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# Second Order Inductive Logic and Wilmers' Principle

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We extend the framework of Inductive Logic to Second Order languages and introduce Wilmers' Principle, a rational principle for probability functions on Second Order languages. We derive a representation theorem for functions satisfying this principle and investigate its relationship to the first order principles of Regularity and Super Regularity.

Key words: Universal Certainty, Second Order Logic, Inductive Logic, Probability Logic, Uncertain Reasoning.

## 1 Introduction

In the framework of Pure Inductive Logic, a rational agent's belief function is usually regarded as a probability function on the set of first order sentences of a certain, fixed language  $L$ . This language contains constants representing the objects of the universe

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and predicates representing the properties of these objects. This allows an agent to express statements about the universe.

So far such statements have, to our knowledge, been limited to first order expressions, allowing the agent to make existential or universal statements about the objects. As the famous Geach-Kaplan statement shows (see e.g. [1]), an agent could increase her expressive power if she were to extend the set of expressions available to include second order statements.

As we identify an agent's belief in a statement with the value the agent's belief function assigns to it, allowing the agent to use second order expressions will require her to extend the domain of the her belief function to include second order sentences.

Unsurprisingly, this leads to a number of complications. Since Second Order logic is inherently incomplete (see e.g. [10]), we will have to be careful picking a suitable framework for Second Order Inductive Logic. At the same time, we would want to have a suitable interpretation of universal and existential quantification over the predicates in  $L$ .

In this paper we intend to provide such a framework, allowing an agent to extend her expressive power to second order logic. Once such a framework is given, we can study rational principles involving second order logic. We will give an example of one such principle, called Wilmers' Principle, and provide a representation theorem for second order belief functions that satisfy this principle. We will then consider the consequences of this principle for the thorny question of Universal Truth for first order statements.

## 2 Second Order probability functions

As with traditional Inductive Logic<sup>1</sup> for example [2], we will work in a unary language  $L$  with predicate symbols  $P_i$  and constants  $a_i$  for  $i \in \{1, 2, 3, \dots\} = \mathbb{N}^+$  but without function symbols or equality. Let  $F_1L$ ,  $S_1L$ ,  $QFS_1L$  respectively denote the first order formulae, sentences and quantifier free sentences of  $L$ .

Let  $\mathcal{T}L$  denote the set of structures  $M$  for  $L$  in which the constants  $a_i$  are interpreted as themselves and  $|M| = \{a_i \mid i \in \mathbb{N}^+\}$ , so every element of the universe of  $M$  is denoted by a constant symbol. Similarly we shall use  $P_j$  to denote  $\{a_i \mid M \models P_j(a_i)\}$ , leaving the  $M$  implicit whenever this cannot cause confusion.

We say that  $w : S_1L \rightarrow [0, 1]$  is a *probability function on  $S_1L$* , if for any  $\vartheta, \varphi \in S_1L$ ,  $\psi(x) \in F_1L$ ,

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<sup>1</sup>Actually Inductive Logic is more commonly presented with only finitely many predicate symbols but as we would in any case advocate the rationality of Unary Language Invariance in this context, see for example [8, Chapter 16], this would ultimately lead to the same situation.

(P1) If  $\models \vartheta$ , then  $w(\vartheta) = 1$ .

(P2) If  $\vartheta \models \neg\varphi$ , then  $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi)$ .

(P3)  $w(\exists x \psi(x)) = \lim_{n \rightarrow \infty} w(\bigvee_{i=1}^n \psi(a_i))$ .

To our mind the central problem of (Pure) Inductive Logic can be picturesquely captured as follows: Imagine an agent inhabiting a structure  $M \in \mathcal{TL}$  but having no further knowledge, so in particular the agent has no particular interpretation in mind for the constant and predicate symbols. In that case what probability  $w(\vartheta)$  should the agent rationally, or logically, give to  $\vartheta \in S_1 L$ ? Or more precisely, since we obviously intend for these probability values to be coherent, what probability function  $w$  should the agent rationally or logically adopt?

In the absence of any clear definition of what is meant here by ‘rationally’ (which for the purpose of this paper we identify with ‘logically’) the usual method of tackling this question is by imposing certain ostensibly rational, or at least not irrational, requirements on  $w$  and seeing where that leads. For example the symmetry between the constants  $a_i$ , and between the predicates  $P_j$ , from the agent’s point of view surely requires that  $w$  should satisfy:

**Constant Exchangeability, Ex**

*w satisfies Ex, if for all  $\vartheta \in S_1 L$  and all permutations  $\sigma$  of  $\mathbb{N}^+$ ,*

$$w(\vartheta(a_1, \dots, a_n)) = w(\vartheta(a_{\sigma(1)}, \dots, a_{\sigma(n)})).$$

**Predicate Exchangeability, Px**

*w satisfies Px, if for all  $\vartheta \in S_1 L$ , and all permutations  $\sigma$  of  $\mathbb{N}^+$ ,*

$$w(\vartheta(P_1, \dots, P_m, a_1, \dots, a_n)) = w(\vartheta(P_{\sigma(1)}, \dots, P_{\sigma(m)}, a_1, \dots, a_n))$$

*whenever  $\vartheta(P_{\sigma(1)}, \dots, P_{\sigma(m)}, a_1, \dots, a_n)$  is the result of replacing each predicate symbol,  $P_j$ , occurring in  $\vartheta$  with  $P_{\sigma(j)}$ .*

Given their standing we shall assume throughout that unless specifically indicated all the probability functions we consider satisfy Ex and Px.

Two further principles which we might impose on  $w$  are based on the idea that in the presence of no prior knowledge about the universe it would seem unreasonable to summarily assign zero probability to sentences which could be true.

**Regularity, Reg**

*w satisfies Reg, if for all  $\vartheta \in QFS_1 L$  such that  $\vartheta$  is satisfiable,  $w(\vartheta) > 0$ .*

**Super Regularity, SReg**

*w satisfies SReg, if for all  $\vartheta \in S_1 L$  such that  $\vartheta$  is satisfiable,  $w(\vartheta) > 0$ .*

Note that we have separate principles for quantifier-free sentences and sentences that may contain quantifiers. The reason for this is that functions satisfying Regularity in general do not need to also satisfy Super Regularity.

The principle SReg<sup>2</sup> has been something of a thorn in the side of traditional Inductive Logic. Whilst seemingly quite reasonable it is inconsistent with several other rationally attractive principles such as Johnson's Sufficiency Postulate, see [6], [8], and in consequence is not satisfied by the probability functions comprising Carnap's Continuum of Inductive Methods, though all of them except the pathological  $c_0$  do satisfy Reg. Indeed SReg already fails there for consistent but non-tautologous  $\Pi_1$  sentences where the failure is sometimes referred to as the problem of *Universal Certainty*. In spite of some attempts, see for example [5], to tweak Johnson's Sufficiency Postulate and allow SReg to hold the issue still seems to us problematic. In this paper we shall be investigating a somewhat different approach which under certain circumstance has SReg as a consequence.

Returning again to our agent it is clear that for this unary language  $L$  there is a certain parallel between the constant symbols and the predicate symbols, as indeed is reflected in the principles Ex and Px. This suggests that from the agent's point of view we could just as well have quantifiers ranging over the predicates as over the constants, in other words *second order* quantifiers  $\forall X$  and  $\exists X$  where the second order variable  $X$  is intended to range over the predicates  $P_i$ .

To this end let  $F_2L, S_2L$  respectively denote the second order formulae and sentences of  $L$  and say that  $w : S_2L \rightarrow [0, 1]$  is a *probability function on  $S_2L$* , if for any  $\vartheta, \varphi \in S_2L$ ,  $\psi(x), \eta(X) \in F_2L$ ,

(P1) If  $\models \vartheta$ , then  $w(\vartheta) = 1$ .

(P2) If  $\vartheta \models \neg\varphi$ , then  $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi)$ .

(P3)  $w(\exists x \psi(x)) = \lim_{n \rightarrow \infty} w(\bigvee_{i=1}^n \psi(a_i))$ .

(P4)  $w(\exists X \eta(X)) = \lim_{n \rightarrow \infty} w(\bigvee_{i=1}^n \eta(P_i))$ .

So here we have added (P4) to the original requirements, reflecting the idea that the  $P_i$  exhaust the second order universe just as (P3) reflected the idea that the  $a_i$  exhaust the first order universe.

This extension from first to second order in this way is in fact hardly more than notational (though it will enable us to state and investigate new principles). For example if  $M \in \mathcal{TL}$  then  $M$  automatically becomes a second order structure by taking the second order

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<sup>2</sup>Also referred to as *Cournot's Principle*, see [3].

universe of  $M$  to simply be the set of interpretations of the relation symbols in  $M$ , i.e.

$$\{\{a_i \mid M \models P_j(a_i)\} \mid j \in \mathbb{N}^+\}.$$

For this reason we shall not distinguish between  $M$  as a first or second order structure and similarly we shall use  $\mathcal{TL}$  for ‘both’ versions.

Notice then that the condition  $\vartheta \models \varphi$  in (P1-2) is actually equivalent to

$$\forall M \in \mathcal{TL}, \text{ if } M \models \vartheta \text{ then } M \models \varphi$$

since if a sentence  $\psi \in S_2L$  has a model then it has a model with denumerably many elements and subsets in its universe and hence by suitable naming a model in  $\mathcal{TL}$ .

With this extended notion of probability function all the standard properties can be proved just as before (e.g. see [7, Proposition 2.1], [4] or [8, Lemma 3.8]):

**Lemma 1.** *Let  $w$  be a probability function on  $S_2L$ . Then for  $\vartheta, \varphi \in S_2L$ ,*

- (a)  $w(\neg\vartheta) = 1 - w(\vartheta)$ .
- (b) *If  $\models \neg\vartheta$ , then  $w(\vartheta) = 0$ .*
- (c) *If  $\vartheta \models \varphi$ , then  $w(\vartheta) \leq w(\varphi)$ .*
- (d) *If  $\vartheta \equiv \varphi$ , then  $w(\vartheta) = w(\varphi)$ .*
- (e)  $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi) - w(\vartheta \wedge \varphi)$ .

**Lemma 2.** *Let  $w$  be a probability function on  $S_2L$  and  $\exists x_1 \dots \exists x_k \vartheta(x_1, \dots, x_k, \vec{P}, \vec{a})$ ,  $\exists X_1 \dots \exists X_k \psi(X_1, \dots, X_k, \vec{P}, \vec{a}) \in S_2L$ . Then*

$$\begin{aligned} w(\exists x_1 \dots \exists x_k \vartheta(x_1, \dots, x_k, \vec{P}, \vec{a})) &= \lim_{n \rightarrow \infty} w \left( \bigvee_{i_1, \dots, i_k \leq n} \vartheta(a_{i_1}, \dots, a_{i_k}, \vec{P}, \vec{a}) \right) \\ w(\forall x_1 \dots \forall x_k \vartheta(x_1, \dots, x_k, \vec{P}, \vec{a})) &= \lim_{n \rightarrow \infty} w \left( \bigwedge_{i_1, \dots, i_k \leq n} \vartheta(a_{i_1}, \dots, a_{i_k}, \vec{P}, \vec{a}) \right) \\ w(\exists X_1 \dots \exists X_k \psi(X_1, \dots, X_k, \vec{P}, \vec{a})) &= \lim_{n \rightarrow \infty} w \left( \bigvee_{i_1, \dots, i_k \leq n} \psi(P_{i_1}, \dots, P_{i_k}, \vec{P}, \vec{a}) \right) \\ w(\forall X_1 \dots \forall X_k \psi(X_1, \dots, X_k, \vec{P}, \vec{a})) &= \lim_{n \rightarrow \infty} w \left( \bigwedge_{i_1, \dots, i_k \leq n} \psi(P_{i_1}, \dots, P_{i_k}, \vec{P}, \vec{a}) \right) \quad (1) \end{aligned}$$

Just as structures in  $\mathcal{TL}$  extend naturally from  $S_1L$  to  $S_2L$  so do probability functions on  $S_1L$  extend uniquely to probability functions on  $S_2L$ . Precisely:

**Theorem 3.** *Let  $w$  be a probability function on  $S_1L$ . Then  $w$  extends uniquely to a probability function on  $S_2L$ . Furthermore if  $w$  satisfies  $Ex$  and  $Px$  on  $S_1L$  then they are preserved in this extension to  $S_2L$ .*

**Proof** Let  $w$  be a probability function on  $S_1L$ , let  $\mathcal{A}$  be the subsets of  $\mathcal{TL}$  of the form

$$\{M \in \mathcal{TL} \mid M \models \vartheta\}$$

for  $\vartheta \in QFS_1L$ . Define a finitely additive measure  $\mu$  on  $\mathcal{A}$  by

$$\mu\{M \in \mathcal{TL} \mid M \models \vartheta\} = w(\vartheta).$$

By a compactness argument if  $\varphi$  and  $\vartheta_n$  are in  $QFS_1L$  for  $n \in \mathbb{N}$  and

$$\{M \in \mathcal{TL} \mid M \models \varphi\} = \bigcap_{n \in \mathbb{N}} \{M \in \mathcal{TL} \mid M \models \vartheta_n\}$$

then for some  $k$

$$\{M \in \mathcal{TL} \mid M \models \varphi\} = \bigcap_{n \leq k} \{M \in \mathcal{TL} \mid M \models \vartheta_n\}$$

and from this it follows that  $\mu$  preserves all infs in  $\mathcal{A}$ .

By Carathéodory's Extension Theorem then  $\mu$  extends uniquely to a countably additive measure on the  $\sigma$ -algebra generated by  $\mathcal{A}$ . Since

$$\{M \in \mathcal{TL} \mid M \models \exists x \vartheta(x)\} = \bigcup_{n \in \mathbb{N}^+} \{M \in \mathcal{TL} \mid M \models \vartheta(a_n)\},$$

$$\{M \in \mathcal{TL} \mid M \models \exists X \vartheta(X)\} = \bigcup_{n \in \mathbb{N}^+} \{M \in \mathcal{TL} \mid M \models \vartheta(P_n)\},$$

all the sets

$$\{M \in \mathcal{TL} \mid M \models \vartheta\}$$

for  $\vartheta \in S_2L$  will be in this  $\sigma$ -algebra and if we define  $w^+$  on  $S_2L$  by

$$w^+(\vartheta) = \mu\{M \in \mathcal{TL} \mid M \models \vartheta\}$$

then  $w^+$  will satisfy (P1-4). Since  $w^+$  agrees with  $w$  on  $QFS_1L$ , by a result of Gaifman, [4], (essentially by induction on the length of formulae using (1))  $w^+$  agrees with  $w$  on  $S_1L$  and hence provides the required extension of  $w$  to  $S_2L$ .

Furthermore it is now easy to see by induction on quantifier complexity that this is the only possible extension of  $w$  to  $S_2L$  and if  $w$  satisfies  $Ex$  and  $Px$  on  $QFS_1L$  then it also does so on  $S_2L$  ■

Given this result we shall in future not particularly distinguish between a probability function defined on  $S_1L$  and its extension to  $S_2L$ .

### 3 Wilmers' Principle

So far we have established some initial technical results, allowing us to work with second order sentences in the framework of Pure Inductive Logic. In this section we suggest and discuss a rational principle for Second Order expressions that rational agents may want to accept as defining their beliefs.

The motivation for the principle is the following idea: Suppose we have a First Order formula  $\vartheta(x)$  with just one free variable. Then in the agent's ambient structure  $M$   $\vartheta(x)$  defines the subset of the universe

$$\{a_i \mid M \models \vartheta(a_i)\}.$$

A rational agent then might feel that there should be a *name* for this set in the language, and thus  $\vartheta(x)$  defines not only a subset of the universe, but also a predicate of the language, i.e.

$$\{a_i \mid M \models \vartheta(a_i)\} = \{a_i \mid M \models P_j(a_i)\}$$

for some (unary)  $P_j$  in the agent's language.

The formal definition of this principle in terms of probability functions is given by:

#### Wilmers' Principle, WP

Let  $w$  be a probability function on  $S_2L$ . Then  $w$  satisfies Wilmers' Principle, WP, if

$$w(\exists X \forall x (\vartheta(x) \leftrightarrow X(x))) = 1$$

whenever  $\vartheta(x) \in F_1L$ .<sup>3</sup>

This principle is based on the original suggestion by George Wilmers that it would be rational that  $w$  gave probability 1 to all the tautologies of second order monadic logic. The current version then is but a fragment of what was originally intended.

The main aim of this section is to provide a certain representation theorem for the probability functions satisfying Wilmers' Principle. This will turn out to be useful in the next section when we consider also Reg and SReg. We first need some notation and apparatus.

For each  $n \in \mathbb{N}^+$ , define  $\wp(n)^\infty \times n^\infty$  and structures  $M_{f,g}$  as follows:

- Let  $\wp(n)$  denote the power set of  $\{1, 2, \dots, n\}$ , let  $\wp(n)^\infty$  denote the set of functions  $f : \mathbb{N}^+ \rightarrow \wp(n)$  and let  $n^\infty$  denote the set of function  $g : \mathbb{N}^+ \rightarrow \{1, 2, \dots, n\}$ . So  $\wp(n)^\infty \times n^\infty$  denotes the set of all pairs  $\langle f, g \rangle$  with  $f \in \wp(n)^\infty$  and  $g \in n^\infty$ .

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<sup>3</sup>As far as the actual statements of results in this paper are concerned it would make no difference if we took instead the version of Wilmers' Principle with  $\vartheta(x) \in F_2L$ .



- For each pair  $\langle f, g \rangle \in \wp(n)^\infty \times n^\infty$ , define the structure  $M_{f,g}$  for  $L$  with finite universe  $\{e_1, e_2, \dots, e_n\}$  by  $a_i^{M_{f,g}} = e_{g(i)}$  and  $P_j^{M_{f,g}} = \{e_k \mid k \in f(j)\}$ . So

$$M_{f,g} \models P_j(a_i) \iff M_{f,g} \models P_j(e_{g(i)}) \iff g(i) \in f(j).$$

[Here  $a_i^{M_{f,g}}$  and  $P_j^{M_{f,g}}$  are the interpretations of  $a_i$  and  $P_j$  in  $M_{f,g}$ .]

Let  $n \in \mathbb{N}^+$  and let  $\mu_n$  be a normalized,  $\sigma$ -additive measure on  $\wp(n)^\infty \times n^\infty$ . We say that

- $\mu_n$  is *invariant under Ex* if for any  $\vartheta \in S_2L$  and any permutation  $\sigma : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ ,

$$\mu_n(\{\langle f, g \rangle \mid M_{f,g} \models \vartheta\}) = \mu_n(\{\langle f, g\sigma \rangle \mid M_{f,g} \models \vartheta\}),$$

- $\mu_n$  is *invariant under Px* if for any  $\vartheta \in S_2L$  and any permutation  $\sigma : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ ,

$$\mu_n(\{\langle f, g \rangle \mid M_{f,g} \models \vartheta\}) = \mu_n(\{\langle f\sigma, g \rangle \mid M_{f,g} \models \vartheta\}).$$

We will use the measures  $\mu_n$  to construct probability functions that only give weight to those structures  $M_{f,g}$  that instantiate precisely  $n$  distinguishable constants. For this, it will be convenient to have a single second order sentence expressing this. Let  $\xi_n$  be the sentence

$$\exists x_1 \dots \exists x_n \left[ \bigwedge_{1 \leq i < j \leq n} \exists X \neg(X(x_i) \leftrightarrow X(x_j)) \wedge \forall z \bigvee_{i=1}^n \forall X (X(x_i) \leftrightarrow X(z)) \right] \quad (2)$$

for  $n \in \mathbb{N}^+$ .

**Theorem 4.** *Let  $\mu$  be a normalized,  $\sigma$ -additive measure on  $\wp(n)^\infty \times n^\infty$  invariant under Ex and Px and suppose that*

$$\mu(\{\langle f, g \rangle \mid \text{ran}(f) = \wp(n) \text{ and } \text{ran}(g) = n\}) = 1. \quad (3)$$

*Let  $w_\mu$  be the function on  $S_2L$  defined by*

$$w_\mu(\vartheta) = \mu(\{\langle f, g \rangle \mid M_{f,g} \models \vartheta\})$$

*for  $\vartheta \in S_2L$ . Then*

- (i)  $w_\mu$  is a probability function on  $S_2L$  satisfying  $Px + Ex$ .
- (ii)  $w_\mu$  satisfies Wilmers' Principle, WP, and  $w_\mu(\xi_n) = 1$ .

*Conversely let  $v$  be a probability function on  $S_2L$  satisfying  $Px + Ex + \text{Wilmers' Principle}$  and suppose that  $v(\xi_n) = 1$ . Then  $v = w_\mu$  for some normalized,  $\sigma$ -additive measure  $\mu$  on  $\wp(n)^\infty \times n^\infty$  invariant under  $Ex + Px$  and satisfying (3).*

**Proof** For (i) it is straightforward to check that  $w_\mu$  satisfies (P1-2). For (P3) let  $\psi(x) \in F_2L$  with  $x$  the only free variable. Then

$$\begin{aligned}
w_\mu(\exists x \psi(x)) &= \mu\{\langle f, g \rangle \mid M_{f,g} \models \exists x \psi(x)\} \\
&= \mu\left\{\bigcup_{k=1}^n \{\langle f, g \rangle \mid M_{f,g} \models \psi(e_k)\}\right\} \\
&= \mu\left\{\bigcup_{i=1}^\infty \{\langle f, g \rangle \mid M_{f,g} \models \psi(a_i)\}\right\}, \text{ since } g \text{ is onto for almost all } \langle f, g \rangle, \\
&= \lim_{m \rightarrow \infty} \mu\left\{\langle f, g \rangle \mid M_{f,g} \models \bigvee_{i=1}^m \psi(a_i)\right\}, \text{ by } \sigma\text{-additivity of } \mu, \\
&= \lim_{m \rightarrow \infty} w_\mu\left(\bigvee_{i=1}^m \psi(a_i)\right),
\end{aligned}$$

and thus (P3) holds for  $w_\mu$ . (P4) now follows using an analogous argument, but without the requirement of  $f$  being onto, for a sentence  $\exists X \eta(X) \in S_2L$ .

Since  $\mu_n$  is invariant under Ex and Px, it follows that  $w_{\mu_n}$  satisfies Ex + Px.

For (ii), we will first show that  $w_\mu$  satisfies Wilmers' Principle. Let  $\vartheta(x) \in F_2L$  with  $x$  the only free variable and let  $\langle f, g \rangle \in \wp^\infty \times n^\infty$  with  $f$  onto. Then there is  $j \in \mathbb{N}^+$  such that

$$\{e_k \mid M_{f,g} \models \vartheta(e_k)\} = \{e_k \mid k \in f(j)\} = \{e_k \mid M_{f,g} \models P_j(e_k)\}.$$

Hence,

$$M_{f,g} \models \forall x (P_j(x) \leftrightarrow \vartheta(x)),$$

and in turn,

$$M_{f,g} \models \exists X \forall x (X(x) \leftrightarrow \vartheta(x)).$$

By (3) then,

$$\mu\{\langle f, g \rangle \mid M_{f,g} \models \exists X \forall x (X(x) \leftrightarrow \vartheta(x))\} = 1$$

and hence

$$w_\mu(\exists X \forall x (X(x) \leftrightarrow \vartheta(x))) = 1.$$

An analogous proof shows that  $w_\mu(\xi_n) = 1$ .

For the converse let  $v$  be a probability function on  $S_2L$  with the properties as given. For  $\varphi \in QFS_1L$  set

$$\mu\{\langle f, g \rangle \in \wp(n)^\infty \times n^\infty \mid M_{f,g} \models \varphi\} = v(\varphi). \quad (4)$$

Then  $\mu$  is a finitely additive normalized measure on the algebra  $\mathcal{A}$  of these subsets of  $\wp(n)^\infty \times n^\infty$ . Furthermore  $\mu$  preserves infs, i.e. is a pre-measure. To see this suppose

that  $\varphi, \vartheta_n \in QFS_1 L$  for  $n \in \mathbb{N}$  and

$$\bigcap_{n \in \mathbb{N}} \{ \langle f, g \rangle \mid M_{f,g} \models \vartheta_n \} = \{ \langle f, g \rangle \mid M_{f,g} \models \varphi \}. \quad (5)$$

Suppose that there is no  $k \in \mathbb{N}$  such that

$$\bigcap_{n \leq k} \{ \langle f, g \rangle \mid M_{f,g} \models \vartheta_n \} = \{ \langle f, g \rangle \mid M_{f,g} \models \varphi \}. \quad (6)$$

Then for each  $k$  we can find a structure  $M_{f_k, g_k}$  to satisfy

$$\{ \vartheta_n \mid n \leq k \} \cup \{ \neg \varphi \}.$$

Taking an ultraproduct of these structures with respect to a non-principle ultrafilter yields a structure  $M$  satisfying the set of sentences

$$\{ \vartheta_n \mid n \in \mathbb{N} \} \cup \{ \neg \varphi \} \quad (7)$$

whose universe is still the set of  $e_k, k = 1, 2, \dots, n$  and, when restricted to the original language, is of the form  $M_{f,g}$  for some  $\langle f, g \rangle \in {}^{\wp(n)} \times n^\infty$ . Furthermore since the  $\vartheta_n, \varphi \in QFS_1 L$  this  $M_{f,g}$  also satisfies (7). But that contradicts (5), so such a  $k$  must exist and the preservation of this inf follows.

By Carathéodory's Extension Theorem we can now uniquely extend  $\mu$  to the  $\sigma$ -algebra generated by  $\mathcal{A}$ . Since  $v$  satisfies Ex and Px then  $\mu$  as defined by (4) will be invariant under Ex and Px and this property will be retained in this extension of  $\mu$ .

We now need to show that

$$\mu \{ \langle f, g \rangle \mid g \text{ is onto} \} = 1. \quad (8)$$

Since  $v(\xi_n) = 1$  for some  $a_{i_1}, a_{i_2}, \dots, a_{i_n}$

$$v \left( \bigwedge_{1 \leq k < j \leq n} \exists X \neg (X(a_{i_k}) \leftrightarrow X(a_{i_j})) \right) > 0.$$

and for some  $P_{q_{k,j}}, 1 \leq k < j \leq n$ ,

$$v \left( \bigwedge_{1 \leq k < j \leq n} \neg (P_{q_{k,j}}(a_{i_k}) \leftrightarrow P_{q_{k,j}}(a_{i_j})) \right) > 0.$$

By definition this left hand side is equal to

$$\mu \left\{ \langle f, g \rangle \mid M_{\langle f, g \rangle} \models \bigwedge_{1 \leq k < j \leq n} \neg (P_{q_{k,j}}(a_{i_k}) \leftrightarrow P_{q_{k,j}}(a_{i_j})) \right\},$$

and similarly for disjunctions of such formulae  $\bigwedge_{1 \leq k < j \leq n} \neg(P_{q_{k,j}}(a_{i_k}) \leftrightarrow P_{q_{k,j}}(a_{i_j}))$ . But since  $v(\xi_n) = 1$ , by (1), the limit probability of these disjunctions is 1, giving (8).

Hence by the first part  $w_\mu$  is a probability function and by definition it agrees with  $v$  on  $QFS_I L$ . So by induction on the quantifier complexity of  $\varphi \in SL2$ ,

$$v(\varphi) = w_\mu(\varphi) = \mu\{\langle f, g \rangle \mid M_{f,g} \models \varphi\}$$

It remains to show that  $\mu$  satisfies (3). Notice that from (8) we are already half way there. Since  $v(WP \wedge \xi_n) = 1$ ,

$$\mu\{\langle f, g \rangle \mid M_{f,g} \models WP \wedge \xi_n\} = 1.$$

Let  $\langle f, g \rangle$  be in this set with  $g$  onto. Then since  $M_{\langle f, g \rangle} \models WP$  we have that for any  $P_i, P_j$  there are  $P_k, P_r, P_m$  such that

$$M_{\langle f, g \rangle} \models \forall x (P_k(x) \leftrightarrow \top)$$

$$M_{\langle f, g \rangle} \models \forall x (P_r(x) \leftrightarrow \neg P_i(x))$$

$$M_{\langle f, g \rangle} \models \forall x (P_m(x) \leftrightarrow (P_i(x) \wedge P_j(x)))$$

Hence the  $P_j^{M_{\langle f, g \rangle}}$  form a Boolean Algebra. Also since  $M_{f,g} \models \xi_n$  every pair  $e_m \neq e_r$  are separated by some  $P_j^{M_{\langle f, g \rangle}}$  so this Boolean Algebra must be all subsets of  $\{e_1, e_2, \dots, e_n\}$ . I.e.  $f$  must be onto and (3) follows. ■

In the case  $n = 1$  there is only one choice for  $\langle f, g \rangle$  and similarly  $\mu$  and in this case  $w_\mu$  is  $c_0^L$  from Carnap's Continuum of Inductive Methods for  $L$ . For  $n > 1$  there are infinitely many choices for  $\mu$  invariant under Ex and Px. For example for  $n = 2$  and  $0 < c < 1$  we can define  $\mu_c$  on the standard basis subsets of  $\wp(2)^\infty \times 2^\infty$  by

$$\mu_c \left\{ \langle f, g \rangle \mid \bigwedge_{k=1}^n f(j_k) = s_k \wedge \bigwedge_{r=1}^m g(i_r) = q_r \right\} = 2^{-m} \prod_{k=1}^m c^{|s_k|} (1-c)^{2-|s_k|}.$$

If we now extend  $\mu_c$  to a normalized  $\sigma$ -additive, Ex and Px invariant measure on  $\wp(2)^\infty \times 2^\infty$  then all the  $w_{\mu_c}$  will be different.

We now give a version of a 'Ladder Theorem' which provides a classification of the probability functions satisfying Wilmers' Principle and will be useful in the next section. In this theorem the  $\xi_n$  are as given in (2).

**Theorem 5 (Ladder Theorem).** *Let  $w$  be a probability function on  $S_2 L$  satisfying Wilmers' Principle. Then  $w$  can be represented as*

$$w = \lambda_0 w_0 + \sum_{n \in \mathbb{N}^+} \lambda_n w_n \tag{9}$$

with the  $w_n$  satisfying Wilmers' Principle,  $w_n(\xi_n) = 1$  and  $w_0(\xi_n) = 0$  for  $n > 0$ ,  $\lambda_n \geq 0$  and  $\sum_n \lambda_n = 1$ .

**Proof** Let  $w$  be a function on  $S_2L$  satisfying Wilmers' Principle. For  $n > 0$ , if  $w(\xi_n) = 0$  let  $\lambda_n = 0$  and take  $w_n$  to be any probability function satisfying Wilmers' Principle and  $w_n(\xi_n) = 1$ . On the other hand if  $w(\xi_n) = \lambda_n > 0$  set  $w_n = w(\cdot | \xi_n)$ . Since  $\xi_n$  does not contain any constant or predicate symbols  $w_n$  satisfies Ex and Px and similarly  $w_n$  inherits Wilmers' Principle from  $w$  since  $w$  gives value 1 to any instance of this principle. Clearly  $w_n(\xi_n) = 1$ .

Now consider

$$\hat{w} = w - \sum_{n \in \mathbb{N}^+} \lambda_n \cdot w_{\mu_n}$$

and let  $\lambda_0 = \hat{w}(\top)$ . If  $\lambda_0 = 0$ , we are done. Otherwise  $w_0 = \lambda_0^{-1} \hat{w}$ , is a probability function satisfying Ex, Px and Wilmers' Principle. Also for  $n > 0$ ,  $w_0(\xi_n) = 0$  since  $w_m(\xi_n) = 0$  for  $0 < m \neq n$  and

$$w(\xi_n) = \lambda_n = \lambda_n w_n(\xi_n).$$

■

We have already seen that there are probability functions  $w$  satisfying Wilmers' Principle in whose Ladder Representation  $\lambda_n > 0$ . The same is true for  $\lambda_0$ :

We are now in a position to show that:

**Proposition 6.** *There are probability functions satisfying Wilmers' Principle with  $\lambda_0 > 0$  in the Ladder Representation.*

**Proof** Let  $\xi_n$  be as above. Since the set  $\{\neg \xi_n | n \in \mathbb{N}^+\}$  together with all instances of Wilmers' Principle is consistent it has a model  $M \in \mathcal{TL}$ . Define the probability function  $V_M : S_2L \rightarrow \{0, 1\}$  by

$$V_M(\vartheta) = \begin{cases} 1 & \text{if } M \models \vartheta, \\ 0 & \text{if } M \models \neg \vartheta. \end{cases} \quad (10)$$

As given  $V_M$  does not satisfy Ex or Px. However by a method introduced into this field by Gaifman, see [4] or [7, Theorem 12.3], we can use  $V_M$  to construct a probability function  $w$  which does satisfy Ex and Px and which gives non-zero probability to the same sentences. Indeed in this case it will continue to give all the instances of Wilmers' Principle and the  $\neg \xi_n$  probability 1 so for this  $w$  we must have that  $\lambda_0 = 1$ . ■

## 4 Regularity and Super Regularity

Concerning the regularity and super regularity<sup>4</sup> of probability functions satisfying Wilmers' Principle the first point to notice is that for  $n > 0$  the  $w_n$  in (9) do not satisfy Reg since, for example, for  $m > n$

$$\bigwedge_{j=1}^m \left( P_j(a_j) \wedge \bigwedge_{\substack{1 \leq i \leq m \\ i \neq j}} \neg P_j(a_i) \right)$$

is inconsistent with  $\xi_n$  and hence gets probability 0 according to  $w_n$ . Hence if  $\lambda_0 = 0$  and only finitely many of the  $\lambda_n$  are non-zero in the representation (9) then  $w$  will not satisfy Reg.

However we do have:

**Theorem 7.** *If  $w$  satisfies Wilmers' Principle and either  $\lambda_0 > 0$  or infinitely many  $\lambda_n$  are non zero in the Ladder Representation of  $w$  then  $w$  satisfies Reg.*

**Proof** We start by considering the case when infinitely many of the  $\lambda_n$  are non-zero.

By the Disjunctive Normal Form Theorem it is enough to show that  $w$  gives non-zero probability to quantifier free sentences of the form

$$\bigwedge_{i=1}^m \bigwedge_{j=1}^k P_j^{\varepsilon_{i,j}}(a_i).$$

Pick  $n$  large (i.e. compared with  $m, k$ ) and such that  $\lambda_n > 0$  and let  $\langle f, g \rangle \in \wp(n)^\infty \times n^\infty$  be such that  $\xi_n$  and Wilmers' Principle hold in  $M_{f,g}$ .

For  $j = 1, \dots, n$  pick distinct  $\vec{\delta}_j = \langle \delta_{1,j}, \delta_{2,j}, \dots, \delta_{n,j} \rangle$  agreeing on the first  $m$  coordinates with  $\langle \varepsilon_{1,j}, \varepsilon_{2,j}, \dots, \varepsilon_{m,j} \rangle$  for  $j \leq k$  and such that the vectors  $\langle \delta_{i,1}, \delta_{i,2}, \dots, \delta_{i,n} \rangle$  are also all distinct (possible since  $n$  is large). From the proof of the converse in Theorem 4  $\text{ran}(f) = \wp(n)$  so for each  $\vec{\delta} = \delta_1, \delta_2, \dots, \delta_n \in \{0, 1\}^n$  there is a  $\kappa(\vec{\delta}) \in \mathbb{N}^+$  such that

$$M_{f,g} \models \bigwedge_{i=1}^n P_{\kappa(\vec{\delta})}^{\delta_i}(e_i).$$

So

$$M_{f,g} \models \bigwedge_{j=1}^k \bigwedge_{i=1}^m P_{\kappa(\vec{\delta}_j)}^{\delta_{i,j}}(e_i),$$

---

<sup>4</sup>Which continue to apply only to first order sentences.

and

$$M_{f,g} \models \exists X_1, \dots, X_n \exists x_1, \dots, x_n \bigwedge_{j=1}^n \bigwedge_{i=1}^n X_j^{\delta_{i,j}}(x_i).$$

Hence from Theorem 4,

$$w_n \left( \exists X_1, \dots, X_n \exists x_1, \dots, x_n \bigwedge_{j=1}^n \bigwedge_{i=1}^n X_j^{\delta_{i,j}}(x_i) \right) = 1.$$

Noticing that by the choice of  $\delta_{i,j}$  these  $X_j$  and  $x_i$  must be distinct, we now obtain by Lemma 2, Ex and Px that

$$w_n \left( \bigwedge_{j=1}^n \bigwedge_{i=1}^n P_j^{\delta_{i,j}}(a_i) \right) > 0, \quad (11)$$

and in turn, as required,

$$w_n \left( \bigwedge_{j=1}^k \bigwedge_{i=1}^m P_j^{\varepsilon_{i,j}}(a_i) \right) > 0.$$

Turning now to the case when  $\lambda_0 > 0$ . Let  $\mu_0$  be the measure for  $w_0$  as given in the proof of Theorem 3, so

$$\mu_0 \left\{ M \in \mathcal{TL} \mid M \models WP \wedge \exists x_1 \dots \exists x_n \bigwedge_{1 \leq i < j \leq n} \exists X \neg (X(x_i) \leftrightarrow X(x_j)) \right\} = 1.$$

Let  $M \in \mathcal{TL}$  be in this measure 1 set, without loss of generality say

$$M \models \bigwedge_{1 \leq i < j \leq n} \exists X \neg (X(a_i) \leftrightarrow X(a_j)).$$

Then since  $M \models WP$ , just as before the sets

$$\{a_i \mid 1 \leq i \leq n, M \models P_j(a_i)\}$$

form a Boolean Algebra which must be all subsets of  $\{a_1, a_2, \dots, a_n\}$ .

The demonstration that Reg holds now proceeds just as in the case above. ■

When infinitely many of the  $\lambda_n$  are non-zero we can improve this conclusion to SReg. First however we need a lemma. In this lemma let  $P_k = P_j$  stand for

$$\forall x (P_k(x) \leftrightarrow P_j(x)).$$

**Lemma 8.** *Let  $k, n \in \mathbb{N}^+$  and suppose that  $w$  satisfies Wilmers' Principle and  $w(\xi_n) = 1$ . Let  $\mu$  be the measure on  $\wp(n)^\infty \times n^\infty$  for  $w$  as in Theorem 4. Then*

$$\mu \left\{ \langle f, g \rangle \mid \exists \text{ distinct } m_1, m_2, \dots, m_k, M_{f,g} \models \bigwedge_{j=1}^k P_1 = P_{m_j} \right\} = 1.$$

**Proof** Suppose not, so

$$\mu \{ \langle f, g \rangle \mid k > |\{m \mid f(m) = f(1)\}| \} > 0.$$

By Invariance under Px the same measure will be given to all the sets

$$\{ \langle f, g \rangle \mid k > |\{m \mid f(m) = f(r)\}| \}$$

for  $r \in \mathbb{N}^+$ . Hence there must be some distinct  $r_1, r_2, \dots, r_{k2^n}$  such that

$$\mu \left\{ \langle f, g \rangle \mid \bigwedge_{s=1}^{k2^n} |\{m \mid f(m) = f(r_s)\}| < k \right\} > 0.$$

But this is impossible since at least  $k$  of the  $f(r_s)$  must be the same subset of  $\{1, 2, \dots, n\}$ . ■

**Theorem 9.** *If  $w$  satisfies Wilmers' Principle and either  $\lambda_0 > 0$  or infinitely many  $\lambda_n$  are non-zero in the Ladder Representation of  $w$  then  $w$  satisfies SReg.*

**Proof** We first give the proof for the case when infinitely many  $\lambda_n$  are non-zero in the Ladder Representation of  $w$ .

Since for this unary language  $L$  any consistent sentence in  $S_1 L$  is logically equivalent to a disjunction of sentences of the form <sup>5</sup>

$$\bigwedge_{i=1}^m P^{\vec{\delta}_i}(a_i) \wedge \bigwedge_{i=1}^s \exists x P^{\vec{\tau}_i}(x) \wedge \bigwedge_{i=1}^q \forall x \neg P^{\vec{\gamma}_i}(x),$$

where  $P^{\vec{\delta}_i}(a_i)$  is short for  $\bigwedge_{j=1}^r P_j^{\delta_{i,j}}(a_i)$  etc., to show that  $w$  satisfies SReg it is enough to show that for a set of  $\langle f, g \rangle \in \wp(n)^\infty \times n^\infty$  with non-zero measure (according to the  $\mu_n$  corresponding to  $w_n$ , where  $\lambda_n > 0$  and  $n$  is large compared to  $m$  and  $r$ ) there is some choice of distinct predicate symbols,  $R_1, \dots, R_r$  say, such that

$$\mu_n \{ \langle f, g \rangle \mid M_{f,g} \models \bigwedge_{i=1}^m R^{\vec{\delta}_i}(a_i) \wedge \bigwedge_{\vec{\delta} \neq \vec{\delta}_1, \dots, \vec{\delta}_m} \forall x \neg R^{\vec{\delta}}(x) \} > 0.$$

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<sup>5</sup>See for example [8, Chapter 10].



Assume that  $\langle 0, 0, \dots, 0 \rangle$  is amongst these  $\vec{\delta}_k$ . [Changing  $\langle 0, 0, \dots, 0 \rangle$  to some other vector  $\vec{\delta}_k$  only requires a minor modification of the proof which follows. Notice that there must be *some*  $\vec{\delta}_k$  otherwise the sentence we are trying to show holds in  $M$  is actually a contradiction!]

By Theorem 7

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \bigwedge_{i=1}^m P^{\vec{\delta}_i}(a_i)\} > 0. \quad (12)$$

Let

$$\psi = \bigwedge_{i=1}^m P^{\vec{\delta}_i}(a_i).$$

From Wilmer's Principle for  $w_n$

$$\mu_n\{\langle f, g \rangle \mid M \models \exists X \forall x (X(x) \leftrightarrow \bigvee_{k=1}^m P^{\vec{\delta}_k}(x))\} = 1$$

so with (12) there is some predicate symbol  $R_{r+1}$  such that

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \psi \wedge \forall x (R_{r+1}(x) \leftrightarrow \bigvee_{k=1}^m P^{\vec{\delta}_k}(x))\} > 0. \quad (13)$$

We now claim that for  $t = 1, \dots, r$  there are distinct predicate symbols  $R_1, \dots, R_t$  such that

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \Phi_t(R_1, R_2, \dots, R_t)\} > 0 \quad (14)$$

where  $\Phi_t(R_1, R_2, \dots, R_t)$  is

$$\psi \wedge \forall x \left( R_{r+1}(x) \leftrightarrow \bigvee_{k=1}^r P^{\vec{\delta}_k}(x) \right) \wedge \bigwedge_{i=1}^t \forall x (R_i(x) \leftrightarrow (R_{r+1}(x) \wedge P_i(x))).$$

To see this suppose that we have such a  $\Phi_t(R_1, R_2, \dots, R_t)$  for some  $t < r$  and distinct  $R_1, \dots, R_t$ . We wish to show it for  $t + 1$ . Since by Wilmer's Principle

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \exists X \forall x (X(x) \leftrightarrow (R_{r+1}(x) \wedge P_{t+1}(x)))\} = 1$$

there is a predicate symbol  $R_{t+1}$  such that

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \zeta_{t+1}(R_1, R_2, \dots, R_t, R_{t+1})\} > 0 \quad (15)$$

where  $\zeta_{t+1}(R_1, R_2, \dots, R_t, R_{t+1})$  is

$$\zeta_t(R_1, R_2, \dots, R_t) \wedge \forall x (R_{t+1}(x) \leftrightarrow (R_{r+1}(x) \wedge P_{t+1}(x)))\} > 0.$$

Unfortunately we have no immediate guarantee that  $R_{t+1}$  differs from  $R_1, \dots, R_t$ .

So suppose that in fact  $R_{t+1} = R_1 \neq R_2, R_3, R_4, \dots, R_t$ . By Px for distinct relation symbols  $S_1, S_2, \dots, S_t$  different from  $P_1, P_2, \dots, P_r, R_1, R_2, \dots, R_t, R_{r+1}$  we have that

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \zeta_{t+1}(R_1, R_2, \dots, R_t, R_{t+1})\} = \mu_n\{\langle f, g \rangle \mid M_{f,g} \models \zeta_{t+1}(S_1, S_2, \dots, S_t, S_1)\} > 0.$$

We can take infinitely many such  $S_1, S_2, \dots, S_t$  so a pair of these must overlap in the sense that their conjunction has non-zero  $\mu_n$  measure. Without loss of generality we may suppose that

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \rho\} > 0$$

where  $\rho$  is

$$\zeta_{t+1}(R_1, R_2, \dots, R_t, R_{t+1}) \wedge \zeta_{t+1}(S_1, S_2, \dots, S_t, S_t).$$

But clearly

$$\rho \models \forall x (R_{t+1}(x) \leftrightarrow (R_{r+1}(x) \wedge P_{t+1}(x))) \wedge \forall x (S_t(x) \leftrightarrow (R_{r+1}(x) \wedge P_{t+1}(x)))$$

so

$$\rho \models \forall x (R_{t+1}(x) \leftrightarrow S_t(x))$$

and in turn

$$\rho \models \zeta_{t+1}(R_1, R_2, \dots, R_t, S_t),$$

so

$$\mu_n\{\langle f, g \rangle \mid M_{f,g} \models \zeta_{t+1}(R_1, R_2, \dots, R_t, S_t)\} \geq \mu_n\{\langle f, g \rangle \mid M_{f,g} \models \rho\} > 0,$$

and the  $R_1, R_2, \dots, R_t, S_t$  now are all different.

Having now found such distinct  $R_1, R_2, \dots, R_r$  let  $\zeta = \zeta_r(R_1, R_2, \dots, R_r)$ , so

$$\mu_n\{\langle f, g \rangle \mid M \models \zeta\} > 0.$$

We now show that

$$\zeta \models \bigwedge_{i=1}^m R^{\vec{\delta}_i}(a_i) \wedge \bigwedge_{\vec{\delta} \neq \vec{\delta}_1, \dots, \vec{\delta}_m} \forall x \neg R^{\vec{\delta}}(x).$$

Since for  $i = 1, \dots, m$

$$\zeta \models \zeta \wedge \psi \models R_{r+1}(a_i) \wedge P^{\vec{\delta}_i}(a_i),$$

it follows that

$$\zeta \models \bigwedge_{i=1}^m R^{\vec{\delta}_i}(a_i),$$

which gets us halfway there.

Now suppose that  $\vec{\delta}$  is not amongst the  $\vec{\delta}_1, \dots, \vec{\delta}_m$ . Notice that by our initial assumption at least one of the coordinates of  $\vec{\delta}$  is 1. In this case

$$\zeta \wedge R^{\vec{\delta}}(x) \models R_{r+1}(x)$$

and since

$$\begin{aligned} \zeta \wedge R_{r+1}(x) &\models P_i(x) \leftrightarrow R_i(x), \\ \zeta \wedge R^{\vec{\delta}}(x) &\models \zeta \wedge R_{r+1}(x) \wedge R^{\vec{\delta}}(x) \models P^{\vec{\delta}}(x) \wedge \bigvee_{k=1}^m P^{\vec{\delta}_k}(x) \models \perp, \end{aligned}$$

which gives that

$$\zeta \models \forall x \neg R^{\vec{\delta}}(x).$$

We now have

$$M_{f,g} \models \bigwedge_{i=1}^m R^{\vec{\delta}_i}(a_i) \wedge \bigwedge_{\vec{\delta} \neq \vec{\delta}_1, \dots, \vec{\delta}_m} \forall x \neg R^{\vec{\delta}}(x). \quad (16)$$

Furthermore all the  $R_1, R_2, \dots, R_r$  are distinct. The result follows.

The corresponding result to Theorem 9 when  $\lambda_0 > 0$  follows similarly. ■

It is worth noting that strengthening Wilmers' Principle to allow the formula  $\vartheta(x)$  to be second order does not provide any correspondingly stronger results than we already have. Firstly Reg is perforce first order and the second order version of SReg trivially cannot be proved because it would require the satisfiable negations of instances of Wilmers' Principle to have non-zero probability in the very presence of Wilmers' Principle!

We have in this note considered one possible second order principle in relation to just one first order consequence i.e. regularity. There are many more such second order principles that one might similarly consider, the only limit being their perceived degree of rationality within the context of Inductive Logic and their interesting first order consequences.

## References

- [1] Boolos, G., To Be is to be a Value of a Variable (or Some Values of Some Variables), *The Journal of Philosophy*, 1984, **81**:436-449
- [2] Carnap, R., A Basic System of Inductive Logic, in *Studies in Inductive Logic and Probability, Volume I*, Eds. R.Carnap & R.C.Jeffrey, University of California Press, 1971, pp33-165.

- [3] Cournot, A.A., *Exposition de la théorie des chances et des probabilités*, L.Hachette, Paris, 1843.
- [4] Gaifman, H., Concerning measures on first order calculi, *Israel Journal of Mathematics*, 1964, **2**:1-18.
- [5] Hintikka, J. & Niiniluoto, I., An Axiomatic Foundation for the Logic of Inductive Generalization, in *Studies in Inductive Logic and Probability, Volume II*, ed. R. C. Jeffrey, University of California Press, 1980, pp157-192
- [6] Johnson, W.E., Probability: The Deductive and Inductive Problems, *Mind*, 1932, **41**:409-423.
- [7] Paris, J.B. *The Uncertain Reasoner's Companion*, Cambridge University Press, 1994.
- [8] Paris, J.B. and Vencovská, A., Pure Inductive Logic, Monograph, to appear
- [9] Paris, J.B. and Vencovská, A., A Note on Irrelevance in Inductive Logic, *Journal of Philosophical Logic*, 2011, **40**:357-370
- [10] Shapiro, S., *Foundations without Foundationalism*, Oxford University Press, 1991