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ON THE VALUE OF OPTIMAL STOPPING GAMES

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Abstract. We show, under weaker assumptions than in the previous literature, that a perpetual optimal stopping game always has a value. We also show that there exists an optimal stopping time for the seller, but not necessarily for the buyer. Moreover, conditions are provided under which the existence of an optimal stopping time for the buyer is guaranteed. The results are illustrated explicitly in two examples.

1. Introduction

In this paper we study a perpetual optimal stopping game between two players, the "buyer" and the "seller". Both players choose a stopping time each, say τ and γ, and at the time $\tau \wedge \gamma := \min\{\tau, \gamma\}$ the seller pays the amount

$$Y_1(\tau)1\!\!1_{\{\tau\leq\gamma\}} + Y_2(\gamma)1\!\!1_{\{\tau>\gamma\}} \tag{1}$$

to the buyer. Here Y_1 and Y_2 are two stochastic processes satisfying $0 \leq Y_1(t) \leq Y_2(t)$ for all t almost surely. Clearly, the seller wants to minimize the amount in (1) and the buyer wants to maximize this amount.

We consider discounted optimal stopping games defined in terms of two continuous contract functions g_1 and g_2 satisfying $0 \leq g_1 \leq g_2$ and a diffusion process $X(t)$. More precisely, given a constant discounting rate $\beta > 0$, let

$$Y_1(t) = e^{-\beta t} g_1(X(t))$$

and

$$Y_2(t) = e^{-\beta t} g_2(X(t)).$$

Define the mapping R_x from the set of pairs (τ, γ) of stopping times to the set $[0, \infty]$ by

$$R_x(\tau, \gamma) := \mathbb{E}_x e^{-\beta\tau\wedge\gamma}\Big(g_1\big(X(\tau)\big)1\!\!1_{\{\tau\leq\gamma\}} + g_2\big(X(\gamma)\big)1\!\!1_{\{\tau>\gamma\}}\Big). \tag{2}$$

Thus $R_x(\tau, \gamma)$ is the expected discounted pay-off when the players use the stopping times τ and γ as stopping strategies. Here the index x indicates that the diffusion X is started at x at time 0. In (2), and in similar situations below, we use the convention that

$$f(X(\sigma)) = 0 \quad \text{on } \{\sigma = \infty\},$$

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where f is a function and σ is a random time. Next define the lower value $\underline{V}$ and the upper value $\overline{V}$ as

$$\underline{V}(x) := \sup_{\tau} \inf_{\gamma} R_x(\tau, \gamma)$$

and

$$\overline{V}(x) := \inf_{\gamma} \sup_{\tau} R_x(\tau, \gamma),$$

respectively, where the supremums and the infimums are taken over random times τ and γ that are stopping times. It is clear that

$$g_1(x) \leq \underline{V}(x) \leq \overline{V}(x) \leq g_2(x)$$

(the first and the last inequality follow from choosing $\tau = 0$ and $\gamma = 0$ in the definitions of $\underline{V}$ and $\overline{V}$, respectively). If, in addition, the inequality

$$\underline{V}(x) \geq \overline{V}(x)$$

holds, i.e. if $\underline{V}(x) = \overline{V}(x)$, then the stochastic game is said to have a value. In such cases, we denote the common value $\underline{V}(x) = \overline{V}(x)$ by $V(x)$. If there exist two stopping times τ' and γ' such that

$$R_x(\tau, \gamma') \leq R_x(\tau', \gamma') \leq R_x(\tau', \gamma) \tag{3}$$

for all stopping times τ and γ, then the pair (τ', γ') is referred to as a saddle point for the stochastic game. It is clear that if there exists a saddle point for the stochastic game, then the game also has a value.

It is well-known, compare [2], [3], [10], [11], [13] and [15], that under the integrability condition

$$\mathbb{E}_x\Big(\sup_{0 \leq t < \infty} e^{-\beta t} g_2\big(X(t)\big)\Big) < \infty \tag{4}$$

and the condition

$$\lim_{t \to \infty} e^{-\beta t} g_2\big(X(t)\big) = 0,$$

the stochastic game has a value V. Moreover, the two stopping times

$$\tau^* := \inf\Big\{t : V\big(X(t)\big) = g_1\big(X(t)\big)\Big\} \tag{5}$$

and

$$\gamma^* := \inf\Big\{t : V\big(X(t)\big) = g_2\big(X(t)\big)\Big\} \tag{6}$$

together form a saddle point for the game. Below we prove the existence of a value under no integrability conditions at all. To do this we use the connection between excessive functions and concave functions, compare [6] and [7]. More specifically, using concave functions we produce a candidate V^* for the value function, and then we prove that $\underline{V} \geq V^* \geq \overline{V}$. Thus there exists a value of the game, and this value is given by the candidate function V^*. One should note that we prove the existence of a value for perpetual optimal stopping games, i.e. when there is no upper bound on the stopping times τ and γ. It remains an open question if all optimal stopping games with a finite time horizon have values.

One easily finds examples of optimal stopping games where the pair (τ^*, γ^*) of stopping times defined by (5) and (6) is not a saddle point, compare for instance the examples in Section 5.1. We prove below, however, that γ^* is always optimal for the seller. More precisely, we deal with the following concepts closely related to the notion of a saddle point: a stopping time τ' is optimal for the buyer if

$$R_x(\tau', \gamma) \geq \overline{V}(x)$$

for all stopping times γ, and a stopping time γ' is optimal for the seller if

$$R_x(\tau, \gamma') \leq \underline{V}(x)$$

for all stopping times τ. Note that

$$\begin{gathered} \tau' \text{ is optimal for the buyer and } \gamma' \text{ is optimal for the buyer} \\ \iff \\ (\tau', \gamma') \text{ is a saddle point.} \end{gathered}$$

Also note that if τ' is optimal for the buyer, then

$$\overline{V}(x) \leq \inf_{\gamma} R_x(\tau', \gamma) \leq \underline{V}(x) \leq \overline{V}(x),$$

so the game has a value $V(x)$ which is given by

$$V(x) = \inf_{\gamma} R_x(\tau', \gamma).$$

Similarly, if γ' is optimal for the seller, then the existence of a value $V(x)$ follows, and

$$V(x) = \sup_{\tau} R_x(\tau, \gamma').$$

The outline of the paper is as follows. In Section 2 we specify the assumptions on the diffusion X and we show that a stochastic game with an infinite time horizon always has a value. This is done without the integrability condition (4), compare Theorem 2.5. We also show that γ^* is an optimal stopping time for the seller. The method used in the proof of Theorem 2.5 also gives a characterization of the value function in terms of concave functions. As a straightforward consequence of this characterization, the Smooth-Fit Principle is deduced in Section 3. In Section 4 we provide additional conditions under which τ^* is optimal for the buyer, i.e. (τ^*, γ^*) is a saddle point. Finally, in Section 5 we explicitly determine the value of two different game options, both of which may be regarded game versions of the American call option. In these examples, the integrability condition (4) is not fulfilled, so they are not covered by the theory in previous literature.

2. The value of a stochastic differential game

Let X be a stochastic process with dynamics

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dW(t), \tag{7}$$

where μ and σ are given functions and W is a standard Brownian motion. We assume that the two end-points of the state space of X are 0 and ∞, and we assume for simplicity that both these end-points are natural. We also assume that the functions $\mu(\cdot)$ and $\sigma(\cdot)$ are continuous and that $\sigma(x) > 0$ for all $x \in (0, \infty)$. It follows that the equation (7) has a (weak) solution which is unique in the sense of probability law. Moreover, X is a regular diffusion, i.e. for all $x, y \in (0, \infty)$ we have that y is reached in finite time with a positive probability if the diffusion is started from x.

The second order ordinary differential equation

$$\mathcal{L}u(x) := \frac{\sigma^2(x)}{2}u_{xx} + \mu(x)u_x - \beta u = 0 \tag{8}$$

has two linearly independent solutions $\psi, \varphi : (0,\infty) \to \mathbb{R}$ which are uniquely determined (up to multiplication with positive constants) by requiring one of them to be positive and strictly increasing and the other one to be positive and strictly decreasing, compare [5]. We let ψ be the increasing solution and φ the decreasing solution. We also let $F : (0,\infty) \to (0,\infty)$ be the strictly increasing positive function defined by

$$F(x) := \frac{\psi(x)}{\varphi(x)}.$$

Recall that a function $u : (0,\infty) \to \mathbb{R}$ is said to be F-concave in an interval $J \subset (0,\infty)$ if

$$u(x) \geq u(l)\frac{F(r) - F(x)}{F(r) - F(l)} + u(r)\frac{F(x) - F(l)}{F(r) - F(l)}$$

for all $l, x, r \in J$ with $l < x < r$. F-convexity of a function is defined similarly.

Below we need the following two theorems relating concave and convex functions to the value functions of optimal stopping problems. The first one is Proposition 4.2 in [6]. The proof of the second one follows along the lines of the proofs of Propositions 3.2 and 4.2 in [6] and is therefore omitted.

Theorem 2.1. *Let l, r be such that $0 < l < r < \infty$, let $g : [l,r] \to [0,\infty)$ be measurable and bounded, and let*

$$U(x) := \sup_{\tau \leq \tau_{l,r}} \mathbb{E}_x e^{-\beta\tau} g(X(\tau)),$$

where

$$\tau_{l,r} := \inf\{t : X(t) \notin (l,r)\}.$$

Then U is the smallest majorant of g such that U/φ is F-concave on $[l,r]$.

Theorem 2.2. *Let l, r be such that $0 < l < r < \infty$, let $g : [l,r] \to [0,\infty)$ be measurable and bounded, and let*

$$U(x) := \inf_{\gamma \leq \gamma_{l,r}} \mathbb{E}_x e^{-\beta\gamma} g(X(\gamma)).$$

Then U is the largest minorant of g such that U/φ is F-convex on $[l,r]$.

Remark Note that it is important in Theorem 2.2 that the stopping times γ are to be chosen among stopping times not exceeding the first exit time $\gamma_{l,r}$ of $X(t)$ from the interval (l,r). If, for example, the choice $\gamma = \infty$ would be included, then U would be identically 0.

Below we find our candidate value function V^* in the set

$$\begin{aligned} \mathbb{F} \;=\; & \big\{f : (0,\infty) \to [0,\infty) : f \text{ is continuous}, \ g_1 \leq f \leq g_2, \\ & \quad f/\varphi \text{ is } F\text{-concave in every interval in which } f < g_2\big\}. \end{aligned}$$

Note that $\mathbb{F}$ is non-empty since $g_2 \in \mathbb{F}$. Since 0 and ∞ are assumed to be natural boundaries of X, we have $\psi(0+) = 0 = \varphi(\infty)$. Thus the inverse

$F^{-1} : (0,\infty) \to (0,\infty)$ of F exists. We work below with the functions $H_i : (0,\infty) \to [0,\infty)$, $i = 1, 2$, defined by

$$H_i(y) := \frac{g_i(F^{-1}(y))}{\varphi(F^{-1}(y))} \tag{9}$$

and the set

$$\begin{aligned} \mathbb{H} \;=\; & \big\{h : (0,\infty) \to [0,\infty) : h \text{ is continuous, } H_1 \le h \le H_2, \\ & \quad h \text{ is concave in every interval in which } h < H_2\big\}. \end{aligned}$$

Note that the functions in $\mathbb{F}$ are precisely the functions $\varphi \cdot (h \circ F)$ for some function $h \in \mathbb{H}$.

Lemma 2.3. *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathbb{H}$. Then the function h defined by*

$$h(y) := \inf_n h_n(y)$$

is an element of $\mathbb{H}$.

Proof. It is straightforward to check that the minimum of two functions in $\mathbb{H}$ is again in $\mathbb{H}$. Thus we may without loss of generality assume that $h_{n+1} \le h_n$ for all n. Let

$$U := \{y : h(y) < H_2(y)\},$$

and note that h, being the infimum of continuous functions, is upper semi-continuous, so U is open. Choose two points $l, r \in U$ with $l < r$ and $[l, r] \subset U$. The interval $[l, r]$ is compact, and it is covered by the increasing family $\{U_n\}_{n=1}^{\infty}$ of open sets

$$U_n := \{y : h_n(y) < H_2(y)\}.$$

Hence there exists an integer N such that $[l, r] \subset U_n$ for all $n \ge N$. For such n, h_n is concave on $[l, r]$, and therefore also h is concave on this interval. Consequently, h is concave on each interval contained in U, and thus also continuous at all points in U.

To show that $h \in \mathbb{H}$, it remains to check that h is continuous also at all boundary points of U. Let $l \in \overline{U} \setminus U$, where $\overline{U}$ is the closure of U in $(0,\infty)$, and let $\{l_k\}_{k=1}^{\infty}$ be a sequence of points in U converging to l from the right (left-continuity is dealt with similarly). Assume first that $(l, l + \epsilon_0) \subset U$ for some $\epsilon_0 > 0$. Since h is concave on $(l, l + \epsilon_0)$, the limit

$$\lim_{0<\delta\to 0} \frac{h(l + \delta) - h(l+)}{\delta}$$

exists and is $> -\infty$. Using this, it is straightforward to check that $h(l) = h(l+)$. Thus, if $(l, r) \subset U$, then h is continuous on $[l, r]$.

Next, if there does not exist an $\epsilon_0 > 0$ such that $h < H_2$ in $(l, l + \epsilon_0)$, then the previous case can be applied to deduce right-continuity of h at l. Indeed, for $\epsilon > 0$, choose $\delta > 0$ such that

$$|H_2(y) - H_2(l)| \le \epsilon \text{ for } y \in [l, l + \delta].$$

Without loss of generality, we may assume that $H_2 = h$ at $l+\delta$ (since points with $H_2 = h$ exist arbitrarily close to l). Now, for a point $l_k \in (l, l + \delta)$ there exists a maximal surrounding interval in which $h < H_2$. We know from above that h is concave in the closure of this interval, and thus $h \ge$

$H_2(l)-\epsilon$ in the interval. In particular, $h(l_k)\geq H_2(l)-\epsilon$. Since we also have $h(l_k)\leq H_2(l_k)\leq H_2(l)+\epsilon$, and since ϵ is arbitrary, it follows that $h(l_k)\to H_2(l)=h(l)$ as $k\to\infty$. Hence h is continuous at l, and thus we have shown that $h\in\mathbb{H}$. □

Lemma 2.4. *There exists a smallest element $V^*\in\mathbb{F}$. Moreover, the function V^*/φ is F-convex in every interval in which $V^*>g_1$.*

Proof. Since the functions in $\mathbb{F}$ are precisely the functions $\varphi\cdot(h\circ F)$ for some function $h\in\mathbb{H}$, it suffices to show that there exists a smallest element in $\mathbb{H}$ and that this smallest element is convex in every interval of strict majorization of H_1. In order to do this, define

$$W(y):=\inf_{h\in\mathbb{H}}h(y).$$

Being the infimum of continuous functions, W is itself upper semi-continuous. Let $\{y_k\}_{k=1}^\infty$ be a dense sequence of points in $(0,\infty)$, and for each k, let

$$\{h_n^k\}_{n=1}^\infty\subseteq\mathbb{H}$$

be a sequence of functions in $\mathbb{H}$ such that $\inf_n h_n^k(y_k)=W(y_k)$. Next, define the function W^* by

$$W^*(y)=\inf_k\inf_n h_n^k(y).$$

According to Lemma 2.3, $W^*\in\mathbb{H}$. Moreover, the non-negative function W^*-W is upper semi-continuous and vanishes on a dense subset of $(0,\infty)$. It follows that $W\equiv W^*$, so $W\in\mathbb{H}$, which finishes the first part of the proof.

To show the convexity on each interval in which $W>H_1$, let I be such an interval and fix $y'\in I$. By continuity of H_1, H_2 and W, we can find $\delta>0$ so that

$$\inf_{y\in I^\delta}\min\big\{W(y),H_2(y)\big\}\geq\sup_{y\in I^\delta}H_1(y),$$

where $I^\delta:=[y'-\delta,y'+\delta]$. Assume there exist points $y_1,y_2\in I^\delta$ with $y_1<y'<y_2$ and

$$W(y')>W(y_1)\frac{y_2-y'}{y_2-y_1}+W(y_2)\frac{y'-y_1}{y_2-y_1}. \tag{10}$$

By continuity, we may assume that y_1 and y_2 are chosen so that

$$W(y)>W(y_1)\frac{y_2-y}{y_2-y_1}+W(y_2)\frac{y-y_1}{y_2-y_1}$$

for all $y\in(y_1,y_2)$. It is now straightforward to check that the function

$$h(y):=\begin{cases}W(y_1)\frac{y_2-y}{y_2-y_1}+W(y_2)\frac{y-y_1}{y_2-y_1} & \text{if } y\in[y_1,y_2]\\ W(y) & \text{if } y\notin(y_1,y_2)\end{cases}$$

satisfies $h\in\mathbb{H}$. However, $h<W$ in $y\in(y_1,y_2)$ contradicts the minimality of W, and thus (10) is not true. This means that W is convex at the point y', so by continuity W is convex on I, which finishes the second part of the proof. □

Theorem 2.5. *For any starting point $x > 0$, the perpetual optimal stopping game described above has a value $V(x) := \underline{V}(x) = \overline{V}(x)$. Moreover, $V \equiv V^*$, where V^* is the function appearing in Lemma 2.4, and the stopping time*

$$\gamma^* := \inf\{t : V(X(t)) = g_2(X(t))\}$$

is an optimal stopping time for the seller.

Proof. Let V^* be the function in Lemma 2.4, and choose $x \in (0, \infty)$. To prove the existence of a value we will show that

$$\overline{V}(x) \leq V^*(x) \leq \underline{V}(x). \tag{11}$$

To prove the first inequality, assume that the maximal interval containing x in which $V^* < g_2$ is (l, r) for some points $l < r$ (if $V^*(x) = g_2(x)$, then the first inequality obviously holds since $\overline{V} \leq g_2$). Assume also, for the moment, that $0 < l$ and $r < \infty$. It follows that $V^*(l) = g_2(l)$ and $V^*(r) = g_2(r)$. Inserting $\gamma = \gamma_{l,r}$ in the definition of $\overline{V}$ yields

$$\begin{aligned}\overline{V}(x) &\leq \sup_\tau \mathbb{E}_x e^{-\beta\tau\wedge\gamma_{l,r}}\Big(g_1\big(X(\tau)\big)\mathbb{1}_{\{\tau\leq\gamma_{l,r}\}} + g_2\big(X(\gamma_{l,r})\big)\mathbb{1}_{\{\tau>\gamma_{l,r}\}}\Big) \\ &= \sup_{\tau\leq\gamma_{l,r}} \mathbb{E}_x e^{-\beta\tau} g^*(X(\tau))\end{aligned} \tag{12}$$

where the function g^* is defined by

$$g^*(x) = \begin{cases} g_1(x) & \text{if } x \in (l, r) \\ g_2(x) & \text{if } x \in \{l, r\}. \end{cases} \tag{13}$$

Note that V^* majorizes g^* and that V^*/φ is F-concave on $[l, r]$. Moreover, by construction, V^* is the smallest such function. It follows from Theorem 2.1 that

$$\sup_{\tau\leq\gamma_{l,r}} \mathbb{E}_x e^{-\beta\tau} g^*(X(\tau)) = V^*(x).$$

Thus

$$\overline{V}(x) \leq V^*(x). \tag{14}$$

Now, if we instead have $0 = l$ and/or $r = \infty$, then the above reasoning again applies if we plug in $\gamma = \gamma_r$, $\gamma = \gamma_l$ or $\gamma = \infty$ in the definition of $\overline{V}$ and use Propositions 5.3 or 5.11 in [6] instead of Theorem 2.1.

To show the second inequality in (11), we argue similarly. Choose an x and let (l, r) be a maximal interval containing x in which $V^* > g_1$. As above, let us first assume that

$$0 < l \quad \text{and} \quad r < \infty. \tag{15}$$

Inserting $\tau = \tau_{l,r}$ in the definition of $\underline{V}$ gives

$$\begin{aligned}\underline{V}(x) &\geq \inf_\gamma \mathbb{E}_x e^{-\beta\tau_{l,r}\wedge\gamma}\Big(g_1\big(X(\tau_{l,r})\big)\mathbb{1}_{\{\tau_{l,r}\leq\gamma\}} + g_2\big(X(\gamma)\big)\mathbb{1}_{\{\tau_{l,r}>\gamma\}}\Big) \\ &= \inf_{\gamma\leq\tau_{l,r}} \mathbb{E}_x e^{-\beta\gamma} g_*(X(\gamma)),\end{aligned}$$

where the function g_* is given by

$$g_*(x) = \begin{cases} g_2(x) & \text{if } x \in (l, r) \\ g_1(x) & \text{if } x \in \{l, r\}. \end{cases}$$

Thus, since V^*/φ is F-convex in $[l, r]$ (see Lemma 2.4), it follows from Theorem 2.2 that $\underline{V}(x) \geq V^*(x)$. Thus we have shown the second inequality in (11) under the assumption (15).

Now, if (15) is not the case, then the second inequality in (11) requires some slightly more involved analysis. For example, assume that

$$0 < l \quad \text{and} \quad r = \infty. \tag{16}$$

To prove $\underline{V}(x) \geq V^*(x)$ in this case we do not plug in $\tau = \tau_l$ in the definition of $\underline{V}$, but we rather use the stopping times $\tau_{l,N}$ for different $N \geq l$ (compare the remark following the current proof). Thus, for any $N \geq x$, choosing $\tau = \tau_{l,N}$ in the definition of $\underline{V}$ gives

$$\underline{V}(x) \geq \inf_{\gamma} R_x(\tau_{l,N}, \gamma) = \inf_{\gamma \leq \tau_{l,N}} \mathbb{E}_x e^{-\beta\gamma} g_*(X(\gamma)) =: V_N(x) \tag{17}$$

where

$$g_*(x) = \begin{cases} g_2(x) & \text{if } x \in (l, N) \\ g_1(x) & \text{if } x = \{l, N\}. \end{cases}$$

¿From Theorem 2.2 it follows that V_N is majorized by g_*, that V_N/φ is F-convex on $[l, N]$, and that V_N is the largest function with these properties. It is clear from (14) and (17) that

$$\sup_{N \geq x} V_N(x) \leq \underline{V}(x) \leq V^*(x).$$

We show below that we in fact have

$$\sup_{N \geq x} V_N(x) = V^*(x). \tag{18}$$

Note that (18) implies that

$$\underline{V}(x) = V^*(x)$$

and therefore also the existence of a value. To prove (18) we will work in the coordinates y defined by $y = F(x)$.

Let H_i, $i = 1, 2$ be defined by $H_i = \frac{g_i}{\varphi} \circ F^{-1}$. Then

$$W_{N'} := \frac{V_N}{\varphi} \circ F^{-1} : [l', N'] \to \mathbb{R}$$

is the largest convex function majorized by the function

$$H(y) := \begin{cases} H_2(y) & \text{if } y \in (l', N') \\ H_1(y) & \text{if } y \in \{l', N'\}, \end{cases}$$

where $l' := F(l)$ and $N' := F(N)$. Let $W := \frac{V^*}{\varphi} \circ F^{-1}$ (thus W is the function defined in the proof of Lemma 2.4). The conditions $0 < l$ and $r = \infty$ translate to $l' > 0$, $W(l') = H_1(l')$ and $W(y) > H_1(y)$ for all $y > l'$. Next, for $y > l'$, define

$$\hat{W}(y) := \sup_{N' \geq y} W_{N'}(y).$$

We need to show that $\hat{W} \geq W$. To do this, note that since $W > H_1$ in the interval $[l', \infty)$ we know from Lemma 2.4 that W is convex in this interval. Choose $y_0 > l'$, let

$$k := \lim_{\epsilon \searrow 0} \frac{W(y_0 + \epsilon) - W(y_0)}{\epsilon}$$

be the right derivative of W at y_0, and let $L(y) = k(y - y_0) + W(y_0)$ be the steepest tangential of W at y_0. Note that $L(y) \leq W(y) \leq H_2(y)$. Now we consider two cases.

First, assuming the existence of a point $N' > y_0$ such that $L(N') = H_1(N')$, the function

$$h(y) = \begin{cases} W(y) & \text{if } y \in [l', y_0] \\ L(y) & \text{if } y \in [y_0, N'] \end{cases}$$

is convex and dominated by H_2 in (l', N') and by H_1 at the points l' and N'. Therefore $h \leq W_{N'}$ by Theorem 2.2, so

$$\hat{W}(y_0) \geq W(y_0).$$

Second, assume that there is no point $N' > y_0$ such that $L(N') = H_1(N')$. Note that the function

$$h(y) = \begin{cases} W(y) & \text{if } y \in (0, y_0] \\ L(y) & \text{if } y \in [y_0, \infty) \end{cases}$$

is an element of the set $\mathbb{H}$. Since W is the smallest function in this set it follows that we must have $W(y) = L(y)$ for all $y \geq y_0$. Moreover, for each $\epsilon > 0$ there exists a point of intersection (to the right of y_0) between the line $L^\epsilon(y) := (k - \epsilon)(y - y_0) + W(y_0)$ and H_1 (otherwise a function in $\mathbb{H}$ can be constructed which is strictly smaller than W in some interval). Now, let $z < W(y_0)$, and consider the straight lines through (y_0, z) that are below W in the interval $[l', y_0]$. Let k' be the slope of the largest such straight line, i.e. k' is the smallest possible slope. Since W is convex in $[l', \infty)$, we have that $k' < k$, and thus the straight line through $(y_0, W(y_0))$ with slope k' and the function H_1 have a point $(N', H_1(N'))$ of intersection, where $N' > y_0$. It is now straightforward to check that, for this N', the corresponding function $W_{N'}$ satisfies $W_{N'}(y_0) > z$. Since $z < W(y_0)$ is arbitrary, it follows that $\hat{W}(y_0) \geq W(y_0)$.

Thus we have shown under the assumption (16) that (18) holds, implying the second inequality in (11). By symmetry, the above argument also applies in the case when $l = 0$ and $r < \infty$. The remaining case, i.e. when $l = 0$ and $r = \infty$, can be handled with similar methods (we omit the details).

Finally, since we have shown that the inequality in (12) actually is an equality, it follows that γ^* is optimal for the seller. □

Remark Note that the function W in the proof of Lemma 2.4 is the smallest function in the set

$$\begin{aligned} \mathbb{H} \;=\; & \{h : (0,\infty) \to [0,\infty) : h \text{ is continuous, } H_1 \leq h \leq H_2, \\ & h \text{ is concave in every interval in which } h < H_2\}, \end{aligned}$$

whereas, in general, it is not the largest function in the set

$$\begin{aligned} & \{h : (0,\infty) \to [0,\infty) : h \text{ is continuous, } H_1 \leq h \leq H_2, \\ & h \text{ is convex in every interval in which } h > H_1\} \end{aligned}$$

(although W is a member also of this set). This asymmetry of the function W (and the corresponding one for the function V^*) may be regarded the underlying reason for the asymmetry in the proof of the first and the second inequality in (11).

Remark Let us introduce the perpetual American option value V_∞ associated with the payoff g_1, that is

$$V_\infty(x) := \sup_\tau \mathbb{E}_x e^{-\beta\tau} g_1(X(\tau)).$$

Obviously, $V \leq V_\infty$. An immediate consequence of Theorem 2.5 is that the implication

$$V_\infty(x_0) \geq g_2(x_0) \text{ for some } x_0 \in (0,\infty) \implies \{x : V(x) = g_2(x)\} \neq \emptyset$$

holds. Indeed, assume that $V_\infty(x_0) \geq g_2(x_0)$ for some x_0 and that $V(x) < g_2(x)$ for all $x \in (0,\infty)$. Then $\gamma^* = \infty$, so $V \equiv V_\infty$ by Theorem 2.5. It follows that $V(x_0) \geq g_2(x_0)$, which is a contradiction.

3. The smooth-fit principle

In the following proposition, let H_1 and H_2 be the functions defined in (9) and let W be the smallest element in the set $\mathbb{H}$. Moreover, let $\frac{d^-}{dy}$ and $\frac{d^+}{dy}$ denote the left and the right differential operators, respectively, i.e.

$$\frac{d^-}{dy}h(y_0) := \lim_{\epsilon \searrow 0} \frac{h(y_0) - h(y_0 - \epsilon)}{-\epsilon}$$

and

$$\frac{d^+}{dy}h(y_0) := \lim_{\epsilon \searrow 0} \frac{h(y_0 + \epsilon) - h(y_0)}{\epsilon}.$$

Proposition 3.1. *Assume that $y_1 \in (0,\infty)$ is such that $H_1(y_1) = W(y_1) < H_2(y_1)$. Also assume that the left and right derivatives $\frac{d^-}{dy}H_1$ and $\frac{d^+}{dy}H_1$ exist at y_1. Then*

$$\frac{d^-}{dy}H_1(y_1) \geq \frac{d^-}{dy}W(y_1) \geq \frac{d^+}{dy}W(y_1) \geq \frac{d^+}{dy}H_1(y_1). \tag{19}$$

Similarly, if $y_2 \in (0,\infty)$ is such that $H_2(y_2) = W(y_2)$ and $\frac{d^-}{dy}H_2$ and $\frac{d^+}{dy}H_2$ exist at y_2, then

$$\frac{d^-}{dy}H_1(y_2) \leq \frac{d^-}{dy}W(y_2) \leq \frac{d^+}{dy}W(y_2) \leq \frac{d^+}{dy}H_1(y_2). \tag{20}$$

Proof. Since $W(y_1) = H_1(y_1)$, the first and the third inequality in (19) follow from $V \geq H_1$. Since $W(y_1) < H_2(y_1)$, we know that W is concave in a neighborhood of y_1. From this the second inequality follows.

The inequalities in (20) follow similarly. □

Remark Note that for the middle inequalities in (19) and (20) to hold it is essential that $W(y_1) < H_2(y_1)$ and $H_1(y_2) < W(y_2)$, respectively. Indeed, (19) is for example not true at the point $y_1 = K_1$ if

$$H_1(y) = (y \wedge K_3 - K_2)^+$$

and

$$H_2(y) = (y - K_1)^+$$

for some constants $K_3 > K_2 > K_1 > 0$.

After a change of coordinates, Proposition 3.1 translates to the following Smooth-Fit Principle. Note that, in line with the above results, no integrability conditions are assumed.

Corollary 3.2. (Smooth-Fit Principle). *Let $x_0 \in (0,\infty)$ and assume that $V(x_0) = g_i(x_0)$, where either $i = 1$ or $i = 2$. Assume also that $g_1(x_0) < g_2(x_0)$ and that g_i is differentiable at x_0. Then also V is differentiable at x_0 and*

$$\frac{d}{dx}V(x_0) = \frac{d}{dx}g_i(x_0).$$

4. Existence of a saddle point

According to Theorem 2.5, γ^* is an optimal stopping time for the seller. It turns out, however, that

$$\tau^* := \inf\{t : V(X(t)) = g_1(X(t))\}$$

in general need not be optimal for the buyer, compare the examples in Section 5. A necessary condition for (τ^*, γ^*) to be a saddle point is that

$$\mathbb{P}(\tau^* < \infty) > 0,$$

or equivalently, that the set

$$E_1 := \Big\{x \in (0,\infty) : V(x) = g_1(x)\Big\}$$

is non-empty. Indeed, $R_x(\infty,\infty) = 0$, and thus $\tau^* = \infty$ cannot be optimal for the buyer (at least not if $g_1 \not\equiv 0$). Below we give an analytical criterion in terms of the differential operator

$$\mathcal{L} := \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x} - \beta$$

ensuring that the set E_1 is empty. To this end, we restrict the class of payoff functions by requiring some additional regularity conditions.

Hypothesis 4.1. *Let $D = \{a_1, \dots, a_n\}$ where $n \in \mathbb{N}$ and a_i are positive real numbers with $a_1 < a_2 < \ldots < a_n$. Suppose that g_1 is a continuous function on $(0,\infty)$ such that g_1' and g_1'' exist and are continuous on $(0,\infty) \setminus D$ and that the limits*

$$g_1'(a_i\pm) := \lim_{x\to a_i\pm} g_1'(x), \quad g_1''(a_i\pm) := \lim_{x\to a_i\pm} g_1''(x)$$

exist and are finite.

Proposition 4.2. *Assume that the function g_1 satisfies Hypothesis 4.1 and that $g_2 > g_1$ on some open interval $\mathcal{I} \subset (0,\infty)$. If $\mathcal{L}g_1$ is a non-zero positive measure on $\mathcal{I}$, then $V(x) > g_1(x)$ for every $x \in \mathcal{I}$. Thus, if $\mathcal{I} = (0,\infty)$, then the set E_1 is empty, and consequently τ^* is not optimal for the buyer (provided $g_1 \not\equiv 0$).*

Similarly, if $\mathcal{L}g_2$ is a non-zero negative measure on $\mathcal{I}$, then $V(x) < g_2(x)$ for all $x \in \mathcal{I}$.

Proof. Fix $x \in \mathcal{I}$ and choose $l, r \in \mathcal{I}$ with $l < x < r$ so that $\mathcal{L}g_1$ is a non-zero positive measure on $(l, r) \subset \mathcal{I}$.
According to Theorem 2.5, $V(x) = \sup_\tau R_x(\tau, \gamma^*)$ and thus

$$V(x) \geq R_x(\tau_{l,r}, \gamma^*) \geq \mathbb{E}_x\left(e^{-\beta(\tau_{l,r}\wedge\gamma^*)} g_1\big(X(\tau_{l,r} \wedge \gamma^*)\big)\right). \tag{21}$$

Note that if $\mathbb{P}_x(\gamma^* < \tau_{l,r}) > 0$, then the second inequality in (21) is strict. Because g_1 satisfies Hypothesis 4.1, the Itô-Tanaka formula (see Theorem 3.7.1, page 218 in [12]) gives

$$\mathbb{E}_x\left(e^{-\beta(\tau_{l,r}\wedge\gamma^*)} g_1\big(X(\tau_{l,r} \wedge \gamma^*)\big)\right) = g_1(x) + \mathbb{E}_x\left(\int_0^{\tau_{l,r}\wedge\gamma^*} e^{-\beta s}\mathcal{L}g_1\big(X(s)\big)\,ds\right)$$
$$+ \sum_{a_i \in (l,r)} \big(g_1'(a_i+) - g_1'(a_i-)\big)\mathbb{E}_x\left(\int_0^{\tau_{l,r}\wedge\gamma^*} e^{-\beta s}\,dL^i(s)\right),$$

where L^i is the local time of X at a_i. Now, since $\mathcal{L}g_1$ is non-negative on (l, r), we find that

$$\mathbb{E}_x\left(e^{-\beta(\tau_{l,r}\wedge\gamma^*)} g_1\big(X(\tau_{l,r} \wedge \gamma^*)\big)\right) \geq g_1(x).$$

Moreover, if $\gamma^* \geq \tau_{l,r}$ a.s., then this inequality is strict. Indeed, since $\mathcal{L}g_1$ is a non-zero positive measure on (l, r), we have that either $\mathcal{L}g_1(y) > 0$ for some $y \in (l, r)$ where g_1 is differentiable (implying that the middle term is strictly positive), or $g_1'(a_i+) > g_1'(a_i-)$ for some $a_i \in (l, r)$ (implying that the last term is strictly positive). Thus, in view of (21) we have $V(x) > g_1(x)$, which finishes the proof of the first part of the proposition.

As for the second claim, by Proposition 4.4 in [6] (note that it is also valid for contracts functions of the type (13)) we may replace (21) with

$$V(x) \leq \sup_\tau R_x(\tau, \gamma_{l,r}) = R_x(\hat{\tau}, \gamma_{l,r})$$

for some stopping time $\hat{\tau}$. The proof now follows as above. □

Below we provide conditions under which τ^* is optimal for the buyer. Following [1] and [6], the conditions are expressed in terms of the two quantities

$$l_0 := \limsup_{x\to 0} \frac{g_1(x)}{\varphi(x)} \quad \text{and} \quad l_\infty := \limsup_{x\to\infty} \frac{g_1(x)}{\psi(x)}.$$

Proposition 4.3. *Assume that both l_0 and l_∞ are finite. Also assume that the non-negative local martingales $e^{-\beta t}\varphi(X(t))$ and $e^{-\beta t}\varphi(X(t))$ satisfy*

$$\mathbb{E}_x\Big(\sup_{0\leq s\leq t} e^{-\beta s}\varphi\big(X(s)\big)\Big) < \infty \quad \text{and} \quad \mathbb{E}_x\Big(\sup_{0\leq s\leq t} e^{-\beta s}\psi\big(X(s)\big)\Big) < \infty \tag{22}$$

for all times t. Then the process $e^{-\beta t\wedge\tau^*}V(X(t \wedge \tau^*))$ is a sub-martingale.

Proof. We know from Theorem 2.5 that V/φ is F-convex in all intervals where $V > g_1$. Arguing as in the proof of Proposition 5.1 in [6], it can therefore be shown that $Z(t) := e^{-\beta t\wedge\tau^*}V(X(t \wedge \tau^*))$ is a sub-martingale provided

$$E_x\Big(\sup_{0\leq s\leq t} Z(s)\Big) < \infty$$

(this is needed for the use of Fatou's lemma). ¿From the results in [6], compare Propositions 5.4 and 5.12 of that paper, we know that

$$\limsup_{x\to 0}\frac{V(x)}{\varphi(x)} = l_0 \quad \text{and} \quad \limsup_{x\to\infty}\frac{V(x)}{\psi(x)} = l_\infty.$$

Thus there exist constants C and D with

$$V(x) \leq C\varphi(x) + D\psi(x)$$

for all $x \in (0,\infty)$. From the assumption (22) it therefore follows that $\sup_{0\leq s\leq t} Z(s)$ is integrable, which finishes the proof. □

Remark Without the assumption (22), Proposition 4.3 would not be true. Also note that to show that the process $e^{-\beta t\wedge\gamma^*}V(X(t\wedge\gamma^*))$ is a supermartingale, neither the finiteness of l_0 and l_∞ nor the condition (22) is needed.

The following two results may be viewed as the game versions of Proposition 5.13 and 5.14 in [6].

Theorem 4.4. *Assume (22) and that*

$$l_0 = l_\infty = 0. \tag{23}$$

Then (τ^*,γ^*) is a saddle point.

Proof. ¿From Proposition 4.3 it follows that

$$\begin{aligned} V(x) &\leq \mathbb{E}_x\left(e^{-\beta(t\wedge\tau^*\wedge\gamma)}V(X(t\wedge\tau^*\wedge\gamma))\right) \\ &\leq \mathbb{E}_x\left(e^{-\beta\tau^*}V(X(\tau^*))1\!\!1_{\{\tau^*\leq t\wedge\gamma\}} + e^{-\beta\gamma}V(X(\gamma))1\!\!1_{\{\gamma<t\wedge\tau^*\}}\right) \\ &\quad +\mathbb{E}_x\left(e^{-\beta t}V(X(t))1\!\!1_{\{t\leq\tau^*\wedge\gamma\}}\right) \end{aligned}$$

for any stopping time γ. We first prove that the last term converges to zero when t tends to $+\infty$. To do this, recall that the assumption (23) implies

$$\lim_{x\to 0}\frac{V(x)}{\varphi(x)} = \lim_{x\to\infty}\frac{V(x)}{\psi(x)} = 0.$$

Thus, given a constant $\delta > 0$, there exists a constant M such that $V(x) \leq \delta\varphi(x)+\delta\psi(x)+M$ for all x. Using the fact that $e^{-\beta t}\psi(X(t))$ and $e^{-\beta t}\psi(X(t))$ are non-negative local martingales, and hence supermartingales, we find

$$\begin{aligned} \mathbb{E}_x\left(e^{-\beta t}V(X(t))1\!\!1_{\{t\leq\tau^*\wedge\gamma\}}\right) &\leq Me^{-\beta t} + \delta\mathbb{E}_x e^{-\beta t}\varphi(X(t)) + \delta\mathbb{E}_x e^{-\beta t}\psi(X(t)) \\ &\leq Me^{-\beta t} + \delta\varphi(x) + \delta\psi(x). \end{aligned}$$

Since δ can be chosen arbitrarily, we conclude the first step. Next, the monotone convergence theorem yields

$$\begin{aligned} V(x) &\leq \lim_{t\to\infty}\mathbb{E}_x\left(e^{-\beta\tau^*}V(X(\tau^*))1\!\!1_{\{\tau^*\leq t\wedge\gamma\}} + e^{-\beta\gamma}V(X(\gamma))1\!\!1_{\{\gamma<t\wedge\tau^*\}}\right) \\ &\leq \lim_{t\to\infty}\mathbb{E}_x\left(e^{-\beta\tau^*}g_1(X(\tau^*))1\!\!1_{\{\tau^*\leq t\wedge\gamma\}} + e^{-\beta\gamma}g_2(X(\gamma))1\!\!1_{\{\gamma<t\wedge\tau^*\}}\right) \\ &= \mathbb{E}_x\left(e^{-\beta\tau^*}g_1(X(\tau^*))1\!\!1_{\{\tau^*\leq\gamma\}} + e^{-\beta\gamma}g_2(X(\gamma))1\!\!1_{\{\gamma<\tau^*\}}\right) \\ &= R_x(\tau^*,\gamma), \end{aligned}$$

i.e. τ^* is optimal for the buyer. This finishes the proof. □

Theorem 4.5. *Assume (22) and that l_0 and l_∞ are both finite. Then, for any starting point x the pair (τ^*,γ^*) is a saddle point if and only if*

$$\left\{\begin{array}{c}\text{there is no } l>0 \text{ such that}\\ g_1(x)<V(x) \text{ for all } x\leq l\\ \text{if } l_0>0\end{array}\right\} \text{ and } \left\{\begin{array}{c}\text{there is no } r>0 \text{ such that}\\ g_1(x)<V(x) \text{ for all } x\geq r\\ \text{if } l_\infty>0.\end{array}\right\}$$

Proof. If $l_0=l_\infty=0$, then the result follows from Theorem 4.4. Therefore we assume that $l_\infty>0$ (the case $l_0>0$ can be treated similarly).

To prove the sufficiency of the condition, fix a starting point $x\in(0,\infty)$. If $V(x)=g_1(x)$, then $\tau^*=0$ is clearly optimal for the buyer, and thus we are finished. If $V(x)>g_1(x)$, let $I:=(a,b)\subset(0,\infty)$ be a maximal interval containing x such that $V>g_1$ in I. Note that

$$\tau^*=\inf\{t:X(t)\notin I\},$$

and that $b<\infty$ by assumption. Moreover, given $\delta>0$ there exists a constant M such that $V\leq M+\delta\varphi$ in I. Indeed, if $a>0$, then V is bounded in I, and if $a=0$, then $l_0=0$ by assumption. Thus, proceeding analogously as in the proof of Theorem 4.4, we obtain

$$\begin{aligned}V(x) &\leq \lim_{t\to\infty}\mathbb{E}_x\left(e^{-\beta(t\wedge\tau^*\wedge\gamma)}V(X(t\wedge\tau^*\wedge\gamma))\right)\\ &\leq \lim_{t\to\infty}\mathbb{E}_x\left(e^{-\beta\tau^*}V(X(\tau^*))1\!\!1_{\{\tau^*\leq t\wedge\gamma\}}+e^{-\beta\gamma}V(X(\gamma))1\!\!1_{\{\gamma<t\wedge\tau^*\}}\right)\\ &\quad +\delta\varphi(x)+\lim_{t\to\infty}Me^{-\beta t}\leq R_x(\tau^*,\gamma)+\delta\varphi(x)\end{aligned}$$

for a stopping time γ. Since δ is arbitrary, this shows that τ^* is optimal for the buyer.

Conversely, assume that (τ^*,γ^*) is a saddle point for each starting point x and that $V(x)>g_1(x)$ for $x\geq r$. Then, for $x\geq r$, the stopping time $\tau^*\geq\tau_r$ *a.s.* The definition of a saddle point and the optional sampling theorem applied to the non-negative supermartingale $e^{-\beta t}V_\infty(X_t)$ give

$$\begin{aligned}V(x) &= R_x(\tau^*,\gamma^*)\leq R_x(\tau^*,\infty)\mathbb{E}_x\left(e^{-\beta\tau^*}g_1(X(\tau^*))\right)\\ &\leq \mathbb{E}_x\left(e^{-\beta\tau^*}V_\infty(X(\tau^*))\right)\leq\mathbb{E}_x\left(e^{-\beta\tau_r}V_\infty(X(\tau_r))\right)=\frac{\varphi(x)}{\varphi(r)}V_\infty(r),\end{aligned}$$

where we in the last used equation (2.6) in [6]. Proposition 5.4 in [6] then implies that

$$l_\infty=\limsup_{x\to\infty}\frac{V(x)}{\psi(x)}\leq\frac{V_\infty(r)}{\varphi(r)}\lim_{x\to\infty}\frac{\varphi(x)}{\psi(x)}=0,$$

which contradicts $l_\infty>0$. □

5. Two examples of game options

In this section we study two examples motivated by applications in finance. In both examples we assume that $\mu(x)=\beta x$, where β is the discounting rate. Thus the diffusion X solves

$$dX(t)=\beta X(t)\,dt+\sigma(X(t))\,dW(t),$$

and V may be interpreted as the arbitrage free price of a game option written on a non-dividend paying stock, compare [14]. Note that the functions ψ and φ are given (up to multiplication with a positive constant) by

$$\psi(x) = x$$

and

$$\varphi(x) = x\int_x^\infty \frac{1}{u^2}\exp\{-\int_1^u \frac{2\beta z}{\sigma^2(z)}\,dz\}\,du. \tag{24}$$

5.1. **The game version of a call option.** In this subsection we study the game version of a call option, i.e.

$$g_1(x) = (x-K)^+ \quad \text{and} \quad g_2(x) = (x-K)^+ + \epsilon$$

for some positive constants K and ϵ. If $\epsilon \geq K$, then one can show that the game option reduces to an ordinary perpetual American call option. Therefore we consider the case with $\epsilon < K$.

The functions $H_i := (\frac{g_i}{\varphi}) \circ F^{-1}$, $i = 1, 2$, are given by

$$H_1(y) = (y - \frac{K}{\varphi(F^{-1}(y))})^+$$

and

$$H_2(y) = (y - \frac{K}{\varphi(F^{-1}(y))})^+ + \frac{\epsilon}{\varphi(F^{-1}(y))}.$$

First we claim that the function

$$w(y) := \frac{1}{\varphi(F^{-1}(y))}$$

is concave. To see this, note that by letting $y = F(x)$ we find that

$$w(y) = \frac{1}{\varphi(F^{-1}(y))} = \frac{1}{\varphi(x)} = \frac{F(x)}{x} = \frac{y}{F^{-1}(y)},$$

where we have used $F(x) = x/\varphi(x)$. Straightforward calculations yield that

$$w''(y) - \frac{\varphi''(x)}{\varphi^3(x)(F'(x))^2}.$$

Using (24) one can check that $\varphi''(x) \geq 0$, so it follows that w is concave. Since w is concave, H_1 is 0 on $(0, F(K))$ and convex in $(F(K), \infty)$, and H_2 is concave in $(0, F(K))$ and convex in $(F(K), \infty)$. This, together with the easily checked facts

$$\lim_{y\to\infty} \frac{H_1(y)}{y} = 1, \qquad H_2'(y) < 1$$

and

$$H_2'(F(K)+) = \frac{\epsilon}{K} + \frac{(K-\epsilon)F(K)}{K^2F'(K)} > \frac{\epsilon}{K} = \frac{H_2(F(K))}{F(K)},$$

implies that the smallest function W in $\mathbb{H}$ is given by

$$W(y) = \begin{cases} \frac{\epsilon y}{K} & \text{if } y \in (0, F(K)] \\ H_2(y) & \text{if } y \in (F(K), \infty). \end{cases}$$

In the usual coordinates this means that the value V of the game version of a call option written on a no-dividend paying stock is

$$V(x) = \begin{cases} \frac{\epsilon x}{K} & \text{if } x \in (0, K] \\ x - K + \epsilon & \text{if } x \in (K, \infty). \end{cases}$$

According to Theorem 2.5, an optimal stopping time for the seller is given by

$$\gamma^* := \inf\{t : X(t) \geq K\}.$$

Also note that the corresponding stopping time $\tau^* = \infty$ is not optimal for the buyer.

5.2. **An example in which convexity is lost.** In this subsection we consider another possible generalization of the American call option. More precisely, let

$$g_1(x) = (x - K)^+ \quad \text{and} \quad g_2(x) = C(x - K)^+$$

for some constant $C > 1$. Moreover, assume for simplicity that the diffusion X is a geometric Brownian motion, i.e. that

$$dX(t) = \beta X(t)\, dt + \sigma X(t)\, dW(t)$$

for some constant $\sigma > 0$. Then the functions ψ and φ are given by

$$\psi(x) = x \quad \text{and} \quad \varphi(x) = x^{-2\beta/\sigma^2},$$

and the functions H_i, $i = 1, 2$ are given by

$$H_1(y) = \big(y - Ky^{2\beta/(2\beta+\sigma^2)}\big)^+ \quad \text{and} \quad H_2(y) = C\big(y - Ky^{2\beta/(2\beta+\sigma^2)}\big)^+.$$

We need to consider two different cases.

5.2.1. *Case 1.* First assume that $C \geq 1 + 2\beta/\sigma^2$. Then it is straightforward to check that $W(y) = (y - K^{(2\beta+\sigma^2)/\sigma^2})^+$, i.e. the value V of the option is given by

$$V(x) = \varphi(x) W(F(x)) = \big(x - K^{(2\beta+\sigma^2)/\sigma^2} x^{-2\beta/\sigma^2}\big)^+.$$

Moreover, Theorem 2.5 tells us that $\gamma^* := \inf\{t : X(t) \leq K\}$ is an optimal stopping time for the seller.

5.2.2. *Case 2.* Now assume that $1 < C < 1 + 2\beta/\sigma^2$. Then one can check that

$$W(y) = \begin{cases} H_2(y) & \text{if } y \in (0, y') \\ H_2(y') + y - y' & \text{if } y \in [y', \infty) \end{cases}$$

where y' is given by

$$y' = \Big(\frac{2\beta CK}{(2\beta + \sigma^2)(C - 1)}\Big)^{(2\beta+\sigma^2)/\sigma^2}.$$

It follows that

$$V(x) = \begin{cases} C(x - K)^+ & \text{if } x \in (0, x') \\ x - \frac{CK\sigma^2}{2\beta+\sigma^2}(\frac{x'}{x})^{2\beta/\sigma^2} & \text{if } x \in [x', \infty) \end{cases}$$

where

$$x' = \frac{2\beta CK}{(2\beta + \sigma^2)(C - 1)}.$$

According to Theorem 2.5, $\gamma^* := \inf\{t : X(t) \leq x'\}$ is optimal for the seller. As in the previous example, however, $\tau^* = \inf\{t : X(t) \leq K\}$ is not optimal for the buyer.

Remark The above example shows, perhaps surprisingly, that game options are not convexity preserving. More precisely, although both contract functions g_1 and g_2 are convex, the value of the game option need not necessarily be convex. This is in contrast to options of European and American style, both of which are known to be convexity preserving, compare for example [4] or [9] and the references therein.

Remark The method to determine the value of an optimal stopping game used in this section is also used in [8]. In that paper the construction of the value using concave functions is shown to be valid under the assumption of the existence of a value and a saddle point of the form (τ^*, γ^*). In the present paper we start with the construction of a natural candidate for the value function (without knowing a priori that such a value function exists), and then we show that this function indeed has to be the value of the game. This allows us to weaken the assumptions under which a game is known to have a value. Also note that the integrability condition (4) is satisfied in neither of the two examples provided in this section.

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