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How chaotic are strange nonchaotic attractors?

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Abstract

We show that the classic examples of quasi-periodically forced maps with strange nonchaotic attractors described by Grebogi *et al* and Herman in the mid-1980s have some chaotic properties. More precisely, we show that these systems exhibit sensitive dependence on initial conditions, both on the whole phase space and restricted to the attractor. The results also remain valid in more general classes of quasiperiodically forced systems. Further, we include an elementary proof of a classic result by Glasner and Weiss on sensitive dependence, and we clarify the structure of the attractor in an example with two-dimensional fibers also introduced by Grebogi *et al*.

1 Introduction

Strange nonchaotic attractors (SNA) are attractors of dynamical systems which have some form of local contraction, but which also have a complicated or fractal structure (hence the word 'strange'). In the context of quasiperiodically forced maps, i.e. maps of the form

$$f(\theta, \xi) = (\theta + \omega, f_\theta(\xi)) \tag{1.1}$$

for $\theta \in \mathbb{T}$, ω irrational, ξ in some suitable metric space M and fiber maps f_θ defined by $f_\theta(\xi) = \pi_2 \circ f(\theta, \xi)$, this notion is used for compact invariant sets which are the topological closure of a non-continuous invariant graph with negative Lyapunov exponents in the fibers (see Remark 3.1 for the precise definition). The negative Lyapunov exponent in the $\xi-$direction[1] provides local contraction and the topological entropy of the system is zero. These two conditions are generally considered sufficient to justify calling the attractors nonchaotic, and the first one ensures that there is local exponential convergence to the attractor in almost all fibers of constant θ. Moreover, many authors remark that this implies that there is no exponential sensitivity to initial conditions. It has also been observed that the existence of a SNA implies that finite time Lyapunov exponents can be positive [20], and that this chaotic-like property is responsible for the lack of smooth invariant curves.

In this paper we consider the chaotic-like properties of SNA in quasiperiodically forced systems in more detail. We focus on the property of sensitive dependence on initial conditions (*sdic*), which

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[1]The Lyapunov exponent in the $\theta-$direction associated to the corresponding invariant measure is always zero.

has been regarded as one of the hallmarks of chaos, and show that many SNA have this property. By *sdic* we mean the classic topological definition as in Devaney [6] which does not impose any conditions on rates of separation. To be precise, the standard definition is as follows.

Definition 1.1
*Let X be a metric space with metric d. A map $g : X \to X$ has sensitive dependence on initial conditions (*sdic*) iff there exists $\varepsilon > 0$ such that for all $\delta > 0$ and $x \in X$ there exists $n \geq 0$ and $y \in X$ depending on x and δ such that $d(x, y) < \delta$ and $d(g^n(x), g^n(y)) > \varepsilon$.*

This was one of the three conditions for chaos introduced by Devaney [6], although it was later shown that it is implied by the other two conditions (transitive and dense periodic orbits) [2, 8] and so, as Glasner and Weiss observe [8], Devaney's definition is too weak to be considered as a good definition of chaotic dynamics. On the other hand it is certainly a feature associated with chaos, and the presence of this property in SNAs emphasizes their position on the cusp between regular and chaotic systems. It is also worth noting that quasiperiodically forced systems cannot be chaotic in the sense of Devaney because since ω is irrational there are no periodic orbits. The definition of *sdic* given above leaves some latitude in the choice of the space X. First of all, it is natural to consider the dynamics restricted to the attractor, thus choosing $X = \mathcal{A}$, and this is investigated in section three. Another obvious choice is to set X to be the whole space on which the system is defined, i.e. $X = \mathbb{T} \times M$, and this will be treated in section four. The difference between these choices is reflected in changes in the set of points in a neighbourhood of any point.

It is one of the most interesting aspects of SNAs that the measure-theoretic and topological point of view often separate, and properties which are generic in the one sense are degenerate in the other and vice versa. For example, it usually makes a great difference whether the measure-theoretic or the topological support of an invariant measure is considered, and there are situations where the former has a very complicated structure while the later is just a smooth torus. In order to fully understand the behavior of quasiperiodically forced maps it is often necessary to combine both viewpoints, and this is explains that while the focus of this paper lies on the topological side, measures will inevitably make an appearance. (On a technical level this happens via the results of Glasner and Weiss [8], for which we included a version of the proof, with a strongly simplified measure-theoretic part in section two.)

One of the most studied classes of SNA arise in pinched skew products. These are systems (1.1) for which there exists at least one value of θ, θ^* say, such that $f(\theta^*, \xi) = 0$ for all $\xi \in M$. In other words, at least one fiber of constant θ is mapped to a single point, the pinched point. These systems include some of the original examples suggested by [10], and are one of the few classes of systems for which it is possible to prove rigorous results about the existence and structure of SNAs [17, 9, 12]. It is not hard to adapt the results of [8] to pinched skew products which satisfy three natural conditions to prove:

> *If $\mathcal{A}$ is the attractor of a pinched skew product $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ satisfying the conditions (3.1) – (3.3) of section three and $\mathcal{A}$ is not a continuous graph then f has* sdic *on $\mathcal{A}$. In particular, if $\mathcal{A}$ is a SNA of a pinched skew product then f has* sdic *on $\mathcal{A}$.*

See Corollary 3.3. The results of Glasner and Weiss [8] give even more information about the structure of points in an SNA of a pinched skew product. A point $x \in X$ is *Lyapunov stable* for the map $g : X \to X$ if for all $\varepsilon > 0$ there exist $\delta > 0$ such that $d(g^n(x), g^n(y)) < \varepsilon$ for all $n \geq 0$ and $y \in X$ with $d(x, y) < \delta$. The existence or nonexistence of Lyapunov stable points effectively determines the dynamics of the pinched skew product.

If $\mathcal{A}$ is the attractor of a pinched skew product which satisfies the conditions (3.1) – (3.3) *of section three then the following are equivalent*

1. *$\mathcal{A}$ contains a Lyapunov stable point;*
2. *$\mathcal{A}$ is a continuous graph;*
3. *$\mathcal{A}$ does not have* sdic.

This can be reinterpreted as saying that $\mathcal{A}$ contains no Lyapunov stable points if and only if $\mathcal{A}$ is strange if and only if $\mathcal{A}$ has *sdic*. These statements are direct consequences of more general results in section three: Corollary 3.3 proves that (2) and (3) are equivalent, and the equivalence of (1) then follows from the dichotomy (2.3). Note that the two results stated above hold for the SNA in the classic example of Grebogi *et al* [10] which has $M = \mathbb{R}$ and

$$f_\theta(\xi) = B\cos 2\pi\theta \tanh \xi \tag{1.2}$$

in (1.1). This has a pinched SNA if $B > 2$ [10, 17].

In section four we turn to the question of *sdic* on the whole phase space, using techniques based only on the dynamics of quasiperiodically forced one-dimensional maps. For the case of pinched skew products, we thus obtain *sdic* on the whole phase space whenever the attractor is not a continuous graph, similar to the results above.

It appears harder to prove the existence of SNA in non-pinched cases. The most prominent and for a long time also the only class of quasiperiodically forced systems where this was possible are quasiperiodic matrix cocycles [11], with quasiperiodic Schrödinger cocycles as a special case. Only recently more general approaches have been developed which are at least in principle applicable to a much broader class of systems ([3, 5, 15]) and thus confirm the strong numerical evidence for the widespread existence of SNA in quasiperiodically forced maps. The application of our results to these examples is discussed in more detail in sections three and four. One important concept in this context is the rotation number of a quasiperiodically forced circle homeomorphism, which Herman has shown to exist in [11] and which has been investigated further by many authors. In section 4 we discuss convergence properties of the rotation number and their implications for *sdic*.

Finally, in section 5 we return to the second example from the original paper of Grebogi et al [10], which does not appear to have been considered further in the literature so far. This system has two-dimensional fibers and a non-pinched attractor, such that our previous results do not apply directly. But after passing to projective polar coordinates in the fibers we are able to clarify the structure of the attractor and to relate its dynamics to those of a matrix cocycle, which makes it possible to prove *sdic* both on the attractor and on the whole phase space. See Figure 1.

The existence of *sdic* in SNAs will not come as a complete surprise to the more applied community, although there has clearly been some confusion. Pikovsky and Feudel [20] define a quantity which measures local separation due to changes in θ for a given orbit which is a function of the orbit and the number of iterates, N, on which separation is considered.

Since SNA have non-positive Lyapunov exponents their measure of separation cannot increase exponentially, but their careful numerical experiments suggest that the maximum separation over N iterates grows roughly linearly with N (to be more precise, their experiments give a growth rate of N^μ with $\mu \approx 0.97$ [20]). In some sense our results can be seen as confirming that their phase sensitivity exponent reflects *sdic* in the system. Of course, for forced differential equations this implies sensitive dependence with respect to small changes in the initial time of a solution as well as with respect to the phase space.

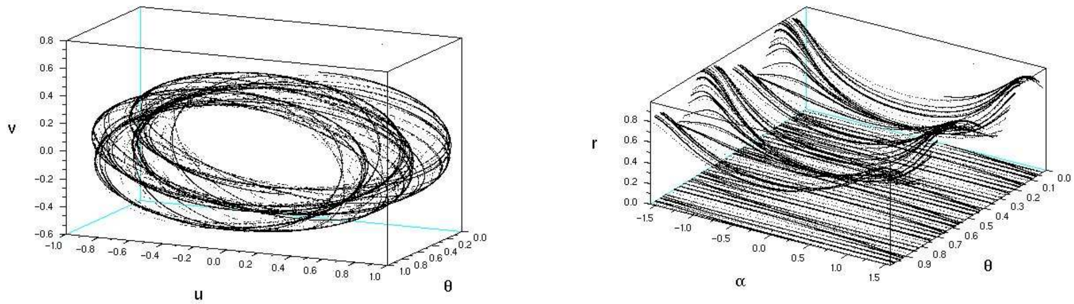

Figure 1: The figure to the left shows the attractor of the map $\Lambda : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$, $\Lambda(\theta, u, v) := \left(\theta + \omega, \frac{\beta}{1+u2+v2} \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} R_\theta \begin{pmatrix} u \\ v \end{pmatrix}\right)$ where R_θ is the rotation matrix with angle $2\pi\theta$. The figure to the right shows the same attractor when projective polar coordinates (α, r) are used in the fibers. It is plotted together with its projection to the (θ, α)-plane, which arises as the attractor of a quasiperiodically matrix cocycle of the type discussed by Herman. Details are given in Section 5.

Notation: We reserve the letter f to denote quasiperiodic maps of the form (1.1), often with the choice of M fixed to be $\mathbb{R}$ or $\mathbb{T}$. General maps of a metric space X will usually be denoted by g (as in this introduction). In particular, the results of section two are all in terms of such general maps g.

2 Sensitivity and equicontinuity

Let $g : X \to X$ be a continuous map of a compact metric space (X, d) which has no isolated points. For $x \in X$ and $r > 0$ let $B_r(x) := \{y \in X : d(x, y) \leq r\}$ and denote by $\overline{O(x)}$ the closure of the orbit $\{g^n(x) : n \in \mathbb{N}\}$ of x. The set of *transitive* points, i.e. the set of points x for which $\overline{O(x)} = X$ is denoted by Tr. If $Tr \neq \emptyset$ one says that g is transitive.

We are interested in the *sensitive* and in the *Lyapunov stable* points of g. To this end we introduce, for each $\epsilon > 0$, the two sets

$$SD_\epsilon := \{x \in X : \forall \delta > 0\, \exists y, z \in B_\delta(x)\, \exists n \in \mathbb{N} \text{ s.t. } d(g^n y, g^n z) \geq \epsilon\} \tag{2.1}$$

$$LS_\epsilon := \{x \in X : \exists \delta = \delta(\epsilon, x) > 0\, \forall y, z \in B_\delta(x)\, \forall n \in \mathbb{N} : d(g^n y, g^n z) < \epsilon\} \tag{2.2}$$

Evidently $LS_\epsilon = X \setminus SD_\epsilon$. Let

$$LS := \bigcap_{\epsilon > 0} LS_\epsilon \text{ and } SD := \bigcup_{\epsilon > 0} SD_\epsilon \text{ , so } LS = X \setminus SD \text{ .}$$

LS is the set of *Lyapunov stable* points, SD that of *sensitive* points. One says that the map g has *sensitive dependence*, if $SD_\epsilon = X$ for some $\epsilon > 0$.[2] In that case, each point of X is sensitive, but the converse is not necessarily true. However, it follows immediately from these definitions that each SD_ϵ is closed and forward invariant under g. Therefore, if SD contains a transitive point x, then $X = \overline{O(x)} \subseteq SD_\epsilon$ for some $\epsilon > 0$.

[2] It is easily seen that this definition is equivalent to the one given in Def 1 in the introduction.

On the other hand, if no sensitive point is transitive, i.e. if $Tr \subseteq LS$, and if $Tr \neq \emptyset$, then actually $Tr = LS$.[3] Hence we have the following dichotomy for transitive systems:[4]

$$\text{If } g \text{ is transitive, then either } g \text{ has sensitive dependence or } Tr = LS. \tag{2.3}$$

Note that if $LS = X$, then the family $(g^n)_{n\in\mathbb{N}}$ is actually *equicontinuous*, i.e.

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall y, z \in X\, \forall n \in \mathbb{N}: \ d(y,z) \leq \delta \Rightarrow d(g^n y, g^n z) \leq \epsilon\ . \tag{2.4}$$

This is an immediate consequence of the compactness of X. So, if g is minimal (i.e. if $Tr = X$), then either g has sensitive dependence or $(g^n)_{n\in\mathbb{N}}$ is equicontinuous, see e.g. [1] where these and related questions are treated systematically. Glasner and Weiss [8] showed that this dichotomy remains true if the assumption of minimality is replaced by the weaker one that f is transitive and admits a finite invariant measure with full topological support.[5] In the rest of this section we will rederive this result with a completely elementary self-contained proof that does neither use Birkhoff's ergodic theorem nor any knowledge about syndetic sets as does the proof in [8].

Recall that a point x is *nonwandering*, if

$$\forall \delta > 0\, \exists n > 0 \text{ s.t. } B_\delta(x) \cap g^n(B_\delta(x)) \neq \emptyset\ .$$

It follows at once that, if x is nonwandering, then for each $\delta > 0$ the set

$$R_\delta(x) := \{n > 0 : B_\delta(x) \cap g^n(B_\delta(x)) \neq \emptyset\} \quad \text{is infinite.}$$

Observe also that, by definition, each transitive point is nonwandering. Recall further that $g : X \to X$ is called *uniformly rigid* if there exists a sequence n_k of integers going to infinity, such that g^{n_k} converges uniformly to the identity map on X. Obviously, every uniformly rigid map must be a homeomorphism.

Lemma 2.1
If x is a nonwandering Lyapunov point with $\delta = \delta(\epsilon/2, x)$ as in (2.2), then

$$\forall \epsilon > 0\, \forall n \in R_{\delta(\epsilon/2,x)}(x)\, \forall y \in \overline{O(x)}: \ d(g^n y, y) \leq \epsilon\ .$$

In particular, $g|_{\overline{O(x)}}$ is uniformly rigid.

Proof: Let $B = B_{\delta(\epsilon/2,x)}$ and $n \in R_{\delta(\epsilon/2,x)}(x)$. Let $u \in B \cap g^n B$, $v \in B \cap g^{-n}\{u\}$. Then $v, g^n(v) \in B = B_{\delta(\epsilon/2,x)}$ so that, for all $k \in \mathbb{N}$,

$$d(g^{k+n}x, g^k x) \leq d(g^{k+n}x, g^{k+n}v) + d(g^k(g^n v), g^k x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ .$$

Hence $d(f^n y, y) \leq \epsilon$ for all $y \in \overline{O(x)}$. q.e.d.

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[3] Let $y \in Tr$, $x \in LS_\epsilon$ for some $\epsilon > 0$, and let $z \in X$ be arbitrary. There is $k \in \mathbb{N}$ such that $d(g^k y, x) < \delta(\frac{\epsilon}{2}, x)$, δ as in (2.2). As $X = \overline{O(y)}$ has no isolated point, there is $n > k$ such that $d(g^n y, z) < \frac{\epsilon}{2}$. Hence $d(g^{n-k}x, z) \leq d(g^{n-k}x, g^{n-k}(g^k y)) + d(g^n y, z) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore $z \in \overline{O(x)}$.

[4] Note that in transitive systems the set Tr is *residual*, i.e. it contains a countable intersection of dense open sets. In particular it is dense by Baire's category theorem.

[5] They also provide examples showing that transitivity alone is not sufficient for the dichotomy.

To get more out of this one needs to control the sets $R_\delta(x)$ in the previous lemma. More precisely, one needs to make sure that $R_\delta(x)$ is *syndetic*, i.e. has bounded gaps:

$$\exists s \in \mathbb{N}\, \forall n \in \mathbb{N}: \ [n, n+s] \cap R_\delta(x) \neq \emptyset \,. \tag{2.5}$$

In this case we say that the gaps are bounded by s.

Lemma 2.2
If x is a nonwandering Lyapunov point for which all sets $R_\delta(x)$ ($\delta > 0$) have bounded gaps, then the family $(g^n|_{\overline{O(x)}})_{n\in\mathbb{N}}$ is equicontinuous.

Proof: Let $\epsilon > 0$ and choose $\delta = \delta(\epsilon/6, x)$ as in (2.2). We may assume that $\delta \in (0, \epsilon/3)$. Let the gaps of $R_\delta(x)$ be bounded by s. As all f^j are continuous, there is some $\eta \in (0, \delta]$ such that

$$\forall y, z \in X\, \forall j \in \{0, \dots, s\}: \ d(y,z) \leq \eta \Rightarrow d(g^j y, g^j z) < \epsilon/3 \,.$$

Let $y, z \in \overline{O(x)}$. Any n can be written as $n = k + j$ with $k \in R_\delta(x)$ and $j \in \{0, \dots, s\}$. Hence, if $d(y,z) \leq \eta$ then, by Lemma 2.1,

$$d(g^n y, g^n z) \leq d(g^k(g^j y), g^j y) + d(g^j y, g^j z) + d(g^j z, g^k(g^j z)) < \epsilon \,.$$

q.e.d.

It remains to give a condition ensuring that the sets $R_\delta(x)$ have bounded gaps. Since this is a kind of uniform recurrence condition, the following lemma is not too surprising.

Lemma 2.3
If x is a nonwandering Lyapunov point which belongs to the topological support of a g-invariant finite measure μ on X[6]*, then $R_\delta(x)$ has bounded gaps for each $\delta > 0$.*

Proof: As μ has full topological support, the ball $B := B_\delta(x)$ has positive μ-measure for every $\delta > 0$. Denote $\hat{B} := \bigcup_{j=0}^{\infty} g^{-j} B$. Trivially, $B \subseteq \hat{B}$, and, by σ-additivity of the measure μ, there is $s \in \mathbb{N}$ such that $\mu(\bigcup_{j=0}^{s} g^{-j} B) > \mu(\hat{B}) - \mu(B)$. As μ is invariant under g this implies, for all $r \in \mathbb{N}$,

$$\mu(B) + \mu\left(\bigcup_{j=0}^{s} g^{-r-j} B\right) = \mu(B) + \mu\left(\bigcup_{j=0}^{s} g^{-j} B\right) > \mu(\hat{B})$$

But $B \cup \bigcup_{j=0}^{s} g^{-r-j} B \subseteq \hat{B}$ so that $B \cap \bigcup_{j=0}^{s} g^{-r-j} B \neq \emptyset$. We conclude that for each $r \in \mathbb{N}$ there is $j_r \in \{0, \dots, s\}$ such that $B \cap g^{-r-j_r} B \neq \emptyset$ and hence $B \cap g^{r+j_r} B \neq \emptyset$. So $R_\delta(x)$ has gaps bounded by s. q.e.d.

Combining the last two lemmas with the elementary dichotomy (2.3) we arrive at the following conclusion which is essentially Proposition 1 in [8].

Proposition 2.4 ([8])
Suppose that X has no isolated points and that g is transitive and has a finite invariant measure with full topological support. Then either

[6]This means $\mu(U) > 0$ for each open neighbourhood U of x.

- *g has sensitive dependence; or*
- *the family $(g^n)_{n\in\mathbb{N}}$ is equicontinuous and $g : X \to X$ is a minimal homeomorphism. In this case also the family $(g^n)_{n\in\mathbb{Z}}$ is equicontinuous.*

Observe that in the second case g is uniquely ergodic.[7]

Proof: Let $x \in X$ be a transitive (and hence nonwandering) point and suppose that g does not have sensitive dependence. In view of the dichotomy (2.3), x is a Lyapunov point. Therefore the equicontinuity follows from Lemmas 2.2 and 2.3. g is minimal, i.e. $Tr = X$, because $X = LS$ by equicontinuity and $LS = Tr$ by (2.3). Further, g is uniformly rigid by Lemma 2.1, such that it must be a homeomorphism. Finally, the equicontinuity of $(g^n)_{n\in\mathbb{Z}}$ follows again from the uniform rigidity of g: The iterates of any two points cannot come arbitrarily close to each other, as they become seperated again when g^n is sufficiently close to the identity. But this implies the equicontinuity of the backwards iterates. q.e.d.

3 Sensitivity on SNAs

Now let $X = \mathbb{T} \times M$, M a metric space, and let $f : X \to X$ be a continuous quasiperiodically forced map. We assume throughout this section that

- ▷ $\mathcal{A}$ is a compact f-invariant subset of X which has no isolated points, (3.1)
- ▷ that $f|_{\mathcal{A}}$ is transitive, and (3.2)
- ▷ that $\mathcal{A}$ is the topological support of a finite f-invariant measure. (3.3)

Remark 3.1 *An* invariant graph *is usually defined as a measurable function $\varphi : \mathbb{T} \to M$ that satisfies*

$$f_\theta(\varphi(\theta)) \;=\; \varphi(\theta+\omega) \qquad \forall \theta \in \mathbb{T}\ ,$$

with Lyapunov exponent $\lambda(\varphi) = \int_{\mathbb{T}} \log|f'_\theta(\varphi(\theta))|\ d\theta$ in case the fiber maps are one-dimensional and differentiable.

However, there is a subtle issue concerning this defintion which we do not want to treat systematically here (this is done e.g. in [12] at the end of section two), but nevertheless feel obliged to mention: We do not want to distinguish between two invariant graphs which coincide Lebesgue-a.e., and in particular we do not want to call an invariant graph non-continuous if it is Lebesgue-a.s. equal to a continuous one. Thus we will implicitely consider an invariant graph to be an equivalence class of Lebesgue-a.s. equal graphs, and by the topological closure of such an equivalence class we mean the smallest compact set that can be obtained as the topological closure of a representitive. This set coincides with the topological support of the measure μ_φ, which is obtained by projecting the Lebesgue measure onto the invariant graph (here it does not matter which representative is chosen). Note that μ_φ is ergodic w.r.t. f.

If we now call the topological closure (in the above sense) of an invariant graph with negative Lyapunov exponent a SNA, then it becomes clear that this satisfies the assumptions (3.1)– (3.3).

[7] This is well known and follows from the fact that ergodic limits of continuous functions are continuous (by equicontinuity) and hence constant (by transitivity).

For any set $\mathcal{A} \subseteq X$, we denote by $\mathcal{A}_\theta$ its intersection with the θ-fiber, more precisely $\mathcal{A}_\theta := \{\xi \in M \mid (\theta,\xi) \in \mathcal{A}\}$. The following concept turned out to be very important in the study of quasiperiodically forced maps (see: [23]):

Definition 3.2
$\mathcal{A} \subseteq X$ is called pinched*, if for some $\theta \in \mathbb{T}$ the set $\mathcal{A}_\theta$ consists only of a single point. In this case we call $\mathcal{A}$ pinched at θ.*

Obviously, if $\mathcal{A}$ is invariant and pinched, then it is pinched on a whole dense set, namely on the forward orbit of a pinched fiber. If in addition $\mathcal{A}$ is compact then the set of θ at which $\mathcal{A}$ is pinched is even residual. This follows quite easily from a Baire argument, as in this case all sets $B_n := \{\theta \in \mathbb{T} \mid \mathrm{diam}(\mathcal{A}_\theta) < \frac{1}{n}\}$ are open and dense, and their intersection gives exactly the set of θ where $\mathcal{A}$ is pinched.

The next two results follow from the more general results of section two.

Corollary 3.3
Suppose $\mathcal{A}$ satisfies assumptions (3.1) – (3.3)*. If $\mathcal{A}$ is pinched, then either*

- *$f|_\mathcal{A}$ has sensitive dependence; or*
- *$\mathcal{A}$ is the graph of a continuous function.*

Proof: If $f|_\mathcal{A}$ does not have sensitive dependence, then $(f^n|_\mathcal{A})_{n\in\mathbb{N}}$ is equicontinuous. Let $\epsilon > 0$. There is $\delta = \delta(\epsilon) > 0$ such that for all fibers $\mathcal{A}_\theta$ of diameter less than δ all their images $f^n\mathcal{A}_\theta = \mathcal{A}_{\theta+n\omega}$ have diameter less than ϵ. As the set of all θ with $\mathrm{diam}(\mathcal{A}_\theta) < \delta$ is open and nonempty, the minimality of the rotation $\theta \mapsto \theta + \omega$ implies that all fibers have diameter less than ϵ. As $\epsilon > 0$ was arbitrary, it follows that $\mathcal{A}$ is the graph of a function $\psi : \mathbb{T} \to M$. As $\mathcal{A}$ is compact, the function ψ is continuous. q.e.d.

Sometimes the case where $\mathcal{A}$ is not necessarily pinched can also be dealt with easily. Here is an example.

Corollary 3.4
Suppose that M is a compact interval and that all fiber maps are monotone increasing and let $\mathcal{A}$ be as before. Then either

- *$f|_\mathcal{A}$ has sensitive dependence; or*
- *$\mathcal{A}$ is the graph of a continuous function.*

Proof: Suppose that $f|_\mathcal{A}$ does not have sensitive dependence so that $(f^n|_\mathcal{A})_{n\in\mathbb{N}}$ is equicontinuous and minimal by Corollary 3.3. Let $\mathcal{A}^+ := \{(\theta, \sup \mathcal{A}_\theta) : \theta \in \mathbb{T}\}$, $\mathcal{A}^- := \{(\theta, \inf \mathcal{A}_\theta) : \theta \in \mathbb{T}\}$. As f has monotone fiber maps, both, $\mathcal{A}^+$ and $\mathcal{A}^-$ are f-invariant subsets of X. Hence, by minimality of f, $\overline{\mathcal{A}^+} = X = \overline{\mathcal{A}^-}$. But this implies that $\mathcal{A}$ is pinched[8] so that Corollary 3.3 applies again. q.e.d.

Also in more delicate situations the dichotomy of Proposition 2.4 can be useful.

Proposition 3.5
Suppose that $M = \mathbb{T}$ and that all fiber maps are orientation preserving circle homeomorphisms. Let $\mathcal{A}$ be a proper subset of X which satisfies assumptions (3.1) – (3.3)*. Then either*

[8] See [7, Lemma 4.3(i)] for the elementary proof.

- $f|_{\mathcal{A}}$ *has sensitive dependence; or*

- $\mathcal{A}$ *is the disjoint union of a finite number of disjoint curves which are cyclically permuted by the action of* f.

Information on situations where $\mathcal{A} = X$ will be provided in section 4.

Proof: Suppose that $f|_{\mathcal{A}}$ does not have sensitive dependence. Then the family $(f^n|_{\mathcal{A}})_{n\in\mathbb{Z}}$ is equicontinuous and $f|_{\mathcal{A}}$ is a minimal homeomorphism by Proposition 2.4.

We introduce some more notation: Let $f_\theta^n(\xi) := \pi_2 \circ f^n(\theta, \xi)$. Further, for $\theta \in \mathbb{T}$ let $\mathcal{J}_\theta$ be the family of all connected components of $\mathbb{T} \setminus \mathcal{A}_\theta$. So each $J \in \mathcal{J}_\theta$ is a maximal interval in the complement of $\mathcal{A}_\theta$. Note that $f_\theta^n J \in \mathcal{J}_{\theta+n\omega}$ if and only if $J \in \mathcal{J}_\theta$.

For $\theta \in \mathbb{T}$ and $J \in \mathcal{J}_\theta$ let $s(J) := \sup_{n\in\mathbb{Z}} |f_\theta^n J|$. As the endpoints of such intervals J belong to $\mathcal{A}$ and as the family $(f^n|_{\mathcal{A}})_{n\in\mathbb{Z}}$ is equicontinuous, there is an increasing function $\delta : (0,1] \to (0,1]$ such that $|J| \geq \delta(t)$ whenever $s(J) \geq t > 0$.

For $t \geq 0$ let

$$N_t(\theta) := \begin{cases} \operatorname{card}\{J \in \mathcal{J}_\theta : s(J) \geq t\} & \text{if } t > 0 \\ +\infty & \text{if } t = 0\ . \end{cases}$$

Note that for any $t > 0$ there holds $0 \leq N_t \leq \delta(t)^{-1}$ and that, for each fixed θ, $t \mapsto N_t(\theta)$ is a decreasing function continuous from the left. As $s(f_\theta J) = s(J)$ for all $J \in \mathcal{J}_\theta$, we see that $N_t(\theta + \omega) = N_t(\theta)$.

Next, for $p = 1, 2, 3, \ldots$, let

$$\gamma_p(\theta) := \sup\{t \in \mathbb{R} : N_t(\theta) \geq p\}\ .$$

Obviously $0 \leq \gamma_p \leq 1$ and $\gamma_p(\theta + \omega) = \gamma_p(\theta)$. Further, as $t \mapsto N_t(\theta)$ is continuous from the left we have $N_{\gamma_p(\theta)}(\theta) \geq p$.

The function $\gamma_1 : \mathbb{T} \to \mathbb{R}$ plays a special role: Observe first that

$$\gamma_1(\theta) = \sup_{n\in\mathbb{Z}} \ell(\theta + n\omega) \quad \text{where} \quad \ell(\theta) := \max\{|J| : J \in \mathcal{J}_\theta\}\ .$$

As $\mathcal{A} \subseteq X$ is closed, the function $\ell : \mathbb{T} \to \mathbb{R}$ is lower semicontinuous and so are the functions $\theta \mapsto \ell(\theta + n\omega)$. Hence, as a supremum of lower semicontinuous functions, also γ_1 is lower semicontinuous, and as γ_1 is invariant under rotation by the irrational ω, it must be constant. As $\mathcal{A} \neq X$ by assumption, we have $\gamma_1 > 0$. We turn to the other functions $\gamma_p : \mathbb{T} \to [0,1]$. Let $\delta_0 := \delta(\frac{\gamma_1}{2}) \leq \frac{\gamma_1}{2}$.

Claim: The sets $\{\gamma_p \leq c\}$ are closed for all $c \in (\gamma_1 - \delta_0, \gamma_1]$.

Indeed, consider a sequence of $\theta_k \in \{\gamma_p \leq c\}$ which converge to some $\theta \in \mathbb{T}$. Let $t \in (c, \gamma_1]$ and denote $q := N_t(\theta)$. Then $q \geq 1$ and there are pairwise disjoint $J_1, \ldots, J_q \in \mathcal{J}_\theta$ with $s(J_i) \geq t$ for all i. Consider compact subintervals $\hat{J}_i \subset J_i$. As $X \setminus \mathcal{A}$ is open there are, for all sufficiently large $k \in \mathbb{N}$, intervals $J_1^k, \ldots, J_q^k \in \mathcal{J}_{\theta_k}$ such that $\hat{J}_i \subset J_i^k$ for $i = 1, \ldots, q$. Since this holds for all choices of the compact subintervals $\hat{J}_i$, we conclude

$$\liminf_{k\to\infty} |f_\theta^n J_i^k| \geq |f_\theta^n J_i| \qquad \text{for all } i = 1, \ldots, q \text{ and all } n \in \mathbb{N}$$

and therefore

$$\liminf_{k\to\infty} s(J_i^k) \geq s(J_i) \geq t > c \qquad \text{for all } i = 1, \ldots, q.$$

This allows the conclusion $N_t(\theta) = q \le \liminf_{k\to\infty} N_u(\theta_k) \le p-1$ for all $t > u > c$ so that $\gamma_p(\theta) \le c$ – and thus proves the claim – once we have shown that the intervals $J_1^k, \dots, J_q^k$ are pairwise distinct for large k. Suppose for a contradiction that this is not the case. Then, w.l.o.g., $J_1^k = J_2^k$ for infinitely many k. Hence $\hat{J}_1 \cup \hat{J}_2 \subseteq J_1^k$ for infinitely many k and so

$$\gamma_1 \ge \limsup_{k\to\infty} |f_\theta^n J_1^k| \ge |f_\theta^n \hat{J}_1| + |f_\theta^n \hat{J}_2| \quad \text{for all } n \in \mathbb{Z}.$$

Again this holds for all choices of the compact subintervals $\hat{J}_1, \hat{J}_2$, so that

$$\gamma_1 \ge |f_\theta^n J_1| + |f_\theta^n J_2| \quad \text{for all } n \in \mathbb{Z}.$$

But $s(f_\theta^n J_2) = s(J_2) \ge t > \gamma_1 - \delta_0 \ge \frac{\gamma_1}{2}$, whence $|f_\theta^n J_2| \ge \delta(\frac{\gamma_1}{2}) = \delta_0$. This yields the contradiction

$$\gamma_1 \ge s(J_1) + \delta_0 \ge t + \delta_0 > c + \delta_0 > \gamma_1$$

and finishes the proof of the claim.

We can summarize that for $c \in (\gamma_1 - \delta_0, \gamma_1]$ the closed sets $\{\gamma_p \le c\}$ are invariant under rotation by the irrational ω, so for all these c and for all $p \in \{1, 2, 3, \dots\}$, either $\{\gamma_p \le c\} = \emptyset$ or $\{\gamma_p \le c\} = \mathbb{T}$. Let $q := \max\{p \in \mathbb{N} : \exists \theta \in \mathbb{T} \text{ s.t. } \gamma_p(\theta) = \gamma_1\}$. Then $\gamma_q = \gamma_1$ is constant and there is $\eta > 0$ such that $\gamma_{q+1} \le \gamma_q - \eta$. This means that for each θ there are q intervals $J \in \mathcal{J}_\theta$ with $s(J) = \gamma_1$ and the s-value of all other intervals is at most $\gamma_1 - \eta$.

Now we are ready to finish the proof of the proposition. Let $E_\theta^\pm$ be the sets of the q "upper" respectively "lower" endpoints of those intervals $J \in \mathcal{J}_\theta$ with $s(J) = \gamma_1$. We show that the set valued maps $\theta \mapsto E_\theta^+$ and $\theta \mapsto E_\theta^-$ are continuous: As above consider a sequence of $\theta_k \in \mathbb{T}$ which converge to some $\theta \in \mathbb{T}$. There are pairwise disjoint intervals $J_1, \dots, J_q \in \mathcal{J}_\theta$ with $s(J_i) = \gamma_1$ for all i. Consider compact subintervals $\hat{J}_i \subset J_i$. As $X \setminus \mathcal{A}$ is open there are, for all sufficiently large $k \in \mathbb{N}$, intervals $J_1^k, \dots, J_q^k \in \mathcal{J}_{\theta_k}$ such that $\hat{J}_i \subset J_i^k$ for $i = 1, \dots, q$. Now let $\epsilon \in (0, \eta)$ and choose $\delta \in (0, \eta)$ as in the definition of equicontinuity (2.4). As $s(J_i) = \gamma_1$ for all i, there are $n_1, \dots, n_q \in \mathbb{Z}$ such that $|f^{n_i} J_i| \ge \gamma_1 - \frac{\delta}{2}$, while $|f^{n_i} J_i|, |f^{n_i} J_i^k| \le \gamma_1$ for all i and k by definition of γ_1. We can choose the intervals $\hat{J}_i \subset J_i$ such that $|f^{n_i} \hat{J}_i| > \gamma_1 - \delta$. So also $|f^{n_i} J_i^k| > \gamma_1 - \delta$ for large k. This has two implications which together yield the continuity of the set valued maps: $s(J_i^k) > \gamma_1 - \eta$ and hence $s(J_1^k) = \dots = s(J_q^k) = \gamma_1$ for large k, and second, the corresponding endpoints of the intervals $f^{n_i} J_i$ and $f^{n_i} J_i^k$ have distance less than δ, so that the corresponding endpoints of the intervals J_i and J_i^k have distance at most ϵ.

The graphs of the maps $\theta \mapsto E_\theta^+$ and $\theta \mapsto E_\theta^-$ are thus closed invariant subsets of $\mathcal{A}$ so that, by minimality of $f|_\mathcal{A}$, both graphs are identical and coincide with $\mathcal{A}$. It follows that $\text{card}(\mathcal{A}_\theta) = \gamma_1$ and $\inf\{d(\xi, \zeta) : \xi, \zeta \in \mathcal{A}_\theta, \xi \ne \zeta\} \ge \delta(\gamma_1) > 0$. From this the second alternative of the proposition follows. q.e.d.

Remark 3.6 *The results of this section remain valid if the rotation $\theta \mapsto \theta + \omega$ which forces the system is replaced by any transitive continuous map R on a compact metric space Z which has no isolated points and which admits a finite invariant measure with full topological support. Indeed, if the forced system does not have* sdic*, then also $R : Z \to Z$ does not have* sdic *so that R is a minimal homeomorphism of Z and $(R^n)_{n\in\mathbb{Z}}$ is equicontinuous, see Proposition 2.4. But only these two properties of the rotation, minimality and equicontinuity, were used in the proofs of this section, so the proofs carry over without changes to the more general R.*

4 Sensitivity on the whole phase space

In this section we turn to the question of sensitive dependence on the whole phase space. In order to do so, we restrict to two classes of quasiperiodically forced systems, namely quasiperiodically forced circle homeomorphisms and quasiperiodically forced monotone interval map.

As in the last section, we will use the notation $f_\theta^n(\xi) := \pi_2 \circ f^n(\theta, \xi)$. We say f is a *quasiperiodically forced circle homeomorphism* if $M = \mathbb{T}$ and each fiber map f_θ is a homeomorphism of the circle. By F we will denote a continuous lift of f to $\mathbb{T} \times \mathbb{R}$. First of all, the case where f is not homotopic to the identity can be treated quite easily:

Proposition 4.1
Suppose f is a quasiperiodically forced circle homeomorphism which is not homotopic to the identity. Then f has sdic *on $\mathbb{T}^2$.*

Proof:
As f is not homotopic to the identity, it is transitive (see [18]), such that we can apply Lemma 2.1 to see that either f has *sdic* on $\mathbb{T}^2$ or f is uniformly rigid. Suppose f is uniformly rigid. Then f^n is arbitrarily close to the identity for infinitly many $n \in \mathbb{N}$, and in particular the image of a constant line $\Gamma = \mathbb{T} \times \{\xi\}$ is mapped arbitrarily close to itself by f^n. However, this would imply that Γ and $f^n(\Gamma)$ are in the same homotopy class, contradicting the fact that f is not homotopic to the identity. Therefore f must have *sdic*.

q.e.d.

Now suppose f is homotopic to the identity. In this case, Herman showed in [11] that similar to the unforced case the limit

$$\rho_F := \lim_{n\to\infty} \frac{1}{n}(F_\theta^n(\xi) - \xi) \tag{4.1}$$

exists for any continuous lift F of f and is independent of θ and ξ. Further, $\rho_f := \rho_F \bmod 1$ does not depend on the choice of the lift F. However, unlike unforced circle homeomorphisms the so-called *deviations from the constant rotation*

$$|F_\theta^n(\xi) - \xi - n\rho_F| \ , \tag{4.2}$$

need not be bounded uniformly in θ, ξ and n, and in fact an important distinction can be made with respect to this: f is called ρ-bounded if the quantities in (4.2) are uniformly bounded and ρ-unbounded otherwise. If the systems is ρ-bounded, the dynamics can be understood quite easily: In this case an analogue to Poincaré's famous classification of the dynamics of circle homeomorphisms holds, such that the system is either semi-conjugate to an irrational translation of the torus and ρ_f is not rationally related to the rotation number ω on the base, or there exists an invariant strip, which is the suitable analogue for a fixed or periodic point in this setting (see [16]), and the rotation numbers ρ_f and ω are rationally related.

The more interesting case, which does not occur in the one-dimensional situation and which we will consider in the following, is the ρ-unbounded one. Here neither of the two above alternatives can occur, the system is always topologically transitive (see [16]), and as we will see below it also has *sdic* on the whole phase space. However, before we can show this we need the following statement, which is contained in [24]:

Lemma 4.2
Suppose F is the lift of a quasiperiodically forced circle homeomorphism homotopic to the identity which is ρ-unbounded. Then there exists a residual set of θ such that the deviations

$$F^n_\theta(\xi) - \xi - n\rho_F \tag{4.3}$$

are unbounded both above and below (independent of ξ), but at the same time there exist two disjoint dense sets of θ such that the deviations (4.3) are bounded uniformly from above, respectively below. (However, there exists no orbit on which the deviations are bounded both above and below.)

Now we can prove the following:

Proposition 4.3
Suppose f is a quasiperiodically forced circle homeomorphism, homotopic to the identity, which is ρ-unbounded. Then f has sdic *on $\mathbb{T}^2$.*

Proof:
Let $F : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ be a lift of f and choose $\epsilon \in (0, \frac{1}{4})$ such that $d(\tilde{x}, \tilde{y}) < \epsilon$ implies $d(F(\tilde{x}), F(\tilde{y})) < \frac{1}{4}$ for all $\tilde{x}, \tilde{y} \in \mathbb{T}\times\mathbb{R}$. Note that $d(\tilde{x}, \tilde{y}) < \frac{1}{4}$ implies $d(\pi(\tilde{x}), \pi(\tilde{y})) = d(\tilde{x}, \tilde{y})$. We will now show that f is ϵ-sensitive on $\mathbb{T}^2$, that is $SD_\epsilon = \mathbb{T}^2$.

To that end, choose any $x \in \mathbb{T}^2$ and $\delta > 0$. Let $\tilde{x} \in \mathbb{T}\times\mathbb{R}$ be a lift of x, i.e. $\pi(\tilde{x}) = x$. As both the fibers which are ρ-bounded above and those which are ρ-unbounded above are dense, we can find both a point $\tilde{y}$ which is ρ-bounded above and a point $\tilde{z}$ which is ρ-unbounded above in $B_\delta(\tilde{x})$. As $\sup_{n\in\mathbb{N}} d(F^n\tilde{y}, F^n\tilde{z}) = \infty$ and due to the choice of ϵ, this means that for some $m \in \mathbb{N}$ we must have $d(F^m\tilde{y}, F^m\tilde{z}) \in [\epsilon, \frac{1}{4})$. Let $y = \pi(\tilde{y})$ and $z = \pi(\tilde{z})$. Then $d(F^m y, F^m z) \in [\epsilon, \frac{1}{4})$ as well, and as $y, z \in B_\delta(x)$ this completes the proof.

q.e.d.

It is this proposition, which applies to Herman's examples mentioned in the introduction. In [11] Herman studies $\mathrm{SL}(2,\mathbb{R})$-cocycles over irrational rotations, that is mappings $(\omega, A) : \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2$, $(\theta, v) \mapsto (\theta+\omega, A(\theta)v)$ where $\omega \in \mathbb{T}$ is irrational and $A : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ is a continuous matrix-valued function. By their action on the real projective space and subsequent identification of $\mathbb{P}(\mathbb{R}^2)$ with $\mathbb{T}$, such a cocycle (ω, A) induces a quasiperiodically forced circle homeomorphism f_A. If the Lyapunov exponent $\lambda(\omega, A) := \liminf_{n\to\infty} \frac{1}{n}\int_{\mathbb{T}} \log \|A(\theta+(n-1)\omega)\circ\ldots\circ A(\theta)\|\, d\theta$ of such a cocycle is positive, the corresponding cocycle f_A will have exactly two invariant graphs, one with positive and one with negative Lyapunov exponent (corresponding to the stable and unstable subspaces in Oseledets Multiplicative Ergodic Theorem). Herman showed that for any pair of rotation numbers ω and ρ there exist cocycles (ω, A) such that f_A has fiberwise rotation number ρ and the Lyapunov exponent of the cocycle is positive (Proposition 4.6 in [11]). If the rotations numbers are chosen rationally independent, the corresponding map f_A must be ρ-unbounded: As there exist invariant graphs it cannot be semi-conjugate to an irrational torus translation, and as the rotation numbers are not rationally dependend there cannot be any invariant strips. Thus, both alternatives in the ρ-bounded case are ruled out, so these examples are ρ-unbounded, topologically transitive, and by the preceding proposition they have *sdic* on X. The only candidates for proper minimal subsets of the whole space are the essential closures of the two invariant graphs[9] and restricted to these f_A has *sdic* as well by Proposition 3.5. However, it should be mentioned that it is still an open question

[9] The topological support of the measures which are obtained by projecting the Lebesgue measure on the base onto the invariant graphs.

whether transitive but non-minimal dynamics do really exist in this setting. To the knowledge of the authors, the only examples where the topological dynamics of such cocycles has been clarified so far are certain quasiperiodic Schrödinger cocycles, for which Bjerklöv proved under some additional assumptions that the dynamics are minimal [4].

We now consider the case in which the fiber maps f_θ are maps of the interval. In this case, continuous and non-continuous invariant graphs may coexist, and consequently there might be different regions in the phase space with and without *sdic*. Hence, instead of looking at the whole phase space we concentrate on the 'domain of attraction' of a non-continuous invariant graph.

Lemma 4.4
Suppose f is a quasiperiodically forced monotone interval map and φ is an upper semi-continuous, non-continuous invariant graph. Then there exists $\epsilon > 0$ such that the following inclusion holds:

$$A_\varphi \ := \ \{(\theta,\xi) \mid \xi > \varphi(\theta) \textit{ and } \inf_{n\in\mathbb{N}} |f^n_\theta(\xi) - \varphi(\theta+n\omega)| = 0\} \ \subseteq \ SD_\epsilon \ .$$

Proof:
Let $\Phi := \{(\theta,\varphi(\theta) \mid \theta \in \mathbb{T}\}$ and denote its topological closure by $\overline{\Phi}$. Let $\varphi^-(\theta) := \inf\{\xi \mid (\theta,\xi) \in \overline{\Phi}\}$. Then φ^- is a lower semi-continuous invariant graph and the set $[\varphi^-,\varphi] := \{(\theta,\xi) \mid \varphi^-(\theta) \leq \xi \leq \varphi(\theta)]$ is pinched, i.e. $\varphi^-(\theta) = \varphi(\theta)$ for a residual set of θ.[10] Choose some $\theta_0 \in \mathbb{T}$ with $\varphi^-(\theta_0) \neq \varphi(\theta_0)$ and let $\epsilon := \frac{1}{4}(\varphi(\theta_0) - \varphi^-(\theta_0))$. Further, let $x = (\theta,\xi) \in A_\varphi$ and $\delta > 0$ be given. As φ is upper semi-continuous, we can assume w.l.o.g (by decreasing δ if necessary) that $\varphi(\tilde\theta) < \xi \ \forall \tilde\theta \in B_\delta(\theta)$.

Due to the definition of φ^-, we can find θ_1 in $B_{\delta/2}(\theta_0)$ with $\varphi(\theta_1) - \varphi^-(\theta_0) \leq \epsilon$, such that $\varphi(\theta_1) \ \leq \ \varphi(\theta_0) - 3\epsilon$. Further, as φ is upper semi-continuous there exists some $\eta \in (0,\delta/2)$ such that

$$\varphi(\tilde\theta) \ \leq \ \varphi(\theta_1) + \epsilon \ \leq \ \varphi(\theta_0) - 2\epsilon \quad \forall \tilde\theta \in B_\eta(\theta_1) \ . \tag{4.4}$$

Now we choose some $n \in \mathbb{N}$ which satisfies $\theta + n\omega \in B_\eta(\theta_1)$ and

$$|f^n_\theta(\xi) - \varphi(\theta+n\omega)| \ \leq \ \epsilon \ . \tag{4.5}$$

Such an integer exists because the set $\{k \in \mathbb{N} \mid \theta + k\omega \in B_\eta(\theta_1)\}$ has bounded gaps (in the sense of (2.5)) and the orbit of $x \in A_\varphi$ will stay ϵ-close to φ for arbitrarily long time intervals due to the definition of A_φ and the continuity of f. Consequently we obtain

$$f^n_\theta(\xi) \ \leq \ \varphi(\theta+n\omega) + \epsilon \ \leq \ \varphi(\theta_0) - \epsilon \tag{4.6}$$

by (4.5) and (4.4). At the same time $f^n_{\theta_0 - n\omega}(\xi) \geq \varphi(\theta_0)$, such that $|f^n_\theta(\xi) - f^n_{\theta_0-n\omega}(\xi)| \geq \epsilon$. As $y = (\theta_0 - n\omega,\xi) \in B_\delta(x)$ (note that $\theta + n\omega \in B_\eta(\theta_1) \subseteq B_\delta(\theta_0)$, such that $\theta_0 - n\omega \in B_\delta(\theta)$), this completes the proof.

q.e.d.

Obviously, an analogous statement holds for the region below a lower semi-continuous invariant graph. As an application we obtain the following proposition, which in particular contains the second statement about pinched systems mentioned in the introduction.

[10]This is quite easy to see using a Baire argument, see [7, Lemma 4.3(i)] or [23] for details.

Corollary 4.5
Suppose f is a quasiperiodically forced monotone interval map, such that the global attractor $\mathcal{K} := \bigcap_{n\in\mathbb{N}} f^n(\mathbb{T}\times[a,b])$ is pinched and the upper and lower bounding graphs $\varphi^+(\theta) := \sup \mathcal{K}_\theta$ and $\varphi^-(\theta) := \inf \mathcal{K}_\theta$ are non-continuous. Then f has sdic *on $\mathbb{T}\times[a,b]$. The same is true if one of the bounding graphs is continuous, but coincides with one of the boundaries of the annulus.*

Proof:
We treat the case of two non-continuous bounding graphs, the second case is similar. Any point (θ,ξ) above φ^+ is necessarily contained in A_{φ^+}. Thus we can apply the above Lemma 4.4 to see that for some suitable $\epsilon > 0$ we have $\{(\theta,\xi) \mid \xi > \varphi^+(\theta)\} \subseteq SD_\epsilon$. Similarly, we can assume $\{\theta,\xi) \mid \xi < \varphi^-(\theta)\} \subseteq SD_\epsilon$, such that together we have $\mathcal{K}^c \subseteq SD_\epsilon$. But as $\mathcal{K}$ is pinched and therefore has empty interior and SD_ϵ is closed, this implies $\mathbb{T}\times[a,b] \subseteq SD_\epsilon$.

q.e.d.

5 A final example

Given the great attention pinched skew products have received after they had been introduced by Grebogi et al. in [10], it is rather surprising that the second type of model system which was proposed in the very same paper has been completely neglected so far. As Example 2 in [10] the authors consider the map $\Lambda : \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2$ depending on parameters β and γ and given by

$$\Lambda(\theta,\xi) \;:=\; \left(\theta+\omega, \frac{\beta}{1+\parallel \xi \parallel_2^2}\cdot\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}\cdot R_\theta\cdot\xi\right) \;. \tag{5.1}$$

Here R_θ denotes the rotation matrix

$$R_\theta \;:=\; \begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}$$

and $\xi = \binom{u}{v}$ is a vector as in Figure 1. In order to obtain a compact phase space, we choose a sufficiently large constant $C > 0$ such that $X := \mathbb{T}\times B_C(0)$ is mapped strictly inside itself (i.e. $\Lambda(X) \subseteq \mathrm{int}(X)$) and consider Λ restricted to X.

Similar to pinched skew products, the 0-line $\xi = 0$ is invariant. Further, as the action of Λ on any continuous curve that does not intersect the 0-line increases the number of lefthand turns around the 0-line there can be no other continuous invariant curve (in other words, the projective action of Λ is not homotopic to the identity). The numerical results in [10] indicate that for the considered parameter values ($\beta = 2,\ \gamma = 0.5$ and ω the golden mean) the system exhibits an SNA. This SNA seems to be a quasiperiodic two-point attractor (i.e. a two-valued measurable invariant graph) which attracts Lebesgue-a.e. initial condition. In the following, we will give a rigorous proof of this observation and show in addition that the attractor is embedded in a two-dimensional torus $\mathcal{T}_0$, which is the boundary of the global attractor $\mathcal{G} := \bigcap_{n\in\mathbb{N}} \Lambda^n(X)$ in the three-dimensional phase space. Further, Λ has *sdic* both restricted to the attractor and on the whole phase space. These results remain valid as long as $1 < \beta \leq 2$, $\gamma \in (0,1)$, and $\beta\gamma \geq 1$.

A two-to-one factor. In order to analyze the dynamics of Λ, it turns out to be more convenient to use polar coordinates in $\mathbb{R}^2 \setminus \{0\}$, and to consider directions only projectively, rather than use the standard Cartesian coordinates in $\mathbb{R}^2$. Therefore, we will now introduce a map $\tilde{\Lambda} : \mathbb{T}^2 \times [0,C] \to \mathbb{T}^2 \times [0,C]$ which is a two-to-one factor of Λ.[11] It will turn out that there exists an attracting invariant graph for $\tilde{\Lambda}$, and the preimage of this graph under the factor map then gives the two-point-attractor for Λ. However, we will have to leave open here whether this attractor can further be decomposed into two one-valued invariant graphs or not.

Let $b(x) := \frac{x}{1+x2}$ and

$$A(\theta) \ := \ \begin{pmatrix} \gamma^{-\frac{1}{2}} & 0 \\ 0 & \gamma^{\frac{1}{2}} \end{pmatrix} \cdot R_\theta \ \in \ \mathrm{SL}(2,\mathbb{R}) \ . \tag{5.2}$$

Then (5.1) becomes

$$\Lambda(\theta,\xi) \ = \ \left(\theta+\omega, \beta\gamma^{\frac{1}{2}} \cdot b(\| \xi \|_2) \cdot A(\theta) \cdot \frac{\xi}{\|\xi\|_2}\right) \ . \tag{5.3}$$

As mentioned, we will consider projective polar coordinates $\alpha = \frac{1}{\pi}\cot^{-1}(\frac{u}{v}) \in \mathbb{T}$ and $r = \| \xi \|_2$ for $\xi = \binom{u}{v} \in \mathbb{R}^2 \setminus \{0\}$. The reason for doing so is the fact that the action of Λ on α does not depend on r, such that the system becomes a skew product over a skew product. Further, the dynamics of α are determined by the projective action of the quasiperiodic $\mathrm{SL}(2,\mathbb{R})$-cocycle

$$(\omega, A) : \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2 \quad , \quad (\theta,\xi) \mapsto (\theta+\omega, A(\theta)\cdot\xi) \ , \tag{5.4}$$

which induces a quasiperiodically forced circle homeomorphisms $f = f_A$.[12] Such cocycles present one of the few classes of quasiperiodically forced systems which are already well-understood, and in particular we can apply results from [11] and [26] to our problem.

If we now let $\Theta = (\theta,\alpha)$, we obtain a map $\tilde{\Lambda} : \mathbb{T}^2 \times [0,C] =: Y \to Y$ given by

$$\tilde{\Lambda}(\Theta, r) \ := \ (f(\Theta), g_\Theta(r)) \tag{5.5}$$

where g_Θ is defined by the dynamics of Λ on r. More precisely, suppose $\xi = \binom{u}{v} \in \mathbb{R}^2 \setminus \{0\}$ is a vector with $\frac{1}{\pi}\cot^{-1}\left(\frac{u}{v}\right) = \alpha$ and length $\| \xi \|_2 = 1$ and let

$$a(\Theta) \ := \| \beta\gamma^{\frac{1}{2}} \cdot A(\theta)\cdot\xi \|_2 \ \ . \tag{5.6}$$

Then it is easy to see from (5.3) that

$$g_\Theta(r) \ = \ a(\Theta)\cdot b(r) \tag{5.7}$$

and a depends continuously on Θ. Further, let $X' := X \setminus (\mathbb{T}\times\{0\})$ and $Y' := Y \setminus (\mathbb{T}^2\times\{0\})$. Then, as mentioned before, $\tilde{\Lambda}_{|Y'}$ is a two-to-one factor of $\Lambda_{|X'}$ with factor map

$$h : (\theta,\xi) \mapsto \left(\theta, \frac{1}{\pi}\cot^{-1}\left(\frac{u}{v}\right), \| \xi \|_2\right) \ . \tag{5.8}$$

[11] To be absolutely precise, $\tilde{\Lambda}_{|\mathbb{T}^2\times(0,C]}$ will be a two-to-one factor of $\Lambda_{|\mathbb{T}\times\mathbb{R}^2\setminus\{0\}}$, whereas the 0-line is 'blown up' into the 0-torus $S = \mathbb{T}^2 \times \{0\}$. However, as the 0-line is invariant and we are only interested in the dynamics off the 0-line, this is sufficient for our purposes.

[12] If $A(\theta) = \begin{pmatrix} a_\theta & b_\theta \\ c_\theta & d_\theta \end{pmatrix}$, then we can first define a map $\tilde{f}_A : \mathbb{T}\times\overline{\mathbb{R}} \to \mathbb{T}\times\overline{\mathbb{R}}$ by

$$\tilde{f}_A(\theta,x) \ = \ \left(\theta+\omega, \frac{a_\theta x + b_\theta}{c_\theta x + d_\theta}\right) \ ,$$

and identification of $\overline{\mathbb{R}}$ with $\mathbb{T}$ via $x \mapsto \frac{1}{\pi}\cot^{-1}(x)$ yields $f = f_A$.

Base dynamics of $\tilde{\Lambda}$. First we analyze the dynamics of the driving homeomorphism f. As we have already argued above, f is not homotopic to the identity and therefore topologically transitive [18] and has *sdic* due to Lemma 4.1. Further, the Lyapunov exponent of the cocycle (ω, A) is defined as

$$\lambda(\omega, A) \ := \ \lim_{n\to\infty} \int_{\mathbb{T}} \log \parallel A_n(\theta) \parallel \ d\theta \ , \tag{5.9}$$

where $A_n(\theta) = A(\theta + (n-1)\omega) \circ \ldots \circ A(\theta)$. Section 4.1 in [11] provides a lower bound for the Lyapunov exponent, namely

$$\lambda(\omega, A) \ \geq \ \log\left(\frac{\sqrt{\gamma}}{2} + \frac{1}{2\sqrt{\gamma}}\right) \ . \tag{5.10}$$

This means that for $\gamma \neq 1$ the Lyapunov exponent is always positive, and Oseledet's Multiplicative Ergodic Theorem then implies that there exists an invariant splitting of $\mathbb{R}^2$ into a stable and an unstable subspace. This in turn is equivalent to the existence of exactly two invariant graphs φ^s and φ^u for the induced map f with positive and negative Lyapunov exponent, respectively.[13] Note that the invariant graph φ^u corresponding to the unstable subspace is the one with negative Lyapunov exponent and attracts Lebesgue-a.e. initial condition.

Ergodic invariant measures and vertical Lyapunov exponents. In order to obtain more information about our system, we have to characterize the ergodic invariant measures for $\tilde{\Lambda}$. Further, we have to determine their *'radial'* Lyapunov exponents, which are defined as

$$\lambda_{\mathrm{rad}}(\nu) \ := \ \int_Y \log Dg_\Theta(r) \ d\nu(\Theta, r) \ , \tag{5.11}$$

where ν is the invariant measure and Dg_Θ denotes the derivative of g_Θ w.r.t. r.[14] First of all, for the base dynamics given by f there exist exactly two ergodic invariant measures μ^s and μ^u which are associated to the invariant graphs by

$$\mu^i(A) \ := \ m(\{\theta \in \mathbb{T} \mid (\theta, \varphi^i(\theta)) \in A\}) \ , \tag{5.12}$$

where m denotes the Lebesgue measure on $\mathbb{T}$. These two measures μ^i can be naturally identified with $\tilde{\Lambda}$-invariant measures $\tilde{\mu}^i$ by embedding them into the invariant 0-torus $S := \mathbb{T}^2 \times \{0\}$ in the canonical way. Their Lyapunov exponents are then given by

$$\lambda_{\mathrm{rad}}(\tilde{\mu}^i) \ = \ \int_{\mathbb{T}} \log Dg_{(\theta,\varphi^i(\theta))}(0) \ d\theta \ = \ \int_{\mathbb{T}} \log a(\theta, \varphi^i(\theta)) \ d\theta \ . \tag{5.13}$$

It is not hard to see that for the considered parameter values $1 < \beta \leq 2$, $\beta\gamma \geq 1$, both exponents are positive: From (5.2) and (5.6) we deduce that $1 \leq a \leq 2$ and $a(\theta, \alpha) = 1$ if and only if $\beta\gamma = 1$ and $\alpha = \frac{1}{2} - \theta$. As the invariant graphs are non-continuous and therefore $\varphi^i(\theta) = \frac{1}{2} - \theta$ cannot hold m-a.s., this implies that $\lambda_{\mathrm{rad}}(\tilde{\mu}^i) > 0$.

Any other ergodic invariant measure ν must project down to an ergodic measure for the base dynamics, that is either μ^s or μ^u. Thus, in order to study ν we can restrict the base dynamics to

[13] In fact, we have $\lambda(\varphi^s) = 2\lambda(\omega, A)$ and $\lambda(\varphi^u) = -2\lambda(\omega, A)$ where φ^s is the invariant graph corresponding to the stable direction and φ^u the one corresponding to the unstable direction.

[14] We denote these Lyapunov exponents by λ_{rad} in order to distinguish them from the Lyapunov exponents in two-dimensional skew products, thus avoiding ambiguities.

the respective invariant graph Φ^i. But this means we obtain a system which can be viewed as a two-dimensinal skew product $h^{(i)} : \mathbb{T} \times [0,C]$ again, with fiber maps

$$h_\theta^{(i)}(r) \;=\; g_{(\theta,\varphi^i(\theta))}(r) \;=\; a(\theta,\varphi^i(\theta)) \cdot b(r) \; . \tag{5.14}$$

As $1 \leq a \leq 2$, $h_\theta^{(i)}(\mathbb{T} \times [0,C]) \subseteq \mathbb{T} \times [0,1]$, and so we can and will assume from now on that $C = 1$. Due to the non-continuity of φ^i the map $h^{(i)}$ is not continuous, but it still has continuous, strictly monotonically increasing and strictly concave fiber maps (observe that $C = 1$). For such systems a basic classification was given in [17] (the continuity assumption made there is not relevant for the facts we are going to state and use in the following):

- Ergodic invariant measures correspond to invariant graphs, in the same sense as in (5.12).
- There are at most two invariant graphs, one of which is the 0-line. If the Lyapunov exponent of the 0-line is non-positive then this is the only invariant graph, if its Lyapunov exponent is positive there exists exactly one other invariant graph ρ^i, which has a negative Lyapunov exponent.

Obviously, the Lyapunov exponent of the 0-line in the system $h^{(i)}$ is equal to $\lambda(\tilde{\mu}^i)$, such that in our situation there always exists one more $h^{(i)}$-invariant graph ρ^i. By

$$\Gamma^i(\theta) \;=\; (\varphi^i(\theta), \rho^i(\theta)) \tag{5.15}$$

we can then define a $\tilde{\Lambda}$-invariant graph, which must be the support of the measure ν. Again, the Lyapunov exponent $\lambda_{\mathrm{rad}}(\nu)$ is equal to the Lyapunov exponent of ρ^i for the system $h^{(i)}$, and therefore strictly negative.

Summarizing we have found that there exist exactly four ergodic invariant measures for $\tilde{\Lambda}$: $\tilde{\mu}^s$ and $\tilde{\mu}^u$, which are embedded in the 0-torus $S = \mathbb{T}^2 \times \{0\}$ and have positive radial Lyapunov exponents, and two measures ν^s and ν^u which are associated to the $\tilde{\Lambda}$-invariant graphs Γ^s and Γ^u and have negative radial Lyapunov exponents. Among these four measures only ν_s has negative exponents in the base, i.e. in α-direction, and also in radial direction.

The global and one-point attractor for $\tilde{\Lambda}$. The 0-torus S is a compact $\tilde{\Lambda}$-invariant set, and all ergodic invariant measures supported on this set have strictly positive vertical Lyapunov exponents. Therefore it follows from the Uniform Ergodic Theorem (in fact from a slight generalization, see [25]) that some iterate of $\tilde{\Lambda}$ is uniformly expanding in the vertical direction on a neighborhood of S. Consequently, for sufficiently small ϵ and suitable $n \in \mathbb{N}$ we have $\tilde{\Lambda}^n(\mathbb{T}^2 \times [\epsilon,1]) \subseteq \mathbb{T}^2 \times (\epsilon,1]$. Let $K := \mathbb{T}^2 \times [\epsilon,1]$. K is compact and forward invariant, and all ergodic invariant measures supported on K (namely ν^s and ν^u) have strictly negative Lyapunov exponents. Therefore the convergence of the ergodic limits is again uniform, and a suitable iterate of $\tilde{\Lambda}$ is a uniform vertical contraction on K. But this implies immediately that $\mathcal{T} := \bigcap_{n\in\mathbb{N}} \tilde{\Lambda}^n(K)$ is homeomorphic to the driving space $\mathbb{T}^2$, i.e. can be represented as the graph of a continuous function $T : \mathbb{T}^2 \to [\epsilon,1]$ (in fact T will be Hölder continuous, see [22]). Evidently $\mathcal{T}$ is the boundary of the global attractor $\mathcal{G}$ and for all $(\Theta,r) \in Y \setminus S$ there holds

$$|\pi_3(\tilde{\Lambda}^n(\Theta,r)) - T(f^n\Theta)| \;\to\; 0 \quad (n \to \infty) \; . \tag{5.16}$$

The one-point attractor mentioned in the beginning is the graph Γ^u: The fact that it attracts Lebesgue-a.e. initial condition follows from the fact that on the base this is true for the graph φ^u, and in the additional third coordinate the convergence is given by (5.16). Figure 1 shows the graph of Γ^u embedded in the manifold $\mathcal{T}$ and also the graph of φ^u, its projection to the 2-dimensional base.

Finally note that $\tilde{\Lambda}$ has *sdic*, both on the whole phase space and on $\mathcal{A} = \mathrm{cl}(\Gamma^u)$. For the whole phase space, this follows from the fact that the base map already has *sdic* on $\mathbb{T}^2$ by Lemma 4.1 . On the other hand, as the attractor is embedded in $\mathcal{T}$, the dynamics of $\tilde{\Lambda}_{|\mathcal{A}}$ are equivalent to the dynamics of $f_{|\mathrm{cl}(\Phi^u)}$ (note that $\mathrm{cl}(\Phi^u) = \pi(\mathcal{A})$). Therefore *sdic* on $\mathcal{A}$ follows either from Proposition 3.5 (if $\mathrm{cl}(\Gamma^u) \neq \mathbb{T}^2$) or Proposition 4.1 (if $\mathcal{A} = \mathbb{T}^2$).

The original system Λ*.* Now we can use the results on $\tilde{\Lambda}$ to describe the dynamics of its extension Λ. The preimage of Γ^u under the factor map h is invariant, consists of exactly two points on every fiber and attracts Lebesgue-a.e. initial condition. As mentioned, the only question we have to leave open here is whether this two-point attractor further (measurably) decomposes into two one-valued invariant graphs.

As $\tilde{\Lambda}$, the map Λ has *sdic* on the whole phase space and on the attractor. For the attractor this is immediate as it is embedded in the two-dimensional torus $\mathcal{T}_0 := h^{-1}(\mathcal{T})$ such that the dynamics on $\mathcal{T}_0$ are a two-to-one extension of f, and $f_{|\pi(\mathcal{A})}$ has *sdic*. On the whole phase space the only problem is that in a neighborhood of the 0-line the metric on the factor space Y' is not equivalent to the usual euclidean metric on X'. However, the two metrics are equivalent if we restrict to the compact and Λ-invariant set $h^{-1}(K)$, and as any open set U which is bounded away from the 0-line ends up in $h^{-1}(K)$ after a finite number of iterates we obtain that $X' \subseteq SD_\epsilon$ for a suitable $\epsilon > 0$. $X \subseteq SD_\epsilon$ then follows again from the fact that SD_ϵ is closed.

References

[1] E. Akin, J. Auslander, and K. Berg (1996) *When is a transitive map chaotic?*, Convergence in Ergodic Theory and Probability, Walter de Gruyter & Co., Berlin , 25-40.

[2] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey (1992) *On Devaney's definition of chaos*, Amer. Math. Monthly **99** 332–334.

[3] K. Bjerklöv (2003) *Dynamical Properties of Quasi-periodic Schrödinger equations*, PhD thesis, Royal Institute of Technology, Stockholm.

[4] K. Bjerklöv (2005), *Positive lyapunov exponent and minimality for a class of one-dimensional quasi-periodic schrödinger equations*, Ergodic Theory and Dynamical Systems **25** 1015–1045.

[5] K. Bjerklöv (2006) *Dynamics of the quasiperiodic Schrödinger cocycle at the lowest energy in the spectrum*, Preprint.

[6] R.L. Devaney (1989) *An Introduction to Chaotic Dynamical Systems (Second Edition)*, Addison-Wesley, California.

[7] R. Fabbri, T. Jäger, R. Johnson, G. Keller (2005) *A Sharkovskii-type theorem for minimally forced interval maps,* Topological Methods in Nonlinear Analysis **26** 163–188.

[8] E. Glasner and B. Weiss (1993) *Sensitive dependence on initial conditions*, Nonlinearity **6** 1067–1075.

[9] P. Glendinning (2002) *Global attractors of pinched skew products*, Dyn. Syst. Int. J. **17** 287–197.

[10] C. Grebogi, E. Ott, S. Pelikan and J.A. Yorke (1984) *Strange attractors that are not chaotic*, Physica D **13** 261–268.

[11] M.R. Herman (1983), *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2*, Commentarii Mathematici Helvetici **58** 453–502.

[12] T. Jäger (2003) *Quasiperiodically forced interval maps with negative schwarzian derivative*, Nonlinearity **16** 1239–1255.

[13] T. Jäger (2004) *On the structure of strange non-chaotic attractors in pinched skew products*, Preprint, to appear in Ergodic Theory and Dynamical Systems.

[14] T. Jäger (2005) *Existence and structure of strange non-chaotic attractors*, PhD thesis, FAU Erlangen-Nürnberg.

[15] T. Jäger (2006) *The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations*, Preprint.

[16] T. Jäger and J. Stark (2005) *Towards a classification of quasiperiodically forced circle homeomorphisms*, Preprint, to appear in Journal of the LMS.

[17] G. Keller (1996), *A note on strange nonchaotic attractors*, Fund. Math. **151** 139–148.

[18] J.-W. Kim, S.-Y. Kim, B. Hunt, and E. Ott (2003), *Fractal properties of robust strange nonchaotic attractors in maps of two or more dimensions*, Physical Review E **67** Article 036211.

[19] J. Milnor (1985) *On the concept of an attractor*, Comm. Math. Phys. **99** 177–195.

[20] A.S. Pikovsky and U. Feudel (1995), *Characterizing strange nonchaotic attractors*, Chaos **5** 253–260.

[21] W. Rudin (1974) *Real and complex analysis, 2nd ed.*, McGraw-Hill, New York.

[22] J. Stark (1999), *Regularity of invariant graphs for forced systems*, Ergodic Theory and Dynamical Systems **19**(1) 155–199.

[23] J. Stark (2003), *Transitive sets for quasiperiodically forced monotone maps*, Dynamical Systems **18**(4) 351–364.

[24] J. Stark, U. Feudel, P.A. Glendinning, and A. Pikovsky (2002), *Rotation numbers for quasiperiodically forced monotone circle maps*, Dynamical Systems **17**(1) 1–28.

[25] J. Stark and R. Sturman (2000), *Semi-uniform ergodic theorems and applications to forced systems*, Nonlinearity **13**(1) 113–143.

[26] P. Thieullen (1997), *Ergodic reduction of random products of two-by-two matrices*, Journal d'Analyse Mathématique **73** 19–64.