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PROPERTIES OF OPTION PRICES IN A JUMP DIFFUSION MODEL

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Abstract. We study convexity and monotonicity properties of option prices in a jump-diffusion model using the fact that these prices satisfy certain parabolic integro-differential equations. Conditions are provided under which preservation of convexity holds, i.e. under which the value, calculated under a chosen martingale measure, of an option with a convex contract function is convex as a function of the underlying stock price. The preservation of convexity is then used to derive monotonicity properties of the option value with respect to the different parameters of the model, such as the volatility, the jump size and the jump intensity.

1. Introduction

The implications of parameter mis-specifications for option pricing in continuous diffusion models have been studied rather extensively in the literature, compare for example [2], [4], [5], [6], [9], [10] and the references therein. A general result of these papers is that in the Black-Scholes model, with the volatilities of the stocks being possibly time- and level-dependent, a European option price is monotonically increasing in the volatility if and only if the option price, at each fixed time prior to maturity, is convex as a function of the price of the underlying assets. We refer to a model as being "convexity preserving" if, for any convex contract function, the corresponding option price is convex as a function of the underlying asset at all times prior to maturity. Before proceeding, we should note that there are several examples of models in finance that are not convexity preserving. For example, general stochastic volatility models, compare [2] and [6], and also several models in higher dimensions (even in the case of time- and level-dependent volatility), compare [5].

In this paper we investigate properties of option prices in a one-dimensional jump-diffusion model (for a nice introduction to financial models with jumps we refer to the book [3]). When introducing jumps into the model, completeness of the model is in general lost, so there is not a unique equivalent martingale measure that can be used for arbitrage free pricing of options. There are at least two different natural questions along the lines of the

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above mentioned monotonicity results. First, given a model specified under the physical measure, which of two given martingale measures gives the higher option price? Second, what are the implications of a possible misspecification of models when using a fixed martingale measure for pricing options? The first issue is dealt with in [8]. In the current paper we focus on the second question. Thus, instead of considering different possible choices of martingale measures, we take the standpoint of having one equivalent martingale measure as given, and we will investigate properties of the option value calculated under this fixed measure. This corresponds to an agent modeling the stock price process, not under the physical measure, but rather directly under a martingale measure, and with a possible uncertainty about the parameters.

The literature about properties of option prices in jump-diffusion models is, at least to the best of our knowledge, not as extensive as the literature about models without jumps. As mentioned above, Henderson and Hobson [8] prove monotonicity results for the option value with respect to different martingale measures. Their results can also be interpreted as a monotonicity result in the intensity of the underlying Poisson process for a *fixed* measure, compare Remark 6.2 in that paper. For the case of American options, see also Pham [12].

To remain in a Markovian setting, all parameters of the model are assumed to be merely time- and level-dependent in this article. Thus the only source of randomness in the parameters is through their dependences on the current stock price. In contrast to the case without jumps, it is easy to construct examples of one-dimensional Markovian models with jumps where convexity is not preserved; such an example is given in Section 4. Below we provide a sufficient condition for a jump-diffusion model to be convexity preserving. We also show, by analogy with the case without jumps, that convexity of the value function implies certain monotonicity results in the different parameters of the model. More precisely, we derive conditions under which the option value of a convex claim is increasing in (i) the diffusion coefficient measuring the continuous fluctuations of the diffusion, (ii) the jump intensity of the underlying Poisson process and (iii) the possible jump sizes. In the proofs of the preservation of convexity and the monotonicity results we use a characterization of the option value as the unique viscosity solution of a certain parabolic integro-differential equation. The methods of proofs are adapted from techniques used to study preservation of convexity of solutions to parabolic partial differential equations, see [11].

The present paper is organized as follows. In Section 2 we introduce the financial model. In Section 3 a regularity result for the value function of the option is provided. In Sections 4 and 5 we formulate our main results. We give a sufficient condition under which the model is convexity preserving, and we apply this to establish monotonicity results in the volatility, the jump size and the jump intensity.

2. The financial model

We consider a financial market with a finite time horizon T. The market consists of a bank account with deterministic interest rate and a risky asset with a positive price process $X(t)$. For our purposes there is no restriction to assume that the price of the risky asset is quoted in terms of the non-risky asset, i.e. that the bank account serves as a numeraire.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_T = \mathcal{F}$ and satisfying the "usual conditions", i.e. $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all P-negligible events in $\mathcal{F}$. Let W be a Brownian motion and let v be a homogeneous Poisson random measure on $[0,T]\times[0,1]$ with intensity measure

$$q(dt, dz) = \lambda(t)\, dt\, dz$$

for some deterministic function $\lambda(t) \geq 0$. Define the compensated jump martingale measure $\tilde{v}$ by $\tilde{v}(dt, dz) = (v - q)(dt, dz)$. Let the risky asset be modeled by a stochastic process $X(t)$ satisfying the stochastic differential equation

$$dX = \beta(X(t-), t)\, dW + \int_0^1 \phi(X(t-), t, z)\, \tilde{v}(dt, dz). \tag{1}$$

The interpretation that should be given to the model is that

- β (or rather $\frac{|\beta|}{x}$) represents the volatility of the Brownian part of the stock price; note that β is possibly time- and level-dependent.
- the jump intensity of the stock price is λ; associated to each jump there is a label z with the interpretation that a jump at time t with label z is of size $\phi(X(t-), t, z)$.

The following are the minimal assumptions used in this paper.

(M1) The diffusion coefficient $\beta : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ and the jump intensity $\lambda : [0,T] \to \mathbb{R}$ are both continuous, and the jump size $\phi : \mathbb{R}^+ \times [0,T] \times [0,1] \to \mathbb{R}$ is measurable and for each fixed $z \in [0,1]$, the function $(x,t) \mapsto \phi(x,t,z)$ is continuous.

Moreover, there exist constants $C > 0$ and $\gamma > -1$ with

(M2) $\beta^2(x,t) + \phi^2(x,t,z) \leq Cx^2$

(M3) $|\beta(x,t) - \beta(y,t)| + |\phi(x,t,z) - \phi(y,t,z)| \leq C|x-y|$

(M4) $\phi(x,t,z) > \gamma x$

for all x, t and z. Under these assumptions there exists a unique solution to the stochastic differential equation (1) for any starting point $x > 0$, compare for example [7]. This solution satisfies

$$P\big(X(t) \leq 0 \text{ for some } t \in [0,T]\big) = 0,$$

i.e. $X(t)$ remains positive at all times with probability 1.

For a continuous contract function g, the value at time t of a European option that at time T pays the amount $g(X(T))$ is $u(X(t), t)$, where

$$u(x,t) = E_{x,t}\, g(X(T)).$$

Here the indices indicate that $X(t) = x$. In Lemma 3.1 in [13], estimates of the second moment of $X(T)$ are given. In our setting with a bounded intensity of jumps, estimates on higher moments than two can be obtained in the same way as in the well-known case of diffusion processes using Gronwall's lemma. Thus the value function u is well-defined for contract functions satisfying

(M5) $g \in C_{pol}(\mathbb{R}^+)$,

where $C_{pol}(\mathbb{R}^+)$ is defined in Definition 3.1 below.

Remark As remarked in the introduction, there is in general no unique risk-neutral measure that could be used for arbitrage free pricing of options in the above jump-diffusion model. In this paper we do not deal with the issue of choosing an appropriate measure for pricing, but we rather assume that the measure has already been chosen. Moreover, we specify our model, not under a physical measure, but directly under this chosen martingale measure. Thus there is no need of changing measures when pricing options.

The pricing function u can under appropriate conditions be characterized as the unique viscosity solution of a certain parabolic integro-differential equation. In fact, after a standard change $t \to T-t$ of the direction of time, u satisfies

$$u_t = \mathcal{L}u$$

with initial condition

$$u(x,0) = g(x).$$

Here $\mathcal{L}$ is the elliptic integro-differential operator

$$\mathcal{L}u := au_{xx} + \mathcal{B}u, \tag{2}$$

where

$$a(x,t) := \frac{\beta^2(x,t)}{2}$$

and

$$\mathcal{B}u = \lambda(t)\int_0^1 \Big(u(x+\phi(x,t,z),t) - u(x,t) - \phi(x,t,z)u_x(x,t)\Big)\,dz.$$

We will see below that, under some conditions, a viscosity solution also is a classical solution. Therefore we do not formally introduce the concept of viscosity solutions here, but we rather refer the reader to the definition in [13].

3. Regularity of the value function

In this section we provide some regularity results for the value function u. A priori, viscosity solutions are merely continuous, but by arguing similarly as in Section 5.2 in [13], higher order regularity can be obtained (see also Theorem 4, page 296 in [7]). To do this we first transform the equation into an equation with coefficients satisfying some standard assumptions in the theory of partial differential equations. We would like to point out that the results in [13] easily extend to the case of a possibly time-dependent jump-intensity λ.

We begin with introducing a few definitions and assumptions that are used below.

Definition 3.1. *(i) For a set $E \subset \mathbb{R}$ we denote by $C^p(E)$ the set of functions $f : E \to \mathbb{R}$ such that the derivatives $\partial_x^k f$ with $k \leq p$ exist in the interior of E and have continuous extensions to E.*

(ii) For a set $E \subset \mathbb{R} \times [0,T]$ we denote by $C^{p,q}(E)$ the set of functions $f : E \to \mathbb{R}$ such that the derivatives $\partial_x^k \partial_t^l f$ with $k + 2l \leq p$ and $l \leq q$ exist in the interior of E and have continuous extensions to E.

(iii) For a set $E \subset \mathbb{R} \times [0,T]$ we denote by $C_{pol}(E)$ the set of functions of at most polynomial growth in x. More explicitly,

$$C_{pol}(E) = \bigcup_{C>0,m>0} \big\{ f : E \to \mathbb{R} : |f(x,t)| \le C(1+|x|^m) \text{ for } (x,t) \in E \big\}.$$

(iv) For $E \subset \mathbb{R} \times [0,T]$ and $\alpha \in (0,1)$ we denote by $C_\alpha(E)$ the set of locally Hölder(α) functions, i.e.

$$C_\alpha(E) := \{ f : E \to \mathbb{R} : \sup_{p,q \in K} \frac{|f(p)-f(q)|}{d(p,q)^\alpha} < \infty \text{ for each compact } K \subset E \}.$$

Here d is the parabolic distance $d\big((x_1,t_1),(x_2,t_2)\big) = (|x_1-x_2|^2+|t_1-t_2|)^{1/2}$.

(v) For $E \subset \mathbb{R} \times [0,T]$, the spaces $C^{p,q}_{pol}(E)$ and $C^{p,q}_\alpha(E)$ are the spaces of functions $f \in C^{p,q}(E)$ for which all the derivatives $\partial^k_x \partial^l_t f$ with $k+2l \le p$ and $l \le q$ belong to $C_{pol}(E)$ and $C_\alpha(E)$, respectively.

(vi) For a set $E \subset \mathbb{R}$, the spaces $C_{pol}(E)$, $C^{p,q}_{pol}(E)$, $C_\alpha(E)$ and $C^{p,q}_\alpha(E)$ are defined similarly.

To prove the regularity result Theorem 3.2 below, we need some additional assumptions beyond the minimal assumptions (M1)-(M5). We will say that assumptions (A1)-(A5) hold if there exist constants C, $\gamma > 0$ and $\alpha \in (0,1)$ such that

(A1) $\gamma x^2 \le \beta^2(x,t)$ for all t and all $x \le \gamma$;

(A2) $\lambda \in C_\alpha([0,T])$;

(A3) $\beta \in C^{2,1}_\alpha(\mathbb{R}^+ \times [0,T])$ with $|\beta_t(x,t)| \le Cx$ and $|\beta_{xx}(x,t)| \le C/x$;

(A4) $\phi(\cdot,\cdot,z) \in C^{2,1}_\alpha(\mathbb{R}^+ \times [0,T])$ (with the Hölder continuity being uniform in z), and

$$|\phi_t(x,t,z)| \le Cx$$

and

$$|\phi_{xx}(x,t,z)| \le Cx^{-1}$$

for all (x,t,z);

(A5) g is Lipschitz continuous, i.e. $|g(x_2)-g(x_1)| \le C|x_2-x_1|$ for all $x_1, x_2 \in \mathbb{R}^+$. Moreover, $g \in C^3_{pol}(\mathbb{R}^+)$.

Theorem 3.2. *In addition to the minimal assumptions (M1)-(M5), assume that (A1)-(A5) hold. Then the value function u is in $C^{4,1}(\mathbb{R}^+ \times (0,T)) \cap C^{2,1}(\mathbb{R}^+ \times [0,T])$. Moreover, there exist constants $m > 0$ and $K > 0$ such that*

$$|u_{xx}(x,t)| \le K(x^{-m} + x^m)$$

for all $(x,t) \in \mathbb{R}^+ \times [0,T]$.

Proof. Let $Y(t) := \Psi\big(X(t)\big)$ where $\Psi : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function with $\Psi(x) = -1/x$ for $x \in (0,1]$, $\Phi(x) = x$ for $x \ge 2$ and $\Psi'(x) > 0$ for all $x \in \mathbb{R}^+$. Applying the Ito formula for diffusions with jumps, compare for example Chapter 8 in [3], it follows that Y solves the stochastic differential equation

$$dY = \tilde{b}\big(Y(t-),t\big)\,dt + \tilde{\beta}\big(Y(t-),t\big)\,dW + \int_0^1 \tilde{\phi}\big(Y(t-),t,z\big)\,\tilde{v}(dt,dz)$$

in $\mathbb{R}\times[0,T]$, where

$$\begin{aligned}\tilde{b}(y,t) &= \frac{1}{2}\Psi''\big(\Psi^{-1}(y)\big)\beta^2\big(\Psi^{-1}(y),t\big)\\ &\quad+\lambda(t)\int_0^1\Big(\tilde{\phi}(y,t,z)-y-\Psi'\big(\Psi^{-1}(y)\big)\phi\big(\Psi^{-1}(y),t,z\big)\Big)\,dz,\end{aligned}$$

$$\tilde{\beta}(y,t)=\Psi'\big(\Psi^{-1}(y)\big)\beta\big(\Psi^{-1}(y),t\big)$$

and

$$\tilde{\phi}(y,t,z)=\Psi\Big(\Psi^{-1}(y)+\phi\big(\Psi^{-1}(y),t,z\big)\Big)-y,$$

and $\tilde{v}$ is the same compensated jump martingale measure as in (1). It is straightforward to check that the coefficients $\tilde{b}$, $\tilde{\beta}$ and $\tilde{\phi}$ together with the initial condition

$$\tilde{g}(y):=g(\Psi^{-1}(y))$$

satisfy the conditions (2.2)-(2.5) in [13]. Therefore, it follows from Theorem 3.1 in [13] that

$$v(y,t):=u\big(\Psi^{-1}(y),t\big),$$

in the viscosity sense, solves the equation

$$v_t=\frac{\tilde{\beta}^2}{2}v_{yy}+\tilde{b}v_y+\lambda\int_0^1 v(y+\tilde{\phi},t)-v-\tilde{\phi}v_y\,dz$$

in $\mathbb{R}\times(0,T]$, with initial condition $v(y,0)=\tilde{g}(y)$. Reasoning as on page 22 in [13], one finds that v also solves, again in the viscosity sense, the equation

$$\begin{cases} v_t=\frac{\tilde{\beta}^2}{2}v_{yy}+\big(\tilde{b}-\lambda\int_0^1\tilde{\phi}\,dz\big)v_y+h\\ v(y,0)=\tilde{g}(y),\end{cases}\tag{3}$$

where

$$h(y,t)=\lambda(t)\int_0^1\big(v(y+\tilde{\phi},t)-v(y,t)\big)\,dz.$$

Moreover, Proposition 3.3 in [13] gives that the function $v(y,t)$ satisfies

$$|v(y_1,t_1)-v(y_2,t_2)|\leq C\Big((1+|y_1|)|t_1-t_2|^{1/2}+|y_1-y_2|\Big)\tag{4}$$

for some constant C (Theorem 3.1 and Proposition 3.3 in [13] are stated for a problem of optimally stopping of a controlled jump diffusion process, but the corresponding proofs also work in our simpler setting).

Using (4) and the assumptions on ϕ it follows that h is in $C_\alpha(\mathbb{R}\times[0,T])\cap C_{pol}(\mathbb{R}\times[0,T])$. From Theorem A.20 in [11] we thus have the existence of a unique solution w to equation (3) with $w\in C^{2,1}_{pol}(\mathbb{R}\times[0,T])$ (note that the proof of that theorem also works with the current weaker condition on h), and from Theorem A.18 in [11] we find that $w\in C^{2,1}_\alpha(\mathbb{R}\times[0,T])$. Moreover, this function w also satisfies the inequality (4) (this can be seen by noting that the classical solution w to (3) also is a stochastic solution, and thus Proposition 3.3 in [13] can be applied also to w).

Since w is a classical solution to (3), it is also a viscosity solution of this equation. It then follows from the uniqueness result Theorem 4.1 in [13] that $v\equiv w$. Consequently, $v\in C^{2,1}_{pol}(\mathbb{R}\times[0,T])\cap C^{2,1}_\alpha(\mathbb{R}\times[0,T])$. It follows that $h \in C^{2,0}_\alpha(\mathbb{R} \times [0,T])$, so applying Theorem A.11 in [11] yields $v \in C^{4,1}(\mathbb{R} \times (0,T))$.

Changing back to the original coordinates, it follows that $u \in C^{4,1}(\mathbb{R}^+ \times (0,T)) \cap C^{2,1}(\mathbb{R}^+ \times [0,T])$ and that there exists a constant m such that $u_{xx} = \mathcal{O}(x^{-m})$ for x close to 0 and $u_{xx} = \mathcal{O}(x^{-m})$ for large x, uniformly in t. □

Remark The reason to use the change of coordinates $y = -1/x$ for small x, and not the more standard change $y = \ln x$, is to be able to use Theorem A.20 in [11]. With the logarithmic coordinate change, the condition (A4) of that theorem would not be fulfilled.

4. Preservation of convexity

In this section we provide a sufficient condition on ϕ under which the model is convexity preserving, see condition (10) below. The methods used are adapted from [11], in which the same problem is studied for parabolic equations. We begin, however, with an example showing that not all models are convexity preserving.

Example (A model which is not convexity preserving.) Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-negative Lipschitz function satisfying $\phi(x) = 0$ for $x \notin (1/2, 3/4)$ and $\phi(x) > 1$ for $x \in (x_1, x_2)$, where x_1, x_2 satisfy $1/2 < x_1 < x_2 < 3/4$. Further, let $g(x) = (1-x)^+$ and the stock price dynamics be given by

$$dX = \phi(X(t-))\,(dN - dt),$$

where N is a Poisson process with intensity 1. Now, since $X(t)$ is a martingale, and since g is convex, it is easy to check that the option value $u(x,0)$ at time 0 satisfies

$$u(x,0) = E_{x,0}\,g(X(T)) \geq g(x),$$

with strict inequality for $x \in (x_1, x_2)$. Moreover, for $x \notin (1/2, 3/4)$ the inequality reduces to an equality, and thus $u(x,0)$ is not convex in the interval $[1/2, 1]$.

Definition 4.1. *We say that a model is convexity preserving if, for each convex contract function $g \in C_{pol}(\mathbb{R}^+)$, the corresponding value function $u(x,t)$ is convex in x for each fixed time $t \in [0,T]$.*

Along the lines of the analysis in [11], we make the following definition. The differential operator $\mathcal{L}$ is defined as in (2).

Definition 4.2. *In addition to the minimal assumptions (M1)-(M5) of Section 2, also assume that (A1)-(A5) hold. Then we say that the model is locally convexity preserving (LCP) at a point $(x_0, t_0) \in C(\mathbb{R}^+ \times [0,T])$ if*

$$\partial_x^2(\mathcal{L}f)(x_0, t_0) \geq 0 \tag{5}$$

holds for any convex function $f \in C^4(\mathbb{R}^+) \cap C^2_{pol}(\mathbb{R}^+)$ with $f_{xx}(x_0) = 0$. We simply say that a model is LCP if it is LCP at all points.

Remark Note that the condition (M2) ensures the integral term $\mathcal{B}f$ in $\mathcal{L}f$ to be well-defined for any function $f \in C^1(\mathbb{R}^+) \cap C_{pol}(\mathbb{R}^+)$. Similarly, (M2), (M3) and (A4) together ensure that the integral term in $\partial_x^2(\mathcal{L}f)$ is well-defined for any function $f \in C^3(\mathbb{R}^+) \cap C^2_{pol}(\mathbb{R}^+)$.

It is intuitively clear that the LCP-condition is a natural condition to impose for preservation of convexity. Indeed, if spatial convexity of u is almost lost at some point (x_0, t_0), then the infinitesimal change of u in the time interval $[t_0, t_0 + \Delta t]$ is given by $\Delta t\, \mathcal{L}u$, which is spatially convex if the LCP-condition is satisfied. Below we show that a model which is LCP in fact also is convexity preserving.

Theorem 4.3. *Let the assumptions of Theorem 3.2 hold, and assume that the model is LCP. Then the model is convexity preserving.*

Proof. We know from Theorem 3.2 that the value function u is in $C^{4,1}(\mathbb{R}^+ \times [0,T])$ and that there exist constants K and m such that

$$|u_{xx}(x,t)| \leq K(x^m + x^{-m}) \tag{6}$$

for all $(x,t) \in \mathbb{R}^+ \times [0,T]$. Define the function $h : \mathbb{R}^+ \to \mathbb{R}$ by $h(x) := x^{m+3} + x^{-m+1}$. Then

$$h_{xx} = (m+2)(m+3)x^{m+1} + m(m-1)x^{-m-1} \tag{7}$$

and

$$\begin{aligned} \partial_x^2(\mathcal{L}h) &= a h_{xxxx} + 2a_x h_{xxx} + a_{xx} h_{xx} \\ &\quad + \lambda \int_0^1 \Big((1+\phi_x)^2 h_{xx}(x+\phi) + \phi_{xx} h_x(x+\phi) - \phi h_{xxx} \\ &\qquad - (1+2\phi_x) h_{xx} - \phi_{xx} h_x \Big)\, dz. \end{aligned}$$

The assumptions (A3)-(A4) on β and on ϕ imply the existence of a large positive constant M such that

$$M\partial_x^2 h - \partial_x^2(\mathcal{L}h) \geq 1 \tag{8}$$

for all x and t. For $\epsilon > 0$, define the function u^ϵ by

$$u^\epsilon(x,t) := u(x,t) + \epsilon e^{Mt} h(x),$$

and assume, to reach a contradiction, that the set

$$E := \{(x,t) : u \text{ is not convex in the spatial variable at } (x,t)\}$$

is non-empty. Since u_{xx} satisfies (6), it follows from (7) that $E \subset (\rho^{-1}, \rho) \times [0,T]$ for some $\rho \in \mathbb{R}^+$. Thus E is bounded, so $\overline{E}$ is compact. Therefore the infimum

$$t_0 := \inf\{t \geq 0 : (x,t) \in \overline{E} \text{ for some } x \in \mathbb{R}^+\}$$

is attained, and there exists $x_0 \in \mathbb{R}^+$ with $(x_0, t_0) \in \overline{E}$. At this point we have by continuity $u^\epsilon_{xx} = 0$, and therefore $t_0 > 0$ (since $u^\epsilon_{xx}(x,0) \geq \epsilon h_{xx}(x) > 0$). By the definition of t_0, $u^\epsilon_{xx}(x_0, t) \geq 0$ for $t \leq t_0$, so

$$\partial_t u^\epsilon_{xx}(x_0, t_0) \leq 0.$$

Moreover, since $u^\epsilon_{xx}(x_0, t_0) = 0$ and u^ϵ is spatially convex at $t = t_0$, the LCP-assumption yields

$$\partial_x^2(\mathcal{L}u^\epsilon) \geq 0$$

at (x_0, t_0). Thus we find that

$$\partial_x^2\big(\partial_t u^\epsilon - \mathcal{L}u^\epsilon\big)(x_0, t_0) \leq 0.$$

On the other hand, using (8) and the fact that u solves the equation $u_t = \mathcal{L}u$, we have

$$\partial_x^2\big(\partial_t u^\epsilon - \mathcal{L}u^\epsilon\big) = \epsilon e^{Mt}\partial_x^2(Mh - \mathcal{L}h) \geq \epsilon e^{Mt} > 0,$$

so we have reached a contradiction. Therefore the set E is empty, and thus u^ϵ is spatially convex at all times. By letting ϵ tend to 0 it follows that also u is spatially convex, finishing the proof. □

Theorem 4.4. *Let the assumptions of Theorem 3.2 hold. Also assume that*

$$\phi_{xx}(x,t,z)\phi(x,t,z) \geq 0 \tag{9}$$

for all x, t, z. Then the model is LCP, and thus also convexity preserving.

Proof. Let $f \in C^4(\mathbb{R}^+) \cap C^2_{pol}(\mathbb{R}^+)$. Then,

$$\begin{aligned}\partial_x^2(\mathcal{L}f) &= af_{xxxx} + 2a_x f_{xxx} + a_{xx} f_{xx} \\ &\quad +\lambda\int_0^1 \Big((1+\phi_x)^2 f_{xx}(x+\phi) + \phi_{xx} f_x(x+\phi) - \phi f_{xxx}(x) \\ &\qquad -(1+2\phi_x)f_{xx}(x) - \phi_{xx} f_x(x)\Big)\,dz.\end{aligned}$$

Now, assuming that f is convex and satisfies $f_{xx}(x_0) = 0$ at some point x_0, f_{xx} has a local minimum at x_0. Thus $f_{xxx}(x_0) = 0$ and $f_{xxxx}(x_0) \geq 0$, so, at a point (x_0, t_0),

$$\begin{aligned}\partial_x^2(\mathcal{L}f) &= af_{xxxx} \\ &\quad +\lambda\int_0^1 \Big((1+\phi_x)^2 f_{xx}(x_0+\phi) + \phi_{xx} f_x(x_0+\phi) - \phi_{xx} f_x(x_0)\Big)\,dz \\ &\geq \lambda\int_0^1 \Big(\phi_{xx} f_x(x_0+\phi) - \phi_{xx} f_x(x_0)\Big)\,dz.\end{aligned}$$

It follows from the assumption (9) and the convexity of f that

$$\phi_{xx} f_x(x+\phi) - \phi_{xx} f_x(x) \geq 0$$

for all (x, t, z). Consequently $\partial_x^2(\mathcal{L}f) \geq 0$ at the point (x_0, t_0), so the model is LCP. □

By approximation we can relax the conditions of Theorem 4.4 as follows.

Theorem 4.5. *Assume the minimal conditions (M1)-(M5) and that*

(10) *ϕ is convex (concave) in x at all points where $\phi(x,t,z) > 0$ (< 0).*

Then the model is convexity preserving.

Proof. Let (β, ϕ) be a model satisfying (M1)-(M4) and (10), and let X and u be the corresponding stock and option prices, respectively. First choose a contract function g satisfying (A5), and let (β_n, ϕ_n) be a sequence of models satisfying (M1)-(M4) uniformly in n, i.e. (M2)-(M4) hold for all models with the same constants C and γ. Also assume that each model (β_n, ϕ_n) satisfies the conditions (A1)-(A4) (not necessarily uniformly in n) and (10) (or equivalently, (9)), and that for each $N > 0$ and $t \in [0, T]$ we have

$$\lim_{n\to\infty} \sup_{x\in(0,N]} |\beta_n(x,t) - \beta(x,t)| = 0,$$

$$\lim_{n\to\infty} \sup_{x\in(0,N]} \int_0^1 |\phi_n(x,t,z)-\phi(x,t,z)|^2\,dz = 0,$$

and

$$\lim_{n\to\infty} |\lambda_n(t)-\lambda(t)| = 0.$$

Then the option price u_n corresponding to the model (β_n, ϕ_n) is spatially convex by Theorem 4.3, and the stock price $X_n(T)$, corresponding to the model (β_n, ϕ_n), converges in L^2 to $X(T)$, compare Part II, §8, Theorem 3 in [7] (this theorem is stated for a constant jump-intensity, but it readily extends to our setting). This implies that $u_n(\cdot,t)$ converges pointwise to $u(\cdot,t)$. Since the pointwise limit of a sequence of convex functions is convex, we find that $u(\cdot,t)$ is convex.

Finally, for a convex contract function g not necessarily satisfying the Lipschitz condition (A5), but merely the weaker condition (M5) allowing polynomial growth, one may approximate g with a sequence of Lipschitz functions. It is straightforward to check that the corresponding prices converges to the correct limit, i.e. the result about preservation of convexity extends to contract functions g of possibly polynomial growth. □

5. Monotonicity in the model parameters

In this section we demonstrate how preservation of convexity can be used to derive monotonicity properties of the option value with respect to the different parameters of the model. To do this we consider two different models, i.e. two sets (β, ϕ, λ) and $(\tilde{\beta}, \tilde{\phi}, \tilde{\lambda})$ of parameters, and we denote by u, $\tilde{u}$ and $\mathcal{L}$, $\tilde{\mathcal{L}}$ the corresponding option values and integro-differential operators, respectively.

Theorem 5.1. *Assume that both models satisfy the minimal assumptions (M1)-(M4) stated in Section 2, and that the contract function g satisfies (M5). Also assume that $|\tilde{\beta}(x,t)| \le |\beta(x,t)|$ and $\tilde{\lambda}(t) \le \lambda(t)$ for all x and t, and that*

$$\frac{\phi(x,t,z)}{\tilde{\phi}(x,t,z)} \ge 1 \tag{11}$$

for all x,t,z with $\tilde{\phi}(x,t,z) \neq 0$. If either ϕ or $\tilde{\phi}$ satisfies the condition (10), and if the contract function g is convex, then

$$\tilde{u}(x,t) \le u(x,t)$$

for all x and t.

Remark Theorem 5.1 extends a result of Henderson and Hobson [8]. In Theorem 6.1 of that paper it is shown that, for any convex contract function g, the corresponding option price u is increasing in the intensity λ provided all parameters of the model are deterministic.

Also note that a consequence of Theorem 5.1 is that if the contract function g is convex, then the Black-Scholes price (corresponding to $\phi \equiv 0$) gives a lower bound for the set of possible arbitrage free option prices. This is also proven by Bellamy and Jeanblanc [1].

Proof. We assume the assumptions (A1)-(A4) to hold for both models and that g satisfies (A5); the general case then follows by an approximation argument similar to the one in the proof of Theorem 4.5. Under these assumptions it follows from Theorem 3.2 that there exist large positive numbers m and K such that

$$\max\{|u(x,t)|,|\tilde{u}(x,t)|\}\leq K(x^m+x^{-m}) \tag{12}$$

for all $(x,t)\in\mathbb{R}^+\times[0,T]$. Let $h=x^{m+1}+x^{-m-1}$, and choose the constant M large so that

$$Mh-\mathcal{L}h\geq 1 \tag{13}$$

for all x and t. Define

$$u^\epsilon(x,t):=u(x,t)+\epsilon e^{Mt}h(x),$$

and suppose that the set

$$E:=\{(x,t)\in\mathbb{R}^+\times[0,T]:u^\epsilon(x,t)<\tilde{u}(x,t)\}.$$

is non-empty. It follows from (12) that there exists $\rho>0$ with $E\subseteq(\rho^{-1},\rho)\times[0,T]$. Thus E is bounded, so $\overline{E}$ is compact. Hence there exists a point $(x_0,t_0)\in\overline{E}$ where

$$t_0=\inf\{t:(x,t)\in\overline{E}\text{ for some }x\in(0,\infty)\}.$$

By continuity, $u^\epsilon(x_0,t_0)=\tilde{u}(x_0,t_0)$, so $u^\epsilon(x,0)-\tilde{u}(x,0)\geq\epsilon h(x)>0$ implies that $t_0>0$. It is therefore clear that

$$\partial_t(u^\epsilon-\tilde{u})\leq 0 \tag{14}$$

at the point (x_0,t_0). On the other hand, at this point we also have

$$\begin{aligned}
\partial_t(u^\epsilon-\tilde{u}) &= \mathcal{L}u^\epsilon-\tilde{\mathcal{L}}\tilde{u}+\epsilon e^{Mt_0}(Mh-\mathcal{L}h)\\
&= \frac{\beta^2}{2}u^\epsilon_{xx}-\frac{\tilde{\beta}^2}{2}\tilde{u}_{xx}+\epsilon e^{Mt_0}(Mh-\mathcal{L}h)\\
&\quad+\int_0^1\Big(\lambda(t_0)\big(u^\epsilon(x_0+\phi,t_0)-u^\epsilon(x_0,t_0)-\phi u^\epsilon_x(x_0,t_0)\big)\\
&\qquad-\tilde{\lambda}(t_0)\big(\tilde{u}(x_0+\tilde{\phi},t_0)-\tilde{u}(x_0,t_0)-\tilde{\phi}\tilde{u}_x(x_0,t_0)\big)\Big)\,dz\\
&> (\frac{\beta^2}{2}-\frac{\tilde{\beta}^2}{2})u^\epsilon_{xx}+\frac{\tilde{\beta}^2}{2}(u^\epsilon_{xx}-\tilde{u}_{xx})\\
&\quad+\int_0^1\Big(\lambda(t_0)\big(u^\epsilon(x_0+\phi,t_0)-u^\epsilon(x_0,t_0)-\phi u^\epsilon_x(x_0,t_0)\big)\\
&\qquad-\tilde{\lambda}(t_0)\big(\tilde{u}(x_0+\tilde{\phi},t_0)-\tilde{u}(x_0,t_0)-\tilde{\phi}\tilde{u}_x(x_0,t_0)\big)\Big)\,dz,
\end{aligned} \tag{15}$$

where we have used the inequality (13). Assume first that ϕ satisfies (10). From Theorem 4.3 it follows that u is spatially convex, and therefore also u_ϵ is spatially convex. Using $u^\epsilon=\tilde{u}$ and $u^\epsilon_x=\tilde{u}_x$ at (x_0,t_0) and the condition (11) we find that

$$u^\epsilon(x_0+\phi,t_0)-u^\epsilon(x_0,t_0)-\phi u^\epsilon_x(x_0,t_0)\geq\tilde{u}(x_0+\tilde{\phi},t_0)-\tilde{u}(x_0,t_0)-\tilde{\phi}\tilde{u}_x(x_0,t_0).$$

Since the expression on the left hand side of this inequality due to convexity is non-negative, we also have

$$\lambda(t_0)\Big(u^\epsilon(x_0+\phi,t_0)-u^\epsilon(x_0,t_0)-\phi u^\epsilon_x(x_0,t_0)\Big)\geq \tilde\lambda(t_0)\Big(\tilde u(x_0+\tilde\phi,t_0)-\tilde u(x_0,t_0)-\tilde\phi\tilde u_x(x_0,t_0)\Big),$$

so it follows from $\beta^2\geq\tilde\beta^2$ and $u^\epsilon_{xx}(x_0,t_0)\geq\tilde u_{xx}(x_0,t_0)$ that

$$\partial_t(u^\epsilon-\tilde u)>0.$$

This contradicts (14). Thus the set E is empty, and so $\tilde u(x,t)\leq u^\epsilon(x,t)$ for all x and t. Now the desired monotonicity result follows by letting $\epsilon\to 0$.

If instead $\tilde\phi$ satisfies (10), then $\tilde u$ is spatially convex, so we can argue similarly as above if the expression in (15) is replaced by

$$\frac{\beta^2}{2}(u^\epsilon_{xx}-\tilde u_{xx})+(\frac{\beta^2}{2}-\frac{\tilde\beta^2}{2})\tilde u_{xx}.$$

□

Remark Also for options of American type, the results corresponding to Theorems 4.5 and 5.1 hold. This can be seen by approximating the American option price with a sequence of Bermudan option prices, all of which are spatially convex and increasing in the model parameters. The details work as in [4], where models without jumps are treated.

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