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Abstract

We investigate base change and automorphic induction $\mathbb{C}/\mathbb{R}$ at the level of K-theory for the general linear group $\mathrm{GL}_n(\mathbb{R})$. In the course of this study, we compute in detail the C^*-algebra K-theory of this disconnected group. We investigate the interaction of base change with the Baum-Connes correspondence for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. This article is the archimedean companion of our previous article [11].

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1 Introduction

In the general theory of automorphic forms, an important role is played by *base change* and *automorphic induction*, two examples of the principle of functoriality in the Langlands program [3]. Base change and automorphic induction have a global aspect and a local aspect [1][7]. In this article, we focus on the archimedean case of base change and automorphic induction for the general linear group $\mathrm{GL}(n,\mathbb{R})$, and we investigate these aspects of functoriality at the level of K-theory.

For $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ we have the Langlands classification and the associated L-parameters [8]. We recall that the domain of an L-parameter of $\mathrm{GL}_n(F)$ over an archimedean field F is the Weil group W_F. The Weil groups are given by

$$W_{\mathbb{C}} = \mathbb{C}^\times$$

and

$$W_{\mathbb{R}} = \langle j \rangle \mathbb{C}^\times$$

where $j^2 = -1 \in \mathbb{C}^\times$, $jc = \bar{c}j$ for all $c \in \mathbb{C}^\times$. Base change is defined by restriction of L-parameter from $W_{\mathbb{R}}$ to $W_{\mathbb{C}}$.

An L-parameter ϕ is *tempered* if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.

In this article, we investigate the interaction of base change with the Baum-Connes correspondence for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$.

Let F denote $\mathbb{R}$ or $\mathbb{C}$ and let $G = G(F) = \mathrm{GL}_n(F)$. Let $C_r^*(G)$ denote the reduced C^*-algebra of G. The Baum-Connes correspondence is a canonical isomorphism [2][5][9]

$$\mu_F : K_*^{G(F)}(\underline{E}G(F)) \to K_* C_r^*(G(F))$$

where $\underline{E}G(F)$ is a universal example for the action of $G(F)$.

The noncommutative space $C_r^*(G(F))$ is strongly Morita equivalent to the commutative C^*-algebra $C_0(\mathcal{A}_n^t(F))$ where $\mathcal{A}_n^t(F)$ denotes the tempered dual of $G(F)$, see [12, §1.2][13]. As a consequence of this, we have

$$K_* C_r^*(G(F)) \cong K^* \mathcal{A}_n^t(F).$$

This leads to the following formulation of the Baum-Connes correspondence:

$$K_*^{G(F)}(\underline{E}G(F)) \cong K^* \mathcal{A}_n^t(F).$$

Base change and automorphic induction $\mathbb{C}/\mathbb{R}$ determine maps

$$\mathcal{BC}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$$

and

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{C}) \to \mathcal{A}_{2n}^t(\mathbb{R}).$$

This leads to the following diagrams

$$\begin{array}{ccc} K_*^{G(\mathbb{C})}(\underline{E}G(\mathbb{C})) & \xrightarrow{\mu_{\mathbb{C}}} & K^* \mathcal{A}_n^t(\mathbb{C})) \\ \downarrow & & \downarrow \mathcal{BC}^*_{\mathbb{C}/\mathbb{R}} \\ K_*^{G(\mathbb{R})}(\underline{E}G(\mathbb{R})) & \xrightarrow[\mu_{\mathbb{R}}]{} & K^* \mathcal{A}_n^t(\mathbb{R}). \end{array}$$

and

$$\begin{array}{ccc} K_*^{G(\mathbb{R})}(\underline{E}G(\mathbb{R})) & \xrightarrow{\mu_{\mathbb{R}}} & K^* \mathcal{A}_{2n}^t(\mathbb{R})) \\ \downarrow & & \downarrow \mathcal{AI}^*_{\mathbb{C}/\mathbb{R}} \\ K_*^{G(\mathbb{C})}(\underline{E}G(\mathbb{C})) & \xrightarrow[\mu_{\mathbb{C}}]{} & K^* \mathcal{A}_n^t(\mathbb{C}). \end{array}$$

where the left-hand vertical maps are the unique maps which make the diagrams commutative.

In section 2 we describe the tempered dual $\mathcal{A}_n^t(F)$ as a locally compact Hausdorff space.

In section 3 we compute the K-theory for the reduced C^*-algebra of $\mathrm{GL}(n, \mathbb{R})$. The real reductive Lie group $\mathrm{GL}(n, \mathbb{R})$ is not connected. If n is even our formulas show that we always have non-trivial K^0 and K^1. We also recall the K-theory for the reduced C^*-algebra of the complex reductive group $\mathrm{GL}(n, \mathbb{C})$, see [13]. In section 4 we recall the Langlands parameters for $\mathrm{GL}(n)$ over archimedean local fields, see [8]. In section 5 we compute the base change map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ and prove that $\mathcal{BC}$ is a continuous proper map. At the level of K-theory, base change is the zero map for $n > 1$ (Theorem 5.4) and is nontrivial for $n = 1$ (Theorem 5.5). In section 6, we compute the automorphic induction map $\mathcal{AI} : \mathcal{A}_n^t(\mathbb{C}) \to \mathcal{A}_{2n}^t(\mathbb{R})$. Contrary to base change, at the level of K-theory, automorphic induction is nontrivial for every n (Theorem 6.5). In section 7, where we study the case $n = 1$, base change for K^1 creates a map

$$\mathcal{R}(\mathrm{U}(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathrm{U}(1))$ is the representation ring of the circle group $\mathrm{U}(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $\mathrm{U}(1)$ to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $\mathrm{U}(1)$ to zero.

This map has an interpretation in terms of K-cycles. The K-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\partial\!\!\!/_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(\underline{E}\mathbb{C}^\times)$ and a class $\partial\!\!\!/_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(\underline{E}\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\partial\!\!\!/_{\mathbb{C}} \mapsto (\partial\!\!\!/_{\mathbb{R}}, \partial\!\!\!/_{\mathbb{R}})$$

This extends the results of [11] to archimedean fields.

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2 On the tempered dual of $\mathrm{GL}(n)$

Let $F = \mathbb{R}$. In order to compute the K-theory of the reduced C^*-algebra of $\mathrm{GL}(n, F)$ we need to parametrize the tempered dual $\mathcal{A}_n^t(F)$ of $GL(n, F)$.

Let M be a standard Levi subgroup of $\mathrm{GL}(n, F)$, i.e. a block-diagonal subgroup. Let ${}^0M$ be the subgroup of M such that the determinant of each block-diagonal is ± 1. Denote by $X(M) = \widehat{M/{}^0M}$ the group of *unramified characters* of M, consisting of those characters which are trivial on ${}^0M$.

Let $W(M) = N(M)/M$ denote the Weyl group of M. $W(M)$ acts on the discrete series $E_2({}^0M)$ of ${}^0M$ by permutations.

Now, choose one element $\sigma \in E_2({}^0M)$ for each $W(M)$-orbit. The *isotropy subgroup* of $W(M)$ is defined to be

$$W_\sigma(M) = \{\omega \in W(M) : \omega.\sigma = \sigma\}.$$

Form the disjoint union

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_{M} \bigsqcup_{\sigma \in E_2({}^0M)} X(M)/W_\sigma(M). \tag{1}$$

The disjoint union has the structure of a locally compact, Hausdorff space and is called the *Harish-Chandra parameter space*. The parametrization of the tempered dual $\mathcal{A}_n^t(\mathbb{R})$ is due to Harish-Chandra, see [10].

Proposition 2.1. *There exists a bijection*

$$\begin{array}{ccc} \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) & \longrightarrow & \mathcal{A}_n^t(\mathbb{R}) \\ \chi^\sigma & \mapsto & i_{GL(n),MN}(\chi^\sigma \otimes 1), \end{array}$$

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ *for all* $x \in M$.

In view of the above bijection, we will denote the Harish-Chandra parameter space by $\mathcal{A}_n^t(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting with $GL(n, \mathbb{R})$. Since the discrete series of $GL(n, \mathbb{R})$ is empty for $n \geq 3$, we only need to consider partitions of n into 1's and 2's. This allows us to to decompose n as $n = 2q + r$, where q is the number of 2's and r is the number of 1's in the partition. To this decomposition we associate the partition

$$n = (\underbrace{2, ..., 2}_{q}, \underbrace{1, ..., 1}_{r}),$$

which corresponds to the Levi subgroup

$$M \cong \underbrace{GL(2, \mathbb{R}) \times ... \times GL(2, \mathbb{R})}_{q} \times \underbrace{GL(1, \mathbb{R}) \times ... \times GL(1, \mathbb{R})}_{r}.$$

Varying q and r we determine a representative in each equivalence class of Levi subgroups. The subgroup ${}^0M$ of M is given by

$${}^0M \cong \underbrace{SL^\pm(2, \mathbb{R}) \times ... \times SL^\pm(2, \mathbb{R})}_{q} \times \underbrace{SL^\pm(1, \mathbb{R}) \times ... \times SL^\pm(1, \mathbb{R})}_{r},$$

where
$$SL^{\pm}(m,\mathbb{R}) = \{g \in GL(m,\mathbb{R}) : |det(g)| = 1\}$$
is the *unimodular subgroup* of $GL(m,\mathbb{R})$. In particular, $SL^{\pm}(1,\mathbb{R}) = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

The representations in the discrete series of $GL(2,\mathbb{R})$, denoted $\mathcal{D}_\ell$ for $\ell \in \mathbb{N}$ ($\ell \geq 1$) are induced from $SL(2,\mathbb{R})$ [8, p.399]:

$$\mathcal{D}_\ell = ind_{SL^{\pm}(2,\mathbb{R}),SL(2,\mathbb{R})}(\mathcal{D}_\ell^{\pm}),$$

where $\mathcal{D}_\ell^{\pm}$ acts in the space

$$\{f : \mathcal{H} \to \mathbb{C} | f \text{ analytic }, \|f\|^2 = \int\int |f(z)|^2 y^{\ell-1}) dxdy < \infty\}.$$

Here, $\mathcal{H}$ denotes the Poincaré upper half plane. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by
$$\mathcal{D}_\ell^{\pm}(g)(f(z)) = (bz+d)^{-(\ell+1)} f(\frac{az+c}{bz+d}).$$

More generally, an element σ from the discrete series $E_2({}^0M)$ is given by

$$\sigma = i_{G,MN}(\mathcal{D}_{\ell_1}^{\pm} \otimes ... \otimes \mathcal{D}_{\ell_q}^{\pm} \otimes \tau_1 \otimes ... \otimes \tau_r \otimes 1), \tag{2}$$

where $\mathcal{D}_{\ell_i}^{\pm}$ ($\ell_i \geq 1$) are the discrete series representations of $SL^{\pm}(2,\mathbb{R})$ and τ_j is a representation of $SL^{\pm}(1,\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $id = (x \mapsto x)$ or $sgn = (x \mapsto \frac{x}{|x|})$.

Finally we will compute the unramified characters $X(M)$, where M is the Levi subgroup associated to the partition $n = 2q + r$.

Let $x \in GL(2,\mathbb{R})$. Any character of $GL(2,\mathbb{R})$ is given by

$$\chi(det(x)) = (sgn(det(x)))^{\varepsilon} |det(x)|^{it}$$

($\varepsilon = 0, 1, t \in \mathbb{R}$) and it is unramified provided that

$$\chi(det(g)) = \chi(\pm 1) = (\pm 1)^{\varepsilon} = 1, \text{ for all } g \in SL^{\pm}(2,\mathbb{R}).$$

This implies $\varepsilon = 0$ and any unramified character of $GL(2,\mathbb{R})$ has the form

$$\chi(x) = |det(x)|^{it}, \text{ for some } t \in \mathbb{R}. \tag{3}$$

Similarly, any unramified character of $GL(1,\mathbb{R}) = \mathbb{R}^\times$ has the form

$$\xi(x) = |x|^{it}, \text{ for some } t \in \mathbb{R}. \tag{4}$$

Given a block diagonal matrix $diag(g_1, ..., g_q, \omega_1, ..., \omega_r) \in M$, where $g_i \in GL(2, \mathbb{R})$ and $\omega_j \in GL(1, \mathbb{R})$, we conclude from (3) and (4) that any unramified character $\chi \in X(M)$ is given by

$$\chi(diag(g_1, ..., g_q, \omega_1, ..., \omega_r)) =$$
$$= |det(g_1)|^{it_1} \times ... \times |det(g_q)|^{it_q} \times |\omega_1|^{it_{q+1}} \times ... \times |\omega_r|^{it_{q+r}},$$

for some $(t_1, ..., t_{q+r}) \in \mathbb{R}^{q+r}$. We can denote such element $\chi \in X(M)$ by $\chi_{(t_1,...,t_{q+r})}$. We have the following result.

Proposition 2.2. *Let M be a Levi subgroup of $GL(n, \mathbb{R})$, associated to the partition $n = 2q + r$. Then, there is a bijection*

$$X(M) \to \mathbb{R}^{q+r} \ , \ \chi_{(t_1,...,t_{q+r})} \mapsto (t_1, ..., t_{q+r}).$$

Let us consider now $\mathrm{GL}(n, \mathbb{C})$. The tempered dual of $\mathrm{GL}(n, \mathbb{C})$ comprises the *unitary principal series* in accordance with Harish-Chandra [6, p. 277]. The corresponding Levi subgroup is a maximal torus $T \cong (\mathbb{C}^\times)^n$. It follows that ${}^0T \cong \mathbb{T}^n$ the compact n-torus.

The principal series representations are given by

$$\pi_{\ell,it} = i_{G,TU}(\sigma \otimes 1), \tag{5}$$

where σ=$\sigma_1 \otimes ... \otimes \sigma_n$ and $\sigma_j(z) = (\frac{z}{|z|})^{\ell_j} |z|^{it_j}$ ($\ell_j \in \mathbb{Z}$ and $t_j \in \mathbb{R}$).

An unramified character is given by

$$\chi \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = |z_1|^{it_1} \times ... \times |z_n|^{it_n}$$

and we can represent χ as $\chi_{(t_1,...,t_n)}$. Therefore, we have the following result.

Proposition 2.3. *Denote by T the standard maximal torus in $GL(n, \mathbb{C})$. There is a bijection*

$$X(T) \to \mathbb{R}^n \ , \ \chi_{(t_1,...,t_n)} \mapsto (t_1, ..., t_n).$$

3 K-theory for $GL(n)$

Using the Harish-Chandra parametrization of the tempered dual of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the K-theory of the reduced C^*-algebras $C_r^*\mathrm{GL}(n, \mathbb{R})$ and $C_r^*\mathrm{GL}(n, \mathbb{C})$.

3.1 K-theory for $\mathrm{GL}(n, \mathbb{R})$

We exploit the strong Morita equivalence described in [12, §1.2]. We infer that

$$\begin{aligned} K_j(C_r^*\mathrm{GL}(n,\mathbb{R})) &= K^j(\textstyle\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M)) \\ &= \textstyle\bigoplus_{(M,\sigma)} K^j(X(M)/W_\sigma(M)) \\ &= \textstyle\bigoplus_{(M,\sigma)} K^j(\mathbb{R}^{n_M}/W_\sigma(M)), \end{aligned} \tag{6}$$

where $n_M = q + r$ if M is a representative of the equivalence class of Levi subgroup associated to the partition $n = 2q+r$. Hence the K-theory depends on n and on each Levi subgroup.

To compute (6) we have to consider the following orbit spaces:

(i) $\mathbb{R}^n$, in which case $W_\sigma(M)$ is the trivial subgroup of the Weil group $W(M)$;

(ii) $\mathbb{R}^n/S_n$, where $W_\sigma(M) = W(M)$ (this is one of the possibilities for the partition of n into 1's);

(iii) $\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k})$, where $W_\sigma(M) = S_{n_1} \times ... \times S_{n_k} \subset W(M)$ (see the examples below).

Definition 3.1. *An orbit space as indicated in* (*ii*) *and* (*iii*) *is called a closed cone.*

The K-theory for $\mathbb{R}^n$ may be summarized as follows

$$K^j(\mathbb{R}^n) = \begin{cases} \mathbb{Z} \text{ if } n = j \mod 2 \\ 0 \text{ otherwise }. \end{cases}$$

The next results show that the K-theory of a closed cone vanishes.

Lemma 3.2. $K^j(\mathbb{R}^n/S_n) = 0, j = 0, 1.$

Proof. We need the following definition. A point $(a_1, ..., a_n) \in \mathbb{R}^n$ is called normalized if $a_j \le a_{j+1}$, for $j = 1, 2, ..., n-1$. Therefore, in each orbit there is exactly one normalized point and $\mathbb{R}^n/S_n$ is homeomorphic to the subset of $\mathbb{R}^n$ consisting of all normalized points of $\mathbb{R}^n$. We denote the set of all normalized points of $\mathbb{R}^n$ by $N(\mathbb{R}^n)$.

In the case of $n = 2$, let (a_1, a_2) be a normalized point of $\mathbb{R}^2$. Then, there is a unique $t \in [1, +\infty[$ such that $a_2 = ta_1$ and the map

$$\mathbb{R} \times [1, +\infty[\to N(\mathbb{R}^2), (a, t) \mapsto (a, ta)$$

is a homeomorphism.

If $n > 2$ then the map

$$N(\mathbb{R}^{n-1}) \times [1, +\infty[\to N(\mathbb{R}^n), (a_1, ..., a_{n-1}, t) \mapsto (a_1, ..., a_{n-1}, ta_n)$$

is a homeomorphism. Since $[1, +\infty[$ kills both the K-theory groups K^0 and K^1, the result follows by applying Künneth formula. □

The symmetric group S_n acts on $\mathbb{R}^n$ by permuting the components. This induces an action of any subgroup $S_{n_1} \times ... \times S_{n_k}$ of S_n on $\mathbb{R}^n$. Write

$$\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times ... \times \mathbb{R}^{n_k} \times \mathbb{R}^{n-n_1-...-n_k}.$$

If $n = n_1 + ... + n_k$ then we simply have $\mathbb{R}^n \cong \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$.

The group $S_{n_1} \times ... \times S_{n_k}$ acts on $\mathbb{R}^n$ as follows.
S_{n_1} permutes the components of $\mathbb{R}^{n_1}$ leaving the remaining fixed;
S_{n_2} permutes the components of $\mathbb{R}^{n_2}$ leaving the remaining fixed;
and so on. If $n > n_1 + ... + n_k$ the components of $\mathbb{R}^{n-n_1-...-n_k}$ remain fixed. This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identify the orbit spaces

$$\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k}) \cong \mathbb{R}^{n_1}/S_{n_1} \times ... \times \mathbb{R}^{n_k}/S_{n_k} \times \mathbb{R}^{n-n_1-...-n_k}$$

Lemma 3.3. *$K^j(\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k}) = 0, j = 0, 1,$ where $S_{n_1} \times ... \times S_{n_k} \subset S_n$.*

Proof. It suffices to prove for $\mathbb{R}^n/(S_{n_1} \times S_{n_2})$. The general case follows by induction on k.

Now, $\mathbb{R}^n/(S_{n_1} \times S_{n_2}) \cong \mathbb{R}^{n_1}/S_{n_1} \times \mathbb{R}^{n-n_1}/S_{n_2}$. Applying the Künneth formula and Lemma 3.2, the result follows. □

We give now some examples by computing $K_j C_r^* GL(n, \mathbb{R})$ for small n.

Example 3.4. *We start with the case of $GL(1, \mathbb{R})$. We have:*

$$M = \mathbb{R}^\times \ , \ {}^0M = \mathbb{Z}/2\mathbb{Z} \ , \ W(M) = 1 \text{ and } X(M) = \mathbb{R}.$$

Hence,

$$\mathcal{A}_1^t(\mathbb{R}) \cong \bigsqcup_{\sigma \in \widehat{(\mathbb{Z}/2\mathbb{Z})}} \mathbb{R}/1 = \mathbb{R} \sqcup \mathbb{R}, \tag{7}$$

and the K-theory is given by

$$K_j C_r^* GL(1, \mathbb{R}) \cong K^j(\mathcal{A}_1^t(\mathbb{R})) = K^j(\mathbb{R} \sqcup \mathbb{R}) = K^j(\mathbb{R}) \oplus K^j(\mathbb{R}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & , j = 1 \\ 0 & , j = 0. \end{cases}$$

Example 3.5. *For $GL(2,\mathbb{R})$ we have two partitions of $n = 2$ and the following data*

Partition	M	${}^0M$	$W(M)$	$X(M)$	$\sigma \in E_2({}^0M)$
2+0	$GL(2,\mathbb{R})$	$SL^{\pm}(2,\mathbb{R})$	1	$\mathbb{R}$	$\sigma = i_{G,P}(\mathcal{D}_\ell^+), \ell \in \mathbb{N}$
1+1	$(\mathbb{R}^\times)^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{R}^2$	$\sigma = i_{G,P}(id \otimes sgn)$

Then the tempered dual is parameterized as follows

$$\mathcal{A}_2^t(\mathbb{R}) \cong \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = (\bigsqcup_{\ell\in\mathbb{N}} \mathbb{R}) \sqcup (\mathbb{R}^2/S_2) \sqcup (\mathbb{R}^2/S_2) \sqcup \mathbb{R}^2,$$

and the K-theory groups are given by

$$K_jC_r^*GL(2,\mathbb{R}) \cong K^j(\mathcal{A}_2^t(\mathbb{R})) \cong (\bigoplus_{\ell\in\mathbb{N}} K^j(\mathbb{R})) \oplus K^j(\mathbb{R}^2) = \begin{cases} \bigoplus_{\ell\in\mathbb{N}} \mathbb{Z} & ,j=1 \\ \mathbb{Z} & ,j=0. \end{cases}$$

Example 3.6. *For $GL(3,\mathbb{R})$ there are two partitions for $n = 3$, to which correspond the following data*

Partition	M	${}^0M$	$W(M)$	$X(M)$
2+1	$GL(2,\mathbb{R}) \times \mathbb{R}^\times$	$SL^{\pm}(2,\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$	1	$\mathbb{R}^2$
1+1+1	$(\mathbb{R}^\times)^3$	$(\mathbb{Z}/2\mathbb{Z})^3$	S_3	$\mathbb{R}^3$

For the partition $3 = 2 + 1$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_\ell^+ \otimes \tau) \ , \ \ell \in \mathbb{N} \text{ and } \tau \in (\widehat{\mathbb{Z}/2\mathbb{Z}}).$$

For the partition $3 = 1 + 1 + 1$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\bigotimes_{i=1}^{3} \tau_i) \ , \ \tau_i \in (\widehat{\mathbb{Z}/2\mathbb{Z}}).$$

The tempered dual is parameterized as follows

$$\mathcal{A}_3^t(\mathbb{R}) \cong \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_{\mathbb{N}\times(\mathbb{Z}/2\mathbb{Z})} (\mathbb{R}^2/1) \bigsqcup_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{R}^3/S_3).$$

The K-theory groups are given by

$$K_jC_r^*GL(3,\mathbb{R}) \cong K^j(\mathcal{A}_3^t(\mathbb{R})) \cong \bigoplus_{\mathbb{N}\times(\mathbb{Z}/2\mathbb{Z})} K^j(\mathbb{R}^2) \oplus 0 = \begin{cases} \bigoplus_{\mathbb{N}\times(\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} & ,j=0 \\ 0 & ,j=1. \end{cases}$$

The general case of $GL(n, \mathbb{R})$ will now be considered. It can be split in two cases: n even and n odd.

• $n = 2q$ even

Suppose n is even. For every partition $n = 2q + r$, either $W_\sigma(M) = 1$ or $W_\sigma(M) \neq 1$. If $W_\sigma(M) \neq 1$ then $\mathbb{R}^{n_M}/W_\sigma(M)$ is a cone and the K-groups K^0 and K^1 both vanish. This happens precisely when $r > 2$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the K-theory with non-zero K-groups

Partition	M	${}^0M$	$W(M)$
$2q$	$GL(2,\mathbb{R})^q$	$SL^{\pm}(2,\mathbb{R})^q$	S_q
$2(q-1)+2$	$GL(2,\mathbb{R})^{q-1} \times (\mathbb{R}^\times)^2$	$SL^{\pm}(2,\mathbb{R})^{q-1} \times (\mathbb{Z}/2\mathbb{Z})^2$	$S_{q-1} \times (\mathbb{Z}/2\mathbb{Z})$

We also have $X(M) \cong \mathbb{R}^q$ for $n = 2q$, and $X(M) \cong \mathbb{R}^{q+1}$, for $n = 2(q-1)+2$.

For the partition $n = 2q$ ($r = 0$), an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}^+_{\ell_1} \otimes ... \otimes \mathcal{D}^+_{\ell_q}) \ , \ (\ell_1, ...\ell_q) \in \mathbb{N}^q \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

For the partition $n = 2(q-1)+2$ ($r = 2$), an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}^+_{\ell_1} \otimes ... \otimes \mathcal{D}^+_{\ell_{q-1}} \otimes id \otimes sgn) \ , \ (\ell_1, ...\ell_{q-1}) \in \mathbb{N}^{q-1} \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

Therefore, the tempered dual has the following form

$$\mathcal{A}^t_n(\mathbb{R}) = \mathcal{A}^t_{2q}(\mathbb{R}) = (\bigsqcup_{\ell \in \mathbb{N}^q} \mathbb{R}^q) \sqcup (\bigsqcup_{\ell' \in \mathbb{N}^{q-1}} \mathbb{R}^{q+1}) \sqcup \mathcal{C}$$

where $\mathcal{C}$ is a disjoint union of closed cones as in Definition 3.1.

Theorem 3.7. *Suppose $n = 2q$ is even. Then the K-groups are*

$$K_j C^*_r GL(n,\mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in \mathbb{N}^q} \mathbb{Z} & , j \equiv q (mod 2) \\ \bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z} & , otherwise. \end{cases}$$

If $q = 1$ then the direct sum $\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z}$ will denote a single copy of $\mathbb{Z}$.

• $n = 2q + 1$ odd

If n is odd only one partition contributes to the K-theory of $GL(n, \mathbb{R})$ with non-zero K-groups:

Partition	M	${}^0M$	$W(M)$	$X(M)$
$2q+1$	$GL(2,\mathbb{R})^{q+1}\times\mathbb{R}^\times$	$SL^{\pm}(2,\mathbb{R})^q\times(\mathbb{Z}/2\mathbb{Z})$	S_q	$\mathbb{R}^{q+1}$

An element $\sigma\in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}^+_{\ell_1}\otimes ...\otimes\mathcal{D}^+_{\ell_q}\otimes\tau)\ ,\ (\ell_1,...\ell_q,\tau)\in\mathbb{N}^q\times(\mathbb{Z}/2\mathbb{Z}) \text{ and } \ell_i\neq\ell_j \text{ if } i\neq j.$$

The tempered dual is given by

$$\mathcal{A}^t_n(\mathbb{R}) = \mathcal{A}^t_{2q+1}(\mathbb{R}) = (\bigsqcup_{\ell\in(\mathbb{N}^q\times(\mathbb{Z}/2\mathbb{Z}))}\mathbb{R}^{q+1})\sqcup\mathcal{C}$$

where $\mathcal{C}$ is a disjoint union of closed cones as in Definition 3.1.

Theorem 3.8. *Suppose $n=2q+1$ is odd. Then the K-groups are*

$$K_jC^*_rGL(n,\mathbb{R})\cong\begin{cases}\bigoplus_{\ell\in\mathbb{N}^q\times(\mathbb{Z}/2\mathbb{Z})}\mathbb{Z} & ,j\equiv q+1(mod2)\\ 0 & ,otherwise.\end{cases}$$

Here, we use the following convention: if $q=0$ then the direct sum is $\bigoplus_{\mathbb{Z}/2\mathbb{Z}}\mathbb{Z}\cong\mathbb{Z}\oplus\mathbb{Z}$.

We conclude that the K-theory of $C^*_rGL(n,\mathbb{R})$ depends on essentially one parameter q given by the maximum number of $2's$ in the partitions of n into $1's$ and $2's$. If n is even then $q=\frac{n}{2}$ and if n is odd then $q=\frac{n-1}{2}$.

3.2 K-theory for $\mathrm{GL}(n,\mathbb{C})$

Let T° be the maximal compact subgroup of the maximal compact torus T of $\mathrm{GL}(n,\mathbb{C})$. Let σ be a unitary character of T°. We note that $W=W(T)$, $W_\sigma=W_\sigma(T)$. If $W_\sigma=1$ then we say that the orbit $W\cdot\sigma$ is *generic*.

Theorem 3.9. *The K-theory of $C^*_r\mathrm{GL}(n,\mathbb{C})$ admits the following description. If $n=j \mod 2$ then K_j is free abelian on countably many generators, one for each generic W- orbit in the unitary dual of T°, and $K_{j+1}=0$.*

Proof. We exploit the strong Morita equivalence described in [13, Prop. 4.1]. We have a homeomorphism of locally compact Hausdorff spaces:

$$\mathcal{A}^t_n(\mathbb{C})\cong\bigsqcup X(T)/W_\sigma(T)$$

by the Harish-Chandra Plancherel Theorem for complex reductive groups [6], and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [13] . The result now follows from Lemma 4.3. □

4 Langlands parameters for $GL(n)$

The Weil group of $\mathbb{C}$ is simply

$$W_{\mathbb{C}} \cong \mathbb{C}^{\times},$$

and the Weil group of $\mathbb{R}$ can be written as disjoint union

$$W_{\mathbb{R}} \cong \mathbb{C}^{\times} \sqcup j\mathbb{C}^{\times},$$

where $j^2 = -1$ and $jcj^{-1} = \overline{c}$ ($\overline{c}$ denotes complex conjugation). We shall use this disjoint union to describe the representation theory of $W_{\mathbb{R}}$.

Definition 4.1. *An L- parameter is a continuous homomorphism*

$$\phi : W_F \to GL(n, \mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$.

L-parameters are also called Langlands parameters. Two L-parameters are equivalent if they are cojugate under $GL(n, \mathbb{C})$. The set of equivalence classes of L-parameters is denoted by $\mathcal{G}_n$. And the set of equivalence classes of L-parameters whose image is bounded is denoted by $\mathcal{G}_n^t$.

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}_n(F)$ (resp. $\mathcal{A}_n^t(F)$) denote the smooth dual (resp. tempered dual) of $GL(n, F)$. The local Langlands correspondence is a bijection

$$\mathcal{G}_n(F) \to \mathcal{A}_n(F).$$

In particular,

$$\mathcal{G}_n^t(F) \to \mathcal{A}_n^t(F)$$

is also a bijection.

We are only interested in L-parameters whose image is bounded. In the sequel we will refer to them, for simplicity, as L-parameters.

L-parameters for $W_{\mathbb{C}}$

A 1-dimensional L-parameter for $W_{\mathbb{C}}$ is simply a character of $\mathbb{C}^{\times}$ (i.e. a unitary quasicharacter):

$$\chi(z) = (\frac{z}{|z|})^{\ell} \otimes |z|^{it}$$

where $|z| = |z|_{\mathbb{C}} = z\overline{z}$, $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$. To emphasize the dependence on parameters (ℓ, t) we write sometimes $\chi = \chi_{\ell,t}$.

An n-dimensional L-parameter can be written as a direct sum of n 1-dimensional characters of $\mathbb{C}^\times$:

$$\phi = \phi_1 \oplus ... \oplus \phi_n,$$

with $\phi_k(z) = (\frac{z}{|z|})^{\ell_k} \otimes |z|^{t_k}, \ell_k \in \mathbb{Z}, t_k \in \mathbb{R}, k = 1, ..., n.$

L-parameters for $W_\mathbb{R}$

The 1-dimensional L-parameters for $W_\mathbb{R}$ are as follows

$$\begin{cases} \phi_{\varepsilon,t}(z) = |z|_\mathbb{R}^{it} \\ \\ \phi_{\varepsilon,t}(j) = (-1)^\varepsilon \end{cases}, \varepsilon \in \{0,1\}, t \in \mathbb{R}.$$

We may now describe the local Langlands correspondence for $GL(1,\mathbb{R})$:

$$\phi_{0,t} \mapsto 1 \otimes |.|_\mathbb{R}^{it}$$
$$\phi_{1,t} \mapsto sgn \otimes |.|_\mathbb{R}^{it}$$

Now, we consider 2-dimensional L-parameters for $W_\mathbb{R}$:

$$\phi_{\ell,t}(z) = \begin{pmatrix} \chi_{\ell,t}(z) & 0 \\ 0 & \overline{\chi}_{\ell,t}(z) \end{pmatrix}, \phi_{\ell,t}(j) = \begin{pmatrix} 0 & (-1)^\ell \\ 1 & 0 \end{pmatrix}.$$

with $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$.

and

$$\phi_{m,t,n,s}(z) = \begin{pmatrix} \chi_{0,t}(z) & 0 \\ 0 & \chi_{0,s}(z) \end{pmatrix}, \phi_{m,t,n,s}(j) = \begin{pmatrix} (-1)^m & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

with $m, n \in \{0,1\}$ and $t, s \in \mathbb{R}$.

The local Langlands correspondence for $GL(2,\mathbb{R})$ may be descibed as follows.

The L-parameter ϕ_{m_1,t_1,m_2,t_2} corresponds, via Langlands correspondence, to the unitary principal series:

$$\phi_{m_1,t_1,m_2,t_2} \mapsto \pi(\mu_1, \mu_2),$$

where μ_i is the character of $\mathbb{R}^\times$ given by

$$\mu_i(x) = (\frac{x}{|x|})^{m_i} |x|^{it}, m_i \in \{0,1\}, t_i \in \mathbb{R}.$$

The L-parameter $\phi_{\ell,t}$ corresponds, via the Langlands correspondence, to the discrete series:

$$\phi_{\ell,t} \mapsto D_\ell \otimes |det(.)|_{\mathbb{R}}^{it} \ , \ \text{ with } \ \ell \in \mathbb{N}, t \in \mathbb{R}.$$

Proposition 4.2. *(i)* $\phi_{\ell,t} \cong \phi_{-\ell,t}$;

(ii) $\phi_{\ell,t,m,s} \cong \phi_{m,s,\ell,t}$;

(iii) $\phi_{0,t} \cong \phi_{1,t,0,t} \cong \phi_{0,t,1,t}$;

The proof is elementary. We now quote the following result.

Lemma 4.3. *[8] Every finite-dimensional semi-simple representation ϕ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.*

5 Base change

We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change $\mathbb{C}/\mathbb{R}$. We have $W_{\mathbb{C}} \subset W_{\mathbb{R}}$ and there is a natural map

$$Res_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} : \mathcal{G}_n(\mathbb{R}) \longrightarrow \mathcal{G}_n(\mathbb{C}) \tag{8}$$

called *restriction*. By the local Langlands correspondence for archimedean fields [3, Theorem 3.1, p.236][8], there is a base change map

$$\mathcal{BC} : \mathcal{A}_n(\mathbb{R}) \longrightarrow \mathcal{A}_n(\mathbb{C}) \tag{9}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_n(\mathbb{R}) & \xrightarrow{\mathcal{BC}} & \mathcal{A}_n(\mathbb{C}) \\ {}_{\mathbb{R}}\mathcal{L}_n \uparrow & & \uparrow {}_{\mathbb{C}}\mathcal{L}_n \\ \mathcal{G}_n(\mathbb{R}) & \xrightarrow[Res_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}]{} & \mathcal{G}_n(\mathbb{C}) \end{array}$$

Arthur and Clozel's book [1] gives a full treatment of base change for $GL(n)$. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let χ_i $(i = 1, ..., q)$ be a ramified character of $\mathbb{C}^\times$ and let ξ_j $(j = 1, ..., r)$ be a ramified character of $\mathbb{R}^\times$. Since the χ_i's are ramified, $\chi_i(z) \neq \chi_i^\tau(z) = \chi_i(\overline{z})$, where τ is a generator of $Gal(\mathbb{C}/\mathbb{R})$. By Langlands classification [8], each χ_i defines a discrete series representation $\pi(\chi_i)$ of $GL(2,\mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i^\tau)$. Denote by $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ the *generalized principal series representation* of $GL(n,\mathbb{R})$

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{GL(n,\mathbb{R}),MN}(\pi(\chi_1)\otimes ... \otimes\pi(\chi_q)\otimes\xi_1\otimes ... \otimes\xi_r\otimes 1). \quad (10)$$

The base change map for the general principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

$$\mathcal{BC}(\pi) = \Pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{GL(n,\mathbb{C}),B(\mathbb{C})}(\chi_1, \chi_1^\tau, ..., \chi_q, \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N), \quad (11)$$

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ is the norm map defined by $z \mapsto z\overline{z}$.

We illustrate the base change map with two simple examples.

Example 5.1. *For $n = 1$, base change is simply composition with the norm map*

$$\mathcal{BC} : \mathcal{A}_1^t(\mathbb{R}) \to \mathcal{A}_1^t(\mathbb{C}) \ , \ \mathcal{BC}(\chi) = \chi \circ N.$$

Example 5.2. *For $n = 2$, there are two different kinds of representations, one for each partition of 2. According to (10), $\pi(\chi)$ corresponds to the partition $2 = 2 + 0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1 + 1$. Then the base change map is given, respectively, by*

$$\mathcal{BC}(\pi(\chi)) = i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\chi, \chi^\tau),$$

and

$$\mathcal{BC}(\pi(\xi_1, \xi_2)) = i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N).$$

5.1 The base change map

Now, we define base change as a map of topological spaces and study the induced K-theory map.

Proposition 5.3. *The base change map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ is a continuous proper map.*

Proof. First, we consider the case $n = 1$. As we have seen in Example 5.1, base change for GL(1) is the map given by $\mathcal{BC}(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}_1^t(\mathbb{R})$, where $N : \mathbb{C}^\times \to \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

$$\mathcal{BC}(\chi)(z) = \chi(|z|^2) = |z|^{2it}. \quad (12)$$

A generic element from $\mathcal{A}_1^t(\mathbb{C})$ has the form

$$\mu(z) = (\frac{z}{|z|})^{\ell}|z|^{it}, \tag{13}$$

where $\ell \in \mathbb{Z}$ and $t \in S^1$, as stated before. Viewing the Pontryagin duals $\mathcal{A}_1^t(\mathbb{R})$ and $\mathcal{A}_1^t(\mathbb{C})$ as topological spaces by forgetting the group structure, and comparing (12) and (13), the base change map can be defined as the following continuous map

$$\begin{array}{rcl} \varphi : \mathcal{A}_1^t(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathcal{A}_1^t(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z} \\ \chi = (t, \varepsilon) & \mapsto & (2t, 0) \end{array}$$

A compact subset of $\mathbb{R} \times \mathbb{Z}$ in the connected component $\{\ell\}$ of $\mathbb{Z}$ has the form $K \times \{\ell\} \subset \mathbb{R} \times \mathbb{Z}$, where $K \subset \mathbb{R}$ is compact. We have

$$\varphi^{-1}(K \times \{\ell\}) = \begin{cases} \emptyset & \text{, if } \ell \neq 0 \\ \frac{1}{2}K \times \{\varepsilon\} & \text{, if } \ell = 0, \end{cases}$$

where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\varphi^{-1}(K \times \{\ell\})$ is compact and φ is proper.

The Case $n > 1$. Base change determines a map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ of topological spaces. Let $X = X(M)/W_\sigma(M)$ be a connected component of $\mathcal{A}_n^t(\mathbb{R})$. Then, X is mapped under BC into a connected component $Y = Y(T)/W_{\sigma'}(T)$ of $\mathcal{A}_n^t(\mathbb{C})$. Given a generalized principal series representation

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$$

where the χ_i's are ramified characters of $\mathbb{C}^\times$ and the ξ's are ramified characters of $\mathbb{R}^\times$, then

$$\mathcal{BC}(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, ..., \chi_q, \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N).$$

Here, $N = N_{\mathbb{C}/\mathbb{R}}$ is the norm map and τ is the generator of $Gal(\mathbb{C}/\mathbb{R})$.

We associate to π the usual parameters uniquely defined for each character χ and ξ. For simplicity, we write the set of parameters as a $(q + r)$-uple:

$$(t, t') = (t_1, ..., t_q, t'_1, ..., t'_r) \in \mathbb{R}^{q+r} \cong X(M).$$

Now, if $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ lies in the connected component defined by the fixed parameters $(\ell, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r$, then

$$(t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T)$$

is a continuous proper map.

It follows that

$$\mathcal{BC} : X(M)/W_\sigma(M) \to Y(T)/W_{\sigma'}(T)$$

is continuous and proper since the orbit spaces are endowed with the quotient topology. □

Theorem 5.4. *The functorial map induced by base change*

$$K_j(C_r^*GL(n,\mathbb{C})) \stackrel{K_j(\mathcal{BC})}{\longrightarrow} K_j(C_r^*GL(n,\mathbb{R}))$$

is zero for $n > 1$.

Proof. We start with the case $n > 2$. Let $n = 2q + r$ be a partition and M the associated Levi subgroup of $GL(n,\mathbb{R})$. Denote by $X_\mathbb{R}(M)$ the unramified characters of M. As we have seen, $X_\mathbb{R}(M)$ is parametrized by $\mathbb{R}^{q+r}$. On the other hand, the only Levi subgroup of $GL(n,\mathbb{C})$ for $n = 2q+r$ is the diagonal subgroup $X_\mathbb{C}(M) = (\mathbb{C}^\times)^{2q+r}$.

If $q = 0$ then $r = n$ and both $X_\mathbb{R}(M)$ and $X_\mathbb{C}(M)$ are parametrized by $\mathbb{R}^n$. But then in the real case an element $\sigma \in E_2(^0M)$ is given by

$$\sigma = i_{GL(n,\mathbb{R}),P}(\chi_1 \otimes ... \otimes \chi_n),$$

with $\chi_i \in \widehat{\mathbb{Z}/2\mathbb{Z}}$. Since $n \geq 3$ there is always repetition of the χ_i's. It follows that the isotropy subgroups $W_\sigma(M)$ are all nontrivial and the quotient spaces $\mathbb{R}^n/W_\sigma$ are closed cones. Therefore, the K-theory groups vanish.

If $q \neq 0$, then $X_\mathbb{R}(M)$ is parametrized by $\mathbb{R}^{q+r}$ and $X_\mathbb{C}(M)$ is parametrized by $\mathbb{R}^{2q+r}$ (see Propositions 2.2 and 2.3).

Base change creates a map

$$\mathbb{R}^{q+r} \longrightarrow \mathbb{R}^{2q+r}.$$

Composing with the stereographic projections we obtain a map

$$S^{q+r} \longrightarrow S^{2q+r}$$

between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced K-theory map

$$K^j(S^{2q+r}) \longrightarrow K^j(S^{q+r})$$

is the zero map.

The Case $n = 2$. For $n = 2$ there are two Levi subgroups of $GL(2,\mathbb{R})$, $M_1 \cong GL(2,\mathbb{R})$ and the diagonal subgroup $M_2 \cong (\mathbb{R}^\times)^2$. By Proposition 2.2

$X(M_1)$ is parametrized by $\mathbb{R}$ and $X(M_2)$ is parametrized by $\mathbb{R}^2$. The group $GL(2,\mathbb{C})$ has only one Levi subgroup, the diagonal subgroup $M \cong (\mathbb{C}^\times)^2$. From Proposition 2.3 it is parametrized by $\mathbb{R}^2$.

Since $K^1(\mathcal{A}_2^t(\mathbb{C})) = 0$ by Theorem 5.1, we only have to consider the K^0 functor. The only contribution to $K^0(\mathcal{A}_2^t(\mathbb{R}))$ comes from $M_2 \cong (\mathbb{R}^\times)^2$ and we have (see Example 3.5)

$$K^0(\mathcal{A}_2^t(\mathbb{R})) \cong \mathbb{Z}.$$

For the Levi subgroup $M_2 \cong (\mathbb{R}^\times)^2$, base change is

$$\begin{array}{rcc} \mathcal{BC} : \mathcal{A}_2^t(\mathbb{R}) & \longrightarrow & \mathcal{A}_2^t(\mathbb{C}) \\ \pi(\xi_1, \xi_2) & \mapsto & i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N), \end{array}$$

Therefore, it maps a class $[t_1, t_2]$, which lies in the connected component $(\varepsilon_1, \varepsilon_2)$, into the class $[2t_1, 2t_2]$, which lies in the connect component $(0,0)$. In other words, base change maps a generalized principal series $\pi(\xi_1, \xi_2)$ into a nongeneric point of $\mathcal{A}_2^t(\mathbb{C})$. It follows from Theorem 3.9 that

$$K^0(\mathcal{BC}) : K^0(\mathcal{A}_2^t(\mathbb{R})) \to K^0(\mathcal{A}_2^t(\mathbb{C}))$$

is the zero map. □

5.2 Base change in one dimension

In this section we consider base change for GL(1).

Theorem 5.5. *The functorial map induced by base change*

$$K_1(C_r^*\mathrm{GL}(1,\mathbb{C})) \xrightarrow{K_1(\mathcal{BC})} K_1(C_r^*\mathrm{GL}(1,\mathbb{R}))$$

is given by $K_1(\mathcal{BC}) = \Delta \circ Pr$*, where* Pr *is the projection of the zero component of* $K^1(\mathcal{A}_1^t(\mathbb{C}))$ *into* $\mathbb{Z}$ *and* Δ *is the diagonal* $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$*.*

Proof. For GL(1), base change

$$\chi \in \mathcal{A}_1^t(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1^t(\mathbb{C})$$

induces a map

$$K^1(\mathcal{BC}) : K^1(\mathcal{A}_1^t(\mathbb{C})) \to K^1(\mathcal{A}_1^t(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1^t(\mathbb{R})$ is uniquely determined by a pair of parameters $(t, \varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1^t(\mathbb{C})$ is uniquely determined by a pair of parameters $(t, \ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter ε (resp., ℓ) labels each connected component of $\mathcal{A}_1^t(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1^t(\mathbb{C}) = \bigsqcup_{\mathbb{Z}} \mathbb{R}$).

Base change maps each component ε of $\mathcal{A}_1^t(\mathbb{R})$ into the component 0 of $\mathcal{A}_1^t(\mathbb{C})$, sending $t \in \mathbb{R}$ to $2t \in \mathbb{R}$. The map $t \mapsto 2t$ is homotopic to the identity. At the level of K^1, the base change map is given by $K_1(BC) = \Delta \circ Pr$, where Pr is the projection of the zero component of $K^1(\mathcal{A}_1^t(\mathbb{C}))$ into $\mathbb{Z}$ and Δ is the diagonal $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

$\square$

6 Automorphic induction

We begin this section by describing the action of $Gal(\mathbb{C}/\mathbb{R})$ on $\widehat{W_{\mathbb{C}}} = \widehat{\mathbb{C}^\times}$. Take $\chi = \chi_{\ell,t} \in \widehat{\mathbb{C}^\times}$ and let τ denote the nontrivial element of $Gal(\mathbb{C}/\mathbb{R})$. Then, $Gal(\mathbb{C}/\mathbb{R})$ acts on $\widehat{\mathbb{C}^\times}$ as follows:

$$\chi^\tau(z) = \chi(\overline{z}).$$

Hence,

$$\chi^\tau_{\ell,t}(z) = (\frac{\overline{z}}{|z|})^\ell |z|^{it}_{\mathbb{C}} = (\frac{z}{|z|})^{-\ell} |z|^{it}_{\mathbb{C}}$$

and we conclude that

$$\chi^\tau_{\ell,t}(z) = \chi_{-\ell,t}(z).$$

In particular,

$$\chi^\tau = \chi \Leftrightarrow \ell = 0 \Leftrightarrow \chi = |.|^{it}_{\mathbb{C}}$$

i.e, χ is unramified.

Note that $W_{\mathbb{C}} \subset W_{\mathbb{R}}$, with index $[W_{\mathbb{R}} : W_{\mathbb{C}}] = 2$. Therefore, there is a natural induction map

$$Ind_{\mathbb{C}/\mathbb{R}} : \mathcal{G}_1^t(\mathbb{C}) \to \mathcal{G}_2^t(\mathbb{R}).$$

By the local Langlands correspondence for archimedean fields [3, 8], there exists an automorphic induction map $\mathcal{AI}_{\mathbb{C}/\mathbb{R}}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_1^t(\mathbb{C}) & \xrightarrow{\mathcal{AI}_{\mathbb{C}/\mathbb{R}}} & \mathcal{A}_2^t(\mathbb{R}) \\ {\scriptstyle \mathbb{C}\mathcal{L}_1} \uparrow & & \uparrow {\scriptstyle \mathbb{R}\mathcal{L}_2} \\ \mathcal{G}_1^t(\mathbb{C}) & \xrightarrow[Ind_{\mathbb{C}/\mathbb{R}}]{} & \mathcal{G}_2^t(\mathbb{R}) \end{array}$$

The next result describes reducibility of induced representations.

Proposition 6.1. *Let χ be a character of $W_{\mathbb{C}}$. We have:*

(i) If $\chi \neq \chi^{\tau}$ then $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ is irreducible;

(ii) If $\chi = \chi^{\tau}$ then $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ is reducible. Moreover, there exist $\rho \in \widehat{W_{\mathbb{R}}}$ such that

$$Ind_{\mathbb{C}/\mathbb{R}}(\chi) = \rho \oplus \rho^{\tau} = \rho \oplus sgn.\rho,$$

where $\rho_{|W_{\mathbb{C}}} = \chi$;

(iii) $Ind_{\mathbb{C}/\mathbb{R}}(\chi_1) \cong Ind_{\mathbb{C}/\mathbb{R}}(\chi_2)$ if and only if $\chi_1 = \chi_2$ of $\chi_1 = \chi_2^{\tau}$.

Proof. Apply Frobenius reciprocity

$$Hom_{W_{\mathbb{R}}}(Ind_{\mathbb{C}/\mathbb{R}}(\chi_1), Ind_{\mathbb{C}/\mathbb{R}}(\chi_2)) \cong Hom_{W_{\mathbb{C}}}(\chi_1, \chi_2).$$

Now, $W_{\mathbb{R}} = W_{\mathbb{C}} \sqcup jW_{\mathbb{C}}$. Therefore, the restriction of $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ to $W_{\mathbb{C}}$ is $\chi \oplus \chi^{\tau}$. The result follow since $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ is semi-simple. □

Proposition 6.2. *A finite dimensional continuous irreducible representation of $W_{\mathbb{R}}$ is either a character or isomorphic to some $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$, with $\chi \neq \chi^{\tau}$.*

Proof. It follows immediately from Lemma 4.3 □

6.1 The automorphic induction map

In this section we describe automorphic induction map as a map of topological spaces. We begin by considering $n = 1$.

Let $\chi = \chi_{\ell,t}$ be a character of $W_{\mathbb{C}}$. If $\chi \neq \chi^{\tau}$, by proposition 4.2, $\phi_{\ell,t} \simeq \phi_{-\ell,t}$. Hence,

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}({}_{\mathbb{C}}\mathcal{L}_1(\chi_{\ell,t})) = D_{|\ell|} \otimes |det(.)|^{it}.$$

On the other hand, if $\chi = \chi^{\tau}$ then $\chi = \chi_{0,t}$ and $\chi(z) = |z|_{\mathbb{C}}^{it}$. Therefore,

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}({}_{\mathbb{C}}\mathcal{L}_1(|.|_{\mathbb{C}}^{it})) = {}_{\mathbb{R}}\mathcal{L}_2(\rho \oplus sgn.\rho) = \pi(\rho, \rho^{-1}),$$

where $\pi(\rho, \rho^{-1})$ is a reducible principal series and ρ is the character of $\mathbb{R}^{\times} \simeq W_{\mathbb{R}}^{ab}$ associated with $\chi_{0,t} = |.|_{\mathbb{C}}^{it}$ via class field theory, i.e. $\rho_{|W_{\mathbb{C}}} = \chi$.

Recall that

$$\mathcal{A}_1^t(\mathbb{C}) \cong \bigsqcup_{\ell \in \mathbb{Z}} \mathbb{R}$$

and

$$\mathcal{A}_2^t(\mathbb{R}) \cong \left(\bigsqcup_{\ell \in \mathbb{N}} \mathbb{R} \right) \bigsqcup (\mathbb{R}^2/S_2) \bigsqcup (\mathbb{R}^2/S_2) \bigsqcup \mathbb{R}^2.$$

As a map of topological spaces, automorphic induction for $n = 1$ may be described as follows:

$$(t, \ell) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, |\ell|) \in \mathbb{R} \times \mathbb{N}, \text{ if } \ell \neq 0 \tag{14}$$

$$(t, 0) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, t) \mapsto \mathbb{R}^2, \text{ if } \ell = 0. \tag{15}$$

More generally, let $\chi_1 \oplus ... \oplus \chi_n$ be an n dimensional L-parameter of $W_{\mathbb{C}}$. Then, either $\chi_k \neq \chi_k^\tau$ for every k, in which case automorphic induction is

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}({}_{\mathbb{C}}\mathcal{L}_n(\chi_1 \oplus ... \oplus \chi_n)) = D_{|\ell_1|} \otimes |det(.)|^{it_1} \oplus ... \oplus D_{|\ell_n|} \otimes |det(.)|^{it_n} \tag{16}$$

or for some k (possibly more than one), $\chi_k = \chi_k^\tau$, in which case we have

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}({}_{\mathbb{C}}\mathcal{L}_n(\chi_1 \oplus ... \oplus |.|_{\mathbb{C}}^{it_k} \oplus ... \oplus \chi_n)) = D_{|\ell_1|} \otimes |det(.)|^{it_1} \oplus ... \tag{17}$$

$$\oplus \pi(\rho_k, \rho_k^{-1}) \oplus ... \oplus D_{|\ell_n|} \otimes |det(.)|^{it_n}.$$

In order to describe automorphic induction as a map of topological spaces, it is enough to consider components of $\mathcal{A}_n^t(\mathbb{C})$ with generic W-orbit. For convenience, we introduce the following notation:

if $(t_1, ..., t_n)$ is in the component of $\mathcal{A}_n^t(\mathbb{C})$ labeled by $(\ell_1, ..., \ell_n)$, i.e

$$(t_1, ..., t_n) \in (\mathbb{R} \times \{\ell_1\}) \times ... \times (\mathbb{R} \times \{\ell_n\})$$

we write simply

$$(t_1, ..., t_n) \in \mathbb{R}^n_{(\ell_1,...,\ell_n)}, \quad \ell_i \in \mathbb{Z}.$$

There are two cases:

Case 1: $\chi_k \neq \chi_k^\tau$, i.e. $\ell_k \neq 0$, for every k,

$$\mathcal{AI} : (t_1, ..., t_n) \in \mathbb{R}^n_{(\ell_1,...,\ell_n)} \longmapsto (t_1, ..., t_n) \in \mathbb{R}^n_{(|\ell_1|,...,|\ell_n|)} \tag{18}$$

So, $(|\ell_1|, ..., |\ell_n|) \in \mathbb{N}^n$.

Case 2: if there are $0 < m < n$ characters such that $\chi_k = \chi_k^\tau$, then

$$\mathcal{AI} : (t_1, ..., t_k, ..., t_n) \in \mathbb{R}^n_{(\ell_1,...,0,...,\ell_n)} \longmapsto (t_1, ..., t_k, t_k, ..., t_n) \in (\mathbb{R}^n/W)_{(|\ell_1|,...,|\ell_n|)^*} \tag{19}$$

where $(|\ell_1|, ..., |\ell_n|)^* \in \mathbb{N}^{n-m}$ means that we have deleted the m labels corresponding to $\ell_k = 0$. Note that if $m > 1$, necessarily $W \neq 0$.

We have the following result

Proposition 6.3. *The automorphic induction map*

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{C}) \to \mathcal{A}_{2n}^t(\mathbb{R})$$

is a continuous proper map.

The proof follows from the above discussion and is similar to that of proposition 5.3.

Example 6.4. *Consider $n = 3$. Then,*

$$\mathcal{A}_3^t(\mathbb{C}) \simeq \bigsqcup_{\sigma} \mathbb{R}^3/W_\sigma$$

and

$$\mathcal{A}_6^t(\mathbb{R}) \simeq (\bigsqcup_{\ell \in \mathbb{N}^3} \mathbb{R}^3) \sqcup (\bigsqcup_{\ell' \in \mathbb{N}^2} \mathbb{R}^4) \sqcup \mathcal{C},$$

where $\mathcal{C}$ is a disjoint union of cones.

Let $\chi_1 \oplus \chi_2 \oplus \chi_3$ denote a 3-dimensional L-parameter of $W_\mathbb{C}$. We have the following description of $\mathcal{AI}_{\mathbb{C}/\mathbb{R}}$ as a map of topological spaces:

- $\chi_1 \neq \chi_1^\tau, \chi_2 \neq \chi_2^\tau, \chi_3 \neq \chi_3^\tau$

$$(t_1, t_2, t_3) \in \mathbb{R}^3_{(\ell_1,\ell_2,\ell_3)} \longmapsto (t_1, t_2, t_3) \in \mathbb{R}^3_{(|\ell_1|,|\ell_2|,|\ell_3|)}$$

with $\ell_i \in \mathbb{Z}\backslash\{0\}$.

- $\chi_1 = \chi_1^\tau, \chi_2 \neq \chi_2^\tau, \chi_3 \neq \chi_3^\tau$

$$(t_1, t_2, t_3) \in \mathbb{R}^3_{(0,\ell_2,\ell_3)} \longmapsto (t_1, t_1, t_2, t_3) \in (\mathbb{R}^4/W)_{(|\ell_2|,|\ell_3|)}$$

with $\ell_i \in \mathbb{Z}\backslash\{0\}$. Similar for the cases $(\ell_1, 0, \ell_3)$ and $(\ell_1, \ell_2, 0)$.

- $\chi_1 = \chi_1^\tau, \chi_2 = \chi_2^\tau, \chi_3 \neq \chi_3^\tau$

$$(t_1, t_2, t_3) \in \mathbb{R}^3_{(0,0,\ell_3)} \longmapsto (t_1, t_1, t_2, t_2, t_3) \in (\mathbb{R}^5/W)_{(|\ell_3|)}$$

with $\ell_3 \in \mathbb{Z}\backslash\{0\}$. Similar for the cases $(\ell_1, 0, 0)$ and $(0, \ell_2, 0)$.

- $\chi_1 = \chi_1^\tau, \chi_2 = \chi_2^\tau, \chi_3 = \chi_3^\tau$

$$(t_1, t_2, t_3) \in \mathbb{R}^3_{(0,0,0)} \longmapsto (t_1, t_1, t_2, t_2, t_3, t_3) \in (\mathbb{R}^6/W)$$

6.2 Automorphic induction in one dimension

Automorphic induction $\mathcal{AI}$ induces a K-theory map at the level of K-theory groups K^1:

$$K^1(\mathcal{AI}) : K^1(\mathcal{A}_2^t(\mathbb{R})) \to K^1(\mathcal{A}_1^t(\mathbb{C})). \tag{20}$$

We have

$$K^1(\mathcal{A}_2^t(\mathbb{R})) \cong \bigoplus_{\ell\in\mathbb{N}} \mathbb{Z}.$$

Each class of 1-dimension L-parameters of $W_\mathbb{C}$ (characters of $\mathbb{C}^\times$)

$$[\chi] = [\chi_{\ell,t}] = [\chi_{-\ell,t}] \ (\ell \neq 0)$$

contributes with one generator to $K^1(\mathcal{A}_2^t(\mathbb{R}))$. Note that, under $\mathcal{AI}$, this is precisely the parametrization given by the discrete series $D_{|\ell|}$.

On the other hand,

$$K^1(\mathcal{A}_1^t(\mathbb{C})) \cong \bigoplus_{\ell\in\mathbb{Z}} \mathbb{Z}.$$

Again, each class (of characters of $\mathbb{C}^\times$) $[\chi]$ contributes with a generator to $K^1(\mathcal{A}_1^t(\mathbb{C}))$, only this time $[\chi_{\ell,t}] \neq [\chi_{-\ell,t}]$, i.e, ℓ and $-\ell$ belong to different classes.

Note that we may write

$$K^1(\mathcal{A}_2^t(\mathbb{R})) \cong \bigoplus_{\text{Discrete series}} \mathbb{Z} = \bigoplus_{[D_{|\ell|}]} \mathbb{Z} = \bigoplus_{\ell\in\mathbb{N}} \mathbb{Z}$$

and

$$K^1(\mathcal{A}_1^t(\mathbb{C})) \cong \bigoplus_{\widehat{\mathbb{C}^\times}} \mathbb{Z} = \bigoplus_{[\chi_\ell]} \mathbb{Z} = \bigoplus_{\ell\in\mathbb{Z}} \mathbb{Z}.$$

The automorphic induction map

$$K^1(\mathcal{AI}) : K^1(\mathcal{A}_2^t(\mathbb{R})) \to K^1(\mathcal{A}_1^t(\mathbb{C}))$$

may be interpreted, at the level of K^1, as a kind of "shift" map

$$[D_{|\ell|}] \mapsto [\chi_{|\ell|}]$$

More explicitly, the "shift" map is

$$\bigoplus_{\ell\in\mathbb{N}} \mathbb{Z} \to \bigoplus_{\ell\in\mathbb{Z}} \mathbb{Z}, ([D_1], [D_2], ...) \mapsto (..., 0, 0, [\chi_1], [\chi_2], ...)$$

where the image under $K^1(\mathcal{AI})$ on each component of the right hand side with label $\ell \leq 0$ is zero (because $K^1(\mathcal{AI})$ is a group homomorphism so it must map zero into zero).

6.3 Automorphic induction in n dimensions

In this section we consider automorphic induction for $GL(n)$. Contrary to base change (see theorems 5.4 and 5.5), the K-theory map of automorphic induction is nonzero for every n.

Theorem 6.5. *The functorial map induced by automorphic induction*

$$K_j(C_r^*\mathrm{GL}(2n,\mathbb{R})) \overset{K_j(\mathcal{AI})}{\longrightarrow} K_j(C_r^*\mathrm{GL}(n,\mathbb{C}))$$

is given by

$$[D_{|\ell_1|} \otimes ... \otimes D_{|\ell_n|}] \longmapsto [\chi_{|\ell_1|} \oplus ... \oplus \chi_{|\ell_n|}]$$

if $n \equiv j\,(mod\,2)$ *and* $\chi_k \neq \chi_k^\tau$ *for every* k*, and is zero otherwise.*

Here, $[D_{|\ell_1|}\otimes...\otimes D_{|\ell_n|}]$ *denotes the generator of the component* $\mathbb{Z}_{(|\ell_1|,...,|\ell_n|)}$ *of* $K_j(C_r^*\mathrm{GL}(2n,\mathbb{R}))$ *and* $[\chi_{|\ell_1|} \oplus ... \oplus \chi_{|\ell_n|}]$ *is the generator of the component* $\mathbb{Z}_{(|\ell_1|,...,|\ell_n|)}$ *of* $K_j(C_r^*\mathrm{GL}(n,\mathbb{C}))$

Proof. Let $0 \leq m < n$ be the number of characters χ_k with $\chi_k = \chi_k^\tau$.

Case 1**:** $m = 0$

In this case, $\chi_k \neq \chi_k^\tau$ for every k. Each character χ_{ℓ_k}, $\ell_k \neq 0$, is mapped via the local langlands correspondence into a discrete series $D_{|\ell_k|}$. At the level of K-theory, a generator $[D_{|\ell_k|}]$ is mapped into $[\chi_{|\ell_k|}]$. The result follows from (16).

Case 2**:** $m > 0$ odd

Then, if $n \equiv j\,(mod\,2)$, $K^j(\mathbb{R}^{n+m}) = 0$ and $K_j(\mathcal{AI})$ is zero.

Case 3**:** $m > 0$ is even

In this case $K^j(\mathbb{R}^n) = K^j(\mathbb{R}^{n+m})$. However, $X_\mathbb{R}(M) \simeq \mathbb{R}^{n+m}$ corresponds precisely to the partition of $2n$ into $1's$ and $0's$ given by

$$2n = 2(n-m) + 2m$$

Hence, the number of $1's$ in the partition is $2m \geq 4$. It follows that $(t_1, ..., t_n)$ is mapped into a cone and, as a consequence, $K_j(\mathcal{AI})$ is zero.

This concludes the proof. □

7 Connections with the Baum-Connes correspondence

The standard maximal compact subgroup of $\mathrm{GL}(1,\mathbb{C})$ is the circle group $U(1)$, and the maximal compact subgroup of $\mathrm{GL}(1,\mathbb{R})$ is $\mathbb{Z}/2\mathbb{Z}$. Base change for K^1 creates a map

$$\mathcal{R}(\mathrm{U}(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathrm{U}(1))$ is the representation ring of the circle group U(1) and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of U(1) to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of U(1) to zero.

This map has an interpretation in terms of K-cycles. The real line $\mathbb{R}$ is a universal example for the action of $\mathbb{R}^\times$ and $\mathbb{C}^\times$. The K-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\partial\!\!\!/_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(\underline{E}\mathbb{C}^\times)$ and a class $\partial\!\!\!/_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(\underline{E}\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\partial\!\!\!/_{\mathbb{C}} \mapsto (\partial\!\!\!/_{\mathbb{R}}, \partial\!\!\!/_{\mathbb{R}})$$

It would be interesting to interpret the automorphic induction map at the level of representation rings:

$$\mathcal{AI}^* : \mathcal{R}(O(2n)) \longrightarrow \mathcal{R}(U(n)).$$

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