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GEOMETRIC STRUCTURE FOR THE PRINCIPAL SERIES OF A SPLIT REDUCTIVE p-ADIC GROUP WITH CONNECTED CENTRE

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ABSTRACT. Let $\mathcal{G}$ be a split reductive p-adic group with connected centre. We show that each Bernstein block in the principal series of $\mathcal{G}$ admits a definite geometric structure, namely that of an extended quotient. For the Iwahori-spherical block, this extended quotient has the form $T//W$ where T is a maximal torus in the Langlands dual group of $\mathcal{G}$ and W is the Weyl group of $\mathcal{G}$.

CONTENTS

1. INTRODUCTION

Let $\mathcal{G}$ be a split reductive p-adic group with connected centre, and let $G = \mathcal{G}^\vee$ denote the Langlands dual group. Then G is a complex reductive group. Let T be a maximal torus in G and let W be the common Weyl group of $\mathcal{G}$ and G. We can form the quotient variety

$$T/W$$

and, familiar from noncommutative geometry [Kha, p.77], the *noncommutative quotient algebra*

$$\mathcal{O}(T) \rtimes W.$$

Within periodic cyclic homology (a noncommutative version of de Rham theory) there is a canonical isomorphism

$$\mathrm{HP}_*(\mathcal{O}(T) \rtimes W) \simeq \mathrm{H}^*(T//W; \mathbb{C})$$

where

$$T//W$$

denotes the *extended quotient* of T by W, see §2. In this sense, the extended quotient $T//W$, a complex algebraic variety, is a more concrete version of the noncommutative quotient algebra $\mathcal{O}(T) \rtimes W$.

Returning to the p-adic group $\mathcal{G}$, let $\mathbf{Irr}(\mathcal{G})^{\mathrm{i}}$ denote the subset of the smooth dual $\mathbf{Irr}(\mathcal{G})$ comprising all the irreducible smooth Iwahori-spherical representations of $\mathcal{G}$. We prove in this article that there is a continuous bijective map, satisfying several constraints, as follows:

$$T//W \simeq \mathbf{Irr}(\mathcal{G})^{\mathrm{i}}.$$

We note that there is nothing in the classical representation theory of $\mathcal{G}$ to indicate that $\mathbf{Irr}(\mathcal{G})^{\mathrm{i}}$ admits such a geometric structure. Nevertheless, such a structure was conjectured by the present authors in [ABPS], and so this article is a confirmation of that conjecture, for the single point i in the Bernstein spectrum of $\mathcal{G}$. We prove, more generally, that, subject to constraints itemized in [ABPS], and subject to the Condition 3.1 on the residual characteristic, there is a continuous bijective map

$$T^{\mathfrak{s}}//W^{\mathfrak{s}} \simeq \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

for each point $\mathfrak{s}$ in the Bernstein spectrum for the principal series of $\mathcal{G}$, see Theorem 5.2. Here, $T^{\mathfrak{s}}$ and $W^{\mathfrak{s}}$ are the complex torus and the finite group attached to $\mathfrak{s}$. This, too, is a confirmation of the geometric conjecture in [ABPS].

Let $\mathbf{W}_F$ denote the Weil group of F. A Langlands parameter Φ for the principal series of $\mathcal{G}$ should have $\Phi(\mathbf{W}_F)$ contained in a maximal torus of G. In particular, it should suffice to consider parameters Φ such that $\Phi\big|_{\mathbf{W}_F}$ factors through $\mathbf{W}_F^{\mathrm{ab}} \cong F^\times$, that is, such that Φ factors as follows:

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

Such a parameter is *enhanced* in the following way. Let ρ be an irreducible representation of the component group of the centralizer of the image of Φ:

$$\rho \in \mathbf{Irr}\,\pi_0(Z_G(\mathrm{im}\ \Phi)).$$

The pair (Φ, ρ) will be called an *enhanced Langlands parameter*.

We rely on Reeder's classification of the constituents of a given principal series representation of $\mathcal{G}$, see [Ree, Theorem 1, p.101-102]. Reeder's theorem amounts to a local Langlands correspondence for the principal series of $\mathcal{G}$. Reeder uses only enhanced Langlands parameters with a particular geometric origin, namely those which occur in the homology of a certain variety of Borel subgroups of G. This condition is essential, see, for example, the discussion, in [ABP2], of the Iwahori-spherical representations of the exceptional group G_2.

In Theorem 4.7 we show how to replace the enhanced Langlands parameters of this kind, namely those of geometric origin, by the *affine Springer*

parameters defined in §4.3. These affine Springer parameters are defined in terms of data attached to the complex reductive group G – in this sense, the affine Springer parameters are independent of the cardinality q of the residue field of F. The scene is now set for us to prove the first theorem of geometric structure, namely Theorem 5.1, from which our main structure theorem, Theorem 5.2 follows.

We also relate our basic structure theorem with L-packets in the principal series of $\mathcal{G}$, see Theorem 6.2.

An earlier, less precise version of our conjecture was formulated in [ABP1]. That version was proven in [Sol1] for Bernstein components which are described nicely by affine Hecke algebras. These include the principal series of split groups (with possibly disconnected centre), symplectic and orthogonal groups and also inner forms of GL_n.

2. EXTENDED QUOTIENT

Let Γ be a finite group acting on a complex affine variety X by automorphisms,

$$\Gamma \times X \to X.$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Let $c(\Gamma_x)$ denote the set of conjugacy classes of Γ_x. The extended quotient is obtained from X/Γ by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}$. It is an affine variety and a subvariety of $\Gamma \times X$. The group Γ acts on $\widetilde{X}$:

$$\Gamma \times \widetilde{X} \to \widetilde{X}$$

$$\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}.$$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X/\!/\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$. The projection

$$\widetilde{X} \to X, \ (\gamma, x) \mapsto x$$

is Γ-equivariant and so passes to quotient spaces to give a morphism of affine varieties

$$\rho \colon X/\!/\Gamma \to X/\Gamma.$$

This map will be referred to as the projection of the extended quotient onto the ordinary quotient. The inclusion

$$X \hookrightarrow \widetilde{X}$$

$$x \mapsto (e, x) \qquad e = \text{identity element of } \Gamma$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion of affine varieties $X/\Gamma \hookrightarrow X//\Gamma$.

This article will be dominated by extended quotients of the form $T//W$ or, more generally, extended quotients of the form $T^{\mathfrak{s}}//W^{\mathfrak{s}}$.

3. The group $W^{\mathfrak{s}}$ as a Weyl group

Let $\mathcal{G}$ be a connected reductive p-adic group over F, which is F-split and has connected centre. Let $\mathcal{T}$ be a F-split maximal torus in $\mathcal{G}$. Let G, T denote the Langlands dual groups of $\mathcal{G}$, $\mathcal{T}$. The principal series consists of all $\mathcal{G}$-representations that are obtained with parabolic induction from characters of $\mathcal{T}$. We will suppose that the residual characteristic p of F satisfies the hypothesis in [Roc, p. 379], for all reductive subgroups $H \subset G$ containing T:

Condition 3.1. *If the root system $R(H,T)$ is irreducible, then the restriction on the residual characteristic p of F is as follows:*

- *for type A_n $p > n+1$*
- *for types B_n, C_n, D_n $p \neq 2$*
- *for type F_4 $p \neq 2, 3$*
- *for types G_2, E_6 $p \neq 2, 3, 5$*
- *for types E_7, E_8 $p \neq 2, 3, 5, 7$.*

If $R(H,T)$ is reducible, one excludes primes attached to each of its irreducible factors.

Since $R(H,T)$ is a subset of $R(G,T) \cong R(\mathcal{G},\mathcal{T})^{\vee}$, these conditions are fulfilled when they hold for $R(\mathcal{G},\mathcal{T})$.

We denote the collection of all Bernstein components of $\mathcal{G}$ of the form $\mathfrak{s} = [\mathcal{T},\chi]_{\mathcal{G}}$ by $\mathfrak{B}(\mathcal{G},\mathcal{T})$ and call these the Bernstein components in the principal series. The union

$$\mathbf{Irr}(\mathcal{G},\mathcal{T}) := \bigcup_{\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

is by definition the set of all irreducible subquotients of principal series representations of $\mathcal{G}$.

Choose a uniformizer $\varpi_F \in F$. There is a bijection $t \mapsto \nu$ between points in T and unramified characters of $\mathcal{T}$, determined by the relation

$$\nu(\lambda(\varpi_F)) = \lambda(t)$$

where $\lambda \in X_*(\mathcal{T}) = X^*(T)$. The space $\mathbf{Irr}(\mathcal{T})^{[\mathcal{T},\chi]_{\mathcal{T}}}$ is in bijection with T via $t \mapsto \nu \mapsto \chi \otimes \nu$. Hence Bernstein's torus $T^{\mathfrak{s}}$ is isomorphic to T. However, because the isomorphism is not canonical and the action of the group $W^{\mathfrak{s}}$ depends on it, we prefer to denote it $T^{\mathfrak{s}}$.

The uniformizer ϖ_F gives rise to a group isomorphism $\mathfrak{o}_F^{\times} \times \mathbb{Z} \to F^{\times}$, which sends $1 \in \mathbb{Z}$ to ϖ_F. Let $\mathcal{T}_0$ denote the maximal compact subgroup of $\mathcal{T}$. As the latter is F-split,

$$(1) \qquad \mathcal{T} \cong F^{\times} \otimes_{\mathbb{Z}} X_*(\mathcal{T}) \cong (\mathfrak{o}_F^{\times} \times \mathbb{Z}) \otimes_{\mathbb{Z}} X_*(\mathcal{T}) = \mathcal{T}_0 \times X_*(\mathcal{T}).$$

Because $\mathcal{W}^G = W(\mathcal{G},\mathcal{T})$ does not act on $F^{\times}$, these isomorphisms are $\mathcal{W}^G$-equivariant if we endow the right hand side with the diagonal $\mathcal{W}^G$-action.

Thus (1) determines a $\mathcal{W}^G$-equivariant isomorphism of character groups

(2) $$\mathbf{Irr}(\mathcal{T}) \cong \mathbf{Irr}(\mathcal{T}_0) \times \mathbf{Irr}(X_*(\mathcal{T})) = \mathbf{Irr}(\mathcal{T}_0) \times X_{\mathrm{unr}}(\mathcal{T}).$$

Lemma 3.2. *Let χ be a character of $\mathcal{T}$, and let*

(3) $$\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$$

be the inertial class of the pair $(\mathcal{T}, \chi)$. Then $\mathfrak{s}$ determines, and is determined by, the $\mathcal{W}^G$-orbit of a smooth morphism

$$c^{\mathfrak{s}} \colon \mathfrak{o}_F^{\times} \to T.$$

Proof. There is a natural isomorphism

$$\mathbf{Irr}(\mathcal{T}) = \mathrm{Hom}(F^{\times} \otimes_{\mathbb{Z}} X_*(\mathcal{T}), \mathbb{C}^{\times}) \cong \mathrm{Hom}(F^{\times}, \mathbb{C}^{\times} \otimes_{\mathbb{Z}} X^*(\mathcal{T})) = \mathrm{Hom}(F^{\times}, T).$$

Together with (2) we obtain isomorphisms

$$\mathbf{Irr}(\mathcal{T}_0) \cong \mathrm{Hom}(\mathfrak{o}_F^{\times}, T),$$
$$X_{\mathrm{unr}}(\mathcal{T}) \cong \mathrm{Hom}(\mathbb{Z}, T) = T.$$

Let $\hat{\chi} \in \mathrm{Hom}(F^{\times}, T)$ be the image of χ under these isomorphisms. By the above the restriction of $\hat{\chi}$ to $\mathfrak{o}_F^{\times}$ is not disturbed by unramified twists, so we take that as $c^{\mathfrak{s}}$. Conversely, by (2) $c^{\mathfrak{s}}$ determines χ up to unramified twists. Two elements of $\mathbf{Irr}(\mathcal{T})$ are $\mathcal{G}$-conjugate if and only if they are $\mathcal{W}^G$-conjugate so, in view of (3), the $\mathcal{W}^G$-orbit of the $c^{\mathfrak{s}}$ contains the same amount of information as $\mathfrak{s}$. $\qquad\square$

We define

(4) $$H := Z_G(\mathrm{im}\ c^{\mathfrak{s}}).$$

The following crucial result is due to Roche, see [Roc, p. 394 – 395].

Lemma 3.3. *The group $H^{\mathfrak{s}}$ is connected, and the finite group $W^{\mathfrak{s}}$ is the Weyl group of $H^{\mathfrak{s}}$:*

$$W^{\mathfrak{s}} = \mathcal{W}^{H^{\mathfrak{s}}}$$

4. COMPARISON OF DIFFERENT PARAMETERS

4.1. **Varieties of Borel subgroups.** We clarify some issues with different varieties of Borel subgroups and different kinds of parameters arising from them.

Let $\mathbf{W}_F$ denote the Weil group of F, let $\mathbf{I}_F$ be the inertia subgroup of $\mathbf{W}_F$. Let $\mathbf{W}_F^{\mathrm{der}}$ denote the closure of the commutator subgroup of $\mathbf{W}_F$, and write $\mathbf{W}_F^{\mathrm{ab}} = \mathbf{W}_F/\mathbf{W}_F^{\mathrm{der}}$. The group of units in $\mathfrak{o}_F$ will be denoted $\mathfrak{o}_F^{\times}$.

Next, we consider conjugacy classes in G of continuous morphisms

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

which are rational on $\mathrm{SL}_2(\mathbb{C})$ and such that $\Phi(\mathbf{W}_F)$ consists of semisimple elements in G.

Let B_2 be the upper triangular Borel subgroup in $\mathrm{SL}_2(\mathbb{C})$. Let $\mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}$ denote the variety of Borel subgroups of G containing $\Phi(\mathbf{W}_F \times B_2)$. The variety $\mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}$ is non-empty if and only if Φ factors through $\mathbf{W}_F^{\mathrm{ab}}$, see [Ree, §4.2]. In that case, we view the domain of Φ to be $F^{\times} \times \mathrm{SL}_2(\mathbb{C})$:

$$\Phi \colon F^{\times} \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

In Section 4.2 we show how such a Langlands parameter Φ can be enhanced with a parameter ρ.

We start with the following data: a point $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ and an L-parameter

$$\Phi \colon F^{\times} \times \mathrm{SL}_2(\mathbb{C}) \to G$$

for which

$$\Phi|\mathfrak{o}_F^{\times} = c^{\mathfrak{s}}.$$

This data creates the following items:

$$
\begin{aligned}
(5) \qquad & t := \Phi(\varpi_F, I), \\
& x := \Phi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right), \\
& M := \mathrm{Z}_H(t).
\end{aligned}
$$

We note that $\Phi(\mathfrak{o}_F^{\times}) \subset \mathrm{Z}(H)$ and that t commutes with $\Phi(\mathrm{SL}_2(\mathbb{C})) \subset M$.

For $\alpha \in \mathbb{C}^{\times}$ we define the following matrix in $\mathrm{SL}_2(\mathbb{C})$:

$$Y_\alpha = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right).$$

For any $q^{1/2} \in \mathbb{C}^{\times}$ the element

$$(6) \qquad t_q := t\Phi\left(Y_{q^{1/2}}\right)$$

satisfies the familiar relation $t_q x t_q^{-1} = x^q$. Indeed

$$
\begin{aligned}
(7) \qquad t_q x t_q^{-1} &= t\Phi(Y_{q^{1/2}})\Phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\Phi(Y_{q^{1/2}}^{-1})t^{-1} \\
&= t\Phi\left(Y_{q^{1/2}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)Y_{q^{1/2}}^{-1}\right)t^{-1} \\
&= t\Phi\left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)t^{-1} = x^q.
\end{aligned}
$$

Notice that $\Phi(\mathfrak{o}_F^{\times})$ lies in every Borel subgroup of H, because it is contained in $\mathrm{Z}(H)$. We abbreviate $\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(\mathrm{im}\,\Phi)$ and similarly for other groups.

Lemma 4.1. *The inclusion map $\mathrm{Z}_H(\Phi) \to \mathrm{Z}_H(t, x)$ is a homotopy equivalence.*

Proof. Our proof depends on [ChGi, Prop. 3.7.23]. There is a Levi decomposition

$$\mathrm{Z}_H(x) = \mathrm{Z}_H(\Phi(\mathrm{SL}_2(\mathbb{C})))U_x$$

where $\mathrm{Z}_H(\Phi(\mathrm{SL}_2(\mathbb{C}))$ a maximal reductive subgroup of $\mathrm{Z}_H(x)$ and U_x is the unipotent radical of $\mathrm{Z}_H(x)$. Therefore

$$(8) \qquad \mathrm{Z}_H(t, x) = \mathrm{Z}_H(\Phi)\mathrm{Z}_{U_x}(t)$$

We note that $\mathrm{Z}_{U_x}(t) \subset U_x$ is contractible, because it is a unipotent complex group. It follows that

$$(9) \qquad \mathrm{Z}_H(\Phi) \to \mathrm{Z}_H(t, x)$$

is a homotopy equivalence. $\qquad\qquad\square$

If a group A acts on a variety X, let $\mathcal{R}(A, X)$ denote the set of irreducible representations of A appearing in the homology $H_*(X)$.

The variety of Borel subgroups of G which contain $\Phi(\mathbf{W}_F \times B_2)$ will be denoted $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ and the variety of Borel subgroups of H containing $\{t, x\}$ will be denoted $\mathcal{B}_H^{t,x}$.

Lemma 4.1 allows us to define

$$A := \pi_0(Z_H(\Phi)) = \pi_0(Z_H(t, x)).$$

Theorem 4.2. *We have*

$$\mathcal{R}(A, \mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}) = \mathcal{R}(A, \mathcal{B}_H^{t,x}).$$

Proof. This statement is equivalent to [Ree, Lemma 4.4.1] with a minor adjustment in his proof. To translate into Reeder's paper, write

$$t_q = \tau, Y_q = \tau_u, x = u, t = s.$$

The adjustment consists in the observation that the Borel subgroup B of H contains $\{x, t_q, Y_q\}$ if and only if B contains $\{x, t, Y_q\}$. This is because $t = t_q Y_q^{-1}$. Therefore, in the conclusion of his proof, $\mathcal{B}_H^{\tau,u}$, which is $\mathcal{B}_H^{t_q,x}$, can be replaced by $\mathcal{B}_H^{t,x}$. $\square$

In the following sections we will make use of two different but related kinds of parameters.

4.2. Enhanced Langlands parameters.
Let $\mathbf{W}_F$ denote the Weil group of F. Via the Artin reciprocity map a Langlands parameter Φ for the principal series of $\mathcal{G}$ will factor through $F^\times \times \mathrm{SL}_2(\mathbb{C})$:

$$(10) \qquad \Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

Such a parameter is *enhanced* in the following way. Let ρ be an irreducible representation of the component group of the centralizer of the image of Φ:

$$\rho \in \mathbf{Irr}\, \pi_0(Z_G(\mathrm{im}\ \Phi)).$$

The pair (Φ, ρ) will be called an *enhanced Langlands parameter.*

We rely on Reeder's classification of the constituents of a given principal series representation of $\mathcal{G}$, see [Ree, Theorem 1, p.101-102]. Reeder's theorem amounts to a local Langlands correspondence for the principal series of $\mathcal{G}$. Reeder uses only enhanced Langlands parameters with a particular geometric origin, namely those which occur in the homology of a certain variety of Borel subgroups of G.

Let B_2 denote the standard Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$. For a Langlands parameter as in (10), the variety of Borel subgroups $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is nonempty, and the centralizer $Z_G(\Phi)$ of the image of Φ acts on it. Hence the group of components $\pi_0(Z_G(\Phi))$ acts on the homology $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C})$. We call an irreducible representation ρ of $\pi_0(Z_G(\Phi))$ *geometric* if

$$\rho \in \mathcal{R}\left(\pi_0(Z_G(\Phi)), \mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}\right).$$

Consider the set of enhanced Langlands parameters (Φ, ρ) for which ρ is geometric. The group G acts on these parameters by

$$(11) \qquad g \cdot (\Phi, \rho) = (g\Phi g^{-1}, \rho \circ \mathrm{Ad}_g^{-1})$$

and we denote the corresponding equivalence class by $[\Phi, \rho]_G$.

Definition 4.3. *Let* $\Psi(G)_{\mathrm{en}}^\mathfrak{s}$ *denote the set of H-conjugacy classes of enhanced parameters (Φ, ρ) for $\mathcal{G}$ such that*

- *ρ is geometric;*

- $\Phi|\mathfrak{o}^{\times} = c^{\mathfrak{s}}$.

Let us define a topology on $\Psi(G)_{\mathrm{en}}^{\mathfrak{s}}$. For any $(\Phi, \rho) \in \Psi(G)_{\mathrm{en}}^{\mathfrak{s}}$ the element $x = \Phi(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) \in H$ is unipotent and $t = \Phi(\varpi_F, I) \in H$ is semisimple. By the Jacobson–Morozov Theorem the H-conjugacy class of Φ is determined completely by the H-conjugacy class of (t, x), see [Ree, §4.2]. We endow the finite set $\mathfrak{U}^{\mathfrak{s}}$ of unipotent conjugacy classes in H with the discrete topology and we regard the space of semisimple conjugacy classes in H as the algebraic variety $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. On $T^{\mathfrak{s}}/W^{\mathfrak{s}} \times \mathfrak{U}^{\mathfrak{s}}$ we take the product topology and we endow $\Psi(G)_{\mathrm{en}}^{\mathfrak{s}}$ with the pullback topology from $T^{\mathfrak{s}}/W^{\mathfrak{s}} \times \mathfrak{U}^{\mathfrak{s}}$, with respect to the map $(\Phi, \rho) \mapsto (t, x)$.

Notice that for this topology the ρ does not play a role, two elements of $\Psi(G)_{\mathrm{en}}^{\mathfrak{s}}$ with the same Φ are inseparable.

Theorem 4.4. [Ree] *Suppose that the residual characteristic of F satisfies Condition 3.1.*

(1) *There is a canonical continuous bijection*

$$\Psi(G)_{\mathrm{en}}^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

(2) *This bijection maps the set of enhanced Langlands parameters (Φ, ρ) for which $\Phi(F^{\times})$ is bounded onto $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cap \mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$.*

(3) *If $\sigma \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ corresponds to (Φ, ρ), then the cuspidal support $\pi^{\mathfrak{s}}(\sigma) \in T^{\mathfrak{s}}/W^{\mathfrak{s}}$, considered as a semisimple conjugacy class in $H^{\mathfrak{s}}$, equals $\Phi(\varpi_F, Y_{q^{1/2}})$.*

Proof. (1) The canonical bijection is Reeder's classification of the constituents of a given principal series representation, see [Ree, Theorem 1, p.101 – 102].

First he associates to Φ a finite length "standard" representation of $\mathcal{G}$, say $M_{t,x}$, with a unique maximal semisimple quotient $V_{t,x}$. Then (Φ, ρ) is mapped to an irreducible constituent of $V_{t,x}$. To check that the bijection is continuous with respect to the above topology, it suffices to see that $M_{t,x}$ depends continuously on t when x is fixed. This property is clear from [Ree, §3.5].

(2) Reeder's work is based on that of Kazhdan–Lusztig, and it is known from [KaLu, §8] that the tempered $\mathcal{G}$-representations correspond precisely to the set of bounded enhanced L-parameters in the setting of [KaLu]. As the constructions in [Ree] preserve temperedness, this characterization remains valid in Reeder's setting.

(3) The element $\Phi(\varpi_F, Y_{q^{1/2}}) \in H$ is the same as t_q in (6), up to H-conjugacy. In the setting of Kazhdan–Lusztig, it is known from [KaLu, 5.12 and Theorem 7.12] that property (3) holds. As in (2), this is respected by the constructions of Reeder that lead to (1).   □

4.3. **Affine Springer parameters.** As before, suppose that $t \in H$ is semisimple and that $x \in Z_H(t)$ is unipotent. Then $Z_H(t, x)$ acts on $\mathcal{B}_H^{t,x}$ and $\pi_0(Z_H(t, x))$ acts on the homology of this variety. In this setting we say that $\rho_1 \in \mathbf{Irr}(\pi_0(Z_H(t, x)))$ is *geometric* if it belongs to $\mathcal{R}\left(\pi_0(Z_H(t, x)), \mathcal{B}_H^{t,x}\right)$.

For the affine Springer parameters it does not matter whether we consider the total homology or only the homology in top degree. Indeed, it follows from [Sho, bottom of page 296 and Remark 6.5] that any irreducible

representation ρ_1 which appears in $H_*\big(\mathcal{B}_H^{t,x},\mathbb{C}\big)$, already appears in the top homology of this variety. Therefore, we may refine Theorem 4.2 as follows:

Theorem 4.5.

$$\mathcal{R}(A,\mathcal{B}^{\Phi(\mathbf{W}_F\times B_2)}) = \mathcal{R}^{\text{top}}(A,\mathcal{B}_H^{t,x}),$$

where top *refers to highest degree in which the homology is nonzero, the real dimension of $\mathcal{B}_H^{t,x}$.*

We call such triples (t,x,ρ_1) affine Springer parameters for H, because they appear naturally in the representation theory of the affine Weyl group associated to H. The group H acts on such parameters by conjugation, and we denote the conjugacy classes by $[t,x,\rho_1]_H$.

Definition 4.6. *The set of H-conjugacy classes of affine Springer parameters will be denoted $\Psi(H)_{\text{aff}}$.*

Notice that the projection on the first coordinate is a canonical map $\Psi(H)_{\text{aff}} \to T/\mathcal{W}^H$. We endow $\Psi(H)_{\text{aff}}$ with a topology in the same way as we did for $\Psi(G)_{\text{en}}^{\mathfrak{s}}$, as the pullback of the product topology on $T/\mathcal{W}^H \times \mathfrak{U}^{\mathfrak{s}}$ via the map $[t,x,\rho_1]_H \mapsto (t,x)$.

For use in Theorem 5.1 we recall the parametrization of irreducible representations of $X^*(T) \rtimes \mathcal{W}^H$ from [Kat]. Let $t \in T$ and let $x \in M^\circ = \mathrm{Z}_H(t)^\circ$ be unipotent. Kato defines an action of $X^*(T) \rtimes \mathcal{W}^H$ on the top homology $H_{d(x)}(\mathcal{B}_H^{t,x},\mathbb{C})$, which commutes with the action of $\mathrm{Z}_H(t,x)$ induced by conjugation of Borel subgroups. By [Kat, Proposition 6.2] there is an isomorphism of $X^*(T) \rtimes \mathcal{W}^H$-representations

$$(12) \qquad H_{d(x)}(\mathcal{B}_H^{t,x},\mathbb{C}) \cong \mathrm{ind}_{X^*(T) \rtimes \mathcal{W}^{M\circ}}^{X^*(T) \rtimes \mathcal{W}^H}\big(\mathbb{C}_t \otimes H_{d(x)}(\mathcal{B}_{M\circ}^x,\mathbb{C})\big).$$

Here $H_{d(x)}(\mathcal{B}_{M\circ}^x,\mathbb{C})$ is a representation occurring in the Springer correspondence for $\mathcal{W}^{M_0}$, promoted to a representation of $X^*(T) \rtimes \mathcal{W}^H$ by letting $X^*(T)$ act trivially. Hence (12) has central character $\mathcal{W}^H t$. We note that the underlying vector space of this representation does not depend on t, and that this determines an algebraic family of $X^*(T) \rtimes \mathcal{W}^H$-representations parametrized by $T^{\mathcal{W}^{M_0}}$. Let $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_H(t,x))\big)$. By [Kat, Theorem 4.1] the $X^*(T) \rtimes \mathcal{W}^H$-representation

$$(13) \qquad \mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho_1, H_{d(x)}(\mathcal{B}_H^{t,x},\mathbb{C})\big)$$

is either irreducible or zero. Moreover every irreducible representation of $X^*(T) \rtimes \mathcal{W}^H$ is obtained in this way, and the data (t,x,ρ_1) are unique up to H-conjugacy. So Kato's results provide a natural bijection

$$(14) \qquad \Psi(H)_{\text{aff}} \to \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H).$$

This generalizes the Springer correspondence for finite Weyl groups, which can be recovered by considering the representations on which $X^*(T)$ acts trivially.

In [KaLu, Ree] there are some indications that the above kinds of parameters are essentially equivalent. The next result allows us to make this precise in the necessary generality.

Theorem 4.7. *Let $\mathfrak{s}$ be a Bernstein component in the principal series, associate $c^{\mathfrak{s}} \colon \mathfrak{o}_F^{\times} \to T$ to it as in Lemma 3.2 and let H be as in (5). There are natural bijections between H-equivalence classes of:*

- *enhanced Langlands parameters (Φ, ρ) for $\mathcal{G}$, with ρ geometric and $\Phi\big|_{\mathfrak{o}_F^{\times}} = c^{\mathfrak{s}}$;*
- *affine Springer parameters for H.*

In other words we have a homeomorphism

$$\Psi(G)_{\mathrm{en}}^{\mathfrak{s}} \simeq \Psi(H)_{\mathrm{aff}}.$$

Proof. An L-parameter gives rise to the ingredients t, x in an affine Springer parameter in the following way. For an L-parameter

$$\Phi \colon F^{\times} \times \mathrm{SL}_2(\mathbb{C}) \to G$$

we set $t = \Phi(\varpi_F, 1)$ and $x = \Phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$.

Conversely, we work with the Jacobson–Morozov Theorem [ChGi, p. 183]. Let x be a unipotent element in M^0. There exist rational homomorphisms

$$(15) \qquad \gamma \colon \mathrm{SL}_2(\mathbb{C}) \to M^0 \quad \text{with} \quad \gamma\big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) = x,$$

see [ChGi, §3.7.4]. Any two such homomorphisms γ are conjugate by elements of $Z_{M^{\circ}}(x)$. Define the Langlands parameter Φ as follows:

$$(16) \qquad \Phi \colon F^{\times} \times \mathrm{SL}_2(\mathbb{C}) \to G, \qquad (u\varpi_F^n, Y) \mapsto c^{\mathfrak{s}}(u) \cdot t^n \cdot \gamma(Y)$$

for all $u \in \mathfrak{o}_F^{\times}$, $n \in \mathbb{Z}$, $Y \in \mathrm{SL}_2(\mathbb{C})$.

Note that the definition of Φ uses the appropriate data: the semisimple element $t \in T$, the map $c^{\mathfrak{s}}$, and the homomorphism γ (which depends on x).

Since x determines γ up to M°-conjugation, $c^{\mathfrak{s}}, x$ and t determine Φ up to conjugation by their common centralizer in G. Notice also that one can recover $c^{\mathfrak{s}}, x$ and t from Φ and that

$$(17) \qquad h(\alpha) := \Phi(1, Y_{\alpha})$$

defines a cocharacter $\mathbb{C}^{\times} \to T$.

To complete Φ or (t, x) to a parameter of the appropriate kind, we must add an irreducible representation ρ or ρ_1. Then the bijectivity follows from Theorem 4.5.

It is clear that the above correspondence between Φ and (t, x) is continuous in both directions. In view of the chosen topologies on $\Psi(G)_{\mathrm{en}}^{\mathfrak{s}}$ and $\Psi(H^{\mathfrak{s}})_{\mathrm{aff}}$, this implies that the bijection is a homeomorphism. $\qquad \square$

5. Structure theorem

Let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G}, \mathcal{T})$ and construct $c^{\mathfrak{s}}$ as in Lemma 3.2. We note that the set of enhanced Langlands parameters $\Phi(G)_{\mathrm{en}}^{\mathfrak{s}}$ is naturally labelled by the unipotent classes in H:

$$(18) \qquad \Phi(G)_{\mathrm{en}}^{\mathfrak{s},[x]} := \big\{ (\Phi, \rho) \in \Phi(G)_{\mathrm{en}}^{\mathfrak{s}} \mid \Phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) \text{ is conjugate to } x \big\}.$$

Via Theorem 4.7 and (14) the sets $\Phi(G)_{\mathrm{en}}^{\mathfrak{s}}$ and $\mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H)$ are naturally in bijection with $\Psi(H)_{\mathrm{aff}}$. In this way we can associate to any of these

parameters a unique unipotent class in H:

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} = \bigcup_{[x]} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s},[x]},$$

(19) $$\Psi(H)_{\mathrm{aff}} = \bigcup_{[x]} \Psi(H)_{\mathrm{aff}}^{[x]},$$

$$\mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) = \bigcup_{[x]} \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H)^{[x]}.$$

As $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} = \mathbf{Irr}(\mathcal{H}^{\mathfrak{s}})$ and $\mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) = \mathbf{Irr}(\mathbb{C}[X^*(T) \rtimes \mathcal{W}^H])$, these spaces are endowed with the Jacobson topology from the respective algebras $\mathcal{H}^{\mathfrak{s}}$ and $\mathbb{C}[X^*(T) \rtimes \mathcal{W}^H]$.

Recall from Section 2 that

$$\widetilde{T^{\mathfrak{s}}} = \{(w,t) \in W^{\mathfrak{s}} \times T^{\mathfrak{s}} \mid wt = t\}$$

and $T^{\mathfrak{s}}//W^{\mathfrak{s}} = \widetilde{T^{\mathfrak{s}}}/W^{\mathfrak{s}}$. We endow $\widetilde{T^{\mathfrak{s}}}$ with the product of the Zariski topology on $T^{\mathfrak{s}} \cong T$ and the discrete topology on $W^{\mathfrak{s}} = \mathcal{W}^H$. Then $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ with the quotient topology from $\widetilde{T^{\mathfrak{s}}}$ becomes a disjoint union of algebraic varieties. The following result enables us to transfer the labellings (19) to $T^{\mathfrak{s}}//W^{\mathfrak{s}}$.

Theorem 5.1. *There exists a bijection $\tilde{\mu}^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}})$ such that:*

(1) *$\tilde{\mu}^{\mathfrak{s}}$ respects the projections to $T^{\mathfrak{s}}/W^{\mathfrak{s}}$;*
(2) *for every unipotent class x of H, the inverse image $(\tilde{\mu}^{\mathfrak{s}})^{-1}\mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}})^{[x]}$ is a union of connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$.*

Proof. Consider the ring $R(X^*(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}})$ of virtual finite dimensional complex representations $X^*(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$. Its canonical $\mathbb{Z}$-basis is

$$\mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}}) = \{\tau(t, x, \rho_1) : (t, x, \rho_1) \in \Psi(H)_{\mathrm{aff}}\}.$$

The $\mathbb{Q}$-vector space

$$R_{\mathbb{Q}}(X^*(T) \rtimes W^{\mathfrak{s}}) := \mathbb{Q} \otimes_{\mathbb{Z}} R(X^*(T) \rtimes W^{\mathfrak{s}})$$

possesses another useful basis coming from $T^{\mathfrak{s}}//W^{\mathfrak{s}}$. Given $w \in W^{\mathfrak{s}}$, let C_w be the cyclic subgroup it generates. We define a character χ_w of C_w by the formula

$$\chi_w(w^n) = \exp(2\pi i n/|C_w|).$$

For any $t \in (T^{\mathfrak{s}})^w$ we obtain a character $\mathbb{C}_t \otimes \chi_w$ of $X^*(T^{\mathfrak{s}}) \rtimes C_w$. We induce that to a $X^*(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$-representation

$$\chi(w,t) := \mathrm{ind}_{X^*(T) \rtimes C_w}^{X^*(T) \rtimes W^{\mathfrak{s}}}(\mathbb{C}_t \otimes \chi_w)$$

with central character $W^{\mathfrak{s}}t$. The representation $\chi(w,t)$ is irreducible whenever t is a generic point of $(T^{\mathfrak{s}})^w$, which in this case means simply that t is not fixed by any element of $W^{\mathfrak{s}} \setminus C_w$. It is easy to see that $\chi(w,t) \cong \chi(w',t')$ if and only if (w,t) and (w',t') are $W^{\mathfrak{s}}$-associate (which means that they determine the same point of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$). Moreover, it follows from Clifford theory and Artin's theorem [Ser, Theorem 17] that

$$\{\chi(w,t) : [w,t] \in T^{\mathfrak{s}}//W^{\mathfrak{s}}\}$$

is a $\mathbb{Q}$-basis of $R_{\mathbb{Q}}(X^*(T) \rtimes W^{\mathfrak{s}})$, see [Sol2, (40)].

Now we construct the desired map $\tilde{\mu}^{\mathfrak{s}}$, with a recursive procedure. Take $0 \leq d \leq \dim_{\mathbb{C}}(T)$. With $w \in W^{\mathfrak{s}}$, define

$$(T^{\mathfrak{s}})^w := \{t \in T^{\mathfrak{s}} : wt = t\}.$$

Suppose that we already have defined $\tilde{\mu}^{\mathfrak{s}}$ on all connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ of dimension $< d$, and that

(20)
$$\operatorname{span} \tilde{\mu}^{\mathfrak{s}}\big(\{[w,t] \in T^{\mathfrak{s}}//W^{\mathfrak{s}} : \dim(T^{\mathfrak{s}})^w < d\}\big) \cap$$
$$\operatorname{span} \{\chi(w,t) : (w,t) \in \widetilde{T^{\mathfrak{s}}}, \dim(T^{\mathfrak{s}})^w \geq d\} = 0.$$

Fix $t_1 \in T^{\mathfrak{s}}$. Since (12) has central character $W^{\mathfrak{s}}t$, both $\{\tau(t,x,\rho_1) : (t_1,x,\rho_1) \in \Psi(H)_{\mathrm{aff}}$ and $\{\chi(w,t_1) : [w,t_1] \in T^{\mathfrak{s}}//W^{\mathfrak{s}}\}$ are bases of the finite dimensional $\mathbb{Q}$-vector space $R_{\mathbb{Q}}(X^*(T) \rtimes W^{\mathfrak{s}})_{W^{\mathfrak{s}}t_1}$ spanned by the $X^*(T) \rtimes W^{\mathfrak{s}}$-representations which admit the central character $W^{\mathfrak{s}}t_1$. From this and the assumption (20) we see that we can find, for very $w \in W^{\mathfrak{s}}$ fixing t_1, an irreducible constituent $\tilde{\mu}^{\mathfrak{s}}([w,t_1])$ of $\chi(w,t_1)$ such that

$$\operatorname{span} \tilde{\mu}^{\mathfrak{s}}\big(\{[w,t_1] \in T^{\mathfrak{s}}//W^{\mathfrak{s}} : \dim(T^{\mathfrak{s}})^w \leq d\}\big)$$
$$\cup \operatorname{span} \{\chi(w,t_1) : (w,t_1) \in \widetilde{T^{\mathfrak{s}}}, \dim(T^{\mathfrak{s}})^w > d\}$$

is again a $\mathbb{Q}$-basis of $R_{\mathbb{Q}}(X^*(T) \rtimes W^{\mathfrak{s}})_{W^{\mathfrak{s}}t_1}$. In this way we construct $\tilde{\mu}^{\mathfrak{s}}$ on the d-dimensional connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$, such that (20) becomes valid for $d+1$. Thus we obtain a bijection $\tilde{\mu}^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}})$ which satisfies (1).

It remains to check (2). Fix $w \in W^{\mathfrak{s}}$ and consider a connected component $(T^{\mathfrak{s}})_i^w$ of $(T^{\mathfrak{s}})^w$. For generic $t \in (T^{\mathfrak{s}})_i^w$, $\chi(w,t) = \tilde{\mu}^{\mathfrak{s}}([w,t])$ is irreducible. We note that both $\{\chi(w,t) : t \in (T^{\mathfrak{s}})_i^w\}$ and (12) (with fixed x) are algebraic families of $X^*(T) \rtimes W^{\mathfrak{s}}$-representations parametrized by $(T^{\mathfrak{s}})_i^w$. That set is an irreducible algebraic variety because it is a coset of the neutral component of $(T^{\mathfrak{s}})^w$, which is a subtorus of $T^{\mathfrak{s}}$. It follows that the irreducible $\chi(w,t)$ are all contained in $\mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}})^{[x]}$ for one x. By continuity $\chi(w,t)$ is a subrepresentation of $H_{d(x)}(\mathcal{B}_H^{t,x}, \mathbb{C})$ for all $t \in (T^{\mathfrak{s}})_i^w$, which implies that the subquotient $\tilde{\mu}^{\mathfrak{s}}([w,t])$ of $\chi(w,t)$ has the form $\tau(t,x,\rho_1)$ for the same x. Hence $\tilde{\mu}^{\mathfrak{s}}([w,(T^{\mathfrak{s}})_i^w]) \subset \mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}})^{[x]}$. $\qquad\square$

We remark that with more effort it is possible to refine the above construction so that $\tilde{\mu}^{\mathfrak{s}}$ becomes continuous. But since we do not need that refinement, we refrain from writing it down here.

Theorem 5.2. *Let $\mathcal{G}$ be a split reductive p-adic group with connected centre, such that the residual characteristic satisfies Condition 3.1. Then, for each point $\mathfrak{s}$ in the principal series of $\mathcal{G}$, we have a continuous bijection*

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

It maps $T_{\mathrm{cpt}}^{\mathfrak{s}}//W^{\mathfrak{s}}$ onto $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cap \mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$.

Proof. To get the bijection $\mu^{\mathfrak{s}}$, apply Theorems 4.7, 4.4.(1) and 5.1. The properties (1) and (2) in Theorem 5.1 ensure that the composed map

$$T^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathbf{Irr}(X^*(T) \rtimes W^{\mathfrak{s}}) \to \Psi(H)_{\mathrm{aff}}$$

is continuous, so $\mu^{\mathfrak{s}}$ is continuous as well.

By Theorem 5.1 $T_{\mathrm{cpt}}^{\mathfrak{s}}//W^{\mathfrak{s}}$ is first mapped bijectively to the set of parameters in $\Psi(H)_{\mathrm{aff}}$ with t compact. From the proof of Theorem 4.7 we see that the latter set is mapped onto the set of enhanced Langlands parameters (Φ,ρ) with $\Phi\big|_{\mathfrak{o}_F^\times}=c^{\mathfrak{s}}$ and $\Phi(\varpi_F)$ compact. These are just the bounded enhanced Langlands parameters, so by Theorem 4.4.(2) they correspond to $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}\cap\mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$. □

6. Correcting cocharacters and L-packets

In this section we construct "correcting cocharacters" on the extended quotient $T^{\mathfrak{s}}//W^{\mathfrak{s}}$. These measure the difference between the canonical projection $T^{\mathfrak{s}}//W^{\mathfrak{s}}\to T^{\mathfrak{s}}/W^{\mathfrak{s}}$ and the composition of $\mu^{\mathfrak{s}}$ (from Theorem 5.2) with the cuspidal support map $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}\to T^{\mathfrak{s}}/W^{\mathfrak{s}}$. As conjectured in [ABPS], they show how to determine when two elements of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ give rise to $\mathcal{G}$-representations in the same L-packet.

Every enhanced Langlands parameter (Φ,ρ) naturally determines a cocharacter h_Φ and elements $\theta(\Phi,\rho,z)\in T^{\mathfrak{s}}$ by

(21)
$$h_\Phi(z)=\Phi\big(1,\big(\begin{smallmatrix}z&0\\0&z^{-1}\end{smallmatrix}\big)\big),$$
$$\theta(\Phi,\rho,z)=\Phi\big(\varpi_F,\big(\begin{smallmatrix}z&0\\0&z^{-1}\end{smallmatrix}\big)\big)=\Phi(\varpi_F)h_\Phi(z).$$

Although these formulas obviously do not depend on ρ, it turns out to be convenient to include it in the notation anyway. However, in this generality we would end up with infinitely many correcting cocharacters, most of them with range outside T. To reduce to finitely many cocharacters with values in T, we must fix some representatives for $\mathfrak{U}^{\mathfrak{s}}$ in H.

Fix a Borel subgroup B_H of H containing T. Following the recipe from the Bala–Carter classification [Car, Theorem 5.9.6] we choose a set of representatives $\mathfrak{U}^{\mathfrak{s}}\subset B_H$ for the unipotent classes of H.

Lemma 6.1. *Every commuting pair (t,x) with $t\in H$ semisimple and $x\in H$ unipotent is conjugate to one with $x\in\mathfrak{U}^{\mathfrak{s}}$ and $t\in T$.*

Proof. Obviously we can achieve that $x\in\mathfrak{U}^{\mathfrak{s}}$ via conjugation in H. Choose a homomorphism of algebraic groups $\gamma:\mathrm{SL}_2(\mathbb{C})\to H$ with $\gamma(\big(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\big))=x$. As noted in (15), such a γ exists and is unique up to conjugation by $Z_H(x)$. The constructions for the Bala–Carter theorem in [Car, §5.9] entail that we can choose γ such that $\gamma(Y_\alpha)\in T$ for all $\alpha\in\mathbb{C}^\times$. On the other hand, we can also construct such a γ inside the reductive group $Z_H(t)$. So, upon conjugating t by a suitable element of $Z_H(x)$, we can achieve that the standard maximal torus T_x of $\gamma(\mathrm{SL}_2(\mathbb{C}))$ is contained in T and commutes with t. Let $S\subset H$ be a maximal torus containing T_x and t. Then

$$T=(T\cap Z_G(\mathrm{im}\,\gamma))^\circ Z(G)T_x,$$

and similarly for S. It follows that

$$T\cap Z_G(\mathrm{im}\,\gamma)^\circ\quad\text{and}\quad S\cap Z_G(\mathrm{im}\,\gamma)^\circ$$

are maximal tori of $Z_G(\mathrm{im}\gamma)^\circ$. They are conjugate, which shows that we can conjugate t to an element of T without changing $x\in\mathfrak{U}^{\mathfrak{s}}$. □

Recall that (19) and Theorem 5.1 determine a labelling of the connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ by unipotent classes in H. This enables us to define

the correcting cocharacters: for a connected component $\mathbf{c}$ of $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ with label (represented by) $x \in \mathfrak{U}^{\mathfrak{s}}$ let $\gamma_x = \gamma$ be as in (15) and Φ as in (16). We take the cocharacter

$$(22) \qquad\qquad h_{\mathbf{c}} = h_x : \mathbb{C}^{\times} \to T, \quad h_x(z) = \gamma_x \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix} \right).$$

Let $\widetilde{\mathbf{c}}$ be a connected component of $\widetilde{T^{\mathfrak{s}}}$ that projects onto $\mathbf{c}$ and centralizes x. In view of Lemma 6.1 this can always be achieved by adjusting by an element of $W^{\mathfrak{s}}$. We define

$$(23) \qquad \begin{aligned} \widetilde{\theta}_z : \widetilde{\mathbf{c}} &\to T^{\mathfrak{s}}, & (w,t) &\mapsto t\, h_{\mathbf{c}}(z), \\ \theta_z : \mathbf{c} &\to T^{\mathfrak{s}}/W^{\mathfrak{s}}, & [w,t] &\mapsto W^{\mathfrak{s}} t\, h_{\mathbf{c}}(z). \end{aligned}$$

Theorem 6.2. *Let $[w,t], [w',t'] \in T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$. Then $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet if and only if*

- *$[w,t]$ and $[w',t']$ are labelled by the same unipotent class in H;*
- *$\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^{\times}$.*

Proof. Suppose that the two $\mathcal{G}$-representations $\mu^{\mathfrak{s}}[w,t] = \pi(\Phi, \rho)$ and $\mu^{\mathfrak{s}}[w',t'] = \pi(\Phi', \rho')$ belong to the same L-packet. By definition this means that Φ and Φ' are G-conjugate. Hence they are labelled by the same unipotent class, say $[x]$ with $x \in \mathfrak{U}^{\mathfrak{s}}$. By choosing suitable representatives we may assume that $\Phi = \Phi'$ and that $\{(\Phi, \rho), (\Phi, \rho')\} \subset \Phi(G)_{\mathrm{en}}^{\mathfrak{s},[x]}$. Then

$$\theta(\Phi, \rho, z) = \theta(\Phi, \rho', z) \text{ for all } z \in \mathbb{C}^{\times}.$$

Although in general $\theta(\Phi, \rho, z) \neq \widetilde{\theta}_z(w,t)$, they differ only by an element of $W^{\mathfrak{s}}$. Hence $\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^{\times}$.

Conversely, suppose that $[w,t], [w',t']$ fulfill the two conditions of the lemma. Let $x \in \mathfrak{U}^{\mathfrak{s}}$ be the representative for the unipotent class which labels them. From Lemma 6.1 we see that there are representatives for $[w,t]$ and $[w',t']$ such that $t(T^w)^{\circ}$ and $t'(T^{w'})^{\circ}$ centralize x. Then

$$\widetilde{\theta}_z(w,t) = t\, h_x(z) \quad \text{and} \quad \widetilde{\theta}_z(w',t') = t'\, h_x(z)$$

are $W^{\mathfrak{s}}$ conjugate for all $z \in \mathbb{C}^{\times}$. As these points depend continuously on z and $W^{\mathfrak{s}}$ is finite, this implies that there exists a $v \in W^{\mathfrak{s}}$ such that

$$v(t\, h_x(z)) = t'\, h_x(z) \quad \text{for all } z \in \mathbb{C}^{\times}.$$

For $z = 1$ we obtain $v(t) = t'$, so v fixes $h_x(z)$ for all z.

Consider the minimal parabolic root subsystem R_P of $R(G,T)$ that supports h_x. In other words, the unique set of roots P such that h_x lies in a facet of type P in the chamber decomposition of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We write

$$T^P = \{t \in T \mid \alpha(t) = 1 \ \forall \alpha \in P\}^{\circ}.$$

Then $t(T^w)^{\circ}$ and $t'(T^{w'})^{\circ}$ are subsets of T^P and v stabilizes T^P. It follows from [Opd, Proposition B.4] that $h_x(q^{1/2})tT^P$ and $h_x(q^{1/2})t'T^P$ are residual cosets in the sense of Opdam. By the above, these two residual cosets are conjugate via $v \in W^{\mathfrak{s}}$. Now [Opd, Corollary B.5] says that the pairs $(h_x(q^{1/2})t, x)$ and $(h_x(q^{1/2})t', x)$ are H-conjugate. Hence the associated Langlands parameters are conjugate, which means that $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet. $\qquad\square$

Corollary 6.3. *Properties 1–5 from* [ABPS, §15] *hold for* $\mu^{\mathfrak{s}}$ *as in Theorem 5.2, with the morphism* θ_z *from* (23) *and the labelling by unipotent classes in* $H^{\mathfrak{s}}$ *from* (19) *and Theorem 5.1.*

Together with Theorem 5.2 this proves the conjecture from [ABPS] *for all Bernstein components in the principal series of a split reductive p-adic group with connected centre, such that the residual characteristic satisfies Condition 3.1.*

Proof. Property (1) was shown in Theorem 5.2. By the definition of θ_z (23), property (4) holds. Property (3) is a consequence of property (4), in combination with Theorems 4.4.(3), 5.2 and 5.1. Property (2) follows from Theorem 5.2 and property (3). Property (5) is none other than Theorem 6.2. □

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