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Mendes, Sergio and Plymen, Roger

2015

MIMS EPrint: 2015.93

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ISSN 1749-9097
Functoriality and $K$-theory for $\text{GL}_n(\mathbb{R})$

Sergio Mendes and Roger Plymen

Abstract

We investigate base change and automorphic induction $\mathbb{C}/\mathbb{R}$ at the level of $K$-theory for the general linear group $\text{GL}_n(\mathbb{R})$. In the course of this study, we compute in detail the $C^*$-algebra $K$-theory of this disconnected group. We investigate the interaction of base change with the Baum-Connes correspondence for $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$. This article is the archimedean companion of our previous article [11].


Keywords. General linear group, tempered dual, base change, $K$-theory.

1 Introduction

In the general theory of automorphic forms, an important role is played by base change and automorphic induction, two examples of the principle of functoriality in the Langlands program [3]. Base change and automorphic induction have a global aspect and a local aspect [1][7]. In this article, we focus on the archimedean case of base change and automorphic induction for the general linear group $\text{GL}_n(\mathbb{R})$, and we investigate these aspects of functoriality at the level of $K$-theory.

For $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$ we have the Langlands classification and the associated $L$-parameters [8]. We recall that the domain of an $L$-parameter of $\text{GL}_n(F)$ over an archimedean field $F$ is the Weil group $W_F$. The Weil groups are given by

$$W_C = \mathbb{C}^\times$$

and

$$W_R = \langle j \rangle \mathbb{C}^\times$$

where $j^2 = -1 \in \mathbb{C}^\times$, $jc = \overline{c}j$ for all $c \in \mathbb{C}^\times$. Base change is defined by restriction of $L$-parameter from $W_R$ to $W_C$.

An $L$-parameter $\phi$ is tempered if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.
In this article, we investigate the interaction of base change with the Baum-Connes correspondence for $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$.

Let $F$ denote $\mathbb{R}$ or $\mathbb{C}$ and let $G = G(F) = \text{GL}_n(F)$. Let $C^*_r(G)$ denote the reduced $C^*$-algebra of $G$. The Baum-Connes correspondence is a canonical isomorphism $\mu_F : K^*_G(F) \to K^*_r(C^*_r(G(F)))$

where $EG(F)$ is a universal example for the action of $G(F)$.

The noncommutative space $C^*_r(G(F))$ is strongly Morita equivalent to the commutative $C^*$-algebra $C_0(A^t_n(F))$ where $A^t_n(F)$ denotes the tempered dual of $G(F)$, see [12, §1.2][13]. As a consequence of this, we have

$$K^*_r(C^*_r(G(F))) \cong K^* A^t_n(F).$$

This leads to the following formulation of the Baum-Connes correspondence:

$$K^*_G(F)(EG(F)) \cong K^* A^t_n(F).$$

Base change and automorphic induction $\mathbb{C}/\mathbb{R}$ determine maps

$$\text{BC}_{\mathbb{C}/\mathbb{R}} : A^t_n(\mathbb{R}) \to A^t_n(\mathbb{C})$$

and

$$\text{AI}_{\mathbb{C}/\mathbb{R}} : A^t_n(\mathbb{C}) \to A^t_{2n}(\mathbb{R}).$$

This leads to the following diagrams

$$K^*_G(\mathbb{C}) \xrightarrow{\mu_C} K^* A^t_n(\mathbb{C})$$

$$\downarrow \quad \downarrow \text{BC}_{\mathbb{C}/\mathbb{R}}$$

$$K^*_G(\mathbb{R}) \xrightarrow{\mu_R} K^* A^t_n(\mathbb{R}).$$

and

$$K^*_G(\mathbb{R}) \xrightarrow{\mu_R} K^* A^t_{2n}(\mathbb{R})$$

$$\downarrow \quad \downarrow \text{AI}_{\mathbb{C}/\mathbb{R}}$$

$$K^*_G(\mathbb{C}) \xrightarrow{\mu_C} K^* A^t_n(\mathbb{C}).$$

where the left-hand vertical maps are the unique maps which make the diagrams commutative.

In section 2 we describe the tempered dual $A^t_n(F)$ as a locally compact Hausdorff space.
In section 3 we compute the $K$-theory for the reduced $C^*$-algebra of $GL(n, \mathbb{R})$. The real reductive Lie group $GL(n, \mathbb{R})$ is not connected. If $n$ is even our formulas show that we always have non-trivial $K^0$ and $K^1$. We also recall the $K$-theory for the reduced $C^*$-algebra of the complex reductive group $GL(n, \mathbb{C})$, see [13]. In section 4 we recall the Langlands parameters for $GL(n)$ over archimedean local fields, see [8]. In section 5 we compute the base change map $BC : A^t_n(\mathbb{R}) \to A^t_n(\mathbb{C})$ and prove that $BC$ is a continuous proper map. At the level of $K$-theory, base change is the zero map for $n > 1$ (Theorem 5.4) and is nontrivial for $n = 1$ (Theorem 5.5). In section 6, we compute the automorphic induction map $AI : A^t_n(\mathbb{C}) \to A^t_{2n}(\mathbb{R})$. Contrary to base change, at the level of $K$-theory, automorphic induction is nontrivial for every $n$ (Theorem 6.5). In section 7, where we study the case $n = 1$, base change for $K^1$ creates a map

$\mathcal{R}(U(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$

where $\mathcal{R}(U(1))$ is the representation ring of the circle group $U(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $U(1)$ to $1 \oplus \varepsilon$, where $\varepsilon$ is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $U(1)$ to zero.

This map has an interpretation in terms of $K$-cycles. The $K$-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\phi_C \in K_1^{\mathbb{C}^\times}(\mathbb{C}^\times)$ and a class $\phi_R \in K_1^{\mathbb{R}^\times}(\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\phi_C \mapsto (\phi_R, \phi_R)$$

This extends the results of [11] to archimedean fields.

We thank Paul Baum for a valuable exchange of emails.

2 On the tempered dual of $GL(n)$

Let $F = \mathbb{R}$. In order to compute the $K$-theory of the reduced $C^*$-algebra of $GL(n, F)$ we need to parametrize the tempered dual $A^t_0(F)$ of $GL(n, F)$.

Let $M$ be a standard Levi subgroup of $GL(n, F)$, i.e. a block-diagonal subgroup. Let $^0M$ be the subgroup of $M$ such that the determinant of each block-diagonal is $\pm 1$. Denote by $X(M) = \hat{M}/^0M$ the group of unramified characters of $M$, consisting of those characters which are trivial on $^0M$. 
Let $W(M) = N(M)/M$ denote the Weyl group of $M$. $W(M)$ acts on the
discrete series $E_2^{0(M)}$ of $^0M$ by permutations.

Now, choose one element $\sigma \in E_2^{0(M)}$ for each $W(M)$-orbit. The *isotropy subgroup* of $W(M)$ is defined to be

$$W_\sigma(M) = \{ \omega \in W(M) : \omega.\sigma = \sigma \}.$$

Form the disjoint union

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_{M} \bigsqcup_{\sigma \in E_2^{0(M)}} X(M)/W_\sigma(M). \quad (1)$$

The disjoint union has the structure of a locally compact, Hausdorff space
and is called the *Harish-Chandra parameter space*. The parametrization of
the tempered dual $A^t_n(\mathbb{R})$ is due to Harish-Chandra, see [10].

**Proposition 2.1.** There exists a bijection

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) \longrightarrow A^t_n(\mathbb{R})$$

$$\chi^\sigma \quad \mapsto \quad i_{GL(n),MN}(\chi^\sigma \otimes 1),$$

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ for all $x \in M$.

In view of the above bijection, we will denote the Harish-Chandra param-
eter space by $A^t_n(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting
with $GL(n, \mathbb{R})$. Since the discrete series of $GL(n, \mathbb{R})$ is empty for $n \geq 3$, we
only need to consider partitions of $n$ into 1’s and 2’s. This allows us to to
decompose $n$ as $n = 2q + r$, where $q$ is the number of 2’s and $r$ is the number
of 1’s in the partition. To this decomposition we associate the partition

$$n = (\underbrace{2, \ldots, 2}_{q}, \underbrace{1, \ldots, 1}_{r}),$$

which corresponds to the Levi subgroup

$$M \cong \underbrace{GL(2, \mathbb{R}) \times \ldots \times GL(2, \mathbb{R})}_{q} \times \underbrace{GL(1, \mathbb{R}) \times \ldots \times GL(1, \mathbb{R})}_{r}.$$

Varying $q$ and $r$ we determine a representative in each equivalence class
of Levi subgroups. The subgroup $^0M$ of $M$ is given by

$$^0M \cong \underbrace{SL^+(2, \mathbb{R}) \times \ldots \times SL^+(2, \mathbb{R})}_{q} \times \underbrace{SL^+(1, \mathbb{R}) \times \ldots \times SL^+(1, \mathbb{R})}_{r},$$
where
\[ SL^\pm(m, \mathbb{R}) = \{ g \in GL(m, \mathbb{R}) : |\det(g)| = 1 \} \]
is the unimodular subgroup of \( GL(m, \mathbb{R}) \). In particular, \( SL^\pm(1, \mathbb{R}) = \{ \pm 1 \} \cong \mathbb{Z}/2\mathbb{Z} \).

The representations in the discrete series of \( GL(2, \mathbb{R}) \), denoted \( \mathcal{D}_\ell \) for \( \ell \in \mathbb{N} \ (\ell \geq 1) \) are induced from \( SL^\pm(2, \mathbb{R}) \) \[8, \text{p.399}\]:
\[ \mathcal{D}_\ell = \text{ind}_{SL^\pm(2, \mathbb{R}), SL^\pm(2, \mathbb{R})} (\mathcal{D}_\ell^\pm), \]
where \( \mathcal{D}_\ell^\pm \) acts in the space
\[ \{ f : \mathcal{H} \to \mathbb{C} | f \text{ analytic }, \|f\|^2 = \int \int |f(z)|^2 y^{\ell-1} dx dy < \infty \}. \]
Here, \( \mathcal{H} \) denotes the Poincaré upper half plane. The action of \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) is given by
\[ \mathcal{D}_\ell^\pm(g)(f(z)) = (bz + d)^{-(\ell+1)}f\left( \frac{az + c}{bz + d} \right). \]

More generally, an element \( \sigma \) from the discrete series \( E_2^0(M) \) is given by
\[ \sigma = i_{G,MN}(\mathcal{D}_{\ell_1}^\pm \otimes \ldots \otimes \mathcal{D}_{\ell_q}^\pm \otimes \tau_1 \otimes \ldots \otimes \tau_r \otimes \tau^\dagger \otimes 1), \]
(2)
where \( \mathcal{D}_{\ell_i}^\pm \) (\( \ell_i \geq 1 \)) are the discrete series representations of \( SL^\pm(2, \mathbb{R}) \) and \( \tau_j \) is a representation of \( SL^\pm(1, \mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} \), i.e. \( id = (x \mapsto x) \) or \( \text{sgn} = (x \mapsto \frac{x}{|x|}) \).

Finally we will compute the unramified characters \( X(M) \), where \( M \) is the Levi subgroup associated to the partition \( n = 2q + r \).

Let \( x \in GL(2, \mathbb{R}) \). Any character of \( GL(2, \mathbb{R}) \) is given by
\[ \chi(\det(x)) = (\text{sgn}(\det(x)))^\varepsilon |\det(x)|^t \]
(\( \varepsilon = 0, 1, t \in \mathbb{R} \)) and it is unramified provided that
\[ \chi(\det(g)) = \chi(\pm 1) = (\pm 1)^\varepsilon = 1, \text{ for all } g \in SL^\pm(2, \mathbb{R}). \]
This implies \( \varepsilon = 0 \) and any unramified character of \( GL(2, \mathbb{R}) \) has the form
\[ \chi(x) = |\det(x)|^t, \text{ for some } t \in \mathbb{R}. \]
(3)
Similarly, any unramified character of \( GL(1, \mathbb{R}) = \mathbb{R}^\times \) has the form
\[ \xi(x) = |x|^t, \text{ for some } t \in \mathbb{R}. \]
(4)
Given a block diagonal matrix \( \operatorname{diag}(g_1, \ldots, g_q, \omega_1, \ldots, \omega_r) \in M \), where \( g_i \in \operatorname{GL}(2, \mathbb{R}) \) and \( \omega_j \in \operatorname{GL}(1, \mathbb{R}) \), we conclude from (3) and (4) that any unramified character \( \chi \in X(M) \) is given by
\[
\chi(\operatorname{diag}(g_1, \ldots, g_q, \omega_1, \ldots, \omega_r)) = |\det(g_1)|^{t_1} \times \cdots \times |\det(g_q)|^{t_q} \times |\omega_1|^{t_{q+1}} \times \cdots \times |\omega_r|^{t_{q+r}},
\]
for some \( (t_1, \ldots, t_{q+r}) \in \mathbb{R}^{q+r} \). We can denote such element \( \chi \in X(M) \) by \( \chi_{(t_1, \ldots, t_{q+r})} \). We have the following result.

**Proposition 2.2.** Let \( M \) be a Levi subgroup of \( \operatorname{GL}(n, \mathbb{R}) \), associated to the partition \( n = 2q + r \). Then, there is a bijection
\[
X(M) \to \mathbb{R}^{q+r}, \chi_{(t_1, \ldots, t_{q+r})} \mapsto (t_1, \ldots, t_{q+r}).
\]

Let us consider now \( \operatorname{GL}(n, \mathbb{C}) \). The tempered dual of \( \operatorname{GL}(n, \mathbb{C}) \) comprises the unitary principal series in accordance with Harish-Chandra [6, p. 277]. The corresponding Levi subgroup is a maximal torus \( T \cong (\mathbb{C}^*)^n \). It follows that \( T = T^n \) the compact \( n \)-torus.

The principal series representations are given by
\[
\pi_{\ell, it} = i_{G, TU}(\sigma \otimes 1),
\]
where \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_n \) and \( \sigma_j(z) = \left( \frac{z}{|z|} \right)^t |z|^{t_j} \) (\( \ell_j \in \mathbb{Z} \) and \( t_j \in \mathbb{R} \)).

An unramified character is given by
\[
\chi \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = |z_1|^{\ell_1} \times \cdots \times |z_n|^{\ell_n}
\]
and we can represent \( \chi \) as \( \chi_{(t_1, \ldots, t_n)} \). Therefore, we have the following result.

**Proposition 2.3.** Denote by \( T \) the standard maximal torus in \( \operatorname{GL}(n, \mathbb{C}) \). There is a bijection
\[
X(T) \to \mathbb{R}^n, \chi_{(t_1, \ldots, t_n)} \mapsto (t_1, \ldots, t_n).
\]

### 3 \( K \)-theory for \( \operatorname{GL}(n) \)

Using the Harish-Chandra parametrization of the tempered dual of \( \operatorname{GL}(n, \mathbb{R}) \) and \( \operatorname{GL}(n, \mathbb{C}) \) (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the \( K \)-theory of the reduced \( C^* \)-algebras \( C^*_r \operatorname{GL}(n, \mathbb{R}) \) and \( C^*_r \operatorname{GL}(n, \mathbb{C}) \).
3.1 \textbf{$K$-theory for $\text{GL}(n, \mathbb{R})$}

We exploit the strong Morita equivalence described in \cite[§1.2]{12}. We infer that

\[ K^j(C_\ast^r \text{GL}(n, \mathbb{R})) = K^j\left( \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) \right) \]
\[ = \bigoplus_{(M, \sigma)} K^j(X(M)/W_\sigma(M)) \]
\[ = \bigoplus_{(M, \sigma)} K^j(\mathbb{R}^{n_M}/W_\sigma(M)), \quad (6) \]

where $n_M = q + r$ if $M$ is a representative of the equivalence class of Levi subgroup associated to the partition $n = 2q + r$. Hence the $K$-theory depends on $n$ and on each Levi subgroup.

To compute (6) we have to consider the following orbit spaces:

\begin{enumerate}[(i)]
\item $\mathbb{R}^n$, in which case $W_\sigma(M)$ is the trivial subgroup of the Weil group $W(M)$;
\item $\mathbb{R}^n/S_n$, where $W_\sigma(M) = W(M)$ (this is one of the possibilities for the partition of $n$ into 1’s);
\item $\mathbb{R}^n/(S_{n_1} \times \ldots \times S_{n_k})$, where $W_\sigma(M) = S_{n_1} \times \ldots \times S_{n_k} \subset W(M)$ (see the examples below).
\end{enumerate}

\textbf{Definition 3.1.} An orbit space as indicated in (ii) and (iii) is called a closed cone.

The $K$-theory for $\mathbb{R}^n$ may be summarized as follows

\[ K^j(\mathbb{R}^n) = \begin{cases} 
\mathbb{Z} & \text{if } n = j \mod 2 \\
0 & \text{otherwise} .
\end{cases} \]

The next results show that the $K$-theory of a closed cone vanishes.

\textbf{Lemma 3.2.} $K^j(\mathbb{R}^n/S_n) = 0$, $j = 0, 1$.

\textit{Proof.} We need the following definition. A point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ is called normalized if $a_j \leq a_{j+1}$, for $j = 1, 2, \ldots, n - 1$. Therefore, in each orbit there is exactly one normalized point and $\mathbb{R}^n/S_n$ is homeomorphic to the subset of $\mathbb{R}^n$ consisting of all normalized points of $\mathbb{R}^n$. We denote the set of all normalized points of $\mathbb{R}^n$ by $N(\mathbb{R}^n)$.

In the case of $n = 2$, let $(a_1, a_2)$ be a normalized point of $\mathbb{R}^2$. Then, there is a unique $t \in [1, +\infty[$ such that $a_2 = ta_1$ and the map

\[ \mathbb{R} \times [1, +\infty[ \rightarrow N(\mathbb{R}^2), (a, t) \mapsto (a, ta) \]

is a homeomorphism.
If $n > 2$ then the map
\[
N(\mathbb{R}^{n-1}) \times [1, +\infty[ \to N(\mathbb{R}^n), (a_1, \ldots, a_{n-1}, t) \mapsto (a_1, \ldots, a_{n-1}, ta_n)
\]
is a homeomorphism. Since $[1, +\infty[$ kills both the $K$-theory groups $K^0$ and $K^1$, the result follows by applying Künneth formula. □

The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permuting the components. This induces an action of any subgroup $S_{n_1} \times \ldots \times S_{n_k}$ of $S_n$ on $\mathbb{R}^n$. Write
\[
\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k} \times \mathbb{R}^{n_1-\ldots-n_k}.
\]
If $n = n_1 + \ldots + n_k$ then we simply have $\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}$.

The group $S_{n_1} \times \ldots \times S_{n_k}$ acts on $\mathbb{R}^n$ as follows.

$S_{n_1}$ permutes the components of $\mathbb{R}^{n_1}$ leaving the remaining fixed; $S_{n_2}$ permutes the components of $\mathbb{R}^{n_2}$ leaving the remaining fixed; and so on. If $n > n_1 + \ldots + n_k$ the components of $\mathbb{R}^{n_1-\ldots-n_k}$ remain fixed. This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identify the orbit spaces
\[
\mathbb{R}^n / (S_{n_1} \times \ldots \times S_{n_k}) \cong \mathbb{R}^{n_1} / S_{n_1} \times \ldots \times \mathbb{R}^{n_k} / S_{n_k} \times \mathbb{R}^{n_1-\ldots-n_k}.
\]

Lemma 3.3. $K^j(\mathbb{R}^n / (S_{n_1} \times \ldots \times S_{n_k})) = 0$, $j = 0, 1$, where $S_{n_1} \times \ldots \times S_{n_k} \subset S_n$.

Proof. It suffices to prove for $\mathbb{R}^n / (S_{n_1} \times S_{n_2})$. The general case follows by induction on $k$.

Now, $\mathbb{R}^n / (S_{n_1} \times S_{n_2}) \cong \mathbb{R}^{n_1} / S_{n_1} \times \mathbb{R}^{n_2} / S_{n_2}$. Applying the Künneth formula and Lemma 3.2, the result follows. □

We give now some examples by computing $K_j C^*_r GL(n, \mathbb{R})$ for small $n$.

Example 3.4. We start with the case of $GL(1, \mathbb{R})$. We have:
\[
M = \mathbb{R}^\times, \quad 0M = \mathbb{Z}/2\mathbb{Z}, \quad W(M) = 1 \text{ and } X(M) = \mathbb{R}.
\]
Hence,
\[
\mathcal{A}_1^1(\mathbb{R}) \cong \bigsqcup_{\sigma \in (\mathbb{Z}/2\mathbb{Z})} \mathbb{R}/1 = \mathbb{R} \sqcup \mathbb{R},
\]
and the $K$-theory is given by
\[
K_j C^*_r GL(1, \mathbb{R}) \cong K^j(\mathcal{A}_1^1(\mathbb{R})) = K^j(\mathbb{R} \sqcup \mathbb{R}) = K^j(\mathbb{R}) \oplus K^j(\mathbb{R}) = \begin{cases}
\mathbb{Z} \oplus \mathbb{Z}, & j = 1 \\
0, & j = 0.
\end{cases}
\]

8
Example 3.5. For $GL(2, \mathbb{R})$ we have two partitions of $n = 2$ and the following data

<table>
<thead>
<tr>
<th>Partition</th>
<th>$M$</th>
<th>$^0M$</th>
<th>$W(M)$</th>
<th>$X(M)$</th>
<th>$\sigma \in E_2(0M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2+0$</td>
<td>$GL(2, \mathbb{R}) \times (\mathbb{R}^\times)^2$</td>
<td>$SL^\pm(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>$1$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{R}^2$</td>
</tr>
<tr>
<td>$1+1$</td>
<td>$GL(2, \mathbb{R}) \times (\mathbb{R}^\times)^2$</td>
<td>$SL^\pm(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>$1$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{R}^2$</td>
</tr>
</tbody>
</table>

Then the tempered dual is parameterized as follows

$$A_2^t(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \bigsqcup_{\ell \in \mathbb{N}} \mathbb{R} \sqcup (\mathbb{R}^2 / S_2) \sqcup (\mathbb{R}^2 / S_2) \sqcup \mathbb{R}^2,$$

and the $K$-theory groups are given by

$$K_j C^*_r GL(2, \mathbb{R}) \cong K^j(A_2^t(\mathbb{R})) \cong \bigoplus_{\ell \in \mathbb{N}} K^j(\mathbb{R}) \oplus K^j(\mathbb{R}^2) = \left\{ \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} \right\}, j = 1 \quad \mathbb{Z}, j = 0.$$

Example 3.6. For $GL(3, \mathbb{R})$ there are two partitions for $n = 3$, to which correspond the following data

<table>
<thead>
<tr>
<th>Partition</th>
<th>$M$</th>
<th>$^0M$</th>
<th>$W(M)$</th>
<th>$X(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2+1$</td>
<td>$GL(2, \mathbb{R}) \times (\mathbb{R}^\times)^3$</td>
<td>$SL^\pm(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})^3$</td>
<td>$1$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>$1+1+1$</td>
<td>$GL(2, \mathbb{R}) \times (\mathbb{R}^\times)^3$</td>
<td>$SL^\pm(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})^3$</td>
<td>$1$</td>
<td>$S_3$</td>
</tr>
</tbody>
</table>

For the partition $3 = 2 + 1$, an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(D_\ell^+ \otimes \tau), \ell \in \mathbb{N} \text{ and } \tau \in (\hat{\mathbb{Z}}/2\mathbb{Z}).$$

For the partition $3 = 1 + 1 + 1$, an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(\bigotimes_{i=1}^3 \tau_i), \tau_i \in (\hat{\mathbb{Z}}/2\mathbb{Z}).$$

The tempered dual is parameterized as follows

$$A_3^t(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \bigsqcup_{\mathbb{N} \times (\mathbb{Z}/2\mathbb{Z})} (\mathbb{R}^2 / S_3) \sqcup (\mathbb{R}^3 / S_3).$$

The $K$-theory groups are given by

$$K_j C^*_r GL(3, \mathbb{R}) \cong K^j(A_3^t(\mathbb{R})) \cong \bigoplus_{\mathbb{N} \times (\mathbb{Z}/2\mathbb{Z})} K^j(\mathbb{R}^2) \oplus 0 = \left\{ \bigoplus_{\mathbb{N} \times (\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} \right\}, j = 0 \quad \mathbb{Z}, j = 1.$$

9
The general case of $GL(n, \mathbb{R})$ will now be considered. It can be split in two cases: $n$ even and $n$ odd.

- $n = 2q$ even

Suppose $n$ is even. For every partition $n = 2q + r$, either $W_\sigma(M) = 1$ or $W_\sigma(M) \neq 1$. If $W_\sigma(M) \neq 1$ then $\mathbb{R}^n / W_\sigma(M)$ is a cone and the $K$-groups $K^0$ and $K^1$ both vanish. This happens precisely when $r > 2$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the $K$-theory with non-zero $K$-groups.

<table>
<thead>
<tr>
<th>Partition</th>
<th>$M$</th>
<th>$^0M$</th>
<th>$W(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2q$</td>
<td>$GL(2, \mathbb{R})^q$</td>
<td>$SL^+(2, \mathbb{R})^q$</td>
<td>$S_q$</td>
</tr>
<tr>
<td>$2(q - 1) + 2$</td>
<td>$GL(2, \mathbb{R})^{q-1} \times (\mathbb{R}^\times)^2$</td>
<td>$SL^+(2, \mathbb{R})^{q-1} \times (\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>$S_{q-1} \times (\mathbb{Z}/2\mathbb{Z})$</td>
</tr>
</tbody>
</table>

We also have $X(M) \cong \mathbb{R}^q$ for $n = 2q$, and $X(M) \cong \mathbb{R}^{q+1}$, for $n = 2(q - 1) + 2$.

For the partition $n = 2q$ ($r = 0$), an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(D_{\ell_1}^+ \otimes \ldots \otimes D_{\ell_q}^+) , \ (\ell_1, \ldots, \ell_q) \in \mathbb{N}^q \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$  

For the partition $n = 2(q - 1) + 2$ ($r = 2$), an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(D_{\ell_1}^+ \otimes \ldots \otimes D_{\ell_{q-1}}^+ \otimes id \otimes sgn) , \ (\ell_1, \ldots, \ell_{q-1}) \in \mathbb{N}^{q-1} \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$  

Therefore, the tempered dual has the following form

$$\mathcal{A}_t^q(\mathbb{R}) = \bigoplus_{\ell \in \mathbb{N}^q} \mathbb{R}^\ell \sqcup \bigoplus_{\ell' \in \mathbb{N}^{q-1}} \mathbb{R}^{\ell' \otimes 1} \sqcup C$$

where $C$ is a disjoint union of closed cones as in Definition 3.1.

**Theorem 3.7.** Suppose $n = 2q$ is even. Then the $K$-groups are

$$K^j_C GL(n, \mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in \mathbb{N}^q} \mathbb{Z} , & j \equiv q(\text{mod} 2) \\ \bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z} , & \text{otherwise}. \end{cases}$$

If $q = 1$ then the direct sum $\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z}$ will denote a single copy of $\mathbb{Z}$.

- $n = 2q + 1$ odd

If $n$ is odd only one partition contributes to the $K$-theory of $GL(n, \mathbb{R})$ with non-zero $K$-groups:
<table>
<thead>
<tr>
<th>Partition</th>
<th>$M$</th>
<th>$GL(2, \mathbb{R})^{2q+1} \times \mathbb{R}^\times$</th>
<th>$W(M)$</th>
<th>$X(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2q+1$</td>
<td>$GL(2, \mathbb{R})^{2q+1} \times \mathbb{R}^\times$</td>
<td>$SL^+(2, \mathbb{R})^q \times (\mathbb{Z}/2\mathbb{Z})$</td>
<td>$S_q$</td>
<td>$\mathbb{R}^{2q+1}$</td>
</tr>
</tbody>
</table>

An element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(D_{\ell_1}^+ \otimes \cdots \otimes D_{\ell_q}^+ \otimes \tau), \quad (\ell_1, \ldots, \ell_q, \tau) \in \mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}) \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$ 

The tempered dual is given by

$$A_{n}(\mathbb{R}) = A_{2q+1}(\mathbb{R}) = \bigoplus_{\ell \in (\mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}))} \mathbb{R}^{q+1} \sqcup C,$$

where $C$ is a disjoint union of closed cones as in Definition 3.1.

**Theorem 3.8.** Suppose $n = 2q + 1$ is odd. Then the $K$-groups are

$$K_j C_r^* GL(n, \mathbb{R}) \cong \left\{ \bigoplus_{\ell \in \mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z})} \mathbb{Z}, \quad j \equiv q + 1 (\text{mod2}) \right. \quad 0, \quad \text{otherwise}.$$

Here, we use the following convention: if $q = 0$ then the direct sum is $\bigoplus_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$.

We conclude that the $K$-theory of $C_r^* GL(n, \mathbb{R})$ depends on essentially one parameter $q$ given by the maximum number of 2’s in the partitions of $n$ into 1’s and 2’s. If $n$ is even then $q = \frac{n}{2}$ and if $n$ is odd then $q = \frac{n-1}{2}$.

### 3.2 $K$-theory for $GL(n, \mathbb{C})$

Let $T^o$ be the maximal compact subgroup of the maximal compact torus $T$ of $GL(n, \mathbb{C})$. Let $\sigma$ be a unitary character of $T^o$. We note that $W = W(T)$, $W_\sigma = W_\chi(T)$. If $W_\sigma = 1$ then we say that the orbit $W \cdot \sigma$ is generic.

**Theorem 3.9.** The $K$-theory of $C_r^* GL(n, \mathbb{C})$ admits the following description. If $n = j \mod 2$ then $K_j$ is free abelian on countably many generators, one for each generic $W$-orbit in the unitary dual of $T^o$, and $K_{j+1} = 0$.

**Proof.** We exploit the strong Morita equivalence described in [13, Prop. 4.1].

We have a homeomorphism of locally compact Hausdorff spaces:

$$A_n^i(\mathbb{C}) \cong \bigsqcup X(T)/W_\sigma(T)$$

by the Harish-Chandra Plancherel Theorem for complex reductive groups [6], and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [13]. The result now follows from Lemma 4.3.
4 Langlands parameters for $GL(n)$

The Weil group of $\mathbb{C}$ is simply

$$W_{\mathbb{C}} \cong \mathbb{C}^\times,$$

and the Weil group of $\mathbb{R}$ can be written as disjoint union

$$W_{\mathbb{R}} \cong \mathbb{C}^\times \sqcup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jcj^{-1} = \overline{c}$ ($\overline{c}$ denotes complex conjugation). We shall use this disjoint union to describe the representation theory of $W_{\mathbb{R}}$.

**Definition 4.1.** An $L$- parameter is a continuous homomorphism

$$\phi : W_F \to GL(n, \mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$.

$L$-parameters are also called Langlands parameters. Two $L$-parameters are equivalent if they are conjugate under $GL(n, \mathbb{C})$. The set of equivalence classes of $L$-parameters is denoted by $\mathcal{G}_n$. And the set of equivalence classes of $L$-parameters whose image is bounded is denoted by $\mathcal{G}^t_n$.

Let $F$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}_n(F)$ (resp. $\mathcal{A}^t_n(F)$) denote the smooth dual (resp. tempered dual) of $GL(n, F)$. The local Langlands correspondence is a bijection

$$\mathcal{G}_n(F) \to \mathcal{A}_n(F).$$

In particular,

$$\mathcal{G}^t_n(F) \to \mathcal{A}^t_n(F)$$

is also a bijection.

We are only interested in $L$-parameters whose image is bounded. In the sequel we will refer to them, for simplicity, as $L$-parameters.

**L-parameters for $W_{\mathbb{C}}$**

A 1-dimensional $L$-parameter for $W_{\mathbb{C}}$ is simply a character of $\mathbb{C}^\times$ (i.e. a unitary quasicharacter):

$$\chi(z) = \left(\frac{z}{\overline{z}}\right)^{\ell} \otimes |z|^t$$

where $|z| = |z|_C = z\overline{z}$, $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$. To emphasize the dependence on parameters $(\ell, t)$ we write sometimes $\chi = \chi_{\ell, t}$. 

12
An $n$-dimensional $L$-parameter can be written as a direct sum of $n$ 1-dimensional characters of $\mathbb{C}^\times$:

$$\phi = \phi_1 \oplus \ldots \oplus \phi_n,$$

with $\phi_k(z) = (z_k | z_k^t)^{\ell_k} \otimes |z|^{\ell_k}, \ell_k \in \mathbb{Z}, t_k \in \mathbb{R}, k = 1, \ldots, n$.

**$L$-parameters for $W_\mathbb{R}$**

The 1-dimensional $L$-parameters for $W_\mathbb{R}$ are as follows

$$\begin{cases} 
\phi_{\varepsilon,t}(z) = |z|^t, & \varepsilon \in \{0, 1\}, t \in \mathbb{R}.
\end{cases}$$

We may now describe the local Langlands correspondence for $GL(1, \mathbb{R})$:

$$\begin{align*}
\phi_{0,t} &\mapsto 1 \otimes |z|^t \\
\phi_{1,t} &\mapsto sgn \otimes |z|^t
\end{align*}$$

Now, we consider 2-dimensional $L$-parameters for $W_\mathbb{R}$:

$$\phi_{\ell,t}(z) = \begin{pmatrix} \chi_{\ell,t}(z) & 0 \\ 0 & \chi_{\ell,t}(z) \end{pmatrix}, \quad \phi_{\ell,t}(j) = \begin{pmatrix} 0 & (-1)^{\ell} \\ 1 & 0 \end{pmatrix}.$$

with $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$.

and

$$\phi_{m,t,n,s}(z) = \begin{pmatrix} \chi_{0,t}(z) & 0 \\ 0 & \chi_{0,s}(z) \end{pmatrix}, \quad \phi_{m,t,n,s}(j) = \begin{pmatrix} (-1)^m & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

with $m, n \in \{0, 1\}$ and $t, s \in \mathbb{R}$.

The local Langlands correspondence for $GL(2, \mathbb{R})$ may be described as follows.

The $L$-parameter $\phi_{m_1,t_1,m_2,t_2}$ corresponds, via Langlands correspondence, to the unitary principal series:

$$\phi_{m_1,t_1,m_2,t_2} \mapsto \pi(\mu_1, \mu_2),$$

where $\mu_i$ is the character of $\mathbb{R}^\times$ given by

$$\mu_i(x) = \left(\frac{x}{|x|}\right)^{m_i} |x|^{it_i}, m_i \in \{0, 1\}, t_i \in \mathbb{R}.$$
The $L$-parameter $\phi_{\ell,t}$ corresponds, via the Langlands correspondence, to the discrete series:

$$\phi_{\ell,t} \mapsto D_{\ell} \otimes |\text{det}(\cdot)|_{\mathbb{R}}^{t}, \quad \text{with} \quad \ell \in \mathbb{N}, t \in \mathbb{R}.$$ 

**Proposition 4.2.**

(i) $\phi_{\ell,t} \cong \phi_{-\ell,t}$;

(ii) $\phi_{\ell,m,s} \cong \phi_{m,s,\ell,t}$;

(iii) $\phi_{0,t} \cong \phi_{1,t,0,t} \cong \phi_{0,t,1,t}$;

The proof is elementary. We now quote the following result.

**Lemma 4.3.** [8] Every finite-dimensional semi-simple representation $\phi$ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.

## 5 Base change

We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change $\mathbb{C}/\mathbb{R}$. We have $W_{\mathbb{C}} \subset W_{\mathbb{R}}$ and there is a natural map

$$\text{Res}^{W_{\mathbb{R}}}_{W_{\mathbb{C}}} : \mathcal{G}_n(\mathbb{R}) \longrightarrow \mathcal{G}_n(\mathbb{C})$$

called *restriction*. By the local Langlands correspondence for archimedean fields [3, Theorem 3.1, p.236][8], there is a base change map

$$BC : \mathcal{A}_n(\mathbb{R}) \longrightarrow \mathcal{A}_n(\mathbb{C})$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{A}_n(\mathbb{R}) & \xrightarrow{BC} & \mathcal{A}_n(\mathbb{C}) \\
\downarrow_{\mathbb{C}L_n} & & \downarrow_{\mathbb{C}L_n} \\
\mathcal{G}_n(\mathbb{R}) & \xrightarrow{\text{Res}^{W_{\mathbb{R}}}_{W_{\mathbb{C}}}} & \mathcal{G}_n(\mathbb{C})
\end{array}$$

Arthur and Clozel’s book [1] gives a full treatment of base change for $GL(n)$. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let $\chi_i$ ($i = 1, \ldots, q$) be a ramified character of $\mathbb{C}^\times$ and let $\xi_j$ ($j = 1, \ldots, r$) be a ramified character of $\mathbb{R}^\times$. Since the $\chi_i$’s
are ramified, $\chi_i(z) \neq \chi_i^\tau(z) = \chi_i(\overline{z})$, where $\tau$ is a generator of $\text{Gal}(\mathbb{C}/\mathbb{R})$. By Langlands classification \[8\], each $\chi_i$ defines a discrete series representation $\pi(\chi_i)$ of $\text{GL}(2, \mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i^\tau)$. Denote by $\pi(\chi_1, \ldots, \chi_q, \xi_1, \ldots, \xi_r)$ the generalized principal series representation of $\text{GL}(n, \mathbb{R})$.

\[ \pi(\chi_1, \ldots, \chi_q, \xi_1, \ldots, \xi_r) = i_{\text{GL}(n, \mathbb{R}), MN}(\pi(\chi_1) \otimes \cdots \otimes \pi(\chi_q) \otimes \xi_1 \otimes \cdots \otimes \xi_r \otimes 1). \quad (10) \]

The base change map for the general principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

\[ \text{BC}(\pi) = \Pi(\chi_1, \ldots, \chi_q, \xi_1, \ldots, \xi_r) = i_{\text{GL}(n, \mathbb{C}), B(\mathbb{C})}(\chi_1, \chi_1^\tau, \ldots, \chi_q, \chi_q^\tau, \xi_1 \circ N, \ldots, \xi_r \circ N), \quad (11) \]

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ is the norm map defined by $z \mapsto z\overline{z}$.

We illustrate the base change map with two simple examples.

**Example 5.1.** For $n = 1$, base change is simply composition with the norm map

\[ \text{BC} : \mathcal{A}_1^1(\mathbb{R}) \rightarrow \mathcal{A}_1^1(\mathbb{C}) \ , \ \text{BC}(\chi) = \chi \circ N. \]

**Example 5.2.** For $n = 2$, there are two different kinds of representations, one for each partition of 2. According to (10), $\pi(\chi)$ corresponds to the partition $2 = 2 + 0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1 + 1$. Then the base change map is given, respectively, by

\[ \text{BC}(\pi(\chi)) = i_{\text{GL}(2, \mathbb{C}), B(\mathbb{C})}(\chi, \chi^\tau), \]

and

\[ \text{BC}(\pi(\xi_1, \xi_2)) = i_{\text{GL}(2, \mathbb{C}), B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N). \]

### 5.1 The base change map

Now, we define base change as a map of topological spaces and study the induced $K$-theory map.

**Proposition 5.3.** The base change map $\text{BC} : \mathcal{A}_n^1(\mathbb{R}) \rightarrow \mathcal{A}_n^1(\mathbb{C})$ is a continuous proper map.

**Proof.** First, we consider the case $n = 1$. As we have seen in Example 5.1, base change for $\text{GL}(1)$ is the map given by $\text{BC}(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}_1^1(\mathbb{R})$, where $N : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

\[ \text{BC}(\chi)(z) = \chi(|z|^2) = |z|^{2u}. \quad (12) \]
A generic element from \( \mathcal{A}_1^t(\mathbb{C}) \) has the form

\[
\mu(z) = \left( \frac{z}{|z|} \right)^\ell |z|^t,
\]

where \( \ell \in \mathbb{Z} \) and \( t \in S^1 \), as stated before. Viewing the Pontryagin duals \( \mathcal{A}_1^t(\mathbb{R}) \) and \( \mathcal{A}_1^t(\mathbb{C}) \) as topological spaces by forgetting the group structure, and comparing (12) and (13), the base change map can be defined as the following continuous map

\[
\varphi : \mathcal{A}_1^t(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathcal{A}_1^t(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z}
\]

\[
\chi = (t, \varepsilon) \quad \mapsto \quad (2t, 0)
\]

A compact subset of \( \mathbb{R} \times \mathbb{Z} \) in the connected component \( \{ \ell \} \) of \( \mathbb{Z} \) has the form \( K \times \{ \ell \} \subset \mathbb{R} \times \mathbb{Z} \), where \( K \subset \mathbb{R} \) is compact. We have

\[
\varphi^{-1}(K \times \{ \ell \}) = \begin{cases} 
\emptyset, & \text{if } \ell \neq 0 \\
\frac{1}{2}K \times \{ \varepsilon \}, & \text{if } \ell = 0
\end{cases}
\]

where \( \varepsilon \in \mathbb{Z}/2\mathbb{Z} \). Therefore \( \varphi^{-1}(K \times \{ \ell \}) \) is compact and \( \varphi \) is proper.

The Case \( n > 1 \). Base change determines a map \( BC : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C}) \) of topological spaces. Let \( X = X(M)/W_{\sigma}(M) \) be a connected component of \( \mathcal{A}_n^t(\mathbb{R}) \). Then, \( X \) is mapped under \( BC \) into a connected component \( Y = Y(T)/W_{\sigma'}(T) \) of \( \mathcal{A}_n^t(\mathbb{C}) \). Given a generalized principal series representation

\[
\pi(\chi_1, \ldots, \chi_q, \xi_1, \ldots, \xi_r)
\]

where the \( \chi_i \)'s are ramified characters of \( \mathbb{C}^\times \) and the \( \xi_i \)'s are ramified characters of \( \mathbb{R}^\times \), then

\[
BC(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, \ldots, \chi_q, \chi_q^\tau, \xi_1 \circ N, \ldots, \xi_r \circ N).
\]

Here, \( N = N_{\mathbb{C}/\mathbb{R}} \) is the norm map and \( \tau \) is the generator of \( Gal(\mathbb{C}/\mathbb{R}) \).

We associate to \( \pi \) the usual parameters uniquely defined for each character \( \chi \) and \( \xi \). For simplicity, we write the set of parameters as a \((q + r)\)-uple:

\[
(t, t') = (t_1, \ldots, t_q, t_1', \ldots, t_r') \in \mathbb{R}^{q+r} \cong X(M).
\]

Now, if \( \pi(\chi_1, \ldots, \chi_q, \xi_1, \ldots, \xi_r) \) lies in the connected component defined by the fixed parameters \( (t, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r \), then

\[
(t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T)
\]

is a continuous proper map.
It follows that
\[
\mathcal{BC} : X(M)/W_\sigma(M) \to Y(T)/W_\sigma(T)
\]
is continuous and proper since the orbit spaces are endowed with the quotient topology.

\textbf{Theorem 5.4.} The functorial map induced by base change
\[
K_j(C^*_r GL(n, \mathbb{C})) \xrightarrow{K_j(\mathcal{BC})} K_j(C^*_r GL(n, \mathbb{R}))
\]
is zero for \( n > 1 \).

\textit{Proof.} We start with the case \( n > 2 \). Let \( n = 2q + r \) be a partition and \( M \) the associated Levi subgroup of \( GL(n, \mathbb{R}) \). Denote by \( X_\mathbb{R}(M) \) the unramified characters of \( M \). As we have seen, \( X_\mathbb{R}(M) \) is parametrized by \( \mathbb{R}^{q+r} \). On the other hand, the only Levi subgroup of \( GL(n, \mathbb{C}) \) for \( n = 2q + r \) is the diagonal subgroup \( X_\mathbb{C}(M) = (\mathbb{C}^\times)^{2q+r} \).

If \( q = 0 \) then \( r = n \) and both \( X_\mathbb{R}(M) \) and \( X_\mathbb{C}(M) \) are parametrized by \( \mathbb{R}^n \). But then in the real case an element \( \sigma \in E_2^0(M) \) is given by
\[
\sigma = i_{GL(n, \mathbb{R}), P}(\chi_1 \otimes \cdots \otimes \chi_n),
\]
with \( \chi_i \in \mathbb{Z}/2\mathbb{Z} \). Since \( n \geq 3 \) there is always repetition of the \( \chi_i \)'s. It follows that the isotropy subgroups \( W_\sigma(M) \) are all nontrivial and the quotient spaces \( \mathbb{R}^n/W_\sigma \) are closed cones. Therefore, the \( K \)-theory groups vanish.

If \( q \neq 0 \), then \( X_\mathbb{R}(M) \) is parametrized by \( \mathbb{R}^{q+r} \) and \( X_\mathbb{C}(M) \) is parametrized by \( \mathbb{R}^{2q+r} \) (see Propositions 2.2 and 2.3).

Base change creates a map
\[
\mathbb{R}^{q+r} \to \mathbb{R}^{2q+r}.
\]
Composing with the stereographic projections we obtain a map
\[
S^{q+r} \to S^{2q+r}
\]
between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced \( K \)-theory map
\[
K^j(S^{2q+r}) \to K^j(S^{q+r})
\]
is the zero map.

The Case \( n = 2 \). For \( n = 2 \) there are two Levi subgroups of \( GL(2, \mathbb{R}) \), \( M_1 \cong GL(2, \mathbb{R}) \) and the diagonal subgroup \( M_2 \cong (\mathbb{R}^\times)^2 \). By Proposition 2.2
$X(M_1)$ is parametrized by $\mathbb{R}$ and $X(M_2)$ is parametrized by $\mathbb{R}^2$. The group $GL(2, \mathbb{C})$ has only one Levi subgroup, the diagonal subgroup $M \cong (\mathbb{C}^*)^2$. From Proposition 2.3 it is parametrized by $\mathbb{R}^2$.

Since $K^1(\mathcal{A}_2^i(\mathbb{C})) = 0$ by Theorem 5.1, we only have to consider the $K^0$ functor. The only contribution to $K^0(\mathcal{A}_2^i(\mathbb{R}))$ comes from $M_2 \cong (\mathbb{R}^*)^2$ and we have (see Example 3.5)

$$K^0(\mathcal{A}_2^i(\mathbb{R})) \cong \mathbb{Z}.$$ 

For the Levi subgroup $M_2 \cong (\mathbb{R}^*)^2$, base change is

$$\text{BC} : \mathcal{A}_2^i(\mathbb{R}) \rightarrow \mathcal{A}_2^i(\mathbb{C})$$

$$\pi(\xi_1, \xi_2) \mapsto i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N),$$

Therefore, it maps a class $[t_1, t_2]$, which lies in the connected component $(\varepsilon_1, \varepsilon_2)$, into the class $[2t_1, 2t_2]$, which lies in the connected component $(0, 0)$. In other words, base change maps a generalized principal series $\pi(\xi_1, \xi_2)$ into a nongeneric point of $\mathcal{A}_2^i(\mathbb{C})$. It follows from Theorem 3.9 that

$$K^0(\text{BC}) : K^0(\mathcal{A}_2^i(\mathbb{R})) \rightarrow K^0(\mathcal{A}_2^i(\mathbb{C}))$$

is the zero map.

\[ \square \]

## 5.2 Base change in one dimension

In this section we consider base change for $GL(1)$.

**Theorem 5.5.** The functorial map induced by base change

$$K_1(C^* GL(1, \mathbb{C})) \xrightarrow{K_1(\text{BC})} K_1(C^* GL(1, \mathbb{R}))$$

is given by $K_1(\text{BC}) = \Delta \circ \text{Pr}$, where $\text{Pr}$ is the projection of the zero component of $K^1(\mathcal{A}_1^i(\mathbb{C}))$ into $\mathbb{Z}$ and $\Delta$ is the diagonal $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$.

**Proof.** For $GL(1)$, base change

$$\chi \in \mathcal{A}_1^i(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1^i(\mathbb{C})$$

induces a map

$$K^1(\text{BC}) : K^1(\mathcal{A}_1^i(\mathbb{C})) \rightarrow K^1(\mathcal{A}_1^i(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1^i(\mathbb{R})$ is uniquely determined by a pair of parameters $(t, \varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1^i(\mathbb{C})$ is uniquely determined by a pair of parameters $(t, \ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter $\varepsilon$ (resp., $\ell$) labels each connected component of $\mathcal{A}_1^i(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1^i(\mathbb{C}) = \bigcup_{\varepsilon} \mathbb{R}$).
Base change maps each component $\varepsilon$ of $A_t^1(\mathbb{R})$ into the component 0 of $A_t^1(\mathbb{C})$, sending $t \in \mathbb{R}$ to $2t \in \mathbb{R}$. The map $t \mapsto 2t$ is homotopic to the identity.

At the level of $K^1$, the base change map is given by $K^1(BC) = \Delta \circ Pr$, where $Pr$ is the projection of the zero component of $K^1(A_t^1(\mathbb{C}))$ into $\mathbb{Z}$ and $\Delta$ is the diagonal $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

\[\square\]

## 6 Automorphic induction

We begin this section by describing the action of $Gal(\mathbb{C}/\mathbb{R})$ on $\hat{W}_C = \hat{\mathbb{C}}^\times$. Take $\chi = \chi_{\ell,t} \in \hat{\mathbb{C}}^\times$ and let $\tau$ denote the nontrivial element of $Gal(\mathbb{C}/\mathbb{R})$.

Then, $Gal(\mathbb{C}/\mathbb{R})$ acts on $\hat{\mathbb{C}}^\times$ as follows:

$$\chi^\tau(z) = \chi(\overline{z}).$$

Hence,

$$\chi_{\ell,t}^\tau(z) = \left(\frac{\overline{z}}{|z|}\right)^\ell |z|_C^\ell = \left(\frac{\overline{z}}{|z|}\right)^{-\ell} |z|_C^\ell$$

and we conclude that

$$\chi_{\ell,t}^\tau(z) = \chi_{-\ell,t}(z).$$

In particular,

$$\chi^\tau = \chi \iff \ell = 0 \iff \chi = |.|_C^\ell$$

i.e, $\chi$ is unramified.

Note that $W_C \subset W_\mathbb{R}$, with index $[W_\mathbb{R} : W_C] = 2$. Therefore, there is a natural induction map

$$Ind_{\mathbb{C}/\mathbb{R}} : G_1^1(\mathbb{C}) \to G_2^1(\mathbb{R}).$$

By the local Langlands correspondence for archimedean fields $[3, 8]$, there exists an automorphic induction map $\mathcal{A}L_{\mathbb{C}/\mathbb{R}}$ such that the following diagram commutes

\[
\begin{array}{ccc}
A_1^1(\mathbb{C}) & \xrightarrow{\mathcal{A}L_{\mathbb{C}/\mathbb{R}}} & A_1^1(\mathbb{R}) \\
\varepsilon \downarrow & & \downarrow \\
G_1^1(\mathbb{C}) & \xrightarrow{Ind_{\mathbb{C}/\mathbb{R}}} & G_2^1(\mathbb{R})
\end{array}
\]

The next result describes reducibility of induced representations.

19
Proposition 6.1. Let $\chi$ be a character of $W_C$. We have:

(i) If $\chi \neq \chi^\tau$ then $\text{Ind}_{C/R}(\chi)$ is irreducible;

(ii) If $\chi = \chi^\tau$ then $\text{Ind}_{C/R}(\chi)$ is reducible. Moreover, there exist $\rho \in \hat{W}_R$ such that

$$\text{Ind}_{C/R}(\chi) = \rho \oplus \rho^\tau = \rho \oplus \text{sgn.}\rho,$$

where $\rho|_{W_C} = \chi$;

(iii) $\text{Ind}_{C/R}(\chi_1) \cong \text{Ind}_{C/R}(\chi_2)$ if and only if $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^\tau$.

Proof. Apply Frobenius reciprocity

$$\text{Hom}_{W_R}(\text{Ind}_{C/R}(\chi_1), \text{Ind}_{C/R}(\chi_2)) \cong \text{Hom}_{W_C}(\chi_1, \chi_2).$$

Now, $W_R = W_C \sqcup jW_C$. Therefore, the restriction of $\text{Ind}_{C/R}(\chi)$ to $W_C$ is $\chi \oplus \chi^\tau$. The result follow since $\text{Ind}_{C/R}(\chi)$ is semi-simple. □

Proposition 6.2. A finite dimensional continuous irreducible representation of $W_R$ is either a character or isomorphic to some $\text{Ind}_{C/R}(\chi)$, with $\chi \neq \chi^\tau$.

Proof. It follows immediately from Lemma 4.3 □

6.1 The automorphic induction map

In this section we describe automorphic induction map as a map of topological spaces. We begin by considering $n = 1$.

Let $\chi = \chi_{\ell,t}$ be a character of $W_C$. If $\chi \neq \chi^\tau$, by proposition 4.2, $\phi_{\ell,t} \simeq \phi_{-\ell,t}$. Hence,

$$\mathcal{A}_C(\mathcal{L}_1(\chi_{\ell,t})) = D_{|\ell|} \otimes |\text{det}(\cdot)|^{|t|}.$$

On the other hand, if $\chi = \chi^\tau$ then $\chi = \chi_{0,t}$ and $\chi(z) = |z|^{|t|}_C$. Therefore,

$$\mathcal{A}_C(\mathcal{L}_1(|\ell|)) = \mathbb{R}\mathcal{L}_2(\rho \oplus \text{sgn.}\rho) = \pi(\rho, \rho^{-1}),$$

where $\pi(\rho, \rho^{-1})$ is a reducible principal series and $\rho$ is the character of $\mathbb{R}^\times \simeq W_R^{ab}$ associated with $\chi_{0,t} = |\ell|_C^{|t|}$ via class field theory, i.e. $\rho|_{W_C} = \chi$.

Recall that

$$\mathcal{A}_1'(\mathbb{C}) \cong \bigsqcup_{\ell \in \mathbb{Z}} \mathbb{R}$$

and

$$\mathcal{A}_2'(\mathbb{R}) \cong \left( \bigsqcup_{\ell \in \mathbb{N}} \mathbb{R} \right) \bigsqcup (\mathbb{R}^2/S_2) \bigsqcup (\mathbb{R}^2/S_2) \bigsqcup \mathbb{R}^2.$$
As a map of topological spaces, automorphic induction for \( n = 1 \) may be described as follows:

\[
(t, \ell) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, |\ell|) \in \mathbb{R} \times \mathbb{N}, \text{ if } \ell \neq 0
\]

\[
(t, 0) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, t) \rightarrow \mathbb{R}^2, \text{ if } \ell = 0.
\]

More generally, let \( \chi_1 \oplus \ldots \oplus \chi_n \) be an \( n \) dimensional \( L \)-parameter of \( W_\mathbb{C} \). Then, either \( \chi_k \neq \chi_\tau^k \) for every \( k \), in which case automorphic induction is

\[
\mathcal{AI}_{C/R}(\mathcal{L}_n(\chi_1 \oplus \ldots \oplus \chi_n)) = D_{|\ell_1|} \otimes |\text{det}(\cdot)|^{\mu_1} \oplus \ldots \oplus D_{|\ell_n|} \otimes |\text{det}(\cdot)|^{\mu_n}
\]

or for some \( k \) (possibly more than one), \( \chi_k = \chi_\tau^k \), in which case we have

\[
\mathcal{AI}_{C/R}(\mathcal{L}_n(\chi_1 \oplus \ldots \oplus |\chi^k_1 \oplus \ldots \oplus \chi_n|)) = D_{|\ell_1|} \otimes |\text{det}(\cdot)|^{\mu_1} \oplus \ldots \oplus \pi(\rho_k, \rho_k^{-1}) \oplus D_{|\ell_n|} \otimes |\text{det}(\cdot)|^{\mu_n}.
\]

In order to describe automorphic induction as a map of topological spaces, it is enough to consider components of \( \mathcal{A}_n^i(\mathbb{C}) \) with generic \( W \)-orbit. For convenience, we introduce the following notation:

if \((t_1, \ldots, t_n)\) is in the component of \( \mathcal{A}_n^i(\mathbb{C}) \) labeled by \((\ell_1, \ldots, \ell_n)\), i.e.

\[
(t_1, \ldots, t_n) \in (\mathbb{R} \times \{\ell_1\}) \times \ldots \times (\mathbb{R} \times \{\ell_n\})
\]

we write simply

\[
(t_1, \ldots, t_n) \in \mathbb{R}^n_{(\ell_1, \ldots, \ell_n)}; \quad \ell_i \in \mathbb{Z}.
\]

There are two cases:

**Case 1:** \( \chi_k \neq \chi_\tau^k \), i.e. \( \ell_k \neq 0 \), for every \( k \),

\[
\mathcal{AI} : (t_1, \ldots, t_n) \in \mathbb{R}^n_{(\ell_1, \ldots, \ell_n)} \mapsto (t_1, \ldots, t_n) \in \mathbb{R}^n_{(|\ell_1|, \ldots, |\ell_n|)}
\]

So, \((|\ell_1|, \ldots, |\ell_n|)\) \( \in \mathbb{N}^n \).

**Case 2:** if there are \( 0 < m < n \) characters such that \( \chi_k = \chi_\tau^k \), then

\[
\mathcal{AI} : (t_1, \ldots, t_k, \ldots, t_n) \in \mathbb{R}^n_{(\ell_1, \ldots, 0, \ldots, \ell_n)} \mapsto (t_1, \ldots, t_k, t_k, \ldots, t_n) \in (\mathbb{R}^n/W)_{(|\ell_1|, \ldots, |\ell_n|)}^*\]

where \((|\ell_1|, \ldots, |\ell_n|)^* \in \mathbb{N}^{n-m}\) means that we have deleted the \( m \) labels corresponding to \( \ell_k = 0 \). Note that if \( m > 1 \), necessarily \( W \neq 0 \).

We have the following result

**Proposition 6.3.** The automorphic induction map

\[
\mathcal{AI}_{C/R} : \mathcal{A}_n^i(\mathbb{C}) \rightarrow \mathcal{A}_{2n}^i(\mathbb{R})
\]

is a continuous proper map.
The proof follows from the above discussion and is similar to that of proposition 5.3.

**Example 6.4.** Consider $n = 3$. Then,

$$\mathcal{A}^t_3(\mathbb{C}) \simeq \bigcup_{\sigma} \mathbb{R}^3/W_{\sigma}$$

and

$$\mathcal{A}^t_6(\mathbb{R}) \simeq (\bigsqcup_{\ell \in \mathbb{N}^3} \mathbb{R}^3) \sqcup (\bigsqcup_{\ell' \in \mathbb{N}^2} \mathbb{R}^4) \sqcup \mathcal{C},$$

where $\mathcal{C}$ is a disjoint union of cones.

Let $\chi_1 \oplus \chi_2 \oplus \chi_3$ denote a 3-dimensional $L$-parameter of $W_{\mathbb{C}}$. We have the following description of $\mathcal{AI}_{\mathbb{C}/\mathbb{R}}$ as a map of topological spaces:

- $\chi_1 \neq \chi_1^\tau, \chi_2 \neq \chi_2^\tau, \chi_3 \neq \chi_3^\tau$

  $$(t_1, t_2, t_3) \in \mathbb{R}^3_{(\ell_1, \ell_2, \ell_3)} \longmapsto (t_1, t_2, t_3) \in \mathbb{R}^3_{(|\ell_1|, |\ell_2|, |\ell_3|)}$$

  with $\ell_i \in \mathbb{Z}\{0\}$.

- $\chi_1 = \chi_1^\tau, \chi_2 \neq \chi_2^\tau, \chi_3 \neq \chi_3^\tau$

  $$(t_1, t_2, t_3) \in \mathbb{R}^3_{(0, \ell_2, \ell_3)} \longmapsto (t_1, t_1, t_2, t_3) \in (\mathbb{R}^4/W)_{(|\ell_2|, |\ell_3|)}$$

  with $\ell_i \in \mathbb{Z}\{0\}$. Similar for the cases $(\ell_1, 0, \ell_3)$ and $(\ell_1, \ell_2, 0)$.

- $\chi_1 = \chi_1^\tau, \chi_2 = \chi_2^\tau, \chi_3 \neq \chi_3^\tau$

  $$(t_1, t_2, t_3) \in \mathbb{R}^3_{(0, 0, \ell_3)} \longmapsto (t_1, t_1, t_2, t_2, t_3) \in (\mathbb{R}^5/W)_{(|\ell_3|)}$$

  with $\ell_3 \in \mathbb{Z}\{0\}$. Similar for the cases $(\ell_1, 0, 0)$ and $(0, \ell_2, 0)$.

- $\chi_1 = \chi_1^\tau, \chi_2 = \chi_2^\tau, \chi_3 = \chi_3^\tau$

  $$(t_1, t_2, t_3) \in \mathbb{R}^3_{(0, 0, 0)} \longmapsto (t_1, t_1, t_2, t_2, t_2, t_3) \in (\mathbb{R}^6/W)$$

**6.2 Automorphic induction in one dimension**

Automorphic induction $\mathcal{AI}$ induces a $K$-theory map at the level of $K$-theory groups $K^1$:

$$K^1(\mathcal{AI}) : K^1(\mathcal{A}^t_2(\mathbb{R})) \to K^1(\mathcal{A}^t_1(\mathbb{C})).$$ (20)
We have
\[ K^1(\mathcal{A}_2^i(\mathbb{R})) \cong \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z}. \]

Each class of 1-dimension \(L\)-parameters of \(W_C\) (characters of \(\mathbb{C}^\times\))
\[[\chi] = [\chi_{\ell,t}] = [\chi_{-\ell,t}] \quad (\ell \neq 0) \]
contributes with one generator to \(K^1(\mathcal{A}_2^i(\mathbb{R}))\). Note that, under \(\mathcal{A}\mathcal{I}\), this is precisely the parametrization given by the discrete series \(D_{[\ell]}\).

On the other hand,
\[ K^1(\mathcal{A}_1^i(\mathbb{C})) \cong \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}. \]

Again, each class (of characters of \(\mathbb{C}^\times\)) \([\chi]\) contributes with a generator to \(K^1(\mathcal{A}_1^i(\mathbb{C}))\), only this time \([\chi_{\ell,t}] \neq [\chi_{-\ell,t}]\), i.e., \(\ell\) and \(-\ell\) belong to different classes.

Note that we may write
\[ K^1(\mathcal{A}_2^i(\mathbb{R})) \cong \bigoplus_{\text{Discrete series}} \mathbb{Z} = \bigoplus_{[D_{[\ell]}]} \mathbb{Z} = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} \]
and
\[ K^1(\mathcal{A}_1^i(\mathbb{C})) \cong \bigoplus_{\mathbb{C}^\times} \mathbb{Z} = \bigoplus_{[\chi]} \mathbb{Z} = \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}. \]

The automorphic induction map
\[ K^1(\mathcal{A}\mathcal{I}) : K^1(\mathcal{A}_2^i(\mathbb{R})) \to K^1(\mathcal{A}_1^i(\mathbb{C})) \]
may be interpreted, at the level of \(K^1\), as a kind of “shift” map
\[ [D_{[\ell]}] \mapsto [\chi_{[\ell]}] \]

More explicitly, the “shift” map is
\[ \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} \to \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}, \quad ([D_1], [D_2], ...) \mapsto (..., 0, 0, [\chi_1], [\chi_2], ...) \]
where the image under \(K^1(\mathcal{A}\mathcal{I})\) on each component of the right hand side with label \(\ell \leq 0\) is zero (because \(K^1(\mathcal{A}\mathcal{I})\) is a group homomorphism so it must map zero into zero).
6.3 Automorphic induction in $n$ dimensions

In this section we consider automorphic induction for $GL(n)$. Contrary to base change (see theorems 5.4 and 5.5), the $K$-theory map of automorphic induction is nonzero for every $n$.

**Theorem 6.5.** The functorial map induced by automorphic induction

$$K_j(C^*_rGL(2n, \mathbb{R})) \xrightarrow{K_j(\mathcal{A}T)} K_j(C^*_rGL(n, \mathbb{C}))$$

is given by

$$[D_{|\ell_1|} \otimes \ldots \otimes D_{|\ell_n|}] \mapsto [\chi_{|\ell_1|} \oplus \ldots \oplus \chi_{|\ell_n|}]$$

if $n \equiv j \pmod{2}$ and $\chi_k \neq \chi_{\tau_k}$ for every $k$, and is zero otherwise.

Here, $[D_{|\ell_1|} \otimes \ldots \otimes D_{|\ell_n|}]$ denotes the generator of the component $\mathbb{Z}_{(|\ell_1|, \ldots, |\ell_n|)}$ of $K_j(C^*_rGL(2n, \mathbb{R}))$ and $[\chi_{|\ell_1|} \oplus \ldots \oplus \chi_{|\ell_n|}]$ is the generator of the component $\mathbb{Z}_{(|\ell_1|, \ldots, |\ell_n|)}$ of $K_j(C^*_rGL(n, \mathbb{C}))$.

**Proof.** Let $0 \leq m < n$ be the number of characters $\chi_k$ with $\chi_k = \chi_{\tau_k}$.

**Case 1:** $m = 0$

In this case, $\chi_k \neq \chi_{\tau_k}$ for every $k$. Each character $\chi_{\ell_k}$, $\ell_k \neq 0$, is mapped via the local langlands correspondence into a discrete series $D_{|\ell_k|}$. At the level of $K$-theory, a generator $[D_{|\ell_k|}]$ is mapped into $[\chi_{|\ell_k|}]$. The result follows from (16).

**Case 2:** $m > 0$ odd

Then, if $n \equiv j \pmod{2}$, $K_j(\mathbb{R}^{n+m}) = 0$ and $K_j(\mathcal{A}T)$ is zero.

**Case 3:** $m > 0$ is even

In this case $K_j(\mathbb{R}^n) = K_j(\mathbb{R}^{n+m})$. However, $X_\mathbb{R}(M) \simeq \mathbb{R}^{n+m}$ corresponds precisely to the partition of $2n$ into 1’s and 0’s given by

$$2n = 2(n - m) + 2m$$

Hence, the number of 1’s in the partition is $2m \geq 4$. It follows that $(t_1, \ldots, t_n)$ is mapped into a cone and, as a consequence, $K_j(\mathcal{A}T)$ is zero.

This concludes the proof. \qed

7 Connections with the Baum-Connes correspondence

The standard maximal compact subgroup of $GL(1, \mathbb{C})$ is the circle group $U(1)$, and the maximal compact subgroup of $GL(1, \mathbb{R})$ is $\mathbb{Z}/2\mathbb{Z}$. Base change for $K^1$ creates a map

$$\mathcal{R}(U(1)) \rightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$
where $\mathcal{R}(U(1))$ is the representation ring of the circle group $U(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $U(1)$ to $1 \oplus \varepsilon$, where $\varepsilon$ is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $U(1)$ to zero.

This map has an interpretation in terms of $K$-cycles. The real line $\mathbb{R}$ is a universal example for the action of $\mathbb{R}^\times$ and $\mathbb{C}^\times$. The $K$-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\varphi_C \in K^{\mathbb{C}^\times}(L^2(\mathbb{C}^\times))$ and a class $\varphi_R \in K^{\mathbb{R}^\times}(L^2(\mathbb{R}^\times))$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\varphi_C \mapsto (\varphi_R, \varphi_R)$$

It would be interesting to interpret the automorphic induction map at the level of representation rings:

$$\mathcal{AT}^*: \mathcal{R}(O(2n)) \longrightarrow \mathcal{R}(U(n)).$$

**References**


S. Mendes, ISCTE, Av. das Forças Armadas, 1649-026, Lisbon, Portugal
Email: sergio.mendes@iscte.pt

R.J. Plymen, Southampton University, Southampton SO17 1BJ, England
and School of Mathematics, Manchester University, Manchester M13 9PL, England
Email: r.j.plymen@soton.ac.uk plymen@manchester.ac.uk