} J & 0 & \\ 0 & J & \\ \hline & & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} J & 0 & \\ 0 & -J & \\ \hline & & 0 \end{array}\right).$$

Where J is the matrix $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$. Yet in $\mathfrak{so}(4)$ they are not conjugate (since they occupy the same Weyl chamber and Adjoint orbits must only intersect in a single point),

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \nsim \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

This reflects the fact that in a (necessarily) even dimensional real vector space the space of complex structures has two connected components,

$$\text{Complex structures on } \mathbb{R}^{2n} = \frac{GL(2n)}{U(n)},$$

corresponding to the two components of $GL(2n)$. This space contracts onto the manifold of orthogonal complex structures,

$$\frac{O(2n)}{U(n)} = \frac{SO^+(2n)}{U(n)} \bigsqcup \frac{SO^-(2n)}{U(n)}.$$

The two components correspond to oriented complex structures for two choices of orientation on $\mathbb{R}^{2n}$. That is, each component is mapped onto the other by conjugating with a linear map with determinant -1. This fact is manifested by the fact that the two matrices $J^+ := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, $J^- := \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$ the first oriented and the second not (in the sense given in (3.5)), are not in the same Adjoint orbit.

However, the orbits are diffeomorphic and this is not a coincidence; for $\mathfrak{h} \subset \mathfrak{so}(4)$ their is an isomorphism (namely reflection in the ρ-axis) which preserves the roots and swaps J^+ with J^-. It is a theorem ([5]p. 173) that any isomorphism of $\mathfrak{h}$ preserving the root system may be extended to an isomorphism over all of $\mathfrak{g}$. Therefore although this reflection is not generated by the Adjoint action (and therefore is not an inner automorphism) it is in fact an outer automorphism of $\mathfrak{so}(4)$ which swaps orientation preserving elements with orientation reversing ones.

Proposition 4.2.2. Consider two points H_1, H_2 in $\mathfrak{h}$ and suppose there is an automorphism φ of $\mathfrak{g}$ for which $\varphi(H_1) = H_2$. Then the Adjoint orbits $\mathcal{O}(H_1)$ and $\mathcal{O}(H_2)$ are diffeomorphic.

Proof. Let G_1, G_2 be the stabilizers for H_1, H_2 and $\mathfrak{g}_1$,$\mathfrak{g}_2$ their respective Lie algebras. In the proof of Proposition 4.2.1 we showed that,

$$\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [X, H_i] = 0\}, \quad (i = 1, 2).$$

We claim that φ restricts to an isomorphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$; Let X belong to $\mathfrak{g}_1$. Then $0 = \varphi([X, H_1]) = [\varphi(X), H_2]$ which implies that $\varphi(\mathfrak{g}_1) \subseteq \mathfrak{g}_2$. Replacing φ with φ^{-1} and swapping H_1 with H_2 in the above argument gives us $\varphi(\mathfrak{g}_1) = \mathfrak{g}_2$. Now let $\widetilde{G}$ be the universal covering group of G and $\widetilde{G}_1$, $\widetilde{G}_2$ the subgroups in $\widetilde{G}$ with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Then φ lifts to an isomorphism $\tilde{\varphi}$ of $\widetilde{G}$ ([4] p.119) such that $\tilde{\varphi}(\widetilde{G}_1) \cong \widetilde{G}_2$. By Proposition 2.1.1, $\mathcal{O}(H_i)$ is diffeomorphic to the Adjoint orbit of $\widetilde{G}$ through H_i with isotropy subgroup $\widetilde{G}_i$. It follows then that there is a diffeomorphism defined coset-wise between the Adjoint orbits viewed as homogenous spaces given by sending $g\widetilde{G}_1$ to $\tilde{\varphi}(g)\tilde{\varphi}(\widetilde{G}_1) = \tilde{\varphi}(g)\widetilde{G}_2$;

$$\mathcal{O}(H_1) = \frac{G}{G_1} = \frac{\widetilde{G}}{\widetilde{G}_1} \xrightarrow{\cong} \frac{\widetilde{G}}{\widetilde{G}_2} = \frac{G}{G_2} = \mathcal{O}(H_2).$$

□

We can establish a nice correspondence between Adjoint orbits of $\mathfrak{g}$ and subsets of the Dynkin diagram; each node in a Dynkin diagram represents a simple root. Given an orbit in $\mathfrak{g}$ let H be the point intersecting the principal Weyl chamber Δ. For each hyperplane Π_α orthogonal to a simple root vector $\widetilde{H}_\alpha$ that contains H we colour the node of the Dynkin diagram corresponding to α. In this way we see that generic orbits, those passing through the interior of Δ have no nodes coloured whereas the trivial point orbit corresponds to the diagram with all nodes coloured.

For example consider the Dynkin diagram $A_1 \times A_1$ for $\mathfrak{so}(4)$. We list below the orbit types with with Dynkin diagrams.

Typical element H	Dynkin diagram
(0)	● ●
$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$	● ○
$\begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix}$	○ ●
$\begin{pmatrix} \rho & 0 \\ 0 & \kappa \end{pmatrix}$	○ ○

It is a fact (see [5]p. 423) that the outer automorphisms modulo inner automorphisms are isomorphic to the automorphism group of the Dynkin diagram. That is,

$$\frac{\mathrm{Aut}(\mathfrak{g})}{\mathrm{Int}(\mathfrak{g})} \cong \mathrm{Aut}\left(\mathrm{Dynkin}(\mathfrak{g})\right).$$

Our previous remark concerning the bijection between the two connected components of the space of complex structures can then be thought of as a consequence of the non-trivial degree two automorphism of the Dynkin diagram $A_1 \times A_1$.

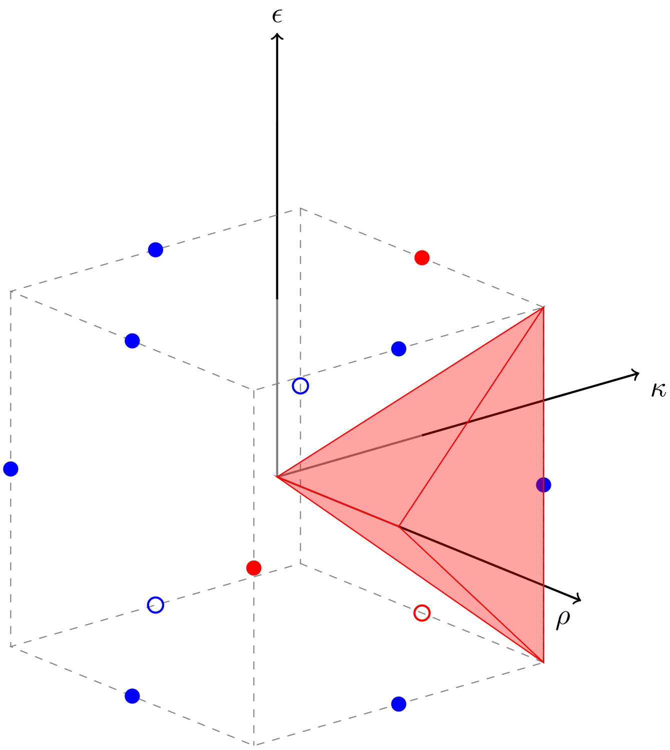


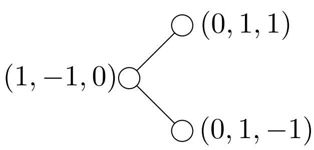
Figure 4.4: Root diagram for $\mathfrak{so}(6)$. The three simple roots are in red; $(1,-1,0),\ (0,1,1),\ (0,1,-1)$, the others in blue and the principal Weyl chamber in pink.

This compact classification of Adjoint orbits via Dynkin diagrams helps us to understand the intersection of orbits with Δ when the rank of $\mathfrak{g}$ becomes large. We now give two examples of rank 3 Lie groups $SO(6)$ and $SO(7)$ where the Weyl chamber is a three dimensional polytope.

Example 10. (Adjoint orbits of $SO(6)$) For $\mathfrak{so}(6)$ we have the following Cartan subalgebra (where we use the ρ-block notation as in (4.11)),

$$\mathfrak{h} = \left\{ \begin{pmatrix} \rho & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \;\middle|\; \rho, \kappa, \epsilon \in \mathbb{R} \right\}$$

The root diagram and Weyl chamber Δ are shown in Figure 4.4 We will denote vectors in $\mathfrak{h}$ by the triple $(\rho, \kappa, \epsilon) \in \mathbb{R}^3$. Below we have the Dynkin diagram along with the simple roots to which each node corresponds.



The task of finding the Adjoint orbit types then becomes that of solving the intersection of the planes orthogonal to the simple roots. Figure 4.5 shows visually how points lying in the

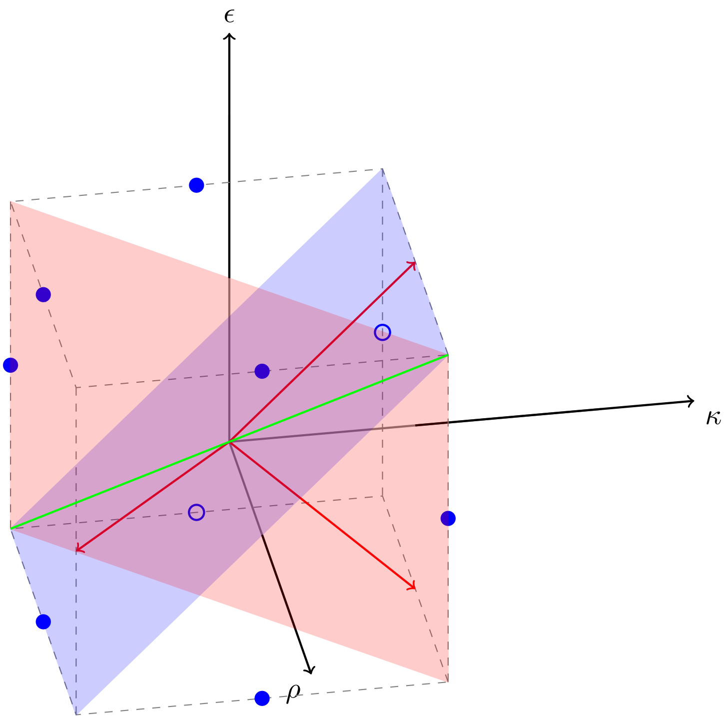


Figure 4.5: This picture attempts to show geometrically the set of points in $\mathfrak{h}$ orthogonal to the simple roots $(1, -1, 0)$ and $(0, 1, -1)$. The solution set is given by the line (ρ, ρ, ρ). Therefore for $\rho \neq 0$ we have an orbit type corresponding to the Dynkin diagram D_3 with the bottom two adjacent nodes coloured. The points have stabilizer $U(3)$ and the orbits are diffeomorphic to $\mathcal{HF}(6)$, the manifold of orthogonal and orientable complex structures on $\mathbb{R}^6$.

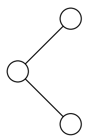
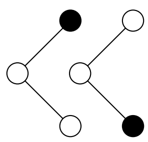
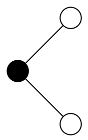
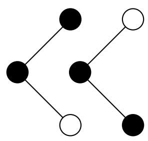
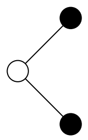
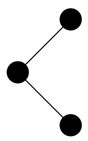
intersection of the planes orthogonal to $(0, 1, -1)$ and $(1, -1, 0)$ have typical orbit element $\begin{pmatrix} \rho & & \\ & \rho & \\ & & \rho \end{pmatrix}$. Table 4.4 provides a full list of orbit types along with a description of the orbit and Dynkin diagram. We remark that $\mathfrak{so}(6) = \mathfrak{su}(4)$. This gives us curious accidental diffeomorphisms between real and complex flags (since by Remark 2.1.1 both Adjoint actions of $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$ are equivalent),

$$\mathcal{HF}(2, 2, 2) \cong \mathcal{F}_{\mathbb{C}}(1, 1, 1, 1)$$
$$\mathcal{HF}(2, 4) \cong \mathcal{F}_{\mathbb{C}}(1, 1, 2)$$
$$\mathcal{HF}(6) \cong \mathrm{Gr}_{\mathbb{C}}(1; 3)$$
$$\widetilde{\mathrm{Gr}}_{\mathbb{R}}(2; 4) \cong \mathrm{Gr}_{\mathbb{C}}(2; 2)$$

Example 11. (Adjoint orbits of $SO(7)$) Our Cartan subalgebra of $\mathfrak{so}(7)$ will be,

$$\mathfrak{h} = \left\{ \left(\begin{array}{ccc|c} \rho & & & \\ & \kappa & & \\ & & \epsilon & \\ \hline & & & 0 \end{array} \right) \;\middle|\; \rho, \kappa, \epsilon \in \mathbb{R} \right\}$$

Table 4.4: Orbit characterization for $\mathfrak{so}(6)$. Distinct orbits related by an outer automorphism induced by the degree two automorphism of the Dynkin diagram share the same row. Note how in this example such related orbits are identical. Note: ρ, κ and ϵ distinct and non-zero.

Dynkin diagram	Typical element H	G_H	$\dim(G_H)$	Orbit
	$\begin{pmatrix}\rho & & \\ & \kappa & \\ & & \epsilon\end{pmatrix}$	$U(1)\times U(1)\times U(1)$	3	$\mathcal{HF}(2,2,2)$
	$\begin{pmatrix}\rho & & \\ & \kappa & \\ & & -\kappa\end{pmatrix}, \begin{pmatrix}\rho & & \\ & \kappa & \\ & & \kappa\end{pmatrix}$	$U(2)\times U(1)$	5	$\mathcal{HF}(2,4)$
	$\begin{pmatrix}\rho & & \\ & \rho & \\ & & \epsilon\end{pmatrix}$	$U(2)\times U(1)$	5	$\mathcal{HF}(2,4)$
	$\begin{pmatrix}\rho & & \\ & \rho & \\ & & -\rho\end{pmatrix}, \begin{pmatrix}\rho & & \\ & \rho & \\ & & \rho\end{pmatrix}$	$U(3)$	9	$\mathcal{HF}(6)$
	$\begin{pmatrix}\rho & & \\ & 0 & \\ & & 0\end{pmatrix}$	$U(1)\times SO(4)$	7	$\widetilde{\mathcal{F}}(2,4)$
	(0)	$SO(6)$	15	$\{0\}$

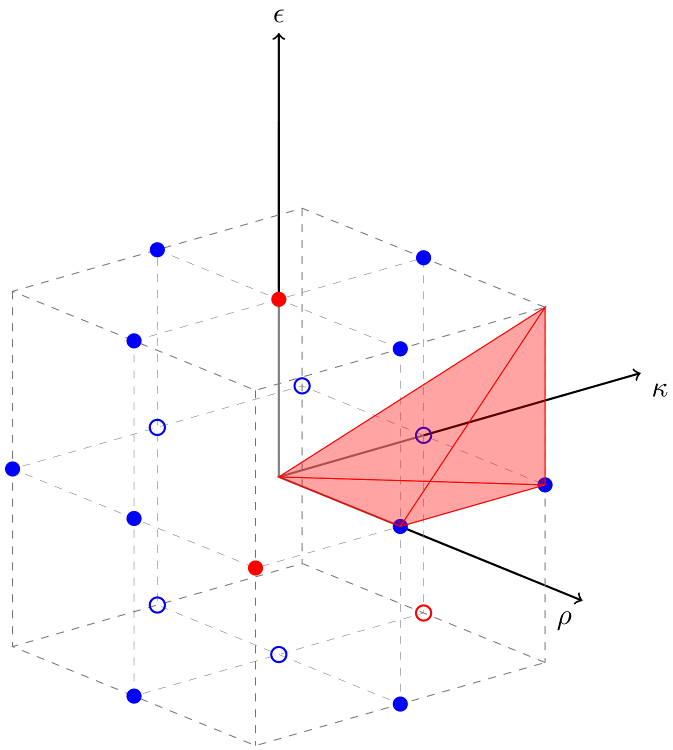
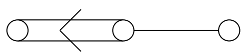


Figure 4.6: Root diagram for $\mathfrak{so}(7)$. The simple roots $(0,0,1)$, $(1,-1,0)$ and $(0,1,-1)$ are in red, the others in blue. Notice how the Weyl chamber is that of $\mathfrak{so}(6)$ chopped in half. The same observation goes for when $n = 3, 4$. This reflects our previous comments regarding outer automorphisms induced by automorphisms of the Dynkin diagram; the Weyl chamber for $\mathfrak{so}(n)$ when n is odd has no symmetry and so neither does its Dynkin diagram B_n.

The root diagram and Weyl chamber are given in Figure 4.6. The Dynkin diagram with simple roots specified is also given below. Note that we continue to write elements in $\mathfrak{h}$ by coordinates as (ρ, κ, ϵ).

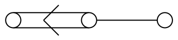
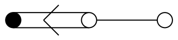
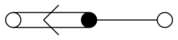
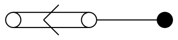
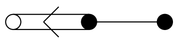
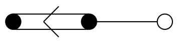
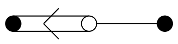
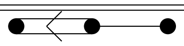


$(0,0,1) \qquad (0,1,-1) \qquad (1,-1,0)$

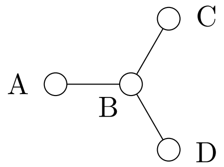
In Table 4.5 we record the Adjoint orbit types via the Dynkin diagram. Notice that this example provides us with an instance where two Adjoint orbits correspond to different non-isomorphic Dynkin diagram subsets and yet have diffeomorphic orbits. This tells us that we cannot distinguish the diffeomorphism type of the orbit by the Dynkin diagram alone.

Example 12. (Adjoint orbits of $SO(8)$) We conclude this section with a final example of $SO(8)$ for two reasons; firstly it is an example where we cannot draw a picture of the root diagram (it has rank 4) and so we must entirely use the Dynkin diagram: secondly the Dynkin diagram D_4 for $\mathfrak{so}(8)$ has the most symmetry out of any Dynkin diagram - its automorphism group is S_3. This high level of symmetry gives us more curious accidental relations between flag manifolds.

Table 4.5: Orbit characterization for $\mathfrak{so}(7)$.

Dynkin diagram	Typical element H	G_H	$\dim(G_H)$	Orbit
	$\left(\begin{array}{ccc\|c} \rho & & & \\ & \kappa & & \\ & & \epsilon & \\ \hline & & & 0 \end{array}\right)$	$U(1) \times U(1) \times U(1)$	3	$\widetilde{\mathcal{F}}(1,2,2,2)$
	$\left(\begin{array}{cc\|c} \rho & & \\ & \kappa & \\ \hline & & 0 \end{array}\right)$	$U(1) \times U(1) \times SO(3)$	5	$\widetilde{\mathcal{F}}(2,2,3)$
	$\left(\begin{array}{ccc\|c} \rho & & & \\ & \epsilon & & \\ & & \epsilon & \\ \hline & & & 0 \end{array}\right)$	$U(1) \times U(2)$	5	$\mathcal{F}(\tilde{1}, 2^{\mathbb{C}}, 4^{\mathbb{C}})$
	$\left(\begin{array}{ccc\|c} \rho & & & \\ & \rho & & \\ & & \epsilon & \\ \hline & & & 0 \end{array}\right)$	$U(2) \times U(1)$	5	$\mathcal{F}(\tilde{1}, 4^{\mathbb{C}}, 2^{\mathbb{C}})$
	$\left(\begin{array}{ccc\|c} \rho & & & \\ & \rho & & \\ & & \rho & \\ \hline & & & 0 \end{array}\right)$	$U(3)$	9	$\mathcal{F}(\tilde{1}, 6^{\mathbb{C}})$
	$\left(\begin{array}{c\|c} \rho & \\ \hline & 0 \end{array}\right)$	$U(1) \times SO(5)$	11	$\widetilde{\mathcal{F}}(2,5)$
	$\left(\begin{array}{cc\|c} \rho & & \\ & \rho & \\ \hline & & 0 \end{array}\right)$	$U(2) \times SO(3)$	7	$\mathcal{F}(4^{\mathbb{C}}, \tilde{3})$
	(0)	$SO(7)$	21	$\{0\}$

For $\mathfrak{so}(8)$ we have as Cartan subalgebra $\mathfrak{h} = \{\rho_1 J_1 + \rho_2 J_2 + \rho_3 J_3 + \rho_4 J_4 \mid (\rho_1, \rho_2, \rho_3, \rho_4) \in \mathbb{R}^4\}$. In this notation J_k is the matrix block $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$ on the main diagonal starting $2k$ places down . We now give the Dynkin diagram and the simple roots.



The simple roots A, B, C and D above are given as,

$$A = J_1 - J_2$$

$$B = J_2 - J_3$$

$$C = J_3 - J_4$$

$$D = J_3 + J_4$$

Obviously the generic orbit is through the point $(\rho_1, \rho_2, \rho_3, \rho_4)$ for $\rho_i \neq \rho_j,\ \forall i \neq j$. To calculate the remaining orbits we need to know the Killing form. The key fact is that $\{J_1, J_2, J_3, J_4\}$ forms an orthogonal basis. Suppose then we want to describe a typical point H in $\mathfrak{h}$ orthogonal to the simple root $A = J_1 - J_2$; then,

$$0 = \langle H, J_1 - J_2 \rangle = \langle \rho_1 J_1 + \rho_2 J_2 + \rho_3 J_3 + \rho_4 J_4, J_1 - J_2 \rangle = \rho_1 \langle J_1, J_1 \rangle - \rho_2 \langle J_2, J_2 \rangle.$$

Therefore a typical point orthogonal to A will have $\rho_1 = \rho_2$. Proceeding in this way we produce the results collected in Table 4.6 for the orbit types of $\mathfrak{so}(8)$.

This example provides us with instances whereby two separate Adjoint orbit types are related by an outer automorphism. Recall that in Proposition 4.2.2 we proved that outer automorphism related orbits had to be diffeomorphic. $\mathfrak{so}(8)$ provides us with two interesting examples:

The orbits through the points $(\rho, \rho, \epsilon, \epsilon)$ and $(\rho, \kappa, 0, 0)$ are related by an outer automorphism given by rotating the Dynkin diagram. They have isotropy subgroups $U(2) \times U(2)$ and $U(1) \times U(1) \times SO(4)$ respectively. These groups are not isomorphic (in fact there is a 2-1 covering map into $U(1) \times U(1) \times SO(4)$ following from $U(n) = U(1) \times SU(n)$ and the accidental isomorphism $\mathrm{Spin}(4) = SU(2) \times SU(2)$). Yet despite having non-isomorphic isotropy groups it follows from Proposition 4.2.2 that the orbits are diffeomorphic,

$$\widetilde{\mathcal{F}}(2,2,4) \cong \mathcal{HF}(4,4).$$

The second case is for the points $(\rho, 0, 0, 0)$ and (ρ, ρ, ρ, ρ) with respective isotropy subgroups $U(1) \times SO(6)$ and $U(4)$. Here thanks to the accidental isomorphism $\mathrm{Spin}(6) = SU(4)$ the subgroups are not isomorphic and are also related by a 2-1 covering. We thus have that the following orbits are diffeomorphic,

$$\widetilde{\mathcal{F}}(2,6) = \widetilde{Gr}_{\mathbb{R}}(2;6) \cong \mathcal{HF}(8).$$

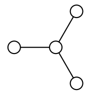
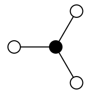
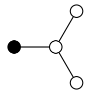
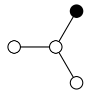
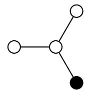
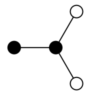
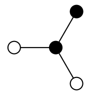
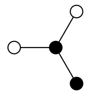
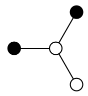
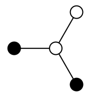
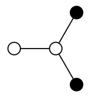
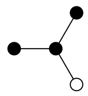
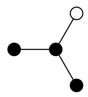
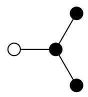
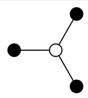
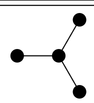
Remark 4.2.3. We highlight here an area of possible confusion. According to Table 4.6 the orbits through the points $(\rho,\rho,\rho,\epsilon)$ and $(\epsilon,\rho,\rho,\rho)$ correspond to different albeit isomorphic Dynkin diagram subsets. From this we may erroneously conclude that the two matrices with these type decompositions are not conjugate, indeed if they were then surely they would be on the same orbit and correspond to the same Dynkin diagram? Yet it's not to difficult to see that they are conjugate. The subtle point here is that the typical points corresponding to the shaded Dynkin diagram are for points contained in the principal Weyl chamber Δ - not just for typical points orthogonal to certain simple roots. In other words it may be the case that two points, A and B say, belong to different faces Π_α and Π_β of the principal Weyl chamber and are therefore not conjugate. However it may also be the case that the orbit of A is conjugate to a point not in the principal Weyl chamber but also in the hyperplane Π_β. The resolution to this seeming contradiction is that we have not yet incorporated the fact that the points $(\rho,\rho,\rho,\epsilon)$ and $(\epsilon,\rho,\rho,\rho)$ lie inside the principal Weyl chamber. This being the case then implies that for $(\rho,\rho,\rho,\epsilon)$ we require $\rho-\epsilon>0$ and for $(\epsilon,\rho,\rho,\rho)$ we need $\epsilon-\rho>0$. From this we see that each value for ρ and ϵ cannot coincide for two such points in the principal Weyl chamber and so they are certainly not conjugate.

Question: It follows from Remark 4.2.1 that the modulus of an orbit type is equal to the number $k-d$ where k is the rank of $\mathfrak{g}$ and d is the number of coloured nodes of the Dynkin diagram. It also follows from (4.6) that the dimension of an orbit is $\leq n-(k+2d)$ where n is the dimension of $\mathfrak{g}$. The examples given in the tables show that equality is certainly not always achieved. It would appear that the difference in dimension of the orbit from $n-(k+2d)$ has something to do with the number of adjacent coloured nodes. For instance, the inequality is strict in our examples for when no two coloured nodes are adjacent. I ask whether the dimension of the orbit can be deduced by the coloured Dynkin diagram?

Question: So far all of our results have been for *compact* simple Lie groups. If we drop the compactness requirement we have seen that this method doesn't work since Theorem 2 doesn't hold (see the example given after the theorem of $\mathfrak{sl}(2;\mathbb{R})$). Can we use the Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ into semisimple, abelian and nilpotent parts to find a subalgebra(s) analagous to the Cartan subalgebra through which every orbit must intersect? This approach would be similar in spirit to that in [2] where for all simple real Lie groups their

Lie algebra elements are decomposed into semisimple and nilpotent parts to provide type decompositions describing the dimension and modulus for all Co/Adjoint orbits for the classical simple real Lie groups. Such a result would apply to the indefinite orthogonal groups $SO(p,q)$ such as the Lorentz group. We note that for $SO(2,2)$ we can already establish the topology of its orbits; $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2;\mathbb{R}) \oplus \mathfrak{sl}(2;\mathbb{R})$ and so from Example 2 the orbits are the product of two manifolds from the set {point,cone−vertex,1-sheeted hyperboloid,2-sheeted hyperboloid}.

Table 4.6: Orbit characterization for $\mathfrak{so}(8)$. Note that ρ, ϵ, κ are distinct non-zero real numbers. Also note that the typical elements given are for those contained in the principal Weyl chamber Δ (see Remark 4.2.3 for clarification).

Dynkin diagram	Typical element H	G_H	$\dim(G_H)$	Orbit
	$(\rho_1, \rho_2, \rho_3, \rho_4)$	$U(1) \times U(1) \times U(1) \times U(1)$	4	$\mathcal{HF}(2,2,2,2)$
	$(\epsilon, \rho, \rho, \kappa)$	$U(1) \times U(1) \times U(2)$	6	$\mathcal{HF}(2,2,4)$
	$(\rho, \rho, \kappa, \epsilon)$	$U(1) \times U(1) \times U(2)$	6	$\mathcal{HF}(2,2,4)$
	$(\rho, \kappa, \epsilon, \epsilon)$	"	"	"
	$(\rho, \kappa, \epsilon, -\epsilon)$	"	"	"
	$(\rho, \rho, \rho, \epsilon)$	$U(3) \times U(1)$	10	$\mathcal{HF}(2,6)$
	$(\epsilon, \rho, \rho, \rho)$	"	"	"
	$(\rho, \epsilon, \epsilon, -\epsilon)$	"	"	"
	$(\rho, \rho, \epsilon, \epsilon)$	$U(2) \times U(2)$	8	$\mathcal{HF}(4,4)$
	$(\rho, \rho, \epsilon, -\epsilon)$	"	"	"
	$(\rho, \kappa, 0, 0)$	$U(1) \times U(1) \times SO(4)$	8	$\widetilde{\mathcal{F}}(2,2,4)$
	(ρ, ρ, ρ, ρ)	$U(4)$	16	$\mathcal{HF}(8)$
	$(\rho, \rho, \rho, -\rho)$	"	"	"
	$(\rho, 0, 0, 0)$	$U(1) \times SO(6)$	16	$\widetilde{\mathcal{F}}(2,6)$
	$(\rho, \rho, 0, 0)$	$U(2) \times SO(4)$	10	$\mathcal{F}(\tilde{4}, 4^{\mathbb{C}})$
	$(0,0,0,0)$	$SO(8)$	28	$\{0\}$

Chapter 5

Affine Lie groups

5.1 Co/Adjoint actions of affine groups

Having considered compact and semisimple Lie groups where the Adjoint and Coadjoint actions coincide we now turn to non-compact and non-semisimple groups for which this no longer necessarily holds. We will be considering a simple class of such groups by taking the semidirect product of a compact semisimple Lie group with a group of translations. Let H be a real compact semisimple Lie group and $\rho : H \hookrightarrow GL(V)$ a faithful representation. We define the *affine Lie group* G to be the semidirect product $G := \rho(H) \ltimes_\rho V$ (our definition has nothing to do with the affine Lie algebras associated with Kac-Moody algebras). Since $\rho(H) \cong H$ we will from now on write H to mean $\rho(H)$ and $\mathfrak{h}$ to mean $\rho_*(\mathfrak{h})$. So our affine group is $G = H \ltimes V$ with Lie algebra $\mathfrak{g} = \mathfrak{h} \times V$. (We have suppressed the choice of representation ρ from our notation for convenience - it is important to bear in mind however that everything to follow depends entirely on such choice).

There is a faithful matrix representation of G which allows us to write out the Adjoint action;

$$G \hookrightarrow GL(V \times \mathbb{F});\ (a, v) \longmapsto \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{g} = \mathfrak{h} \times V \hookrightarrow \mathfrak{gl}(V \times \mathbb{F});\ (A, X) \longmapsto \begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix}.$$

The Adjoint action of $(a, v) \in H \times V = G$ on $(A, X) \in \mathfrak{h} \times V = \mathfrak{g}$ is then given by,

$$\mathrm{Ad}_{(a,v)}(A, X) = (\mathrm{Ad}_a A,\ aX - (\mathrm{Ad}_a A)v)\,. \tag{5.1}$$

We now wish to define the isotropy subgroup $G_{A,X}$ for an arbitrary point $(A, X) \in \mathfrak{g}$. Clearly we first of all require $a \in H_a = \{a \mid \mathrm{Ad}_a A = A\}$. It remains to satisfy,

$$aX - X = Av. \tag{5.2}$$

We claim that there exists an H-invariant inner product $\langle\ ,\ \rangle$ on V; since H is compact we may use the averaging method to integrate an arbitrary inner product over the group H. It then follows that with respect to this inner product that $A \in \mathfrak{h}$ is skew self-adjoint,

$$\langle Ax, y\rangle + \langle x, Ay\rangle = 0. \tag{5.3}$$

This implies that A is a semisimple operator which in turn tells us that the space V decomposes as,

$$V = \operatorname{Ker} A \oplus \operatorname{Im} A. \tag{5.4}$$

With respect to this decomposition we will write X as $\overline{X} + Ay$, where $\overline{X}$ is the projection of X onto $\operatorname{Ker} A$ and y any suitable vector in V. (Note that the action of $\mathrm{Ad}_{(I,y)}$ sends (A, X) to $(A, \overline{X})$). Since $a \in H_A$, $aA = Aa$ and so $a \cdot \operatorname{Ker} A \subseteq \operatorname{Ker} A$ and $a \cdot \operatorname{Im} A \subseteq A$. Therefore we may rewrite the left hand side of (5.2) with respect to (5.4),

$$\underbrace{(a\overline{X} - \overline{X})}_{\in \operatorname{Ker} A} + \underbrace{(aAy - Ay))}_{\in \operatorname{Im} A} = Av.$$

We therefore must have $a \in H_{\overline{X}} = \{a \mid a\overline{X} = \overline{X}\}$, and $aAy - y = Av$. We have therefore proved that $G_{A,X}$ sits inside an exact sequence,

$$0 \longrightarrow \operatorname{Ker} A \xrightarrow{\alpha} G_{A,X} \xrightarrow{\beta} H_{\overline{X}} \cap H_A \longrightarrow I \tag{5.5}$$

Here the maps are $\alpha(v) = (I, v)$ and $\beta\left((a, v)\right) = a$. When X is inside the kernel $\operatorname{Ker} A$, that is $\overline{X} = X$ this sequence is split; there is a splitting map $\gamma : H_{\overline{X}} \cap H_A \longrightarrow G_{A,X}$ given by $\gamma(a) = (a, 0)$. We have thus proved:

Proposition 5.1.1. For any point $(A, X) \in \mathfrak{h} \times V = \mathfrak{g}$ the stabilizer $G_{A,X}$ of the Adjoint action is an extension of $H_{\overline{X}} \cap H_A$ by $\operatorname{Ker} A$. If X belongs to $\operatorname{Ker} A$ (that is $\overline{X} = X$) then the stabilizer is the subgroup $\left(H_{\overline{X}} \cap H_A\right) \ltimes \operatorname{Ker} A \subseteq G$. Moreover the two points (A, X) and $(A, \overline{X})$ belong to the same orbit.

We shall now carry out the same task but for the Coadjoint action. Denote a typical element in the dual of the Lie algebra by $(\Omega, \zeta) \in \mathfrak{h}^* \times V^* = \mathfrak{g}^*$. By the definition of Coadjoint action we have,

$$\langle \mathrm{Coad}_{(a,v)}(\Omega, \zeta), (A, X)\rangle = \langle (\Omega, \zeta), \mathrm{Ad}_{(a,v)^{-1}}(A, X)\rangle \quad \forall (A, X) \in \mathfrak{g}.$$

Here $\langle\ ,\ \rangle$ is for a moment just the standard pairing between a space and its dual. Using equation (5.1) we get,

$$\begin{aligned}\langle \mathrm{Coad}_{(a,v)}(\Omega, \zeta), (A, X)\rangle &= \langle (\Omega, \zeta), \mathrm{Ad}_{(a^{-1}, -a^{-1}v)}(A, X)\rangle \\ &= \langle \Omega, \mathrm{Ad}_{a^{-1}} A\rangle + \langle \zeta, a^{-1}X\rangle + \langle \zeta, a^{-1}Av\rangle.\end{aligned}$$

We now introduce pairings between spaces and their duals. For $\mathfrak{h}$ we let $\langle\ ,\ \rangle$ be the Killing form (noting that this is non-degenerate since $\mathfrak{h}$ is semisimple). For V, $\langle\ ,\ \rangle$ will be the H-invariant inner product defined earlier. We can therefore now write $\langle\Omega, \operatorname{Ad}_{a^{-1}} A\rangle = \langle\operatorname{Ad}_a \Omega, A\rangle$, $\langle\zeta, a^{-1}X\rangle = \langle a\zeta, X\rangle$ and $\langle\zeta, a^{-1}Av\rangle = \langle a\zeta, Av\rangle$.

We now turn our attention to the term $\langle a\zeta, Av\rangle$. Let $\mu : T^*V \longrightarrow \mathfrak{h}^*$ be the moment map for the action of H on V lifted to T^*V. Then μ satisfies,

$$\langle\mu(v,\eta), A\rangle = \langle\eta, Av\rangle, \quad \forall v, \in V, \eta \in V^*, A \in \mathfrak{h}. \tag{5.6}$$

Identifying T^*V with $V \times V$ and $\mathfrak{h}^*$ with $\mathfrak{h}$ using our relations above allow us to write the term $\langle a\zeta, Av\rangle$ as $\langle\mu(v, a\zeta), A\rangle$. We have thus shown that,

$$\langle\operatorname{Coad}_{(a,v)}(\Omega,\zeta), (A,X)\rangle = \langle\operatorname{Ad}_a \Omega + \mu(v, a\zeta), A\rangle + \langle a\zeta, X\rangle, \quad \forall (A,X) \in \mathfrak{g}$$

and so therefore we have the following expression for the Coadjoint action of G on $\mathfrak{g}^*$ identified with $\mathfrak{g}$,

$$\operatorname{Coad}_{(a,v)}(\Omega,\zeta) = \left(\operatorname{Ad}_a \Omega + \mu(v, a\zeta),\ a\zeta\right). \tag{5.7}$$

The aim now is to describe the stabilizer $G_{\Omega,\zeta}$ of the point (Ω,ζ). Clearly we first require $a \in H_\zeta = \{a \mid a\zeta = \zeta\}$. It then remains to satisfy $\operatorname{Ad}_a \Omega + \mu(v,\zeta) = \Omega$. We would like to get a handle on the possible values of $\mu(v,\zeta)$. For fixed ζ define the linear map $\tau : V \longrightarrow \mathfrak{h};\ v \mapsto \mu(v,\zeta)$. Let $\mathfrak{h}_\zeta = \operatorname{Lie}(H_\zeta) = \{A \in \mathfrak{g} \mid A\zeta = 0\}$.

Claim 5.1.1. $(\operatorname{Im}\tau)^\perp = \mathfrak{h}_\zeta$.

Proof. Let $B \in (\operatorname{Im}\tau)^\perp$. Then $\langle\mu(v,\zeta), B\rangle = 0$ for all $v \in V$. From (5.3) we have $0 = \langle\zeta, Bv\rangle = -\langle B\zeta, v\rangle$ so $\langle B\zeta, v\rangle = 0$ for all $v \in V$ which implies $B\zeta = 0$ and so $B \in \mathfrak{h}_\zeta$, hence $(\operatorname{Im}\tau)^\perp \subseteq \mathfrak{h}_\zeta$. The implications in this argument may be reversed giving $\mathfrak{h}_\zeta \subseteq (\operatorname{Im}\tau)^\perp$. □

Recall that we require $\Omega - \operatorname{Ad}_a \Omega$ to belong to $\operatorname{Im}\tau = \mathfrak{h}_\zeta^\perp$, i.e. $\langle\Omega - \operatorname{Ad}_a \Omega,\ B\rangle = 0$ for all $B \in \mathfrak{h}_\zeta$. With respect to the orthogonal decomposition $\mathfrak{h} = (\operatorname{Im}\tau)^\perp \oplus \operatorname{Im}\tau = \mathfrak{h}_\zeta \oplus \mathfrak{h}_\zeta^\perp$ write $\Omega = \overline{\Omega} + \tau(v_\circ)$ for some suitable $v_\circ \in V$. The Adjoint action preserves the Killing form and so respects this orthogonal decomposition. Therefore our requirement is equivalent to

$$\langle\overline{\Omega} - \operatorname{Ad}_a \overline{\Omega},\ B\rangle = 0, \quad \forall B \in \mathfrak{h}_\zeta.$$

Since the Killing form is non-degenerate on $\mathfrak{h}$ (as it is semisimple) this implies that our requirement is that $\operatorname{Ad}_a \overline{\Omega} = \overline{\Omega}$, and so $a \in H_{\overline{\Omega}} = \{a \mid \operatorname{Ad}_a \overline{\Omega} = \overline{\Omega}\}$. Note also that we may act by a suitable (I, v) to send (Ω,ζ) to $(\overline{\Omega},\zeta)$. We have now proved that $G_{\Omega,\zeta}$ also fits inside an exact sequence:

$$0 \longrightarrow \operatorname{Ker}\tau \xrightarrow{\alpha} G_{\Omega,\zeta} \xrightarrow{\beta} H_{\overline{\Omega}} \cap H_\zeta \longrightarrow I$$

In this sequence the maps α and β are the obvious inclusions and projections respectively. When $\overline{\Omega} = \Omega$ we have a splitting map $\gamma : H_{\overline{\Omega}} \cap H_\zeta \longrightarrow G_{\Omega,\zeta}$ given by inclusion $\gamma(a) = (a, 0)$. This gives us the following result dual to Proposition 5.1.1

Proposition 5.1.2. For any point $(\Omega, \zeta) \in \mathfrak{h}^* \times V^* = \mathfrak{g}^*$ the stabilizer $G_{\Omega,\zeta}$ of the Coadjoint action is an extension of $H_{\overline{\Omega}} \cap H_\zeta$ by $\operatorname{Ker}\tau$. If $\overline{\Omega}$ belongs to $\mathfrak{h}_\zeta$ (that is $\overline{\Omega} = \Omega$) then the stabilizer is the subgroup $\left(H_{\overline{\Omega}} \cap H_\zeta\right) \ltimes \operatorname{Ker}\tau \subseteq G$. Moreover the two points (Ω, ζ) and $(\overline{\Omega}, \zeta)$ belong to the same orbit.

We will define a pair $(A, X) \in \mathfrak{g}$ to be *proper* if $\overline{X} = X$. Similarly the element $(\Omega, \zeta) \in \mathfrak{g}^*$ will be called proper if $\overline{\Omega} = \Omega$. Propositions 5.1.1 and 5.1.2 show that every orbit contains a proper point and that the stabilizers of proper points are semidirect products. The notion of proper coincides for both Adjoint and Coadjoint orbits, that is if $(A, X) \in \mathfrak{g}$ proper then so is $(A, X) = (\Omega, \zeta) \in \mathfrak{g}^*$ since $\Omega\zeta = 0$ as $X \in \operatorname{Ker} A$ and so $\Omega \in \mathfrak{h}_\zeta \implies \overline{\Omega} = \Omega$.

5.2 Bijections between Adjoint and Coadjoint orbits

Since these affine groups are neither compact nor semisimple there appears on the face of it no reason why the Adjoint and Coadjoint actions should share any similarity. We will in this section however show that there are situations where the two actions have a lot in common. We firstly show that there are instances where the two representations are in fact equivalent, in particular for groups that we refer to as affine adjoint groups. We secondly establish a bijection between orbit types of the special Euclidean group $SE(n)$. This bijection between Adjoint and Coadjoint orbits respects their homotopy type and modulus.

Given a Lie group H recall that the *adjoint group*, $\mathrm{Ad}(H)$ is the image of H under $\mathrm{Ad} : H \longrightarrow GL(\mathfrak{h})$. For H semisimple the adjoint representation $\mathrm{ad} = d\,(\mathrm{Ad})_e : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{h})$ is injective (ref. [4]p. 124) and so $\mathrm{Ad}(H)$ is, up to isomorphism, the only centerless group with Lie algbera $\mathfrak{h}$, (so any other Lie group $\widetilde{H}$ with Lie algebra $\mathfrak{h}$ is a covering space for $\mathrm{Ad}(H)$). We define the *affine adjoint group* G of H to be $G := \mathrm{Ad}(H) \ltimes_{\mathrm{Ad}} \mathfrak{h}$.

Theorem 4. The Coadjoint and Adjoint representations of the affine adjoint group $\mathrm{Ad}(H) \ltimes \mathfrak{h}$ are equivalent if H is semisimple and compact.

Proof. First note that $\mathfrak{g} = \mathrm{ad}(\mathfrak{h}) \times \mathfrak{h} \cong \mathfrak{h} \times \mathfrak{h}$ since ad is injective by semisimplicity. This ensures that the map swapping the two factors $\varphi : \mathfrak{g} \to \mathfrak{g}^*(\cong \mathfrak{g})$; $(A, X) \mapsto (\Omega, \zeta) = (X, A)$ makes sense. We will show that φ is a G-equivariant map from $\mathfrak{g}$ into itself. Since $\varphi \circ \varphi = Id_{\mathfrak{g}}$ this will mean that φ is a G-equivariant isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ and therefore an interwining map between both representations.

In our discussion on Co/Adjoint actions above there were two actions of the group H; one was the action aX on V and the other was the Adjoint action $\mathrm{Ad}_a A$ on $\mathfrak{h}$. For the affine adjoint group $\mathfrak{h}$ and V are the same and so are both actions. We therefore will write $a \cdot X$ for both $\mathrm{Ad}_a X$ and aX. The moment map $\mu : T^*V \longrightarrow \mathfrak{h}^*$ for the action of H on its Lie algebra by the Adjoint action is just the commutator, $\mu(X, Y) = [X, Y]$, (identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ via Killing form). For when elements of $\mathfrak{h}$ act on V the corresponding notion for the affine adjoint group is $A \cdot X = \mathrm{ad}_A X = [A, X]$. Plugging these facts into equations (5.1) and (5.7) gives,

$$\begin{aligned} \mathrm{Ad}_{(a,v)}(A, X) &= (a \cdot A,\ a \cdot X - [a \cdot A, v]) \\ \mathrm{Coad}_{(a,v)}(\Omega, \zeta) &= (a \cdot \Omega + [v, a \cdot \zeta],\ a \cdot \zeta)\,. \end{aligned}$$

From this we see that the map φ sending (A, X) to $(\Omega, \zeta) = (X, A)$ is an equivariant involutive map on $\mathfrak{g}$ and hence an interwining map between representations. $\square$

For $SE(n)$ we have already seen in Example 3 for $SE(2)$ that the Adjoint and Coadjoint actions are not equivalent. We now prove that although they are indeed inequivalent (with the fluke exception of $SE(3)$) that there exists a bijection between orbit types which in some sense preserves the structure of the orbits.

Before we begin our next theorem we change slightly our derivation of the Coadjoint action for $SE(n)$. The inner product on V had to be $SO(n)$ invariant. For this we make the obvious choice and make this inner product the standard Euclidean metric, $\langle v, w\rangle = v^T w$. For the inner product on $\mathfrak{so}(n)$ we chose the Killing form. Instead we will choose the form $\langle A,\ B\rangle = \frac{1}{2}\operatorname{Tr}\left(A^T B\right)$. For this form the adjoint operator of Ad remains Ad (using the fact that $aa^T = 1$ for $SO(n)$) and so therefore equation (5.7) remains valid.

As for the moment map we now have an explicit description; $\langle \mu(v, w),\ A\rangle = \langle w,\ Av\rangle = w^T Av = \operatorname{Tr}\left(w^T Av\right)$, which by using the cyclic property of the trace and $A + A^T = 0$, $\forall A \in \mathfrak{so}(n)$ gives us,

$$\mu(w, v) = v \wedge w := vw^T - wv^T.$$

From this we can show that $\operatorname{Ker}\tau = \operatorname{Span}\{\zeta\}$ and that for $\zeta = \lambda e_1$ the image $\operatorname{Im}\tau$ is,

$$\tau(v) = \zeta \wedge v = \lambda\, e_1 \wedge v = \left(\begin{array}{c|c} 0 & v_2 \cdots v_n \\ \hline -v_2 & \\ \vdots & \text{\Large 0} \\ -v_n & \end{array}\right), \quad \text{where } v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \tag{5.8}$$

Theorem 5. There is a structure preserving relation between Adjoint and Coadjoint orbits of $SE(n)$ in the following sense: Let $\mathcal{A}$, $\mathcal{C}$ denote the sets of Adjoint and Coadjoint orbits.

There exists a bijection $\mathcal{A} \leftrightarrow \mathcal{C}$ such that for any two related orbits, one is a vector bundle over the other and the moduli of both orbits are the same.

Proof. Take some $(A', X) \in \mathfrak{g}$ and consider the Adjoint orbit through it. On the same orbit is the proper point $(A', \overline{X})$. Now act on this by $(a, 0)$ where a is a suitable rotation sending the point to $(A, \lambda e_1)$ where $\lambda = |\overline{X}| \in \mathbb{R}$. It is not too difficult to show that $(A, \lambda e_1)$ is also proper. By the definition of proper we must have $Ae_1 = 0$ if $\lambda \neq 0$. Since A is skew symmetric this means that it must be of the form,

$$\left(\begin{array}{c|c} 0 & 0\cdots 0 \\ \hline 0 & \\ \vdots & \\ 0 & \end{array}\right) \quad \text{for when } \lambda \neq 0 \tag{5.9}$$

Now consider the Coadjoint orbit through the point $(\Omega, \zeta) = (A, \lambda e_1)$. We claim that this is also proper; the pair (Ω, ζ) is proper iff $\Omega \in \mathfrak{h}_\zeta = \{A \mid A\zeta = 0\}$ for $\zeta = \lambda e_1$ which is true since $\Omega = A$ and $\lambda e_1 \in \operatorname{Ker} A$ since $(A, \lambda e_1)$ is proper.

Next we aim to show that one of the isotropy groups $G_{A,X}$, $G_{\Omega,\zeta}$ is a subgroup of the other, (now writing $(A, \lambda e_1) = (A, X) = (\Omega, \zeta) = (\overline{\Omega}, \lambda e_1)$).

$$\begin{aligned} G_{A,X} &= \left(H_A \cap H_{\overline{X}}\right) \ltimes \operatorname{Ker} A, \quad & H_A \cap H_{\overline{X}} &= \{a \mid ae_1 = e_1,\ \operatorname{Ad}_a A = A\} \\ G_{\Omega,\zeta} &= \left(H_{\overline{\Omega}} \cap H_\zeta\right) \ltimes \operatorname{Ker} \tau, \quad & H_{\overline{\Omega}} \cap H_\zeta &= \{a \mid ae_1 = e_1,\ \operatorname{Ad}_a \overline{\Omega} = \overline{\Omega}\} \end{aligned}$$

Since $\overline{\Omega} = \Omega$ and $\overline{X} = X$ as (Ω, ζ) and (A, X) are proper, we see that $H_A \cap H_{\overline{X}} = H_{\overline{\Omega}} \cap H_\zeta$. Now $\operatorname{Ker} \tau = \operatorname{Span}\{e_1\}$ if $\lambda \neq 0$ and $\operatorname{Ker} \tau = \mathbb{R}^n$ otherwise. Therefore when $\lambda = 0$, i.e. when $X = \zeta = 0$, $G_{\Omega,\zeta} = \left(H_{\overline{\Omega}} \cap H_\zeta\right) \ltimes \mathbb{R}^n$ and so we have the following vector bundle,

$$\mathcal{O}((A, 0)) \longrightarrow \mathcal{O}((\Omega, 0)), \quad \text{with fibre } \frac{\mathbb{R}^n}{\operatorname{Ker} A}.$$

For $\lambda \neq 0$ we have $\operatorname{Ker} \tau \subseteq \operatorname{Ker} A$ as $Ae_1 = 0$. In this case we have the vector bundle,

$$\mathcal{O}((\Omega, \zeta)) \longrightarrow \mathcal{O}((A, X)), \quad \text{with fibre } \frac{\operatorname{Ker} A}{\operatorname{Ker} \tau}.$$

In particular when $\operatorname{Ker} \tau = \operatorname{Ker} A$ the orbits through (Ω, ζ) and (A, X) are the same. The diffeomorphism types of both orbits are parametrized by the type decomposition of A and $\lambda \in \mathbb{R}$ and so both have the same moduli.

Conversely had we started with an orbit through a point (Ω', ζ) in $\mathfrak{g}^*$ we can send it to a proper element and then apply a rotation to get to $(\Omega, \lambda e_1)$. We can then show that this point remains proper. Therefore observe that the orbit is in the image of our orbit relation $\mathcal{A} \to \mathcal{C}$. Injectivity of the map comes from noting that if two points are sent to the same

Coadjoint orbit then they have to belong to the same orbit to start with. This establishes the orbit bijection and finishes the proof. $\square$

Remark 5.2.1. The key ingredients to the proof above were an explicit expression for τ and transitivity of H on V. This was possible since the moment map for $SO(n)$ acting on $\mathbb{R}^n$ is well known. Similarly we can prove the same theorem above for the affine special unitary group $SU(n) \ltimes \mathbb{C}^n$. It's moment map is $\mu(v, w) = w \wedge v^\dagger - v \wedge w^\dagger$. It would be interesting to know what the moment map is for the action of $SO(n)$ (or indeed any semisimple Lie group) on a different vector space V induced by a different irreducible representation of $SO(n)$.

5.3 Affine flag manifolds as orbits

We will now give a detailed description for $SE(n)$, highlighting the orbit bijection and demonstrating that these orbits are flags and affine flags.

Firstly note that any Adjoint orbit contains a proper point of the form $(A, \lambda e_1)$, where since A is proper it must be of the form given in (5.9) for when $\lambda \neq 0$. The same goes for Coadjoint orbits, they contain a proper point $(\Omega, \lambda e_1)$. In identifying Adjoint orbits with Coadjoint orbits we will write Ω and A to denote the same matrix element in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. For convenience we will write $\overline{A}$ and $\overline{\Omega}$ to denote the entries in the bottom right of the matrix, that is $\overline{A} \in \mathfrak{so}(n-1)$ and $\overline{\Omega} \in \mathfrak{so}(n-1)^*$, this will be helpful whenever $\lambda \neq 0$ and so when Ω, A are in the form given in (5.9). To be clear then our typical proper elements are of the form $(A, \lambda e_1) \in \mathfrak{g}$, $(\Omega, \lambda e_1) \in \mathfrak{g}^*$ where,

$$A = \left(\begin{array}{c|c} 0 & \cdots \\ \hline \vdots & \overline{A} \end{array}\right), \quad \Omega = \left(\begin{array}{c|c} 0 & \cdots \\ \hline \vdots & \overline{\Omega} \end{array}\right), \quad \lambda \in \mathbb{R},\ \overline{A} \in \mathfrak{so}(n-1),\ \overline{\Omega} \in \mathfrak{so}(n-1)^*$$

Consider first then the task of finding $G_{A,X}$ for a typical proper point $(A, \lambda e_1)$ of the Adjoint action. We split into two cases;

1. $\lambda \neq \mathbf{0}$: In this case A must be singular since $Ae_1 = 0$. The subgroup $H_A \cap H_X = H_A \cap H_{e_1}$ is then isomorphic to the isotropy group $H_{\overline{A}} \in SO(n-1)$ for $\overline{A} \in \mathfrak{so}(n-1)$. The stabilizer $G_{A,X}$ is then $H_{\overline{A}} \ltimes \operatorname{Ker} A$.

2. $\lambda = \mathbf{0}$: In this case $H_A \cap H_X = H_A$ since $X = 0$. The stabilizer is then $G_{A,X} = H_A \ltimes \operatorname{Ker} A$.

Now to describe the groups $G_{\Omega,\zeta}$ for the proper points $(\Omega, \lambda e_1)$ for the Coadjoint action. It to splits into two cases;

1. $\lambda \neq \mathbf{0}$: In this case Ω must be singular since $\Omega e_1 = 0$. The subgroup $H_\Omega \cap H_\zeta = H_\Omega \cap H_{e_1}$ is then isomorphic to the isotropy group $H_{\overline{\Omega}} \in SO(n-1)$ for $\overline{\Omega} \in \mathfrak{so}(n-1)^*$. The stabilizer $G_{\Omega,\zeta}$ is then $H_{\overline{\Omega}} \ltimes \mathbb{R} \cdot \zeta$.

2. $\lambda = \mathbf{0}$: In this case $H_\Omega \cap H_\zeta = H_\Omega$ since $\zeta = 0$. The stabilizer is then $G_{\Omega,\zeta} = H_\Omega \ltimes \mathbb{R}^n$ as $\operatorname{Ker} \tau = \mathbb{R}^n$ since $\zeta = 0$.

We collect these results in Table 5.1 which highlights the bijection between orbit types.

With reference to Table 5.1 we draw up other tables showing the orbit types of $SE(n)$ for $n = 3, 4, 5, 6$. The groups H_Ω and $H_{\overline{\Omega}}$ are determined by the type decompositions Ω and $\overline{\Omega}$ respectively. Therefore we write the typical proper elements (Ω, ζ) supposing wlog that Ω belongs to the Cartan subalgebra and therefore as a matrix is described entirely by its type decomposition. The tables are composed so that orbits in the same row are related by the bijection. Note that the Adjoint orbits are not always flag manifolds wheras the Coadjoint orbits always are.

Remark 5.3.1. Observe that for $H = SO(3)$ the adjoint group is the same , $Ad(G) \cong G$ since $SO(3)$ is already centerless. Also note that by identifying $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the affine adjoint group $SO(3) \ltimes \mathfrak{so}(3)$ is isomorphic to the group $SE(3) = SO(3) \ltimes \mathbb{R}^3$. By Theorem 4 the Adjoint and Coadjoint actions are therefore equivalant. Note that from Table 5.2 we indeed see that the orbits are the same yet are not always preserved by the bijection.

Theorem 6. The Coadjoint orbits of $SE(n)$ through a proper point (Ω, ζ) are diffeomorphic to the Co/Adjoint orbits of $SO(n)$ through $\Omega \in \mathfrak{so}(n)$ when $\zeta = 0$. For when $\zeta \neq 0$ the Coadjoint orbits fibre over the manifold of directed lines $\mathrm{Aff}\widetilde{\mathcal{F}}(1; n-1)$ in $\mathbb{R}^n$ with fibre diffeomorphic to the Co/Adjoint orbit of $SO(n-1)$ through $\overline{\Omega}$. In particular the affine flag manifold $\mathrm{Aff}\mathcal{F}(\tilde{1}; d_1, ..., d_r)$ is symplectic whenever $\mathcal{F}(d_1, ..., d_r)$ is diffeomorphic to a Co/Adadjoint orbit of $SO(n-1)$.

Question: For what other affine groups would a result like this still hold? We have seen in Theorem 4 that $\rho(H) \ltimes V$ has equivalent Adjoint and Coadjoint representations for when ρ is the Adjoint representation. For $H = SO(n), SU(n)$ the representations are no longer equivalent but there is still a shared structure given by a bijection between orbits. What happens for other higher dimensional irreducible representations of these groups? For the Poincaré group it is shown in [3] that an orbit bijection holds, (indeed the method we have outlined may be tweaked to show this). Could it hold for other affine simple Lie groups? What is the nature of the bijection in these cases?

Table 5.1: Generic isotropy subgroups for proper points from $\mathfrak{g}$ and $\mathfrak{g}^*$ which are in bijection.

Adjoint orbit through proper $(A, \lambda e_1)$			Coadjoint orbit through proper $(\Omega, \lambda e_1)$		
	Type decomposition	$G_{A,X}$		Type decomposition	$G_{\Omega,\zeta}$
$\lambda \neq 0$	$\Delta_{\overline{A}}$	$H_{\overline{A}} \ltimes \mathrm{Ker}\, A$	$\lambda \neq 0$	$\Delta_{\overline{\Omega}}$	$H_{\overline{\Omega}} \ltimes \mathbb{R} \cdot \zeta$
$\lambda = 0$	Δ_A	$H_A \ltimes \mathrm{Ker}\, A$	$\lambda = 0$	Δ_Ω	$H_\Omega \ltimes \mathbb{R}^n$

Table 5.2: Orbits for $SE(3)$

	Type decomp.	$G_{A,X}$	Adjoint orbit	$G_{\Omega,\zeta}$	Coadjoint orbit
$\lambda \neq 0$, $\Delta_{\overline{\Omega}}$	(0)	$SO(2) \ltimes \mathbb{R}^3$	S^2	$SO(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;2)$
	(ρ)	$SO(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;2)$	$SO(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;2)$
$\lambda \neq 0$, Δ_Ω	(0)	$SO(3) \ltimes \mathbb{R}^3$	pt.	$SO(3) \ltimes \mathbb{R}^3$	pt.
	$\left(\begin{array}{c\|c} 0 & \\ \hline & \rho \end{array}\right)$	$SO(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;2)$	$SO(2) \ltimes \mathbb{R}^3$	S^2

Table 5.3: Orbits for $SE(4)$

	Type	$G_{A,X}$	Adjoint orbit	$G_{\Omega,\zeta}$	Coad. orbit
$\lambda \neq 0$, $\Delta_{\overline{\Omega}}$	(0)	$SO(3) \ltimes \mathbb{R}^4$	S^3	$SO(3) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;3)$
	$\left(\begin{array}{c\|c} 0 & \\ \hline & \rho \end{array}\right)$	$SO(2) \ltimes \mathbb{R}^2$	$\mathrm{Aff}\widetilde{\mathcal{F}}(2;1,1)$	$SO(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;1,2)$
$\lambda = 0$, $\Delta_\Omega =$	(0)	$SO(4) \ltimes \mathbb{R}^4$	pt.	$SO(4) \ltimes \mathbb{R}^4$	pt.
	$\begin{pmatrix} 0 & \\ & \rho \end{pmatrix}$	$SO(2) \times U(1) \ltimes \mathbb{R}^2$	$\mathrm{Aff}\mathcal{HF}(2,2)$	$SO(2) \times U(1) \ltimes \mathbb{R}^4$	$\mathcal{HF}(2,2)$
	$\begin{pmatrix} \rho & \\ & \rho \end{pmatrix}$	$U(2)$	$\mathcal{HF}(4) \times \mathbb{R}^4$	$U(2) \ltimes \mathbb{R}^4$	$\mathcal{HF}(4)$
	$\begin{pmatrix} \rho & \\ & \kappa \end{pmatrix}$	$U(1) \times U(1)$	$\mathcal{HF}(2,2) \times \mathbb{R}^4$	$U(1) \times U(1) \ltimes \mathbb{R}^4$	$\mathcal{HF}(2,2)$

Table 5.4: Orbits for $SE(5)$, (* indicates an orbit which is not a flag manifold).

	Type	$G_{A,X}$	Adjoint orbit	$G_{\Omega,\zeta}$	Coad. orbit
	(0)	$SO(4) \ltimes \mathbb{R}^5$	S^4	$SO(4) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;4)$
$\lambda \neq 0,$	$\begin{pmatrix} 0 & \\ & \rho \end{pmatrix}$	$U(1) \times SO(2) \ltimes \mathbb{R}^3$	*	$U(1) \times SO(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;2,2)$
$\Delta_{\overline{\Omega}} =$	$\begin{pmatrix} \rho & \\ & \rho \end{pmatrix}$	$U(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};4^{\mathbb{C}})$	$U(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};4^{\mathbb{C}})$
	$\begin{pmatrix} \rho & \\ & \kappa \end{pmatrix}$	$U(1) \times U(1) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};2^{\mathbb{C}},2^{\mathbb{C}})$	$U(1) \times U(1) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};2^{\mathbb{C}},2^{\mathbb{C}})$
	(0)	$SO(5) \ltimes \mathbb{R}^5$	pt.	$SO(5) \ltimes \mathbb{R}^5$	pt.
$\lambda = 0,$	$\left(\begin{array}{c\|c} \rho & \\ \hline & 0 \end{array}\right)$	$U(1) \times SO(3) \ltimes \mathbb{R}^3$	$\mathrm{Aff}\widetilde{\mathcal{F}}(3;2)$	$U(1) \times SO(3) \ltimes \mathbb{R}^5$	$\widetilde{\mathcal{F}}(3,2)$
$\Delta_{\Omega} =$	$\left(\begin{array}{cc\|c} \rho & & \\ & \rho & \\ \hline & & 0 \end{array}\right)$	$U(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};4^{\mathbb{C}})$	$U(2) \ltimes \mathbb{R}^5$	$\mathcal{F}(\tilde{1},4^{\mathbb{C}})$
	$\left(\begin{array}{cc\|c} \rho & & \\ & \kappa & \\ \hline & & 0 \end{array}\right)$	$U(1) \times U(1) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};2^{\mathbb{C}},2^{\mathbb{C}})$	$U(1) \times U(1) \ltimes \mathbb{R}^5$	$\widetilde{\mathcal{F}}(\tilde{1};2,2)$

Table 5.5: Orbits for $SE(6)$, (* indicates an orbit which is not a flag manifold).

	Type	$G_{A,X}$	Adjoint orbit	$G_{\Omega,\zeta}$	Coadjoint orbit
	(0)	$SO(5) \ltimes \mathbb{R}^6$	S^5	$SO(5) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;5)$
$\lambda \neq 0,$	$\left(\begin{array}{c\|c} \rho & \\ \hline & 0 \end{array}\right)$	$U(1) \times SO(3) \ltimes \mathbb{R}^4$	*	$U(1) \times SO(3) \ltimes \mathbb{R}$	$\mathrm{Aff}\widetilde{\mathcal{F}}(1;2,3)$
$\Delta_{\overline{\Omega}} =$	$\left(\begin{array}{cc\|c} \rho & & \\ & \rho & \\ \hline & & 0 \end{array}\right)$	$U(2) \ltimes \mathbb{R}^2$	*	$U(2) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};\tilde{1},4^{\mathbb{C}})$
	$\left(\begin{array}{cc\|c} \rho & & \\ & \kappa & \\ \hline & & 0 \end{array}\right)$	$U(1) \times U(1) \ltimes \mathbb{R}^2$	*	$U(1) \times U(1) \ltimes \mathbb{R}$	$\mathrm{Aff}\mathcal{F}(\tilde{1};\tilde{1},2^{\mathbb{C}},2^{\mathbb{C}})$
	(0)	$SO(6) \ltimes \mathbb{R}^6$	pt.	$SO(6) \ltimes \mathbb{R}^6$	pt.
$\lambda = 0,$	$\begin{pmatrix} \rho & & \\ & 0 & \\ & & 0 \end{pmatrix}$	$U(1) \times SO(4) \ltimes \mathbb{R}^6$	$\widetilde{\mathcal{F}}(2,4)$	$U(1) \times SO(4) \ltimes \mathbb{R}^6$	$\widetilde{\mathcal{F}}(2,4)$
	$\begin{pmatrix} \rho & & \\ & \rho & \\ & & 0 \end{pmatrix}$	$U(2) \times SO(2) \ltimes \mathbb{R}^2$	$\mathrm{Aff}\mathcal{HF}(2,4)$	$U(2) \times SO(2) \ltimes \mathbb{R}^6$	$\mathcal{HF}(2,4)$
$\Delta_{\Omega} =$	$\begin{pmatrix} \rho & & \\ & \rho & \\ & & \rho \end{pmatrix}$	$U(3)$	$\mathcal{HF}(6) \ltimes \mathbb{R}^6$	$U(3) \ltimes \mathbb{R}^6$	$\mathcal{HF}(6)$
	$\begin{pmatrix} \rho & & \\ & \kappa & \\ & & \kappa \end{pmatrix}$	$U(1) \times U(2)$	$\mathcal{HF}(2,4) \ltimes \mathbb{R}^6$	$U(1) \times U(2) \ltimes \mathbb{R}^6$	$\mathcal{HF}(2,4)$
	$\begin{pmatrix} \rho & & \\ & \kappa & \\ & & \epsilon \end{pmatrix}$	$U(1) \times U(1) \times U(1)$	$\mathcal{HF}(2,2,2) \ltimes \mathbb{R}^6$	$U(1) \times U(1) \times U(1) \ltimes \mathbb{R}^6$	$\mathcal{HF}(2,2,2)$

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