The point-line collinearity graph of the *Fi'*$_{24} maximal 2-local geometry - the first three discs$

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The point-line collinearity graph of the Fi_{24}' maximal 2-local geometry - the first three discs

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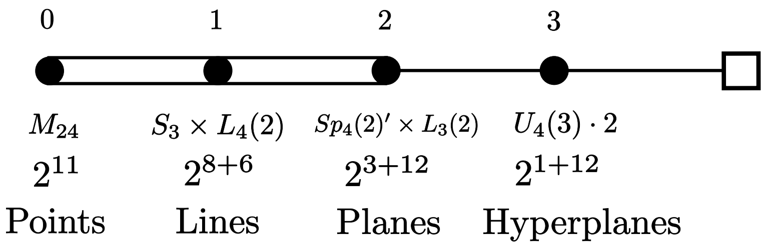
Abstract

The disc structure of the point-line collinearity graph for the maximal 2-local geometry associated with the largest simple Fischer group is investigated. For an arbitrary vertex of this graph the first three discs are determined. Additionally a fragment of the fourth disc is uncovered.

1 Introduction and main results

The investigations of Fischer [5] into groups generated by 3-transpositions not only had an influence upon certain later work related to the classification of the finite simple groups but also unearthed three previously unknown sporadic groups, Fi_{22}, Fi_{23} and Fi_{24}. The first two of these are simple while Fi_{24}, though not simple, has a simple subgroup Fi_{24}' of index 2. For more on these groups and 3-transposition groups in general see the book by Aschbacher [1].

Along with many of the other sporadic simple groups, Fi_{22}, Fi_{23} and Fi'_{24} possess minimal parabolic geometries and maximal 2-local geometries (see [9] and [10]). In the present paper we study the point-line collinearity graph $\mathcal{G}$ of Γ, the maximal 2-local geometry for Fi'_{24}. This geometry has rank 4 and its associated diagram is



Many properties of Γ are itemized in Section 2. We recall that the vertices of $\mathcal{G}$ are Γ_0, the points of Γ and two points are adjacent in $\mathcal{G}$ if they are incident with a common line. In [11],[12],[13] and [14] complete and detailed descriptions of the corresponding point-line collinearity graphs for Fi_{22} and Fi_{23} are presented.

For $x \in \Gamma_0$ and $i \in \mathbb{N}$, $\Delta_i(x)$ denotes the set of points of Γ_0 distance i from x. Let $G = Fi'_{24}$. Now G acts flag transitively on Γ and so, in studying $\mathcal{G}$, there is no loss in choosing and fixing a point a of Γ. Here we shall obtain properties of the first three discs of $\mathcal{G}$ around a (that is, of $\Delta_1(a), \Delta_2(a)$ and $\Delta_3(a)$) as well as describing a certain fragment of $\Delta_4(a)$. In a subsequent paper [16], a complete description of $\mathcal{G}$ is obtained - however the work in [16] is exclusively computer based, whereas this paper does not rely on any machine calculations. It is worth remarking that the notation and conventions used here and in [16] are compatible so as to allow a smooth transition between the two viewpoints. Earlier in [17], the second author obtained results on the structure of the first three discs of $\mathcal{G}$. The arguments given here will differ to some extent from those in [17] as we may now call upon results in [12],[13] and [14]. Further we are able to give more detail on adjacency within $\Delta_3(a)$.

We now present our main results - for notation we refer the reader to Section 2.

Theorem 1. *(i) $\Delta_1(a)$ is a G_a-orbit of size 1518;*

(ii) $\Delta_2(a)$ is the union of three G_a-orbits $\Delta_2^i(a)$ $(i = 1, 2, 3)$ and $|\Delta_2(a)| = 1,560,504$;

(iii) $\Delta_3(a)$ is the union of ten G_a-orbits $\Delta_3^i(a)$ $(i = 1, ..., 10)$ and $|\Delta_3(a)| = 1,400,874,432$; and

(iv) $\Delta_4(a) \cap \{x \in \Gamma_0 | \Omega_x \cap \Omega_a \neq \emptyset\}$ is the union of six G_a-orbits $\Delta_4^i(a)$ $(i = 1, ..., 6)$ and consists of 3,992,911,872 points.

Tables 1 and 2 list the sizes of the above mentioned G_a-orbits $\Delta_j^i(a)$.

$\Delta_j^i(a)$	Size of $\Delta_j^i(a)$
$\Delta_1(a)$	$2.3.11.23 = 1518$
$\Delta_2^1(a)$	$2^5.3.7.11.23 = 170,016$
$\Delta_2^2(a)$	$2^8.3.7.11.23 = 1,360,128$
$\Delta_2^3(a)$	$2^3.3.5.11.23 = 30,360$
$\Delta_3^1(a)$	$2^{12}.11.23 = 1,036,288$
$\Delta_3^2(a)$	$2^{10}.3^2.5.11.23 = 11,658,240$
$\Delta_3^3(a)$	$2^{12}.3.7.11.23 = 21,762,048$
$\Delta_3^4(a)$	$2^{12}.3.23 = 282,624$
$\Delta_3^5(a)$	$2^{15}.3^2.7.11.23 = 522,289,152$
$\Delta_3^6(a)$	$2^{12}.3.5.7.11.23 = 108,810,240$
$\Delta_3^7(a)$	$2^9.3^2.5.7.11.23 = 40,803,840$
$\Delta_3^8(a)$	$2^6.5.7.11.23 = 566,720$
$\Delta_3^9(a)$	$2^{13}.3^2.5.7.11.23 = 652,861,440$
$\Delta_3^{10}(a)$	$2^9.3^2.5.7.11.23 = 40,803,840$

Table 1

$\Delta_j^i(a)$	Size of $\Delta_j^i(a)$
$\Delta_4^1(a)$	$2^{16}.3^2.5.11.23 = 746,127,360$
$\Delta_4^2(a)$	$2^{15}.3.11.23 = 24,870,912$
$\Delta_4^3(a)$	$2^{15}.3.5.7.11.23 = 870,481,920$
$\Delta_4^4(a)$	$2^{19}.3^2.7.23 = 759,693,312$
$\Delta_4^5(a)$	$2^{18}.3.11.23 = 198,967,296$
$\Delta_4^6(a)$	$2^{18}.3.7.11.23 = 1,392,771,072$

Table 2

Theorem 2. *Let $x \in \Delta_1(a)$. Then $G_{ax} \sim 2^{10}2^4A_8$ (with $G_{ax}^{*x} = (G_{ax}^{*x})_{x+a} \sim 2^4A_8$, an octad stabilizer) has 4 orbits on $\Gamma_1(x)$ with point distribution as follows.*

Orbit	*Size*	*Point distribution*
$\{x+a\}$	*1*	$\{a\}2\Delta_1$
$\alpha_0(x, x+a)$	*30*	$\Delta_1 2\Delta_2^3$
$\alpha_2(x, x+a)$	*448*	$\Delta_1 2\Delta_2^2$
$\alpha_4(x, x+a)$	*280*	$\Delta_1 2\Delta_2^1$

Theorem 3. *Let $x \in \Delta_2^1(a)$. Then $G_{ax} \sim 2^7 2^6(3 \times S_5)$ (with $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^6(3 \times S_5)$, where $\Lambda_1 = \Omega_a \cap \Omega_x$ is a tetrad and Λ_2 is the unique sextet of Ω_x containing Λ_1. Also $G_{ax} \leq G_{aX}$ where X is the unique hyperplane incident with both a and x. Further, G_{ax} has 6 orbits on $\Gamma_1(x)$ with point distribution as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_{4,4^2}(x, \Lambda_1, \Lambda_2)$	*5*	$\Delta_1 2\Delta_2^1$
$\alpha_{0,4^2}(x, \Lambda_1, \Lambda_2)$	*10*	$\Delta_2^1 2\Delta_3^8$
$\alpha_{1,31^5}(x, \Lambda_1, \Lambda_2)$	*320*	$\Delta_2^1 2\Delta_3^6$
$\alpha_{2,2^4}(x, \Lambda_1, \Lambda_2)$	*240*	$\Delta_2^1 2\Delta_3^2$
$\alpha_{0,2^4}(x, \Lambda_1, \Lambda_2)$	*120*	$\Delta_2^1 2\Delta_3^7$
$\alpha_{3,31^5}(x, \Lambda_1, \Lambda_2)$	*64*	$\Delta_2^1 2\Delta_3^1$

Theorem 4. *Let $x \in \Delta_2^2(a)$. Then $G_{ax} \sim 2^5 2^4 S_6$, $|\{a,x\}^\perp| = 1$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^4 S_6$, where Λ_1 is the octad of Ω_x corresponding to $x+b$ (where $b = \{a,x\}^\perp$) and $\Lambda_2 = \Omega_a \cap \Omega_b \cap \Omega_x$, a duad contained in Λ_1. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 8 with point distribution as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_{8,2}(x,\Lambda_1,\Lambda_2) = \{x+b\}$	*1*	$\Delta_1 2\Delta_2^2$
$\alpha_{2,2}(x,\Lambda_1,\Lambda_2)$	*16*	$\Delta_2^2\Delta_3^3\Delta_3^4$
$\alpha_{4,2}(x,\Lambda_1,\Lambda_2)$	*60*	$\Delta_2^2 2\Delta_3^2$
$\alpha_{4,1}(x,\Lambda_1,\Lambda_2)$	*160*	$\Delta_2^2 2\Delta_3^6$
$\alpha_{2,1}(x,\Lambda_1,\Lambda_2)$	*192*	$\Delta_2^2 2\Delta_3^5$
$\alpha_{4,0}(x,\Lambda_1,\Lambda_2)$	*60*	$\Delta_2^2 2\Delta_3^{10}$
$\alpha_{2,0}(x,\Lambda_1,\Lambda_2)$	*240*	$\Delta_2^2 2\Delta_3^9$
$\alpha_{0,0}(x,\Lambda_1,\Lambda_2)$	*30*	$\Delta_2^2 2\Delta_3^7$

Theorem 5. *Let $x \in \Delta_2^3(a)$. Then $G_{ax} \sim 2^9 2^6 (L_3(2) \times 3)$ and $G_{ax}^{*x} \sim 2^6(L_3(2) \times 3)$, the derived subgroup of $Stab_{G_x^{*x}}\{\Lambda_1\}$ where Λ_1 is a trio of Ω_x. Also $G_{ax} \le G_{a\pi}$ where π is the unique plane incident with both a and x. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 3 with point distribution as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_{80^2}(x,\Lambda_1)$	*3*	$\Delta_1 2\Delta_2^3$
$\alpha_{4^2}(x,\Lambda_1)$	*84*	$\Delta_2^3 2\Delta_3^8$
$\alpha_{42^2}(x,\Lambda_1)$	*672*	$\Delta_2^3 2\Delta_3^{10}$

Now we move onto $\Delta_3(a)$ the third disc of a; we caution that in the following results the point distribution is incomplete.

Theorem 6. *Let $x \in \Delta_3^1(a)$. Then $G_{ax} \sim 2^2 L_3(4) S_3$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1\} \sim L_3(4) : S_3$ where Λ_1 is a triad of Ω_x. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 4, the point distribution of 3 of them are as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_3(x, \Lambda_1)$	*21*	$\Delta_2^1 2\Delta_3^1$
$\alpha_2(x, \Lambda_1)$	*168*	$\Delta_3^1 2\Delta_3^3$
$\alpha_1(x, \Lambda_1)$	*360*	$\Delta_3^1 2\Delta_4^1$

Theorem 7. *Let $x \in \Delta_3^2(a)$. Then $G_{ax} \sim 2^4 2^3 : (L_3(2) \times 2)$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^3 : (L_3(2) \times 2)$ where Λ_1 is an octad and Λ_2 is a duad of Ω_x and $\Lambda_1 \cap \Lambda_2 = \emptyset$. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 11, the point distribution of 6 of them are as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_{0,2}(x, \Lambda_1, \Lambda_2)$	*7*	$\Delta_2^1 2\Delta_3^2$
$\alpha_{0,1}(x, \Lambda_1, \Lambda_2)$	*16*	$\Delta_3^2 2\Delta_4^2$
$\alpha_{4,2}(x, \Lambda_1, \Lambda_2)$	*14*	$\Delta_2^2 2\Delta_3^2$
$\alpha_{2,2}(x, \Lambda_1, \Lambda_2)$	*56*	$\Delta_3^2 2\Delta_3^3$
$\alpha_{4,1}(x, \Lambda_1, \Lambda_2)$	*112*	$\Delta_3^2 2\Delta_4^3$
$\alpha_{2,1}(x, \Lambda_1, \Lambda_2)$	*224*	$\Delta_3^2 2\Delta_4^1$

Theorem 8. *Let $x \in \Delta_3^3(a)$. Then $G_{ax} \sim 22^4 : S_6$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^4 : S_6$ where Λ_1 is an octad and Λ_2 is a duad of Ω_x and $\Lambda_2 \subseteq \Lambda_1$. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 8, the point distribution of 5 of them are as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_{8,2}(x, \Lambda_1, \Lambda_2) = \{\Lambda_1\}$	*1*	$\Delta_2^2 \Delta_3^3 \Delta_3^4$
$\alpha_{2,2}(x, \Lambda_1, \Lambda_2)$	*16*	$\Delta_3^1 2\Delta_3^3$
$\alpha_{4,2}(x, \Lambda_1, \Lambda_2)$	*60*	$\Delta_3^2 2\Delta_3^3$
$\alpha_{4,1}(x, \Lambda_1, \Lambda_2)$	*160*	$\Delta_3^3 2\Delta_4^6$
$\alpha_{2,1}(x, \Lambda_1, \Lambda_2)$	*192*	$\Delta_3^3 2\Delta_4^4$

Theorem 9. *Let $x \in \Delta_3^4(a)$. Then $G_{ax} \sim 2 : M_{22} : 2$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1\} \sim M_{22} : 2$ where Λ_1 is a duad of Ω_x. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 3, the point distribution of 2 of them are as follows.*

Orbit	Size	Point distribution
$\alpha_2(x,\Lambda_1)$	*77*	$\Delta_2^2\Delta_3^3\Delta_3^4$
$\alpha_1(x,\Lambda_1)$	*352*	$\Delta_3^4 2\Delta_4^5$

Theorem 10. *Let $x \in \Delta_3^5(a)$. Then $G_{ax} \cong G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4 : A_5$ where $\Lambda_1 = O_1$, $\Lambda_2 = \{\infty\}$ and $\Lambda_3 = \{14\}$. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 13, the point distribution of 9 of them are as follows.*

Orbit	Size	Point distribution
$\alpha_{8,1,1}(x,\Lambda_1,\Lambda_2,\Lambda_3) = \{O_1\}$	*1*	$\Delta_2^2 2\Delta_3^5$
$\alpha_{2,1,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*16*	$\Delta_3^5\Delta_4^4\Delta_4^5$
$\alpha_{4,1,0}^{(1)}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*40*	$\Delta_3^5 2\Delta_4^3$
$\alpha_{4,1,0}^{(2)}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*40*	$\Delta_3^5\Delta_4^5\Delta_4^6$
$\alpha_{4,0,1}^{(1)}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*40*	$\Delta_3^5 2\Delta_4^3$
$\alpha_{4,0,1}^{(2)}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*40*	$\Delta_3^5\Delta_4^5\Delta_4^6$
$\alpha_{4,1,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*60*	$\Delta_3^5\Delta_4^1\Delta_4^3$
$\alpha_{2,1,0}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*96*	$2\Delta_3^5\Delta_4^6$
$\alpha_{2,0,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*96*	$2\Delta_3^5\Delta_4^6$

Theorem 11. *Let $x \in \Delta_3^6(a)$. Then $G_{ax} \sim 2^6 : 3S_4$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4 : 3S_4$ where Λ_1 is an octad of Ω_x, Λ_2 a tetrad contained in Λ_1 and Λ_3 a 1-element subset of Λ_2. The number of G_{ax}-orbits on $\Gamma_1(x)$ is 16, the point distribution of 7 of them are as follows.*

Orbit	Size	Point distribution
$\{\Lambda_1\}$	*1*	$\Delta_2^1 2\Delta_3^6$
$\alpha_{4,4,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*4*	$\Delta_2^2 2\Delta_3^6$
$\alpha_{4,1,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*16*	$\Delta_3^6\Delta_4^2\Delta_4^3$
$\alpha_{2,2,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*48*	$\Delta_3^6 2\Delta_4^3$
$\alpha_{4,3,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*48*	$\Delta_3^6 2\Delta_4^1$
$\alpha_{2,1,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*64*	$\Delta_3^6\Delta_4^5\Delta_4^6$
$\alpha_{4,2,1}(x,\Lambda_1,\Lambda_2,\Lambda_3)$	*72*	$\Delta_3^6 2\Delta_4^1$

Theorem 12. *(i) Let $x \in \Delta_3^7(a)$. Then $G_{ax} \sim [2^9]S_4$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim [2^6]S_4$ where $\Lambda_1 = O_1$, $\Lambda_2 = \mathcal{T}_0$ and Λ_3 is the partition of O_1 given by $\{\infty, 14\}, \{0, 8\}, \{3, 20\}, \{15, 18\}$. The point distribution of 2 of the G_{ax}-orbits on $\Gamma_1(x)$ are as follows.*

Orbit	*Size*	*Point distribution*
$\alpha_{8,8,2^4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	*1*	$\Delta_2^1 2\Delta_3^7$
$\alpha_{0,8,0^4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	*1*	$\Delta_2^2 2\Delta_3^7$
$\alpha_{0,0,0^4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	*1*	$\Delta_2^2 2\Delta_3^7$

(ii) For $x \in \Delta_3^8(a)$, $G_{ax} \sim 2^{13} : 3.3^2 : 4$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^6.3.3^2 : 4$ where $\Lambda_1 = S_0$ and Λ_2 is the partition given by

$$\Sigma = \{\infty, 14, 0, 8, 3, 20, 15, 18, 17, 4, 16, 10\} \text{ and } \Omega_x \backslash \Sigma.$$

The point distribution of 2 of the G_{ax}-orbits on $\Gamma_1(x)$ are as follows.

Orbit	*Size*	*Point distribution*
$\alpha_{4^2,8}(x, \Lambda_1, \Lambda_2)$	*6*	$\Delta_2^3 2\Delta_3^8$
$\alpha_{4^2,4^2}(x, \Lambda_1, \Lambda_2)$	*9*	$\Delta_2^1 2\Delta_3^8$

(iii) For $x \in \Delta_3^9(a)$, $G_{ax} \cong G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2.2^4 : S_4$ where $\Lambda_1 = O_1$, $\Lambda_2 = \{\infty, 14\}$ and $\Lambda_3 = \mathcal{T}_0$. The point distribution of the G_{ax}-orbit $\alpha_{8,2,8}(x, \Lambda_1, \Lambda_2, \Lambda_3)$ is $\Delta_2^2 2\Delta_3^9$.

(iv) For $x \in \Delta_3^{10}(a)$, $G_{ax} \sim [2^9] : S_4$ and $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim [2^5] : S_4$ where Λ_1 is the tetrad $\{\infty, 0, 3, 15\}$, Λ_2 is the duad $\{14, 8\}$ and Λ_3 is the duad $\{20, 18\}$. The point distributions of 2 of the G_{ax}-orbits on $\Gamma_1(x)$ are as follows.

Orbit	*Size*	*Point distribution*
$\alpha_{4,2,2}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	*1*	$\Delta_2^3 2\Delta_3^{10}$
$\alpha_{4,0,0}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	*4*	$\Delta_2^2 2\Delta_3^{10}$

Theorem 13. *(i) For $x \in \Delta_4^1(a)$, $G_{ax} \sim 2L_3(2)2$ and $G_{ax}^{*x} \sim L_3(2)2$.*

(ii) For $x \in \Delta_4^2(a)$, $G_{ax} \cong G_{ax}^{*x} \cong A_8$.

(iii) For $x \in \Delta_4^3(a)$, $G_{ax} \cong G_{ax}^{*x} \sim 2^6 3^2$.

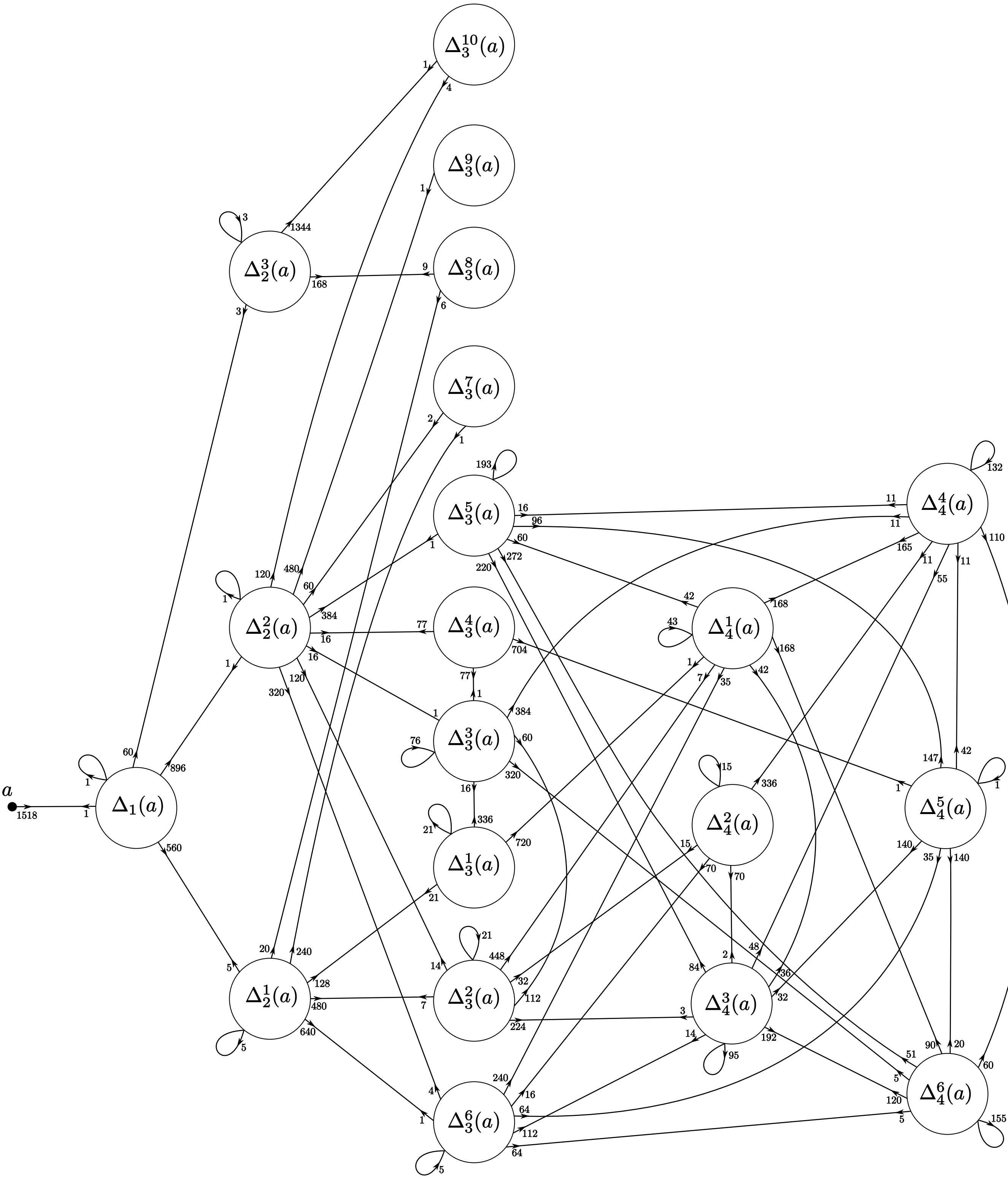
(iv) For $x \in \Delta_4^4(a)$, $G_{ax} \cong G_{ax}^{*x} \cong L_2(11)$.

(v) For $x \in \Delta_4^5(a)$, $G_{ax} \cong G_{ax}^{*x} \cong A_7$.

(vi) For $x \in \Delta_4^6(a)$, $G_{ax} \cong G_{ax}^{*x} \sim (3 \times A_5)2$.

Since, for $t \in \Omega_a$, $G_{ax} = G_{ax}^t$ for all $x \in \Delta_4^i(a)$ $(i = 1, ..., 6)$, the point distributions given in Theorems 11-16 of [12] may be directly translated to give the point distributions for G_{ax}-orbits on $\Gamma_1(x)$ of those lines within Γ_0^t.

We close this section by summarizing the collapsed adjacencies established in the above results.



a
$\Delta_1(a)$
$\Delta_2^1(a)$
$\Delta_2^2(a)$
$\Delta_2^3(a)$
$\Delta_3^1(a)$
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$\Delta_4^6(a)$

2 Notation and Γ

The maximal 2-local geometry Γ for $G = Fi'_{24}$ has rank 4 and we use Γ_i ($i = 0, 1, 2, 3$) to denote the objects of type i in Γ; objects of type 0 (respectively 1,2,3) will be referred to as points (respectively lines, planes, hyperplanes). For $x \in \Gamma$, the residue of x, Γ_x, is defined to be $\{y \in \Gamma \mid x * y\}$ where $*$ is the symmetric incidence relation of Γ. Also, for $x \in \Gamma$, we set

$$Q(x) = \{g \in G_x \mid g \text{ fixes all objects in } \Gamma_x\},$$

and for $H \leqslant G_x$ we write H^{*x} for $HQ(x)/Q(x)$. If $\Sigma \subseteq \Gamma$ and $i \in \{0, 1, 2, 3\}$, then we set $\Gamma_i(\Sigma) = \{x \in \Gamma_i \mid x * y \text{ for all } y \in \Sigma\}$. The point-line collinearity graph $\mathcal{G}$ of Γ has Γ_0 as its vertex set and for $x, y \in \Gamma_0$, x and y are adjacent in $\mathcal{G}$ if they are collinear, that is if $\Gamma_1(x, y) \neq \emptyset$. For $x, y \in \Gamma_0$, put $\{x, y\}^\perp = \Delta_1(x) \cap \Delta_1(y)$. Also for $x \in \Gamma_0$, we define $Z_1(x) = \{g \in G \mid g \text{ fixes } \{x\} \cup \Delta_1(x) \text{ pointwise}\}$ - note that $Z_1(x) \trianglelefteq G_x$.

We take as our starting point the following properties of Γ.

(2.1)(i) *G acts flag transitively on Γ.*

(ii) *Γ is a string geometry.*

(iii) *For $l \in \Gamma_1$, $|\Gamma_0(l)| = 3$ and if $x, y \in \Gamma_0(l)$ with $x \neq y$, then $\Gamma_1(x, y) = \{l\}$.*

(iv) *For $x \in \Gamma_0$, $G_x \sim 2^{11\cdot}M_{24}$ with $Q(x) \cong 2^{11}$, the dual of the Golay code module and $G_x^{*x} \cong M_{24}$. Moreover, Γ_x is isomorphic to the M_{24} maximal 2-local geometry.*

(v) *For $X \in \Gamma_3$, $G_X \sim 2_+^{1+12}.3.U_4(3).2$ with $Q(X) \sim 2_+^{1+12}.3$, $Z(G_X) = Z(O_2(Q(X))) \cong 2$ and $G_X^{*X} \sim U_4(3).2$. Also, Γ_X is isomorphic to a geometry for $U_4(3).2$ which is a subgeometry of the unitary geometry for $U_6(2)$.*

In (2.1) and elsewhere we follow the ubiquitous ATLAS [2] in describing group structures - it is also a convenient source for information about subgroups of M_{24} and $U_4(3).2$. In the situation of (2.1) we shall frequently denote ℓ by $x+y$ (to indicate we are viewing ℓ in Γ_x) or $y+x$ (to indicate we are viewing ℓ in Γ_y). See Section 3 for further details on the residue geometry in (2.1)(v).

Let $x \in \Gamma_0$ and let l, π, X be, respectively, a line, plane and hyperplane in Γ_x. We remark that ℓ corresponds to an octad, π to a trio and X to a sextet (see [9] and [4]). For a further discussion of Γ_x and Γ_X see Section 3. Other details of these geometries may be found in [6] and [17].

Before introducing an alternative way of viewing Γ we note, in passing, that $|\Gamma_0| = 2,503,413,946,215$ and that the permutation rank of G on Γ_0 is 120 [7].

Let $\mathcal{T}$ denote the set of transpositions in Fi_{24}. It is a fact that a maximal set B of pairwise commuting transpositions has $|B| = 24$ and $Stab_G(B) \sim 2^{11} \cdot M_{24}$. Such a set is called a base in [2] and G is transitive on the set of bases. Since Fi'_{24} has only one conjugacy class of subgroups isomorphic to $2^{11} \cdot M_{24}$ we may identify Γ_0 with the set of bases in a way which is compatible with the G-action. For $x \in \Gamma_0$ we use Ω_x to denote the base identified with x. Now Ω_x carries a copy of the Steiner system $S(24, 8, 5)$ preserved by $Stab_G(\Omega_x)$. Indeed an octad of Ω_x corresponds to a line in Γ_x (such an octad is contained in precisely three bases and incidence between points and lines corresponds to containment of bases and octads). Therefore $x, y \in \Gamma_0$ are adjacent in $\mathcal{G}$ if and only if $\Omega_x \cap \Omega_y$ is an octad of both Ω_x and Ω_y.

For $t \in \mathcal{T}$ put $\Gamma_0^t = \{x \in \Gamma_0 | t \in \Omega_x\}$. So the points in Γ_0^t correspond to all the bases which contain the fixed transposition t. Also put $G^t = C_G(t)$. Then $G^t \cong Fi_{23}$ and Γ_0^t is the set of points of the Fi_{23} geometry scrutinized in [11],[12],[13] (see especially Section 1). Further, if $\mathcal{G}^t$ denotes the point-line collinearity graph of this Fi_{23} geometry, then we see that for $x, y \in \Gamma_0^t$, x and

y are adjacent in $\mathcal{G}^t$ if and only if x and y are adjacent in $\mathcal{G}$. This observation gives us access to a rich vein of geometric information from [12],[13],[14]. So, in studying $\mathcal{G}$, we may view Γ geometrically working within residues or regard Γ_0 as living in the world of transpositions. In our arguments we adopt whichever viewpoint is the most efficacious. We shall also frequently call upon data given in [15] and accordingly will denote result (i.j) in [15] by $\mathbf{O}$(i.j). We carry along the notational conventions of [4]. So $\mathcal{S}_0$ and $\mathcal{T}_0$ denote the standard sextet and standard trio and O_1, O_2, O_3 are the heavy blocks of the MOG. Additionally we adapt the notation in [15] in the following manner. Let $x \in \Gamma_0$. In Γ_x the lines correspond to the octads of the M_{24} maximal 2-local geometry so to indicate we are working in Γ_x we write $\alpha_i(x, \Lambda_1)$ instead of just α_i (see $\mathbf{O}(2.1)$), with a similar convention for the other orbits itemized in [15].

(2.2) Let x be a point in Γ.

(i) $\Delta_2^1(x) = \{y \in \Gamma_0 |$ *there exists* $b \in \{x, y\}^\perp$ *such that* $b+y \in \alpha_4(b, b+x)\}$.

(ii) $\Delta_2^2(x) = \{y \in \Gamma_0 |$ *there exists* $b \in \{x, y\}^\perp$ *such that* $b+y \in \alpha_2(b, b+x)\}$.

(iii) $\Delta_2^3(x) = \{y \in \Gamma_0 |$ *there exists* $b \in \{x, y\}^\perp$ *such that* $b+y \in \alpha_0(b, b+x)\}$.

(iv) $\Delta_3^1(x) = \{y \in \Gamma_0 |$ *there exists* $c \in \Delta_2^1(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{3,31^5}(c, \Omega_x \cap \Omega_c, \mathcal{S}_{cx})\}$.

(v) $\Delta_3^2(x) = \{y \in \Gamma_0 |$ *there exists* $c \in \Delta_2^2(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{4,2}(c, c+b, \mathcal{D}_{cx})$, *where* $\{b\} = \{x, c\}^\perp\}$.

(vi) $\Delta_3^3(x) = \{y \in \Gamma_0 |$ *there exists* $c \in \Delta_2^2(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{2,2}(c, c+b, \mathcal{D}_{cx})$, *where* $\{b\} = \{x, c\}^\perp$ *and for* $t \in \mathcal{D}_{cx}, c$ *is the unique point in* $\Gamma_0^t \cap \Delta_2^2(x) \cap \Delta_1(y)\}$.

(vii) $\Delta_3^4(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^2(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{2,2}(c, c+b, \mathcal{D}_{cx})$, *where* $\{b\} = \{x, c\}^\perp$ *and for* $t \in \mathcal{D}_{cx}$, *there are 77 points in* $\Gamma_0^t \cap \Delta_2^2(x) \cap \Delta_1(y)\}$.

(viii) $\Delta_3^5(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^2(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{2,1}(c, c+b, \mathcal{D}_{cx})$, *where* $\{b\} = \{x, c\}^\perp\}$.

(ix) $\Delta_3^6(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^2(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{4,1}(c, c+b, \mathcal{D}_{cx})$, *where* $\{b\} = \{x, c\}^\perp\}$.

(x) $\Delta_3^7(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^1(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{0,2^4}(c, \Omega_x \cap \Omega_c, \mathcal{S}_{cx})\}$.

(xi) $\Delta_3^8(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^3(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_4(c, \mathcal{T}_{cx})\}$.

(xii) $\Delta_3^9(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^2(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{2,0}(c, c+b, \mathcal{D}_{cx})$, *where* $\{b\} = \{x, c\}^\perp\}$.

(xiii) $\Delta_3^{10}(x) = \{y \in \Gamma_0|$ *there exists* $c \in \Delta_2^3(x) \cap \Delta_1(y)$ *such that* $c + y \in \alpha_{42^2}(c, \mathcal{T}_{cx})\}$.

(xiv) $\Delta_4^1(x) = \{y \in \Gamma_0|$ *there exists* $d \in \Delta_3^1(x) \cap \Delta_1(y)$ *such that* $d + y \in \alpha_1(d, \mathcal{T}_{dx})\}$.

(xv) $\Delta_4^2(x) = \{y \in \Gamma_0|$ *there exists* $d \in \Delta_3^2(x) \cap \Delta_1(y)$ *such that* $d + y \in \alpha_{0,1}(d, \mathcal{O}_{dx}, \mathcal{D}_{dx})\}$.

(xvi) $\Delta_4^3(x) = \{y \in \Gamma_0|$ *there exists* $d \in \Delta_3^2(x) \cap \Delta_1(y)$ *such that* $d + y \in \alpha_{4,1}(d, \mathcal{O}_{dx}, \mathcal{D}_{dx})\}$.

(xvii) $\Delta_4^4(x) = \{y \in \Gamma_0|$ *there exists* $d \in \Delta_3^3(x) \cap \Delta_1(y)$ *such that* $d + y \in \alpha_{2,1}(d, d+b, \mathcal{D}_{dx})$, *where* $\{b\} = \Delta_1(d) \cap \Delta_2^2(x)\}$.

(xviii) $\Delta_4^5(x) = \{y \in \Gamma_0|$ *there exists* $d \in \Delta_3^4(x) \cap \Delta_1(y)$ *such that* $d + y \in \alpha_1(d, \mathcal{D}_{dx})\}$.

(xix) $\Delta_4^6(x) = \{y \in \Gamma_0|$ *there exists* $d \in \Delta_3^3(x) \cap \Delta_1(y)$ *such that* $d + y \in \alpha_{4,1}(d, d+b, \mathcal{D}_{dx})$*, where* $\{b\} = \Delta_1(d) \cap \Delta_2^2(x)\}$.

In (2.2) the letters $\mathcal{O}, \mathcal{D}, \mathcal{S}, \mathcal{T}$ (with appropriate subscripts) stand for, respectively, particular octads, duads, sextets and trios of certain bases. Their exact description will emerge later, and will tie in with the data given in [15].

Remark

In fact

$$\Delta_3^8(x) = \bigcup_{X \in \Gamma_3(x)} \Gamma_0(X) \cap \Delta_3(x).$$

See [17] for further details.

Let $x \in \Gamma_0$ and $t \in \Omega_x$. Set $\Delta_1(x)^t = \Delta_1(x) \cap \Gamma_0^t$ and for $i = 1, 2$, let $\Delta_2^i(x)^t = \Delta_2^i(x) \cap \Gamma_0^t$. For $i = 1, ..., 6$ we set

$$\Delta_3^i(x)^t = \Delta_3^i(x) \cap \Gamma_0^t \qquad \text{and}$$

$$\Delta_4^i(x)^t = \Delta_4^i(x) \cap \Gamma_0^t.$$

Further we put $Q(x)^t = Q(x) \cap G^t$. The above notation is set up so as $\Delta_j^i(x)^t$ corresponds to the $\Delta_j^i(x)$ as given in [12;(2.15)] for the point-line collinearity graph $\mathcal{G}^t$.

(2.3) Let $x \in \Gamma_0$.

(i) $\Delta_1(x) = \bigcup_{t \in \Omega_x} \Delta_1(x)^t$, $\Delta_2^i(x) = \bigcup_{t \in \Omega_x} \Delta_2^i(x)^t$ *(*$i = 1, 2$*) and* $\Delta_j^i(x) = \bigcup_{t \in \Omega_x} \Delta_j^i(x)^t$ *(*$i = 1, ..., 6, j = 3, 4$*).*

(ii) For each $t \in \Omega_x$, $Q(x) = Q(x)^t$.

(iii) $\Delta_1(x), \Delta_2^1(x), \Delta_2^2(x), \Delta_3^i(x)$ *and* $\Delta_4^i(x)$ $(i = 1, ..., 6)$ *are all distinct* G_x*-orbits.*

(iv) If $t \in \Omega_x$ *and* $y \in \Gamma_0^t$*, then* $[G_{xy} : G_{xy}^t] \leq 24$.

Proof. Part (i) follows from (2.2) and (ii) holds because $Q(x)$ centralizes all transpositions t in Ω_x. Since G_x acts transitively on the 24 transpositions in Ω_x and , by [12], $\Delta_1(x)^t, \Delta_2^1(x)^t, \Delta_2^2(x)^t, \Delta_3^i(x)^t, \Delta_4^i(x)^t$ are all G_x^t-orbits (of differing sizes) we infer that (iii) holds. Because $|\Omega_x| = 24$ the G_{xy}-orbit of t can have size at most 24, whence we have (iv). □

3 The point and hyperplane residues

Recall that we shall employ the same notational conventions as in [15] for the subscripts of α. Suppose that $x \in \Gamma_0, \ell \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. Hence by (2.1) we may identify ℓ with an octad of Ω_x and X with a sextet of Ω_x. So, for example, $\alpha_{4^2}(x, X)$ denotes the set of octads (lines) which cuts the sextet X in 4^2, and $\alpha_2(x, \ell)$ is the set of octads (lines) which intersects the octad ℓ in two elements. Also we define $\beta_0(x, X), \beta_1(x, X), \beta_3(x, X)$ to be the set of sextets of Ω_x (not equal to X) which have, respectively, exactly 0, 1 and 3 octads which are also incident with X. Additionally we define the following subsets of $\Gamma_3(x)$:-

$$\delta_1(x, \ell) = \{Y \in \Gamma_3(x) \mid \ell \in \alpha_{4^2}(x, Y)\}$$

$$\delta_2(x, \ell) = \{Y \in \Gamma_3(x) \mid \ell \in \alpha_{2^4}(x, Y)\}$$

$$\delta_3(x, \ell) = \{Y \in \Gamma_3(x) \mid \ell \in \alpha_{1^5 3}(x, Y)\}.$$

Lemma 3.1. *Let* $x \in \Gamma_0$, $\ell \in \Gamma_1(x)$ *and* $X, \in \Gamma_3(x)$.

(i) The $G_{x\ell}$*-orbits on* $\Gamma_1(x)$ *are* $\{\ell\}$, $\alpha_0(x, \ell)$, $\alpha_2(x, \ell)$ *and* $\alpha_4(x, \ell)$ *where* $|\alpha_0(x, \ell)| = 30$, $|\alpha_2(x, \ell)| = 448$ *and* $|\alpha_4(x, \ell)| = 280$

(ii) *The* $G_{x\ell}$*-orbits on* $\Gamma_3(x)$ *are* $\delta_1(x,\ell)$, $\delta_2(x,\ell)$ *and* $\delta_3(x,\ell)$ *where* $|\delta_1(x,\ell)| = 35$, $|\delta_2(x,\ell)| = 840$ *and* $|\delta_3(x,\ell)| = 896$

Proof. See [3] or [4]. □

Lemma 3.2. *Let* $x \in \Gamma_0$ *and* $X \in \Gamma_3(x)$ *(so in* Γ_x*,* X *may be identified with a sextet in* Ω_x*). Then the orbits of* G_{xX} *on* $\Gamma_1(x)$ *(the octads of* Ω_x*) are* $\alpha_{4^2}(x,X), \alpha_{1^53}(x,X)$ *and* $\alpha_{2^4}(x,X)$*. Moreover* $|\alpha_{4^2}(x,X)| = 15, |\alpha_{1^53}(x,X)| = 384$ *and* $|\alpha_{2^4}(x,X)| = 360$.

Proof. Since $G_{xX}^{*x} \sim 2^6 3S_6$, the stabilizer of the sextet X, this follows from [3]. □

Lemma 3.3. *For* $x \in \Gamma_0$ *and* $X \in \Gamma_3(x)$*, the* $G_{xX}-$*orbits on* $\Gamma_3(x)$ *are* $\{X\}, \beta_0(x,X), \beta_1(x,X)$ *and* $\beta_3(x,X)$*. Further* $|\beta_0(x,X)| = 1440, |\beta_1(x,X)| = 240$ *and* $|\beta_3(x,X)| = 90$.

Proof. See [3]. □

Lemma 3.4. *Let* $x \in \Gamma_0$ *and* $X, Y \in \Gamma_3(x)$.

(i) *Suppose* $Y \in \beta_3(x,X)$*. Of the fifteen octads in* Ω_x *incident with* X*, three are in* $\alpha_{4^2}(x,Y)$ *and twelve are in* $\alpha_{2^4}(x,Y)$.

(ii) *Suppose* $Y \in \beta_1(x,X)$*. Of the fifteen octads in* Ω_x *incident with* X*, one is in* $\alpha_{4^2}(x,Y)$*, six are in* $\alpha_{2^4}(x,Y)$ *and eight are in* $\alpha_{1^53}(x,Y)$.

(iii) *Suppose* $Y \in \beta_0(x,X)$*. Of the fifteen octads in* Ω_x *incident with* X*, seven are in* $\alpha_{2^4}(x,Y)$ *and eight are in* $\alpha_{1^53}(x,Y)$.

Proof. Since G_x is transitive on $\Gamma_3(x)$ we may suppose X is the standard sextet. Then, in view of Lemma 3.2, for parts (i) (ii) and (iii) respectively we may take

$$Y = \begin{array}{|cc|cc|cc|} \hline \times & \times & - & - & * & * \\ \times & \times & - & - & * & * \\ \circ & \circ & + & + & \Box & \Box \\ \circ & \circ & + & + & \Box & \Box \\ \hline \end{array}, \quad Y = \begin{array}{|cc|cc|cc|} \hline \times & \circ & - & - & - & - \\ \circ & \times & + & + & + & + \\ \circ & \times & * & * & * & * \\ \circ & \times & \Box & \Box & \Box & \Box \\ \hline \end{array}$$

$$\text{and} \quad Y = \begin{array}{|cc|cc|cc|} \hline \times & \times & \times & - & * & + \\ \times & - & - & - & \circ & \Box \\ \circ & + & * & \Box & * & \circ \\ \Box & * & + & \circ & \Box & + \\ \hline \end{array}.$$

It is now straightforward to check the result. $\Box$

Lemma 3.5. *Let $x \in \Gamma_0, m \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. If $m \notin \Gamma_1(X)$, then there exists $Y \in \beta_3(x, X) \cup \{X\}$ such that $m \in \alpha_{1^5 3}(x, Y)$.*

Proof. Since $m \notin \Gamma_1(X), m \notin \alpha_{4^2}(x, X)$. Hence, by Lemma 3.2, $m \in \alpha_{1^5 3}(x, X) \cup \alpha_{2^4}(x, X)$. If $m \in \alpha_{1^5 3}(x, X)$, then we let $Y = X$. So now we assume that $m \in \alpha_{2^4}(x, X)$. Let t_1 and t_2 be tetrads of X such that $|t_1 \cap m| = 2$ and $|t_2 \cap m| = 0$. Now choose a tetrad t_3 such that $|t_3 \cap t_1| = |t_3 \cap t_2| = 2$ and $|t_3 \cap m| = 1$. Letting Y be the unique sextet containing t_3, we have $Y \in \beta_3(x, X)$ and $m \in \alpha_{1^5 3}(x, Y)$, so proving the lemma. $\Box$

The balance of this section considers the hyperplane residue of Γ. Set $H = U_4(3).2(\cong G_X^{*X}$ where $X \in \Gamma_3)$. We consider H as a subgroup of $U_6(2)$, and let V denote the 6-dimensional $GF(4)$ unitary module. Now there are 693 isotropic 1-subspaces of V (see [2]) and H has two orbits on these 1-spaces, say, $\mathcal{P}$ and $\mathcal{Q}$ with $|\mathcal{P}| = 567$ and $|\mathcal{Q}| = 126$. Of the 6237 isotropic 2-subspaces of V, 2835 of them have three 1-subspaces in $\mathcal{P}$ and two 1-subspaces in $\mathcal{Q}$ – denote this set by $\mathcal{L}$. Among the 891 isotropic 3-subspaces, 567 contain exactly one 1-subspace in $\mathcal{Q}$; call this set $\mathcal{R}$. We define a geometry $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ where $\Lambda_0 = \mathcal{P}, \Lambda_1 = \mathcal{L}$ and $\Lambda_2 = \mathcal{R}$ where incidence is symmetrized inclusion. This geometry is an example of a GAB (see [6]) and we have

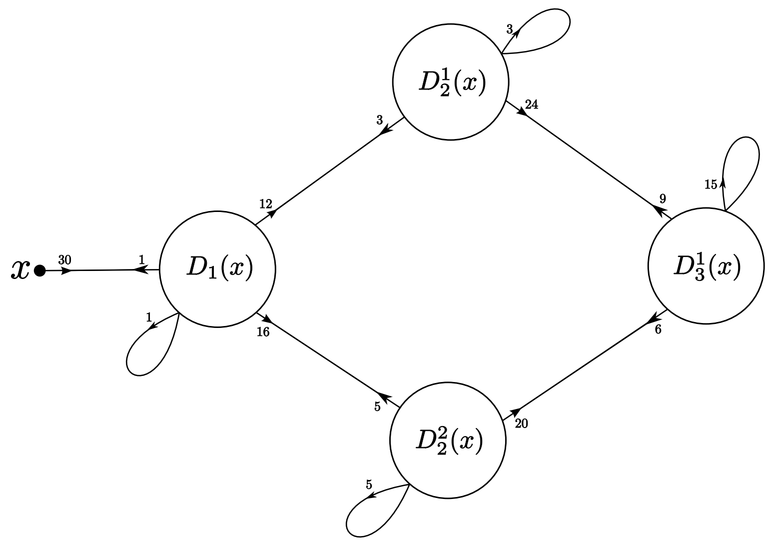
Lemma 3.6. *For $X \in \Gamma_3$, Γ_X is isomorphic to Λ.*

Our next result lists some properties of Λ we shall require later on.

Lemma 3.7. *Let $x \in \Lambda_0$.*

(i) The H_x-orbits on Λ_0 are $D_1(x), D_2^1(x), D_2^2(x)$ and $D_3^1(x)$ where $|D_1(x)| = 30$, $|D_2^1(x)| = 120$, $|D_2^2(x)| = 96$ and $|D_3^1(x)| = 320$.

(ii) The point-line collinearity graph of Λ is as follows



(iii) We have $H_x \sim 2^4 S_6$ with $O_2(H) \cong 2^4$.

(iv) If $g \in O_2(H_x)$, $g \neq 1$, then g interchanges $\Lambda_0(l) \backslash \{x\}$ for 8 lines l incident with x and fixes $\Lambda_0(l)$ for the other 7 lines incident with x.

Proof. See either [6], [8] or Section 3 of [17]. $\square$

4 Involutions

In this section we explore the combinatorial relationship between G and the residue geometries as it relates to the action of G on Γ.

Lemma 4.1. *Let $x \in \Gamma_0$ and $X \in \Gamma_3(x)$. Then*

(i) $Q(x) \cap Q(X) \cong 2^7$ *and* $Q(x)^{*X}(\cong 2^4) \trianglelefteq G_{xX}^{*X} \sim 2^4 S_6$*; and*

(ii) $Z_1(x) = 1$.

Proof. First we note that $Q(x) \nleqslant Q(X)$. For $Q(x) \leqslant Q(X)$ gives, by (2.1)(v), $Q(x) \leqslant O_2(Q(X)) \cong 2_+^{1+12}$. Since $Q(x)$ is elementary abelian of order 2^{11}, this is impossible. So $1 \neq Q(x)^{*X} \trianglelefteq G_{xX}^{*X} \cong 2^4 S_6$, using Lemma 3.7(iii). Since the 2^4 is an irreducible S_6-module we must have $Q(x)^{*X} \cong 2^4$. Hence $Q(x) \cap Q(X) \cong 2^7$ and part (i) holds.

Since $Q(x)$ is an irreducible G_x-module and $Z_1(x) \trianglelefteq G_x$, either $Z_1(x) = 1$ or $Z_1(x) = Q(x)$. If $Z_1(x) = Q(x)$, then $Z_1(x)^{*X} = O_2(G_{xX}^{*X})$ by part (i). However, from Lemma 3.7(iv), every non-trivial element of $O_2(G_{xX}^{*X})$ moves some point in $\Gamma_X \cap \Delta_1(x)$ whereas $Z_1(x)$ fixes all points in $\Delta_1(x)$ by definition, a contradiction. Thus $Z_1(x) = 1$. □

For $X \in \Gamma_3$, we use $\tau(X)$ to denote the involution in $Z(G_X)$; recall that $|Z(G_X)| = 2$ by (2.1)(v). Now let $x \in \Gamma_0(X)$. In Γ_x we may identify X with a sextet (of Ω_x) whose tetrads are $T_1, ..., T_6$, and we have, for each $i \in \{1, ..., 6\}$,

$$\tau(X) = \prod_{t \in T_i} t$$

(We note that $\tau(X)$ is a tetra-transposition in the language of [2;p207].) Also observe, as $C_G(\tau(X)) = G_X$, for $X, Y \in \Gamma_3$, $\tau(X) = \tau(Y)$ if and only if $X = Y$.

Let $x \in \Gamma_0$. In Ω_x consider a duad (that is, a 2-element subset), say $D = \{t_1, t_2\}$. Then $\delta(D) = t_1 t_2$ is referred to as a bi-transposition in [2]. Every involution in G is conjugate in G to either $\tau(X)$ or $\delta(D)$.

Lemma 4.2. *Let $x \in \Gamma_0$, $X \in \Gamma_3(x)$ and D be a duad of Ω_x. Then*

(i) $\tau(X), \delta(D) \in Q(x)$;

(ii) $C_G(\tau(X)) \sim 2^{1+12}3U_4(3)2$, $C_G(\delta(D)) \sim 2{\cdot}F_{22} : 2$ *and*

(iii) $Q(x) \backslash \{1\} = \tau(X)^{G_x} \cup \delta(D)^{G_x}$ *with* $|\tau(X)^{G_x}| = 1771$ *and* $|\delta(D)^{G_x}| = 276$.

Proof. The definitions of $\tau(X), \delta(D)$ and (2.1)(iv),(v) give part (i). For part (ii) see [2]. Part (iii) follows from the definition of $\tau(X), \delta(D)$ and properties of the Golay co-code. □

Our next lemma concerns sextet lines whose definition we recall. For $x \in \Gamma_0$, let $X_1, X_2, X_3 \in \Gamma_3(x)$, if for all $i, j, 1 \leq i < j \leq 3$ we have $X_i \in \beta_3(x, X_j)$, then $\{X_1, X_2, X_3\}$ is called a sextet line of Ω_x.

Lemma 4.3. *Suppose that $x \in \Gamma_0$ and $\{X_1, X_2, X_3\}$ is a sextet line of Ω_x. Then $\tau(X_1)\tau(X_2) = \tau(X_3)$.*

Proof. Since, for $X \in \Gamma_3$,

$$\tau(X) = \prod_{t \in T} t$$

for any tetrad T of X, the lemma follows immediately. □

Lemma 4.4. *Let $x \in \Gamma_0$, $l \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. Then $\tau(X)$ interchanges the points in $\Gamma_0(l) \backslash \{x\}$ if and only if $l \in \alpha_{1^5 3}(x, X)$.*

Proof. Since G_x is transitive on $\Gamma_3(x)$ we may in Γ_x, without loss of generality, suppose X is the standard sextet. Now let Y be the sextet

$$\begin{array}{|cc|cc|cc|}
\hline
\times & \circ & - & - & - & - \\
\circ & \times & + & + & + & + \\
\circ & \times & * & * & * & * \\
\circ & \times & \square & \square & \square & \square \\
\hline
\end{array}.$$

By Lemma 3.4(ii), of the 15 octads incident with Y, one is in $\alpha_{4^2}(x, X)$, eight are in $\alpha_{1^53}(x, X)$ and six are in $\alpha_{2^4}(x, X)$. Since $\tau(X) \in Z(G_{xX})$, if $\tau(X)$ fixes $\Gamma_0(l)$ (point-wise) for some $l \in \alpha_{4^2}(x, X)$ (respectively $\alpha_{1^53}(x, X)$, $\alpha_{2^4}(x, X)$), then, by Lemma 3.2 $\tau(X)$ fixes $\Gamma_0(l)$ (point-wise)) for all $l \in \alpha_{4^2}(x, X)$ (respectively $\alpha_{1^53}(x, X)$, $\alpha_{2^4}(x, X)$). Because G_x is transitive on $\Gamma_3(x)$ and, by Lemma 4.1(ii), $Z_1(x) = 1$, $\tau(X)^{*Y} \neq 1$. So, by Lemmas 4.1(i) and 4.2(i), $1 \neq \tau(X)^{*Y} \in Q(x)^{*Y} = O_2(G_{xY}^{*Y})$. Then $\tau(X)^{*Y}$ (and $\tau(X)$) fixes $\Gamma_0(l)$ (point-wise) for exactly 7 of the lines $l \in \Gamma_1(x, Y)$ by Lemma 3.7(iv). Therefore $\tau(X)$ interchanges the points in $\Gamma_0(l)\backslash\{x\}$ only when $l \in \alpha_{1^53}(x, X)$. $\square$

Lemma 4.5. *Let $x \in \Gamma_0$, $l \in \Gamma_1(x)$ and D be a duad in Ω_x. Then $\delta(D)$ interchanges the points in $\Gamma_0(l)\backslash\{x\}$ if and only if $l \in \alpha_1(x, D)$.*

Proof. INSERT DOESN'T LOOK LIKE WE NEED $\delta(D)$. $\square$

Lemma 4.6. *Let $x \in \Gamma_0$ and $X, Y \in \Gamma_3(x)$ with $X \neq Y$. Then $Y \in \beta_3(x, X)$ if and only if $\tau(Y) \in Q(X)$.*

Proof. If $Y \in \beta_0(x, X) \cup \beta_1(x, X)$, then there exists $l \in \alpha_{1^53}(x, Y)$ by consulting the MOG in [4], and so, by Lemma 4.4, $\tau(Y)$ does not fix $\Gamma_0(l)$ point-wise. Therefore $\tau(Y) \notin Q(X)$. While if $Y \in \beta_3(x, X)$, then $\Gamma_1(x, X) \subseteq \alpha_{4^2}(x, Y) \cup \alpha_{2^4}(x, Y)$ and hence $\tau(Y)$ fixes $\Gamma_0(l)$ point-wise for all $l \in \Gamma_1(x, X)$ by Lemma 4.4. Since, by Lemmas 4.1(i) and 4.2(i), $\tau(Y)^{*X} \in Q(x)^{*X} = O_2(G_{xX}^{*X})$, Lemma 3.7(iv) implies $\tau(Y)^{*X} = 1$. So $\tau(Y) \in Q(X)$ as desired. $\square$

Lemma 4.7. *Let x, y, z be distinct points of Γ_0 such that $\{x, y, z\}$ is a triangle in $\mathcal{G}$. Then $z \in \Gamma_0(x+y)$ (or, in other words, $\{x, y, z\} = \Gamma_0(l)$ for some $l \in \Gamma_1$).*

Proof. We have that $\Omega_x \cap \Omega_y$ and $\Omega_z \cap \Omega_y$ are octads in Ω_y. Let $t \in \Omega_x \cap \Omega_z$. Then t centralizes the transpositions in $\Omega_x \cap \Omega_y$ and $\Omega_z \cap \Omega_y$ and so either $\Omega_x \cap \Omega_y = \Omega_z \cap \Omega_y$ or $t \in \Omega_y$. In either case we get $\Omega_x \cap \Omega_y = \Omega_y \cap \Omega_z = \Omega_x \cap \Omega_z$. □

Lemma 4.8. *(i) $|\Delta_1(a)| = 1518 = 2.3.11.23$;*

(ii) $\Delta_1(a)$ is a G_a-orbit; and

(iii) if $x \in \Delta_1(a)$, then $G_{ax} \sim 2^{10}2^4A_8$ (with $G_{ax}^{*x} = G_{xx+a}^{*x}$, an octad stabilizer).

Proof. (i) Since $|\Gamma_0(l) \backslash \{a\}| = 2$ for any $l \in \Gamma_1(x)$, $|\Delta_1(a)| = 2|\Gamma_1(a)| = 1518$.

(ii) For $l \in \Gamma_1(a)$ we can find $X \in \Gamma_3(a)$ such that $l \in \alpha_{1^53}(a, X)$. Hence by Lemma 4.4 , $Q(a)$ is transitive on $\Gamma_0(l)\backslash\{a\}$. Since G_a is transitive on $\Gamma_1(a)$, (ii) holds.

(iii) We have $G_{ax} \leqslant G_{xx+a}$ because $x + a$ is the unique line in $\Gamma_1(a, x)$ and $[G_{xx+a} : G_{ax}] \leq 2$ as $|\Gamma_0(x+a)\backslash\{x\}| = 2$. Hence as $Q(a)$ is transitive on $\Gamma_0(x + a)\backslash\{x\}$ we obtain (iii). □

Combining Lemma 4.8 and $\mathbf{O}(2.1)$ with the definitions of $\Delta_2^1(a)$, $\Delta_2^2(a)$ and $\Delta_2^3(a)$ given in (2.2) we obtain Theorem 2.

Lemma 4.9. *Let $y \in \Delta_1(x)$ where $x \in \Gamma_0$. Then*

(i) $|Q(x) \cap Q(y)| = 2^6$; and

(ii) for $X \in \Gamma_3(x)$, $\tau(X) \in Q(y)$ if and only if $X \in \Gamma_3(y)$.

Proof. Since $O_2(G_{xy}^{*y})$ is an irreducible 4-dimensional A_8-module over $GF(2)$, $Q(x)_y^{*y} = 1$ or $O_2(G_{xy})^{*y}$. Suppose $Q(x)_y^{*y} = 1$ and so $Q(x)_y = Q(x) \cap Q(y)$.

Let $X \in \Gamma_3(x)$ with $x+y \in \alpha_{2^4}(x,X)$. Then $\tau(X) \in Q(x)_y \subseteq Q(y)$. Therefore

$$|Q(y)| \geq 1771 + 840 = 2611$$

by Lemma 3.1(ii). This contradicts $|Q(y)| = 2^{11}$ from (2.1)(iv). So $|Q(x)_y^{*b}| = 2^4$ and then part (i) follows from Lemma 4.8(iii). For part (ii), if $X \in \Gamma_3(y)$ then $\tau(X) \in Q(y)$ by Lemma 4.2(i). Suppose that $X \notin \Gamma_3(y)$ and $\tau(X) \in Q(y)$. Since $x+y \notin \Gamma_1(X)$, we then have $x+y \in \alpha_{2^4}(x,X) \cup \alpha_{1^5 3}(x,X)$. Suppose that $x+y \in \alpha_{2^4}(x,X)$. Since G_{xy}^{*x} is transitive on the set of hyperplanes $\delta_2(x,x+y) = \{Y \in \Gamma_3(x) | x+y \in \alpha_{2^4}(x,Y)\}$ by Lemma 3.1(ii) and $\tau(X) \in Q(y)$ we have $\tau(Y) \in Q(y)$ for all $Y \in \delta_2(x,x+y)$. Then

$$|Q(x) \cap Q(y)| \geq 35 + 840 = 875.$$

This contradicts part (i). By a similar argument, if $x+y \in \alpha_{1^5 3}(x,X)$ we get

$$|Q(x) \cap Q(y)| \geq 35 + 896 = 933,$$

again giving a contradiction. This proves part (ii). $\square$

5 The Second Disc of a

We begin by defining certain subsets of $\Delta_2(a)$ as follows.

$$\widetilde{\Delta}_2^1(a) = \{x \in \Delta_2(a) \,|\, \Gamma_3(a,x) \neq \emptyset = \Gamma_2(a,x)\}$$

$$\widetilde{\Delta}_2^2(a) = \{x \in \Delta_2(a) \,|\, \Gamma_3(a,x) = \emptyset\}$$

$$\widetilde{\Delta}_2^3(a) = \{x \in \Delta_2(a) \,|\, \Gamma_3(a,x) \neq \emptyset \neq \Gamma_2(a,x)\}.$$

An immediate consequence of these definitions is

Lemma 5.1. *For* $1 \leq j < k \leq 3$, $\widetilde{\Delta}_2^j(a) \cap \widetilde{\Delta}_2^k(a) = \emptyset$ *and* $\bigcup_{i=1}^3 \widetilde{\Delta}_2^i(a) =$

$\Delta_2(a)$.

Lemma 5.2. *Suppose $x \in \Delta_2(a)$ with $X \in \Gamma_3(a,x)$. Then $\{a,x\}^\perp \subseteq \Gamma_0(X)$.*

Proof. Let $b \in \{a,x\}^\perp$ and assume that $b \notin \Gamma_0(X)$. Then $a+b \notin \Gamma_1(X)$ as Γ is a string geometry. Using Lemma 3.5, we can find $Y \in \beta_3(a,X) \cup \{X\}$ for which $a+b \in \alpha_{1^5 3}(a,Y)$. By Lemma 4.6, $\tau(Y) \in Q(X)$ which implies that $\tau(Y) \in Q(a)_x$. Since $\tau(Y)$ does not fix b by Lemma 4.4 we get a triangle $\{x, b, b^{\tau(Y)}\}$ which then forces $a = x$ by Lemma 4.7. From this contradiction we infer that $b \in \Gamma_0(X)$, so proving the lemma. $\square$

Lemma 5.3. *For $i = 1,2,3$, $\widetilde{\Delta}_2^i(a) = \Delta_2^i(a)$.*

Proof. Let $b \in \{a,x\}^\perp$. Using MOG information in Ω_b, Lemma 5.2 implies that $\widetilde{\Delta}_2^i(a) = \Delta_2^i(a)$ for $i = 1,2,3$. $\square$

Lemma 5.4. *Let $x \in \Delta_2^1(a)$. Then there is a unique hyperplane in $\Gamma_3(a,x)$.*

Proof. Let $X, Y \in \Gamma_3(a,x)$ and $b \in \{a,x\}^\perp$. Then $b \in \Gamma_0(X) \cap \Gamma_0(Y)$ by Lemma 5.2. If $X \neq Y$, then $b+x, b+a \in \Gamma_0(X) \cap \Gamma_0(Y)$ and $\Gamma_2(b+x, b+a) \neq \emptyset$ by considering MOG information in Γ_b. Hence $x \notin \widetilde{\Delta}_2^1(x)$, whereas $\widetilde{\Delta}_2^1(x) = \Delta_2^1(x)$ by Lemma 5.3. Thus we conclude $X = Y$ and the lemma is proved. $\square$

Let the unique hyperplane in Lemma 5.4 be denoted by $X(a,x)$ (respectively $X(x,a)$) if we regard $X(a,x) \in \Gamma_3(a)$ (respectively $X(x,a) \in \Gamma_3(x)$). Of course $X(a,x) = X(x,a)$.

Lemma 5.5. *Let $x \in \Delta_2^1(a)$. Then $|\{a,x\}^\perp| = 5$ and, for each $b \in \{a,x\}^\perp$, the octad $a+b$ in Ω_a contains a fixed tetrad of the sextet $X(a,x)$.*

Proof. By Lemma 5.2, for every $b \in \{a,x\}^\perp$, $b \in \Gamma_0(X(a,x))$ and so $a+b \in \Gamma_1(X(a,x))$. Working in the residue geometry of $X(a,x)$ and using Lemma 3.7(ii) we get $|\{a,x\}^\perp| = 5$. Since $\Gamma_2(a,x) = \emptyset$ by Lemma 5.3, in Ω_a, the five octads $\{a+b | b \in \{a,x\}^\perp\}$ must intersect in the same tetrad of the sextet $X(a,x)$. $\square$

Note that $x \in \Delta_2^1(a)$ implies $a \in \Delta_2^1(x)$. We denote the fixed tetrad in Ω_a (respectively Ω_x) described in Lemma 5.5 by $t(a,x)$ (respectively $t(x,a)$).

Lemma 5.6. *(i)* $|\Delta_2^1(a)| = 2^5.3.7.11.23$.

(ii) $\Delta_2^1(a)$ *is a* G_a*-orbit.*

(iii) For $x \in \Delta_2^1(a)$ *and* $G_{ax}^{*x} \sim 2^6(3 \times S_5)$ *is the stabilizer in* G_x^{*x} *of* $X(x,a)$ *and* $t(x,a)$ *and* $|Q(x)_a| = 2^7$.

Proof. By Lemma 3.7(i), for any $X \in \Gamma_3(a)$, $|\Gamma_0(X) \cap \Delta_2^1(a)| = 96$ and so by Lemma 5.4 we get $|\Delta_2^1| = 96.|\Gamma_3(a)| = 2^5.3.7.11.23$, proving part (i).

For part (ii), let $b \in \Delta_1(a)$ and $x \in \Delta_2^1(a) \cap \Delta_1(b)$. Then in Ω_b, $b + a \in \alpha_4(b, b+x)$. Since $\alpha_4(b, b+x)$ is a G_{ab}^{*b}-orbit it is enough to show that there exists $g \in G_{ab}$ with $x^g = x'$ where $\Gamma_0(b+x) = \{b, x, x'\}$. In Ω_b we can choose a sextet Y incident with the octad $b+a$ such that $b + x \in \alpha_{1^5 3}(x, Y)$. Then by Lemma 4.4, $\tau(Y) \in (Q(a) \cap Q(b)) \backslash G_x$ and so $\tau(Y)$ is the required element of G_{ab}.

For $t \in \Omega_a \cap \Omega_b \cap \Omega_x$, $a, x \in V(\mathcal{G}^t)$ with $x \in \Delta_2^1(a)^t$. Hence $Q(x)_a = Q(x)_a^t \cong 2^7$ by Theorem 3 of [12]. Since, by parts (i) and (ii), $|G_{ax}| = 2^{16}.3^2.5$, Lemmas 5.4 and 5.5 yield part (iii).

□

We now turn to $\Delta_2^2(a)$.

Lemma 5.7. *Let* $x \in \Delta_2^2(a)$ *and* $b \in \{a,x\}^\perp$. *Then*

(i) $|\Delta_1(b) \cap \Delta_2^2(a)| = 2^7.7$ *with* G_{ab} *transitive on* $\Delta_1(b) \cap \Delta_2^2(a)$*; and*

(ii) $|\{a,x\}^\perp| = 1$ *or 2.*

Proof. Since $\Gamma_3(a,x) = \emptyset$ by Lemma 5.3, we have $b + a \in \alpha_2(b, b+x)$. So by Lemma3.1(i), $|\Delta_1(b) \cap \Delta_2^2(a)| = 2 \times 448 = 2^7.7$. Let $x' \in \Gamma_0(b+x) \backslash \{b, x\}$. We can choose $Y \in \Gamma_3(b+a)$ with $b + x \in \alpha_{1^5 3}(b, Y)$. By Lemma 4.4 $x^{\tau(Y)} = x'$

and so G_{ab} is transitive on $\Delta_1(b) \cap \Delta_2^2(a)$ because $\alpha_2(b, b+x)$ is a G_{ab}-orbit by Lemma 3.1(i).

Using (i), [2] and the fact that $G_{ab} \sim 2^{14}A_8$ by Lemma 4.8(iii) we must have $G_{abx} \sim 2^9 S_6$ or $2^{10}A_6$. In either case G_{abx}^{*a} is contained in the stabilizer in Ω_a of a duad δ contained in the octad $a+b$. We now show that for every $c \in \{a, x\}^{\perp}$, the octad $a+c$ in Ω_a contains δ. Assume, for a contradiction that for some $c \in \{a, x\}^{\perp}$, $a+c$ does not contain δ. Since $\Gamma_3(a, x) = \emptyset$, we must have $a+c \in \alpha_2(a, a+b)$. Using MOG information there are exactly 15 sextets in $\Gamma_3(a, b)$ that each have a tetrad containing δ. Let T denote this set of 15 sextets. We can take $Y_1, Y_2, Y_3 \in T$ forming a sextet line. Since $\tau(Y_1)\tau(Y_2) = \tau(Y_3)$ by Lemma 4.3 we must have $\tau(Y_i) \in G_x$ for each $i = 1, 2, 3$. Since G_{abx} is transitive on T we must have $\tau(Y) \in G_x$ for each $Y \in T$ Since $a+c$ does not contain δ we must have $a+c \in \alpha_{1^53}(Y)$ for some $Y \in T$ and then $x^{\tau(Y)} \neq x$. Lemma 4.6 now implies that $a = x$, a contradiction. Part (ii) follows because we cannot find three octads in Ω_a, intersecting pairwise in exactly δ. $\square$

Lemma 5.8. *Let $x \in \Delta_2^2(a)$. Then*

(i) $\Delta_2^2(a)$ *is a* G_a*-orbit;*

(ii) $|\{a, x\}^{\perp}| = 1$, $|\Delta_2^2(a)| = 2^8.3.7.11.23$ *and* G_{ab} *is transitive on* $\Delta_1(b) \cap \Delta_2^2(a)$ *where* $\{a, x\}^{\perp} = \{b\}$*; and*

(iii) $G_{ax}^{*x} \sim 2^4 S_6$ *is the stabilizer in* Ω_x *of the octad* $x+b$ *and the duad* $\Omega_a \cap \Omega_b \cap \Omega_x$ *where* $\{a, x\}^{\perp} = \{b\}$.

Proof. Part (i) follows from Lemma 5.7(i) and the fact that $\Delta_1(a)$ is a G_a-orbit.

Suppose that $|\{a, x\}^{\perp}| \neq 1$. Then $\{a, x\}^{\perp} = \{b, c\}$ with $b \neq c$ by Lemma 5.7(ii). Lemma 4.6 rules out $d(b, c) = 1$. If $c \in \Delta_2^1(b) \cup \Delta_2^3(b)$ $(= \widetilde{\Delta}_2^1(b) \cup \widetilde{\Delta}_2^3(b))$, then $b, c \in \Gamma_0(X)$ for some $X \in \Gamma_3$ whence, by Lemma 5.2, $a, x \in$

$\Gamma_0(X)$. However $\Gamma_3(a,x) = \emptyset$, and therefore $x \in \Delta_2^2(b)$. Hence $a + c \in \alpha_2(a, a+b)$. From Theorem 4 of [12] $Q(a)_x \cong 2^5$, as $G_{abx} \sim 2^9 S_6$ or $2^{10} A_6$, $G^{*a}_{abx} \sim 2^4 S_6$ or $2^5 A_6$. In particular $2^8 \big| |G^{*a}_{abx}|$. Clearly $G_{abx} = G_{abxc}$ and so $G^{*a}_{abxc} = G^{*a}_{abx}$. Since $a+c \in \alpha_2(a, a+b)$, G^{*a}_{abxc} leaves a dodecad of Ω_a invariant whence G^{*a}_{abxc} is isomorphic to a subgroup of M_{12}. But $2^8 \big| |G^{*a}_{abxc}|$ yields a contradiction. Thus we conclude that $|\{a,x\}^\perp| = 1$, and consequently for $b \in \{a,x\}^\perp$

$$|\Delta_2^2(a)| = \frac{|\Delta_1(b) \cap \Delta_2^2(a)||\Delta_1(a)|}{|\{a,x\}^\perp|} = 2^8.3.7.11.13.$$

Part (iii), using $Q(x)_a \cong 2^5$, follows readily.

□

Lemma 5.9. *Let $x \in \Delta_2^3(a)$. Then there is a unique element $\Lambda(a,x) \in \Gamma_2(a,x)$ and for every $b \in \{a,x\}^\perp$, $b \in \Gamma_0(\Lambda(a,x))$.*

Proof. By definition, $\Gamma_2(a,x) \neq \emptyset$. Let $b \in \{a,x\}^\perp$ with $b + a \in \alpha_0(b, b+x)$ and let $\Lambda(a,x)$ be the unique element of $\Gamma_2(b+a, b+x)$. Suppose $b' \in \{a,x\}^\perp$ with $b' \notin \Gamma_0(\Lambda(a,x))$. In Ω_b there are seven sextets X_i $(i = 1, ..., 7)$ in $\Gamma_3(b+a, b+x)$ and by Lemma 5.2 $b' \in \Gamma_0(X_i)$ for each $i = 1, ..., 7$. Therefore, in $\Omega_{b'}$ there exists a trio $\Lambda \in \Gamma_2(b'+a, b'+x, X_i)$ for each $i = 1, ..., 7$. Considering the situation in Ω_a we must have $\Lambda = \Lambda(a,x)$ and the lemma is proved.

□

We follow our earlier notational convention and also denote the unique plane in Lemma 5.9 by $\Lambda(x,a)$ if we are viewing $\Lambda(x,a)$ as a trio in Γ_x.

Lemma 5.10. *Let $x \in \Delta_2^3(a)$. Then $|\Gamma_3(a,x)| = 7$ and $|\{a,x\}^\perp| = 3$.*

Proof. By Lemma 5.2, for $X \in \Gamma_3$, $X \in \Gamma_3(a,x)$ if and only if $X \in \Gamma_3(\Lambda(a,x))$. The result now follows from Lemma 5.9 because in Γ_X there are three points collinear with a and x and in Γ_a, $|\Gamma_3(\Lambda(a,x))| = 7$.

□

Lemma 5.11. *Let $x \in \Delta_2^3(a)$. Then*

(i) $|\Delta_2^3(a)| = 2^3.3.5.11.23$*;*

(ii) $\Delta_2^3(a)$ *is a* G_a*-orbit;*

(iii) $G_{ax}^{*x} \sim 2^6(L_3(2) \times 3)$ *is a subgroup of index 2 of the stabilizer in* Ω_x *of the trio* $\Lambda(x,a)$ *and* $|Q(x)_a| = 2^9$.

Proof. Since $|\{a,x\}^\perp| = 3$ by Lemma 5.10, $|\alpha_0(b, b+a)| = 30$ ($b \in \{a,x\}^\perp$) and, by Lemma 4.8(i), $|\Delta_1(a)| = 2.3.11.23$, we calculate that $|\Delta_2^3(a)| = 2^3.3.5.11.23$.

For part (ii), let $b \in \Delta_1(a)$ with $\Lambda \in \Gamma_2(a,b)$ and $X \in \Gamma_3(\Lambda)$. Then $G_{aX\Lambda}^{*X} \sim 2^4(S_4 \times 2)$ and is transitive on the four points in $\Delta_2^3(a) \cap \Delta_1(b) \cap \Gamma_0(\Lambda)$. Then G_a is transitive on $\Delta_2^3(a)$ because $\Gamma_2(a)$ and $\Delta_1(a)$ are G_a-orbits.

By Lemma 5.10 $\{a,x\}^\perp = \{b_1, b_2, b_3\}$. Also, using Lemma 5.9, $G_{ax}^{*a} \leq G_{ax\Lambda(a,x)}^{*a} \sim 2^6(L_3(2) \times S_3)$. Let $1 \leq i < j \leq 3$. Then $a + b_i$ and $a + b_j$ are disjoint octads as they are both incident with the trio $\Lambda(a,x)$. Choose a tetrad δ of Ω_a which intersects $a+b_i$ in two elements and $a+b_j$ in one element, and let Y denote the sextet of Ω_a with δ a tetrad of Y. Then $a+b_i \in \alpha_{2^4}(a,Y)$ and $a+b_j \in \alpha_{1^5 3}(a,Y)$. Hence, by Lemma 4.4, $\tau(Y) \in Q(a)_{b_i} \setminus Q(a)_{b_j}$. Thus $Q(a)_{b_i} \neq Q(a)_{b_j}$ for $1 \leq i < j \leq 3$. Further $Q(a)_x \leq Q(a)_{b_i}$ ($1 \leq i \leq 3$), for $Q(a)_x \not\leq Q(a)_{b_i}$ yields that $|\{a,x\}^\perp \cap \Gamma_0(a+b_i)| = 2$ whereas no two points of $\{a,x\}^\perp$ are colinear. So, as $[Q(a) : Q(a)_{b_i}] = 2$ and $Q(a)_{b_i} \neq Q(a)_{b_j}$ for $i \neq j$, we have $[Q(a) : Q(a)_x] \geq 2^2$. Consequently using part (i) either $G_{ax}^{*a} \sim 2^6(L_3(2) \times 3)$ with $|Q(a)_x| = 2^9$ or $G_{ax}^{*x} \sim 2^6(L_3(2) \times S_3)$ with $|Q(a)_x| = 2^8$. Suppose the latter holds. Let ξ be the element of order 3 in the S_3 direct factor of G_{ax}^{*x}. Then, as ξ permutes the three octads $\{a + b_i \big| i = 1,2,3\}$ and $Q(a)_{b_i} \neq Q(a)_{b_j}$ ($i \neq j$), ξ must act non-trivially on $Q(a)/Q(a)_x$. But then λ centralizes $Q(a)/Q(a)_x$ where λ is an element of

G_{ax}^{*x} of order 7, a contradiction as $|C_{Q(a)}(\lambda)| = 2^2$. Thus, as $a \in \Delta_2^3(x)$, we obtain $G_{ax}^{*x} \sim 2^6(L_3(2) \times S_3)$ and $|Q(x)_a| = 2^9$, so proving (iii).

□

Lemma 5.6 combined with (2.2) proves Theorem 3 except for the octad orbits $\alpha_{2,2^4}(x, \Lambda_1, \Lambda_2)$, $\alpha_{1,31^5}(x, \Lambda_1, \Lambda_2)$ and $\alpha_{0,4^2}(x, \Lambda_1, \Lambda_2)$. The first two will be settled by Theorems 7 and 11 and the data in **O**(2.2), while the last one follows from Theorem 12(ii). Theorem 4, apart from the octad orbits $\alpha_{0,0}(x, \Lambda_1, \Lambda_2)$ and $\alpha_{4,0}(x, \Lambda_1, \Lambda_2)$, follows from Lemma 5.8 and (2.2). The remaining two orbits are dealt with by Theorem 12(i),(iv) and **O**(2.3). Finally Lemma 5.11 and (2.2) deliver Theorem 5.

6 Theorems 6-11 and 13

Lemma 6.1. *Suppose that $x \in \Gamma_0$ and that $\Omega_a \cap \Omega_x \neq \emptyset$. Let $t \in \Omega_a$ and let Δ denote the G_a-orbit of x. Set $k = |\{s \in \Omega_a | x \in \Gamma_0^s\}|$. Then*

$$k|\Delta| = 24|\Delta \cap \Gamma_0^t|.$$

Proof. Since Δ is a G_a-orbit and G_a acts transitively on Ω_a, $|\Delta \cap \Gamma_0^s|$ is the same for all $s \in \Omega_a$. Furthermore we also have that $|\{s \in \Omega_a | y \in \Gamma_0^s\}|$ is the same for all $y \in \Delta$. Because $\Omega_a \cap \Omega_x \neq \emptyset$ we note that $k \neq 0$. Now counting in two ways the number of elements in

$$|\{(s, y) \in \Omega_a \times \Delta | y \in \Gamma_0^s\}|$$

yields, as $|\Omega_a| = 24$, the
lemma. □

For $x \in \Gamma_0$ and $s \in \Omega_x$, G_x^s denotes the stabilizer of x in $G^s \cong Fi_{23}$. So $G_x^s \sim 2^{11} M_{23}$. Also recall that $Q(x)^s$ denotes the normal elementary abelian subgroup of G_x^s of order 2^{11}.

Lemma 6.2. *For $x \in \Gamma_0$ and $s \in \Omega_x$, $Q(x)^s = Q(x)$.*

Proof. Since $2^{11}M_{23} \sim G_x^s \leqslant G_x \sim 2^{11}M_{24}$, the subgroup structure of M_{24} forces $Q(x)^s = Q(x)$. $\square$

Lemma 6.3. *(i) If $x \in \Delta_2^1(a)$, then $|\{s \in \Omega_a | x \in \Gamma_0^s\}| = 4$.*

(ii) If $x \in \Delta_2^2(a)$, then $|\{s \in \Omega_a | x \in \Gamma_0^s\}| = 2$.

Proof. Let $x \in \Delta_2^1(a)$ and set $k = |\{s \in \Omega_a | x \in \Gamma_0^s\}|$. Observe that, for $t \in \Omega_a$, $\Delta_2^1(a) \cap \Gamma_0^t = \Delta_2^1(a)^t$. Since G_a is transitive on $\Delta_2^1(a)$, Lemmas 5.6(ii) and 6.1 imply that

$$k|\Delta_2^1(a)| = 24|\Delta_2^1(a)^t|,$$

where t is some fixed transposition in Ω_a. From Lemma 5.6(i) and Table 1 of [12], $|\Delta_2^1(a)| = 2^5.3.7.11.23$ and $|\Delta_2^1(a)^t| = 2^4.7.11.23$, and therefore $k = 4$.

A similar argument, using Lemma 5.8 instead of Lemma 5.6, establishes part (ii). $\square$

Lemma 6.4. *For $i = 1, ..., 6$, $\Delta_3^i(a)$ is a G_a-orbit and, for $t \in \Omega_a$, $\Delta_3^i(a) \cap \Gamma_0^t = \Delta_3^i(a)^t$.*

Proof. Let $x \in \Delta_2^1(a)$ and $t \in \{s \in \Omega_a | x \in \Gamma_0^s\} = \Omega_a \cap \Omega_x$. From Lemma 5.6 and Theorem 3 of [12], $|G_{ax}| = 2^{16}.3.5$ and $|G_{ax}^t| = 2^{14}.3.5$. So $[G_{ax} : G_{ax}^t] = 4$ and hence, by 6.3(i), G_{ax} is transitive on $\{s \in \Omega_a | x \in \Gamma_0^s\}$. Because G_{ax}^t is transitive on $\Delta_3^1(a)^t = \Delta_3^1(a) \cap \Gamma_0^t$, we conclude that $G_a$is transitive on $\Delta_3^1(a)$.

The remaining sets $\Delta_3^i(a)$ $(i = 2, ..., 6)$ are defined from $\Delta_2^2(a)$. Now similar arguments may be employed for these sets as $[G_{ax} : G_{ax}^t] = 2$ for $x \in \Delta_2^2(a)$ (where $t \in \{s \in \Omega_a | x \in \Gamma_0^s\}$) and, by 6.3(ii) $|\{s \in \Omega_a | x \in \Gamma_0^s\}| = 2$. $\square$

Theorem 6.5. *Let $x \in \Delta_3^i(a)$.*

(i) If $i = 1$, then $G_{ax} \sim 2^2L_3(4)S_3$, $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1\} \sim L_3(4)S_3$ where Λ_1 is a triad of Ω_x and $|\Delta_3^1(a)| = 2^{12}.11.23$.

(ii) *If* $i = 2$*, then* $G_{ax} \sim 2^4 2^3(L_3(2) \times 2)$*,* $G^{*x}_{ax} = Stab_{G^{*x}_x}\{\Lambda_1, \Lambda_2\} \sim 2^3(L_3(2) \times 2)$ *where* Λ_1 *is an octad,* Λ_2 *a duad of* Ω_x *with* $\Lambda_1 \cap \Lambda_2 = \emptyset$ *and* $|\Delta^2_3(a)| = 2^{10}.3^2.5.11.23$.

(iii) *If* $i = 3$*, then* $G_{ax} \sim 22^4 S_6$*,* $G^{*x}_{ax} = Stab_{G^{*x}_x}\{\Lambda_1, \Lambda_2\} \sim 2^4 S_6$ *where* Λ_1 *is an octad,* Λ_2 *a duad of* Ω_x *with* $\Lambda_2 \subseteq \Lambda_1$*, and* $|\Delta^3_3(a)| = 2^{12}.3.7.11.23$.

(iv) *If* $i = 4$*, then* $G_{ax} \sim 2M_{22}2$*,* $G^{*x}_{ax} = Stab_{G^{*x}_x}\{\Lambda_1\} \cong M_{22}2$ *where* Λ_1 *is a duad of* Ω_x *and* $|\Delta^4_3(a)| = 2^{12}.3.23$.

(v) *If* $i = 5$*, then* $G_{ax} \cong G^{*x}_{ax} = Stab_{G^{*x}_x}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4 A_5$ *where* Λ_1 *is an octad of* Ω_x*,* $|\Lambda_2| = |\Lambda_3| = 1$ *with* $\Lambda_2 \cup \Lambda_3 \subseteq \Lambda_1$*, and* $|\Delta^5_3(a)| = 2^{15}.3^2.7.11.23$.

(vi) *If* $i = 6$*, then* $G_{ax} \sim 2^6 3S_4$*,* $G^{*x}_{ax} = Stab_{G^{*x}_x}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4 3S_4$ *where* Λ_1 *is an octad of* Ω_x*,* $|\Lambda_2| = 4, |\Lambda_3| = 1$*,* $\Lambda_3 \subseteq \Lambda_2 \subseteq \Lambda_1$*, and* $|\Delta^6_3(a)| = 2^{12}.3.5.7.11.23$.

Proof. (i) Let $t \in \Omega_a$. From Lemma 6.4 $\Delta^1_3(a)$ is a G_a-orbit and $\Delta^1_3(a) \cap \Gamma^t_0 = \Delta^1_3(a)^t$. For $x \in \Delta^1_3(a)$, let $k = |\{s \in \Omega_a | x \in \Gamma^s_0\}|$. Using Lemma 6.1 we obtain

$$k|\Delta^1_3(a)| = 24|\Delta^1_3(a)^t|.$$

By the definition of $\Delta^1_3(a)$, there exists $y \in \Delta^1_2(a)$ such that $y + x \in \alpha_{3,31^5}(x, \Lambda_1, \Lambda_2)$. Now consulting Theorem 3, we see that $\Lambda_1 = \Omega_a \cap \Omega_y$, and hence $|\Omega_a \cap \Omega_x| \geq 3$. So $k \geq 3$. Therefore, as $|\Delta^1_3(a)^t| = 2^9.11.23$ by Table 1 of [12],

$$\begin{aligned} |\Delta^1_3(a)| &= \frac{24|\Delta^1_3(a)^t|}{k} = \frac{24.2^9.11.23}{k} \\ &\leq \frac{24.2^9.11.23}{3} = 2^{12}.11.2. \end{aligned}$$

Supposing that $x \in \Gamma^t_0$. Then $G_{ax} \sim 2^2 L_3(4)2$ by 5 of [12]. Since $\Delta^1_3(a)$ is a G_a-orbit, $|\Delta^1_3(a)|$ must divide $[G_a : G^t_{ax}] = 2^1 2.3.11.23$. Bearing in mind the possible overgroups of $L_3(4)2$ in $M_{24} \cong G_x^{*x}$ and Lemma 6.2, we get that $[G_{ax} : G_{ax}^t] = 3$. Thus $|\Delta_3^1(a)| = 2^{12}.11.23$ with $k = 3$ and $G_{ax} \sim 2^2L_3(4)S_3$ with $G_{ax}^{*x} = Stab_{G_x^{*x}}\{\Lambda_1\}$, Λ_1 being the triad $\{t\} \cup D(x,a)$. (With $D(a,x)$ as in Theorem 5 of [12].) This establishes (i).

Parts (ii)-(vi) may be proved in a similar fashion. For these cases we may extract $k = |\{s \in \Omega_a | x \in \Gamma_0^s\}|$ (for $x \in \Delta_3^i(x)$, $i = 2, ..., 6$) from [12]. Recall that in the Fi_{23} geometry, a hyperplane is just a transposition with points of this geometry being sets of 23 pairwise commuting transpositions. For $x \in \Delta_3^i(a)^t$, $t \in \Omega_a$ where $i \in \{2,3,4\}$, a and x are incident with a unique hyperplane of the Fi_{23} geometry (see Section 1 of [12]) - so for $i \in \{2,3,4\}$, $k = 2$. Whereas, for $x \in \Delta_3^i(a)^t$, $i \in \{5,6\}$, a and x are not incident with a common hyperplane of the Fi_{23} geometry. Thus $k = 1$ for $i \in \{5,6\}$. So knowing k we can make effective use of Lemma 6.1. We observe that for $x \in \Delta_3^i(a) \cap \Gamma_0^t$ ($t \in \Omega_a$) we have $G_{ax} = G_{ax}^t$ for $i = 3,5,6$. While $[G_{ax} : G_{ax}^t] = 2$ for $i = 2,3,4$. In these latter cases we must also call on the services of Lemma 6.2 in order to deduce that G_{ax} has shape, respectively, $2^4 2^3(L_3(2) \times 2)$, $2^2 2^4 S_6$ and $2M_{22}2$.

We are now in a position to verify Theorems 6-13. For Theorem 6, Theorem 6.5(i) gives G_{ax} and G_{ax}^{*x} for $x \in \Delta_3^1(a)$. We must discover the point distribution of the G_{ax} line orbits $\alpha_i(x,\Lambda_1)$ ($i = 1,2,3$), three of the G_{ax}^{*x}-orbits on lines - see [15]. Let $y \in \Delta_1(x)$ be such that $x + y \in \alpha_1(x,\Lambda_1)$. Now we may further assume y is chosen so as $x, y \in \Gamma_0^t$ for some $t \in \Omega_a$. Then, by Theorem 5 of [12], $x + y \in \alpha_0(x, D(x,a))$ (seen within Γ_0^t) with $x + y$ having point distribution $\Delta_3^{1t} 2\Delta_4^{1t}$. Since $\alpha_1(x,\Lambda_1)$ is a G_{ax}-orbit and $\Delta_4^1(a)^t \subseteq \Delta_4^1(a)$, we conclude that lines in $\alpha_1(x,\Lambda_1)$ have point distribution $\Delta_3^1 2\Delta_4^1$. Similarly we see that $\alpha_2(x,\Lambda_1)$ has point distribution $\Delta_3^1 2\Delta_3^3$ and $\alpha_3(x,\Lambda_1)$ has point distribution $\Delta_2^1 2\Delta_3^1$.

The same kind of arguments work for $\Delta_3^2(a), \Delta_3^3(a), \Delta_3^4(a), \Delta_3^5(a)$ and $\Delta_3^6(a)$, so we omit the details.

$\square$

The same strategy as employed in this section will reveal G_{ax} and orbit sizes for $x \in \Delta_4^i(a)$, $i = 1, ..., 6$. Note that in all these cases $k = |\{s \in \Omega_a | x \in \Gamma_0^s\}| = 1$ as a and x cannot be incident with a common hyperplane in the Fi_{23} geometry, as the point-line collinearity graph of the Fi_{22} geometry has diameter 3 (see Appendix 1 of [11]).

7 Proof of Theorem 12

The orbits considered in Theorem 12 do not lie within a Fi_{23} residue and so we cannot apply the same reasoning as in Section 6. Recall that for any $X \in \Gamma_3$, Γ_X is isomorphic to the geometry for $U_4(3).2$ described in [6].

We define

$$\widetilde{\Delta_3^8}(a) = \{x \in \Gamma_0 \mid \Gamma_3(a, x) \neq \emptyset \text{ and } d(a, x) = 3\}.$$

Lemma 7.1. $\Delta_3^8(a) = \widetilde{\Delta_3^8}(a)$.

Proof. If $x \in \widetilde{\Delta_3^8}(a)$ and $X \in \Gamma(a, x)$, using information about the geometry Γ_X given in Lemma 3.7(ii), there exists $c \in \Delta_2(a) \cap \Delta_1(a)$ with $c + x \in \alpha_{4^2}(c, \mathcal{T})$ where $\mathcal{T} \in \Gamma_2(a, c)$. By (2.2) $c \in \Delta_2^3(a)$ and $x \in \Delta_3^8(a)$. Conversely if $x \in \Delta_3^8(a)$ we must have $\Gamma_3(a, x) \neq \emptyset$ by $\mathbf{O}(2.4)$ and $d(a, x) = 3$ by Lemma 5.2. So $x \in \widetilde{\Delta_3^8}(a)$ as required. $\square$

Lemma 7.2. *If $x \in \Delta_3^8(a)$, then $|\Gamma_3(a, x)| = 1$.*

Proof. Let $x \in \Delta_3^8(a)$ and assume that $X, Y \in \Gamma_3(a, x)$ with $X \neq Y$. Using information about the $U_4(3).2$ geometry described in Lemma 3.7(??), for every $l \in \Gamma_1(a, X)$, there exists $b \in \Gamma_0(l)$ with $b \in \Delta_2(x) \cap \Delta_1(a)$. If $Y \notin \beta_3(a, X)$, then there is some $b \in \Delta_2(x) \cap \Delta_1(a)$ with $a + b \in \alpha_{1^5 3}(a, Y)$ by Lemma 3.4. Therefore Lemma 4.4 implies that $\tau(Y)$ does not fix b. Since

$\tau(Y) \in Q(a)_x$, $b^{\tau(Y)} \in \Gamma_0(a+b) \cap \Delta_2(x)$. However, as $\Gamma_0(a+b) \subseteq \Gamma_0(X)$, Lemma 3.7(ii) implies that $a \in \Delta_1(x)$, a contradiction. Hence $Y \in \beta_3(a, X)$. In Γ_a, there are three octads l incident with X and Y and for one of these, we can find $y \in \Gamma_0(l) \cap \Delta_2^1(a)$. Since $X, Y \in \Gamma_3(a, y)$ we now have a contradiction to Lemma 5.4, and so $X = Y$ as asserted. □

Lemma 7.3. *Let* $c_1 \in \Delta_2^2(a)$ *and* $c_2 \in \Delta_2(a) \cap \Delta_1(c_1)$. *Then*

(i) $c_2 \in \Delta_2^2(a)$*; and*

(ii) if $y \in \Gamma_0(c_1+c_2) \backslash \{c_1, c_2\}$, *then* $y \in \Delta_1(a)$.

Proof. (i) Suppose that $c_2 \in \Delta_2^1(a) \cup \Delta_2^3(a)$, and argue for a contradiction. Then, by definition of $\Delta_2^1(a)$ and $\Delta_2^3(a)$, there exists $X \in \Gamma_3(a, c_2)$. Since $c_1 \in \Delta_2^2(a)$, $|\{a, c_1\}^\perp| = 1$. Let $\{a, c_1\}^\perp = \{b\}$. If $b \in \Gamma_0(X)$, then, using Lemma 5.2, $c_1 \in \{b, c_2\}^\perp \subseteq \Gamma_0(X)$ and so $X \in \Gamma_3(a, c_2)$, whereas $\Gamma_3(a, c_2) = \emptyset$. Thus $b \notin \Gamma_0(X)$ and as a consequence $a+b \notin \Gamma_1(X)$. Hence $a+b \in \alpha_{2^4}(a, X) \cup \alpha_{1^5 3}(a, X)$. Assume that $a+b \in \alpha_{2^4}(a, X)$. Then $\tau(X) \in Q(a)_b$ by Lemma 4.4. Since $X \notin \Gamma_3(b)$, $\tau(X) \notin Q(b)$ by Lemma 4.9(ii). So $\tau(X) \in Q(a)_b \backslash Q(b)$ and hence $\tau(X)^{*b} \in Q(a)^{*b} = O_2(G_{ba}^{*b})$. Since $b+c_1 \in \alpha_2(b, b+a)$ we then infer that $\tau(X)^{*b}$ does not leave the octad $b+c$ invariant. Hence $\tau(X) \notin G_{c_1}$. However $\tau(X) \in Q(c_2)$ and so we obtain a triangle $\{b, c_1, c_1^{\tau(X)}\}$ with $c_1^{\tau(X)} \in \Gamma_0(c_1+c_2)$. Lemma 4.7 forces $b = c_2$, a contradiction. Thus we have shown that $a+b \notin \alpha_{2^4}(a, X)$ and so $a+b \in \alpha_{1^5 3}(a, X)$. By Lemma 4.4, $b^{\tau(X)} \neq b$. If $c_1^{\tau(X)} = c_1$, then $\{b, b^{\tau(X)}, c_1\}$ is a triangle, whence $a = c_1$ by Lemma 4.7. Thus $c_1^{\tau(X)} \neq c_1$. Since $c_1^{\tau(X)} \in \Gamma_0(c_1+c_2)$, this gives $\{b, c_1^{\tau(X)}\} \subseteq \{b^{\tau(X)}, c_1\}^\perp$ which, as $b^{\tau(X)} \in \Delta_2^2(c_1)$, contradicts Lemma 5.8(ii) (note that $b = c_1^{\tau(X)}$ would give $c_2 \in \Gamma_0(b+c_1)$ and then $c_2 \in \Delta_2^2(a)$). With this contradiction we have established part (i).

(ii) Let $\{a, c_i\}^\perp = \{b_i\}$ for $i = 1, 2$. Suppose (ii) is false and argue for a contradiction. We first claim that $d(b_1, c_2) = 2 = d(b_2, c_1)$. If, say, $d(b_1, c_2) =$

1, then $\{b_1, c_1, c_2\}$ is a triangle and so, as $c_1, c_2 \in \Delta_2(a)$, Lemma 4.7 yields that $y = b_1 \in \Delta_1(a)$. Thus $d(b_1, c_2) = 1$ and, similarly, $d(b_2, c_1) = 2$. In particular, this gives $b_1 \neq b_2$. Further, $d(b_1, b_2) = 2$. For $d(b_1, b_2) = 1$ implies $b_2 \in \Gamma_0(a + b_1)$ by Lemma 4.7 and then $\{b_1, c_2\} \subseteq \{b_2, c_1\}^{\perp}$. This contradicts Lemma 5.8(ii) as $b_2 \in \Delta_2^2(c_1)$.

If $b_1 \in \Delta_2^1(b_2) \cup \Delta_2^3(b_2)$, then by part (i) (with b_1 in place of a) $c_1 \notin \Delta_2^2(b_2)$. Therefore $c_1 \in \Delta_2^1(b_2) \cup \Delta_2^3(b_2)$. Consequently $a \in \Delta_2^2(c_1)$ and $b_2 \in \Delta_2^1(c_1) \cup \Delta_2^3(c_1)$ which is contrary to part (i) (with c_1 in place of a). Thus $b_1 \notin \Delta_2^1(b_2) \cup \Delta_2^3(b_2)$ and hence $b_1 \in \Delta_2^2(b_2)$. Similar arguments show that $c_1 \in \Delta_2^2(b_2)$ and $c_2 \in \Delta_2^2(b_1)$. By considering the elements of $\Gamma_3(b_1, c_1)$ as sextets in Ω_{b_1} and using Lemma 4.4 there exists $Y \in \Gamma_3(b_1, c_1)$ with $\tau(Y) \in G_a$. Suppose that $\tau(Y) \notin G_{c_2}$. Since $\tau(Y)$ fixes the line $c_1 + c_2$, Lemma 4.7 implies that $b_2^{\tau(Y)} \neq b_2$ and $\tau(Y) \notin Q(a)$. Therefore $1 \neq \tau(Y)^{*a} \in O_2(G_{ab_1}^{*a})$. This means that, in Ω_a, the octads $a + b_2^{\tau(Y)}$, $a + b_1$ and $a + b_2$ interest pairwise in the same duad. However we see from the MOG [4] that this is impossible. Thus we have shown that $\tau(Y) \in G_{c_2}$. Since $b_1 \in \Delta_2^2(c_2)$, $Y \notin \Gamma_3(c_2)$ and so $\tau(Y) \notin Q(c_2)$ by Lemma 4.9(ii). Then $1 \neq \tau(Y)^{*c_2} \in O_2(G_{c_2c_1}^{*c_2})$. Since $c_2 + b_2 \in \alpha_2(c_2, c_2 + c_1)$, $\tau(Y)^{*c_2}$ does not fix $c_2 + b_2$. This contradicts Lemma 5.8(ii) and this gives part (ii), and completes the proof of Lemma 7.3.

$\square$

Lemma 7.4. *(i) $\Delta_3^8(a)$ is a G_a-orbit and $|\Delta_3^8(a)| = 2^6.5.7.11.23$.*

(ii) For $x \in \Delta_3^8(a)$, $G_{ax} \sim 2^{13} : 3.3^2 : 4$ and $G_{ax}^{*x} \sim 2^6 : 3.3^2 : 4$ is the stabilizer in G_x^{*x} of the sextet $X \in \Gamma_3(a, x)$ and the partition of Ω_x into $\Sigma = \{\infty, 14, 0, 8, 3, 20, 15, 18, 17, 4, 16, 10\}$ and its complement (where X is identified with a standard sextet in Ω_x).

(iii) $|\Delta_2^1(a) \cap \Delta_1(x)| = 6$ and $|\Delta_2^3(a) \cap \Delta_1(x)| = 9$.

(iv) Let $x \in \Delta_3^8(a)$ and $\{X\} = \Gamma_3(a, x)$. If $\{a, b, c, x\}$ is a path of length 3 in $\mathcal{G}$, then $b, c \in \Gamma_0(X)$. Moreover $\Delta_2^2(a) \cap \Delta_1(x) = \emptyset$.

Proof. Let $x \in \Delta_3^8(a)$. By Lemma 7.2 $\Gamma_3(a,x) = \{X\}$. Observe that $\Gamma_0(X) \cap \Delta_3^8(a) = D_3^1(a)$ by Lemmas 7.1 and 3.7(ii). Since G_a is transitive on $\Gamma_3(a)$ and by Lemma 3.7(i), $D_3^1(a)$ is a G_{aX}-orbit, we see that $\Delta_3^8(a)$ is a G_a-orbit. Also, as $|D_3^1(a)| = 320$ by Lemma 3.7(i),

$$\begin{aligned}|\Delta_3^8(a)| &= |\Gamma_3(a)||\Gamma_0(X) \cap \Delta_3^8(a)| \\ &= 7.11.23.320 = 2^6.5.7.11.23.\end{aligned}$$

So (i) holds.

Clearly we have $G_{ax} \leq G_{axX}$ and so $G_{ax}^{*x} \leq G_{axX}^{*x} \sim 2^6 3S_6$. Also, by part (i), $|G_{ax}| = 2^{15}.3^3$. We now look at $Q(a)_x$. Using Lemma 4.6, as $a, x \in \Gamma_0(X)$, gives $\langle \tau(Y) | Y \in \beta_3(a,X) \rangle \leq Q(a)_x$. Hence, by Lemma 3.3, $|Q(a)_x| \geq 2^7$. Now select $y \in \Delta_2^1(a) \cap \Gamma_0(X)$ $(= D_2^2(a))$ with $y \in \Delta_1(x)$. Suppose $Q(a)_x \not\leq Q(a)_y$, and let $g \in Q(a)_x \setminus Q(a)_y$. Then $y^g \neq y$ and $y^g \in \Delta_1(x) \cap \Delta_2^1(a) \cap \Gamma_0(X)$. Let $b \in \{a,y\}^\perp$ (and note that $y \in \Gamma_0(X)$. Since $y \in Q(a)$, $b^g \in \Gamma_0(a+b)$. If $b \neq b^g$, then Lemma 3.7(ii) forces $a \in \Delta_1(x)$ whereas $d(a,x) = 3$. Thus $b = b^g$ and consequently $\{a,y\}^\perp = \{a,y^g\}^\perp$. Looking in $\Gamma_0(X)$ we see this is impossible [*****NEED GOOD REASON*****] Hence we infer that $Q(a)_x \leq Q(a)_y$. By Theorem 3 $|Q(a)'_y| = 2^7$ and therefore $|Q(a)_x| = 2^7$. Since $a \in \Delta_3^8(x)$, we also get $|Q(x)_a| = 2^7$, and so $|G_{ax}^{*x}| = 2^8.3^3$. Since G_{ax}^{*x} contains a Sylow 3-subgroup of G_{axX}^{*x} and the only subgroup of S_6 of order $3^2 2^\alpha$ are subgroups of $3^2 : 4$ we see that $G_{ax}^{*x} \sim 2^6 : 3.3^2 : 4$ which completes the proof of (ii).

Consulting Lemma 3.7(ii) we see $|\Delta_2^1(a) \cap \Delta_1(x) \cap \Gamma_0(X)| = 6$ and $|\Delta_2^3(a) \cap \Delta_1(x) \cap \Gamma_0(X)| = 9$. If $|\Delta_2^3(a) \cap \Delta_1(x)| > 9$ then for $y \in \Delta_2^3(a)$ the lines in $\alpha_{42^2}(y, \tau(a,y))$ must be incident with at least one point in $\Delta_3^8(a)$. Let $k = |\Delta_1(x) \cap \Delta_2^3(a)|$. Using part (ii), Lemma 5.11 and $\mathbf{O}(2.4)$ we calculate that $k = 36+9$ or $72+9$. Now, by $\mathbf{O}(2.11)$, there are no line orbits (apart from $\alpha_{4^2,8}(x,X)$ and $\alpha_{4^2,4^2}(x,X)$) of size ≤ 72. Thus we conclude that $|\Delta_2^3(a) \cap$

$\Delta_1(x)| = 9$. A similar argument, using $|\Delta_3^8(a)|$, $|\Delta_2^1(a)|$ and $\mathbf{O}(2.11)$ shows that $|\Delta_2^1(a) \cap \Delta_1(x)| = 6$ - note that all the line orbits from $y \in \Delta_2^1(a)$ have already been accounted for except $\alpha_{0,2^4}(y, \Lambda_1, \Lambda_2)$.
Suppose (iv) is false, and argue for a contradiction. Then, by Lemma 5.2, $b, c \notin \Gamma_0(X)$. By Lemma 3.5 there exists $Y \in \beta_3(a, X) \cup \{X\}$ with $a + b \in \alpha_{1^5 3}(a, Y)$. Set $\tau = \tau(Y)$. By Lemma 4.6 $\tau \in Q(X)$ and so $a^\tau = a$ and $x^\tau = x$. Also, from Lemma 4.4, $b \neq b^\tau \in \Gamma_0(a + b)$. Note that $b, b^\tau \in \Delta_2(x)$ and that b and b^τ are in the same G_x-orbit. Lemma 7.1 implies that $a \in \Delta_3^8(x)$. If $b \in \Delta_2^1(x) \cup \Delta_2^3(x)$, then part (iii) (with a and x interchanged) yields that $b \in \Gamma_0(X)$. Thus $b, b^\tau \in \Delta_2^2(x)$. Using Lemma 7.3(ii) (with x in place of a) we infer that $a \in \Delta_1(x)$, a contradiction. That $\Delta_2^2(a) \cap \Delta_1(x) = \emptyset$ follows from Lemma 3.7(ii). $\square$

We now consider the set

$$\Delta_3^{10}(a) = \{x \in \Gamma_0 |\text{ there exists } c \in \Delta_2^3(a) \cap \Delta_1(x) \text{ such that } c + x \in \alpha_{42^2}(c, \mathcal{T}_{ca})\}$$

where $\mathcal{T}_{ca}$ is the unique element of $\Gamma_2(a, c)$.

Lemma 7.5. *$\Delta_3^{10}(a) \subseteq \Delta_3(a)$ and $\Delta_3^{10}(a) \cap \Delta_3^8(a) = \emptyset$ and so $\Gamma_3(a, x) = \emptyset$.*

Proof. Let $x \in \Delta_3^{10}(a)$ and $c \in \Delta_2^3(a) \cap \Delta_1(x)$ such that $c + x \in \alpha_{42^2}(c, \mathcal{T}_{ca})$. By Lemma 7.4(iv), if $x \in \Delta_3^8(a)$, then there exists $X \in \Gamma_3(a, x)$ and in Ω_c the octad $c + x$ would intersect $\mathcal{T}_{ca}$ in 4^2, a contradiction. So $\Delta_3^{10}(a) \cap \Delta_3^8(a) = \emptyset$ and $\Gamma_3(a, x) = \emptyset$. If $x \in \Delta_1(a)$, then $x \in \{a, c\}^\perp$ and so $x \in \Gamma_0(X)$ for each $X \in \Gamma_3(a, c)$, a contradiction. Suppose that $x \in \Delta_2(a)$. Then Lemma 7.3 gives that $x \in \Delta_2^1(a) \cup \Delta_2^3(a)$. However this contradicts $\Gamma_3(a, x) = \emptyset$ again. Therefore $x \in \Delta_3(a)$ by definition. $\square$

We now turn to $\Delta_3^7(a)$. Recall from (2.2) that

$$\Delta_3^7(a) = \{x \in \Gamma_0 |\text{ there exists } c \in \Delta_2^1(a) \cap \Delta_1(x) \text{ such that } c + y \in \alpha_{0,2^4}(c, \Omega_c \cap \Omega_a, \mathcal{S}_{ca}\}$$

where $\Omega_c \cap \Omega_a$ is the tetrad of Ω_a described in Lemma 5.5 and $\mathcal{S}_{ca}$ is the sextet in Ω_a corresponding to the unique element of $\Gamma_3(a, c)$.

The next result shows the link between $\Delta_3^7(a)$ and $\Delta_3^{10}(a)$.

Lemma 7.6. *For any $x \in \Gamma_0$, $x \in \Delta_3^{10}(a)$ if and only if $a \in \Delta_3^7(a)$.*

Proof. Let $x \in \Delta_3^{10}(a)$ and let $c \in \Delta_2^3(a) \cap \Delta_1(x)$ with $c + x \in \alpha_{4^2}(c, \mathcal{T}_{ca})$. If $\{a, c\}^\perp = \{b_1, b_2, b_3\}$ we may suppose that $b_1 \in \Delta_2^1(x)$ and $b_2, b_3 \in \Delta_2^2(a)$. In Ω_{b_1}, the octad $b_1 + c$ is incident with the sextet $X(b_1, x)$, where $X(b_1, x)$ is the unique element of $\Gamma_3(b_1, x)$ (see Lemma 5.4). Also $(b_1 + a) \cap (b_1 + c) = \emptyset$ as octads because $c \in \Delta_2^3(a)$. Therefore $b_1 + a \in \alpha_{2^4}(b_1, X(b_1, x))$ and $|(b_1 + a) \cap t(b_1, x)| = 0$ where T is the tetrad contained in $b_1 + d$ for all $d \in \{b_1, x\}^\perp$. Therefore $a \in \Delta_3^7(x)$ by definition.

Conversely assume $a \in \Delta_3^7(x)$ and let $b \in \Delta_2^1(x) \cap \Delta_1(a)$ with $b + a \in \alpha_{2^4}(b, X)$ where X is the unique element of $\Gamma_3(b, x)$ and $|(b+a) \cap t(b, x)| = 0$ in Ω_b where $t(b, x) = \Omega_b \cap \Omega_x$. Then there exists $d \in \{b, x\}^\perp$ such that $b + x \in \Gamma_1(X)$ and $(b + d) \cap (b + a) = \emptyset$ in Ω_b. Hence $d \in \Delta_2^3(a)$ and now $x \in \Delta_3^{10}$ by definition. □

Lemma 7.7. *Suppose that $x_1, x_2 \in \Delta_2(a)$ and $x_1 \in \Delta_1(x_2)$. Let $\Gamma_0(x_1 + x_2) = \{x_1, x_2, x\}$. Then $x_1, x_2 \in \Delta_2^i(a)$ for the same $i \in \{1, 2, 3\}$ and $x \in \Delta_1(a)$.*

Proof. If $x_1 \in \Delta_2^3(a)$, the lemma follows from Lemma 7.3. So we may assume $x_1 \in \Delta_2^1(a) \cup \Delta_2^3(a)$. The point distributions [****GIVE REFS****] of lines from $\Delta_2^1(a) \cup \Delta_2^3(a)$ are all known with the exception of $\alpha_{0,2^4}(x_1, \Lambda_1, \Lambda_2)$ when $x_1 \in \Delta_2^1(a)$. From Lemmas 7.5 and 7.6, we deduce that $\Delta_3^7(a) \subseteq \Delta_3(a)$. In particular, for $\ell \in \alpha_{0,2^4}(x_1, \Lambda_1, \Lambda_2)$, $\Lambda_0(\ell) \cap \Delta_2^1(a) = \{x_1\}$, so completing the proof of Lemma 7.7. □

Lemma 7.8. *Let $x \in \Delta_3^{10}(a)$ and $c \in \Delta_2^3(a) \cap \Delta_1(x)$.*

(i) We have $\Delta_2(x) \cap \Delta_1(a) = \{a, c\}^\perp$ with $|\Delta_2^1(x) \cap \Delta_1(a)| = 1$, $|\Delta_2^2(x) \cap \Delta_1(a)| = 2$ and $|\Delta_2^3(x) \cap \Delta_1(a)| = 0$.

(ii) If $b \in \Delta_2^1(x) \cap \Delta_1(a)$, then $\Delta_2(a) \cap \Delta_1(x) = \{b, x\}^\perp$ with $|\Delta_2^1(x) \cap \Delta_1(a)| = 0$, $|\Delta_2^2(x) \cap \Delta_1(a)| = 4$ and $|\Delta_2^3(x) \cap \Delta_1(a)| = 1$.

Proof. In Ω_c, for every $b \in \{a, c\}^\perp$, the octad $c+b$ is incident with the trio $\mathcal{T}_{ca}$ and since $c+x \in \alpha_{42^2}(c, \mathcal{T}_{ca})$ we get $|\Delta_2^1(x) \cap \{a, c\}^\perp| = 1$, $|\Delta_2^2(x) \cap \{a, c\}^\perp| = 2$ and $|\Delta_2^3(x) \cap \{a, c\}^\perp| = 0$ from the definitions of $\Delta_2^i(x)$, $i = 1, 2, 3$. Let $\{b\} = \Delta_2^1(x) \cap \{a, c\}^\perp$. In Ω_b, the two octads $b+a$ and $b+c$ are incident with the trio $\mathcal{T}_{ca}$ and so the octads are disjoint. Let X be the unique element of $\Gamma_3(b, x)$. Then $b+a \in \alpha_{2^4}(b, X)$. Therefore, for every $d \in \{b, x\}^\perp$, the octads $b+a$ and $b+x$ intersect in exactly two elements of Ω_b. So $|\Delta_2^1(x) \cap \{b, x\}^\perp| = 0$, $|\Delta_2^2(x) \cap \{b, x\}^\perp| = 4$ and $|\Delta_2^3(x) \cap \{b, x\}^\perp| = 1$.

To complete the proof by Lemma 5.8(ii), it is enough to show that $\Delta_2(x) \cap \Delta_1(a) = \{a, c\}^\perp$. Assume that $b_1 \in \Delta_2(x) \cap \Delta_1(a)$ with $b_1 \notin \{a, c\}^\perp$. If $a + b_1 \in \Gamma_1(X)$ for some $X \in \Gamma_3(a, c)$, then $a + b_1 \in \Gamma_1(Y) \cup \alpha_{2^4}(a, Y)$ for every $Y \in \Gamma_3(a, c)$ and so $\tau(Y) \in Q(a)_b$ by Lemmas 4.2(i) and 4.4. By the definition of $\Delta_3^7(a)$ we can find $Y \in \Gamma_3(a, c)$ with $c + x \in \alpha_{1^53}(c, Y)$ and then $\tau(Y) \notin G_x$ by Lemma 4.4. So $x, x^{\tau(Y)} \in \Delta_2(b_1)$ and Lemma 7.7 gives $c \in \Delta_1(b_1)$, contrary to the choice of b_1. Therefore $a + b_1 \notin \Gamma_1(X)$ for all $X \in \Gamma_3(a, c)$ and so in Ω_a, the octad $a + b_1$ intersects the trio $\mathcal{T}_{ca}$ in 42^2.

We now show that $b_1 \in \Delta_2^2(x)$. Let X be the unique element of $\Gamma_3(b, x)$. Assume $b_1 \notin \Delta_2^2(x)$ for a contradiction. Then there exists $Y \in \Gamma_3(x, b_1)$. If $X \in \beta_i(x, Y)$ for $i = 1, 0$, then there exists $d \in \{x, b_1\}^\perp$ with $x + d \in \alpha_{1^53}(x, X)$. By Lemma 4.4 $d^{\tau(X)} \neq d$. Since $b + a \in \alpha_{2^4}(b, X)$, using Lemma 4.4 again we have $a^{\tau(X)} = a$. Using Lemma 7.7 with d and $d^{\tau(X)}$ we get $y \in \Delta_1(a)$. So we must have $X \in \beta_3(x, Y)$. We can choose $d \in \{d, x\}^\perp$ with $x + d \in \alpha_{42^2}(x, Y)$. So $d \in \Gamma_0(Y)$. If $d \in \Delta_1(b_1)$ then $d \notin \Delta_2^2(a)$ by Lemma 5.8(ii). Then $d = c$ from the first part of the proof. This contradicts the fact that $b_1 \notin \{a, c\}^\perp$. If $d \in \Delta_2(b_1)$, then Lemma 7.7 implies that the point in $\Gamma_0(a + b_1) \backslash \{a, b_1\}$ lies in $\Delta_1(x)$ and using Lemma 7.7 again we get $x \in \Delta_1(a)$. So $d \in \Delta_3^8(b_1)$ because $\Gamma_3(d, b_1) \neq \emptyset$. However Lemma 7.4(iv) now yields $\Gamma_3(a,x) \neq \emptyset$ which contradicts Lemma 7.5. Hence we have shown that $b_1 \in \Delta_2^2(x)$.

Let d be the unique point in $\{x, b_1\}^\perp$. We can choose $Y \in \Gamma_3(a,c)$ such that $c+x \in \alpha_{2^4}(c,Y)$. Then $\tau(Y)$ fixes x by Lemma 4.4. Assume $d \notin \Delta_2^2(a)$ and let $Z \in \Gamma_3(a,d)$. If $Z \notin \Gamma_3(\mathcal{T}_{ca})$ we could choose $Y_1 \in \Gamma_3(\mathcal{T}_{ca})$ such that $Y_1 \in \beta_i(a,Z)$ for $i=0$ or 1 and $b' \in \{a,d\}^\perp$ with $a+b' \in \alpha_{1^53}(a,Y_1)$. So $\tau(Y_1)$ does not fix b' by Lemma 4.4 and then Lemma 7.7 gives $a \in \Delta_1(x)$, a contradiction. Therefore $Z \in \Gamma_3(\mathcal{T}_{ca})$. Applying a similar argument to the one used to show $b_1 \in \Delta_2^2(x)$, we can prove that $d \in \Delta_2^2(a)$.

Since $b_1 \notin \{a,c\}^\perp$, the octad $a+b_1$ in Ω_a is not incident with the trio $\mathcal{T}_{ca}$. Therefore we can choose $Y \in \Gamma_3(a,c)$ with $\tau(Y) \in G_x$ and $b_1 \notin \Gamma_0(Y)$. If $\tau(Y)$ does not fix b_1 Lemma 7.7 would imply that $a \in \Delta_1(x)$ and so $\tau(Y) \in G_{b_1}$. ***SHOW THAT $d^{\tau(Y)} \neq d$***. We now have $|\{b_1,x\}^\perp| > 1$ which contradicts Lemma 5.8(ii). This completes the proof of the lemma. □

Lemma 7.9. *(i)* $|\Delta_3^{10}(a)| = 2^9.3^2.5.7.11.23$.

(ii) G_a *is transitive on* $\Delta_3^{10}(a)$.

(iii) For $x \in \Delta_3^{10}(a)$, $G_{ax} \sim 2^9 : S_4$ *and* $G_{ax}^{*x} \sim 2^6 : S_4$ *is the stabilizer in* G_x^{*x} *of the tetrad* $t(x,b)$ *(where* b *is the unique element of* $\Delta_1(a) \cap \Delta_2^1(x)$*) and a partition of* $x + c \backslash t(x,b)$ *into two pairs of elements.*

Proof. Let $x \in \Delta_3^{10}(a)$ and c be the unique point in $\Delta_2^3(a) \cap \Delta_1(x)$ (c exists by Lemma 7.8). Then $|\Delta_3^{10}(a) \cap \Delta_1(c)| = 2|\alpha_{42^2}(c, \mathcal{T}_{ca}| = 2.672$ by **O**(2.4). By the uniqueness of c and Lemma 5.11(i) we have

$$|\Delta_3^2(a)| = 2.672.|\Delta_2^3(a)| = 2^9.3^2.5.7.11.23.$$

For part (ii), working in Ω_c, there are four sextets $X \in \Gamma_3(a,c)$ such that $c+x \in \alpha_{1^53}(c,X)$ and so $\tau(X) \notin G_x$ by Lemma 4.4. Therefore G_{ac}

is transitive on $\Gamma_0(c+x)\backslash\{c\}$. Now part (ii) follows because $\alpha_{1^5 3}(c, X)$ is a G_{ac}-orbit on $\Gamma_1(c)$ and $\Delta_2^3(a)$ is a G_a-orbit of points by Lemma 5.11(ii).

Turning to part (iii) we have $G_{ax} \leqslant G_{ac}$. Let $b \in \{a,c\}^\perp \cap \Delta_2^1(x)$ (b exists and is unique by Lemma 7.8(i)). By Lemma 7.8(ii) there exists $c_1, c_2 \in \{b,x\}^\perp \cap \Delta_2^2(a)$ with $c_1 \neq c_2$. We show that $|Q(a)_x| \leq 2^3$ by first proving that $Q(a)_x \leqslant Q(a)_{c_i}$ for $i = 1$ and 2. Assume $g \in Q(a)x\backslash Q(a)_{c_i}$ for a contradiction. If $b^g = b$, then in Ω_b the octads $b + c_1$ and $b + c_1^g$ contain the same two elements of $b + a$. However Lemma 5.5 implies that $t(b,x) \subseteq b + c$ which gives $(b+a)\cap(b+c) \neq \emptyset$, contrary to Lemma 5.9. So $b^g \neq b$ and we can use Lemma 7.7 to show that $a \in \Delta_1(x)$, a contradiction. So $Q(a)_x \leqslant Q(a)_{c_i}$ for $i = 1, 2$. Since $(b + a) \cap (b + c_i) = \emptyset$ in Ω_b, there are seven hyperplanes $Y_i \in \Gamma_3(a,b)$ $(i = 1, ..., 7)$ with $\tau(Y_i) \in G_{c_1c_2}$ and the subgroup generated by the elements $\tau(Y_i)$ has order at least 2^4. Further we can show that, up to relabelling $Q(a) \cap Q(c_1) = \langle \tau(Y_1)\tau(Y_2)\tau(Y_3)\rangle \leqslant Q(a)_{c_1c_2}$. (See Lemma 6.15 in [17] for details). Since $Q(a)_{c_1c_2} \neq Q(a)_{c_1}$ we have $|Q(a)_{c_1c_2}| = 2^4$ by Theorem 4. Therefore $|Q(a)^{*c_1}_{c_1c_2}| = 2^3$. In Ω_{c_1} the octads $c_1 + b$ and $c_1 + x$ intersect in four elements and the subgroup of $O_2(G^{*c_1}_{c_1b})$ fixing $c_1 + x$ is of order 2^2. Therefore $|Q(a)^{*c_1}_{c_1c_2x}| \leq 2^2$ and so $|Q(a)_x| \leq 2^3$, as required.

By parts (i) and (ii), $[G_{ac} : G_{ax}] = 2^6.3.7$. Since $|Q(a)_c| = 2^9$ by Theorem 5 we must have $|Q(a)_x| \leq 2^3$ and so $|Q(a)_x| = 2^3$ and $[G^{*a}_{ac} : G^{*a}_{ax}] = 3.7$. Using the ATLAS [2] and Theorem 5 we get $G^{*a}_{ax} \sim 2^6 : S_4$. This completes the proof of the lemma. $\square$

Lemmas 7.6 and 7.8 now imply

Lemma 7.10. *Let $x \in \Delta_3^7(a)$ and $c \in \Delta_2^1(a) \cap \Delta_1(x)$. Then*

(i) $\Delta_2(x) \cap \Delta_1(a) = \{a,c\}^\perp$ *with* $|\Delta_2^1(x) \cap \Delta_1(a)| = 2$, $|\Delta_2^2(x) \cap \Delta_1(a)| = 0$ *and* $|\Delta_2^3(x) \cap \Delta_1(a)| = 1$.

(ii) If $b \in \Delta_2^3(x) \cap \Delta_1(a)$, *then* $\Delta_2(a) \cap \Delta_1(x) = \{b,x\}^\perp$ *with* $|\Delta_2^1(a) \cap \Delta_1(x)| = 4$, $|\Delta_2^2(a) \cap \Delta_1(x)| = 1$ *and* $|\Delta_2^3(a) \cap \Delta_1(x)| = 0$.

Lemma 7.11. *(i)* $|\Delta_3^7(a)| = 2^9.3^2.5.7.11.23$.

(ii) G_a *is transitive on* $\Delta_3^7(a)$.

(iii) For $x \in \Delta_3^7(a)$, $G_{ax} \sim 2^9 : S_4$ *and* $G_{ax}^{*x} \sim 2^5 : S_4$ *is the stabilizer in* G_x^{*x} *of the octad* $x+d$ *(where* d *is the unique element of* $\Delta_2^2(a) \cap \Delta_1(x)$*), the trio* $\mathcal{T}_0$ *which is the unique element of* $\Gamma_3(b,x)$ *for* $b \in \Delta_2^3(x) \cap \Delta_1(a)$ *and a partition of the octad* $x+d$ *into four 2-element sets.*

Proof. Let $x \in \Delta_3^7(a)$ and $c \in \Delta_2^1(a) \cap \Delta_1(x)$. Then $|\Delta_3^3(a) \cap \Delta_1(c)|$ is twice the number of octads in Ω_c lying in $\alpha_{2^4}(c, X(c,a))$ that have an empty intersection with $t(c,a)$. This number is 240. Therefore Lemmas 5.6 and 7.10 give

$$|\Delta_3^7| = 2^9.3^2.5.7.11.23.$$

Let $x' \in \Gamma_0(c+x)\backslash\{c,x\}$. Then by definition $x' \in \Delta_3^7(a)$. Since $c+x \notin \Gamma_1(X(c,a))$, there exists $Y \in \beta_3(c, X(c,a))$ with $c+x \in \alpha_{1^5 3}(c,Y)$. For this Y we have $x^{\tau(Y)} = x'$ by Lemma 4.4. By $\mathbf{O}(2.2)$ and Lemma 5.5(iii) G_{ca}^{*c} is transitive on the lines in $\alpha_{2^4}(c, X(c,a))$ that have an empty intersection with $t(c,a)$ and so part (ii) follows from the transitivity of G_a on $\Delta_2^2(a)$ (see Lemma 5.6(ii)).

For part (iii) we know that $a \in \Delta_3^{10}(x)$ by Lemma 7.6 and hence $G_{ax} \sim 2^9 : S_4$ by Lemma 7.9(iii). Let $b \in \Delta_2^3(x) \cap \Delta_1(a)$ and $e_1, e_2 \in \{b,x\}^\perp \cap \Delta_2^2(a)$ with $e_1 \neq e_2$. (Such points exist by Lemma 7.10.) Assume $Q(a)_x \not\leqslant Q(a)_{e_1}$ and let $g \in Q(a)_y \backslash Q(a)_{e_1}$. If $b^g = b$, then $g^{*b} \in O_2(G_{ba}^{*b})$ and so in Ω_b, the octads $b+e_1$ and $b+e_1^g$ intersect $b+a$ in the same two elements. However $e_1^g \in \{b,x\}^\perp$ and so $(b+e_1)\cap(b+e_1^g) = \emptyset$ because $b \in \Delta_2^3(x)$. Therefore $b^g \neq b$. Since $b^g \in \Gamma_0(a+b)$, Lemma 7.7 implies that $a \in \Delta_1(x)$, a contradiction. Therefore $Q(a)_x \leqslant Q(a)_{e_1}$ and similarly $Q(a)_x \leqslant Q(a)_{e_2}$. Using an argument similar to that in the proof of Lemma 7.9(iii), we get $|Q(a)_x| \leq 2^4$.

Since c is the unique point in $\Delta_2^1(a) \cap \Delta_1(x)$, $G_{ax} \leqslant G_{ac}$. By Lemma 5.6(iii) we have $|Q(c)_a| = 2^7$. Therefore $Q(c)_a^{*a} \leqslant O_2(G_{ac}^{*a})$. Since $c+x \in$

$\alpha_{2^4}(c, X(c,a))$, there exists $Y \in \beta_3(c, X(c,a))$ such that $c + x \in \alpha_{1^5 3}(c, Y)$. Then Lemma 4.4 implies that $\tau(Y) \notin G_x$. However $\tau(Y) \in Q(c) \cap Q(X(c,a))$ and $\tau(Y) \notin Q(a)$. Therefore $|O_2(G^{*x}_{xa}| \leq 2^5$ and so $G^{*x}_{xa} \sim 2^5 S_4$ and $|Q(x)_a| = 2^4$.

□

We end this section by examining the set

$$\Delta_3^9(a) = \{x \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(a) \cap \Delta_1(x) \text{ such that } c + x \in \alpha_{2,0}(c, c+b, \mathcal{D}_{ca}), \text{ where } \{b\} = \{a,c\}^\perp\}.$$

Lemma 7.12. $\Delta_3^9(a) \cap \Delta_3^i(a) = \emptyset$ *for* $i = 1, ..., 8$ *and* $i = 10$.

Proof. Since $\Omega_a \cap \Omega_x = \emptyset$ by definition, $\Delta_3^9(a) \cap \Delta_3^i(a) = \emptyset$ for $i = 1, ..., 6$. By Lemma 7.4(iii), $\Delta_3^9(a) \cap \Delta_3^8(a) = \emptyset$. By Lemmas 7.8 and 7.10 and [RW5; (2.3)] if $x \in \Delta_3^7(a) \cup \Delta_3^{10}(a)$, then $|c + x \cap c + b| = 0$ or 4 in Ω_c for any $c \in \Delta_2^2(a) \cap \Delta_1(x)$. Therefore $\Delta_3^9(a) \cap \Delta_3^i(a) = \emptyset$ for $i = 7, 10$ as required. □

Lemma 7.13. *Let* $x \in \Delta_3^9(a)$. *Then there exists a unique path of length three between* a *and* x *in* $\mathcal{G}$.

Proof. Let $c \in \Delta_2^2(a) \cap \Delta_1(x)$ with $c + x \in \alpha_{2,0}(c, c+b, \mathcal{D}_{ca})$ and $\{b\} = \{a,c\}^\perp$. Then $b \in \Delta_2^2(x)$ by definition. Assume that a, b_1, c_1, x is another path of length three in $\mathcal{G}$. By Lemmas 7.8, 7.10 and 7.4 and [RW5;(2.2) and (2.4)] we must have $c_1 \in \Delta_2^2(a)$ and $b_1 \in \Delta_2^2(x)$. It then follows from Lemma 5.8(ii) that $b_1 \neq b$ and $c_1 \neq c$. Therefore $b_1 \in \Delta_2(b)$ and we consider the three possible choices separately.

First assume that $b_1 \in \Delta_2^3(b)$. Notice that $c_1 \notin \Delta_1(b) \cup \Delta_2(b)$ by Lemma 7.7 and Lemma 5.8(ii) and so $c_1 \in \Delta_3(b)$. Therefore $c_1 \in \Delta_3^8(b) \cup \Delta_3^{10}(b)$ by [RW5;(2.4)]. However Lemma 7.4(iii) implies that $c_1 \in \Delta_3^{10}(b)$. We now have $c \in \Delta_2^1(c_1) \cap \{b, b_1\}^\perp$ by Lemma 7.8 and so $c, c_1 \in \{x, b_1\}^\perp$, contrary to Lemma 5.8(ii).

Next suppose that $b_1 \in \Delta_2^1(b)$. Therefore $c \in \Delta_3^7(b_1) \cup \Delta_3^8(b_1)$. Using Lemma 7.4(iii) we must have $c \in \Delta_3^7(b_1)$. This again leads to the contradiction that $c, c_1 \in \{x, b_1\}^\perp$.

Therefore we must have $b_1 \in \Delta_2^2(b)$. In Ω_c, $c + x \cap \mathcal{D}_{ca} = \emptyset$ and $|c + x \cap c + b| = 2$. Using the MOG in [C1] and Lemma 4.4 we can find a sextet $Y \in \Gamma_3(c, b)$ with $\tau(Y) \in G_a x$. Since $\tau(Y) \notin Q(a)$ and $a + b_1 \in \alpha_2(a, a + b)$, $\tau(Y)$ does not fix $a + b_1$. However by the above argument we must have $a + b_1^{\tau(Y)} \alpha_2(a, a + b) \cap \alpha_2(a, a + b_1)$ and $a + b \cap a + b_1 {=} a + b \cap a + b_1^{\tau(Y)}$ in Ω_a. As this cannot occur we again get a contradiction. This completes the proof of the lemma. $\square$

Lemma 7.14. *(i)* $|\Delta_3^9(a)| = 2^{13}.3^2.5.7.11.23$.

(ii) G_a *is transitive on* $\Delta_3^9(a)$.

(iii) For $x \in \Delta_3^9(a)$, $G_{ax} \sim 2^5 : S_4$ *and* $G_{ax}^{*x} \sim 2^4 : S_4$.

Proof. Let $x \in \Delta_3^9(a)$ and let a, b, c, x be the unique path of length three between a and x in $\mathcal{G}$.

(i) From [RW5:(2.4)], $\Delta_1(c) \cap \Delta_3^9(a) = 2 \times 240 = 2^5.3.5$. Using Lemmas 5.8(ii) and Lemma 7.13 we then have $|\Delta_3^9(a)| = 2^13.3^2.5.7.11.23$.

(ii) Since $c + x\alpha_{2,0}(c, c + b, \mathcal{D}_{ca})$ and using Lemma 4.4, we can find $Y \in \Gamma_3(b, c)$ with $\tau(Y) \in G_a \backslash G_x$. Since $\tau(Y)$ fixes $c + x$, $\tau(Y)$ interchanges the points in $\Gamma_0(c+x) \backslash \{c\}$. Since G_a is transitive on $\Delta_2^2(a)$ and $\alpha_{2,0}(c, c+b, \mathcal{D}_{ca})$ is a G_{ac} orbit, G_a is transitive on $\Delta_3^9(a)$.

(iii) We have $G_{ax} \le G_{ac}$. Since $c+x \in \alpha_2(c, c+b)$, then $Q(a)_{cx}^{*c} = 1$ and so $Q(a)_x \le Q(a) \cap Q(c)$. Using the MOG in [C1], there exist $Y_1, Y_2, Y_3 \in \Gamma_(c+b)$ with $Q(a) \cap Q(c) =< \tau(Y_1)\tau(Y_2)\tau(Y_3) >$. Further, if δ is the duad in Ω_c fixed by G_{ca}^{*c} and t_i is the tetrad in Y_i containing δ ($i = 1, 2, 3$), then of the six elements in $(c + a) \backslash \delta$ in Ω_c, three lie in exactly two of the tetrads t_i and three lie in none of the tetrads t_i. (For details see Proposition 8.12 in [W] where $\Delta_2^2(a)$ is denoted by $\Delta_2^3(a)$.) Since $\Omega_a \cap \Omega_x = \emptyset$ we have that

$c+x \in \alpha_{1^5 3}(c, Y_i)$ for precisely two or none of the sextets Y_i, $i = 1, 2, 3$. Therefore $\tau(Y_1)\tau(Y_2)\tau(Y_3) \in G_x$ by Lemma 4.4. Therefore $|Q(a)_x| = 2$ and it follows that $[G^{*x}_{xb} : G^{*x}_{xa}] = 2.3.5.$. Since $G^{*x}_{xb} \sim 2^4 S_6$ by Lemma 5.8(iii) we must have $G^{*x}_{xa} \sim 2^4 S_4$, as required.

□

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