Topics in Finite Groups: Homology Groups, Pi-product Graphs, Wreath Products and Cuspidal Characters

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2015

MIMS EPrint: 2015.65

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ISSN 1749-9097
TOPICS IN FINITE GROUPS: HOMOLOGY GROUPS, $\pi$-PRODUCT GRAPHS, WREATH PRODUCTS AND CUSPIDAL CHARACTERS

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Engineering and Physical Sciences

2015

David Ward
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TOPICS IN FINITE GROUPS: HOMOLOGY GROUPS, $\pi$-PRODUCT GRAPHS, WREATH PRODUCTS AND CUSPIDAL CHARACTERS

This thesis was submitted by David Ward to The University of Manchester for the degree of Doctor of Philosophy on Friday 24th July, 2015.

Abstract: In this thesis we consider four topics relating to finite groups: homology of presheaves of abelian groups, $\pi$-product graphs, wreath products of cyclic $p$-groups and $p$-cuspidal characters.

Let $G$ be a finite group, $p$ a prime divisor of the order of $G$ and $k := GF(p)$ - the field of $p$ elements. In a series of papers in the 1980s, Mark Ronan and Stephen Smith formed the notion of a presheaf of abelian groups for $G$. This mirrored the topological definition of a presheaf and was built on an arbitrary simplicial complex having associated $G$-action. They then proceeded to define a chain complex on a presheaf such that the corresponding homology groups were $kG$-modules. We consider this theory and investigate if every irreducible $kG$-module can be achieved as a quotient of the zero-homology group of a universal panel-irreducible presheaf defined on the simplicial complex of parabolic subgroups of $G$. In general, we see that this is not the case. We also explicitly calculate the zero-homology groups of universal panel-irreducible presheaves defined on $\text{Sym}(6)$, $M_{11}$ and $M_{22}$ over $GF(2)$, together with the irreducible quotients of the zero-homology groups of universal panel-irreducible presheaves defined on $M_{12}$, $M_{23}$ and $M_{24}$ over $GF(2)$.

Suppose that $G$ is a group, $X$ is a subset of $G$ and $\pi$ is a set of natural numbers. The $\pi$-product graph $P_\pi(G,X)$ has $X$ as its vertex set and distinct vertices are joined by an edge if the order of their product is contained in $\pi$. If $X$ is a set of involutions, then $P_\pi(G,X)$ is called a $\pi$-product involution graph. We study the connectivity and diameters of $P_\pi(G,X)$ when $G$ is a finite symmetric group and $X$ is a $G$-conjugacy class of involutions, and fully determine when $P_\pi(G,X)$ is connected. In the case that $\pi = \{4\}$, the diameter of all connected graphs $P_\pi(G,X)$ is shown to equal 2. The connectivity of $P_\pi(G,X)$ is also determined in the cases that $X$ is a class of involutions with support of order $2^a$ for some $a \geq 3$ and $\pi = \{2^a\}$, $X$ is a class of involutions having support of order $p^a - 1$ for some odd prime $p$, some $a \geq 1$ and $\pi = \{p^a\}$ and in the more general case that $X$ is a class of involutions and $\pi = \{b\}$ where $b$ is closely related to the order of the support of the involutions in $X$. Bounds on the diameters of the connected graphs are also obtained. In the case that $X$ is the conjugacy class of involutions with support of order 8 and $G := \text{Sym}(n)$ for $n \geq 10$, the diameter of $P_{\{8\}}(G,X)$ is explicitly determined.

The $\pi$-product involution graph $P_{\{4\}}(G,X)$ may also be viewed as a graph having vertex set $X$, and with two involutions $x, y \in X$ joined by an edge precisely when they generate $\text{Dih}(8)$ - the dihedral group of order 8. A natural generalisation of the results for $\{4\}$-product involution graphs is to consider $p$-elements in place of involutions and the wreath product $C_p \wr C_p$ in place of $\text{Dih}(8)$. Indeed, when do two conjugate $p$-elements of a given symmetric group generate a wreath product of two cyclic groups of order $p^2$? We give necessary and sufficient conditions for this in the case that our $p$-elements have full support. These conditions relate to given matrices that are of circulant or permutation type, and corresponding polynomials that represent these matrices. We also consider the case that the elements do not have full support, and see why generalising our results to such elements would not be a natural generalisation.

The concluding topic of this thesis is $p$-cuspidal characters of finite groups. Given a finite group $G$ containing a subgroup $X$, an $X$-parabolic system of $G$ may be defined. This consists of pairs of subgroups $(P_J, Q_J)$ for all subsets $J$ of a given indexing set such that certain conditions hold. These $X$-parabolic systems mirror some of the properties of parabolic subgroups and unipotent radicals for groups of Lie type, and allow an analogue of cuspidal characters to be defined for an arbitrary finite group. Such cuspidal characters are dependant on the choice of $X$-parabolic system. We consider systems defined by $p$-minimal parabolic systems, and hence obtain $p$-cuspidal characters of $G$ for each prime divisor of the order of $G$. Theoretical results are developed reflecting known results for groups of Lie type. For each sporadic group $G$ and each prime divisor of the order of $G$, we then calculate the $p$-cuspidal characters of $G$ (with the exception of $G = \mathbb{E}$ and $p = 2$).
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Acknowledgements

First and foremost, I would like to thank my Lord and Saviour Jesus Christ for the grace and compassion he has shown to me in my daily life, and for allowing me to discover a small part of his creation.

My sincere thanks and gratitude go to my supervisor Prof. Peter Rowley. He has been a constant guide throughout my research, whilst giving me the freedom to investigate tangential paths at will and for considerable time, even if at first they appeared to be cul-de-sacs. I am also eternally grateful for the financial support of the School of Mathematics through the University of Manchester Faculty of Engineering and Physical Sciences Dean’s Award, and also for the pastoral support that the School has shown me during my studies.

I am indebted to my fellow group theorists; John Ballantyne, Paul Bradley, Tim Crinion, Alex Mcgaw, Athirah Nawawi, Peter Neuhaus, Jamie Phillips, Inga Schwabrow, Paul Taylor and Daniel Vasey. They were a constant source of reference for group-theoretical results and were always happy to listen to my - usually flawed - mathematical reasoning. Thanks also go to Nic Clarke, Andrew Davies, Sian Fryer and Dave Wilding for assisting me with the more algebraic parts of my research.

My research would not have been as enjoyable without the light-hearted environment provided by my office colleagues; Wemedh Mohammed Aeal (a.k.a. Dragon!), Areej Al Muhaimeed, Akram Al-Sabbagh, Maria Alexandrou, Philip Bridge, Alfredo Camacho Valle, Anthony Chiu, Nic Clarke, Marine Fontaine, Mohsen Khani, Akeel Ramadan Mehdi and Amna Shaddad. Special thanks go to Anthony for his never-ending willingness to help me with computer-related issues.

I am indebted to Bob Pegnell for his guiding influence on the start of my mathematical journey many years ago, and for allowing me to take him directly from Ipswich to Woodbridge without visiting Cornwall on the way.

Finally, I would like to thank my parents, sister, brother-in-law and nieces. Their constant encouragement and belief in me, and their all-round support helped me through many a dark day. They also helped me to see outside of my office into the wider world.
Chapter 1

Overview

Since the axiomatisation of the abstract group in the late nineteenth century, mathematicians have used many methods for studying groups. The axiomatisation brought the areas of permutation groups, abelian groups and classical groups under a universal umbrella from which universal properties could be established in the greatest of generality. In subsequent years, the individual areas of group theory have been treated in both universal and individual contexts. Properties and notions readily evident in one area could then be considered in other areas, with varying degrees of success. This is especially evident in the generalisation of ideas concerning groups of Lie type to arbitrary finite groups and vice versa. In this thesis, we consider two such notions; presheaves of abelian groups and cuspidal characters. We also see how considering a specific graph-theoretic approach to involutions in symmetric groups leads to a matrix-orientated approach to $p$-elements.

The thesis has four main chapters, which are preceded in Chapter 2 by a summary of the mathematical background material on which the subsequent chapters build, together with a review of the notation used throughout the thesis. In Chapter 3 we consider the work of Mark Ronan and Stephen Smith on the homology groups of presheaves of abelian groups [RS85], [RS86], [RS89], [Ron89]. Ronan and Smith were motivated by the topological definition of a presheaf, and used parabolic subgroups in place of open sets. They further required an associated group action. Equipped with such an action, their presheaves had nice combinatorial properties which allow a homological approach to be used in considering them. With such an approach, the homology groups are modules for the given group. Moreover, Ronan and Smith were able to define the notion of a universal presheaf, such that any “nice” presheaf could be considered as a quotient of a universal presheaf.

Full details of the work of Ronan and Smith is given in Sections 3.1-3.6. All results in these sections, except for Corollary 3.6.8 and Theorem 3.6.9, are due to Ronan and Smith. Corollary 3.6.8 is an easy deduction from previous results, whilst Theorem 3.6.9 is a corrected version of a comment appearing in [RS89]. In Sections 3.1-3.6 we direct the reader to the literature for proofs of known results where possible. However, we provide proofs for results when no such references exist.

Having given a survey of the work of Ronan and Smith, we pose a question about irreducible quotients of zero-homology groups of universal panel-irreducible presheaves in Section 3.7. A partial answer is given in this section, and the question is reconsidered in Section 3.9. These sections
sandwich our main work in this area, which is given in Section 3.8 namely the calculation of the zero-homology groups of universal panel-irreducible presheaves of the symmetric group Sym(6) and the Mathieu groups $M_{11}$ and $M_{22}$ together with calculations of the irreducible quotients of such zero-homology groups for $M_{12}$, $M_{23}$ and $M_{24}$ all over the field of two elements, $GF(2)$. The zero-homology groups of universal panel-irreducible presheaves of $M_{24}$ over $GF(2)$ were considered by Ronan and Smith in [RSS99, Section 4]. However, this was over a different geometry to that used in this thesis. For the sake of brevity, some calculations of vertex terms of universal panel-irreducible presheaves appear separately as Appendix A.

In Chapter 3 our universal panel-irreducible presheaves are defined on the simplicial complex of parabolic subgroups of a given group. However, presheaves may be defined on any simplicial complex with associated group action. One such simplicial complex is given by the Brown complex $|S_p(G)|$ of a group $G$, which for a given prime $p$ consists of all chains of non-trivial $p$-subgroups of $G$, ordered by inclusion. Let $G$ be the symmetric group of degree $n$ and suppose that $X$ is the conjugacy class of all involutions in $G$ whose support have order $2m$ for some $m \leq n/2$. We may consider all simplices of $|S_2(G)|$ consisting of a single subgroup isomorphic to the dihedral group, $Dih(8)$, of order 8 together with all simplices consisting of a single subgroup generated by an element of $X$. A natural question to ask is when do two such subgroups of order 2 generate a subgroup isomorphic to $Dih(8)$ (or equivalently when does the product of the two generating involutions have order 4)? This is the main topic of chapter 4.

Generalising these ideas, we may define the $\pi$-product graph, $P_\pi(G, X)$, for an arbitrary group $G$, a subset $X \subseteq G$ and for $\pi$ a set of positive integers to be the graph having vertex set $X$ and with two vertices joined by an edge precisely when the order of their product is contained in $\pi$. In all of the cases that we consider, our vertex set $X$ will consist of involutions, and hence the resulting $\pi$-product graph (also know as a $\pi$-product involution graph) is undirected. Taking $\pi$ to be the set of all odd natural numbers and $X$ a $G$-conjugacy class, $P_\pi(G, X)$ becomes the local fusion graph $F(G, X)$ which has featured in [Bal13] and [BGR13].

Chapter 4 contains a survey of $\pi$-product graphs of symmetric groups in the case that $X$ is a conjugacy class of involutions and $\pi$ is a singleton set and is heavily based on the preprint [RW14]. The work relies heavily on the notion of an $x$-graph, a computationally-pleasing graph introduced by Bates, Bundy, Parker and Rowley whilst considering the commuting involution graphs of symmetric groups in [BBPR03b]. We see that $\pi$-product graphs are one of a bountfulness of graphs that may be associated to groups. Further examples are given by local fusion graphs, power graphs and commuting graphs.

The main result of Chapter 4 is that if $G$ is a symmetric group and $X$ is a conjugacy class of involutions, then the $\{4\}$-product graph $P_{\{4\}}(G, X)$ is either disconnected or it is connected and has diameter equal to 2. The proof of this result involves a large number of $x$-graphs and these appear as Appendix B.

The contents of Chapter 4 may be generalised in a number of different ways, and one such way is considered in Chapter 5. Indeed, rather than taking $X$ to be a conjugacy class of involutions within a given symmetric group, we consider the case that $X$ is a conjugacy class of elements of
order $p$. In such a situation, we no longer have the versatile $x$-graph at our disposal. Generalising $x$-graphs from a graph-theoretical viewpoint is impractical and would result in weighted directed graphs. In Chapter 5 we see that we can define certain matrices to be - in some senses - the analogue to $x$-graphs in this more general setting.

The matrices defined may be used in computations, and by considering the dihedral group $\text{Dih}(8)$ to be the wreath product, $C_2 \wr C_2$, of cyclic groups of order 2, we may generalise the results of Chapter 4. In the case that $X$ contains elements of full support, we ask when do two elements of $X$ generate the wreath product $C_p \wr C_p$? We see that in such cases, the aforementioned matrices take the form of block matrices involving both permutation and circulant matrices. A definitive answer to our question may then be given by considering these matrices, the determinants of their blocks and the representative polynomials of the circulant blocks. We also see that considering such an approach when elements of $X$ do not have full support is unrealistic.

To gain a greater understanding of the matrices and associated polynomials arising in Chapter 5 the computer algebra system MAGMA may be used. Further information about MAGMA can be found in the online handbook [Com] or in [BC06], [CPB08] or [CP08]. MAGMA code that may be beneficial to the reader is given in Appendix C.

We note that Chapter 5 has appeared in its entirety in the preprint [War14] and an abridged version is also available as [Warar].

The final main chapter is of a slightly different sapidity. In a style mirroring that of Chapter 3 we take a concept from groups of Lie type and seek to form an analogue in a more general setting. The concept in question is that of being a cuspidal character. This is a property that a character of a group of Lie type in characteristic zero may satisfy, and in that setting, cuspidal characters may be seen - in some senses - as the building blocks which contain all irreducible characters.

We begin Chapter 6 by formulating an $X$-parabolic system of a group $G$ having subgroup $X$. We see that these $X$-parabolic systems are the cognate of the building of a group of Lie type, upon which the cuspidal property is based, and hence they allow us to define cuspidal characters in a more general setting. This is followed by a brief résumé of some elementary definitions and properties of cuspidal characters of groups of Lie type.

Having established the corresponding properties of cuspidal characters in the finite setting, we then proceed to give a full survey of the $p$-cuspidal characters for each of the 26 sporadic simple groups and for each prime divisor, $p$, of the group order. These results are obtained via assimilating known knowledge about the fusion within subgroups of each of the sporadic groups and combining this with information from the ATLAS and direct calculations involving MAGMA. We fully determine all $p$-cuspidal characters of each sporadic group $G$, with the exception of the case when $(G, p) = (\mathbb{B}, 2)$. In this case, it is likely that there are no 2-cuspidal characters of $\mathbb{B}$. However, at the time of writing, we have been unable to prove this.

The thesis concludes in Chapter 7 with a few comments on possible extensions of and future areas of research related to the topics contained in Chapters 3-6.
Chapter 2

Background Material

The aim of this chapter is to introduce the general background material that is required in subsequent chapters. For general group-theoretic definitions and results, we refer the reader to [Asc00], [Gor80], [Isa08] or [Suz82].

We begin by introducing some general notation in Section 2.1 that will be used throughout the thesis. Further notation from the ATLAS of Finite Groups [CCN+09] (subsequently referred to as the ATLAS) is introduced in Section 2.3 together with a summary of the required results on finite simple groups. This is preceded in Section 2.2 with background material on $p$-groups and is followed in Sections 2.4, 2.5 and 2.6 with material on buildings, minimal parabolic systems and geometries.

Throughout the thesis, there are many references to the computer algebra system MAGMA. Further information about MAGMA can be found in the online handbook [Com] or in [BC06], [CPB08] or [CP08]. In a number of places throughout this thesis we have also used information contained in the ATLAS. Such information has been used without proof.

2.1 General Notation and Conventions

Throughout the thesis, unless otherwise stated, $G$ will denote a finite group and $k$ will denote a field of characteristic char $k$. The Galois field of order $p^n$ is denoted by $GF(p^n)$. We will use the notation $|G|$ and ord$(g)$ for the order of $G$ and the order of the element $g \in G$ respectively. If $H$ is a subgroup, proper subgroup or normal subgroup of $G$, then the respective notation $H \leq G$, $H < G$ and $H \trianglelefteq G$ is used. The index of $H$ in $G$ is denoted by $[G : H]$. If $p$ is a prime, we denote the set of Sylow $p$-subgroups of $G$ by $Syl_p(G)$. The trivial group is denoted by 1, the cyclic group of order $r$ is denoted by $C_r$ and the symmetric and alternating groups on $n$ symbols are denoted by Sym$(n)$ and Alt$(n)$ respectively.

Given elements $g, h \in G$, we use $g^h$ to denote the conjugate of $g$ by $h$ and $g^G$ to denote the $G$-conjugacy class of $g$. The commutator of $g$ with $h$ is given by $[g, h] := g^{-1}h^{-1}gh$. The centralizer of $g$ in $G$ is denoted by $C_G(g)$ whilst if $H$ is a subgroup of $G$, the normalizer of $H$ in $G$ is denoted by $N_G(H)$. If $G$ acts on a set $\Delta$ and $\sigma \in \Delta$, we denote the stabilizer in $G$ of $\sigma$ by Stab$(\sigma)$ and the $G$-orbit of $\sigma$ by $\sigma^G$. 
For a given group $G$, we denote the center of $G$ and the derived subgroup of $G$ by $Z(G)$ and $G' = [G, G]$ respectively.

For a ring $R$, if $M$ is an $R$-module and $N$ is an $R$-submodule of $M$, we use the notation $N \leq M$. If $S$ is a subring of $R$, we denote the restriction of $M$ to $S$ by $M_S$. Conversely, if $L$ is an $S$-module, then we denote the induction of $L$ to $R$ by $\text{Ind}_L^R$ or $L^R$. Similarly, if $\chi$ and $\psi$ are characters of $H$ and $G$ respectively with $H \leq G$, then we denote the restriction of $\psi$ to $H$ and the induction of $\chi$ to $G$ by $\psi|_H$ or $\psi_H$ and $\chi_H^G$, $\chi^G$ or $\chi \uparrow^G$ respectively. The set of all irreducible complex characters of $G$ is denoted by $\text{Irr}(G)$, and the inner product on complex characters is denoted $(\cdot, \cdot)$.

For a given group $G$ and field $k$, if there is a unique irreducible $kG$-module of dimension $i$, then we shall denote this by $i_{kG}$, or simply $i_G$ if $k$ is clear from the context. The zero $kG$-module will be denoted by $0_{kG}$ or 0. Given a reducible $kG$-module $M$ having composition series

$$0 = M_{n+1} \leq M_n \leq \cdots \leq M_2 \leq M_1 = M$$

and composition factors $N_i := M_i/M_{i+1}$, we shall sometimes denote $M$ by $N_1/N_2/\cdots/N_n$.

The notation $X := \ldots$ should be read as $X$ is defined to be $\ldots$. Finally, throughout the thesis we shall use the convention that functions are composed from left to right. Thus for functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we denote the image of $x \in X$ under the successive actions of $f$ and $g$ by $xfg$.

### 2.2 $p$-Groups

Let $p$ be a prime. A group $P$ is called a $p$-group if every element of $G$ has order $p^r$ for some $r \geq 0$. For finite groups this is equivalent to saying that $P$ has order $p^r$ for some $r \geq 0$. An abelian $p$-group, $P$, of exponent $p$ is called an elementary abelian $p$-group and we call $\log_p |P|$ the rank of $P$. We write $p^a$ for the elementary abelian $p$-group of rank $a$, namely $C_p^a$. An arbitrary $p$-group of order $p^a$ will be denoted by $[p^a]$. For the rest of this section, we will assume that $P$ is finite.

For an arbitrary $p$-group, $P$, we will often use the notation of $[RSS1]$. Thus we assume that $P$ has a composition series

$$1 = P_0 \lhd P_1 \lhd \cdots \lhd P_n = P$$

having elementary abelian quotients (also known as sections) $P_i/P_{i-1}$ of rank $r_i$ for $i = 1, \ldots, n$. The notation $p^{r_1+\cdots+r_n}$ is then used to represent $P$.

An important class of $p$-groups are the extra-special $p$-groups. Before describing these, we recall that the Frattini subgroup, $\Phi(G)$, of a finite group $G$ is the intersection of all maximal subgroups of $G$.

**Definition 2.2.1.** [$Gor80$] A $p$-group $P$ is called special if either $P$ is elementary abelian or $P$ is of nilpotency class 2 and $P' = Z(P) = \Phi(P)$ is elementary abelian. A special $p$-group, $P$, of nilpotency class 2 satisfying $|P'| = p$ is called extra-special.

There are two isomorphism classes of extra-special groups of order $p^3$. 
Theorem 2.2.2. \textit{[Gor80, Theorem 5.5.1]} A non-abelian $p$-group $P$ of order $p^3$ is extra-special and is isomorphic to one of the following groups if $p = 2$:

(i) $D := \text{Dih}(8) = \langle x, y | x^4 = y^2 = 1, x^y = x^3 \rangle$ - the dihedral group of order 8; or

(ii) $Q := \langle x, y, z | x^2 = y^2 = z, z^2 = 1, y^{-1}xy = x^{-1} \rangle$ - the quaternion group.

Meanwhile, if $p > 2$, then $P$ is isomorphic to one of:

(i) $M := \langle x, y, z | x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, \text{ and } [x, y] = z \rangle$; or

(ii) $N := \langle x, y | x^{p^2} = y^p = 1, x^y = x^{1+p} \rangle$.

We see that when $p = 2$, all extra-special groups of order 8 have exponent 4, whilst if $p > 2$, then the group $M$ has exponent $p$ and the group $N$ has exponent $p^2$. The groups $D$, $M$, $N$ and $Q$ may be used to classify all extra-special $p$-groups. To do so, we require a further definition.

Definition 2.2.3. \textit{[Suz82]} The group $G$ is called the central product of two subgroups $H$ and $K$ if $H$ and $K$ commute element-wise and $G = HK$.

The central product of $H$ and $K$ is denoted $H \ast K$. However, for the sake of brevity, we shall denote the central product of $i$ copies of $N$ (respectively $D$) and $j$ copies of $M$ (respectively $Q$) by $N^iM^j$ (respectively $D^iQ^j$).

Theorem 2.2.4. \textit{[Gor80, Theorem 5.5.2]} An extra-special $p$-group $P$ is the central product of $r \geq 1$ nonabelian subgroups of order $p^3$. Moreover, we have

(i) If $p$ is odd, $P$ is isomorphic to $N^kM^{r-k}$, while if $p = 2$, $P$ is isomorphic to $D^kQ^{r-k}$ for some $k$. In either case, $|P| = p^{2r+1}$.

(ii) If $p$ is odd and $k \geq 1$, $N^kM^{r-k}$ is isomorphic to $NM^{r-1}$, and the groups $M^r$ and $NM^{r-1}$ are not isomorphic.

(iii) If $p = 2$, then $D^kQ^{r-k}$ is isomorphic to $DQ^{r-1}$ if $k$ is odd and to $Q^r$ if $k$ is even, and the groups $Q^r$ and $DQ^{r-1}$ are not isomorphic.

We follow the conventions of \textit{[CCN+09]} and denote the extra-special 2-groups $DQ^{r-1}$ and $Q^r$ by $2_+^{1+2r}$ and $2_+^{1+2r}$ if $r$ is odd and by $2_+^{1+2r}$ and $2_+^{1+2r}$ if $r$ is even. We also denote $M^r$ by $p_+^{1+2r}$ and $NM^{r-1}$ by $p_+^{1+2r}$.

For any group $G$ and any prime $p$, we may construct a $p$-group.

Definition 2.2.5. \textit{[DH92]} Let $G$ be a group and $p$ be a prime. The $p$-core of $G$, $O_p(G)$, is defined to be

$$O_p(G) := \langle N | N \trianglelefteq G, N \text{ is a } p\text{-group} \rangle.$$
Clearly, $O_p(G)$ is the largest normal $p$-subgroup of $G$. In Chapters 3 and 6, we shall use the $p$-cores of the parabolic subgroups of an arbitrary group (defined in Section 2.5) as the analogue of the unipotent radicals of the parabolic subgroups of a finite group of Lie type. Indeed, in Chapter 3 we shall refer to the $p$-cores of certain groups as unipotent radicals (as this mirrors the motivating examples of presheaves from [RS85]).

Another tool that we shall use in Chapter 3 is that of local subgroups.

Definition 2.2.6. [Isa08] Let $G$ be a group and $p$ be a prime. A subgroup $H$ of $G$ is called $p$-local if $H$ is of the form $H = N_G(P)$, where $P$ is some nonidentity $p$-subgroup of $G$. A subgroup is called local if it is $p$-local for some prime $p$.

We shall see that one can use representations on a local level for a given group $G$ - representations that are already known - to obtain representations for the group $G$ itself.

2.3 Finite Simple Groups and ATLAS Notation

2.3.1 Finite Simple Groups

A group $G$ is called simple if it has no non-trivial proper normal subgroups. Simple groups are seen as the building blocks of finite group theory, since every finite group admits a composition series, which - by the Jordan-Hölder Theorem - has unique composition factors up to reordering. It follows that an understanding of finite simple groups is crucial to an understanding of many areas of finite group theory.

During the twentieth century, a project aimed at classifying all finite simple groups was undertaken, and this resulted in the Classification Theorem for Finite Simple Groups.

Theorem 2.3.1 (The Classification Theorem for Finite Simple Groups). [Wil09] Every finite simple group is isomorphic to one of the following:

(i) a cyclic group $C_p$ of prime order $p$;

(ii) an alternating group $\text{Alt}(n)$, for $n \geq 5$;

(iii) a classical group:

- linear: $\text{PSL}_n(q)$, $n \geq 2$, except $\text{PSL}_2(2)$ and $\text{PSL}_2(3)$;
- unitary: $\text{PSU}_n(q)$, $n \geq 3$, except $\text{PSU}_3(2)$;
- symplectic: $\text{PSp}_{2n}(q)$, $n \geq 2$, except $\text{PSp}_4(2)$;
- orthogonal: $\text{PO}_{2n+1}(q)$, $n \geq 3$, $q$ odd;
- $\text{PO}_{2n}^+(q)$, $n \geq 4$;
- $\text{PO}_{2n-1}^-(q)$, $n \geq 4$;

where $q$ is a power $p^a$ of a prime $p$.

As an aside, we note that a subgroup $P$ of $G$ for which $P = O_p(N_G(P))$ is called a radical subgroup or a $p$-radical subgroup.
(iv) an exceptional group of Lie type:

\[ G_2(q), q \geq 3; F_4(q); E_6(q); 2E_6(q); D_4(q); E_7(q); E_8(q) \]

where \( q \) is a prime power, or

\[ ^2B_2(2^{2n+1}), n \geq 1; ^2G_2(3^{2n+1}), n \geq 1; ^2F_4(2^{2n+1}), n \geq 1 \]

or the Tits group \( ^2F_4(2)' \);

(v) one of 26 sporadic simple groups:

- the five Mathieu groups: \( M_{11}, M_{12}, M_{22}, M_{23}, M_{24} \);
- the seven Leech lattice groups: \( Co_1, Co_2, Co_3, McL, HS, Suz, J_2 \);
- the three Fischer groups: \( Fi_{22}, Fi_{23}, Fi_{24}' \);
- the five Monstrous groups: \( M, E, Th, HN, He \);
- the six pariahs: \( J_1, J_3, J_4, O'N, Ly, Ru \).

The only repetitions within the groups in Theorem 2.3.1 are defined by the following six isomorphisms (see [Wil09]):

\[
\begin{align*}
PSL_2(4) & \cong PSL_2(5) \cong \text{Alt}(5); \\
PSL_2(7) & \cong PSL_3(2); \\
PSL_2(9) & \cong \text{Alt}(6); \\
PSL_4(2) & \cong \text{Alt}(8); \\
PSU_4(2) & \cong PSp_4(3). 
\end{align*}
\] (2.3.1)

A good overview of the finite simple groups listed in Theorem 2.3.1 and in (2.3.1) may be found in [Wil09].

The study of finite simple groups is an ongoing area of active research, and the quest to gain a greater understanding of certain groups or families of groups continues. The methods of Chapter 3 will hopefully allow a greater understanding of the representation theory of the sporadic groups over finite fields to be achieved, whilst analysis of the \( p \)-cuspidal characters of these groups found in Chapter 6 may lead to a greater understanding of certain geometries associated to these groups.

### 2.3.2 ATLAS Notation

To an arbitrary finite group \( G \), we may associate a shape. This gives certain information about \( G \) via extensions of groups, and is the notation used throughout this thesis and in the ATLAS. As we shall see, a shape of a group does not define the group up to isomorphism. However, it does give useful information about the group’s structure. We begin by defining the two types of extensions that we will encounter.

**Definition 2.3.2.** [Isa08] Let \( G, H \) and \( N \) be groups. If there exists a subgroup \( N_0 \leq G \) with \( N_0 \cong N \) and \( G/N_0 \cong H \), then we call \( G \) an extension of \( H \) by \( N \), denoted \( G = N.H \) or \( G = NH \).

A special type of extension is a split extension.
Definition 2.3.3. [Isa08] Let $G$, $H$ and $N$ be groups. If there exist subgroups $N_0, H_0 \triangleleft G$ with $N_0 \cong N$, $H_0 \cong H$, $G/N_0 \cong H$ and $N_0 \cap H_0 = 1$, then we call $G$ a split extension of $H$ by $N$, denoted $G = N : H$.

If $G = N : H$, then we also call $H$ a complement to $N$ in $G$.

Another way to view extensions is via the corresponding short exact sequence. Indeed, we have that $G = N : H$ if there exists a short exact sequence
\begin{equation}
1 \longrightarrow N_0 \longrightarrow G \longrightarrow H \longrightarrow 1
\end{equation}
for some normal subgroup $N_0 \cong N$ of $G$. The extension is then a split extension if the exact sequence (2.3.2) is right split (that is there exists a homomorphism $r : H \rightarrow G$ such that $rq = id_H$).

If $G$ is a split extension of $H$ by $N$, then we also refer to $G$ as the (internal) semidirect product of $N$ with $H$.

A specific type of semidirect product is given by the wreath product.

Definition 2.3.6. [DM96] Let $K$ and $H$ be groups and suppose $H$ acts on the nonempty set $\Gamma$. Then the wreath product of $K$ by $H$ with respect to this action is defined to be the semidirect product $\text{Fun}(\Gamma, K) \rtimes H$ where $H$ acts on the group $\text{Fun}(\Gamma, K)$ via
\[ f^x(\gamma) := f(\gamma'^{-1}) \quad \text{for all } f \in \text{Fun}(\Gamma, K), \gamma \in \Gamma \text{ and } x \in H. \]
2.3. FINITE SIMPLE GROUPS AND ATLAS NOTATION

We denote this group by $K \wr_{\Gamma} H$ (or $K \triangleright H$), and call the subgroup
\[ B := \{(f, 1) | f \in \text{Fun}(\Gamma, K)\} \cong \text{Fun}(\Gamma, K) \]
the base group of the wreath product.

Since all the sets that we encounter are finite, we will usually identify $\text{Fun}(\Gamma, K)$ with $\prod_{\gamma \in \Gamma} K_{\gamma}$ - the direct product of $|\Gamma|$ copies of $K$ - with $(k_{\gamma_1}, \ldots, k_{\gamma|\Gamma|}) \in \prod_{\gamma \in \Gamma} K_{\gamma}$ corresponding to the function $f : \Gamma \to K$ given by $f : \gamma \mapsto k_{\gamma}$.

In the case that $\Gamma$ is infinite, the above construction is known as the unrestricted wreath product of $K$ by $H$. If the functions in $\text{Fun}(\Gamma, K)$ are taken to have finite support (or equivalently, we take the direct sum $\bigoplus_{\gamma \in \Gamma} K_{\gamma}$ instead of the direct product), then we obtain a different group known as the restricted wreath product of $K$ by $H$. In the case that the set $\Gamma$ in Definition 2.3.6 is equal to $H$, and the action of $H$ on $\Gamma$ is given by the regular action (so for $x, h \in H$ the action of $h$ on $x$ is given by $x \cdot h = xh$), then the resulting wreath product is known as the regular wreath product of $K$ by $H$.

Example 2.3.7. Let $G$ be the regular wreath product of $C_p$ by $C_p$. The base group of $G$ consists of $p$ copies of $C_p$, which we may generate by the elements
\[ a_i := ((i-1)p + 1, \ldots, (i-1)p + p) \in \text{Sym}(\{(i-1)p + 1, \ldots, (i-1)p + p\}) \]
for $i = 1, \ldots, p$. Thus we may embed the base group within $\text{Sym}(p^2)$. The action of the second cyclic group on this base group is given by the regular action, and hence may be given by conjugation by the element
\[ x := (1, p+1, \ldots, p(p-1)+1)(2, p+2, \ldots, p(p-1)+2) \cdots (p, 2p, \ldots, p(p-1)+p) \in \text{Sym}(p^2). \]
Thus we see that
\[ G \cong \langle a_i | i = 1, \ldots, p \rangle = \langle a_1, x \rangle \leq \text{Sym}(p^2). \]
This is the representation of $C_p \wr C_p$ that we will use frequently in Chapter 3.

We now return to the notion of the shape of a group $G$. The aim is to describe some of the structure of $G$, and is best illustrated with an example.

Example 2.3.8. Let $G := \text{Sym}(6)$ and let $S$ be a self-normalizing Sylow 2-subgroup of $G$. By using MAGMA, we see that there are two isomorphic subgroups of $G$ containing $S$ as a maximal subgroup (see Example 2.5.3). Let $P$ be one of these subgroups. We see that - using the notation of Section 2.3 - $O_2(P) \cong 2^3$. Moreover, the quotient $P/O_2(P)$ is isomorphic to $\text{Sym}(3)$. Thus we say that $P$ has shape $2^3 \cdot \text{Sym}(3)$.

If the group $G$ has shape $X$, then we write $G \sim X$. Thus in the above example, $P \sim 2^3 \cdot \text{Sym}(3)$.

We conclude this section by noting that we will also follow ATLAS-notation when referring to characters of finite groups over the complex numbers in Chapter 3. Indeed, for a given group $G$ the character $\chi_i$ will equal the corresponding character of $G$ given in the character table in [CCN+$09$].
2.4 Complexes and Buildings

Let $G$ be a group of Lie type defined over a field $k$. There is a natural module $V$ associated to $G$, and we may consider this to be a vector space of dimension $n$ over $k$. The (non-zero) subspaces of $V$ have dimensions $1, \ldots, n$, and we may construct a flag of subspaces of $V$ of rank $r$ for each $1 \leq r \leq n$:

$$F : 0 \neq V_{i_1} \subseteq V_{i_2} \subseteq \cdots \subseteq V_{i_r}.$$ 

By constructing the stabilizer of all such flags, we obtain a structure known as a building, first introduced by Jacques Tits.\footnote{Tits gave a rigorous introduction to buildings in his book *Buildings of Spherical Type and Finite BN-pairs*.}

Before giving the definition of a building as formulated by Tits, we first recall a few basic ideas relating to complexes and simplexes. We begin by defining a simplex.

**Definition 2.4.1.** \cite{Suz82} A partially ordered set $X$ is called a simplex of rank $d$ if $X$ is isomorphic to the partially ordered set formed from all the subsets of a set of $d$ elements with respect to the containment relation.

Closely related to simplexes are complexes.

**Definition 2.4.2.** \cite{Suz82} Let $\Delta$ be a set endowed with a partial order which will be denoted by $A \triangleright B$. The set $\Delta$ is said to be a complex if the following two conditions are satisfied:

(a) For any two elements $A$ and $B$ of $\Delta$, there is a greatest lower bound (an element $C$ such that $C \subseteq A, C \subseteq B$, and $C$ is the largest element which satisfies these conditions - $C$ is denoted $A \cap B$).

(b) For any $A \in \Delta$, the subset of $\Delta$ which consists of the elements contained in $A$ forms a simplex.

For example, let $V$ be any vector space of dimension $d + 1$ over a field $k$, and let $\Delta$ denote the set of all subspaces of $V$. Define a partial order $\leq$ on $\Delta$ as follows; for all $A, B \in \Delta$,

$$A \leq B \iff B \subseteq A.$$ 

Given two elements $A, B \in \Delta$, the subspace of $V$ generated by $A$ and $B$ is the greatest lower bound of $A$ and $B$. Moreover, since $\Delta$ consists of all subspaces of $V$, the subset of $\Delta$ consisting of the elements contained in some $A \in \Delta$ (with respect to $\leq$) is naturally isomorphic to the partially ordered set on $\dim_k V - \dim_k A$ elements. Thus $\Delta$ is a complex.

As with most mathematical structures, we have the concept of a subcomplex, and for certain complexes we may also associate a rank.

**Definition 2.4.3.** \cite{Suz82} (i) A subset $\Gamma$ of a complex $\Delta$ is said to be a subcomplex of $\Delta$ if

$$A \in \Gamma, B \subseteq A \implies B \in \Gamma.$$ 

(ii) Given a complex $\Delta$ and $d \in \mathbb{N}$, if any element is contained in a maximal element which is a simplex of rank $d$, then we call $\Delta$ a complex of rank $d$.\footnote{Tits gave a rigorous introduction to buildings in his book *Buildings of Spherical Type and Finite BN-pairs*.}
Returning to our example, where $\Delta$ consisted of the complex of subspaces of a given vector space $V$ partially ordered by reverse inclusion, we see that for any two maximal elements $\alpha, \beta$ of $\Delta$ - so two subspaces of $V$ of dimension 1 over $k$ - there is an element of $\Delta$ of rank $d - 1$, namely the subspace $\gamma$ such that $\gamma \leq \alpha$ and $\gamma \leq \beta$. We conclude that in some sense $\alpha$ and $\beta$ are close. This idea is made rigorous by defining adjacency and connectivity within a complex.

**Definition 2.4.4.** [Suz82] Let $\Delta$ be a complex of rank $d$. Two maximal elements $A$ and $B$ are said to be adjacent if $A \neq B$ and if there is an element, $C$, of rank $d - 1$ such that $C \subseteq A$ and $C \subseteq B$. If $A'$ and $B'$ are arbitrary elements of $\Delta$, then we say that $A'$ and $B'$ are connected if there is a finite sequence $\{C_i\}_{i=1}^n$ of maximal elements $C_i$ of $\Delta$ such that $A' \subseteq C_1$, $B' \subseteq C_m$, and $C_i$ and $C_{i+1}$ are adjacent for each $i$. We say that $\Delta$ is connected if any $A', B' \in \Delta$ are connected.

Before defining a building, we must first give one final technical definition.

**Definition 2.4.5.** [Suz82] Let $\Gamma$ be a complex of rank $d$. If any element of $A$ of rank $d - 1$ is contained in exactly two maximal elements, then $\Gamma$ is said to be a thin complex. If there are at least three maximal elements containing any element of rank $d - 1$, then $\Gamma$ is called a thick complex.

We are now in a position to define a building.

**Definition 2.4.6.** [Suz82] A complex $\Delta$ of rank $d$ is said to be a building of rank $d$ if there is a collection $\mathcal{A}$ of subcomplexes of $\Delta$ such that $(\Delta, \mathcal{A})$ satisfies the following conditions:

(B1) $\Delta$ is a thick complex.

(B2) Every element of $\mathcal{A}$ is a connected, thin complex of rank $d$.

(B3) For any two elements $A$ and $B$ of $\Delta$, there is an element of $\mathcal{A}$ which contains both $A$ and $B$.

(B4) Let $A$ and $A'$ be two elements of $\Delta$. If $\Sigma$ and $\Sigma'$ are elements of $\mathcal{A}$ which contain both $A$ and $A'$, then there is an isomorphism $\phi$ from $\Sigma$ onto $\Sigma'$ which satisfies $\phi(B) = B$ for all $B \subseteq A$ as well as $\phi(B') = B'$ for all $B' \subseteq A'$.

A subcomplex of $\Delta$ which is a member of $\mathcal{A}$ is called an apartment. A maximal element of $\Delta$ is called a chamber.

Let us consider again the case of a finite group $G$ of Lie type, with natural module $V$ of dimension $d + 1$ over some field $k$. For the sake of example, we consider $G := GL_{d+1}(k)$. Let $\mathcal{P}(V)$ denote the set of all subspaces of $V$ and let $\Delta(\mathcal{P})$ denote the set of all flags of subspaces of $V$. Let $\Sigma = \{a_0, a_1, \ldots, a_d\}$ denote a basis for $V$. If

$$\mathcal{F} : 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$$

is a flag of subspaces of $V$ such that for each $V_i$ there is a subset $\Sigma_i$ of $\Sigma$ such that $\Sigma_i$ is a basis for $V_i$, then we say that $\Sigma$ supports $\mathcal{F}$. Let $\Sigma_{\Delta(\mathcal{P})}$ denote the set of all flags of subspaces of $V$ supported

---

3Previously we had denoted this set by $\Delta$. However, it actually corresponds to the projective geometry, $\mathcal{P}(V)$, of $V$. 

2.4. COMPLEXES AND BUILDINGS
by $\Sigma$. Then it can be proved that $\Delta(\mathcal{P})$ forms a complex under inclusion, which has the structure of a building, whose apartments are the complexes $\Sigma_{\Delta(\mathcal{P})}$ for each basis $\Sigma$ of $V$ (see [Suz82] for full details).

For each flag $F$ in $\Delta(\mathcal{P})$, let $G_F := \text{Stab}_G(F)$. Let $\Delta := \{G_F|F \in \Delta(\mathcal{P})\}$, and partially order $\Delta$ by reverse inclusion, denoted $\sqsubseteq$. If

$$F_1 \colon 0 \subseteq V_{i_0} \subseteq V_{i_1} \subseteq \cdots \subseteq V_{i_j} \subseteq \cdots \subseteq V_{i_n}; \quad \text{and} \quad F_2 \colon 0 \subseteq V_{i_0} \subseteq V_{i_1} \subseteq \cdots \subseteq V_{i_{j-1}} \subseteq V_{i_{j+1}} \subseteq \cdots \subseteq V_{i_n},$$

are elements of $\Delta(\mathcal{P})$, then $F_2$ is contained in $F_1$. It follows that

$$G_{F_1} = \text{Stab}_G(F_1) \subseteq \text{Stab}_G(F_2) = G_{F_2},$$

and hence that $G_{F_2} \subseteq G_{F_1}$. It follows that $\Delta$ and $\Delta(\mathcal{P})$ are isomorphic as complexes. For an apartment $\mathcal{A}$ of $\Delta(\mathcal{P})$, let $G_{\mathcal{A}}$ denote the subcomplex of $\Delta$ corresponding to the stabilizers in $G$ of elements of $\mathcal{A}$. Since properties (B1)-(B4) hold for the building $\Delta(\mathcal{P})$, they also hold for the isomorphic complex $\Delta$, with apartments given by the $G_{\mathcal{A}}$ where $\mathcal{A}$ is an apartment for $\Delta(\mathcal{P})$. We call $\Delta$ the building of $G$.

We conclude this section with a brief aside relating to abstract simplicial complexes.

**Definition 2.4.7.** [AB08] An (abstract) simplicial complex with vertex set $\mathcal{V}$ is a nonempty collection $\Delta$ of finite subsets of $\mathcal{V}$ (called simplices) such that every singleton $\{v\}$ is a simplex and every subset of a simplex $A$ is a simplex (called a face of $A$). The cardinality $r$ of $A$ is called the rank of $A$, and $r - 1$ is called the dimension of $A$. A subcomplex of $\Delta$ is a subset $\Delta'$ that contains, for each of its elements $A$, all the faces of $A$; thus $\Delta'$ is a simplicial complex in its own right, with vertex set equal to some subset of $\mathcal{V}$.

We will also require the following associated definition in Chapter 3.

**Definition 2.4.8.** [AB08] Let $\Delta$ be a simplicial complex, and let $\sigma$ be a simplex of $\Delta$. We define the star of $\sigma$, denoted $\text{St} \sigma$, to be the subset of $\Delta$ consisting of the simplices of $\Delta$ having $\sigma$ as a face.

If $\Delta$ is the building of a group $G$ of Lie type with natural module $V$ of dimension $n$ over $k$, then we set $\mathcal{V} := \{\text{Stab}_G(F)|F \in \Delta(\mathcal{P})\}$ to be our vertex set, stabilizers of flags of rank 2 to be edges, $\ldots$, and stabilizers of flags of rank $n$ to be chambers. For simplices $\sigma, \tau \in \Delta$, we define $\tau$ to be a face of $\sigma$ if $\sigma \subseteq \tau$. With this convention, we see that the building $\Delta$ has the structure of a simplicial complex.

### 2.5 Minimal Parabolic Systems

In Chapter 3, we shall see how Mark Ronan and Stephen Smith used the building of a finite Chevalley group $G$ to construct representations for $G$ over a finite field $k$ using structures known as presheaves. Naturally, they wished to construct such presheaves for more general groups as indicated in [Ron84].
2.5. MINIMAL PARABOLIC SYSTEMS

and [RSS5]. They proceeded to formulate a construction on an arbitrary simplicial complex with associated $G$-action (full details of which are given in Chapter 3). If $G$ is a finite group, then a rich source of such simplicial complexes arise from minimal parabolic systems.

The theory of minimal parabolic systems was first introduced by Ronan and Stroth [RSS4] and was later considered by Lempken, Parker and Rowley [LPR98]. Throughout this section, we assume that $G$ is a finite group, $p$ is a prime and that $p$ divides $|G|$.

**Definition 2.5.1.** [LPR98] Let $S \in \text{Syl}_p(G)$ and let $B := N_G(S)$. A minimal parabolic subgroup of $G$ is a subgroup $P \leq G$ such that $B$ is contained in a unique maximal subgroup of $P$. We denote the set of all minimal parabolic subgroups of $G$ with respect to $B$ by $\mathcal{M}(G,B)$.

We note that Ronan and Stroth imposed the additional requirement that $O_p(P) \neq 1$. However - like Lempken, Parker and Rowley - we do not require this.

Given such a set of minimal parabolic subgroups for $G$, it is natural to consider certain subsets of $\mathcal{M}(G,B)$ that generate $G$.

**Definition 2.5.2.** [LPR98]

(i) A set $S = \{P_1, \ldots, P_n\} \subseteq \mathcal{M}(G,B)$ is called a minimal parabolic system for $G$ of rank $n$ or a $p$-minimal parabolic system for $G$ of rank $n$ if $G = \langle S \rangle$ and no proper subset of $S$ generates $G$.

(ii) We call subgroups of the form $P := \langle P_{i_1}, \ldots, P_{i_m} \rangle$ parabolic subgroups of $G$ and denote them by $P_{i_1 \cdots i_m}$. The rank of $P$ is defined to be the minimal such $m$.

A full exposition of the $p$-minimal parabolic systems of the sporadic groups is given in [RS84].

**Example 2.5.3.** Let $G := \text{Sym}(6)$ and let $S \sim 2^{3+1}$ be a self-normalizing Sylow 2-subgroup of $G$. There are two isomorphic (but not conjugate) proper subgroups of $G$ containing $S$, having shape $2^3.\text{Sym}(3)$. It follows that $G$ has a unique 2-minimal parabolic system consisting of these two subgroups.

The parabolic subgroups associated to a minimal parabolic system of $G$ are self-normalizing as we now prove.

**Lemma 2.5.4.** Let $G$ be a finite group, $S \in \text{Syl}_p(G)$, $B = N_G(S)$ and let $S = \{P_1, \ldots, P_n\}$ be a minimal parabolic system for $G$ with respect to $B$. Then the parabolic subgroups of $G$ formed from $S$ are self-normalizing.

**Proof.** Let $P$ be a parabolic subgroup of $G$ formed from $S$, and let $g \in N_G(P)$. As $P$ contains $S$ and $P^g = P$, we deduce that $S^g \leq P^g = P$, and hence $S^g \in \text{Syl}_p(P)$. By Sylow’s Theorems, the Sylow $p$-subgroups of $P$ are conjugate to $S$ in $P$. Thus there exists $h \in P$ such that $S^g = S^h$, and hence $S^{gh^{-1}} = S$. Consequently $gh^{-1} \in N_G(S) = B \leq P$. As $h \in P$, it follows that $g \in P$ and so $N_G(P) \leq P$. Since the reverse inclusion clearly holds, we deduce that $P = N_G(P)$ and $P$ is self-normalizing.
Table 2.1: The shapes of the 2-parabolic subgroups of $\text{McL}$ that do not contain a Sylow 2-subgroup as a maximal subgroup.

By considering the parabolic subgroups associated to given minimal parabolic systems, we see that the set of all minimal parabolic systems for $G$ can be partitioned into two sets; geometric minimal parabolic systems and non-geometric minimal parabolic systems.

**Definition 2.5.5.** \([RS84]\) A minimal parabolic system $S$ is called geometric if for any two associated parabolic subgroups $P$ and $Q$, $P \cap Q$ is also a parabolic subgroup. If $S$ is not geometric, then we call it non-geometric.

**Example 2.5.6.** Let $G := \text{McL}$ and let $S$ be a self-normalizing Sylow 2-subgroup of $G$. There are eight 2-minimal parabolic subgroups of $G$, which we denote by $P_1$, $P_{1a} := P_1^\sigma$, $P_2$, $P_{2a} := P_2^\sigma$, $P_3$, $P_4$, $P_5$ and $P_{5a} := P_5^\sigma$, where $\sigma$ is the non-trivial outer-automorphism of $G$. We see that $S$ is maximal in all 2-minimal parabolic subgroups excluding $P_5$ and $P_{5a}$. These latter two subgroups have a unique maximal subgroup containing $S$, namely $P_4$.

The Hasse diagram of the associated parabolic subgroups is presented in \([RS84\ p79]\). The 2-minimal parabolic subgroups $P_1$ and $P_{1a}$ both have shape $2^{4+2}\text{Sym}(3)$, whilst the other 2-minimal parabolic subgroups containing $S$ as a maximal subgroup are pairwise isomorphic and also have shape $2^{4+2}\text{Sym}(3)$ (although they are not isomorphic to $P_1$). We summarise the shapes of the other parabolic subgroups of $G$ in Table 2.1. There are 16 minimal parabolic systems for $G$ associated to the given minimal parabolic subgroups. Of these, 9 are geometric minimal parabolic systems and 7 are non-geometric. We summarise these systems in Table 2.2.

For a given $p$-minimal parabolic system $S := \{P_1, \ldots, P_m\}$ for a group $G$ with respect to the normalizer, $B$, of a Sylow $p$-subgroup and for any $g \in G$, we see that $S^g = \{P_1^g, \ldots, P_m^g\}$ is a
2.6 Geometries and their Diagrams

<table>
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<tr>
<th>Geometric Minimal Parabolic Systems</th>
<th>Non-Geometric Minimal Parabolic Systems</th>
</tr>
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<tr>
<td>{P_1, P_{1a}, P_2}</td>
<td>{P_1, P_{2a}, P_{5a}}</td>
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</table>

Table 2.2: The geometric and non-geometric 2-minimal parabolic systems for the McLaughlin group, $McL$.

$p$-minimal parabolic system for $G$ with respect to $B^g$. Define $P_\emptyset := B$ and

$$\Delta := \bigcup_{g \in G} \{P_g^J : J \subseteq \{1, \ldots, m\}\}.$$  

We see that $\Delta$ is a simplicial complex. Indeed, to mirror the setting of buildings, we define the vertices of $\Delta$ to be parabolic subgroups of the form $P_g^J$ for some $g \in G$ and some $J \subseteq \{1, \ldots, m\}$ of cardinality $m - 1$ and use reverse inclusion. Thus edges are parabolic subgroups $P_g^J$ corresponding to some $J \subseteq \{1, \ldots, m\}$ of cardinality $m - 2$, and so on. Finally, we define conjugates of $P_\emptyset = B$ to be the chambers of $\Delta$.

The group $G$ has a natural action on $\Delta$ given by conjugation. We shall see in Chapter 3 that we are able to define presheaves on $\Delta$, and thus obtain representations for $G$.

2.6 Geometries and their Diagrams

In many areas of mathematics, we may partition a given set of elements into subsets, each subset containing elements that are in some sense of the same type. For example, if we consider the set of simplices of a 3-dimensional Euclidean tetrahedron, $T$, then we naturally partition the set into 0-, 1- and 2-dimensional simplices known as vertices, edges and faces of $T$. These ideas lead to the concept of an incidence geometry.

**Definition 2.6.1.** [Bue81] Let $\Delta$ be a set. An incidence geometry $\Gamma$ over $\Delta$ is a triple $\Gamma = (S, *, t)$, where $S$ is a set (the elements of $\Gamma$), $*$ is a symmetric and reflexive relation defined on $S$ (the incidence relation of $\Gamma$) and $t$ is a mapping of $S$ onto $\Delta$ (the type function of $\Gamma$) such that:

(TF) The restriction of $t$ to every maximal set of pairwise incident elements is a bijection onto $\Delta$ (transversality property).

For brevity, when the incident relation and type function are clear from the context we will simply refer to the geometry $S$. The transversality property (TF) is sometimes relaxed and replaced by the property:

(TF’’) If $x, y \in S$ ($x \neq y$) with $x * y$, then $t(x) \neq t(y)$. 

Clearly (TF) implies (TF'), however the converse is not true in general. Indeed we can easily extend any geometry $\Gamma$ by adding a single element $\alpha$ to the set $S$ and defining the type of $\alpha$ to be an element not contained in $\Delta$. Moreover, we may define $\alpha$ in such a way that it is not incident to any other element of $S$. Then clearly (TF') will still hold, however (TF) will not hold. Indeed, any maximal set, $\mathcal{M}$, of pairwise incident elements of $S \cup \{\alpha\}$ will either be $\{\alpha\}$ or will not contain $\alpha$, and hence the restriction of $t$ to $\mathcal{M}$ will not be a bijection onto $\Delta \cup \{t(\alpha)\}$.

An important property of a geometry is its rank. If $\Gamma = (S, *, t)$ is a geometry over $\Delta$, then the rank of $\Gamma$ is defined to be $|\Delta|$. Property (TF) illustrates the importance of sets of pairwise incident elements of a geometry, and we define a flag, $F$, of $\Gamma$ to be a (possibly empty) set of pairwise incident elements of $S$. Moreover, the type of $F$ is defined to be the image of $F$ under $t$, whilst the cotype of $F$ is defined to be $\Delta \setminus t(F)$. Clearly a flag, $F$, together with the restriction of $*$ and $t$ to $F$ forms a geometry over $t(F)$. The rank of this geometry is just $|F|$.

Given a flag of a geometry, we may construct a further geometry in the following way:

**Definition 2.6.2.** [Bue81] The residue of a flag $F$ in a geometry $\Gamma = (S, *, t)$ (over $\Delta$) is the geometry $\Gamma_F = (S_F, *, F, t_F)$ over $t(S_F) = \Delta \setminus t(F)$ defined by

- $S_F$ is the set of all elements of $\Gamma$ not in $F$, incident with all elements of $F$;
- $*_F$ is the restriction of $*$ to $S_F$;
- $t_F$ is the restriction of $t$ to $S_F$.

Returning to our example of the Euclidean tetrahedron, $T$, if we set $F$ to be a flag consisting of an incident face and vertex, then $S_F$ consists of the two edges of $T$ that are incident to the given face and vertex.

Given a geometry $\Gamma = (S, *, t)$ over a set $\Delta$, we may construct the incidence graph of $\Gamma$ whose vertex set is $S$ and where $x, y \in S$ are joined by an edge if and only if $x * y$. This is an easy graph to construct from a theoretical point of view, but it can be computationally cumbersome. It is standard to represent a geometry by the diagram of the geometry. This is constructed as follows; for each pair of distinct elements $i, j \in \Delta$, let $F$ be a flag of $\Gamma$ with residue $\Gamma_F$ of type $\{i, j\}$.

Construct the incidence graph $G_F$ of $\Gamma_F$. We define the $i$-diameter of $G_F$, denoted $d_{i,j}$, to be the maximal diameter at an element of type $i$, the $j$-diameter of $G_F$, denoted $d_{j,i}$, to be the maximal diameter at an element of type $j$, and the girth of $G_F$ to be $2g_{i,j}$ (so $g_{i,j} = g_{j,i}$). The diagram of the geometry $\Gamma$ has vertex set $\Delta$, and between each pair of incident vertices $i, j \in \Delta$, there is an undirected edge from $i$ to $j$ labelled $d_{i,j}, g_{i,j}, d_{j,i}$.

**Example 2.6.3.** Let $\Gamma$ be the geometry of vertices, edges and faces of the Euclidean tetrahedron, $T$, as described above. By labelling the vertices of $T$ by $v_1, \ldots, v_4$, the edges by $e_1, \ldots, e_6$ and the faces by $f_1, \ldots, f_4$ we may represent $T$ diagrammatically as in Figure 2.1. Let $F = F_{v_1}$ be a flag consisting of a single vertex. The residue $\Gamma_F$ consists of the three edges and three faces incident to $v_1$, and the incidence graph of $\Gamma_F$ can be seen to have edge-diameter and face-diameter equal to 3, and girth equal to 6. In a similar way if $F = F_{e_1}$ is a flag consisting of a single edge, then the incidence graph of $\Gamma_F$ has vertex-diameter and face-diameter equal to 2, and girth equal to...
2.6. GEOMETRIES AND THEIR DIAGRAMS

4. Finally if $F = F_i$ is a flag consisting of a single face, then the incidence graph of the residue $\Gamma_F$ has vertex-diameter and edge-diameter equal to 3, and girth equal to 6. The diagram of the geometry $\Gamma$ is given in Figure 2.2, where we use $v$, $e$ and $f$ to represent the three types of elements of $T$; vertex, edge and face respectively.

Example 2.6.3 illustrates a common situation. If for each pair of distinct types $i, j \in \Delta$, we have $d_{i,j} = d_{j,i} = g_{i,j}$, then we may simplify the diagram of our geometry to create the Buekenhout diagram of the geometry as follows: we represent each type in $\Delta$ by a circle, and we join $i, j \in \Delta$ by $g_{i,j}$ edges if $g_{i,j} \leq 4$ (noting that $2 \leq g_{i,j}$ by definition), and if $g_{i,j} > 4$, we join $i$ and $j$ by a single edge and label the edge $(n)$ where $n = g_{i,j} - 2$. Using this convention, we see that the Buekenhout diagram of the tetrahedral geometry from Example 2.6.3 is much simpler, and takes the form shown in Figure 2.3.

Our motivation for considering geometries arises from minimal parabolic systems. Indeed, given a minimal parabolic system $S = \{P_1, \ldots, P_n\}$ for a group $G$ with respect to a Sylow normalizer $B$, then we may form a geometry as follows: take $S$ to be the set containing all $G$-conjugates of the maximal parabolic subgroups $P_{\{1, \ldots, n\}\setminus\{i\}}$, $\Delta := \{1, \ldots, n\}$, define $t : S \to \Delta$ by $t : P_{\{1, \ldots, n\}\setminus\{i\}} \mapsto i$, and define the incidence relation $\ast$ by $P_{\{1, \ldots, n\}\setminus\{i\}} \ast P_{\{1, \ldots, n\}\setminus\{j\}}$ precisely when $i \neq j$ and there exists some $k \in G$ for which $B^k \subseteq P_{\{1, \ldots, n\}\setminus\{i\}} \cap P_{\{1, \ldots, n\}\setminus\{j\}}$. Since the transversality property clearly
holds, we conclude that \((S, \ast, t)\) is a geometry. The rank of the resulting geometry is equal to the rank of the minimal parabolic system \(S\). In Chapter 3 we shall consider rank 2 and rank 3 geometries arising from 2-minimal parabolic systems for the Mathieu Groups.

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\footnote{We note that our formulation of the geometry \(S\) differs from that used in \cite{RS84}. Ronan and Stroth defined two maximal parabolic subgroups of differing types to be incident if their intersection was also a parabolic subgroup. Thus they only considered the geometries defined by \textit{geometric} minimal parabolic systems. We have chosen a broader formulation of our geometries to allow all minimal parabolic systems to be considered.}
Chapter 3

Homology of Presheaves of Abelian Groups

The subject of representation theory for finite groups has a long and rich history stretching back to the nineteenth century. One of the first mathematicians to consider the subject was William Burnside (although we note that he referred to a representation of a group as a group of linear representations in his famous treatise Theory of Groups of Finite Order [Bur55]). By the beginning of the twentieth century representation theory was starting to be seen as a powerful tool in proving group-theoretic results, such as Burnside’s $p^aq^b$-Theorem. Further applications came to fruition later in the century, and representation theory lies at the heart of parts of the proof of the classification theorem for finite simple groups. This led to a comprehensive study of representations of finite groups over the complex numbers, and the publication of the ATLAS of character tables for many finite groups. [CCN+09]

Whilst considerable focus was paid to representation theory over the complex numbers, much time and effort was also devoted to modular representation theory. As early as 1902, Leonard Dickson had published a paper considering modular representations [Dic02], and over the course of the twentieth century, much work was undertaken on the subject. However, although many results have been formulated about modular representations and a modular atlas has been published [ABL+], the exact structure of many of the irreducible $kG$-modules for a given finite group $G$ finite field $k$ remain unknown.

The theory of modular representations has been well developed for finite Chevalley groups. For such a group, $G$, we have already seen that it is possible to define a simplicial complex consisting of the parabolic subgroups of $G$, known as the building of $G$. This simplicial complex can then be used to determine the irreducible $kG$-modules for a finite field $k$. In the 1980s, a number of mathematicians considered how the idea of the building of a group could be extended from the Chevalley groups to an arbitrary finite group. This led to the 1984 paper of Mark Ronan and Gernoth Stroth [RSS84], where they defined the notion of minimal parabolic subgroups and minimal parabolic systems of a finite group $G$ with respect to the normalizer of a given Sylow $p$-subgroup of $G$. They considered such a situation for each of the 26 sporadic simple groups, and for the fields $GF(p)$ where $p = 2, 3, 5$. Mark Ronan and Stephen Smith then used such minimal parabolic
By defining chain spaces and boundary maps on such presheaves, they were then able to form homology groups. These homology groups are $kG$-modules, and via their construction, considerable information is known about their structure.

The two mathematicians published a series of three joint papers on the subject between 1985 and 1989 ([RS85], [RS86], [RS89]), in addition to a further paper in 1989 published by Ronan ([Ron89]). Their work culminated in the calculation of the zero-homology groups of presheaves defined over the field $GF(2)$ for the groups $L_4(2), \text{Alt}(7)$ and the triple cover of $Sp_4(2)$ in addition to calculations of quotients of the zero-homology groups for presheaves of $M_{24}$.

Ronan and Smith went on to prove that for a finite Chevalley group $G$, there is a 1-1 correspondence between irreducible $kG$-modules $V$ and presheaves $F_V$, known as the fixed-point presheaf of the module $V$ [RS85, Section 3].

We begin this Chapter in Section 3.1 by introducing the basic motivation behind the subject and the underlying definitions introduced by Ronan and Smith. This is followed in Section 3.2 by a brief discussion about two important classes of presheaves; constant presheaves and fixed-point presheaves and in Section 3.3 by the formulation of the homology and cohomology groups of a given presheaf. The Euler characteristic is also introduced here. In Section 3.4 we consider the universal construction of a presheaf, from a minimal substructure known as a II-stalk. Such presheaves admit all other chamber-generated presheaves as quotients, and hence they are of great interest. Closely connected to universal presheaves are dual presheaves, and these are introduced in Section 3.5. These were first introduced by Ronan in [Ron89], and their importance lies in a theorem of Ronan relating the homology groups of a presheaf to the cohomology groups of the dual presheaf. This result is given - in addition to many other known results - in Section 3.6 as Theorem 3.6.10. In Section 3.7 we briefly consider the question of whether for a finite group $G$ and finite field $k$, every irreducible $kG$-module can occur as the quotient of the zero-homology group of a universal panel-irreducible presheaf. We see that in general this does not occur. The focus of the chapter then changes in Section 3.8 from theoretical to computational, as the zero-homology groups over $GF(2)$ of presheaves of the symmetric group $\text{Sym}(6)$ and the Mathieu groups $M_{11}$ and $M_{22}$ are explicitly calculated, whilst those of the Mathieu group $M_{12}$ are partially calculated. In addition, the irreducible quotients of the zero-homology groups of presheaves of the Mathieu groups $M_{23}$ and $M_{24}$ over $GF(2)$ are determined. These calculations are preceded by a discussion of a method known as geometric spanning, which can be used to obtain upper bounds on the dimensions of zero-homology groups. The chapter concludes in Section 3.9 with a few final remarks on the aforementioned question concerning irreducible $kG$-modules arising as quotients of zero-homology groups of universal panel-irreducible presheaves.

### 3.1 Presheaves and Coordinate Systems

Let $X$ be an arbitrary topological space. Denote by $\mathcal{X}$ the category of open subsets of $X$ with morphisms consisting of inclusion maps, and let $\text{Ab}$ denote the category of abelian groups.

---

1 Ronan and Smith referred to presheaves as sheaves throughout their papers [RS85], [RS86] and [RS89].
Definition 3.1.1. [Bre97] A presheaf $\mathcal{F}$ (of abelian groups) on $X$ is a contravariant functor from $X$ to Ab. Thus for each $U$ an open subset of $X$, we associate an abelian group $\mathcal{F}(U)$, and for each inclusion map $i : U \hookrightarrow V$ we associate a homomorphism of abelian groups $\tau_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that

1. $\tau_{U,U} = \text{id}_U$ for all open subsets $U \subseteq X$; and
2. for $U \subseteq V \subseteq W$ open subsets of $X$, we have $\tau_{U,W} = \tau_{V,W} \circ \tau_{U,V}$.

An example of a presheaf on $X$ is given by the constant presheaf. Let $A$ be an arbitrary abelian group. The constant presheaf of $A$ on $X$ is defined by setting $\mathcal{F}_p U = A$ for all open subsets $U \subseteq X$, and for open subsets $U \subseteq V$, define $\tau_{U,V} = \text{id}_A$. It is clear that this satisfies the given conditions, and hence $\mathcal{F}$ is a well-defined presheaf.

Ronan and Smith extended this notion of presheaf from topological spaces to simplicial complexes. First they defined a coefficient system as follows:

Definition 3.1.2. [RS85] Let $\Delta$ be a simplicial complex. A coefficient system $\mathcal{F}$ for $\Delta$ is a set of abelian groups $\{\mathcal{F}_\sigma \mid \sigma$ is a simplex of $\Delta\}$ together with connecting homomorphisms $\varphi_{\sigma\tau} : \mathcal{F}_\sigma \to \mathcal{F}_\tau$ whenever $\tau$ is a face of $\sigma$, such that if $\gamma$ is also a face of $\tau$, then

$$\varphi_{\sigma\gamma} = \varphi_{\sigma\tau} \circ \varphi_{\tau\gamma}. \quad (3.1.1)$$

Since the parabolic subgroups associated to a geometric minimal parabolic system of a group $G$ form a simplicial complex $\Delta$, Ronan and Smith imposed further conditions on a coefficient system $\mathcal{F}$ for $\Delta$, mirroring those properties found in such a minimal parabolic system. They required that each $\mathcal{F}_\sigma$ is a vector space over a finite field $k$, and that there exists a $G$-action on the formal direct sum of the terms of $\mathcal{F}$, corresponding to a permutation action of $G$ on $\Delta$. For $g \in G$, the restriction of the $g$-action to $\mathcal{F}_\sigma$, denoted $\tilde{g}_\sigma$, is a $k$-homomorphism from $\mathcal{F}_\sigma$ to $\mathcal{F}_{\sigma g}$ (the term at the simplex $\sigma g$). The focus of Ronan and Smith’s work was on $kG$-modules, so it is natural to further assume that the $G$-action commutes with the connecting maps. This means that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{F}_\sigma & \xrightarrow{\tilde{g}_\sigma} & \mathcal{F}_{\sigma g} \\
\varphi_{\sigma\tau} \downarrow & & \downarrow \varphi_{\sigma\tau g} \\
\mathcal{F}_\tau & \xrightarrow{\tilde{g}_\tau} & \mathcal{F}_{\tau g}
\end{array}
$$

whenever $\tau$ is a face of $\sigma$. In such a situation, we call the coefficient system $\mathcal{F}$ a presheaf on $\Delta$.\footnote{We note that for the sake of brevity, Ronan and Smith referred to presheaves as sheaves. However, we will not be adopting this convention.}

If for each simplex $\sigma \in \Delta$, the presheaf term $\mathcal{F}_\sigma$ is spanned by the images of the presheaf terms of the chambers of $\Delta$ containing $\sigma$ under the corresponding connecting maps,

$$\mathcal{F}_\sigma = \langle \mathcal{F}_c \varphi_{ca} \mid c \text{ is a chamber and } \sigma \text{ is a face of } c \rangle \quad \text{ for all } \sigma \in \Delta,$$

then we call $\mathcal{F}$ a chamber-generated presheaf.
Let $\Delta$ be a simplicial complex with associated $G$-action for some group $G$, and let $\mathcal{F}$ be a presheaf defined on $\Delta$. If $\sigma \in \Delta$ is a simplex, then we denote the stabilizer of $\sigma$ in $G$ by $G_\sigma$. For $g \in G_\sigma$, we have $\tilde{g}_\sigma : \mathcal{F}_\sigma \to \mathcal{F}_{\sigma g} = \mathcal{F}_\sigma$. It follows that $\mathcal{F}_\sigma$ has a $kG_\sigma$-module structure. We call a presheaf $\mathcal{F}$ irreducible if the terms of $\mathcal{F}$ at every simplex of $\Delta$ are irreducible modules. Similarly, we call $\mathcal{F}$ panel-irreducible if the terms of $\mathcal{F}$ at every panel and every chamber of $\Delta$ are irreducible modules.

By defining connecting maps $\varphi_{\sigma \tau} = \text{id}_\sigma$ for all simplices $\sigma \in \Delta$, we see that a coefficient system $\mathcal{F}$ on $\Delta$ with $G$-action satisfying (3.1.2) does indeed generalise the notion of a presheaf for a topological space.

As with most algebraic structures, we may define subpresheaves and quotient presheaves in the natural way. Let $\mathcal{F}$ be a presheaf defined on a simplicial complex $\Delta$, with connecting maps $\varphi_{\sigma \tau}$ whenever $\tau$ is a face of $\sigma$. A presheaf $\mathcal{F}$ on $\Delta$ is called a subpresheaf of $\mathcal{F}$ if $G_\tau \subseteq G_\sigma$ for each simplex $\sigma \in \Delta$, and whenever $\tau$ is a face of $\sigma$, the connecting map in $\mathcal{F}$ is given by $\varphi_{\sigma \tau}|_{G_\tau}$. Given a subpresheaf $\mathcal{G}$ of $\mathcal{F}$, we then define the quotient presheaf $\mathcal{F}/\mathcal{G}$, whose term at the simplex $\sigma$ is the quotient space $\mathcal{F}_\sigma/\mathcal{G}_\sigma$, and whose connecting maps are given by the corresponding quotient maps.

It is also natural to consider mappings between presheaves. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves defined on a simplicial complex $\Delta$, with connecting maps $\varphi_{\sigma \tau}$ and $\theta_{\sigma \tau}$ respectively (whenever $\tau$ is a face of $\sigma$), and with $G$-actions denoted by $\tilde{g}$ and $\hat{g}$ respectively (for each $g$ in the defining group $G$). A morphism from $\mathcal{F}$ to $\mathcal{G}$ is a set of $k$-linear maps $m : \mathcal{F}_\sigma \to \mathcal{G}_\sigma$ for each simplex $\sigma \in \Delta$, such that the following two diagrams commute:

(i) if $\tau$ is a face of $\sigma$, then

\[
\begin{array}{ccc}
\mathcal{F}_\sigma & \xrightarrow{m_\sigma} & \mathcal{G}_\sigma \\
\varphi_{\sigma \tau} \downarrow & & \downarrow \theta_{\sigma \tau} \\
\mathcal{F}_\tau & \xrightarrow{m_\tau} & \mathcal{G}_\tau
\end{array}
\]

(ii) for each $\sigma \in \Delta$ and each $g \in G$,

\[
\begin{array}{ccc}
\mathcal{F}_\sigma & \xrightarrow{m_\sigma} & \mathcal{G}_\sigma \\
\tilde{g} \downarrow & & \downarrow \hat{g} \\
\mathcal{F}_{\sigma g} & \xrightarrow{m_{\sigma g}} & \mathcal{G}_{\sigma g}
\end{array}
\]  

We call the morphism $m : \mathcal{F} \to \mathcal{G}$ injective if each $k$-linear map $m_\sigma$ is injective, surjective if each $m_\sigma$ is surjective, and we call $m$ an isomorphism if it is both injective and surjective. From a categorical point-of-view, we see that morphisms of presheaves generalise the notion of natural transformations between presheaves (when viewed as functors). We note that if $m : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then each $m_\sigma \in m$ is a $kG_\sigma$-homomorphism.

Before continuing with the theory of presheaves, we consider the notion of isomorphic presheaves further. If $m : \mathcal{F} \to \mathcal{G}$ is an isomorphism of presheaves, then using the commutativity of (3.1.3)
(resp. (3.1.4)), we see that the connecting maps (resp. $G$-action) in $\mathcal{G}$ are uniquely determined by the connecting maps (resp. $G$-action) in $\mathcal{F}$. The fact that commutativity holds in (3.1.3) is crucial to the notion of an isomorphism. Indeed, let $\Delta$ be the simplicial complex of parabolic subgroups of some group $G$. For each parabolic subgroup $\sigma = P_a \in \Delta$, define $\mathcal{F}_\sigma = \mathcal{G}_\sigma = kP_a$. Define connecting maps in $\mathcal{F}$ by setting $\varphi_{\sigma\tau}$ to be the inclusion map, and in $\mathcal{G}$ by setting $\theta_{\sigma\tau}$ to be the zero map, whenever $\tau$ is a face of $\sigma$. It is easy to check that $\mathcal{F}$ and $\mathcal{G}$ with their connecting maps $\varphi_{\sigma\tau}$ and $\theta_{\sigma\tau}$ respectively are presheaves (where the $G$-action in each presheaf is given by multiplication in $kG$). However, $\mathcal{F}$ and $\mathcal{G}$ are clearly not isomorphic presheaves.

### 3.2 Classes of Presheaves

We now consider two important classes of presheaves that we will use frequently throughout our subsequent work; constant presheaves and fixed-point presheaves. We define constant presheaves for arbitrary simplicial complexes. However, we will only define fixed-point presheaves on the simplicial complex of parabolic subgroups of a group $G$ defined by a given geometric minimal parabolic system. Unless otherwise stated, for the rest of this chapter $k$ will denote a finite field of characteristic $p$.

#### 3.2.1 Constant Presheaves

Let $V$ be a $kG$-module. In Section 3.1 we saw how to define the constant presheaf of $V$ on a topological space $X$. Given a simplicial complex $\Delta$ (with associated $G$-action) we can follow a similar construction.

**Definition 3.2.1.** [RS85] The constant presheaf of $V$ on $\Delta$, denoted $K_V$, is the presheaf on $\Delta$ with

1. $(K_V)_\sigma := V$ for every simplex $\sigma \in \Delta$;
2. for $\tau$ a face of $\sigma$ we define $\varphi_{\sigma\tau}$ to be the identity map on $V$; and
3. the $G$-action in $K_V$ is defined by the $G$-action in the module $V$.

It is clear from the definition that constant presheaves are chamber-generated.

Suppose that $n : V \to W$ is a $kG$-module isomorphism. By setting $m_\sigma = n$ for each $\sigma \in \Delta$ we see that $m := \{m_\sigma | \sigma \in \Delta\}$ is an isomorphism of presheaves from $K_V$ to $K_W$. Hence isomorphic $kG$-modules give rise to isomorphic constant presheaves.

#### 3.2.2 Fixed-Point Presheaves

Let $G$ be a group and let $\Delta$ be the simplicial complex of parabolic subgroups of $G$ defined by a geometric minimal parabolic system. Let $V$ be a $kG$-module. For each parabolic subgroup $\sigma \in \Delta$ we may consider the unipotent radical $U_\sigma := O_p(G_\sigma)$ of $G_\sigma$ (as defined in Chapter 2). Let $V_\sigma$ be the fixed-point subspace of $V$ under $U_\sigma$:

$$V_\sigma := \{v \in V | v \ast u = v \text{ for all } u \in U_\sigma\}.$$
CHAPTER 3. HOMOLOGY OF PRESHEAVES OF ABELIAN GROUPS

Since $U_\sigma$ is a normal subgroup of $G_\sigma$, it follows that $V_\sigma$ is a $kG_\sigma$-module (more precisely, it is a $kG_\sigma$-submodule of $V|_{kG_\sigma}$). Moreover, if $\sigma \subseteq \tau$, then $G_\sigma \subseteq G_\tau$. Since $U_\sigma, U_\tau$ are normal $p$-subgroups of each Sylow $p$-subgroup of $G_\sigma$, it follows that $U_\tau$ is a normal $p$-subgroup of $G_\sigma$. Since $U_\sigma$ is the largest such subgroup, we have that $U_\tau \subseteq U_\sigma$ and hence $V_\tau \subseteq V_\sigma$.

Definition 3.2.2. [RS85] The fixed-point presheaf of $V$ on $\Delta$, denoted $F_V$, is defined by setting:

(i) $(F_V)_\sigma := V_\sigma$ for each $\sigma \in \Delta$;

(ii) for $\tau$ a face of $\sigma$, setting $\varphi_{\sigma\tau}$ to be the inclusion map from $V_\sigma$ to $V_\tau$; and

(iii) for $g \in G$, define the action of $g$ on $F_V$ to be the action defined within $V$.

Since the connecting maps in $F_V$ are inclusion maps, properties (3.1.1) and (3.1.2) clearly hold, and we have a well-defined presheaf. Clearly for a $kG$-module $V$, we have $F_V \subseteq K_V$ and equality holds when $V$ is the trivial $kG$-module.

When $G$ is a finite Chevalley group, we have the following result.

Theorem 3.2.3. [Smi82] Let $G$ be a finite Chevalley group over the finite field $k$, let $P$ be a parabolic subgroup of $G$ having unipotent radical $U$ and complement $L$ and let $V$ be a finite-dimensional irreducible $kG$-module. Then the fixed-point subspace $V^U$ affords an irreducible $kL$-module.

An immediate consequence of this is that for finite Chevalley groups, the fixed-point presheaf $F_V$ is irreducible.

We note that fixed-point presheaves are not necessarily chamber-generated, as illustrated by the following example.

Example 3.2.4. Let $G := McL$. The 2-minimal parabolic subgroups and systems of $G$ were introduced in Example 2.5.6. We note that the 22-dimensional irreducible $GF(2)McL$-module $V$ gives rise to a fixed-point space $F_B \cong 1 \oplus 1 \oplus 1$. From this, we deduce that the $kP$-modules

$$\langle F_c \varphi_{c,P} | c \text{ is a chamber of } P \rangle$$

for $P = P_{11a4}, P_{15a}, P_{1a5}, P_{55a}$ have dimensions 21, 21, 21 and 20 respectively over $GF(2)$. However, the 2-cores of each of these maximal 2-parabolic subgroups is trivial. Hence the presheaf terms $F_{P_{11a4}}, F_{P_{15a}}, F_{P_{1a5}}$ and $F_{P_{55a}}$ are all equal to $V$, and thus have dimension 22 over $GF(2)$. It follows that for any 2-minimal parabolic system of McL containing $P_{11a4}, P_{15a}, P_{1a5}$ or $P_{55a}$, the fixed-point presheaf $F_V$ is not chamber-generated.

3.3 Homology of Presheaves

A simplicial complex, $\Delta$, consists of simplices, which we may partition into sets according to their rank. If a group $G$ acts on $\Delta$ naturally, then it will preserve the rank of simplices. Hence for a presheaf defined on $\Delta$, we may think of the presheaf terms at simplices of equal rank to be - in some senses - similar. We make this idea precise through the notion of chain spaces for a presheaf, from
3.3. HOMOLOGY OF PRESHEAVES

which we may define homology groups. These homology groups will turn out to be $kG$-modules and are of great interest.

For the remainder of this chapter, we will assume that each simplicial complex $\Delta$ has a natural $G$-action for some group $G$.

3.3.1 Homology Groups

Let $\Delta$ be a simplicial complex of dimension $n$ having underlying vertex set $\{v_i\}_{i=0}^m$ for some $m \in \mathbb{N}$ and let $\mathcal{F}$ be a presheaf defined on $\Delta$. For each $r = 0, \ldots, n$ denote by $C_r(\Delta, \mathcal{F})$ the formal direct sum of the presheaf terms at $r$-simplices. As the action of $G$ on $\mathcal{F}$ preserves the simplicial dimension, each $C_r(\Delta, \mathcal{F})$ is a $kG$-module. We call $C_r(\Delta, \mathcal{F})$ the $r$-th chain space of $\mathcal{F}$ with respect to $\Delta$.

The connecting maps in $\mathcal{F}$ map direct summands of $C_r(\Delta, \mathcal{F})$ into $C_{r-1}(\Delta, \mathcal{F})$ for each $r = 1, \ldots, n$. We use these maps to define boundary maps $\partial_r : C_r(\Delta, \mathcal{F}) \to C_{r-1}(\Delta, \mathcal{F})$. Indeed, let $\sigma$ be an $r$-simplex. Then $\sigma = \{v_{i_0}, \ldots, v_{i_r}\} \subseteq \{v_0, \ldots, v_m\}$ with $i_j < i_{j+1}$ for all $j = 0, \ldots, r-1$. Define the type of $\sigma$ to be $v_{i_0} < \cdots < v_{i_r}$ and define $\sigma_j := \{v_{i_0}, \ldots, v_{i_j-1}, v_{i_{j+1}}, \ldots, v_{i_r}\}$ for each $j = 0, \ldots, r$. Thus $\sigma_j$ is a maximal face of $\sigma$ of type $v_{i_0} < \cdots < v_{i_{j-1}} < v_{i_{j+1}} < \cdots < v_{i_r}$. We define $\partial_{\sigma, \sigma_j} := (-1)^j \varphi_{\sigma, \sigma_j}$ and then define $\partial_\sigma := \sum_{j=0}^r \partial_{\sigma, \sigma_j}$. Finally, summing over all simplices of dimension $r$, we obtain a map $\partial_r := \sum_{\dim \sigma = r} \partial_\sigma : C_r(\Delta, \mathcal{F}) \to C_{r-1}(\Delta, \mathcal{F})$. We define $\partial_0$ to be the zero map on $C_0(\Delta, \mathcal{F})$.

Since each $\partial_r$ is a linear combination of connecting maps of the presheaf $\mathcal{F}$, and these connecting maps commute with the $G$-action on $\mathcal{F}$, we see that $\partial_r$ commutes with the $G$-action. Moreover, since the connecting maps are homomorphisms, each map $\partial_r$ is a $kG$-homomorphism. We call the maps $\partial_r$ (for $r = 0, 1, \ldots, n$) boundary maps. The kernel of $\partial_r$ is called the space of $r$-cycles and is denoted $Z_r(\Delta, \mathcal{F})$. The image of $\partial_r$, denoted $B_{r-1}(\Delta, \mathcal{F})$, is called the space of $r$-boundaries. We set $B_n(\Delta, \mathcal{F})$ to be the 0-space.

**Lemma 3.3.1.** The chain spaces and boundary maps as defined above form a chain complex

$$0 \to C_n(\Delta, \mathcal{F}) \xrightarrow{\partial_n} C_{n-1}(\Delta, \mathcal{F}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(\Delta, \mathcal{F}) \xrightarrow{\partial_0} 0$$

**Proof.** Let $r \in \{2, \ldots, n\}$. It suffices to show that $(\partial_r \partial_{r-1})|_{\mathcal{F}_\sigma} = 0$ for an arbitrary $r$-simplex $\sigma \in \Delta$. Indeed, assume that $\sigma$ has type $a_0 < \cdots < a_r$. Let $\sigma_i$ denote its maximal face of type $a_0 < \cdots < a_{i-1} < a_{i+1} < \cdots < a_r$ and for $i \neq j$ let $\sigma_{ij}$ denote the maximal face of $\sigma_i$ of type $a_0 < \cdots < a_{i-1} < a_{i+1} < \cdots < a_{j-1} < a_{j+1} < \cdots < a_r$ or $a_0 < \cdots < a_{j-1} < a_{j+1} < \cdots < a_{i-1} < a_{i+1} < \cdots < a_r$. The connecting maps $\partial_{\sigma, \sigma_i}$ are zero on $\sigma$ because they are zero on all faces of $\sigma$. Moreover, $\partial_r$ is a linear combination of connecting maps $\partial_{\sigma, \sigma_j}$ for all faces $\sigma_j$ of $\sigma$. Therefore, $(\partial_r \partial_{r-1})|_{\mathcal{F}_\sigma} = 0$.


where we have used property (3.1.1) to deduce that 
\( \varphi_m \) is a presheaf defined on \( \Delta \). It follows that \( \sigma_{ij} = \sigma_{ji} \). Thus

\[
(\partial_r \partial_{r-1})|_{\mathcal{F}_\sigma} = \partial_\sigma \partial_{r-1}
\]

\[
= \sum_{i=0}^{r} (-1)^i \varphi_{\sigma_i} \left( \sum_{j=0}^{r} (-1)^j \varphi_{\sigma_{ij}} - \sum_{j=0}^{r} (-1)^j \varphi_{\sigma_{ji}} \right)
\]

\[
= \sum_{i=0}^{r} \sum_{j=0}^{r} (-1)^{i+j} \varphi_{\sigma_i} \varphi_{\sigma_{ij}} - \sum_{i=0}^{r} \sum_{j>0}^{r} (-1)^{i+j} \varphi_{\sigma_i} \varphi_{\sigma_{ji}}
\]

\[
= \sum_{i=0}^{r} \sum_{j=0}^{r} (-1)^{i+j} \varphi_{\sigma_i} \varphi_{\sigma_{ij}} - \sum_{i=0}^{r} \sum_{j<i}^{r} (-1)^{i+j} \varphi_{\sigma_i} \varphi_{\sigma_{ij}}
\]

\[
= \sum_{i=0}^{r} \sum_{j=0}^{r} (-1)^{i+j} \varphi_{\sigma_{ij}} - \sum_{j=0}^{r} \sum_{i<j}^{r} (-1)^{i+j} \varphi_{\sigma_{ij}} = 0,
\]

where we have used property (3.1.1) to deduce that 
\( \varphi_{\sigma_{ij}} \varphi_{\sigma_{ij}} = \varphi_{\sigma_{ij}} \varphi_{\sigma_{ij}} = \varphi_{\sigma_{ij}} \). Thus by our comments above the result holds.

\[\square\]

**Definition 3.3.2.** [RS85] Let \( \mathcal{F} \) be a presheaf defined on \( \Delta \). We define the homology groups, \( H_r(\Delta, \mathcal{F}) \), of \( \mathcal{F} \) to be

\[
H_r(\Delta, \mathcal{F}) := \frac{Z_r(\Delta, \mathcal{F})}{B_r(\Delta, \mathcal{F})}
\]

for \( r = 0, \ldots, n \).

These homology groups are well-defined by Lemma 3.3.1. When the simplicial complex \( \Delta \) is understood, we drop all references to \( \Delta \), and denote our chain spaces, cycle spaces, boundary spaces and homology groups by \( C_r(\mathcal{F}), Z_r(\mathcal{F}), B_r(\mathcal{F}) \) and \( H_r(\mathcal{F}) \) respectively.

It is natural to consider presheaves up to isomorphism. To do so, the following result will be useful:

**Lemma 3.3.3.** Let \( \mathcal{F} \) and \( \mathcal{G} \) be isomorphic presheaves defined on a simplicial complex \( \Delta \). Then \( H_0(\mathcal{F}) \) and \( H_0(\mathcal{G}) \) are isomorphic \( kG \)-modules.

**Proof.** Assume that \( \mathcal{F} \) and \( \mathcal{G} \) are isomorphic via an isomorphism \( m := \{ m_\sigma : \mathcal{F}_\sigma \rightarrow \mathcal{G}_\sigma | \sigma \in \Delta \} \).

Since \( m \) is a presheaf morphism it commutes with connecting maps of \( \mathcal{F} \) and \( \mathcal{G} \). The boundary maps \( \partial_i \) and \( \delta_i \) (of \( \mathcal{F} \) and \( \mathcal{G} \) respectively) are linear combinations of connecting maps, hence also commute with \( m \). Moreover, if we consider \( m \) as acting on the chain spaces \( C_i(\mathcal{F}) \), we have a chain
map between the chain complexes \((C_i(\mathcal{F}), \partial_i)\) and \((C_i(\mathcal{G}), \delta_i)\):

\[
\cdots \longrightarrow C_2(\mathcal{F}) \overset{\delta_2}{\longrightarrow} C_1(\mathcal{F}) \overset{\partial_1}{\longrightarrow} C_0(\mathcal{F}) \overset{\delta_0}{\longrightarrow} 0 \\
\cdots \longrightarrow C_2(\mathcal{G}) \overset{\delta_2}{\longrightarrow} C_1(\mathcal{G}) \overset{\partial_1}{\longrightarrow} C_0(\mathcal{G}) \overset{\delta_0}{\longrightarrow} 0
\] (3.3.5)

By diagram chasing in (3.3.5) we see that \(m\) maps \(\text{im} \partial_1\) bijectively onto \(\text{im} \delta_1\). Moreover, as \(m\) is a presheaf morphism, it commutes with the action of the group algebra \(kG\), and hence is a \(kG\)-isomorphism from \(C_0(\mathcal{F})\) to \(C_0(\mathcal{G})\) which restricts to a \(kG\)-isomorphism from \(\text{im} \partial_1\) to \(\text{im} \delta_1\). It follows that \(m\) induces a \(kG\)-isomorphism from \(H_0(\mathcal{F})\) to \(H_0(\mathcal{G})\) as required.

The argument above can evidently be generalised to show that if \(\mathcal{F}\) and \(\mathcal{G}\) are isomorphic presheaves, then \(H_i(\mathcal{F})\) and \(H_i(\mathcal{G})\) will be isomorphic \(kG\)-modules for all \(i\). We also note that this is a necessary but not sufficient condition. Indeed, let \(\Delta\) be the simplicial complex of parabolic subgroups associated to the unique 2-minimal parabolic system of \(\text{Sym}(6)\) given in Example 2.5.3. Define the presheaf \(\mathcal{F}\) on \(\Delta\) to have 0-dimensional terms at simplices of type \(P_1\) and \(P_2\) and a 1-dimensional irreducible module at presheaf terms of type \(B\). Then clearly \(H_0(\mathcal{F}) \cong H_0(K_0) \cong 0\), but \(\mathcal{F} \not\cong K_0\).

We conclude this subsection by considering a presheaf \(\mathcal{F}\) defined on \(\Delta\). For each vertex \(\sigma \in \Delta\), let \(\varphi_\sigma\) denote the restriction of the quotient map \(\varphi : C_0(\mathcal{F}) \to H_0(\mathcal{F})\) to \(\mathcal{F}_\sigma\). For a simplex \(\gamma \in \Delta\), if \(\gamma\) is not a vertex, then choose a vertex \(\sigma\) of \(\gamma\). We define \(\varphi_\gamma : = \varphi_{\gamma\sigma}\varphi_\sigma\). The definition of \(\varphi_\gamma\) appears to depend on the choice of vertex \(\sigma\), but this is not the case. Indeed, let \(\tau\) be another vertex of \(\gamma\). Since \(\Delta\) is a simplicial complex, the vertices \(\sigma\) and \(\tau\) lie in a common 1-simplex \(\beta\) where \(\partial_\beta = \pm(\varphi_{\beta\sigma} - \varphi_{\beta\tau})\). Since \(\beta\) is a 1-simplex, the image of \(\partial_\beta\) is contained in the image of \(\partial_1\), which in turn is contained in the \(\text{ker} \varphi\). We deduce that \(\varphi_{\beta\sigma}\varphi_\sigma - \varphi_{\beta\tau}\varphi_\tau = 0\). Consequently, using property (3.1.1) for connecting maps we have

\[
\varphi_\gamma \circ \varphi_\sigma = (\varphi_{\gamma\beta} \circ \varphi_{\beta\sigma}) \circ \varphi_\sigma = (\varphi_{\gamma\beta} \circ \varphi_{\beta\tau}) \circ \varphi_\tau = \varphi_{\gamma\tau} \circ \varphi_\tau,
\]

and our maps \(\varphi_\gamma\) for \(\gamma \in \Delta\) are well-defined. We will use these maps to define universal presheaves in Section 3.4.

### 3.3.2 Cohomology Groups

For a presheaf \(\mathcal{F}\) defined on a simplicial complex \(\Delta\) over the field \(k\), the chain spaces \(C_i(\Delta, \mathcal{F})\) may be used to define a co-chain complex. Indeed we take \(C^i(\Delta, \mathcal{F}) := \text{Hom}(C_i(\Delta, \mathcal{F}), k)\) and for \(f \in C^i(\Delta, \mathcal{F})\), define \(f\delta^i := \partial_{i+1}f\). This gives rise to the co-chain complex

\[
0 \longrightarrow C^0(\Delta, \mathcal{F}) \overset{\delta^0}{\longrightarrow} C^1(\Delta, \mathcal{F}) \overset{\delta^1}{\longrightarrow} \cdots \overset{\delta^{n-1}}{\longrightarrow} C^n(\Delta, \mathcal{F}) \longrightarrow 0.
\]

It follows that we may define the \(r\)-th cohomology group of a given presheaf.
Definition 3.3.4. \[\text{[Ron89]}\] Let \( \mathcal{F} \) be a presheaf defined on \( \Delta \). We define the \( r \)-th cohomology group, \( H^r(\Delta, \mathcal{F}) \), of \( \mathcal{F} \) to be

\[
H^r(\Delta, \mathcal{F}) := \begin{cases} 
\ker \delta^0 & \text{if } r = 0; \\
\ker \delta^r & \text{if } r = 1, \ldots, n - 1; \text{ and} \\
\text{im} \delta^{-1} & \text{if } r = n.
\end{cases}
\]

As with homology groups, when the simplicial complex \( \Delta \) is clear, we will suppress it in our notation and we shall respectively write \( C^r(\mathcal{F}) \) and \( H^r(\mathcal{F}) \) for \( C^r(\Delta, \mathcal{F}) \) and \( H^r(\Delta, \mathcal{F}) \).

We shall see in Section 3.6 that there is a close relation between the zero-th and \( n \)-th homology groups of a presheaf \( \mathcal{F} \) and the respective \( n \)-th and zero-th cohomology groups of its dual presheaf \( \mathcal{F}^* \) (as defined in Section 3.5).

### 3.3.3 Euler Characteristic

Closely associated to chain spaces and homology groups is the notion of the Euler characteristic of a presheaf. We assume that a geometric minimal parabolic system of rank \( n + 1 \) for a group \( G \) (and hence a simplicial complex \( \Delta \)) has been fixed.

**Definition 3.3.5.** The Euler characteristic of the presheaf \( \mathcal{F} \), denoted \( \chi(\mathcal{F}) \), is the alternating sum

\[
\chi(\mathcal{F}) := \sum_{i=0}^{n} (-1)^i \dim H_i(\mathcal{F})
\]

The Euler characteristic can also be defined in terms of dimensions of the chain spaces of the presheaf \( \mathcal{F} \) as we now prove.

**Lemma 3.3.6.** For a presheaf \( \mathcal{F} \) we have

\[
\chi(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim H_i(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim C_i(\mathcal{F}).
\]

**Proof.** This is a simple calculation. We note that \( \dim H_i(\mathcal{F}) = \dim Z_i(\mathcal{F}) - \dim B_i(\mathcal{F}) \) for all \( i = 0, \ldots, n \). Moreover, as each boundary map \( \partial_i : C_i(\mathcal{F}) \to C_{i-1}(\mathcal{F}) \) is a \( kG \)-homomorphism, we have \( C_i(\mathcal{F})/\ker \partial_i \cong \text{im} \partial_i \) and hence

\[
\dim C_i(\mathcal{F}) = \dim \text{im} \partial_i + \dim \ker \partial_i = \dim B_{i-1}(\mathcal{F}) + \dim Z_i(\mathcal{F}).
\]

It follows that

\[
\chi(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim H_i(\mathcal{F})
\]

\[
= \sum_{i=0}^{n} (-1)^i \left( \dim Z_i(\mathcal{F}) - \dim B_i(\mathcal{F}) \right)
\]

\[
= \dim Z_0(\mathcal{F}) + (-1)^{n+1} \dim B_n(\mathcal{F}) + \sum_{i=1}^{n} (-1)^i \left( \dim Z_i(\mathcal{F}) + \dim B_{i-1}(\mathcal{F}) \right)
\]

\[
= \dim C_0(\mathcal{F}) + 0 + \sum_{i=1}^{n} (-1)^i \dim C_i(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim C_i(\mathcal{F})
\]

as required. \( \square \)
As the chain spaces $C_i(\mathcal{F})$ are formal direct sums of presheaf terms, their dimensions are easily determined. Lemma 3.3.6 allows us to effortlessly calculate the Euler characteristic of $\mathcal{F}$, and thus we obtain a useful relationship between the dimensions of the homology groups of $\mathcal{F}$. This is a method we will use frequently in Section 3.8, and is very powerful when used in conjunction with the dual presheaf $\mathcal{F}^*$ of a given presheaf $\mathcal{F}$ (as defined in Section 3.5).

### 3.4 Universal Presheaves

So far we have considered presheaves which are characterised by explicitly defining all terms and connecting maps. However, it is possible to construct a presheaf from a minimal amount of data, namely the presheaf terms at a chamber $B$ and at given panels of each type that are faces of $B$. This leads to the notion of universal presheaves, which as their name suggests satisfy a universal property.

Our motivating simplicial complexes arise from geometric minimal parabolic systems. Consequently we will assume that $\Delta$ is the standard $n$-simplex, having vertices $\{v_0, \ldots, v_n\}$ and we set $I = \{0, \ldots, n\}$. For the sake of brevity, if $J = \{i_0, \ldots, i_t\} \subseteq I$, with $i_j < i_{j+1}$ for all $j$, then we say that the simplex $\{v_{i_0}, \ldots, v_{i_t}\}$ has type $J$. Let $\Pi$ be a subset of the power set of $I$ that is closed under taking subsets; if $J \subseteq K \in \Pi$, then $J \in \Pi$.

**Definition 3.4.1.** [RS86] A $\Pi$-presheaf is a collection of $k$-vector spaces $\mathcal{F}_\sigma$ for each $\sigma$ of cotype $J \in \Pi$, connecting maps $\varphi_{\sigma\tau}$ whenever $\tau$ is a face of $\sigma$, and $g$-action for each $g \in G$ on the $\mathcal{F}_\sigma$ such that conditions (3.1.1) and (3.1.2) hold for all simplices of $\Delta$ of cotype $J \in \Pi$.

There may be many presheaves $\mathcal{G}$ whose terms at simplices of cotype $J \in \Pi$ form the same $\Pi$-presheaf $\mathcal{F}$. We call such presheaves extensions of $\mathcal{F}$, and for each such $\mathcal{G}$ we call $\mathcal{F}$ the $\Pi$-restriction of $\mathcal{G}$. Ronan and Smith proved that all extensions of a $\Pi$-presheaf $\mathcal{F}$ that are generated by the terms at simplices of cotype $J \in \Pi$ are quotients of some universal extension.

**Theorem 3.4.2.** [RS86, Theorem 2.1] If $\Pi$ is a non-empty subset of the power set of $I$ that is closed under taking subsets, then each $\Pi$-presheaf $\mathcal{F}$ admits an extension $\mathcal{F}'$ with the following universal property. Suppose $\mathcal{G}$ is a presheaf on $\Delta$ (with connecting maps $\psi_{\sigma\tau}$) satisfying:

(i) the $\Pi$-restriction of $\mathcal{G}$ is isomorphic to $\mathcal{F}$; and

(ii) if cotype $\tau \notin \Pi$, then $\mathcal{G}_\tau$ is generated by the $(\mathcal{F}_\sigma)\psi_{\sigma\tau}$.

Then $\mathcal{G}$ is a quotient of $\mathcal{F}'$.

We call the presheaf $\mathcal{F}'$ from Theorem 3.4.2 a universal presheaf. Below we describe the presheaf terms, connecting maps and $G$-action for $\mathcal{F}'$. We refer the reader to [RS86, p141] for a proof of Theorem 3.4.2 (that is a proof that these terms, maps and actions give a well-defined presheaf $\mathcal{F}'$ and that $\mathcal{F}'$ is indeed universal). The presheaf $\mathcal{F}'$ is defined inductively.

Let $J$ be a subset of $I$ not contained in $\Pi$ such that every proper subset of $J$ is contained in $\Pi$. Let $\sigma$ be a simplex of cotype $J$. By assumption for each simplex $\rho$ having $\sigma$ as a face, the
presheaf term \( \mathcal{F}_\rho \) has been defined. Thus \( \mathcal{F}|St\sigma \) is a well-defined presheaf on the star of \( \sigma \). We define \( \mathcal{F}_\sigma := H_0(\mathcal{F}|St\sigma) \). If \( \sigma \) is a face of \( \rho \), then we have a map

\[ \varphi_\rho : \mathcal{F}_\rho |St\sigma \to H_0(\mathcal{F}|St\sigma) = \mathcal{F}_\sigma, \]

as defined in Section 3.3 and this map satisfies \( \varphi_\tau = \varphi_\rho \varphi_\tau \) whenever \( \rho \) is a face of \( \tau \). We define \( \mathcal{F}_1 \sigma : \mathcal{H}_0 \mathcal{F}_\rho |St\sigma q \). If \( \sigma \) is a face of \( \rho \), then we have a map \( \varphi_\rho : \mathcal{F}_1 \rho |St\sigma \to H_0(\mathcal{F}|St\sigma |St\sigma q \mathcal{F}_1 \sigma, \) as defined in Section 3.3, and this map satisfies \( \varphi_\tau = \varphi_\rho \varphi_\tau \) whenever \( \rho \) is a face of \( \tau \). We define \( \varphi_\sigma \rho : \mathcal{F}_\sigma |St\sigma \to H_0(\mathcal{F}|St\sigma |St\sigma q |St\sigma \mathcal{F}_\sigma \).

Continuing in this way, we define the presheaf \( \mathcal{F}' \).

It is possible to generalise the construction above by considering only terms \( \mathcal{F}_\sigma \) defined at certain faces of single (fixed) chamber \( c \). We then extend these terms to a full \( \Pi \)-presheaf \( \mathcal{F} \), using multiplication within \( G \) to define the other \( \Pi \)-presheaf terms. In such a case, condition (3.1.2) is not relevant, since we only have a single chamber \( c \). It is replaced by the following condition:

\[ (3.4.3.\left[RS86{\text{ Hypothesis 2.2}}\right]) \text{ If } J \subseteq K \in \Pi \text{ and } \sigma_J, \sigma_K \text{ are faces of } c \text{ of cotype } J \text{ and } K \text{ respectively, then } \mathcal{F}_{\sigma_J} \text{ is a } kG_J \text{-module and } \varphi_{\sigma_J\sigma_K} \text{ is a } kG_J \text{-module homomorphism.} \]

We now give an analogue of Definition 3.4.1 in this more general setting.

**Definition 3.4.4.** [RS86] A \( \Pi \)-stalk at \( c \) is a system of coefficients \( \mathcal{F}_{\sigma_J} \) for \( J \in \Pi \) satisfying the composition property (3.1.1) and property (3.4.3).

We note that a \( \Pi \)-presheaf \( \mathcal{F} \), will restrict to a \( \Pi \)-stalk at any one of its chambers, by considering only the presheaf terms corresponding to faces of that chamber. In fact the converse also holds, as proved by Ronan and Smith:

**Theorem 3.4.5.** [RS86] Theorem 2.3] Every \( \Pi \)-stalk is the restriction (to the faces of \( c \)) of a unique \( \Pi \)-presheaf.

An immediate consequence of this theorem is that every \( \Pi \)-stalk can be extended to a unique \( \Pi \)-presheaf, which in turn may be extended to a universal presheaf. As with Theorem 3.4.2, we illustrate the construction of such a \( \Pi \)-presheaf, and refer the reader to [RS86 pp142-144] for a proof that this construction is well-defined and unique.

Continuing the notation of property (3.4.3), assume that \( \{ \mathcal{F}_{\sigma_J} \mid J \in \Pi \} \) is a \( \Pi \)-stalk with connecting maps \( \varphi_{\sigma_J\sigma_K} \). Let \( T_J \) denote a right transversal of \( G_J \) in \( G \). We may choose a transversal such that the elements of \( T_J \) are indexed by the coefficient spaces of cotype \( J \), so that for each \( \sigma_J h \) of cotype \( J \), there is a unique element \( g_{\sigma_J h} \in T_J \) such that \( \sigma_J g_{\sigma_J h} = \sigma_J h \). For \( \sigma_J 1 = \sigma_J \), we take \( g_{\sigma_J 1} := 1 \). Having obtained such a transversal, we define each \( \mathcal{F}_\sigma \) corresponding to a simplex of cotype \( J \) to be a space \( k \)-isomorphic to \( \mathcal{F}_{\sigma_J} \) via a given isomorphism \( \hat{g}_\sigma : \mathcal{F}_{\sigma_J} \to \mathcal{F}_\sigma \). For ease of notation, we take \( \hat{g}_{\sigma_J} = \text{id}_{\mathcal{F}_{\sigma_J}} \).

To define the \( G \)-action on the direct sum of these coefficient spaces, we use the isomorphisms above to map each coefficient space \( \mathcal{F}_\sigma \) of cotype \( J \) back to \( \mathcal{F}_{\sigma_J} \), apply a \( G \)-action on \( \mathcal{F}_{\sigma_J} \), and then apply an isomorphism to the resulting image. More precisely, if \( a \in G \) and \( \sigma \) is a simplex of
cotype \( J \), then we have a \( k \)-homomorphism \( \hat{g}_a : \mathcal{F}_{\sigma J} \rightarrow \mathcal{F}_\sigma \). Since \( T_J \) is a right transversal of \( G_J \) in \( G \), there exists a unique element \( h \in G_J \) satisfying

\[
g_\sigma \cdot a = h \cdot g(\sigma a). \tag{3.4.6}\]

As \( h \in G_J \), and \( \mathcal{F}_{\sigma J} \) is a \( kG_J \)-module (by (3.4.3)), we have a well-defined action of \( h \) on \( \mathcal{F}_{\sigma J} \). It follows that we may define the action, \( \tilde{a} \), of \( a \) on the direct sum of our coefficient spaces of cotype \( J \), so that \( \tilde{a} \) completes the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}_{\sigma J} & \xrightarrow{\hat{g}_a} & \mathcal{F}_\sigma \\
\downarrow{h} & & \downarrow{\tilde{a}} \\
\mathcal{F}_{\sigma J} & \xrightarrow{\hat{g}_{\sigma a}} & \mathcal{F}_{\sigma a}
\end{array}
\]

Thus the action of \( a \in G \) on \( \mathcal{F}_\sigma \) is given by \( \tilde{a} := (\hat{g}_a)^{-1} h \hat{g}_{\sigma a} \).

To define the connecting maps of the \( \Pi \)-presheaf \( \mathcal{F} \), we use a similar approach as above; we map everything back to our \( \Pi \)-stalk and use the \( G \)-action we have just defined. Indeed let \( \sigma \subseteq \tau \) be simplices of cotypes \( J \subseteq K \in \Pi \) respectively, and let \( \sigma, \tau \subseteq \tau_K \) be faces of \( c \) of cotypes \( J \subseteq K \in \Pi \) respectively. Take any \( a \in G \) such that \( \tilde{a} : \mathcal{F}_{\sigma J} \rightarrow \mathcal{F}_\sigma \) and \( \tilde{a} : \mathcal{F}_{\tau K} \rightarrow \mathcal{F}_{\tau} \). As above, we define \( \varphi_{\sigma \tau} : \mathcal{F}_\sigma \rightarrow \mathcal{F}_{\tau} \) to be the homomorphism that completes the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}_{\sigma J} & \xrightarrow{\tilde{a}} & \mathcal{F}_\sigma \\
\varphi_{\sigma J \tau K} \downarrow & & \varphi_{\sigma \tau} \downarrow \\
\mathcal{F}_{\tau K} & \xrightarrow{\tilde{a}} & \mathcal{F}_{\tau}
\end{array}
\]

meaning that \( \varphi_{\sigma \tau} := (\tilde{a})^{-1} \varphi_{\sigma J \tau K} \tilde{a} = (a^{-1})^r \varphi_{\sigma J \tau K} \tilde{a} \).

Theorems 3.4.2 and 3.4.5 illustrate why it suffices to restrict our attention to universal presheaves. This will be the main focus of our attention in Sections 3.7-3.9.

We conclude this section by reconsidering chamber-generated presheaves. As mentioned in Section 3.1 a presheaf \( \mathcal{F} \) is called chamber-generated if for each \( \sigma \in \Delta \), the presheaf term \( \mathcal{F}_\sigma \) is spanned by the images \( \mathcal{F}_c \varphi_{c \sigma} \) for all chambers \( c \) containing \( \sigma \). Clearly such presheaves are nice to work with. Indeed, since the connecting maps of a presheaf \( \mathcal{F} \) are \( k \)-homomorphisms that commute with the \( G \)-action, we see that there is a homomorphism

\[
\bigoplus_{c \text{ a chamber containing } \sigma} \varphi_{c \sigma} : (\mathcal{F}_c)^\pi \rightarrow \mathcal{F}_\sigma.
\]

Thus \( \mathcal{F}_\sigma \) is a quotient of the induced module \( (\mathcal{F}_c)^\pi \).

If we form the universal presheaf from a chamber-generated \( \Pi \)-presheaf or \( \Pi \)-stalk, it follows that the terms of the universal presheaf, not defined by the \( \Pi \)-presheaf/stalk, are homology groups. Hence we obtain the following result:

**Lemma 3.4.6.** If \( \mathcal{F} \) is a \( \Pi \)-stalk or a \( \Pi \)-presheaf, and each term \( \mathcal{F}_\sigma \) of \( \mathcal{F} \) is spanned by the images \( \mathcal{F}_c \varphi_{c \sigma} \) for all chambers \( c \) containing \( \sigma \), then the universal presheaf obtained using Theorems 3.4.2 and 3.4.5 will be a chamber-generated presheaf.
3.5 Dual Presheaves

In Section 3.4 we saw how one could define a presheaf inductively using the presheaf terms at a single chamber, a single panel of each type contained in this chamber and the associated connecting maps. We shall now see that given a panel-irreducible chamber-generated universal presheaf $F$ defined on the simplicial complex $\Delta$ of parabolic subgroups of some group $G$ with respect to some prime divisor $p$ of the order of $G$, we may construct a dual presheaf as originally introduced in \cite{Ron89}. Throughout this section we will set $k := GF(p)$.

Consider the panel-terms of $F$ and appealing to Lemma 2.5.4 we see that chamber-generatedness ensures that for a fixed panel $P \in \Delta$, the panel term $\mathcal{F}_P$ will be generated by the set
\[
\{ \mathcal{F}_{BsB_P} | g \in P \} = \{ \mathcal{F}_{BsB_P} | g \in T \text{ for } T \text{ a transversal of } B \text{ in } P \}.
\]
It follows that $\mathcal{F}_P$ will be a quotient of the induced module $(\mathcal{F}_B)_B^P$, and hence we have a short exact sequence
\[
0 \to M_P \xrightarrow{\vartheta_P} (\mathcal{F}_B)_B^P \xrightarrow{\psi_P} \mathcal{F}_P \to 0, \quad (3.5.8)
\]
for some $kP$-module $M_P$. Applying $\text{Hom}(-, k)$ to (3.5.8), we obtain the short exact sequence
\[
0 \to \text{Hom}(\mathcal{F}_P, k) \xrightarrow{\psi_P} \text{Hom}((\mathcal{F}_B)_B^P, k) \xrightarrow{\vartheta_P'} \text{Hom}(M_P, k) \to 0. \quad (3.5.9)
\]

Denote the rank of $\Delta$ by $n$.

**Definition 3.5.1.** Let $F$ be a universal panel-irreducible presheaf constructed on $\Delta$ from a $\{\emptyset, \{0\}, \{1\}, \ldots, \{n-1\}\}$-stalk $\mathcal{F}$. The dual presheaf is the universal presheaf $\mathcal{F}^*$ constructed from the $\{\emptyset, \{0\}, \{1\}, \ldots, \{n-1\}\}$-stalk $\mathcal{F}^*$ having chamber term $(\mathcal{F}^*)_B := \text{Hom}(\mathcal{F}_B, k)$ and panel terms $(\mathcal{F}^*)_P := \text{Hom}(M_P, k)$ and having connecting maps $\vartheta_p'$ (as defined in (3.5.9)).

We illustrate this with an example.

**Example 3.5.2.** As seen in Example 2.5.3, the symmetric group $\text{Sym}(6)$ admits a unique 2-minimal parabolic system given by $\{P_1 \sim 2^3, \text{Sym}(3), P_2 \sim 2^3, \text{Sym}(3)\}$. Let $k := GF(2)$. There are two non-isomorphic irreducible $kP_i$-modules for $i = 1, 2$ having dimension 1 and 2 over $k$. We denote these by $1_i$ and $2_i$ respectively. In addition, we have a unique irreducible $kB$-module, $1_B$, of dimension 1 over $k$. These modules give rise to four universal panel-irreducible presheaves. We denote by $\lambda_{i,j}$ the universal panel-irreducible presheaf satisfying $(\lambda_{i,j})_{P_i} = 1_i$, $(\lambda_{i,j})_{P_2} = 2_j$ and $(\lambda_{i,j})_B = 1_B$. The presheaf terms of the dual presheaves $\lambda_{i,j}^*$ for $i, j = 1, 2$ are given in Table 3.1.

We conclude this section by considering the case that $G$ is a finite group, $p$ is a prime divisor of $|G|$ and $k := GF(p)$. Given an irreducible $kG$-module $M$, we see that every element of $\text{Hom}(M, k)$ is uniquely defined by the preimage of $1 \in k$. Thus for $m \in M \setminus \{0\}$ we may define $\varphi_m \in \text{Hom}(M, k)$ by $\varphi_m : m \mapsto 1$, whilst taking $\varphi_0 := 0 \in \text{Hom}(M, k)$. This gives a well-defined epimorphism from $M$ to $\text{Hom}(M, k)$ and hence $M \cong_{kG} \text{Hom}(M, k)$. We shall use this fact without further mention when explicitly calculating homology groups in Section 3.8.
3.6 Summary of Known Results

Ronan and Smith first defined a presheaf of abelian groups on the building $\Delta$ of a finite Chevalley group in their 1985 paper [RS85]. In this paper, they looked at basic properties of such presheaves. They proceeded this paper with a further paper on universal presheaves [RS86] in 1986, before using their results to calculate/partially calculate the zero-homology groups for presheaves associated to $L_4(2)$, the $C_3$ geometry of $\text{Alt}(7)$, the triple cover of $\text{Sym}(6)$ and the Mathieu group $M_{24}$, which they published in 1989 (see [RS89]). In this section we present many of the results from these papers.

We give all results for the simplicial complex $\Delta$ of parabolic subgroups arising from a minimal parabolic system of a given group $G$ (mirroring the notion of a building). However, we note that many of the results hold in a more general setting.

**Lemma 3.6.1.** [RS85, (1.1)] If $W$ is a $kG$-module, and $\mathcal{K}_W$ is the constant presheaf on $W$, then $H_0(\mathcal{K}_W) \cong W$.

**Proof.** As $\Delta$ is connected, the copies of $W$ in $\mathcal{K}_W$ at the vertices of $\Delta$ are all identified in $H_0(\mathcal{K}_W)$. $\Box$

When $H$ is a subgroup of $G$, Frobenius-Nakayama reciprocity relates the restriction of $kG$-modules to the induction of $kH$-modules.

**Theorem 3.6.2.** [Ben91, Frobenius-Nakayama Reciprocity, 2.8.3] If $k$ is a field, $H \leq G$ are groups, $M$ is a $kG$-module and $N$ is a $kH$-module, then there is a natural isomorphism

$$\text{Hom}_{kH}(N,M) \cong \text{Hom}_{kG}(N^{kG},M).$$

When considering presheaves on our simplicial complex $\Delta$, we consider all parabolic subgroups. It follows that the process of induction is replaced by taking the zero-homology group.

**Theorem 3.6.3.** [RS85, (1.2)] Let $W$ be a $kG$-module, and $\mathcal{F}$ a presheaf. Then

$$\text{Hom}_k(\mathcal{F},\mathcal{K}_W) \cong_k \text{Hom}_{kG}(H_0(\mathcal{F}),W).$$

For a presheaf $\mathcal{F}$, this result allows us to determine certain irreducible quotients of $H_0(\mathcal{F})$.

**Theorem 3.6.4.** [RS85, (1.3)] Let $W$ be a $kG$-module, and suppose $\mathcal{F}$ is a subpresheaf of $\mathcal{K}_W$, whose terms generate $W$. Then $W$ is a quotient of $H_0(\mathcal{F})$. 

### Table 3.1: The dual presheaves of the universal panel-irreducible presheaves over $GF(2)$ defined on the unique 2-minimal parabolic system of $\text{Sym}(6)$.

<table>
<thead>
<tr>
<th>Presheaf, $\lambda^*_i,j$</th>
<th>$(\lambda^*_{i,j})_B^\lambda$</th>
<th>$(\lambda^*<em>{i,j})</em>{P_1}^\lambda$</th>
<th>$(\lambda^*<em>{i,j})</em>{P_2}^\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^*_{1,1}$</td>
<td>Hom$(1_B, k)$</td>
<td>Hom$(2_1, k)$</td>
<td>Hom$(2_2, k)$</td>
</tr>
<tr>
<td>$\lambda^*_{1,2}$</td>
<td>Hom$(1_B, k)$</td>
<td>Hom$(2_1, k)$</td>
<td>Hom$(1_2, k)$</td>
</tr>
<tr>
<td>$\lambda^*_{2,1}$</td>
<td>Hom$(1_B, k)$</td>
<td>Hom$(1_1, k)$</td>
<td>Hom$(2_2, k)$</td>
</tr>
<tr>
<td>$\lambda^*_{2,2}$</td>
<td>Hom$(1_B, k)$</td>
<td>Hom$(1_1, k)$</td>
<td>Hom$(1_2, k)$</td>
</tr>
</tbody>
</table>
If \( W \) is an irreducible \( kG \)-module, then the terms of any non-zero subpresheaf of \( K_W \) will generate \( W \). Thus we obtain the following result.

**Corollary 3.6.5.** If \( \mathcal{F} \) is a subpresheaf of \( K_W \) for some irreducible \( kG \)-module \( W \), then \( W \) is a quotient of \( H_0(\mathcal{F}) \).

Conversely, if \( W \) is not irreducible, then the terms of any non-zero subpresheaf \( \mathcal{F} \) of \( K_W \) will generate a submodule \( V \) of \( W \). Since all modules we encounter are finite-dimensional \( k \)-vector spaces, we may extend \( V \) to form a composition series for \( W \), in which \( V \) will be one of the terms:

\[
0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_i = V \subseteq W_{i+1} \subseteq \cdots \subseteq W_n = W.
\]

By Theorem 3.6.4 \( V \) is a quotient of \( H_0(\mathcal{F}) \). It follows that \( V/W_{i-1} \) is also a quotient of \( H_0(\mathcal{F}) \). Thus we have the following corollary to Theorem 3.6.4.

**Corollary 3.6.6.** If \( \mathcal{F} \) is a subpresheaf of \( K_W \) for some \( kG \)-module \( W \), then \( W \) is a quotient of \( H_0(\mathcal{F}) \).

There is also a relationship between surjections of presheaves and surjections of the homology groups of presheaves.

**Proposition 3.6.7.** [RS86, (1.4)] A surjection \( p : \mathcal{F} \to \mathcal{G} \) of presheaves induces a surjection \( \hat{p} : H_0(\mathcal{F}) \to H_0(\mathcal{G}) \) in homology groups.

Combining Proposition 3.6.7 and Theorem 3.4.2 we obtain the following result.

**Corollary 3.6.8.** Let \( \mathcal{U} \) be the universal presheaf defined by a \( \Pi \)-presheaf or \( \Pi \)-stalk \( \mathcal{F} \), whose existence is asserted by Theorem 3.4.2. If \( \mathcal{G} \) is a presheaf (with connecting maps \( \psi_{\sigma\tau} \)) satisfying:

(i) the \( \Pi \)-restriction of \( \mathcal{G} \) is isomorphic to \( \mathcal{F} \); and

(ii) if cotype \( \tau \notin \Pi \), then \( \mathcal{G}_\tau \) is generated by the \((\mathcal{G}_\sigma)\psi_{\sigma\tau} \),

then \( H_0(\mathcal{G}) \) is a quotient of \( H_0(\mathcal{U}) \).

**Proof.** By Theorem 3.4.2 the presheaf \( \mathcal{G} \) is a quotient of \( \mathcal{U} \), and thus there is a surjection \( p : \mathcal{U} \to \mathcal{G} \). This surjection of presheaves induces a surjection in homology \( \hat{p} : H_0(\mathcal{U}) \to H_0(\mathcal{G}) \) by Proposition 3.6.7 and hence \( H_0(\mathcal{G}) \) is a quotient of \( H_0(\mathcal{U}) \).

This corollary illustrates the importance of universal presheaves in determining the zero-homology groups of arbitrary presheaves.

A final application of Proposition 3.6.7 is found in the proof of the following result:

**Theorem 3.6.9.** Let \( \mathcal{F} \) be a presheaf. An irreducible module \( V \) is a quotient of \( H_0(\mathcal{F}) \) if and

\[\text{[Corollary 3.6.5]}
\]

\[\text{[Theorem 3.6.4] The analogue of Frobenius reciprocity [11; (1.2)] asserts that a module \( V \) is a quotient of \( H_0(\mathcal{F}) \) precisely when \( \mathcal{F} \) is a subsheaf of the constant sheaf \( K_V \). This statement is actually incorrect, and a counterexample to this situation arises in the presheaf \( \lambda_3 \) of Table (2.1) in the same paper. The presheaf \( \lambda_3 \) has a 5-dimensional term at vertices of a given type, but \( H_0(\lambda_3) \) is a 4-dimensional module. However, Stephen Smith has kindly confirmed the correct formulation of the result as found in Theorem 3.6.9, which we have then been able to prove.}

\[\text{[Remark 3.6.9] The presheaf \( \lambda_3 \) mentioned above does satisfy the hypothesis of the theorem, as there is a map into the constant presheaf of the 4-dimensional irreducible module. However, this map is not faithful.}\]
3.6. SUMMARY OF KNOWN RESULTS

only if there is a surjective morphism of presheaves from \( \mathcal{F} \) onto a (non-zero) subpresheaf of the constant presheaf, \( K_V \), of \( V \).

**Proof.** Let \( \mathcal{F} \) be a presheaf defined on a simplicial complex \( \Delta \) and suppose that \( V \) is a quotient of \( H_0(\mathcal{F}) \) via a quotient map \( q : H_0(\mathcal{F}) \to V \). For each \( \sigma \in \Delta \), there is a map \( \varphi_\sigma : \mathcal{F}_\sigma \to H_0(\mathcal{F}) \) (as described in Section 3.3). Set \( m_\sigma := \varphi_\sigma q \) for each \( \sigma \in \Delta \). Since the map \( \varphi_\sigma \) is the composition of a connecting map in the presheaf \( \mathcal{F} \) and the quotient map \( C_0(\mathcal{F}) \to H_0(\mathcal{F}) \), it commutes with the \( G \)-action. The map \( q \) also commutes with the \( G \)-action, and thus so does the map \( m_\sigma \) for each \( \sigma \in \Delta \). If \( \tau \) is a face of \( \sigma \), then the connecting map \( \theta_\sigma \tau \) in the constant presheaf \( K_V \) is the inclusion map. It follows that

\[
\varphi_\sigma m_\sigma = \varphi_\tau m_\tau = \varphi_\sigma q = m_\sigma = m_\sigma \theta_\sigma \tau.
\]

We conclude that \( m := \{ m_\sigma : \sigma \in \Delta \} \) is a morphism from \( \mathcal{F} \) into the constant presheaf \( K_V \), and hence \( \mathcal{F} \) is mapped onto a subpresheaf of \( K_V \), namely the image of \( m \).

Conversely, suppose that \( \mathcal{F} \) is mapped surjectively onto a subpresheaf, \( \mathcal{G} \), of the constant presheaf \( K_V \). Since \( V \) is irreducible, Corollary 3.6.5 asserts that \( V \) is a quotient of \( H_0(\mathcal{G}) \). Moreover, as \( \mathcal{F} \) is mapped surjectively onto \( \mathcal{G} \), the homology group \( H_0(\mathcal{G}) \) is a quotient of \( H_0(\mathcal{F}) \) by Proposition 3.6.7. It follows that \( V \) is a quotient of \( H_0(\mathcal{F}) \). \( \square \)

The final results that we give in this section were formulated by Ronan in his paper on dual presheaves [Ron89].

**Theorem 3.6.10.** Let \( \mathcal{F} \) be a presheaf defined on the simplicial complex of parabolic subgroups associated to a minimal parabolic system of rank \( n+1 \) of a group \( G \), and let \( \mathcal{F}^* \) be the dual presheaf of \( \mathcal{F} \). Then

(i) [Ron89, Theorem 2] \( H_0(\mathcal{F}) \) and \( H^n(\mathcal{F}^*) \) are isomorphic as \( kG \)-modules.

(ii) [Ron89, Corollary] \( H_n(\mathcal{F}) \) and \( H^0(\mathcal{F}^*) \) are isomorphic as \( kG \)-modules.

We shall sometimes refer to Theorem 3.6.10 as Ronan’s Duality Theorem.

Since the dimension of the \( i \)-th cohomology group and the corresponding \( i \)-th homology group are equal as \( k \)-vector spaces (as they are dual vector spaces), we see that for a given presheaf \( \mathcal{F} \) we have

\[
\dim_k H_0(\mathcal{F}) = \dim_k H^n(\mathcal{F}^*) = \dim_k H_n(\mathcal{F}^*); \quad \text{and} \quad (3.6.10)
\]

\[
\dim_k H_n(\mathcal{F}) = \dim_k H^0(\mathcal{F}^*) = \dim_k H_0(\mathcal{F}^*). \quad (3.6.11)
\]

We will use (3.6.10) and (3.6.11) in conjunction with the Euler characteristic of a given presheaf (as formulated in Section 3.3) to obtain bounds on the possible dimensions of our homology groups. This is a very effective way to obtain such bounds when the rank of the minimal parabolic system is small.
3.7 Homology Groups of Universal Presheaves

We have previously seen in Theorem 3.2.3 that if $G$ is a finite Chevalley group over the finite field $k$, and $V$ is a finite-dimensional irreducible $kG$-module, then the fixed-point presheaf $F_V$ is irreducible. It follows from Corollary 3.6.5 that $V$ is a quotient of $H_0(F_V)$. In particular, every irreducible $kG$-module is the quotient of the zero-homology group of a universal panel-irreducible presheaf. We might hope that this generalises and ask the following question:

**Question 3.7.1.** If $G$ is a group, $p$ a prime dividing $|G|$, $k = GF(p)$, $\Delta$ is the simplicial complex of parabolic subgroups associated to a $p$-minimal parabolic system of $G$ and $V$ is an irreducible $kG$-module, then does there exist a universal panel-irreducible presheaf $\mathcal{F}$ defined on $\Delta$ such that $V$ is a quotient of $H_0(\mathcal{F})$?

In general, there is no guarantee that such a presheaf $\mathcal{F}$ will exist, as we now illustrate.

**Example 3.7.2.** Let $G := \text{Sym}(10)$ and let $B = S \sim 2^{5+2+1} \in \text{Syl}_2(G)$ be a self-normalizing Sylow subgroup. There is a unique 2-minimal parabolic system of $G$ with respect to $S$, given by $\{P_1 \sim 2^5 \text{Sym}(5), P_2 \sim 2^5 \cdot 3^2 \cdot \text{Dih}(8)\}$. Taking $V$ to be the 26-dimensional irreducible $GF(2)G$-module, we see that there is no $v \in V$ such that the $kB$-module $\langle v \rangle_{kB}$ and the $kP_1$-module $\langle v \rangle_{kP_1}$ are both irreducible. It follows that the presheaf $K_V$ does not admit an irreducible subpresheaf, and hence there does not exist a panel-irreducible presheaf $\mathcal{F}$ for which $V$ is a quotient of $H_0(\mathcal{F})$.

We note that in this case $B$ is a maximal subgroup of $P_2$, but it is not maximal in $P_1$. Indeed we may take $B < H_1 < P_1$ with $H_1 \sim 2^{5+2} \cdot \text{Sym}(3)$. We see that there is a unique element $v \in V$ such that the modules $\langle v \rangle_{kB}$, $\langle v \rangle_{kH_1}$ and $\langle v \rangle_{kP_2}$ are all irreducible. Hence, considering the minimal parabolic system $\{H_1, P_2\}$ of $\text{Sym}(8) \times 2 \cong \langle H_1, P_2 \rangle$, we may form a panel-irreducible presheaf $\mathcal{F}$ on this system for which a composition factor of $V$ is a quotient of $H_0(\mathcal{F})$.

In the above example, to prove that no such panel-irreducible presheaf existed, we considered subpresheaves of the constant presheaf $K_V$ whose panel and chamber terms were generated by a given element of $V$. Clearly, panel-irreducibility means that it suffices to check subpresheaves of the fixed-point presheaf $F_V$.

Although in Example 3.7.2 the maximality of the chamber in the minimal parabolic subgroups appeared to be the issue, we shall see in Section 3.8 that maximality alone is not a sufficient condition for Question 3.7.1 to have an affirmative answer. Indeed, we shall exhibit two simplicial complexes - namely $\Delta_{1,3}$ and $\Delta_{2,3}$ - of 2-parabolic subgroups associated to the Mathieu group $M_{11}$ for which there is no universal panel-irreducible presheaf, $\mathcal{F}$, having the irreducible $GF(2)M_{11}$-module $10_{M_{11}}$ as a quotient of $H_0(\mathcal{F})$.

3.8 Calculation of Homology Groups

In Sections 3.1-3.7 we looked at things from a purely theoretical viewpoint. In the current section, we will explicitly calculate the zero-homology groups of some universal panel-irreducible presheaves. In general this is extremely difficult, and at best we can hope to obtain bounds on the dimensions
and possible irreducible quotients of such homology groups. This mirrors the goals of Ronan and Smith in [RS89]. Before giving explicit calculations in Subsection 3.8.2, we describe the method used throughout [RS89] known as geometric spanning.

3.8.1 Geometric Spanning

Given a simplicial complex of parabolic subgroups $\Delta$ associated to some minimal parabolic system $\mathcal{S} = \{P_1, \ldots, P_{n+1}\}$ of a finite group $G$, we recall that the vertices of $\Delta$ are the conjugates of the parabolic subgroups of the form $P_{I(i)}$ for some $i \in I = \{1, \ldots, n+1\}$. We call a conjugate of $P_{I(i)}$, a vertex of type $i$. Suppose we are given a vertex $\sigma = P_{I(j)}^q$ of type $j$ and let $\gamma := P_{I(i)}^p$ for some $i \neq j$. Let $\mathcal{F}$ be a chamber-generated presheaf defined on $\Delta$ with connecting maps $\varphi_{\alpha\tau}$ whenever $\tau$ is a face of $\alpha$. Thus

$$\mathcal{F}_\alpha = \langle \mathcal{F} B^h \varphi B^h \alpha \mid B^h \subseteq \alpha \rangle$$

for each $\alpha \in \Delta$,

where $B$ is the normalizer of the Sylow $p$-subgroup that defines the minimal parabolic system $\mathcal{S}$. For each chamber $B^h \subseteq \sigma$, we have $B^h \subseteq \gamma^h$. Moreover, as $\gamma^h$ (resp. $\sigma$) is the unique vertex of type $i$ (resp. $j$) that is a face of $B^h$, and $P_{I(i,j)}^h$ is the unique 1-simplex of type $\{i, j\}$ that is a face of $B^h$, we have that both $\gamma^h$ and $\sigma$ are faces of $P_{I(i,j)}^h$.

Since $\mathcal{F}$ is chamber generated, there exists $v_1, \ldots, v_r$ in chamber-terms $\mathcal{F} B^{q_1}, \ldots, \mathcal{F} B^{q_r}$ respectively (where $B^{q_l} \subseteq \sigma$ for each $l = 1, \ldots, r$) such that $\mathcal{F}_\sigma$ is generated by $v_1 \varphi B^{q_1} \sigma, \ldots, v_r \varphi B^{q_r} \sigma$. Now setting $\mu_l := P_{I(i,j)}^{q_l}$, we have that

$$v_l \varphi B^{q_l} \sigma = v_l \varphi B^{q_l} \mu_l \varphi_{\mu_l \sigma}$$

for each $l = 1, \ldots, r$.

Defining $w_l := v_l \varphi B^{q_l} \mu_l \in \mathcal{F}_{\mu_l}$ for each $l$, the presheaf term $\mathcal{F}_\sigma$ is generated by $w_1 \varphi_{\mu_1 \sigma}, \ldots, w_r \varphi_{\mu_r \sigma}$. Consider $w_1 \in \mathcal{F}_{\mu_1}$. As $\mu_1 = P_{I(i,j)}^{q_1}$ is a 1-simplex of type $\{i, j\}$, it has precisely two faces, namely $\sigma$ and $\gamma^{q_1}$. Considering the image of $w_1$ under the boundary map $\partial_1 : C_1(\mathcal{F}) \to C_0(\mathcal{F})$ we have

$$w_1 \partial_1 = \pm(w_1 \varphi_{\mu_1 \sigma} - w_1 \varphi_{\mu_1 \gamma^{q_1}}),$$

where the sign is positive if $j < i$ and negative if $i < j$. As $w_1 \varphi_{\mu_1 \sigma}$ and $w_1 \varphi_{\mu_1 \gamma^{q_1}}$ are both contained in $C_0(\mathcal{F})$, their images in the homology group $H_0(\mathcal{F})$ are equal:

$$w_1 \varphi_{\mu_1 \sigma} + \text{Im} \partial_1 = w_1 \varphi_{\mu_1 \gamma^{q_1}} + \text{Im} \partial_1.$$

We may use the same methodology for each $l = 1, \ldots, r$ to see that the image of the presheaf term $\mathcal{F}_\sigma$ in the homology group $H_0(\mathcal{F})$ is contained in the span of the images of the presheaf terms of $\mathcal{F}$ at vertices of type $i$.

Repeating this process for all vertices of type $j \neq i$ we see that $H_0(\mathcal{F})$ is spanned by the images of all presheaf terms at vertices of type $i$. Since $i$ was chosen arbitrarily, we see that $H_0(\mathcal{F})$ is spanned by the images of presheaf terms of $\mathcal{F}$ at vertices of any given type.

Using further applications of this methodology, we can determine subsets of vertices of a given type such that the images of the presheaf terms at these vertices generate the zero-homology group.
parabolic subgroups of $G$ complex, $\Delta$ parabolic system of $G$

Let $\Delta$ be the graph having vertex set the minimal parabolic subgroups of $G$, and with $P_1^0$ and $P_1^3$ incident if they share a common chamber, we may calculate the collapsed adjacency diagram of $\Delta$ with respect to $P_1$. This is given in Figure 3.1. The girth of $\Delta$ is 8. We note that the collapsed adjacency diagram of $\Delta$ with respect to $P_2$ is identical to the collapsed adjacency diagram with respect to $P_1$. Since $\Delta$ may be seen to be an incidence geometry, we will refer to conjugates of $P_1$ and $P_2$ as points and lines respectively.

Let $k = GF(2)$ and let $\mathcal{F}$ denote the universal panel-irreducible presheaf defined on $\Delta$ having 1-dimensional irreducible $k\mathbb{B}$- and $kP_2$-modules at the simplices $B$ and $P_2$ and a 2-dimensional irreducible $kP_1$-module at $P_1$. Define $p_0 := P_1$ and denote the lines in $\Delta_1(p_0)$ by $l_0, l_1, l_2$, the points in $\Delta_2(p_0)$ by $p_1, \ldots, p_6$ and the lines in $\Delta_3(p_0)$ by $l_3, \ldots, l_{14}$ such that the subgraph of $\Gamma$ spanned by these vertices is as given in Figure 3.2. From Figures 3.1 and 3.2 we see that the images of $p_1, \ldots, p_6$ in $H_0(\mathcal{F})$ form a spanning set, since the images of all lines will be contained in this span. It follows that the images of $l_3, \ldots, l_{14}$ also span $H_0(\mathcal{F})$. Considering each point not occurring in Figure 3.2, we note that as the girth of $\Gamma$ is 8, each point must be incident to one line from each of the sets $\{l_3, l_4, l_5, l_6\}$, $\{l_7, l_8, l_9, l_{10}\}$ and $\{l_{11}, l_{12}, l_{13}, l_{14}\}$. Since the image of a point is spanned by the images of any two of its incident lines, we conclude that the images of $l_3, \ldots, l_{10}$ span $H_0(\mathcal{F})$, and hence so do the images of $p_1, \ldots, p_4$.

Finally, denote the non-zero vector in $\mathcal{F}_{l_i}$ by $v_i$ for $i = 3, \ldots, 14$ and let $p_j \in \Delta_4(p_0)$ be incident to $l_9$. Without loss of generality we may assume that $p_j$ is also incident to $l_3$ and $l_{11}$. Thus in $H_0(\mathcal{F})$ the relation

$$v_9 = v_3 + v_{11} \quad (3.8.12)$$

holds. Let $p_k$ be the other point of $\Delta_4(p_0)$ that is incident to $l_{11}$. Without loss of generality, $p_k$ is also incident to $l_5$ and $l_7$. This gives

$$v_{11} = v_5 + v_7 \quad (3.8.13)$$

\[\text{Figure 3.1: The collapsed adjacency diagram of the graph $\Gamma$ from Example 3.8.1 with respect to $P_1$.}\]

\[\text{This method is best illustrated by an example. For the sake of brevity, from this point forward we will often talk about presheaf terms spanning the zero-homology group. However strictly speaking, we mean that the images of the presheaf terms span the zero-homology group.}\]

\[\text{Example 3.8.1. Let $G := \text{Sym}(6)$. We saw in Example 2.5.3 that there is a unique 2-minimal parabolic system of $G$ given by $\{P_1 \sim 2^3, \text{Sym}(3), P_2 \sim 2^3, \text{Sym}(3)\}$, having associated simplicial complex, $\Delta$, of parabolic subgroups. Defining $\Gamma$ to be the graph having vertex set the minimal parabolic subgroups of $G$, and with $P_1^0$ and $P_1^3$ incident if they share a common chamber, we may calculate the collapsed adjacency diagram of $\Gamma$ with respect to $P_1$. This is given in Figure 3.1. The girth of $\Gamma$ is 8. We note that the collapsed adjacency diagram of $\Gamma$ with respect to $P_2$ is identical to the collapsed adjacency diagram with respect to $P_1$. Since $\Gamma$ may be seen to be an incidence geometry, we will refer to conjugates of $P_1$ and $P_2$ as points and lines respectively.}\]

\[\text{Let $k = GF(2)$ and let $\mathcal{F}$ denote the universal panel-irreducible presheaf defined on $\Delta$ having 1-dimensional irreducible $k\mathbb{B}$- and $kP_2$-modules at the simplices $B$ and $P_2$ and a 2-dimensional irreducible $kP_1$-module at $P_1$. Define $p_0 := P_1$ and denote the lines in $\Delta_1(p_0)$ by $l_0, l_1, l_2$, the points in $\Delta_2(p_0)$ by $p_1, \ldots, p_6$ and the lines in $\Delta_3(p_0)$ by $l_3, \ldots, l_{14}$ such that the subgraph of $\Gamma$ spanned by these vertices is as given in Figure 3.2. From Figures 3.1 and 3.2 we see that the images of $p_1, \ldots, p_6$ in $H_0(\mathcal{F})$ form a spanning set, since the images of all lines will be contained in this span. It follows that the images of $l_3, \ldots, l_{14}$ also span $H_0(\mathcal{F})$. Considering each point not occurring in Figure 3.2, we note that as the girth of $\Gamma$ is 8, each point must be incident to one line from each of the sets $\{l_3, l_4, l_5, l_6\}$, $\{l_7, l_8, l_9, l_{10}\}$ and $\{l_{11}, l_{12}, l_{13}, l_{14}\}$. Since the image of a point is spanned by the images of any two of its incident lines, we conclude that the images of $l_3, \ldots, l_{10}$ span $H_0(\mathcal{F})$, and hence so do the images of $p_1, \ldots, p_4$.}\]

\[\text{Finally, denote the non-zero vector in $\mathcal{F}_{l_i}$ by $v_i$ for $i = 3, \ldots, 14$ and let $p_j \in \Delta_4(p_0)$ be incident to $l_9$. Without loss of generality we may assume that $p_j$ is also incident to $l_3$ and $l_{11}$. Thus in $H_0(\mathcal{F})$ the relation}\]

\[v_9 = v_3 + v_{11}\]

\[\text{holds. Let $p_k$ be the other point of $\Delta_4(p_0)$ that is incident to $l_{11}$. Without loss of generality, $p_k$ is also incident to $l_5$ and $l_7$. This gives}\]

\[v_{11} = v_5 + v_7\]
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Figure 3.2: The subgraph of $\Gamma$ spanned by the points $p_0, \ldots, p_6$ and the lines $l_0, \ldots, l_{14}$ from Example 3.8.1.

in $H_0(\mathcal{F})$. Combining (3.8.12) and (3.8.13) we obtain

$$v_9 = v_3 + v_5 + v_7$$

in $H_0(\mathcal{F})$ and hence $\mathcal{F}_{l_9} \subseteq \text{span}\{\mathcal{F}_{p_1}, \mathcal{F}_{p_2}, \mathcal{F}_{p_3}\}$. An analogous argument may be used to show that $\mathcal{F}_{l_{10}} \subseteq \text{span}\{\mathcal{F}_{p_1}, \mathcal{F}_{p_2}, \mathcal{F}_{p_3}\}$. Thus $H_0(\mathcal{F})$ is spanned by $\{\mathcal{F}_{p_1}, \mathcal{F}_{p_2}, \mathcal{F}_{p_3}\}$.

We conclude by noting that

$$H_0(\mathcal{F}) = \text{span}\{\mathcal{F}_{p_1}, \mathcal{F}_{p_2}, \mathcal{F}_{p_3}\} = \text{span}\{\mathcal{F}_{l_0}, \mathcal{F}_{l_1}, \mathcal{F}_{l_3}, \mathcal{F}_{l_5}, \mathcal{F}_{l_7}\},$$

whence

$$\dim(H_0(\mathcal{F})) = \dim(\text{span}\{\mathcal{F}_{l_0}, \mathcal{F}_{l_1}, \mathcal{F}_{l_3}, \mathcal{F}_{l_5}, \mathcal{F}_{l_7}\}) \leq 5. \quad (3.8.14)$$

We shall utilise the bound in (3.8.14) later, to explicitly determine the structure of $H_0(\mathcal{F})$.

We conclude this subsection by noting that geometric spanning arguments are most powerful when dealing with groups of small order.

3.8.2 Explicit Calculations

The explicit calculation of the zero-homology groups of universal panel-irreducible presheaves is typically extremely complex. Often the best that we can hope to obtain is a bound on the dimension of the homology group and/or a list of possible irreducible quotients. We illustrate this complexity by calculating the zero-homology groups of the universal panel-irreducible presheaves of the symmetric group $\text{Sym}(6)$ and the Mathieu groups $M_{11}$ and $M_{22}$ and partially calculating the zero-homology groups of the universal panel-irreducible presheaves of the Mathieu group $M_{12}$. All of
these presheaves are defined on rank 2 minimal parabolic systems. We also calculate the irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on $M_{23}$ and $M_{24}$ over $GF(2)$, noting that the minimal parabolic systems on which they are defined have rank 3. Some of our calculations are heavily based on the geometric spanning method introduced in Subsection 3.8.1 combined with the notion of the dual presheaf and Theorem 3.6.9.

Sym(6)

We continue the notation of Example 3.5.2 for the universal panel-irreducible presheaves of Sym(6) over $GF(2)$. There are four irreducible $GF(2)^{Sym(6)}$-modules having dimensions 1, 4, 4 and 16. We denote these by $1$, $4_a$, $4_b$ and 16 respectively.

$\lambda_{1,1}$

The presheaf $\lambda_{1,1}$ is precisely the constant presheaf $\mathcal{K}_1$. Thus $H_0(\mathcal{F}) \cong 1$ by Lemma 3.6.1.

$\lambda_{2,1}$

In Example 3.8.1 we obtained the bound

$$\dim(H_0(\lambda_{2,1})) \leq 5. \quad (3.8.15)$$

Since $H_0(\lambda_{2,1})$ is spanned by the images at simplices of any given type, we see that $H_0(\lambda_{2,1})$ is simultaneously a quotient of $((\lambda_{2,1})_B)^{Sym(6)}_B$, $((\lambda_{2,1})_{P_1})^{Sym(6)}_{P_1}$ and $((\lambda_{2,1})_{P_2})^{Sym(6)}_{P_2}$. There are three such quotients of dimension at most 5, namely 0, $4_b$ and $4_b/1$.

Consider the $GF(2)^{Sym(6)}$-module $4_b/1$. It contains two vectors $v_1$ and $v_2$ such that the modules $\langle v_i \rangle_{kB}$, $\langle v_i \rangle_{kP_1}$ and $\langle v_i \rangle_{kP_2}$ are all irreducible (for $i = 1, 2$). Of these vectors, one vector generates the universal panel-irreducible subpresheaf $\lambda_{1,1}$ of $\mathcal{K}_{4_b/1}$, whilst the other generates the universal panel-irreducible subpresheaf $\lambda_{2,1}$. Thus we obtain $\lambda_{2,1} \subseteq \mathcal{K}_{4_b/1}$ and since $(\lambda_{2,1})_{P_1}$ is 2-dimensional, the terms of $\lambda_{2,1}$ must generate $4_b/1$. Thus $4_b/1$ is a quotient of $H_0(\lambda_{2,1})$ by Theorem 3.6.4. Combining this with (3.8.15) we obtain $H_0(\lambda_{2,1}) \cong 4_b/1$.

$\lambda_{2,2}$

As $\lambda_{2,2}^{\ast} \cong \lambda_{2,1}$, Ronan’s Duality Theorem asserts that $\dim(H_0(\lambda_{1,2})) = \dim(H_1(\lambda_{2,1}))$. Considering the Euler characteristic of $\lambda_{2,1}$ we obtain

$$\dim(H_0(\lambda_{1,2})) = \dim(H_1(\lambda_{2,1})) = \dim(H_0(\lambda_{2,1})) - \chi(\lambda_{2,1}) = 5 - 0 = 5.$$

Mirroring the approach for $\lambda_{2,1}$, we see that there is a unique 5-dimensional quotient of $((\lambda_{1,2})_B)^{Sym(6)}_B$, $((\lambda_{1,2})_{P_1})^{Sym(6)}_{P_1}$ and $((\lambda_{1,2})_{P_2})^{Sym(6)}_{P_2}$, namely $4_a/1$. Thus $H_0(\lambda_{1,2}) \cong 4_a/1$.

$\lambda_{2,2}$

The presheaves $\lambda_{2,2}$ and $\lambda_{1,1}$ are dual to each other. Thus by Ronan’s Duality Theorem

$$\dim(H_0(\lambda_{2,2})) = \dim(H_1(\lambda_{1,1})) = \dim(H_0(\lambda_{1,1})) - \chi(\lambda_{1,1}) = 1 - (-15) = 16.$$
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We denote by $\lambda_j$ We now consider the Mathieu group $M_{11}$, which admits three 2-minimal parabolic systems, namely

$$S_1 := \{P_1 \sim 2^{1+2} \text{Sym}(3), P_2 \sim 3^2SD_{16}\}, \quad S_2 := \{P_1 \sim 2^{1+2} \text{Sym}(3), P_3 \sim \text{Alt}(6).2\} \quad \text{and} \quad S_3 := \{P_2 \sim 3^2SD_{16}, P_3 \sim \text{Alt}(6).2\}.$$  

Denote the simplicial complexes of parabolic subgroups associated to $S_1$, $S_2$ and $S_3$ by $\Delta_{1,2}$, $\Delta_{1,3}$ and $\Delta_{2,3}$ respectively.

The self-normalizing Sylow 2-subgroup, $B$, of $M_{11}$ admits a unique irreducible $GF(2)B$-module of dimension 1, whilst the minimal parabolic subgroups $P_1$, $P_2$ and $P_3$ admit irreducible $GF(2)P_1$-, $GF(2)P_2$- and $GF(2)P_3$- modules of dimensions 1 and 2, 1 and 8 and 1, 8 and 16 respectively. We denote by $\lambda_{i,j}$ the universal panel-irreducible presheaf defined on $S_1$ having an irreducible 1-dimensional module at chambers, and irreducible $i$- and $j$-dimensional $GF(2)P_1$- and $GF(2)P_2$-modules at simplices of type $P_1$ and $P_2$ respectively. Similarly, denote by $\mu_{i,j}$ (respectively $\gamma_{i,j}$) the universal panel-irreducible presheaf defined on $S_2$ (respectively $S_3$) having irreducible $i$- and $j$-dimensional $GF(2)P_1$- and $GF(2)P_2$-modules (respectively $GF(2)P_2$- and $GF(2)P_3$-modules) at simplices of type $P_1$ and $P_3$ (respectively $P_2$ and $P_3$).

To apply geometric spanning arguments to calculate the zero-homology groups of the $\lambda_{i,j}$, $\mu_{i,j}$ and $\gamma_{i,j}$, we first define the graphs $\Gamma_{i,j}$ for $(i, j) = (1, 2), (1, 3), (2, 3)$. Indeed we take the vertex set of $\Gamma_{i,j}$ to be the union of all conjugates of $P_1$ and all conjugates of $P_j$ in $M_{11}$, and we join $P_i^g$ and $P_j^h$ precisely when they contain a common chamber. The collapsed adjacency diagrams of $\Gamma_{12}$, $\Gamma_{13}$ and $\Gamma_{23}$ are given in Figures 3.3, 3.4 and 3.5. The respective diameters of $\Gamma_{1,2}$, $\Gamma_{1,3}$ and $\Gamma_{2,3}$ are 6, 4 and 3, and the respective girths are 6, 4 and 4.

There are four isomorphism classes of irreducible $GF(2)M_{11}$-modules, having dimensions 1, 10, 32 and 44. Clearly $\lambda_{1,1}$, $\mu_{1,1}$ and $\gamma_{1,1}$ are the constant presheaves of the 1-dimensional irreducible

Figure 3.3: The collapsed adjacency diagrams of $\Gamma_{12}$ with respect to $P_1$ (top) and $P_2$ (bottom).

Considering the irreducible $GF(2)\text{Sym}(6)$-module of dimension 16, we see that it contains a vector $v$ satisfying $\langle v \rangle_{kB} \cong 1_B$, $\langle v \rangle_{kP_1} \cong 2_{P_1}$ and $\langle v \rangle_{kP_2} \cong 2_{P_2}$. Thus we may realise $\lambda_{2,2}$ as a subpresheaf of $K_{16}$ and so 16 is an irreducible quotient of $H_0(\lambda_{2,2})$ by Theorem 3.6.9. We conclude that $H_0(\lambda_{2,2}) \cong 16$.
CHAPTER 3. HOMOLOGY OF PRESHEAVES OF ABELIAN GROUPS

Figure 3.4: The collapsed adjacency diagrams of $\Gamma_{13}$ with respect to $P_1$ (top) and $P_3$ (bottom).

Figure 3.5: The collapsed adjacency diagrams of $\Gamma_{23}$ with respect to $P_2$ (top) and $P_3$ (bottom).
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$GF(2)M_{11}$-module and hence $H_0(\lambda_{1,1}) \cong H_0(\mu_{1,1}) \cong H_0(\gamma_{1,1}) \cong 1$. Moreover, these are the only universal panel-irreducible presheaves for which 1 can be an irreducible quotient.

Considering the 10-dimensional irreducible $GF(2)M_{11}$-module $10_{M_{11}}$, we see that there are two vectors in $10_{M_{11}}$ that generate irreducible $kB_i$, $kP_i$- and $kP_j$-modules for distinct $i, j = 1, 2, 3$. The first vector gives rise to irreducible $kB_i$, $kP_i$- and $kP_j$-modules of dimension 1, 1 and 8, whilst the second gives rise to irreducible $kB_i$, $kP_i$- and $kP_j$-modules of dimension 1, 2 and 1. It follows from Theorem 3.6.9 that $10_{M_{11}}$ is an irreducible quotient of precisely $H_0(\lambda_{1,8})$ and $H_0(\lambda_{2,1})$. Thus, as stated in Section 3.7, there are no universal panel-irreducible presheaves defined on $\Delta_{1,3}$ or $\Delta_{2,3}$, for which $10_{M_{11}}$ is a quotient of their zero-homology group.

In a similar way, calculations in MAGMA show that the 32-dimensional module, $32_{M_{11}}$, is a quotient of the zero-homology group of precisely three universal panel-irreducible presheaves, these being $\lambda_{2,8}$, $\mu_{2,16}$ and $\gamma_{8,16}$ (each arising from the unique element of $32_{M_{11}}$ that generates an irreducible $GF(2)B$-module). We note that such explicit calculations are not required in the case of presheaves defined on $\Delta_{1,2}$ and $\Delta_{1,3}$ having a 1-dimensional term at panels of type $P_1$. Indeed, the zero-homology group of such presheaves would be a quotient of $(1_P)_{P_1}^{M_{11}}$ . However, no such quotient has $32_{M_{11}}$ as an irreducible quotient.

We now consider the $\lambda_{i,j}$, $\mu_{i,j}$ and $\gamma_{i,j}$ for $(i,j) \neq (1,1)$ in turn.

Presheaves Defined on $\Delta_{1,2}$

We note that $\lambda^*_{1,1} \cong \lambda_{2,8}$ and $\lambda^*_{1,8} \cong \lambda_{2,1}$.

$\lambda_{1,8}$ and $\lambda_{2,1}$

The zero-homology group $H_0(\lambda_{2,1})$ is spanned by the images of the presheaf terms at vertices of type $P_2$ and hence is a quotient of the induced module $(1_{P_2})_{P_2}^{M_{11}}$. It is clear from the collapsed adjacency diagram of $\Gamma_{1,2}$ with respect to $P_2$ - seen in Figure 3.3 - that $\dim(H_0(\lambda_{2,1})) \leq 53$ and we have seen that $10_{M_{11}}$ is a quotient of $H_0(\lambda_{2,1})$. Considering the possible quotients of $(1_{P_2})_{P_2}^{M_{11}}$ having dimensions between 10 and 53, we see that there are four such quotients, having dimensions 10, 11, 44 and 45. The quotients of dimension 11 and 45 have a 1-dimensional quotient, whilst the 44-dimensional quotient is irreducible. Thus by our knowledge of irreducible quotients of $H_0(\lambda_{2,1})$, we deduce that $H_0(\lambda_{2,1}) \cong 10_{M_{11}}$.

To calculate the dimension of $H_0(\lambda_{1,8})$ we use the Euler characteristic and Ronan’s Duality Theorem. Indeed

$$\dim(H_0(\lambda_{1,8})) = \chi(\lambda_{1,8}) + \dim(H_1(\lambda_{1,8}))$$
$$= \chi(\lambda_{1,8}) + \dim(H_0(\lambda^*_{1,8}))$$
$$= (1 \cdot 165 + 8 \cdot 55 - 1 \cdot 495) + 10 = 120.$$ 

There are two isomorphism classes of 120-dimensional quotients of $(1_B)_{B}^{M_{11}}$ that do not have $1_{M_{11}}$ as a quotient. They have the structures

$10/10/1/44/1/10/44$ and $44 \oplus 10/1/10/44/1/10$. 
Consideration of the elements of 44|\_P_2 which generate a copy of 8_\_P_2, we see that \( \lambda_{1,8} \) may be constructed as a subpresheaf of \( \mathcal{K}_{44} \). It follows that \( H_0(\lambda_{1,8}) \cong 44 \oplus 10/1/10/44/1/10. \)

\( \lambda_{2,8} \)

The Euler characteristic of \( \lambda_{2,8} \) is

\[
\chi(\lambda_{2,8}) = \dim(C_0(\lambda_{2,8})) - \dim(C_1(\lambda_{2,8})) = (2 \cdot 165 + 8 \cdot 55) - 1 \cdot 495 = 275.
\]

Appealing to Ronan’s Duality Theorem we see that

\[
\dim(H_0(\lambda_{2,8})) = \chi(\lambda_{2,8}) + \dim(H_1(\lambda_{2,8})) = \chi(\lambda_{2,8}) + \dim(H_0(\lambda_{2,8}^\bullet)) = 275 + 1 = 276.
\]

There is a unique isomorphism class of 276-dimensional quotients of \( (2_{P_1})_{M_1} \), and thus \( H_0(\lambda_{2,8}) \) must belong to this class. Hence

\[
H_0(\lambda_{2,8}) \cong 32 \oplus 44/1/44/10/1 \oplus 44/1/10/1/44/44.
\]

**Presheaves Defined on \( \Delta_{1,3} \)**

\( \mu_{1,8} \) and \( \mu_{2,8} \)

The zero-homology groups of \( \mu_{1,8} \) and \( \mu_{2,8} \) are both quotients of the indecomposable induced \( GF(2)M_{11}\)-module \( (8_{P_3})_{P_{31}} \cong 44/44. \)

Let \( T \) denote a transversal in \( M_{11} \) of \( N_{M_{11}}(P_3) \). From the collapsed adjacency diagram of \( \Gamma_{1,3} \) with respect to \( P_3 \) (given in Figure 3.4), we see that

\[
H_0(\mu_{1,8}) \cong \langle (\mu_{1,8})_{P_3^j} + \im \partial_1 | t \in T \setminus \{1\} \rangle.
\]

We also note that the images of any two presheaf terms at vertices of type \( P_3 \) will have 9 non-zero vectors in common. Finally, the image of any presheaf term in the second disc of the adjacency diagram, will intersect the images of the other presheaf terms corresponding to the disc in 36 non-zero vectors. Enumerating the elements of \( T \) by \( \{0, \ldots, 10\} \) with \( t_0 := 1 \), we may assume that

\[
\dim((\mu_{1,8})_{P_3^j} + \im \partial_1) \leq 8, \dim((\mu_{1,8})_{P_3^j} + \im \partial_1) \leq 4 \text{ for } j = 2, \ldots, 9, \text{ and } \dim((\mu_{1,8})_{P_3^{10}} + \im \partial_1) \leq 2.
\]

Hence

\[
\dim(H_0(\mu_{1,8})) \leq 8 + 8 \cdot 4 + 2 = 42.
\]

Since \( H_0(\mu_{1,8}) \) is a quotient of 44/44, we conclude that \( H_0(\mu_{1,8}) \cong 0. \)

Calculations in MAGMA show that 44_{M_{11}} admits \( \mu_{2,8} \). We conclude that \( H_0(\mu_{2,8}) \cong 44_{M_{11}}, \) since clearly \( \dim(H_0(\mu_{2,8})) \neq 88. \)

\( \mu_{1,16} \)

As \( 1_{M_{11}}, 10_{M_{11}}, \) and \( 32_{M_{11}} \) are not quotients of \( H_0(\mu_{1,16}), \) it suffices to consider quotients featuring 44_{M_{11}}. The restriction \( (44_{M_{11}})_{P_3} \) decomposes as a direct sum of 16_{P_3} and a 28-dimensional indecomposable module for which 16_{P_3} is not a submodule. Thus we may explicitly construct
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16_{P_3} within (44_{M_{11}})|_{P_3}. Considering its elements, and the \( GF(2)B \)- and \( GF(2)P_1 \)-modules that they generate, we see that there is a copy of \( \mu_{2,16} \) sitting inside \( K_{44} \), but there is no subpresheaf isomorphic to \( \mu_{1,16} \). Hence \( 44_{M_{11}} \) is not a quotient of \( H_0(\mu_{1,16}) \), meaning that \( H_0(\mu_{1,16}) = 0 \).

\[ \mu_{2,1} \]

The homology group \( H_0(\mu_{2,1}) \) is spanned by the images of the 1-dimensional presheaf terms at vertices of type \( P_3 \), meaning that \( \dim(H_0(\mu_{2,1})) \leq [G : P_3] = 11 \). Hence, as \( 1_{M_{11}} \) and \( 10_{M_{11}} \) are not quotients of \( H_0(\mu_{2,1}) \), it follows that \( H_0(\mu_{2,1}) = 0 \).

\[ \mu_{2,16} \]

We know that \( 32_{M_{11}} \) and \( 44_{M_{11}} \) are both quotients of \( H_0(\mu_{2,16}) \), but that \( 1_{M_{11}} \) and \( 10_{M_{11}} \) are not. Consider the induced module

\[
(16_{P_3})_{P_3}^{M_{11}} \cong 32 \oplus 44/(44 \oplus 1/10/1)/44.
\]

Here the summand \( 1/10/1 \) is a uniserial module.\(^5\) There are nine quotients of \( (16_{P_3})_{P_3}^{M_{11}} \) satisfying our condition on irreducible quotients. Calculations in MAGMA show that of these quotients, only \( 32 \oplus 44 \) and \( 32 \oplus 44/1 \) have a generating element that also generates an isomorphic copy of \( \mu_{2,16} \). It follows from Theorem 3.6.4 that \( H_0(\mu_{2,16}) \cong 32 \oplus 44/1 \).

**Presheaves Defined on \( \Delta_{2,3} \)**

\( \gamma_{1,8} \) and \( \gamma_{1,16} \)

Considering the collapsed adjacency diagram of \( \Gamma_{2,3} \) with respect to \( P_2 \) - as given in Figure 3.5 - we see that the image of the presheaf terms in \( \Delta_3(P_2) \) span a space of dimension at most

\[
2 \cdot \dim((\gamma_{1,*})_{P_3}) - 1 \leq 2 \cdot 16 - 1 = 31.
\]

Consequently, the images of the presheaf terms in \( \Delta_2(P_2) \) span a space of dimension at most 31. However, the images of the presheaf terms in both \( \Delta_1(P_2) \) and \( \Delta_3(P_2) \) are contained in this span. Thus as the images of all presheaf terms of type \( P_3 \) are contained in this span, we conclude that \( \dim(H_0(\gamma_{1,*})) \leq 31 \). Hence as \( 1_{M_{11}} \) and \( 10_{M_{11}} \) are not quotients of \( H_0(\gamma_{1,*}) \), we deduce that \( H_0(\gamma_{1,8}) = H_0(\gamma_{1,16}) = 0 \).

\( \gamma_{8,1} \)

Since \( H_0(\gamma_{8,1}) \) is spanned by the images of presheaf terms at vertices of type \( P_3 \), it follows that \( \dim(H_0(\gamma_{8,1})) \leq [G : P_3] = 11 \). Thus as \( 1_{M_{11}} \) and \( 10_{M_{11}} \) are not quotients of \( H_0(\gamma_{8,1}) \), we conclude that \( H_0(\gamma_{8,1}) = 0 \).

\(^5\)We recall that a module is called *uniserial* if its submodules can be totally ordered. In our case the submodules are given by 0, 1, 10/1 and 1/10/1.
The Euler characteristic of $\gamma_{8,8}$ is 33. Appealing to Lemma 3.3.6 it follows that $H_0(\gamma_{8,8})$ is non-zero. Since it is also a quotient of the indecomposable induced module $(8P_3)_M^{11}$, we deduce that $H_0(\gamma_{8,8}) < 88$ clearly holds (for example using a geometric spanning argument), we deduce that $H_0(\gamma_{8,8}) \cong 44$.

The Euler characteristic of $\gamma_{8,16}$ is 121. Consequently $\dim(H_0(\gamma_{8,16})) \geq 121$. There are 5-possible quotients of $(16P_3)_M^{11}$ satisfying this dimensional bound. Their dimensions are 121, 131, 132, 144 and 176.

To obtain an upper bound for $\dim(H_0(\gamma_{8,16}))$ we use a geometric spanning argument. Indeed, let $t_1, \ldots, t_{10} \in M_{11}$ be such that the 2-disc, $\Delta_2(P_3)$, in $\Gamma_{23}$ is equal to $\{P_3^t | i = 1, \ldots, 10\}$. For each $i = 1, \ldots, 10$, we consider the number of vectors in $(\gamma_{8,16})_{P_3^t} + \im \partial_1$ that are contained in $\bigcup_{i+j=1}^6 (\gamma_{8,16})_{P_3^t} + \im \partial_1$. Since the image of each $(\gamma_{8,16})_{P_3^t}$ contains 7 vectors in the image of presheaf terms of elements of the 3-disc of $P_3$, and these in turn each give rise to an additional 8 vectors in $\bigcup_{i+j=1}^6 (\gamma_{8,16})_{P_3^t} + \im \partial_1$, we see that there are 72 non-zero vectors in $(\gamma_{8,16})_{P_3^t} + \im \partial_1$ that are also contained in $\bigcup_{i+j=1}^6 (\gamma_{8,16})_{P_3^t} + \im \partial_1$. Thus a 7-dimensional subspace of $(\gamma_{8,16})_{P_3^t} + \im \partial_1$ is contained in $\operatorname{span}\{(\gamma_{8,16})_{P_3^t} + \im \partial_1 | j = 1, \ldots, 10, j \neq i\}$. Working sequentially from $i = 1$ to $i = 10$ to calculate the number of vectors in $(\gamma_{8,16})_{P_3^t} + \im \partial_1$ that are contained in $\bigcup_{i+j=1}^6 (\gamma_{8,16})_{P_3^t} + \im \partial_1$ we deduce that
\[
\dim(\operatorname{span}\{(\gamma_{8,16})_{P_3^t} + \im \partial_1 | i = 1, \ldots, 10\}) \leq (16 - 7) + 4 \cdot (16 - 6) + 2 \cdot (16 - 5) + (16 - 4) + 2 \cdot 16 = 115.
\]

Defining $t_0 := 1$, we may repeat the above argument with $P_3^{t_0}$ taking the role of $P_3^{t_1} = P_3$, to see that $(\gamma_{8,16})_{P_3} + \im \partial_1$ contains 72 vectors contained in $\operatorname{span}\{(\gamma_{8,16})_{P_3^t} + \im \partial_1 | i = 1, \ldots, 10\}$. Combining this with (3.8.16) we see that
\[
\dim(H_0(\gamma_{8,16})) = \dim(\operatorname{span}\{(\gamma_{8,16})_{P_3^t} + \im \partial_1 | i = 0, \ldots, 10\}) \leq 115 + (16 - 7) = 124.
\]

We conclude that $H_0(\gamma_{8,16})$ is the unique 121-dimensional quotient of $(16P_3)_M^{11}$, namely
\[
H_0(\gamma_{8,16}) \cong 32 \oplus 44/1/44.
\]

The Mathieu group $M_{12}$ admits a unique 2-minimal parabolic system, given by
\[
\{P_1 \sim 2^{1+4} \cdot \text{Sym}(3), P_2 \sim 4^2 \cdot \text{Sym}(3)\},
\]
which has associated simplicial complex $\Delta$. Both minimal parabolic subgroups give rise to two classes of irreducible $GF(2)P_2$-modules, having dimensions 1 and 2, whilst the self-normalizing
Sylow 2-subgroups of $M_{12}$ admit a unique class of irreducible modules having dimension 1 over $GF(2)$. Consequently, there are four universal panel-irreducible presheaves defined on $\Delta$ which we denote by $\lambda_{i,j}$ for $i, j = 1, 2$. Here the presheaf $\lambda_{i,j}$ has $i$- and $j$-dimensional modules at the minimal parabolic subgroups of type $P_1$ and $P_2$ respectively, and a 1-dimensional irreducible module at each chamber.

Define the graph $\Gamma$ to have vertex set all vertices of $\Delta$, and join a vertex $P_t$ of type $P_1$ to a vertex $P_s$ of type $P_2$ precisely when they intersect in a common chamber. The diameter and girth of $\Gamma$ are 12 and 16 respectively and the collapsed adjacency diagrams of $\Gamma$ with respect to $P_1$ and $P_2$ are given in Figure 3.6. These will be utilised in the geometric spanning arguments used to bound the dimensions of the homology groups $H_0(\lambda_{1,1})$ and $H_0(\lambda_{2,1})$ below.

A consequence of the 2-minimal parabolic subgroups having shape $O_2(P_1).\operatorname{Sym}(3)$ is that the 4 universal panel-irreducible presheaves are in dual pairs. Indeed, $\lambda_{1,1}^{\ast} \cong \lambda_{2,2}$ and $\lambda_{1,2}^{\ast} \cong \lambda_{2,1}$. We shall use this in our calculations below. We shall also utilise the fact that there are five classes of irreducible $GF(2)M_{12}$-modules, having dimensions 1, 10, 32, 44 and 144. Clearly the 1-dimensional irreducible can only be a quotient of $H_0(\lambda_{1,1})$ by Theorem 3.6.9. Indeed, $H_0(\lambda_{1,1}) \cong 1_{M_{12}}$ by Lemma 3.6.1. Moreover, applying the restriction methods used with presheaves of $M_{11}$ together with Theorem 3.6.4, we may fully determine the irreducible quotients of the zero-homology groups of the remaining presheaves. We see that $10_{M_{12}}$ is the unique irreducible quotient of $H_0(\lambda_{1,2})$, etc.

---

**Figure 3.6:** The collapsed adjacency diagrams of the graph $\Gamma$ associated to $M_{12}$ with respect to the 2-minimal parabolic subgroups $P_1$ (top) and $P_2$ (bottom).
32_{M_{12}} and 44_{M_{12}} are both quotients of $H_0(\lambda_{2,1})$ and $144_{M_{12}}$ is the unique irreducible quotient of $H_0(\lambda_{2,2})$.

For the presheaf $\lambda_{i,j}$, we see from the collapsed adjacency diagrams in Figure 3.6 that irrespective of the values of $i, j$ - the homology group $H_0(\lambda_{i,j})$ is spanned by

$$\{(\lambda_{i,j})_P + \text{im } \partial_1 | P \in \Delta_8(P_1)\}.$$ 

It is also spanned by

$$\{(\lambda_{i,j})_P + \text{im } \partial_1 | P \in \Delta_8(P_2) \cup \Delta_9(P_2)\}.$$ 

We shall now apply geometric spanning arguments to obtain upper bounds for the dimensions of the zero-homology groups of $\lambda_{1,2}$ and $\lambda_{2,1}$ in addition to commenting on the dimension of $H_0(\lambda_{2,2})$.

**$\lambda_{1,2}$ and $\lambda_{2,1}$**

The dual presheaves $\lambda_{1,2}$ and $\lambda_{2,1}$ satisfy $\chi(\lambda_{1,2}) = \chi(\lambda_{2,1}) = 0$, and hence by Ronan’s Duality Theorem we deduce that $\dim(H_0(\lambda_{1,2})) = \dim(H_0(\lambda_{2,1}))$.

Consider the presheaf $\lambda_{2,1}$. As previously noted, by analysing the collapsed adjacency diagram of $\Gamma$ with respect to $P_2$, we see that $H_0(\lambda_{2,1})$ is spanned by

$$\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_8(P_2) \cup \Delta_9(P_2)\}.$$ 

For each $P \in \Delta_8(P_2)$ there exists some $Q \in \Delta_9(P_2)$ which is adjacent to $P$ in $\Gamma$. It follows that $(\lambda_{2,1})_P + \text{im } \partial_1 \subseteq (\lambda_{2,1})_Q + \text{im } \partial_1$, and hence $H_0(\lambda_{2,1})$ is spanned by $\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_9(P_2)\}$.

Since each $Q \in \Delta_9(P_2)$ is adjacent to two elements of $\Delta_{10}(P_2)$ and the presheaf terms at elements of $\Delta_9(P_2)$ and $\Delta_{10}(P_2)$ are 2- and 1-dimensional respectively, we conclude that $H_0(\lambda_{2,1})$ is actually spanned by $\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_{10}(P_2)\}$.

Define

$$\Delta_{10}^a(P_2) := \{P \in \Delta_{10}(P_2) \mid \Delta_1(P) \cap \Delta_{11}(P_2) \neq \emptyset\},$$

and set $\Delta_{10}^b(P_2) := \Delta_{10}(P_2) \setminus \Delta_{10}^a(P_2)$. We have that $|\Delta_{10}^a(P_2)| = 96$ and $|\Delta_{10}^b(P_2)| = 64$. The image of the presheaf terms of $\lambda_{2,1}$ at elements of $\Delta_{10}^a(P_2)$ are contained in the span of $D_{11} := \{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_{11}(P_2)\}$. However, each element of $D_{11}$ is spanned by the image of the presheaf term at an element of $\Delta_{12}(P_2)$ and the image of a term at an element of $\Delta_{10}^b(P_2)$. Consequently

$$\dim(D_{11}) \leq |\Delta_{11}(P_2)| + \dim(\text{span}\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_{12}(P_2)\})$$

$$\leq 48 + 16 = 64.$$

We conclude that

$$\dim(H_0(\lambda_{2,1})) = \dim(\text{span}\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_{10}(P_2)\})$$

$$\leq \dim(\text{span}\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_{10}^a(P_2)\})$$

$$+ \dim(\text{span}\{(\lambda_{2,1})_P + \text{im } \partial_1 | P \in \Delta_{10}^b(P_2)\})$$

$$\leq \dim(\text{span } D_{11}) + |\Delta_{10}^b(P_2)|$$

$$\leq 64 + 64 = 128.$$
3.8. Calculation of Homology Groups

<table>
<thead>
<tr>
<th>Presheaf, $\lambda_{i,j}$</th>
<th>Irreducible quotients of $H_0(\lambda_{i,j})$</th>
<th>Notes on $H_0(\lambda_{i,j})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,1}$</td>
<td>1</td>
<td>$H_0(\lambda_{1,1}) \cong 1$</td>
</tr>
<tr>
<td>$\lambda_{1,2}$</td>
<td>10</td>
<td>$76 \leq \dim(H_0(\lambda_{1,2})) = \dim(H_0(\lambda_{2,1})) \leq 128$</td>
</tr>
<tr>
<td>$\lambda_{2,1}$</td>
<td>32, 44</td>
<td>$76 \leq \dim(H_0(\lambda_{2,1})) = \dim(H_0(\lambda_{1,2})) \leq 128$</td>
</tr>
<tr>
<td>$\lambda_{2,2}$</td>
<td>144</td>
<td>$\dim(H_0(\lambda_{2,2})) = 496$</td>
</tr>
</tbody>
</table>

Table 3.2: Dimensional-bounds and irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves of $M_{12}$ over $GF(2)$.

Finally, as $32_{M_{12}}$ and $44_{M_{12}}$ are quotients of $H_0(\lambda_{2,1})$ we see that

$$76 \leq \dim(H_0(\lambda_{1,2})) = \dim(H_0(\lambda_{2,1})) \leq 128.$$  

$\lambda_{2,2}$

As $\lambda_{2,2} \cong \lambda_{1,1}^4$, Ronan’s Duality Theorem asserts that

$$\dim(H_0(\lambda_{2,2})) = \chi(\lambda_{2,2}) + \dim(H_1(\lambda_{2,2})) = \chi(\lambda_{2,2}) + \dim(H_0(\lambda_{1,1})) = 495 + 1 = 496.$$  

Unfortunately, it is unrealistic to fully determine the structure of $H_0(\lambda_{i,j})$ for $(i, j) \in \{(1, 2), (2, 1), (2, 2)\}$. Obtaining bounds on the dimension and the irreducible quotients of the homology group is the best that one can usually hope for. We summarise our findings in Table 3.2

$M_{22}$

The unique 2-minimal parabolic system for $M_{22}$ is given by

$$\{P_1 \sim 2^{4+2} \cdot \text{Sym}(3), P_2 \sim 2^4 \cdot \text{Sym}(5)\}.$$  

There are six classes of irreducible $GF(2)M_{22}$-modules which we denote by $1_{M_{22}}, 10_{M_{22}}, 10_{M_{22}}, 34_{M_{22}}, 98_{M_{22}}$ and $140_{M_{22}}$ (where the number denotes the dimension of the module). Furthermore, setting $B$ to be a self-normalizing Sylow 2-subgroup, we note that there are one, two and three classes of irreducible $GF(2)B$-, $GF(2)P_1$- and $GF(2)P_2$- modules. We denote these by $1_B$, $1_{P_1}$ and $2_{P_1}$ and $1_{P_2}$, $4_{P_2}$ and $4_{P_2}$ respectively. It follows that there are six universal panel-irreducible presheaves defined on $M_{22}$ over $GF(2)$, given in the usual notation by $\lambda_{i,j}$ for $(i, j) \in \{(1, 1), (1, 4), (1, 4), (2, 1), (2, 4), (2, 4)\}$.

For each irreducible $GF(2)M_{22}$-module $V$, we may as in previous cases consider the restriction of $V$ to $P_1$ and $P_2$ to allow us to determine those $\lambda_{i,j}$ for which $V$ is a quotient of $H_0(\lambda_{i,j})$. The modules $1_{M_{22}}, 10_{M_{22}}, 10_{M_{22}}, 98_{M_{22}}$ and $140_{M_{22}}$ are each quotients of the zero-homology group of a unique presheaf, these being $\lambda_{1,1}, \lambda_{1,4}, \lambda_{2,1}, \lambda_{2,4}$ and $\lambda_{2,7}$ respectively. Moreover, $34_{M_{22}}$ is not a quotient of any zero-homology group. It follows that $H_0(\lambda_{1,3}) = 0$. In addition, we trivially have that $H_0(\lambda_{1,1}) \cong 1_{M_{22}}$.

To obtain upper bounds on the dimensions of the zero-homology groups of the remaining four presheaves, we define $\Gamma$ to be the graph having conjugates of $P_1$ and $P_2$ as its vertices and with
\[
\begin{align*}
\text{ CHAPTER 3. HOMOLOGY OF PRESHEAVES OF ABELIAN GROUPS }
\end{align*}
\]

\[
\begin{align*}
&\Delta_0(P_1) & \Delta_1(P_1) & \Delta_2(P_1) & \Delta_3(P_1) & \Delta_4(P_1) & \Delta_5(P_1) \\
&\Delta_0(P_2) & \Delta_1(P_2) & \Delta_2(P_2) & \Delta_3(P_2) & \Delta_4(P_2) & \Delta_5(P_2)
\end{align*}
\]

Figure 3.7: The collapsed adjacency diagrams of the graph \( \Gamma \) associated to \( M_{22} \) with respect to the 2-minimal parabolic subgroups \( P_1 \) (top) and \( P_2 \) (bottom).

\( \{P_1^1, P_2^2\} \) forming an edge of \( \Gamma \) precisely when \( P_1^1 \cap P_2^2 = B^tw \) for some \( t_B \in M_{22} \). The diameter and girth of \( \Gamma \) are both equal to 6 and the collapsed adjacency diagrams of \( \Gamma \) with respect to \( P_1 \) and \( P_2 \) are given in Figure 3.7.

We now consider the four remaining presheaves in turn.

\( \lambda_{1,4} \)

By constructing quotients of the induced module \( (4P_2)^{M_{22}} \) in MAGMA, we obtain seven possibilities for \( H_0(\lambda_{1,4}) \). Indeed, the homology group has 10\( M_{22} \) as a unique irreducible quotient, and there are seven isomorphism classes of quotients of \( (4P_2)^{M_{22}} \) that do likewise. These are 10, 10/\( \mathbb{T} \), 10/\( \mathbb{T}\mathbb{O}/1 \), 10/\( \mathbb{T}\mathbb{O}/1/34 \), 10/\( \mathbb{T}\mathbb{O}/1/34/1 \), 10/\( \mathbb{T}\mathbb{O}/1/34/1/10 \) and 10/\( \mathbb{T}\mathbb{O}/1/34/1/10/\mathbb{T} \), all of which are uniserial modules. Considering the restriction of each module to \( B \), \( P_1 \) and \( P_2 \), we see that the only modules whose constant presheaves admit panel-irreducible subpresheaves are 10, 10/\( \mathbb{O} \) and 10/\( \mathbb{O}/1 \). Thus by Theorem 3.6.4, we conclude that \( H_0(\lambda_{1,4}) \cong 10/\mathbb{T}\mathbb{O}/1 \).

\( \lambda_{2,1} \)

The induced module \( (1P_2)^{M_{22}} \) is 231-dimensional, and hence its quotients may be explicitly constructed in MAGMA. The homology group \( H_0(\lambda_{2,1}) \) is one such quotient, having \( \mathbb{O} \) as a unique irreducible quotient. Considering all such quotients of \( (1P_2)^{M_{22}} \) with this property, we see that there are four possibilities for \( H_0(\lambda_{2,1}) \), namely \( \mathbb{O} \), \( \mathbb{O}/1 \), \( \mathbb{O}/10 \) and \( \mathbb{O}/10/1 \) all of which are indecomposable (with \( \mathbb{O} \), \( \mathbb{O}/1 \) and \( \mathbb{O}/10 \) also being uniserial modules). Using elements of each module to generate subpresheaves of the constant presheaves \( K_{\mathbb{O}} \), \( K_{\mathbb{O}/1} \), \( K_{\mathbb{O}/10} \) and \( K_{\mathbb{O}/10/1} \) and appealing to Theorem 3.6.4 we see that \( \mathbb{O} \) and \( \mathbb{O}/1 \) are quotients of \( H_0(\lambda_{2,1}) \), but \( \mathbb{O}/10 \) and \( \mathbb{O}/10/1 \) are not. Consequently, \( H_0(\lambda_{2,1}) \cong \mathbb{O}/1 \).

\( \lambda_{2,4} \) and \( \lambda_{2,7} \)

For the presheaves \( \lambda_{2,4} \) and \( \lambda_{2,7} \) we begin by using a geometric spanning argument. From the collapsed adjacency diagram of the graph \( \Gamma \) associated to \( M_{22} \) with respect to \( P_2 \) we see that the
image of each presheaf term is contained in $\Delta_4(P_2) + \operatorname{im} \hat{c}_1$ with the possible exception of the 40 elements of $\Delta_3(P_2)$ having no neighbour in $\Delta_4(P_2)$. The image of these latter terms will be at most 2-dimensional, whilst the images of the terms in $\Delta_4(P_2)$ will be at most 4-dimensional. However, the image of each element of $\Delta_5(P_2)$ gives rise to a relation between the images of three non-zero vectors of the presheaf terms at elements of $\Delta_4(P_2)$. At worst, these 800 relations would lead to the images of 67 elements of $\Delta_4(P_2)$ being zero. In practice, they will account for a much larger decrease in the possible dimension of the zero-homology groups. We conclude that a rather crude bound on the dimension of the zero-homology groups of $\lambda_{2,4}$ and $\lambda_{2,7}$ is given by

$$\dim(H_0(\lambda_{2,4})), \dim(H_0(\lambda_{2,7})) \leq 200 \cdot 4 + 40 \cdot 2 - 67 \cdot 4 = 612. \quad (3.8.17)$$

The homology group $H_0(\lambda_{2,4})$ is a quotient of $(4P_2)^{M_{22}}$, which has a unique irreducible quotient, namely $98_{M_{22}}$. There are 481 classes of such quotients satisfying the dimensional bound from (3.8.17). We may consider these classes in MAGMA and appeal to Theorem 3.6.4 and Proposition 3.6.7. However, we do not need to consider all classes explicitly. Indeed for a given quotient $W$, if we determine that it does admit $\lambda_{2,4}$ as a subpresheaf of $K_W$, then we may rule out all classes for which $W$ is a not a quotient. Conversely, if $W$ does not admit $\lambda_{2,4}$, then we may rule out all classes for which $W$ is a quotient. We deduce that

$$H_0(\lambda_{2,4}) \cong 98/1/\left(\frac{34}{(1/10) \oplus (10/1) \oplus (10/1)}\right)/34/1.$$  

We follow an analogous approach to calculate $H_0(\lambda_{2,7})$, which is by definition a quotient of $(\overline{4P}_2)^{M_{22}}$. Using the bound from (3.8.17) together with the fact that $140_{M_{22}}$ is the unique irreducible quotient of $H_0(\lambda_{2,7})$, we are left with 38 possible classes for $H_0(\lambda_{2,7})$. By using MAGMA as above, we may calculate that

$$H_0(\lambda_{2,7}) \cong 140/(1 \oplus 1/34/1/10/34).$$

As we have seen with $M_{22}$, even for small groups it is usually extremely difficult to determine the exact dimension and structure of the zero-homology group of a universal panel-irreducible presheaf. In the case of $M_{23}$ and $M_{24}$ below, we merely determine the irreducible quotients of the homology groups by considering the restriction of modules to minimal parabolic subgroups.

$M_{23}$

The second largest Mathieu group, $M_{23}$, admits seven 2-minimal parabolic subgroups with respect to a self-normalizing Sylow 2-subgroup $B$. In the notation of Ronan and Stroth (see [RS84]) these are $P_i \sim 2^{4+2}.\text{Sym}(3)$ for $i = 1, \ldots, 5$ and $P_j \sim 2^4.\text{Sym}(5)$ for $j = 6, 7$, where $P_5 = P_6 \cap P_7$. These minimal parabolic subgroups give rise to seven 2-minimal parabolic systems for $M_{23}$, namely $\{P_1, P_3, P_7\}$, $\{P_2, P_3, P_7\}$, $\{P_3, P_4, P_7\}$ and $\{P_i, P_6, P_7\}$ for $i = 1, 2, 3, 4$. The systems $\{P_1, P_3, P_7\}$ and $\{P_3, P_4, P_7\}$ are geometric, whilst the remaining systems are non-geometric.

There are ten irreducible $\text{GF}(2)M_{23}$-modules, which we denote by their dimensions as $1_{M_{23}}$, $11_{M_{23}}$, $11\overline{M}_{23}$, $11\overline{M}_{23}$, $120_{M_{23}}$, $220_{M_{23}}$, $220\overline{M}_{23}$, $252_{M_{23}}$ and $1792_{M_{23}}$. For $i = 1, 2, 3, 4$ there
are two classes of irreducible $GF(2)P_i$-modules, $1P_i$ and $2P_i$, whilst for $j = 6, 7$ there are three classes of irreducible $GF(2)P_j$-modules, namely $1P_j$, $4P_j$ and $4P_j$.

Every 2-minimal parabolic system of $M_{23}$ features $P_i$. Consequently, for each irreducible $GF(2)M_{23}$-module, $V_i$, we may consider the elements of $V$ which generate irreducible submodules of the restriction of $V$ to $P_i$. For each such element, we may then consider the submodules of $V|_B$ and $V|_{P_i}$ that the element generates for $i = 1, 2, 3, 4, 6, 7$. In Table 3.3 we list such elements that generate irreducible $GF(2)B$-modules and irreducible $GF(2)P_i$-modules for every $P_i$ featuring in one of the 2-minimal parabolic systems of $M_{23}$. Entries of the form “N/A” indicate that the given module is reducible.

The seven 2-minimal parabolic systems give rise to an abundance of universal panel-irreducible presheaves for $M_{23}$ over $GF(2)$. For ease of notation, we denote the presheaf defined on the minimal parabolic system $\{P_i, P_j, P_k\}$ having a $*_{\dim}$-dimensional irreducible $GF(2)P_*|$ module at panels of type $P_*$ by $\lambda_{\dim, j\dim, k\dim}^{(ijk)}$. An analysis of the vertex terms of these presheaves is given in Appendix A. Using the information in Table 3.3 we may fully determine the irreducible quotients of the zero-homology group of each presheaf. We summarise our findings in Tables 3.4-3.10. In these tables, entries of the form “None” indicate that the given homology group is equal to 0.

In most of the cases mentioned above, the irreducible quotients may be determined by taking an irreducible $GF(2)M_{23}$-module $V$ and finding the socle of its restriction to $P_i$.

---

Table 3.3: The elements of the irreducible $GF(2)M_{23}$-modules which admit universal panel-irreducible presheaves.

| $GF(2)M_{23}$-Module | Element of Module, $v$ | $\langle v \rangle|_B$ | $\langle v \rangle|_{P_1}$ | $\langle v \rangle|_{P_2}$ | $\langle v \rangle|_{P_3}$ | $\langle v \rangle|_{P_4}$ | $\langle v \rangle|_{P_5}$ | $\langle v \rangle|_{P_6}$ | $\langle v \rangle|_{P_7}$ |
|----------------------|------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $1_{M_{23}}$         | $v_1$                  | 1B             | 1P            | 1P            | 1P            | 1P             | 1P             | 1P             | 1P             |
| $11_{M_{23}}$        | $v_{11}$               | 1B             | 1P            | 1P            | 1P            | 1P             | 1P             | 1P             | 4P             |
| $\Pi_{M_{23}}$       | $v_{\Pi_{1}}$          | 1B             | 1P            | 1P            | 2P            | 1P             | 4P             | 1P             | 1P             |
|                     | $v_{\Pi_{2}}$          | 1B             | 2P            | 2P            | N/A           | 2P             | 1P             | 1P             | 1P             |
| $44_{M_{23}}$        | $v_{44}$               | 1B             | 1P            | 1P            | 1P            | N/A            | 1P             | 4P             | 1P             |
| $\Pi_{M_{23}}$       | $v_{\Pi_{1}}$          | 1B             | 2P            | 2P            | 2P            | 2P             | 4P             | 1P             | 1P             |
|                     | $v_{\Pi_{2}}$          | 1B             | 1P            | 1P            | 1P            | N/A            | 1P             | N/A            | 1P             |
| $120_{M_{23}}$       | $v_{120_{1}}$          | 1B             | 1P            | 2P            | 2P            | 2P             | 4P             | 4P             | 4P             |
|                     | $v_{120_{2}}$          | 1B             | 2P            | 2P            | N/A            | 2P             | N/A            | 4P             | 4P             |
|                     | $v_{120_{3}}$          | 1B             | 1P            | 2P            | 2P            | 2P             | N/A            | 4P             | 4P             |
| $220_{M_{23}}$       | $v_{220_{1}}$          | 1B             | 2P            | 1P            | 2P            | 2P             | N/A            | 4P             | 4P             |
|                     | $v_{220_{2}}$          | 1B             | 2P            | 1P            | 2P            | 1P             | N/A            | 4P             | 4P             |
|                     | $v_{220_{3}}$          | 1B             | 2P            | 2P            | N/A            | N/A            | 4P             | 4P             | 4P             |
| $252_{M_{23}}$       | $v_{252_{1}}$          | 1B             | 1P            | 1P            | N/A            | 1P             | N/A            | 4P             | 4P             |
|                     | $v_{252_{2}}$          | 1B             | 2P            | 1P            | N/A            | 2P             | N/A            | 4P             | 4P             |
|                     | $v_{252_{3}}$          | 1B             | 2P            | 2P            | N/A            | N/A            | 4P             | 4P             | 4P             |
|                     | $v_{252_{4}}$          | 1B             | 1P            | 1P            | N/A            | 1P             | N/A            | 4P             | 4P             |
| $1792_{M_{23}}$      | $v_{1792_{1}}$         | 1B             | 2P            | 1P            | N/A            | 1P             | 2P             | 4P             | 4P             |

---

6We recall that the socle of a module $V$, denoted soc($V$), is the sum of all minimal submodules of $V$. [Lam99]
### 3.8. Calculation of Homology Groups

Table 3.4: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_1, P_3, P_7\} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(137)} )</td>
<td>( 1_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,1,1}^{(137)} )</td>
<td>( 11_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,1,4}^{(137)} )</td>
<td>None</td>
</tr>
<tr>
<td>( \lambda_{1,1,7}^{(137)} )</td>
<td>( \Pi_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,2,1}^{(137)} )</td>
<td>( 120_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,2,4}^{(137)} )</td>
<td>( 220_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,3,1}^{(137)} )</td>
<td>None</td>
</tr>
<tr>
<td>( \lambda_{2,1,4}^{(137)} )</td>
<td>None</td>
</tr>
<tr>
<td>( \lambda_{2,1,7}^{(137)} )</td>
<td>None</td>
</tr>
<tr>
<td>( \lambda_{2,2,1}^{(137)} )</td>
<td>( \Pi_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{2,2,4}^{(137)} )</td>
<td>( 252_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{2,3,1}^{(137)} )</td>
<td>( 220_{M_{23}}, 220_{M_{23}} )</td>
</tr>
</tbody>
</table>

Table 3.5: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_1, P_6, P_7\} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(167)} )</td>
<td>( 1_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,1,1}^{(167)} )</td>
<td>( 11_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,1,4}^{(167)} )</td>
<td>None</td>
</tr>
<tr>
<td>( \lambda_{1,1,7}^{(167)} )</td>
<td>( \Pi_{M_{23}}, 44_{M_{23}} )</td>
</tr>
<tr>
<td>( \lambda_{1,4,1}^{(167)} )</td>
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<tr>
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<tr>
<td>( \lambda_{1,4,7}^{(167)} )</td>
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<tr>
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<tr>
<td>( \lambda_{1,7,4}^{(167)} )</td>
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<tr>
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<tr>
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### Table 3.6: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_3, P_7\}$ of $M_{23}$.

<table>
<thead>
<tr>
<th>Presheaf, $\lambda$</th>
<th>Irreducible quotients of $H_0(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{(237)}^{1,1,1}$</td>
<td>$1_{M_{23}}$</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{1,1,4}$</td>
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</tr>
<tr>
<td>$\lambda_{(237)}^{1,1,7}$</td>
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<tr>
<td>$\lambda_{(237)}^{1,2,1}$</td>
<td>$\overline{\Pi}<em>{M</em>{23}}$</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{1,2,4}$</td>
<td>$120_{M_{23}}$</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{1,2,7}$</td>
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</tr>
<tr>
<td>$\lambda_{(237)}^{2,1,1}$</td>
<td>None</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{2,1,4}$</td>
<td>None</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{2,1,7}$</td>
<td>None</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{2,2,1}$</td>
<td>$\overline{\Pi}<em>{M</em>{23}}$</td>
</tr>
<tr>
<td>$\lambda_{(237)}^{2,2,4}$</td>
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</tr>
<tr>
<td>$\lambda_{(237)}^{2,2,7}$</td>
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### Table 3.7: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_6, P_7\}$ of $M_{23}$.

<table>
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<td>$\lambda_{(267)}^{1,1,4}$</td>
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<td>$\lambda_{(267)}^{1,1,7}$</td>
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<tr>
<td>$\lambda_{(267)}^{1,4,1}$</td>
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</tr>
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<td>$\lambda_{(267)}^{1,4,4}$</td>
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</tr>
<tr>
<td>$\lambda_{(267)}^{1,7,1}$</td>
<td>None</td>
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<tr>
<td>$\lambda_{(267)}^{1,7,4}$</td>
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<tr>
<td>$\lambda_{(267)}^{1,7,7}$</td>
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<tr>
<td>$\lambda_{(267)}^{2,1,1}$</td>
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<td>$\lambda_{(267)}^{2,1,4}$</td>
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<td>$\lambda_{(267)}^{2,1,7}$</td>
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<tr>
<td>$\lambda_{(267)}^{2,4,1}$</td>
<td>$44_{M_{23}}, 220_{M_{23}}$</td>
</tr>
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<td>$\lambda_{(267)}^{2,4,4}$</td>
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<td>$220_{M_{23}}$</td>
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</table>
### 3.8. CALCULATION OF HOMOLOGY GROUPS

<table>
<thead>
<tr>
<th>Presheaf, $\lambda$</th>
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<tbody>
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<td>$\lambda_{1,1,1}^{(347)}$</td>
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<td>None</td>
</tr>
<tr>
<td>$\lambda_{1,2,4}^{(347)}$</td>
<td>None</td>
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<tr>
<td>$\lambda_{1,2,4}^{(347)}$</td>
<td>$\Pi_{M_{23}}$</td>
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<tr>
<td>$\lambda_{2,1,1}^{(347)}$</td>
<td>$120_{M_{23}}$</td>
</tr>
<tr>
<td>$\lambda_{2,1,4}^{(347)}$</td>
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<td>$\Pi_{M_{23}}$</td>
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<td>$\lambda_{2,2,4}^{(347)}$</td>
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</tr>
<tr>
<td>$\lambda_{2,2,7}^{(347)}$</td>
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Table 3.8: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_4, P_7\}$ of $M_{23}$.

<table>
<thead>
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<th>Presheaf, $\lambda$</th>
<th>Irreducible quotients of $H_0(\lambda)$</th>
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<td>$\lambda_{1,1,1}^{(367)}$</td>
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<tr>
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<td>$\lambda_{1,7,4}^{(367)}$</td>
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<td>$\lambda_{1,3,1}^{(367)}$</td>
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<td>$\lambda_{3,3,1}^{(367)}$</td>
<td>$\Pi_{M_{23}}$, $44_{M_{23}}$</td>
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<tr>
<td>$\lambda_{2,7,4}^{(367)}$</td>
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Table 3.9: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_6, P_7\}$ of $M_{23}$. 
Table 3.10: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_4, P_6, P_7\} \) of \( M_{23} \).

<table>
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<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
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<tbody>
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<td>( 11_{M_{23}} )</td>
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<tr>
<td>( \lambda_{1,4,1}^{(467)} )</td>
<td>( \Pi_{M_{23}}, 44_{M_{23}} )</td>
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<td>( \lambda_{1,4,4}^{(467)} )</td>
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<td>None</td>
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<tr>
<td>( \lambda_{1,3,3}^{(467)} )</td>
<td>( \Pi_{M_{23}} )</td>
</tr>
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3.8. Calculation of Homology Groups

<table>
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<th>Irreducible quotients of $H_0(\lambda)$</th>
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<tbody>
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<td>$\lambda_{1,1,1}$</td>
<td>$1_{M_{24}}$, $\mathbb{M}<em>{M</em>{24}}$</td>
</tr>
<tr>
<td>$\lambda_{1,1,2}$</td>
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<td>$\lambda_{1,2,2}$</td>
<td>$11_{M_{24}}$, $120_{M_{24}}$</td>
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<td>$\lambda_{2,1,1}$</td>
<td>$120_{M_{24}}$, $220_{M_{24}}$</td>
</tr>
<tr>
<td>$\lambda_{2,1,2}$</td>
<td>$252_{M_{24}}$, $320_{M_{24}}$, $1792_{M_{24}}$, $1242_{M_{24}}$, $1792_{M_{24}}$</td>
</tr>
</tbody>
</table>

Table 3.11: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the unique 2-minimal parabolic system of $M_{24}$.

elements of $\text{soc}(V|_{P_i})$, we may then determine which universal panel-irreducible presheaves admit $V$. In the case of $1792_{M_{24}}$, a more subtle approach is required as $\text{soc}(1792_{M_{24}}|_{P_i})$ is 32-dimensional. However, we may decompose the socle as a direct sum of two semisimple submodules, the first generated by all minimal submodules isomorphic to $4_{P_i}$ and the second generated by all minimal submodules isomorphic to $I_{P_i}$. The former is an 8-dimensional submodule, whilst the latter is 24-dimensional. By considering all elements of each of these summands, we then obtain the results.

$M_{24}$

The simple group $M_{24}$ admits four 2-minimal parabolic subgroups. In the notation of Ronan and Stroth these are $P_i \sim 2^{6k+3}.\text{Sym}(3)$ for $i = 1, 2, 3, 4$. Here $P_1 \cong P_2$ and $P_3 \cong P_4$. These minimal parabolic subgroups give rise to a unique 2-minimal parabolic system $\{P_1, P_3, P_4\}$ of $M_{24}$, which is geometric. Since each minimal parabolic subgroup, $P_i$, has $\text{Sym}(3)$ above its 2-core, there are two classes of irreducible $GF(2)\mathbf{P_i}$-modules, $1_{P_i}$ and $2_{P_i}$. There is also a unique class of irreducible $GF(2)\mathbf{B}$-modules for a self-normalizing Sylow 2-subgroup $B$. These modules are 1-dimensional. It follows that there are eight universal panel-irreducible presheaves for $M_{24}$. We denote the presheaf having a 1-dimensional chamber term and $i$-, $j$- and $k$-dimensional panel terms at panels of type $P_1$, $P_3$ and $P_4$ by $\lambda_{i,j,k}$. An exploration of the vertex terms of these presheaves is given in Section A.2

Aside 3.8.2. We note that the presheaves $\lambda_{i,j,k}$ differ from those considered by Ronan and Smith in [RS89]. Indeed, since the minimal parabolic subgroup $P_2$ does not feature in a minimal parabolic system of $M_{24}$, Ronan and Smith defined $Q_i := \langle P_1, P_2 \rangle$ for $i = 1, 2, 3, 4$ and considered presheaves defined on the simplicial complex having conjugates of $Q_2$ as its chambers, and conjugates of $Q_i$ for $i = 1, 3, 4$ as its panels.

There are thirteen classes of irreducible $GF(2)M_{24}$-modules, denoted by their dimensions as $1_{M_{24}}$, $11_{M_{24}}$, $\mathbb{M}_{M_{24}}$, $44_{M_{24}}$, $11_{M_{24}}$, $120_{M_{24}}$, $220_{M_{24}}$, $252_{M_{24}}$, $320_{M_{24}}$, $320_{M_{24}}$, $1242_{M_{24}}$ and $1792_{M_{24}}$. For each such module, $V$, we mimic the calculations undertaken for $M_{23}$ to determine which homology groups $H_0(\lambda_{i,j,k})$ admit $V$ as a quotient. A summary of these calculations is given in Table 3.11.

To deduce that $1242_{M_{24}}$ admits a unique universal panel-irreducible presheaf, namely $\lambda_{2,2,2}$, we note that the restriction $1242_{M_{24}}|_{P_{34}}$ has three minimal submodules, these being isomorphic to
1_{P_{34}}, 3_{P_{34}} and 8_{P_{34}}. The elements of 1242_{M_{24}} generating 3_{P_{34}} do not give rise to a panel-irreducible presheaf, whilst those generating 8_{P_{34}} give rise to \( \lambda_{2,2,2} \). Consequently, the only remaining presheaf that may have 1242_{M_{24}} as a quotient of its zero-homology group will necessarily have 1-dimensional terms at panels of type \( P_3 \) and \( P_4 \). The only such possibility is \( \lambda_{2,1,1} \). However, consideration of the elements of 1242_{M_{24}} that generate 2-dimensional \( GF(p) \)-modules proves that this possibility cannot occur.

### 3.9 Homology Groups of Universal Presheaves Revisited

We saw in Section 3.8 that the normalizer of a Sylow subgroup being maximal in each minimal parabolic subgroup was not sufficient for an affirmative answer to Question 3.7.1. Indeed, as described in Subsection 3.8.2 there does not exist a universal panel-irreducible presheaf, \( \mathcal{P} \), defined on the minimal parabolic systems \( S_2 \) or \( S_3 \) of \( M_{11} \) for which 10_{M_{11}} is a quotient of \( H_0(\mathcal{P}) \).

A natural evolvement is to ask if there are any types of \( p \)-minimal parabolic system, \( S \), of a group \( G \) for which every irreducible \( GF(p)G \)-module is admitted as a quotient of the zero-homology group of some universal panel-irreducible presheaf? From our calculations in Section 3.8 we see that this may be the case when \( p = 2 \), a Sylow 2-subgroup of \( G \) is self-normalizing and each 2-minimal parabolic subgroup, \( P \), of \( G \) is of shape \( O_2(p) \cdot Sym(3) \). Indeed, we saw that this occurred for the unique 2-minimal parabolic systems of \( Sym(6) \), \( M_{12} \) and \( M_{24} \). It also occurs for the 2-minimal parabolic systems of \( L_4(2) \) and \( Alt(7) \), considered by Ronan and Smith in [RSS9] and for the parabolic subgroups \( P_{13} \) and \( P_{34} \) of \( M_{23} \) (considered in Appendix A). This leads to the following question.

**Question 3.9.1.** Suppose that \( G \) is a group of even order possessing a self-normalizing Sylow 2-subgroup. Moreover, assume that \( G \) admits a 2-minimal parabolic system \( S := \{ P_i \mid i = 1, \ldots, n \} \) such that \( P_i \sim O_2(P_i) \cdot Sym(3) \) for all \( i = 1, \ldots, n \). For each \( GF(2)G \)-module \( V \), does there exist a universal panel-irreducible presheaf, \( \mathcal{P}_V \), defined on \( S \) such that \( V \) is a quotient of \( H_0(\mathcal{P}_V) \)?

If Question 3.9.1 has a positive answer, it would likely rely on the fact that if \( B \) is the self-normalizing Sylow 2-subgroup of \( G \) and \( P \) is a 2-minimal parabolic subgroup containing \( B \), then there is a unique class of irreducible \( GF(2)B \)-modules and every non-trivial proper quotient of \( (1_B)_B^P \) is irreducible. Thus it would suffice to prove that for every irreducible \( GF(2)G \)-module, \( V \), there exists a non-zero element \( v \in V \) satisfying \( \langle v \rangle_{kB} \cong 1_B \) and \( \langle v \rangle_{kP} \neq (1_B)_B^P \) for every minimal parabolic subgroup \( P \).
Chapter 4

π-Product Graphs in Symmetric Groups

There is a cornucopia of combinatorial and geometric structures which are associated with groups. These range from graphs to posets and topological spaces such as simplicial complexes. An example of the latter type arises in a finite group $G$ where for a prime $p$ dividing the order of $G$ we may define the poset of all non-trivial $p$-subgroups of $G$, denoted $S_p(G)$, ordered by inclusion. This poset has a rich structure, as has its associated order complex $|S_p(G)|$ known as the Brown complex, after being studied by - among others - Brown in his paper [Bro75]. An analogous order complex, called the Quillen complex, can be defined for the poset $A_p(G)$ of all non-trivial elementary abelian $p$-subgroups of $G$. Indeed, Quillen showed in [Qui78] that the Brown and Quillen complexes are $G$-homotopy equivalent. Thévenez and Webb later showed that the complexes consisting of chains of normal series of $p$-subgroups, and chains of radical $p$-subgroups are also $G$-homotopy equivalent to the Brown and Quillen complexes (see [Thé87] and [TW91] for full details). In the case when $G$ is a group of Lie type, the order complex $|S_p(G)|$ is the same as the building of $G$.

The aforementioned subgroup complexes arise in many different areas. Brown was motivated by cohomology and the calculation of cohomology groups for discrete groups - the subject of [Bro76]. The complexes are also closely related to fusion in finite groups and the existence of strongly $p$-embedded subgroups in $G$ is equivalent to the disconnectedness of $A_p(G)$ and $S_p(G)$ (further details of which can be found for example in [Asc00]). It is also possible to build modular representations of $G$ by first defining such representations on stabilizers of the simplices of these complexes, following the constructions of Ronan and Smith in [Ron89], [RS85], [RS86] and [RSS99]. A good survey of the versatility of such complexes can be found in [Smi11].

We mention a few graphs among the multitude of such structures that we may associate to a given group $G$. Let $X$ be a subset of $G$. The commuting graph $C(G, X)$ has vertex set $X$ and distinct elements $x, y \in X$ are joined by an edge whenever $xy = yx$. The case when $X = G \setminus Z(G)$, first studied in [BF55], has been the focus of interest recently - see [BG13], [MP13] and [Par13]. When $X$ is taken to be a $G$-conjugacy class of involutions, we get the so-called commuting involution graph, the subject of a number of papers (see [BBHR09], [BBPR03a], [BBPR03b], [BBPR04], [Eve11], [Per06] and [Sal11]).
If \( \pi \) is a set of natural numbers, then the \( \pi \)-product graph \( \mathcal{P}_\pi(G, X) \) again has vertex set \( X \), with distinct vertices \( x, y \in X \) joined by an edge if the order of \( xy \) is in \( \pi \). In the case when \( X \) is a \( G \)-conjugacy class of involutions, we note that \( \mathcal{P}_{\{2\}}(G, X) \) is just a commuting involution graph. Taking \( \pi \) to be the set of all odd natural numbers and \( X \) a \( G \)-conjugacy class, \( \mathcal{P}_\pi(G, X) \) becomes the local fusion graph \( \mathcal{F}(G, X) \) which has featured in [Bal13] and [BGR13].

In the case when \( X \) is a set of involutions we refer to \( \mathcal{P}_\pi(G, X) \) as a \( \pi \)-product involution graph. It is such graphs when \( X \) is a conjugacy class that we consider in this chapter for \( G = \text{Sym}(n) \), the symmetric group of degree \( n \). We use the standard distance metric on \( \mathcal{P}_\pi(G, X) \), which we denote by \( d(\cdot, \cdot) \). For \( x \in X \) and \( i \in \mathbb{N} \) we denote the set of vertices distance \( i \) from \( x \) in \( \mathcal{P}_\pi(G, X) \) by \( \Delta_i(x) \). We also denote by \( \Omega := \{1, \ldots, n\} \) the underlying set upon which \( \text{Sym}(n) \) acts.

We first consider the case when \( \pi = \{4\} \). In other words, two distinct involutions \( x, y \in X \) are joined by an edge whenever \( \langle x, y \rangle \cong \text{Dih}(8) \), the dihedral group of order 8. In considering this, we are in effect looking at a section of the poset \( \mathcal{S}_2(\text{Sym}(n)) \). Our first result determines when \( \mathcal{P}_{\{4\}}(G, X) \) is connected and in such cases, the diameter of \( \mathcal{P}_{\{4\}}(G, X) \) is also determined.

**Theorem 4.0.1.** Suppose \( G = \text{Sym}(n) \), \( t = (1, 2) \cdots (2m - 1, 2m) \in G \), and let \( X \) denote the \( G \)-conjugacy class of \( t \).

(i) The graph \( \mathcal{P}_{\{4\}}(G, X) \) is disconnected if and only if one of the following holds:

(a) \( n = 2m + 1 \);

(b) \( m = 1 \);

(c) \( (n, m) = (4, 2) \) or \( (6, 3) \).

(ii) If \( \mathcal{P}_{\{4\}}(G, X) \) is connected, then \( \text{Diam}(\mathcal{P}_{\{4\}}(G, X)) = 2 \).

In (i)(a) of Theorem 4.0.1 we observe that \( \mathcal{P}_{\{4\}}(G, X) \) consists of \( n \) copies of \( \mathcal{P}_{\{4\}}(\text{Sym}(2m), Y) \) where \( Y \) consists of all involutions of cycle type \( 2^m \). This corresponds to the \( n \) possible fixed points of the involutions of \( X \). Cases (i)(b) and (i)(c) result in totally disconnected graphs.

For symmetric groups, the diameters of the connected \( \pi \)-product involution graphs have been determined when \( \pi = \{2\} \) - that is the commuting involution graphs - and \( \pi = \mathbb{N}_{\text{odd}} \) (=the set of all odd natural numbers) - the local fusion graphs. In the former case the diameter is bounded above by 3 except for three small cases when the diameter is 4. Moreover, the diameter can be 3 infinitely often. In the latter case, the connected local fusion graphs for symmetric groups always have diameter 2. So, from this perspective, \( \mathcal{P}_\pi(G, X) \) for \( \pi = \{4\} \) and \( \pi = \mathbb{N}_{\text{odd}} \) are bed fellows. However, this apparent similarity does not extend to the case that \( \pi = \{2^a\} \) for some \( a \geq 3 \). Indeed, we shall derive the following result.

**Theorem 4.0.2.** Suppose that \( G = \text{Sym}(n) \), \( 2m = 2^a \leq n \) for some \( a \geq 3 \), \( t = (1, 2)(3, 4) \cdots (2m - 1, 2m) \) and \( X \) is the \( G \)-conjugacy class of \( t \). Then

(i) \( \mathcal{P}_{\{2^m\}}(G, X) \) is connected if and only if \( n \geq 2m + 2 \); and

(ii) if \( \mathcal{P}_{\{2^m\}}(G, X) \) is connected, then

\[
\min\{m, [n/2 - m]\} \leq \text{Diam}(\mathcal{P}_{\{2^m\}}(G, X)) \leq 2m - 1
\]
(where \([x]\) denotes the smallest integer greater than or equal to \(x\)). Taking \(n = 4m\) in this theorem gives an infinite family of \(\pi\)-product involution graphs whose diameter is unbounded.

Specialising to the case \(m = 4\) (so \(2m = 8\)) we can give precise values for the diameter of \(\mathcal{P}_{[8]}(G, X)\) in our next theorem.

**Theorem 4.0.3.** Suppose \(G = \text{Sym}(n)\), \(t = (1, 2)(3, 4)(5, 6)(7, 8)\) and let \(X\) be the \(G\)-conjugacy class of \(t\). Then

(i) for \(10 \leq n \leq 14\), \(\text{Diam}(\mathcal{P}_{[8]}(G, X)) = 3\); and

(ii) for \(n \geq 15\), \(\text{Diam}(\mathcal{P}_{[8]}(G, X)) = 4\).

An analogous version of Theorem 4.0.2 also holds for any odd prime power.

**Theorem 4.0.4.** Suppose that \(G = \text{Sym}(n)\), \(p\) is an odd prime and \(q = p^a\) for some \(a \geq 1\). Let \(t = (1, 2) \cdots (q - 2, q - 1)\) and \(X\) be the \(G\)-conjugacy class of \(t\). Then

(i) \(\mathcal{P}_{[q]}(G, X)\) is connected if and only if \(n \geq q\); and

(ii) if \(\mathcal{P}_{[q]}(G, X)\) is connected, then

\[
\min\{q - 1, n + 1 - q\} \leq \text{Diam}(\mathcal{P}_{[q]}(G, X)) \leq q - 1.
\]

Our final result combines Theorems 4.0.2 and 4.0.4.

**Theorem 4.0.5.** Suppose that \(G = \text{Sym}(n)\), and \(p_1, \ldots, p_r\) are distinct primes with \(p_i < p_{i+1}\) for \(i = 1, \ldots, r - 1\). Let \(q = p_1^{a_1} \cdots p_r^{a_r}\) for some \(a_1, \ldots, a_r \geq 1\) with \(a_1 \geq 2\) if \(p_1 = 2\) and set

\[
q_i = \begin{cases} 
p_i^{a_i} & \text{if } p_i = 2; \\
p_i^{a_i} - 1 & \text{otherwise},
\end{cases}
\]

and \(2m = q_1 \cdots q_r\). Assuming \(2m \leq n\), let \(t = (1, 2) \cdots (2m - 1, 2m)\) and \(X\) be the \(G\)-conjugacy class of \(t\).

(i) The graph \(\mathcal{P}_{[q]}(G, X)\) is connected if and only if

\[
n \geq \begin{cases} 
q + 2 & \text{if } p_1 = 2; \\
q & \text{otherwise}.
\end{cases}
\]

(ii) If \(\mathcal{P}_{[q]}(G, X)\) is connected, then

\[
\text{Diam}(\mathcal{P}_{[q]}(G, X)) \leq \max_i q_i.
\]

This chapter is arranged as follows. In Section 4.1 we introduce the notion of the \(x\)-graph of an element of \(X\). These are graphs that encapsulate the \(C_G(x)\)-orbits of \(X\) and were first introduced by Bates, Bundy, Perkins and Rowley in [BBPR03b]. We present a number of their results, and relate the connected components of an \(x\)-graph to the disc \(\Delta_1(t)\) for a fixed involution.
t of X. Section 4.2 begins by considering combinations of connected components of x-graphs, and we show that Theorem 4.0.1 holds when restricted to the supports of such components. In particular we consider the case when our conjugacy class consists of elements of full support in Lemma 4.2.8. We then proceed to give a general proof of Theorem 4.0.1 at the end of this section. The chapter concludes in Section 4.3 with an analysis of π-product graphs when π ≠ {4}. We begin by considering the case when π = {2^n} for some a ≥ 3. Calculations of the sizes of discs Δ_i(t) for certain π-product involution graphs are given and these give a direct proof of Theorem 4.0.3. This is followed by constructive proofs of Theorems 4.0.2, 4.0.4 and 4.0.5. Finally, we consider some smaller symmetric groups and calculate the sizes of discs of the π-product graphs P_π(G, X) when π = {6} or {8}.

4.1 Preliminary Results

Throughout this chapter, we set G = Sym(n) and consider G as acting on a set of n letters (or points), Ω = {1, ..., n}. Let t ∈ G be a fixed involution and let X be the G-conjugacy class of t. For an element g ∈ G, we denote the set of fixed points of g on Ω by fix(g) and define the support of g to be supp(g) := Ω \ fix(g). For the sake of brevity, if x_1, x_2, ..., x_r ∈ G we denote supp(x_1) ∪ supp(x_2) ∪ ... ∪ supp(x_r) by supp(x_1, x_2, ..., x_r).

To study the graph P_{(4)}(G, X), we first introduce another type of graph known as an x-graph. Indeed, let x ∈ X. The x-graph corresponding to x, denoted G_x, has vertex set given by the orbits of Ω under ⟨t⟩. Two vertices σ, γ are joined in G_x if there exists σ_0 ∈ σ and γ_0 ∈ γ such that {σ_0, γ_0} is an orbit of Ω under ⟨x⟩. We call the vertices corresponding to transpositions of t white vertices, denoted 0, and those corresponding to fixed points of t white vertices, denoted O. As an example, let n = 15, t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) and x = (1,7)(2,3)(4,10)(8,9)(11,13)(12,14). The x-graph G_x is given by

We note that we could swap the roles of t and x to produce another x-graph which we denote by G_t^x. In general the x-graph G_x^y has vertices the orbits of Ω under ⟨y⟩, and edges defined by the orbits of Ω under ⟨x⟩.

The concepts of x-graphs were first introduced in [BBPR03] as a tool for studying the commuting involution graphs of the symmetric groups. More recently they have also been used in the study of local fusion graphs for the symmetric groups (see [BGR13] for further details). The versatility of x-graphs in calculations arises from the simple observation that each black vertex has valency at most two and each white vertex has valency at most one. Consequently, we may fully determine the possible connected components of a given x-graph.

Lemma 4.1.1. Let x ∈ X. The possible connected components of G_x are

(i)
4.1. PRELIMINARY RESULTS

(ii) \[\xrightarrow\cdots\bullet\xrightarrow\cdots\bullet\xrightarrow\cdots\bullet\cdots\xrightarrow\bullet;\] and

(iii) \[\xrightarrow\bullet\cdots\xrightarrow\bullet;\]

In the subsequent discussion, we will consider $x$-graphs up to isomorphism. It is implicit that such an isomorphism will preserve vertex colours. We also fix $t = (1, 2) \cdots (2m - 1, 2m) \in G$.

Bates, Bundy, Perkins and Rowley’s interest in $x$-graphs stemmed from the following elementary result.

**Lemma 4.1.2.** (i) Every graph with $b$ black vertices of valency at most two, $w$ white vertices of valency at most one and exactly $b$ edges is the $x$-graph for some $x \in X$ (with $m = b$ and $n = 2b + w$).

(ii) Let $x, y \in X$. Then $x$ and $y$ are in the same $C_G(t)$-orbit if and only if $G_x$ and $G_y$ are isomorphic graphs.

**Proof.** See Lemma 2.1 of [BBPR03b].

Part (i) of Lemma 4.1.2 is of particular interest, as it confirms that when employing a combinatorial approach using the connected components of $x$-graphs, we must consider all possible connected components given in Lemma 4.1.1. This approach will be used repeatedly in the proof of Theorem 4.0.1.

An immediate consequence of the definition of $G_x$ is that the number of black vertices is equal to the number of edges. Consequently the number of connected components of the form $\bullet\cdots\bullet\xrightarrow\bullet$ containing at least one black vertex must be equal to the number of connected components of the form $\xrightarrow\bullet\cdots\xrightarrow\bullet$ and $\xrightarrow\bullet\xrightarrow\bullet$.

Lemma 4.1.1 allows a combinatorial approach to be used when considering conjugate involutions. Indeed, given a connected component $C_i$ of $G_x$, we may define $\Omega_i$ to be the union of all vertices of $C_i$. We may then define the $i$-part of $t$, denoted $t_i$, to be the product of those transpositions of $t$ that occur in $\text{Sym}(\Omega_i)$. We define $x_i$ similarly. By analysing the structure of the connected components given in Lemma 4.1.1 it is possible to relate the order of $tx$ to the $x$-graph $G_x$.

**Lemma 4.1.3.** Suppose that $x \in X$ and that $C_1, \ldots, C_k$ are the connected components of $G_x$. Denote the number of black vertices, white vertices and cycles in $C_i$ by $b_i$, $w_i$ and $c_i$ respectively. Then

(i) the order of $tx$ is the least common multiple of the orders of $t_i x_i$ (for $i = 1, \ldots, k$); and

(ii) the order of $t_i x_i$ is $(2b_i + w_i)/(1 + c_i)$ for each $i = 1, \ldots, k$.

**Proof.** See Proposition 2.2 of [BBPR03b].

We have the following immediate corollary to Lemmas 4.1.1 and 4.1.3.
Corollary 4.1.4. For \( P_{\{4\}}(G, X) \) the disc \( \Delta_1(t) \) consists of all \( x \in X \) whose \( x \)-graphs have at least one connected component of the form \( \bullet - \bullet \), \( \circ - \circ \) or \( \bullet - \bullet - \bullet - \bullet - \bullet \) and all other components have the form \( \circ - \circ - \circ - \circ - \circ - \circ - \circ \) or \( \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet \).

Proof. The element \( x \) lies in \( \Delta_1(t) \) precisely when \( tx \) has order 4. The result then follows from Lemmas 4.1.1 and 4.1.3.

We conclude this section by noting that we can define an \( x \)-graph for any two (not-necessarily conjugate) involutions. This we will do frequently in Section 4.2. However, in such a situation it is no longer the case that the number of edges of \( G_x \) is equal to the number of black vertices.

4.2 Proof of Theorem 4.0.1

In this section, we prove Theorem 4.0.1. Note that for \( m \geq 2 \) and \( t = (1, 2) \cdots (2m - 1, 2m) \), the involution \( x = (1, 3)(2, 4)(5, 6) \cdots (2m - 1, 2m) \in X \) satisfies \( d(t, x) \geq 2 \). Thus it suffices to prove when \( P_{\{4\}}(G, X) \) is connected, that for all \( x \in X \) we have \( d(t, x) \leq 2 \). To do this we consider pairs or triples of connected components \( C_i, C_j \) and \( C_k \) of \( G_x \) and the corresponding parts \( t_i, t_j, t_k, x_i, x_j, x_k \) of \( t \) and \( x \). We then construct an element \( y_{ijk} \in H \), where \( H := \text{Sym}(\text{supp}(t_i, t_j, t_k, x_i, x_j, x_k)) \), which is \( H \)-conjugate to \( t_it_jt_k \) and such that the \( x \)-graphs \( G_{y_{ijk}}^{t_it_jt_k} \) and \( G_{x_i,x_j,x_k}^{y_{ijk}} \) have connected components featuring in Corollary 4.1.4.

We begin by proving a few preliminary results, dealing with the case \( n = 2m \).

Lemma 4.2.1. Let \( m \geq 5 \), \( n = 2m \) and suppose that \( x \in X \) is such that \( G_x \) is connected. Then there exists \( y \in X \) such that \( d(t, y) = d(y, x) = 1 \).

Proof. Without loss of generality we may assume that \( x = (1, 2m)(2, 3) \cdots (2m - 2, 2m - 1) \). If \( m = 5 \), then taking \( y = (1, 10)(2, 6)(3, 4)(5, 8)(7, 9) \) we see that \( G_y \) and \( G_x^y \) are given respectively by

\[
\begin{align*}
\{5, 6\} &
\{1, 2\} \quad \{9, 10\} \quad \{7, 8\} \quad \{3, 4\}
\end{align*}
\]

and

\[
\begin{align*}
\{7, 9\} &
\{2, 6\} \quad \{3, 4\} \quad \{5, 8\} \quad \{1, 10\}
\end{align*}
\]

If \( m = 6 \), we take \( y = (1, 3)(2, 4)(5, 7)(6, 12)(8, 9)(10, 11) \). Then \( G_y \) and \( G_x^y \) are, respectively

\[
\begin{align*}
\{11, 12\} &
\{1, 2\} \quad \{3, 4\} \quad \{5, 6\} \quad \{7, 8\} \quad \{9, 10\}
\end{align*}
\]
In the general case when \( m \geq 7 \), we take

\[
y = (1, 3)(2, 4)(5, 7)(6, 2m)(8, 9)(10, 2m - 1)(11, 2m - 2) \cdots (m + 4, m + 5).
\]

The exact nature of the associated \( x \)-graphs is dependent on the parity of \( m \). If \( m \) is even, then \( G_y \) is given by

![Diagram](image1)

and \( G^y_x \) is given by

![Diagram](image2)

If \( m \) is odd, the graphs \( G_y \) and \( G^y_x \) are, respectively

![Diagram](image3)

and

![Diagram](image4)

In all cases, the given graphs satisfy the conditions of Corollary 4.1.4, whence \( d(t, y) = d(y, x) = 1 \).

The proof of Lemma 4.2.1 illustrates a general feature that the actual \( x \)-graphs constructed may vary depending on the parity and values of the given parameters (such as the parameter \( m \) above). However, in using Corollary 4.1.4 we are only interested in the connected components of the \( x \)-graph. Thus for the sake of brevity, in all future proofs we will only describe the connected components of each \( x \)-graph, relegating the \( x \)-graphs themselves to Appendix B.

**Lemma 4.2.2.** Suppose that \( m = 3, n = 6 \) and \( x \in X \). If \( G_x \) is connected, then there exists \( y \in X \) such that the \( x \)-graphs \( G_y \) and \( G^y_x \) are isomorphic to

![Graph](image5)
Proof. Without loss of generality, we may assume that \( x = (1,6)(2,3)(4,5) \). Then \( y = (1,2)(3,6)(4,5) \) is the required element.

**Lemma 4.2.3.** Let \( m = 4 \text{ and } n = 8 \). Suppose that \( x \in X\setminus\{t\} \) has a disconnected \( x \)-graph, \( G_x \). Then there exists \( y \in X \) such that \( d(t,y) = d(y,x) = 1 \).

Proof. If \( G_x \) has a connected component of the form \( \bullet\bullet \), then we may assume that \( x = (1,6)(2,3)(4,5)(7,8) \). The element \( y = (1,8)(2,4)(3,6)(5,7) \) is then the desired \( y \). The other possibilities occur when \( G_x \) has one or two connected components of the form \( \bullet\bullet \), corresponding respectively to \( x = (1,3)(2,4)(5,6)(7,8) \) and \( x = (1,3)(2,4)(5,7)(6,8) \). The \( y \) satisfying the lemma for both such \( x \) is \( y = (1,8)(2,3)(4,5)(6,7) \).

**Lemma 4.2.4.** Suppose that \( m \geq 5 \), \( n = 2m \) and that \( G_x \) consists entirely of components of the form \( \bullet\bullet \) and \( \bullet\bullet\bullet \). Then there exists \( y \in X \) such that \( d(t,y) = d(y,x) = 1 \).

Proof. We consider three separate cases. First assume that \( G_x \) contains at least two components, \( C_i \) and \( C_j \), of the form \( \bullet\bullet \). Without loss, we may take \( t_i = (1,2)(3,4)(5,6), t_j = (7,8)(9,10)(11,12), x_i = (1,6)(2,3)(4,5) \) and \( x_j = (7,12)(8,9)(10,11) \). Defining \( y_{ij} \in \text{Sym}(\text{supp}(t_i, t_j, x_i, x_j)) \) to be \( y_{ij} = (1,2)(3,4)(5,7)(6,12)(8,9)(10,11) \), we see that both \( G_{y_{ij}}^{t_{ij}} \) and \( G_{x_i,x_j}^{y_{ij}} \) are isomorphic to

Denote the remaining parts of \( t \) and \( x \) by \( t_k \) and \( x_k \). Applying Lemmas 4.2.2 and 4.2.3 to \( t_k \) and \( x_k \) produces an element \( y_k \in \text{Sym}(\text{supp}(t_k, x_k)) \) such that \( y := y_{ij}y_k \) is the desired element of \( X \).

In the case that \( G_x \) contains a unique component, \( C_i \), of the form \( \bullet\bullet \), then there exists at least one component, say \( C_j \), of the form \( \bullet\bullet\bullet \). Taking \( t_i = (1,2)(3,4)(5,6), t_j = (7,8)(9,10), x_i = (1,6)(2,3)(4,5) \) and \( x_j = (7,9)(8,10) \), then the element \( y_{ij} \in \text{Sym}(\text{supp}(t_i, t_j, x_i, x_j)) \) given by \( y_{ij} = (1,2)(3,10)(4,6)(5,8)(7,9) \) results in \( x \)-graphs \( G_{y_{ij}}^{t_{ij}} \) and \( G_{x_i,x_j}^{y_{ij}} \) which are isomorphic to

Denoting the remaining part of \( t \) by \( t_k \) and setting \( y := y_{ij}t_k \in X \) we have that \( d(t,y) = d(y,x) = 1 \) as required.

Finally, assume that all connected components of \( G_x \) are of the form \( \bullet\bullet\bullet \) and let \( C_i \) and \( C_j \) be two such components. Without loss of generality we have that \( t_i = (1,2)(3,4), t_j = (5,6)(7,8), x_i = (1,3)(2,4) \) and \( x_j = (5,7)(6,8) \), and as above denote the remaining part of \( t \) by \( t_k \). Setting \( y_{ij} = (1,5)(2,7)(3,8)(4,6) \), we see that both \( G_{y_{ij}}^{t_{ij}} \) and \( G_{x_i,x_j}^{y_{ij}} \) consist of a single connected component of isomorphism type \( \bullet\bullet\bullet \). Hence, \( y := y_{ij}t_k \) is our desired element of \( X \).
4.2. PROOF OF THEOREM 4.0.1

Lemmas 4.2.1, 4.2.2, 4.2.3, and 4.2.4 combine to prove Theorem 4.0.1 in the case when \( n = 2m \).

**Corollary 4.2.5.** If \( n = 2m \), then Theorem 4.0.1 holds.

**Proof.** Let \( x \in X \). If \( G_x \) has connected components containing precisely 4 black vertices then we leave the parts of \( t \) and \( x \) corresponding to such components alone. We then apply Lemma 4.2.1 to any connected component containing at least 5 black vertices, and Lemma 4.2.2 to any connected component containing 3 black vertices to obtain the desired result. Otherwise all connected components have at most 3 black vertices. Applying Lemmas 4.2.3 and 4.2.4 to a pair of components containing a total of 4, 5 or 6 black vertices, Lemma 4.2.2 to any remaining connected components containing 3 black vertices, and leaving all other connected components invariant gives the result.

Before presenting the proof of Theorem 4.0.1 we give a further three intermediate results.

**Lemma 4.2.6.** Let \( x \in X \). Suppose that \( G_x \) has connected components \( C_i \) and \( C_j \) of the given forms. Then there exists \( y_{ij} \in H \), where \( H := \text{Sym}(\text{supp}(t_i, t_j, x_i, x_j)) \), which is \( H \)-conjugate to \( t_it_j \) and such that the connected components of the \( x \)-graphs \( G_{y_{ij}}^{t_it_j} \) and \( G_{x_itx_j}^{t_itj} \) satisfy the conditions of Corollary 4.1.4.

(i) \( C_i : \bullet \cdots \bullet \) (with \( q \geq 3 \) black vertices), \( C_j : \bigcirc \cdots \bigcirc \) (with \( r \geq 0 \) black vertices);

(ii) \( C_i : \bigcirc \cdots \bigcirc \) (with \( q \geq 2 \) black vertices), \( C_j : \emptyset \);

(iii) \( C_i : \bullet \cdots \bullet \) (with \( q \geq 2 \) black vertices), \( C_j : \emptyset \);

(iv) \( C_i : \emptyset \cdots \emptyset \) (with \( q \geq 1 \) black vertices),

\( C_j : \bigcirc \cdots \bigcirc \) (with \( r \geq 1 \) black vertices); and

(v) \( C_i \) and \( C_j \) are both of the form \( \bullet \cdots \bullet \bigcirc \) (with \( q, r \geq 1 \) black vertices respectively).

**Proof.** For each case, without loss of generality we give explicit formulations of the \( t_i \) and \( x_i \). For ease of notation, where parameters \( q \) and \( r \) have been defined we set \( v = 2(q + r) \).

In case (i) assume that \( t_i = (1, 2) \cdots (2q - 1, 2q), t_j = (2q + 1, 2q + 2) \cdots (v - 1, v)(v + 1)(v + 2), \) and \( x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q) \). We now consider three possibilities. If \( r = 0 \), then we may assume that \( x_j = (2q + 1, 2q + 2) \) and we take

\[
y_{ij} := (1, 2q)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q + 1, 2q + 2).
\]

If \( r = 1 \), then taking \( x_j = (2q + 1, v + 1)(2q + 2, v + 2) \) we define

\[
y_{ij} := (1, 2q)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(v + 1, v + 2)(2q + 1, 2q + 2).
\]

Finally, if \( r > 1 \), then we assume that \( x_j = (2q + 1, v + 1)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, v + 2) \) and define

\[
y_{ij} := (1, 2q)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(v + 1, v + 2)
(2q + 1, v) \cdots (2q + r, 2q + r + 1).
\]
We see that the $x$-graph $G_{y_{i,j}}^{t_{i,j}}$ has connected components of the form \( \bullet - \bullet \), \( \odot - \odot \) and - depending on the values of $q$ and $r$ - also \( \bullet - \bullet \) and \( \bullet \). Similarly $G_{x_{i,j}}^{y_{i,j}}$ has connected components of the form \( \odot - \odot - \odot \), \( \bullet - \bullet \) and in some cases also \( \bullet - \bullet \) and \( \bullet \) as required (see Figures B.1-B.8).

For (ii) we may set $t_i = (1,2) \cdots (2q-1,2q)(2q+1)(2q+2)$ and $x_i = (1,2q+2)(2,3) \cdots (2q,2q+1)$. Then the element

$$y_i = (1)(2,2q-1)(3,2q-2) \cdots (q,q+1)(2q+1,2q+2)$$

results in the $x$-graph $G_{y_{i,j}}^{t_{i}}$ having connected components \( \odot - \odot - \odot \), \( \bullet - \bullet - \bullet \) and in some cases \( \bullet - \bullet \) and \( \bullet \) - depending on the value and parity of $q$. The graph $G_{y_{i,j}}^{x_{i,j}}$ has connected components \( \odot - \odot \) and possibly \( \bullet - \bullet \) and \( \bullet \) (see Figures B.9 and B.10).

Considering case [iii] we take $t_i = (1,2) \cdots (2q-1,2q)(2q+1)$, $x_i = (1,2,3) \cdots (2q,2q+1)$ and $t_j = x_j = (2q+2)$. If $q = 2$, define

$$y_{ij} = (1,4)(2)(3)(5,6),$$

whilst if $q \geq 3$ define

$$y_{ij} = (1,2q)(2)(2q-1)(3,2q-2) \cdots (q,q+1)(2q+1,2q+2).$$

Then the permutation $y_{ij}$ gives the desired $x$-graphs. Indeed, $G_{y_{i,j}}^{t_{i,j}}$ has connected components of the form \( \odot - \odot \) and \( \bullet - \bullet - \bullet \), $G_{x_{i,j}}^{y_{i,j}}$ has components of the form \( \bullet - \bullet - \bullet \) and \( \odot - \odot - \odot \) (with the black vertex omitted if $q = 2$) and both $x$-graphs may also have connected components of the form \( \bullet - \bullet - \bullet \) and \( \bullet \) depending on the value and parity of $q$ (see Figures B.11-B.13).

Turning to [iv] if $q = 1$, then without loss of generality we have that $t_i = (1,2)(v+1)$, $t_j = (3,4) \cdots (v-1,v)(v+2)(v+3)$, $x_i = (1,2,v+1)$ and $x_j = (3,v+2)(4,5) \cdots (v-2,v-1)(v,v+3)$ (take $x_j = (3,6)(4,7)$ if $r = 1$). When $r = 1$, define

$$y_{ij} = (1)(2)(3,5)(4,7)(6).$$

The $x$-graphs $G_{y_{i,j}}^{t_{i,j}}$ and $G_{x_{i,j}}^{y_{i,j}}$ are isomorphic to

\[ \bullet - \bullet - \bullet \quad \odot - \odot \] and \[ \bullet - \bullet - \bullet \quad \odot - \odot \]

respectively as required. If $r > 1$, then

$$y_{ij} = (1)(2)(3,v-2) \cdots (r+1,r+2)(v-1,v+2)(v,v+1)(v+3)$$

is our desired element. Indeed in this case $G_{y_{i,j}}^{t_{i,j}}$ has connected components of the form \( \odot \), \( \bullet \) and \( \odot - \odot - \odot \) in addition to components of the form \( \bullet \) and/or \( \bullet - \bullet - \bullet \) (depending on the value of
r), whilst \( G_{x,2}^{y_i} \) has components of the forms \( \bigcirc \) and \( \bigcirc \longrightarrow \bigcirc \) in addition to components of the form \( \bullet \) and/or \( \bullet \longrightarrow \bullet \) (depending on the value of \( r \) - see Figures B.14, B.16 for full details).

If \( q > 1 \), then we define \( t_i = (1, 2) \cdots (2q - 1, 2q)(v + 1) \), \( t_j = (2q + 1, 2q + 2) \cdots (v - 1, v)(v + 2)(v + 3) \), \( x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q, v + 1) \) and \( x_j = (2q + 1, v + 2)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, v + 3) \). Our desired element is then

\[
y_{ij} = (1)(2, 2q - 1) \cdots (q, q + 1)(2q + 1, v - 2) \cdots \cdots (2q + r - 1, 2q + r)(v - 1, v + 2)(v, v + 1)(v + 3).
\]

It follows that the connected components of \( G_{v_i,j}^{t_i} \) and \( G_{x,x_j}^{y_i} \) have the form \( \bigcirc \longrightarrow \bigcirc \) and \( \bigcirc \) and possibly \( \bigcirc \longrightarrow \bigcirc \), whilst \( G_{v_i,j}^{t_i} \) has an additional connected component of the form \( \bullet \longrightarrow \bullet \) (see Figures B.17, B.20).

For case \( v \) we assume without loss of generality that \( q \geq r \). We consider the subcases \( q = r = 1 \), \( q > r = 1 \), \( q = r > 1 \) and \( q > r > 1 \) in turn. If \( q = r = 1 \), then we take \( t_i = (1, 2)(5), t_j = (3, 4)(6) \), \( x_i = (1)(2, 5) \) and \( x_j = (3)(4, 6) \). Defining

\[
y_{ij} = (1, 3)(5, 6)(2)(4),
\]

we see that \( G_{v_i,j}^{t_i} \) has isomorphism type \( \bullet \longrightarrow \bigcirc \bigcirc \) and \( G_{x,x_j}^{y_i} \) has isomorphism type \( \bigcirc \longrightarrow \bigcirc \bullet \) as required (see Figure B.21).

If \( q > r = 1 \), then setting \( t_i = (1, 2) \cdots (2q - 1, 2q)(v + 1) \), \( t_j = (2q + 1, 2q + 2)(v + 2) \), \( x_i = (1)(2, 3) \cdots (2q, v + 1) \) and \( x_j = (2q + 1)(2q + 2, v + 2) \) we define

\[
y_{ij} = (1, 2(q - 1))(2, 2(q - 1) - 1) \cdots (q - 1, q)(2q - 1, 2q + 1)
(2q)(2q + 2)(v + 1, v + 2).
\]

Consequently \( G_{v_i,j}^{t_i} \) has connected components of the form \( \bullet \longrightarrow \bigcirc \bigcirc \) and \( \bigcirc \longrightarrow \bullet \bullet \) and/or \( \bullet \) (depending on the value of \( q \)), whilst \( G_{x,x_j}^{y_i} \) has components of the form \( \bigcirc \longrightarrow \bigcirc \bullet \bullet \) and \( \bullet \longrightarrow \bullet \) and/or \( \bullet \) (see Figures B.22 and B.23).

When \( r > 1 \) we may assume that \( t_i = (1, 2) \cdots (2q - 1, 2q)(v + 1) \), \( t_j = (2q + 1, 2q + 2) \cdots (v - 1, v)(v + 2) \), \( x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q, v + 1) \) and \( x_j = (2q + 1)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, v + 2) \). Define

\[
y_{ij} = (1, 2q + 1)(2, 2q + 2) \cdots (2q - 1, v - 1)(2q)(v)(v + 1, v + 2)
\]

if \( q = r \) and

\[
y_{ij} = (1, 2(q - r))(2, 2(q - r) - 1) \cdots (q - r, q - r + 1)
(2q - r + 1, 2q + 1) \cdots (2q - 1, v - 1)(2q)(v)(v + 1, v + 2)
\]

if \( q \neq r \). We see that \( G_{v_i,j}^{t_i} \) has connected components \( \bigcirc \longrightarrow \bigcirc \), \( \bullet \longrightarrow \bullet \) and \( \bullet \longrightarrow \bullet \) in addition to \( \bullet \) for some values of \( q \) and \( r \). The \( x \)-graph \( G_{x,x_j}^{y_i} \) also has the desired properties having a component of
We also set \( x \) to satisfy the conditions of Corollary 4.1.4:

**Proof.** We follow the approach of the proof of Lemma 4.2.6 and construct the appropriate \( H \) associated with \( C \) and/or \( \sum \) depending on the values of \( q \) and \( r \) and the parity of \( q - r \) (see Figures B.24-B.26).

We note that in cases \([ii]\) and \([iv]\) above, \( t_it_j \) and \( x_ix_j \) have different cycle types. This is a fact which we will utilise in the proof of Theorem 4.0.1.

In a similar vein to Lemma 4.2.6 we next consider collections of three connected components simultaneously.

**Lemma 4.2.7.** Let \( x \in X \). Suppose that \( G_x \) has connected components \( C_i, C_j \) and \( C_k \) of the given forms and define \( H := \text{Sym}(\text{supp}(t_i,t_j,t_k,x_i,x_j,x_k)) \). Then there exists \( y_{ijk} \in H \) which is \( H \)-conjugate to \( t_it_jt_k \) and such that the connected components of the \( x \)-graphs \( G_{y_{ijk}}^{t_it_jt_k} \) and \( G_{x_i,x_j,x_k}^{y_{ijk}} \) satisfy the conditions of Corollary 4.1.4.

(i) \( C_i, C_j, C_k \) are each of the form \( \circ - \bullet - \bullet - \bullet - \circ \) (having \( q,r,s \geq 1 \) black vertices respectively); and

(ii) \( C_i : \circ - \bullet - \bullet - \bullet - \bullet - \circ \) (with \( q \geq 1 \) black vertices),

\( C_j : \bullet - \bullet - \bullet - \bullet - \bullet - \circ \) (with \( r \geq 1 \) black vertices), \( C_k : \bullet - \bullet - \bullet - \circ \).

**Proof.** We follow the approach of the proof of Lemma 4.2.6 and construct the appropriate \( t_i \) and \( x_i \). We also set \( v = 2(q + r) \) and \( w = 2(q + r + s) \).

For case \([i]\) we may assume without loss of generality that \( q \geq r \geq s \geq 1 \) and set

\[
  t_i = (1, 2) \cdots (2q - 1, 2q)(w + 1), \quad t_j = (2q + 1, 2q + 2) \cdots (v - 1, v)(w + 2) \quad \text{and} \quad t_k = (v + 1, v + 2) \cdots (w - 1, w)(w + 3).
\]

We also set

\[
  x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q, w + 1),
\]

\[
  x_j = (2q + 1)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, w + 2), \quad \text{and}
\]

\[
  x_k = (v + 1)(v + 2, v + 3) \cdots (w - 2, w - 1)(w, w + 3),
\]

taking \( x_i = (1)(2, w + 1) \) in the case when \( q = 1, x_j = (2q + 1)(2q + 2, w + 2) \) when \( r = 1 \) and \( x_k = (v + 1)(v + 2, w + 3) \) when \( s = 1 \). There are three subcases to consider. If \( q = r = s = 1 \), then taking \( y_{ijk} = (1, 4)(5, 8)(6, 7)(2)(3)(9) \) we see that the \( x \)-graphs \( G_{y_{ijk}}^{t_it_jt_k} \) and \( G_{x_i,x_j,x_k}^{y_{ijk}} \) are both isomorphic to

\[
  \circ - \bullet - \bullet - \bullet - \circ
\]

which has the desired form (see Figure B.27). If \( s = 1 \), but \( q \neq 1 \), we set

\[
  y_{ijk} = (1, w + 2)(2, w + 1)(3, 2q)(4, 2q - 1) \cdots (q + 1, q + 2) \quad (2q + 1, v) \cdots (2q + r, 2q + r + 1)(v + 1)(w)(w + 3),
\]
whilst if $s > 1$, we define

$$y_{ijk} = (1, w + 2)(2, w + 1)(3, 2q)(4, 2q - 1) \cdots (q + 1, q + 2)
(2q + 1, v) \cdots (2q + r, 2q + r + 1)
(v + 2, w - 1) \cdots (v + s, v + s + 1)(v + 1)(w + 3).$$

It follows that the $x$-graph $G_{y_{ijk}}^{x_{t_{ij}t_k}}$ has connected components of the form $\circ\circ\circ\circ$ and $\circ$ with further components of the form $\bullet\bullet\bullet\bullet$ (if $s = 1$), $\bullet\bullet\bullet\bullet$ (if $s > 1$) and $\bullet\bullet\bullet\bullet$ and/or $\bullet$ depending on the values of $q$, $r$ and $s$. Meanwhile, $G_{x_{t_{ij}t_k}}^{y_{ijk}}$ has connected components of the form $\bullet\bullet\circ\circ\circ\circ$ and $\circ$, in addition to $\bullet\bullet\bullet\bullet$ and/or $\bullet$ depending on the parameters $q$, $r$, $s$ (see Figures B.38 B.39 for full details).

Finally, we consider case (ii) and note that in this case $w = v + 2$. Assume that $t_i = (1, 2) \cdots (2q - 1, 2q)$, $t_j = (2q + 1, 2q + 2) \cdots (v + 1, v + 1)$, $x_i = (1, 2, 3) \cdots (2q - 2, 2q - 1)(2q)$, $x_j = (2q + 1, 2q + 2, 2q + 3) \cdots (v, v + 1)$ and $x_k = (v + 2, v + 3)$. If $q = r = 1$, then $x = (1, 2, 3)(4, 5)(6, 7)$, and defining

$$y_{ijk} = (1, 3)(2, 4)(5, 6, 7)$$

we see that the $x$-graphs $G_{y_{ijk}}^{x_{t_{ij}t_k}}$ and $G_{x_{t_{ij}t_k}}^{y_{ijk}}$ are of isomorphism type

$$\bullet\bullet\bullet\bullet\circ\circ\circ\circ\bullet\bullet$$

respectively (see Figure B.40). Meanwhile, if $q = 1$ and $r > 1$, then $x = (1, 2, 3)(4, 5, 6, 7)$, and so setting

$$y_{ijk} = (1, 2, v - 3) \cdots (r, r + 1)(v - 1, v + 2)(v + 1)$$

results in the $x$-graph $G_{y_{ijk}}^{x_{t_{ij}t_k}}$ having connected components $\circ\circ\circ\circ\bullet\bullet\bullet\bullet\circ$ and possibly $\bullet\bullet\circ\circ\circ\circ$ and $\bullet$. Moreover, $G_{x_{t_{ij}t_k}}^{y_{ijk}}$ has connected components $\circ\circ\circ\circ\bullet\bullet\bullet\bullet\circ$ and for some values of $r$ also $\bullet\bullet\circ\circ\circ\circ\bullet\bullet$ and/or $\bullet\bullet\bullet\bullet\bullet\bullet$ (see Figures B.41 and B.42).

If $q = 2$, then we take

$$y_{ijk} = (1, v + 1)(2, v + 2)(3, 4)(2q + 1, v) \cdots (2q + r, 2q + r + 1)(v + 3),$$

whilst if $q > 2$ we take

$$y_{ijk} = (1, v + 1)(2, v + 2)(3, 4)(2q + 1, v) \cdots (2q + r, 2q + r + 1)(v + 3).$$

Consequently, the $x$-graph $G_{y_{ijk}}^{x_{t_{ij}t_k}}$ has connected components $\circ\circ\bullet\bullet\circ$ and $\circ$ and possibly also $\bullet\bullet\bullet\bullet\bullet\bullet$ and $\bullet\bullet\bullet\bullet\bullet\bullet$. Meanwhile, $G_{x_{t_{ij}t_k}}^{y_{ijk}}$ has connected components $\circ\circ\bullet\bullet\bullet\bullet\bullet\bullet\circ$ and $\bullet\bullet\bullet\bullet\bullet\bullet$ and in some cases also $\bullet\bullet\bullet\bullet\bullet\bullet$ and/or $\bullet\bullet\bullet\bullet\bullet\bullet$ (see Figures B.43 B.48 for full details).
**Lemma 4.2.8.** Suppose that $m \geq 2$, $n \geq 7$ with $n \neq 2m + 1$ and let $x \in X \setminus \{t\}$ be such that $\text{fix}(t) = \text{fix}(x)$. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.

**Proof.** By considering $t, x \in \text{Sym}(\text{supp}(t))$ and appealing to Corollary 4.2.5 we may assume that $m = 2$ or 3 and hence that $|\text{fix}(t)| \geq 2$. If $G_x$ contains a connected component of the form $\bullet\cdots\bullet$, then without loss of generality we have

$$t = (1, 2)(3, 4)(5, 6)(7) \cdots (n) \quad \text{and} \quad x = (1, 3)(2, 4)(5, 6)(7) \cdots (n)$$

(where the transposition $(5, 6)$ is replaced by $(5)(6)$ if $m = 2$). We take $y = (1, 4)(2, 3)(5, 6)(7) \cdots (n - 2)(n - 1, n)$ (again replacing $(5, 6)$ by $(5)(6)$ if $m = 2$). If $m = 3$ and $G_x$ contains a cycle of three black vertices, then we may assume that

$$t = (1, 2)(3, 4)(5, 6)(7) \cdots (n) \quad \text{and} \quad x = (1, 6)(2, 3)(4, 5)(7) \cdots (n).$$

In this case, we set $y = (1)(2, 3)(4)(5, 6)(7, 8)(9) \cdots (n)$. In all cases we have that $G_y$ has one connected component of the form $\bullet\cdots\bullet$, $G_x$ has one connected component of the form $\bigcirc\cdots\bigcirc$ and all other connected components of these $x$-graphs are of the form $\bigcirc$, $\bigcirc\bigcirc$ and $\bullet$ (see Figures B.49-B.51). Thus $d(t, y) = d(y, x) = 1$ by Corollary 4.1.4.

We are now in a position to prove Theorem 4.0.1 in the general case. For $x \in X$ we proceed by considering collections of connected components $\{C_i\}_{i \in I}$ of $G_x$ for some set $I$, and then finding an element $y_I \in \text{Sym}(\cup_{i \in I} \text{supp}(C_i))$ that is conjugate to $t_I := \prod_{i \in I} t_i$ such that the connected components of $G_{y_I}^I$ and $G_{x_I}^I$ satisfy the conditions of Corollary 4.1.4 (where $x_I := \prod_{i \in I} x_i$). The product of all such $y_I$ will then be our desired element of $X$.

**Proof of Theorem 4.0.1.**

Let $x \in X$.

(i) First assume that $n = 2m + 1$. We observe that the product of two elements of $X$ that fix distinct elements of $\Omega$ cannot have order 4. Thus $\mathcal{P}_{[4]}(G, X)$ consists of $n$ copies of the $\{4\}$-product involution graph $\mathcal{P}_{[4]}(\text{Sym}(2m), Y)$, where $Y$ is the conjugacy class of $\text{Sym}(2m)$ consisting of elements of cycle type $2^m$.

If $m = 1$, then $\mathcal{P}_{[4]}(G, X)$ is clearly totally disconnected.

In the case that $(n, m) = (4, 2)$ (respectively $(n, m) = 6, 3$)), then any $x$-graph will contain 2 (respectively 3) black vertices and 2 (respectively 3) edges. It follows from Corollary 4.1.4 that $\mathcal{P}_{[4]}(G, X)$ is totally disconnected.

(ii) Assume that $m \neq 1$ and $(n, m) \neq (4, 2), (6, 3)$ or $(2m + 1, m)$. We first consider the case that $(n, m) = (6, 2)$. If distinct involutions $t = (1, 2)(3, 4)$ and $x$ of cycle type $2^2$ do not have product of order 4, then the reader may check that the $x$-graph $G_x$ will be isomorphic to one given in Table 4.1 and that the given element $y$ satisfies $d(t, y) = d(y, x) = 1$. Hence $\mathcal{P}_{[4]}(G, X)$ is connected and $\text{Diam}(\mathcal{P}_{[4]}(G, X)) = 2$.

By Corollary 4.2.5 as $n \neq 2m + 1$, we may assume that $|\text{fix}(t)| \geq 2$, and so $G_x$ contains at least 2 white vertices. Moreover, by Lemma 4.2.8 we only need to consider the case when $\text{fix}(t) \neq \text{fix}(x)$.
Let α, β, γ and δ denote the number of connected components (containing at least 1 black vertex and 1 edge) of \(G_x\) of the form

\[
\alpha: \bullet \bullet \bullet \bullet \bullet \\
\beta: \circ \bullet \bullet \bullet \bullet \\
\gamma: \bullet \bullet \bullet \bullet \circ \\
\delta: \bullet \bullet \bullet \bullet \bullet
\]

and let \(\epsilon\) denote the number of connected components of the form \(\circ \circ \circ \circ\). For ease of reading, we shall refer to components of type \(\alpha\) instead of components of the form \(\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \). Similarly for \(\beta, \gamma, \delta\) and \(\epsilon\). Note that \(\alpha \leq \beta + \epsilon\), and as \(\text{fix}(t) \neq \text{fix}(x)\) it follows that \(\beta, \gamma\) and \(\epsilon\) are not all zero.

If \(\gamma \geq 2\), then partitioning the components of type \(\gamma\) into pairs or triples we obtain a suitable \(y_I\) from Lemmas 4.2.1(iv), 4.2.2 and 4.2.7(ii). Indexing the remaining connected components by \(J\), a suitable \(y_J\) such that \(t_{j_1}y_{j_1}\) and \(x_{j_1}y_{j_1}\) have orders 1, 2 or 4 may be constructed using Lemmas 4.2.1(i), 4.2.2 and 4.2.7(ii). In the forthcoming cases, when referring to the construction of \(y_J\), it will be implicit that \(t_{j_1}y_{j_1}\) and \(x_{j_1}y_{j_1}\) have orders 1, 2 or 4.

If \(\gamma = 1\) and \(\beta \neq 0\), then we pair the unique component of type \(\gamma\) with one of type \(\beta\) to obtain an element \(y_I\) via Lemma 4.2.6(iv). In the case that \(\alpha = \beta + \epsilon\), then consider one component of type \(\alpha\). Without loss we may assume that this corresponds to some \(t_i = (1,2) \cdots (2r-1,2r)\) and some \(x_i = (1)(2,3) \cdots (2r-2,2r-1)(2r)\). Setting \(y_i = (1,2)(3,2r)(4,2r-1)(5,2r-2) \cdots (r+1, r+2)\) we have that \(\text{ord}(t_i y_i), \text{ord}(y_i x_i) \in \{1, 2, 4\}\). An element \(y_J\) for the remaining components follows from Lemmas 4.2.1(i), 4.2.2 and 4.2.7(ii).

If \(\gamma = 1\), \(\beta = 0\) and \(\alpha \geq 1\), then \(\epsilon \geq 1\). Hence we may use Lemma 4.2.7(ii) to construct the element \(y_I\) and Lemmas 4.2.1(i), 4.2.2 and 4.2.7(ii) to obtain a suitable \(y_J\).

If \(\gamma = 1\) and \(\alpha = \beta = 0\), then there must be a connected component of \(G_x\) consisting of a single vertex. Assume first that the connected component of type \(\gamma\) contains at least two black vertices. If there is an isolated white vertex in \(G_x\), then the existence of \(y_I\) follows from Lemma 4.2.6(iii). Conversely, if there is an isolated black vertex, then - as the number of black vertices equals the number of edges - there must be a connected component of type \(\epsilon\). Applying Lemma 4.2.7(ii) to

<table>
<thead>
<tr>
<th>Representative (x \in X \backslash (\Delta_1(t) \cup {t}))</th>
<th>(x)-graph, (G_x)</th>
<th>Representative (y \in \Delta_1(t) \cap \Delta_1(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,2)(5,6))</td>
<td>(\bullet \bullet \circ \circ)</td>
<td>((3,6)(4,5))</td>
</tr>
<tr>
<td>((2,5)(4,6))</td>
<td>(\bullet \circ \bullet \circ)</td>
<td>((1,3)(5,6))</td>
</tr>
<tr>
<td>((2,3)(4,5))</td>
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<td>((1,4)(5,6))</td>
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<tr>
<td>((2,5)(3,4))</td>
<td>(\bullet \circ \bullet \circ)</td>
<td>((1,5)(2,6))</td>
</tr>
<tr>
<td>((1,3)(2,4))</td>
<td>(\bullet \bullet \circ \circ)</td>
<td>((1,4)(5,6))</td>
</tr>
</tbody>
</table>

Table 4.1: Representatives of \(X \backslash (\Delta_1(t) \cup \{t\})\) and their corresponding neighbours in \(\Delta_1(t) \cap \Delta_1(x)\) for \(n = 6, m = 2\).
this connected component, the connected component of type \( \gamma \) and an isolated black vertex results in our element \( y_I \). Applying Lemmas 4.2.1 and 4.2.2 to our remaining components as appropriate gives our desired element \( y_J \).

Now assume that our connected component of type \( \gamma \) contains precisely one black vertex. If all other white vertices are isolated, then \( G_x \) contains a connected component consisting of a cycle of \( u \geq 1 \) black vertices. We may consider one such cycle, an isolated white vertex and the connected component of type \( \gamma \) to correspond to those components indexed by \( I \). Thus without loss of generality

\[
t_I = (1, 2) \cdots (2u - 1, 2u)(2u + 1, 2u + 2)(2u + 3)(2u + 4); \quad \text{and} \quad x_I = (1, 2u)(2, 3) \cdots (2u - 2, 2u - 1)(2u + 1)(2u + 2, 2u + 3)(2u + 4),
\]

(4.2.1)

unless \( u = 1 \) in which case we let \( t_I = (1, 2)(3, 4)(5)(6) \) and \( x_I = (1, 2)(3)(4, 5)(6) \). Taking \( y_I = (1)(2)(2u - 1) \cdots (u, u + 1)(2u + 1, 2u + 3)(2u + 2, 2u + 4) \) (or \( y_I = (1)(2)(3, 5)(4, 6) \) if \( u = 1 \)) it follows that \( G_x^{y_I} \) has one connected component of the form \( \circ \underline{\bullet} \bullet \circ \). \( G_x^{y_I} \) has one connected component of the form \( \circ \underline{\bullet} \bullet \) and the remaining components of these \( x \)-graphs are of the form \( \bullet \circ \), \( \bullet \circ \circ \circ \) and \( \circ \bullet \bullet \), thus satisfying the conditions of Corollary 4.1.4 (see Figures B.52-B.54).

Conversely, if there exists a white vertex that is not isolated, then it will be in a component of type \( \epsilon \). Again, as the number of edges and black vertices must be equal, there exists an isolated black vertex. We take the connected component of type \( \gamma \) along with one of type \( \epsilon \) and an isolated black vertex to be those indexed by \( I \). The existence of \( y_I \) then follows from Lemma 4.2.7(ii). Finally applying Lemmas 4.2.1 and 4.2.2 as appropriate to the remaining connected components, we obtain an element \( y_J \) as required.

If \( \gamma = 0 \) but \( \beta \neq 0 \), then consider the connected components of type \( \alpha \). If there exist connected components of type \( \alpha \) containing at least 3 black vertices, then we may pair these up with connected components of type \( \beta \) and \( \epsilon \) and apply Lemma 4.2.1 to obtain our element \( y_I \). If all connected components of type \( \alpha \) contain at most 2 black vertices, then we simply apply Lemma 4.2.6(ii) (if required) to the connected components of type \( \beta \) to obtain \( y_I \). Finally, applying Lemmas 4.2.1 and 4.2.2 to the remaining connected components ensures the existence of \( y_J \).

If \( \beta = \gamma = 0 \) and \( \alpha \neq 0 \), then we apply Lemma 4.2.1 to the connected components of type \( \alpha \) and \( \epsilon \) (if required) to obtain \( y_I \) and Lemmas 4.2.1 and 4.2.2 to the remaining connected components to find a suitable \( y_J \).

If \( \alpha = \beta = \gamma = 0 \), then \( \epsilon \geq 1 \) as by assumption \( \text{fix}(t) \neq \text{fix}(x) \). As the number of edges of \( G_x \) equals the number of black vertices, there exists an isolated black vertex. Moreover, as \( m \geq 2 \), there are two possible cases. If every black vertex is isolated, then there exists \( m \) connected components of type \( \epsilon \). Take two such components and two isolated black vertices as the components corresponding to our indexing set \( I \), and leave all other components of type \( \epsilon \) alone. Without loss of generality, we may assume that the parts of \( t \) and \( x \) corresponding to \( I \) are

\[
t_I = (1, 2)(3, 4)(5)(6)(7)(8) \quad \text{and} \quad x_I = (1)(2)(3)(4)(5, 6)(7, 8).
\]

(4.2.2)
4.3. THE CASES $\pi \neq \{4\}$:

Let $y_I = (1,5)(2,6)(3)(4)(7)(8)$. Then the connected components of $G^I_{y_I}$ and $G^2_{y_I}$ satisfy the conditions of Corollary 4.1.4 (see Figure B.55).

Conversely, if there is only one isolated black vertex, then there exists a connected component which is a cycle of $u \geq 1$ black vertices. Thus taking the components indexed by $I$ to be an isolated black vertex, a cycle of $u \geq 1$ black vertices and a component of type $\epsilon$, and leaving all other components of type $\epsilon$ alone, we may take $t_I$ and $x_I$ to be

$$t_I = (1,2) \cdots (2u-1,2u)(2u+1,2u+2)(2u+3)(2u+4); \quad \text{and}$$

$$x_I = (1,2u)(2,3) \cdots (2u-2,2u-1)(2u+1)(2u+2)(2u+3,2u+4),$$

(4.2.3)

or $t_I = (1,2)(3,4)(5)(6)$ and $x_I = (1,2)(3)(4)(5,6)$ if $u = 1$. Taking $y_I = (1)(2u)(2u-1) \cdots (u,u+1)(2u+1,2u+3)(2u+2,2u+4)$ (or $y_I = (1)(2)(3,5)(4,6)$ if $u = 1$), we see that the connected components of $G^I_{y_I}$ and $G^2_{y_I}$ satisfy the conditions of Corollary 4.1.4 (for full details see Figures B.56-B.58). Finally, applying Lemmas 4.2.1 and 4.2.2 to the remaining components of type $\delta$ (that is cycles of black vertices) gives the desired $y_I$.

Since all possible $x$-graphs have been analysed, this completes the proof of Theorem 4.0.1.

### 4.3 The cases $\pi \neq \{4\}$:

We illustrate the exceptional nature of $P_{\{4\}}(G,X)$ with a brief exploration of other $\pi$-product involution graphs. We begin by considering the case that $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geq 3$. The simplest such case arises when $a = 3$. Thus for $G = \text{Sym}(n)$, we consider the $G$-conjugacy class of $t = (1,2)(3,4)(5,6)(7,8)$, which we denote by $X$. As $\text{supp}(t)$ has size 8, it suffices to consider $8 \leq n \leq 16$. We calculate the sizes of the discs $\Delta_i(t)$ of $P_{\{8\}}(G,X)$ using the computer algebra package MAGMA. Theorem 4.0.3 is an immediate consequence of our calculations, which are summarised in Table 4.2.

| $n$ | $|\Delta_1(t)|$ | $|\Delta_2(t)|$ | $|\Delta_3(t)|$ | $|\Delta_4(t)|$ | $|X|$ | Diam$(P_{\{8\}}(G,X))$ |
|-----|----------------|----------------|----------------|----------------|-----|------------------|
| 8   | 384            | 4308           | 32             |                 | 105 | Totally Disconnected |
| 9   | 1152           | 16076          | 96             |                 | 945 | Totally Disconnected |
| 10  | 2304           | 49382          | 288            |                 | 4725| 3                |
| 11  | 3840           | 123974         | 7320           |                 | 17325| 3            |
| 12  | 5760           | 267014         | 42540          |                 | 51975| 3            |
| 13  | 8064           | 512630         | 154140         | 840            | 135135| 4            |
| 14  | 10752          | 902012         | 431760         | 6825           | 675675| 4            |
| 15  |                |                |                |                | 135135| 4            |
| 16  |                |                |                |                |       |                 |

Table 4.2: The sizes of the discs $\Delta_i(t)$ for $P_{\{8\}}(G,X)$, where $G := \text{Sym}(n)$ and $X$ is the $G$-conjugacy class of $t := (1,2)(3,4)(5,6)(7,8)$.

when $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geq 3$. This leads to the formulation of Theorem 4.0.2, a proof of which we now give.
Proof of Theorem 4.0.2

We note that as the support of $t$ has size $2m$, any $x$ in $X$ - the $G$-conjugacy class of $t$ - will have an $x$-graph, $G_x$, containing $m$ black vertices. If the order of $tx$ is $2m$, then $G_x$ must be of the form

(1) \[
\begin{array}{c}
\circ & \cdots & \circ \\
\end{array}
\]

(2) \[
\begin{array}{c}
\circ & \cdots & \circ & \circ \\
\end{array}
\]

(where additional isolated white vertices may be present). In both cases, we see that $|\text{supp}(t) \cap \text{supp}(x)| = 2m - 2$. It follows that if $n = 2m$ or $2m + 1$, then $P_{(2m)}(G, X)$ is totally disconnected. Thus assume that $n \geq 2m + 2$. Denote the $m$ transpositions that $x$ is comprised of by $x_1, \ldots, x_m$, where $\min \text{supp}(x_i) \leq \min \text{supp}(x_{i+1})$ for all $i = 1, \ldots, m - 1$. We will construct elements $y_i \in X$ for $i = 1, \ldots, m$ such that

$$y_i = x_1 \cdots x_i w_{i+1} \cdots w_m$$

(4.3.4)

for some transpositions $w_{i+1}, \ldots, w_m$ and such that $y_i$ is connected to $t$ in $P_{(2m)}(G, X)$. At each stage, if $y_i$ involves the transposition $x_{i+1}$ we are done, so we will assume that this is not the case.

Assume first that $|\text{supp}(x_1) \cap \text{supp}(t)| = 2$. Then without loss we may assume that $x_1 = (1, 3)$. Define

$$y_1 = (1, 3)(4, 5)(6, 7) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2).$$

(4.3.5)

Thus $G_{y_1}$ has the form [2] (as seen in Figure B.59). If $|\text{supp}(x_1) \cap \text{supp}(t)| = 1$, then we may assume that $x_1 = (1, 2m + 1)$. Taking

$$y_1 = (1, 2m + 1)(2, 3)(4, 5) \cdots (2m - 4, 2m - 3)(2m - 2, 2m + 2),$$

(4.3.6)

it follows that $G_{y_1}$ is isomorphic to [1] (see Figure B.60). Finally, if $|\text{supp}(x_1) \cap \text{supp}(t)| = 0$, then we assume that $x_1 = (2m + 1, 2m + 2)$. The element

$$y_1 = (2, 3)(4, 5) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2)$$

(4.3.7)

results in an $x$-graph $G_{y_1}$ of type [2] (as seen in Figure B.61). In all cases we see that $t$ and $y_1$ are adjacent - and hence connected - vertices of $P_{(2m)}(G, X)$.

Suppose that for some $1 \leq i < m$ an element $y_i$ of the form (4.3.4) exists with $y_i$ connected to $t$ in $P_{(2m)}(G, X)$. Note that $y_i$ fixes at least two elements of $\Omega$ as $n \geq 2m + 2$. We denote these elements by $f_1, f_2$. Define

$$x_{j,1} := \min \text{supp}(x_j) \quad \text{and} \quad x_{j,2} := \max \text{supp}(x_j).$$

(4.3.8)

for $1 \leq j \leq i + 1$ and

$$w_{j,1} := \min \text{supp}(w_j) \quad \text{and} \quad w_{j,2} := \max \text{supp}(w_j).$$

(4.3.9)

for $i + 1 \leq j \leq m$. Set $\alpha = x_{i+1,1}, \beta = x_{i+1,2}$ and $w = w_{i+1} \cdots w_m$. Thus $x_{i+1} = (\alpha, \beta)$. 


4.3. THE CASES $\pi \neq \{4\}$:

We follow an analogous approach to that used to define $y_1$. If $|\text{supp}(x_{i+1}) \cap \text{supp}(w)| = 2$, then without loss we have that $w_{i+1,1} = \alpha$ and $w_{i+2,1} = \beta$. First we construct an element $z_{i+1} \in X$ given by

$$z_{i+1} = (x_{1,1}, w_{m,2})(x_{1,2,2,1})(x_{2,2,1,1})(x_{i-1,2,2,1,1})(x_{i,2,2,1,1})(x_{i,2,2,1,1})$$

(4.3.10)

for $i > 1$ and

$$z_2 = (x_{1,1}, w_{m,2})(x_{1,2,2,1})(x_{1,2,2,1})(x_{1,2,2,1})(x_{1,2,2,1})(x_{1,2,2,1})$$

(4.3.11)

Consequently $G^h_{z_{i+1}}$ has the form given in (2) (see Figures B.62 and B.63). The element

$$y_{i+1} = x_1 x_2 \cdots x_{i+1} w_{i+2,1} \cdots w_m$$

(4.3.12)

results in an $x$-graph $G^h_{y_{i+1}}$ of isomorphism type [1] (see Figures B.62 and B.63). We deduce that $y_{i+1}$ is connected to $y_i$ and hence to $t$, and that $d(y_i, y_{i+1}) \leq 2$.

In the case that $|\text{supp}(x_{i+1}) \cap \text{supp}(w)| = 1$, we may assume that $w_{i+1,1} = \alpha$ and that $\beta = f_1 \in \text{fix}(y_i)$. Taking $z_2 = (\beta, w_{m,2})(w_{2,2,1,1})(x_{m-1,2,2,1,1})(x_{m-1,2,2,1,1})(x_{m-1,2,2,1,1})$,

$$z_{i+1} = (\beta, x_{2,2,1,1})(x_{1,2,2,1,1})(x_{i-1,2,2,1,1})(x_{i,2,2,1,1})(x_{i,2,2,1,1})(x_{i,2,2,1,1})$$

(4.3.13)

for $i > 1$ and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+2} \cdots w_m$$

(4.3.14)

we see that $G^h_{z_{i+1}}$ is of isomorphism type [1] whilst $G^h_{y_{i+1}}$ is of isomorphism type [2] (see Figures B.64 and B.65). Thus $d(y_i, y_{i+1}) \leq 2$ and $y_{i+1}$ is connected to $t$.

The final possibility is that $|\text{supp}(x_{i+1}) \cap \text{supp}(w)| = 0$. Consequently, defining $z_2 = (x_{1,1,2,1})(x_{1,2,2,1})(x_{2,2,1,1})(x_{m-1,2,2,1,1})(x_{m-1,2,2,1,1})(x_{m-1,2,2,1,1})$,

$$z_{i+1} = (x_{1,1,2,1})(x_{1,2,2,1})(x_{i-1,2,2,1,1})(x_{i,2,2,1,1})(x_{i,2,2,1,1})(x_{i,2,2,1,1})$$

(4.3.15)

for $i > 1$ and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+2} \cdots w_m$$

(4.3.16)

we obtain $x$-graphs $G^h_{z_{i+1}}$ and $G^h_{y_{i+1}}$ of types [1] and [2] respectively (see Figures B.66 and B.67). We conclude that $d(y_i, y_{i+1}) \leq 2$, and hence $y_{i+1}$ is connected to $t$ in $P_{\{2m\}}(G, X)$ as required.

(ii) If $P_{\{2m\}}(G, X)$ is connected, then $n \geq 2m + 2$ by (1). Moreover, the above argument shows that $d(t, y_i) \leq 1$ and for $1 \leq i \leq m - 1$ we have $d(y_i, y_{i+1}) \leq 2$. Thus as $x = y_m$, we conclude that $\text{Diam}(P_{\{2m\}}(G, X)) \leq 2m - 1$. For the lower bound, we note that $x \in \Delta_1(t)$ precisely when $G_x$ is of type [1] or [2]. In particular $|\text{supp}(t) \cap \text{supp}(x)| = 2m - 2$. Arguing iteratively we deduce that if $d(t, x) \leq s$, then $|\text{supp}(t) \cap \text{supp}(x)| \geq 2m - 2s$. Since $X$ contains an involution $x$ satisfying $|\text{supp}(t) \cap \text{supp}(x)| = 0, 4m - n$ we deduce that

$$\text{Diam}(P_{\{2m\}}(G, X)) \geq \min\{m, \lfloor n/2 - m \rfloor\}$$
as required.

We note that it is also possible to define \( y_1 \in X \) of the form (4.3.4) which is connected to \( t \) by a path of length 2 in \( P_{|2m|}(G, X) \). Indeed, if \( x_1 = (1, 3) \), we set

\[
\begin{align*}
z_1 &= (2, 5)(4, 2m)(6, 7) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2); \quad \text{and} \\
y_1 &= (1, 3)(4, 2m + 1)(5, 6)(7, 8) \cdots (2m - 1, 2m). 
\end{align*}
\]

(4.3.17)

If \( x_1 = (1, 2m + 1) \), then define

\[
\begin{align*}
z_1 &= (2, 3)(4, 5) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2); \quad \text{and} \\
y_1 &= (1, 2m + 1)(2, 2m + 2)(3, 4) \cdots (2m - 5, 2m - 4)(2m - 3, 2m). 
\end{align*}
\]

(4.3.18)

Finally, if \( x_1 = (2m + 1, 2m + 2) \), then take

\[
\begin{align*}
z_1 &= (1, 2m + 1)(2, 3)(4, 5) \cdots (2m - 4, 2m - 3)(2m - 2, 2m + 2); \quad \text{and} \\
y_1 &= (3, 4)(5, 6) \cdots (2m - 1, 2m)(2m + 1, 2m + 2). 
\end{align*}
\]

(4.3.19)

In each case, the \( x \)-graphs \( G_{z_i} \) and \( G_{y_i}^{\dagger} \) are of isomorphism type (1) or (2) as required (see Figures B.68-B.70 for full details).

We now consider Theorem 4.0.4. A non-constructive proof of the connectivity of \( P_{|q|}(G, X) \) using Jordan’s theorem is contained in the proof of [BGR13, Theorem 4.1]. Here we give a constructive proof in a similar vein to the proof of Theorem 4.0.2 above.

**Proof of Theorem 4.0.4:**

We first note that the elements of the disc \( \Delta_1(t) \) are precisely those elements \( x \in X \) whose \( x \)-graph, \( G_x \), is of isomorphism type \( \bullet \cdots \bullet \cdots \odot \). Consequently, \( P_{|q|}(G, X) \) is totally disconnected if \( n = q - 1 \). Thus assume that \( n \geq q \) and hence that \( |\text{fix}(t)| \geq 1 \).

We proceed as in the proof of Theorem 4.0.2 and set \( 2m = q - 1 \). Let \( x \in X \) be given and denote the transpositions of \( x \) as \( x = x_1 \circ x_2 \circ \cdots \circ x_m \). As in the proof of Theorem 4.0.2, we will construct elements \( y_i \in X \) for \( i = 1, \ldots, m \) such that

\[
y_i = x_1 \cdots x_i w_{i+1} \cdots w_m
\]

(4.3.20)

for some transpositions \( w_j \) and such that \( y_i \) is connected to \( t \) in \( P_{|q|}(G, X) \). Mirroring the situation of the proof of Theorem 4.0.2, we may assume that \( x_{i+1} \) is not a transposition of \( y_i \). We continue to use the notation \( x_{j,1}, x_{j,2} \) and \( w_{j,1}, w_{j,2} \) previously introduced in (4.3.8) and (4.3.9) respectively. For convenience we define \( y_0 := t \) and for each \( y_i \) we consider the cases \( \text{supp}(y_i) = \text{supp}(x) \) and \( \text{supp}(y_i) \neq \text{supp}(x) \) separately.

Assume that \( \text{supp}(t) = \text{supp}(x) \). Thus \( |\text{supp}(t) \cap \text{supp}(x)| = 2 \) and without loss of generality we may take \( x_1 = (1, 3) \). Define \( z_1 \) and \( y_1 \) by

\[
\begin{align*}
z_1 &= (2, 3)(4, 5) \cdots (2m, 2m + 1); \quad \text{and} \\
y_1 &= x_1(2, 4)(5, 6) \cdots (2m - 1, 2m). 
\end{align*}
\]

(4.3.21)
4.3. THE CASES $\pi \neq \{4\}$:

The $x$-graphs $G_G$ and $G_{y_1}^{\infty}$ are both of isomorphism type $\bullet \cdots \bullet \circ$ (see Figure B.71) and hence $t$ and $y_1$ are connected in $P_{\{q\}}(G, X)$ with $d(t, y_1) \leq 2$.

When $\text{supp}(t) \neq \text{supp}(x)$ we consider three subcases. If $\text{supp}(t) \cap \text{supp}(x) = 2$, then without loss we have $x_1 = (1,3)$ and we take

$$y_1 = x_1(4,5)(6,7) \cdots (2m,2m+1).$$  \hfill (4.3.22)

If $\text{supp}(t) \cap \text{supp}(x) = 1$, then we may assume that $x_1 = (1,2m+1)$ and thus take

$$y_1 = x_1(2,3)(4,5) \cdots (2m-2,2m-1).$$  \hfill (4.3.23)

In both cases, $G_{y_1}$ is of the required form (see Figures B.72 and B.73). Consequently $t$ and $y_1$ are adjacent in $P_{\{q\}}(G,X)$. Finally, suppose that $\text{supp}(t) \cap \text{supp}(x) = 0$ and hence that $x_1 = (2m+1,2m+2)$. Defining

$$z_1 = (1,2m+1)(2,3) \cdots (2m-2,2m-1); \text{ and}$$

$$y_1 = (x_1)(1,2)(3,4) \cdots (2m-3,2m-2),$$  \hfill (4.3.24)

we have that $G_G$ and $G_{y_1}^{\infty}$ are of the aforementioned isomorphism type (see Figure B.74), and so $t$ and $y_1$ are connected in $P_{\{q\}}(G,X)$. Moreover, $d(t,y_1) \leq 2$.

Now suppose that a $y_i$ of the form (4.3.20) has been defined for some $i < m$ with $y_i$ connected to $t$ in $P_{\{q\}}(G,X)$. First assume that $\text{supp}(y_i) = \text{supp}(x)$ and that $\alpha \in \text{fix}(y_i) \cap \text{fix}(x)$. Without loss we may assume that $x_{i+1,1} = w_{i+1,1}$ and $x_{i+1,2} = w_{i+2,1}$. Define $z_2 = (x_{1,2}, w_{2,1})(w_{2,2}, w_{3,1}) \cdots (w_{m-2,1}, w_{m,1})(w_{m,2}, \alpha)$,

$$z_{i+1} = (x_{1,2}, w_{2,1})(x_{2,2}, w_{3,1}) \cdots (x_{i-1,2}, w_{i,1})(x_{i,2}, w_{i+1,1})$$

$$(w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-2,1}, w_{m,1})(w_{m,2}, \alpha); \text{ and}$$  \hfill (4.3.25)

$$y_{i+1} = x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2}) w_{i+3} \cdots w_m$$

for $i > 1$. We have that the $x$-graphs $G_{z_{i+1}}^{\infty}$ and $G_{y_{i+1}}^{\infty}$ have the required form and hence $d(y_i,y_{i+1}) \leq 2$.

It remains to consider the case that $\text{supp}(y_i) \neq \text{supp}(x)$. Let $f_i \in \text{fix}(y_i) \setminus \text{fix}(x)$ and $f_x \in \text{fix}(x) \setminus \text{fix}(y_i)$. In the case when $\text{supp}(y_i) \cap \text{supp}(x_{i+1}) = 2$, assume that $x_{i+1} = (w_{i+1,1}, w_{i+2,1})$ and $z_2 = (x_{1,1}, w_{m,2})(x_{1,2}, w_{2,1})(w_{3,1}, f_i)(w_{3,2}, w_{4,1}) \cdots (w_{m-2,1}, w_{m,1})$,

$$z_{i+1} = (x_{1,1}, w_{m,2})(x_{1,2}, w_{2,1}) \cdots (x_{i-1,2}, w_{i,1})(x_{i,2}, w_{i+1,1})(w_{i+2,1}, f_i)$$

$$(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-2,1}, w_{m,1}); \text{ and}$$  \hfill (4.3.26)

$$y_{i+1} = x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2}) w_{i+3} \cdots w_m$$

for $i > 1$. If $\text{supp}(y_i) \cap \text{supp}(x_{i+1}) = 1$, then without loss we have $x_{i+1} = (w_{i+1,1}, f_i)$. Hence we define $z_2 = (x_{1,1}, w_{m,2})(x_{1,2}, w_{2,1})(w_{3,1}, f_i)(w_{3,2}, w_{4,1}) \cdots (w_{m-2,1}, w_{m,1}),$

$$z_{i+1} = (x_{1,1}, w_{m,2})(x_{1,2}, w_{2,1}) \cdots (x_{i-1,2}, w_{i,1})(x_{i,2}, w_{i+1,1})$$

$$(w_{i+2,1}, f_i)(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-2,1}, w_{m,1}); \text{ and}$$  \hfill (4.3.27)

$$y_{i+1} = x_1 x_2 \cdots x_{i+1} w_{i+2} \cdots w_{m-1}(w_{m,1}, w_{i+1,2}).$$
for $i > 1$. Finally, if $x_{i+1}$ and $y_i$ are disjoint we set $z_2 = (x_{1,1}, x_{2,1})(x_{1,2}, w_{1,2})(w_{2,2}, w_{3,1}) \cdots (w_{m-1,2}, w_{m,1})$,

$z_{i+1} = (x_{1,1}, x_{i+1,1})(x_{1,2}, x_{2,1}) \cdots (x_{i-1,1}, x_{i,1})(x_{i,2}, w_{i+1,1})$

$w_{i+1,2}, w_{i+1,2}) \cdots (w_{m-1,2}, w_{m,1});$ and

$y_{i+1} = x_{1,2} \cdots x_{i+1,1} w_{i+1,2} \cdots w_{m-1}$

for $i > 1$. For each pair $(z_{i+1}, y_{i+1})$ the $x$-graphs $G^p_{z_{i+1}}$ and $G^{y_{i+1}}_{qX}$ have isomorphism type $\bullet \cdots \bullet \circ$ (see Figures B.75-B.82 for details). Consequently $d(y_i, y_{i+1}) \leq 2$ and the elements $t$ and $y_{i+1}$ are connected in $P_{[q]}(G, X)$.

(iii) By part (ii) we have that Diam $P_{[q]}(G, x) \leq q - 1$. For the lower bound, we note that for $x \in X$ to be adjacent to $t$ we have $|\text{supp}(t) \cap \text{supp}(x)| = |\text{supp}(t)| - 1$. Taking $x \in X$ such that $|\text{supp}(t) \cap \text{supp}(x)|$ is minimal we have that $|\text{supp}(t) \cap \text{supp}(x)| = \max\{0, 2q - 2 - n\}$ and hence $d(t, x) \geq \min\{q - 1, n + 1 - q\}$ as required.

Following a similar approach to that used when $p = 2$, we define $y_1 \in X$ of the form (4.3.20) such that there is a path in $P_{[q]}(G, X)$ from $t$ to $y_1$ of length 2. These paths will be used in the proof of Theorem 4.0.5. Such paths were defined in the above proof except when $\text{supp}(t) \neq \text{supp}(x)$ and $|\text{supp}(t) \cap \text{supp}(x)| = 1$ or 2. In the latter case, we may assume without loss that $x_1 = (1, 3)$ and define

$z_1 = (2, 3)(4, 5) \cdots (2m, 2m + 1)$; and

$y_1 = (1, 3)(2, 4)(5, 6) \cdots (2m - 1, 2m)$. (4.3.29)

In the former case, we assume that $x_1 = (1, 2m + 1)$ and set

$z_1 = (2, 3)(4, 5) \cdots (2m, 2m + 1)$; and

$y_1 = (1, 2m + 1)(3, 4)(5, 6) \cdots (2m - 1, 2m)$. (4.3.30)

Both cases give rise to $x$-graphs $G_{z_1}$ and $G^{y_1}_{qX}$ of the required form to show that there is a path from $t$ to $y_1$ of length 2 in $P_{[q]}(G, X)$ (see Figures B.83 and B.84).

The proofs of Theorems 4.0.2 and 4.0.4 are utilised in the proof of Theorem 4.0.5.

Proof of Theorem 4.0.5

The $x$-graph $G_x$ of any $x \in \Delta_1(t)$ must consist of a connected component of isomorphism type $\bullet \cdots \circ$ containing $q_i/2$ black vertices for each $i = 2, \ldots, r$. In addition, there will be connected components of types [1] or [2] from the proof of Theorem 4.0.2 if $p_1 = 2$, or a component of type $\bullet \cdots \circ$ containing $q_i/2$ black vertices if $p_1 \neq 2$. It follows that

$$|\text{fix}(t)| \geq \begin{cases} r + 1 & \text{if } p_1 = 2; \text{ and} \\ r & \text{otherwise.} \end{cases}$$ (4.3.31)

We conclude that if

$$n \geq \begin{cases} q + 2 & \text{if } p_1 = 2; \text{ and} \\ q & \text{otherwise} \end{cases}$$ (4.3.32)
4.3. THE CASES $\pi \neq \{4\}$

does not hold, then $\mathcal{P}_{(q)}(G, X)$ is totally disconnected. Thus assume that (4.3.32) holds.

Denote the the transpositions of $x$ by

$$x = x_1^{(1)} \cdots x_1^{(q_1/2)} x_2^{(2)} \cdots x_2^{(q_2/2)} \cdots x_r^{(r)} \cdots x_r^{(q_r/2)}.$$

Moreover, if $\ell = \max\{q_i\}$, define $x_{q_j/2+1}^{(j)} = \cdots = x_{q\ell/2}^{(j)} = 1$ for all $j = 1, \ldots, r$ where these permutations are not already defined.

Using the proofs of Theorems 4.0.1, 4.0.2 and 4.0.4 and the comments following the proofs, we may construct $y_i \in X$ for $i = 0, \ldots, \ell/2$ with $y_0 := t$ satisfying

(i) there is a path in $\mathcal{P}_{(q)}(G, X)$ from $y_i$ to $y_{i+1}$ of length 2 for $i = 0, \ldots, \ell/2 - 1$; and

(ii) $y_i$ contains the transpositions $x_1^{(j)}, \ldots, x_i^{(j)}$ for $j = 1, \ldots, r$.

This is allowable as (4.3.31) holds. The result now follows immediately.

We briefly consider an example of a case when $\pi$ consists of a composite number. The smallest such situation arises when $\pi = \{6\}$. If $t \in G$ is an involution and $X$ is the $G$-conjugacy class of $t$, then any $x \in X$ will be adjacent to $t$ in $\mathcal{P}_{\{6\}}(G, X)$ if the connected components of $G_x$ consist of components of the form

(i) $\bigcirc$, $\bullet$;

(ii) $\bullet$, $\bigcirc$, $\bigcirc$, $\bullet$;

(iii) $\bullet$, $\bigcirc$, $\bullet$, and

(iv) $\bigcirc$, $\bullet$, $\bigcirc$, $\bullet$, $\bullet$, $\bullet$, $\bullet$ (with 6 black vertices).

Moreover, either one component is of isomorphism type (iv) or there exists at least one component of type (iii) and one component of type (iii)

Finally, Table 4.3 gives the sizes of the discs $\Delta_i(t)$ of $\mathcal{P}_{\{6\}}(G, X)$ for the symmetric groups $G := \text{Sym}(n)$ ($6 \leq n \leq 10$), when $X$ is the $G$-conjugacy class of an involution $t \in G$. 

Table 4.3: The sizes of the discs $\Delta_i(t)$ for $P_{\{6\}}(G, X)$, where $X$ is the $G$-conjugacy class of $t = (1, 2) \cdots (2m - 1, 2m) \in G := \text{Sym}(n)$. 

| $n$ | $m$ | $|\Delta_1(t)|$ | $|\Delta_2(t)|$ | $|\Delta_3(t)|$ | $|X|$ | Diam$(P_{\{6\}}(G, X))$ |
|-----|-----|---------------|---------------|---------------|------|------------------|
| 6   | 2   |               |               |               | 45   | 15               |
|     | 3   |               |               |               | 105  | TotallyDisconnected |
| 7   | 2   | 12            | 38            | 54            | 105  | 3                |
|     | 3   | 12            | 60            | 32            | 105  | 3                |
| 8   | 2   | 48            | 158           | 3             | 210  | 3                |
|     | 3   | 72            | 347           |               | 420  | 2                |
|     | 4   |               |               |               | 105  | TotallyDisconnected |
| 9   | 2   | 120           | 242           | 15            | 378  | 3                |
|     | 3   | 216           | 1043          |               | 1260 | 2                |
|     | 4   | 48            | 836           | 60            | 945  | 3                |
| 10  | 2   | 240           | 389           |               | 630  | 2                |
|     | 3   | 624           | 2525          |               | 3150 | 2                |
|     | 4   | 416           | 4308          |               | 4725 | 2                |
|     | 5   | 160           | 784           |               | 945  | 2                |
Chapter 5

Conjugate $p$-elements of Full Support that Generate the Wreath Product $C_p \ltimes C_p$

In Chapter 4 we saw how the rich structure of the poset $S_p(\text{Sym}(n))$ could be used to characterise when two conjugate involutions generated the dihedral group $\text{Dih}(8)$. There are two natural generalisations of this result. The first would consider when two conjugate involutions generate the dihedral group $\text{Dih}(2^m)$ for $m > 3$. An analysis of $x$-graphs would be fundamental in such an approach, which would then become a case-by-case analysis of the possible situations that could arise - similar to that given in the proof of Theorem 4.0.1.

An alternative generalisation involves considering $\text{Dih}(8)$ as the wreath product, $C_2 \ltimes C_2$, of two cyclic groups of order 2. From this viewpoint, a natural generalisation is to consider when two conjugate $p$-elements of $\text{Sym}(n)$ generate the wreath product $C_p \ltimes C_p$. In this chapter, we consider conjugate $p$-elements of full support in $\text{Sym}(n)$ that generate this wreath product. To analyse such situations, it would be desirable to form a generalisation of the $x$-graph. The natural generalisation results in non-planar directed graphs and hence we consider the adjacency matrix of such a graph. Indeed, given conjugate $p$-elements $a$ and $x$ of full support in $\text{Sym}(n)$, we form suitable matrices $A_a^x$ and $A_x^a$. Here $a$ will be the standard $p$-element of $G$ namely $a = (1, 2, \ldots, p)(p + 1, \ldots, 2p) \cdots ((r - 1)p + 1, \ldots, rp)$ where $n = rp$ and we label the $p$-cycles forming $a$ by $\alpha_i = (p(i - 1) + 1, \ldots, pi)$ for $i = 1, \ldots, r$. Similarly if $x$ is a $G$-conjugate of $a$, then we may label its disjoint $p$-cycles by $\chi_1, \ldots, \chi_r$. The matrices $A_a^x$ and $A_x^a$ have $(i, j)$ entries given by $|\text{supp}(\alpha_i^x) \cap \text{supp}(\alpha_j)|$ and $|\text{supp}(\chi_i^a) \cap \text{supp}(\chi_j)|$ respectively.

Throughout this chapter, $p$ will be an odd prime. Our first result considers the case that $n = p^2$. We see that the matrices $A_a^x$ and $A_x^a$ do indeed encode data which we may use to determine when $a$ and $x$ generate the wreath product $W_p := C_p \ltimes C_p$. This encoding involves circulant matrices and their representer polynomials in addition to permutation matrices, all of which are defined in Section 5.1.

**Theorem 5.0.1.** Let $G = \text{Sym}(p^2)$, $a$ be the standard $p$-element of $G$ and let $x$ be a conjugate of
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a. Then $\langle a, x \rangle \cong W_p$ precisely when the following two equivalent conditions hold.

(i) $A^x_a = p \cdot Y_\sigma$ for some $p$-cycle $\sigma \in \text{Sym}(p)$, $A^x_a = \text{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_i \in \mathbb{Z}_p$, $X = 1$ is a simple root of the representer polynomial $f_{A^x_a}(X) \in \mathbb{Z}_p[X]$ and $[a, a^x] = 1$.

(ii) $A^x_a = p \cdot Y_\sigma$ for some $p$-cycle $\sigma \in \text{Sym}(p)$, $A^x_a = \text{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_i \in \mathbb{Z}_p$, $|\det(A^x_a)|_p = p^2$ and $[a, a^x] = 1$.

(or (i) and (ii) hold with the roles of $a$ and $x$ interchanged).

The second result concerns the case when $n = p^3$. It involves the notions of the block sum matrix, $BS(M)$, of a matrix $M$ and of a reduced representer polynomial as given in Definitions 5.4.5 and 5.4.6.

**Theorem 5.0.2.** Let $G = \text{Sym}(p^3)$, $a$ be the standard $p$-element of $G$ and let $x$ be a conjugate of $a$. Moreover, suppose that we cannot decompose $a = a_1 \cdots a_p$ and $x = x_1 \cdots x_p$ with $a_i, x_i$ sitting inside a copy of $\text{Sym}(p^2)$ for each $i$ with one such pair satisfying the conditions of Theorem 5.0.1. Then $\langle a, x \rangle \cong W_p$ precisely when up to a renumbering of the $\alpha_i$ the following conditions hold

(i) $[a, a^x] = 1$ for all $i = 1, \ldots, (p - 1)/2$;

(ii) $A^x_a$ is a block matrix having $p \times p$ blocks, all of which are zero except for those immediately above the leading diagonal. The non-zero $p \times p$ blocks are either equal to $p \cdot Y_\sigma$ for some $\sigma \in \text{Sym}(p)$ or have every entry equal to 1. Moreover, $A^x_a$ contains at least one block whose entries are all equal to 1; and

(iii) $A^x_a$ is a block matrix having $p \times p$ blocks each of which is circulant and the diagonal blocks are all equal. Moreover, either

(c1) $X = 1$ is a root of multiplicity at least 2 of the representer polynomial $f_{BS(A^x_a)}(X) \in \mathbb{Z}_p[X]$ but is not a root of the representer polynomial $f'_{BS(A^x_a)}(X) \in \mathbb{Z}_{p^2}[X]$; or

(c2) $X = 1$ is not a root of the reduced representer polynomial $g'_{A^x_a}(X) \in \mathbb{Z}_{p^2}[X]$;

(or (i), (ii) and (iii) hold - up to a renumbering of the $\chi_i$ - with the roles of $a$ and $x$ interchanged).

Our final result combines Theorems 5.0.1 and 5.0.2 to consider the most general setting. Since $C_p \triangleleft C_p$ is not a subgroup of $\text{Sym}(n)$ for $n < p^2$, this occurs when $G = \text{Sym}(rp)$ for some $r \geq p$.

**Theorem 5.0.3.** Let $r \geq p$, $n = rp$ and $G = \text{Sym}(n)$. If $a$ is the standard $p$-element of $G$ having $G$-conjugacy class $\mathcal{X}$ and $x \in \mathcal{X}$, then $\langle a, x \rangle \cong W_p$ precisely when for a suitable renumbering of the $\alpha_i$ and $\chi_j$ the following conditions hold

(i) $[a, a^x] = 1$ for $i = 1, \ldots, (p - 1)/2$;

(ii) $A^x_a$ is a block diagonal matrix having blocks $D_1$, $D_2$ and $D_3$, where

- $D_1 = p \cdot I$ for some identity matrix $I$;
5.1. PRELIMINARY RESULTS

- $D_2$ is a block diagonal matrix having $p \times p$ blocks, where each block has the form $p \cdot Y_{\sigma}$ for some $p$-cycle $\sigma \in \text{Sym}(p)$; and
- $D_3$ is a block diagonal matrix having $p^2 \times p^2$ blocks having the form of the matrix $A_x^p$ from Theorem 5.0.2.

(iii) $A_x^p$ is a block diagonal matrix having blocks $E_1$, $E_2$ and $E_3$, where

- $E_1 = D_1$;
- $E_2$ is a block diagonal matrix of the same size as $D_2$, where each block is a $p \times p$ circulant matrix having row sum equal to $p$; and
- $E_3$ is a block diagonal matrix of the same size as $D_3$, where each block is a $p^2 \times p^2$ block matrix having $p \times p$ blocks each of which is circulant and the diagonal blocks are all equal.

(iv) One of the following holds

- The representer polynomial $f_B(X) \in \mathbb{Z}_p[X]$ of at least one of the circulant blocks, $B$, of $E_2$ has a simple root at $X = 1$;
- There is at least one block, $C$, of $E_3$ such that $X = 1$ is a root of multiplicity at least 2 over $\mathbb{Z}_p$ of the representer polynomial of the block sum matrix $BS(C)$ but $X = 1$ is not a root of the representer polynomial $f'_{BS(C)}(X) \in \mathbb{Z}_p[X]$;
- There is at least one block, $C$, of $E_3$ such that $X = 1$ is not a root of the reduced representer polynomial $g_C'(X) \in \mathbb{Z}_p[X]$.

(or the conditions (i) – (iv) hold with the roles of $a$ and $x$ interchanged)

This chapter is arranged as follows. In Section 5.1 we look at the matrices $A_x^p$ and $A_a^x$ and give some basic results about the wreath product $C_p \wr C_p$. We also introduce the notion of circulant matrices and illustrate a number of properties that they satisfy, which will be used in our subsequent work. Sections 5.2 and 5.3 consider the case that $n = p^2$. The former section leads up to a proof of the equivalence of $<a, x> \cong W_p$ and part (i) of Theorem 5.0.1 whilst the latter proves the equivalence of parts (i) and (ii) of the same theorem. The case that $n = p^3$ is considered in Section 5.4 in which Theorem 5.0.2 is proved. Due to the technical nature of some of the proofs in Sections 5.2 and 5.4 both sections conclude with worked examples highlighting the key steps in the proofs. The chapter concludes in Section 5.5 where Theorem 5.0.3 is proved.

5.1 Preliminary Results

Throughout this chapter we fix a prime $p \neq 2$ and set $G := \text{Sym}(pr)$ for some $r \geq p$. We shall also use the notation $W_p$ to denote the wreath product $C_p \wr C_p$ of two cyclic groups of order $p$. We consider $G$ to be acting on the set $\Omega := \{1, \ldots, pr\}$ and fix an element $a = \alpha_1 \alpha_2 \cdots \alpha_r$ where $\alpha_i = (p(i - 1) + 1, p(i - 1) + 2, \ldots, pi)$, which we will sometimes refer to as the standard $p$-element of $G$. We denote the $G$-conjugacy class of $a$ by $X = a^G$. Given $x \in X$, it will be useful to denote
the disjoint $p$-cycles forming $x$ by $\chi_1, \ldots, \chi_r$. The $\chi_i$ are defined recursively by setting $\chi_1$ to be the $p$-cycle for which the orbit of 1 under $\langle \chi_1 \rangle$ is non-trivial. Assuming that $\chi_i$ has been defined for $1 \leq i \leq j$ we then define

$$t_j := \min \{ t \in \Omega \mid \text{the orbit of } t \text{ under } \langle \chi_1, \ldots, \chi_j \rangle \text{ is trivial} \}$$

and define $\chi_{j+1}$ to be the $p$-cycle for which the orbit of $t_j$ under $\langle \chi_{j+1} \rangle$ is non-trivial.

In Chapter 4, it was seen that when $X$ was an arbitrary conjugacy class of involutions, then to determine when two conjugate involutions generate the wreath product $W_2 \rtimes \text{Dih}(8)$, we may use graphs known as $x$-graphs. These graphs essentially considered the intersections of the supports of the transpositions in the decomposition of the involutions. Generalising this from a strictly graph-theoretic viewpoint is impractical. However, consideration of the adjacency matrix of such a generalisation does yield results.

**Definition 5.1.1.** Let $x \in \mathcal{X}$ be given. We define the $r \times r$ matrices $A_x^a$ and $A_x^a$ by

$$(A_x^a)_{i,j} := |\text{supp}(\alpha_i^x) \cap \text{supp}(\alpha_j)|$$

and

$$(A_x^a)_{i,j} := |\text{supp}(\chi_i^x) \cap \text{supp}(\chi_j)|.$$  

As with $x$-graphs, since the element $a$ is fixed, we suppress it in our notation and denote $A_x^a$ by $A_x$. The determinant of a matrix $B$ will be denoted by $\det(B)$ and the largest power of $p$ dividing it will be denoted $|\det(B)|_p$.

We illustrate these notions with an example. Let $p = 5$ and let $a$ be the standard 5-element of $G := \text{Sym}(25)$. Taking

$$x = \chi_1 \cdot \chi_2 \cdot \chi_3 \cdot \chi_4 \cdot \chi_5$$

$$= (1, 22, 15, 4, 5)(2, 8, 7, 19, 9)(3, 10, 14, 24, 11)$$

$$(6, 23, 25, 20, 13)(12, 18, 16, 21, 17)$$

in the $G$-conjugacy class of $a$, we see that

$$\alpha_1^x = (1, 22, 8, 10, 5), \quad \alpha_2^x = (2, 14, 23, 19, 7), \quad \alpha_3^x = (3, 18, 6, 24, 4),$$

$$\alpha_4^x = (9, 13, 21, 12, 16) \quad \text{and} \quad \alpha_5^x = (11, 20, 17, 15, 25).$$

Consequently

$$A_x = \begin{pmatrix}
2 & 2 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 \\
2 & 1 & 0 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 2 & 2 & 1
\end{pmatrix}.$$
Similarly

\[ \chi_1^a = (1, 2, 23, 11, 5), \quad \chi_2^a = (3, 9, 8, 20, 10), \quad \chi_3^a = (4, 6, 15, 25, 12), \]
\[ \chi_4^a = (7, 24, 21, 16, 14), \quad \text{and} \quad \chi_5^a = (13, 19, 17, 22, 18). \]

This gives rise to the matrix

\[
A_a^x = \begin{pmatrix}
2 & 1 & 1 & 1 & 0 \\
0 & 2 & 2 & 1 & 0 \\
2 & 0 & 0 & 2 & 1 \\
0 & 1 & 2 & 0 & 2 \\
1 & 1 & 0 & 1 & 2
\end{pmatrix}.
\]

In Section 5.2 we will be concerned with two specific classes of matrices: permutation matrices and circulant matrices. Indeed, if \( \sigma \in \text{Sym}(n) \), then we denote by \( Y_\sigma \) the \( n \times n \) permutation matrix defined by

\[
y_{i,j}^\sigma = \begin{cases} 
1 & \text{if } i\sigma = j; \\
0 & \text{otherwise}.
\end{cases}
\]

In the case that \( \sigma = (1, 2, \ldots, n) \), we set \( \pi := Y_\sigma \). For a given \( n \times n \)-matrix \( A \), multiplying \( A \) on the left by \( \pi \) cyclically shifts the rows of \( A \) up by a row, whilst multiplying by \( \pi \) on the right cyclically shifts the columns of \( A \) to the right by a column. Matrices that are invariant under conjugation by \( \pi \) are known as circulant matrices. Thus any circulant matrix \( C \) satisfies \( C_{i,j} = C_{i+k,j+k} \) for any \( k = 1, \ldots, n \) (where we replace \( i + k \) (respectively \( j + k \)) by \( i + k - n \) (respectively \( j + k - n \)) if \( i + k > n \) (respectively \( j + k > n \))). Consequently a circulant matrix is uniquely determined by its first row, and we denote the circulant matrix

\[
C = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & c_3 & \cdots & c_0
\end{pmatrix},
\]

by \( C = \text{circ}(c_0, c_1, \ldots, c_{n-1}) \).

Associated to any circulant matrix \( C = \text{circ}(c_0, c_1, \ldots, c_{n-1}) \), is the representor polynomial \( f_C(X) \) in the indeterminant \( X \), given by

\[
f_C(X) = c_0 + c_1X + c_2X^2 + \cdots + c_{n-1}X^{n-1}.
\]

Strictly speaking this is a polynomial with coefficients in the underlying ring of \( C \). However, in this chapter the underlying ring of our circulant matrices will be the integers, and it will sometimes be preferable to consider \( f_C(X) \) as an element of \( \mathbb{Z}[X] \) or \( \mathbb{Z}_p[X] \). The following result illustrates this correspondence.

**Lemma 5.1.2.** Let \( X = \text{circ}(x_0, \ldots, x_{p-1}) \) be an integer circulant matrix. Then \( \det(X) \equiv x_0 + \cdots + x_{p-1} \pmod{p} \).
Proof. Expanding \( \det(X) \) gives a series of summands \( \alpha_{a_0 \cdots a_{p-1}} x_0^{a_0} x_1^{a_1} \cdots x_{p-1}^{a_{p-1}} \) where \( a_0 + \cdots + a_{p-1} = p \) and \( a_i \in \mathbb{N} \). Clearly if \( a_i = p \) for some \( i \), then the coefficient \( \alpha_{a_0 \cdots a_{p-1}} \) is equal to 1 and the corresponding summand of \( \det(X) \) is \( x_0^p \). Conversely, consider an occurrence of \( x_0^{a_0} \cdots x_{p-1}^{a_{p-1}} \) in the expansion of \( \det(X) \) where no \( a_j \) is equal to \( p \). Suppose that \( x_0^{a_0} \cdots x_{p-1}^{a_{p-1}} \) occurs from the entries \( X_{i_1,j_1}, \ldots, X_{i_p,j_p} \) of \( X \). Due to the circular nature of entries of \( X \), it will also occur from the entries \( X_{i_1+k,j_1+k}, \ldots, X_{i_p+k,j_p+k} \) for \( k = 1, \ldots, p-1 \) (where we set \( X_{i+p,j}, X_{i,j+p} \) and \( X_{i+p,j+p} \) equal to \( X_{i,j} \) for \( i, j = 1, \ldots, p \)). Since each \( a_j \) is not equal to \( p \), and \( p \neq 2 \) we conclude that
\[
\{X_{i_1+s,j_1+s}, \ldots, X_{i_p+s,j_p+s}\} \neq \{X_{i_1+t,j_1+t}, \ldots, X_{i_p+t,j_p+t}\}
\]
for any distinct \( s, t = 0, \ldots, p-1 \). Moreover, consideration of the signs of each of these occurrences of \( x_0^{a_0} \cdots x_{p-1}^{a_{p-1}} \) shows that they must all be equal. Since there are \( p \) such occurrences, we conclude that if \( a_j \neq p \) for \( j = 0, \ldots, p-1 \), then \( \alpha_{a_0 \cdots a_{p-1}} \) is divisible by \( p \). Consequently, applying Fermat’s Little Theorem we obtain
\[
\det(X) \equiv x_0^p \cdots x_{p-1}^p \equiv x_0 \cdots x_{p-1} \pmod{p}.
\]

Next we consider the structure of \( W_p \cong \Gamma \rtimes C_p \), where \( \Gamma \) is an elementary abelian \( p \)-group of rank \( p \).

**Lemma 5.1.3.** Let \( p \) be an odd prime and let \( G \) be a group having a normal elementary abelian \( p \)-subgroup \( \Gamma \) of index \( p \) in \( G \). Suppose that \( \Gamma \) has rank \( p \). If \( a, x \in G \) satisfy \( \ord(a) = \ord(x) = p \) and \( \langle a, x \rangle = G \), then either \( a \) or \( x \) is contained in \( \Gamma \).

**Proof.** Let \( a, x \in G \setminus \Gamma \) satisfy the given properties. Since \( G/\Gamma \) is a cyclic \( p \)-group, by replacing \( x \) by an appropriate power, we may assume without loss of generality that \( a\Gamma = x\Gamma \) and hence that the action of \( a \) on \( \Gamma \) is equal to the action of \( x \) on \( \Gamma \).

Let \( \Delta := \langle a^{-1}x \rangle \times \langle a^{-2}x^2 \rangle \times \cdots \times \langle ax^{-1} \rangle \). We note that as \( a\Gamma = x\Gamma \) each of \( a^{-i}x^i \) occurs in \( \Gamma \) for \( i = 1, \ldots, p-1 \). Hence \( \Delta \subseteq \Gamma \). Conversely, suppose that \( y \in \Gamma \). As \( y \in G = \langle a, x \rangle \), there exist \( b_1, \ldots, b_{2n} \in \mathbb{Z}_p \) (for some \( n \)) such that
\[
y = a^{b_1} x^{b_2} a^{b_3} x^{b_4} \cdots a^{b_{2n-1}} x^{b_{2n}}.
\]
Note that as \( G/\Gamma = \langle a\Gamma \rangle = \langle x\Gamma \rangle \), we have the relation \( \sum_{i=1}^{2n} b_i \equiv 0 \pmod{p} \).

We now proceed to build \( y \) recursively. We assume that \( y \) is written in the form \((5.1.1)\) such that no \( b_i \) is zero. In the case that \( b_1 = 0 \) or \( b_{2n} = 0 \), an analogous argument holds. Define \( y_1 := a^{b_1} x^{-b_1} \). Thus \( y_1 \) agrees with \( y \) in the first position. Now assume that we have defined \( y_i \in \Delta \) for some \( i < 2n \) such that \( y_i = a^{b_1} x^{b_2} \cdots a^{b_i} x^{d} \) or \( y_i = a^{b_1} x^{b_2} \cdots x^{b_i} a^d \) depending on the parity of \( i \). Here \( d := -\sum_{j=1}^{i} b_j \). Thus \( y_i \) agrees with \( y \) on the first \( i \) entries. In the former case define \( y_{i+1} := y_i x^{b_i+1} d a^{d-b_i+1} \) and in the latter case define \( y_{i+1} := y_i a^{b_i+1} d x^{d-b_i+1} \). Thus \( y_{i+1} \in \Delta \) and \( y_{i+1} \) agrees with \( y \) in the first \( i+1 \) positions. Continuing in this way we may define \( y_i \) for \( i = 1, \ldots, 2n \), where each \( y_i \in \Delta \). However,
\[
y_{2n} = a^{b_1} x^{b_2} a^{b_3} x^{b_4} \cdots a^{b_{2n-1}} x^{b_{2n}} a^{-\sum_{j=1}^{2n} b_j} = y,
\]
and hence \( y \in \Delta \). We conclude that \( \Gamma = \Delta \), from which the result follows immediately. \( \square \)
Applying Lemma 5.1.3 to \(W_p\) we obtain the following result that we will use in Section 5.2.

**Corollary 5.1.4.** Let \(p\) be an odd prime and let \(a,x \in W_p\) be elements of order \(p\) such that \(\langle a,x \rangle = W_p\). Then either \([a,a^x] = 1\) for all \(i \in \{1, \ldots, p\}\) or \([x,x^a] = 1\) for all \(i \in \{1, \ldots, p\}\).

**Proof.** Since \(W_p\) satisfies the hypothesis of Lemma 5.1.3 either \(a\) or \(x\) is contained in the base group of \(W_p\), which is elementary abelian.

We conclude this section with a remark.

**Remark 5.1.5.** We may construct the matrices \(A_x\) and \(A_x^2\) in the computer algebra system MAGMA. The appropriate code for doing so, together with code for constructing arbitrary circulant matrices and for checking if a given matrix is circulant is given in Appendix C.

### 5.2 The \(n = p^2\) Case

In the next two sections, we prove Theorem 5.0.1. This theorem reflects the fact that in the wreath product \(C_p \wr C_p\), the \(p\) copies of \(C_p\) are permuted in a \(p\)-cycle, and elements within each copy of \(C_p\) have a circular orbit.

The proof of Theorem 5.0.1 can be split into two separate cases. In Subsection 5.2.1 we will prove the equivalence of \(\langle a,x \rangle \cong W_p\) with part (i) of Theorem 5.0.1. This is proved in Proposition 5.2.2. Due to the technical nature of the proofs in this subsection, we follow this in Subsection 5.2.2 with a worked example mirroring the proofs. The equivalence of parts (i) and (ii) of Theorem 5.0.1 is then proved in Section 5.3 as Theorem 5.3.2.

#### 5.2.1 The Results

We begin by formulating a result relating the conjugation action of \(x\) on \(a\) to the rank of an elementary abelian \(p\)-group.

**Lemma 5.2.1.** Let \(G = \text{Sym}(p^2)\), \(a\) be the standard \(p\)-element of \(G\) and let \(x\) be a conjugate of \(a\). If \(\alpha_j^{-1} = \alpha_1^{e_{x_j}^{-1}}\) for some \(p\)-cycle \(\sigma \in \text{Sym}(p)\), some non-zero \(e_1, \ldots, e_p \in \mathbb{Z}_p\) and for all \(j = 1, \ldots, p\), then the elementary abelian \(p\)-group \(\Gamma := \langle a^x \mid j = 0, \ldots, p-1 \rangle\) has rank \(p\) if and only if \(e_1 + \cdots + e_p^{-1} \neq 0\) (mod \(p\)).

**Proof.** For ease of notation we will assume without loss of generality that \(\sigma = (1,2,\ldots,p)\). Moreover, for simplicity of subsequent arguments we define \(e_{mp+i} := e_i\) for all \(m \in \mathbb{Z}\). Let \(d_1, \ldots, d_p \in \mathbb{Z}_p\) be such that

\[
\prod_{i=1}^{p} (a^{x_i})^{d_i} = 1. \tag{5.2.2}
\]

As \(\alpha_j^{-1} = \alpha_1^{e_{x_j}^{-1}} = \alpha_j^{e_j}\), it follows that \(a^{x^2} = \prod_{j=1}^{p} \alpha_j^{-1} \alpha_j^{e_j+1}\). Iterating this we have that \(a^{x^2} = \prod_{j=1}^{p} \alpha_j^{-1} \alpha_j^{e_j+1}\) and in general \(a^{x^i} = \prod_{j=1}^{p} \alpha_j^{-1} \alpha_j^{e_j+1}\). Thus we may write (5.2.2) as

\[
\prod_{i=1}^{p} \prod_{j=1}^{p} (\alpha_j^{-1} \alpha_j^{e_j+1})^{d_i} = \prod_{i=1}^{p} \left( \prod_{j=1}^{p} \alpha_j^{-1} \alpha_j^{e_j+1} \right)^{d_i} = 1. \tag{5.2.3}
\]
It follows that for a fixed \( \ell \in \{1, \ldots, p\} \), the exponent of \( \alpha \ell \) in \([5.2.3]\) is congruent to 0 (mod \( p \)). Thus we have

\[
\sum_{i=1}^{p} (e_{\ell-1}^{-1}e_{\ell})d_i \equiv 0 \pmod{p}.
\] (5.2.4)

The equations in \([5.2.4]\) give rise to a homogeneous system of linear equations in the \( d_i \) with coefficient matrix given by \( A = (a_{\ell,i}) \), where \( a_{\ell,i} = e_{\ell-1}^{-1}e_{\ell} \). This matrix is invertible if and only if the matrix \( B = (b_{\ell,i}) \) is invertible, where \( b_{\ell,i} = e_{\ell-i}^{-1} \). We may conclude that there is a non-trivial solution to \([5.2.2]\) meaning that the group \( \Gamma \) has rank less than \( p \), precisely when \( \det(B) \equiv 0 \pmod{p} \). However, \( B = \text{circ}(e_p^{-1}, e_{p-1}^{-1}, \ldots, e_1^{-1}) \) and so appealing to Lemma 5.1.2 we deduce that this occurs precisely when \( e_1^{-1} + \cdots + e_p^{-1} \equiv 0 \pmod{p} \).

Lemma 5.2.1 underpins the proof of the equivalence of \( \langle a, x \rangle \cong W_p \) and part \((i)\) of Theorem 5.0.1 as it allows us to relate the coefficients of the circulant matrix - \( A_a^x \) or \( A_x \) - with the base group of the wreath product \( W_p \).

**Proposition 5.2.2.** Let \( G = \text{Sym}(p^2) \), \( a \) be the standard \( p \)-element of \( G \) and let \( x \) be a conjugate of \( a \). The following are equivalent

1. \( \langle a, x \rangle \cong W_p \);
2. \( A_x = p \cdot Y_{\sigma} \) for some \( p \)-cycle \( \sigma \in \text{Sym}(p) \), \( A_a^x = \text{circ}(0, c_1, \ldots, c_{p-1}) \) for some \( c_i \in \mathbb{Z}_p \), \( X = 1 \)
   - is a simple root of the representer polynomial \( f_{A_a^x}(X) \in \mathbb{Z}_p[X] \) and \( [a, x^a] = 1 \) (or with the roles of \( a \) and \( x \) interchanged).

**Proof.** Throughout the proof we write \( a = \alpha_1 \cdots \alpha_p \) and \( x = \chi_1 \cdots \chi_p \) as in Section 5.1. Assume that \( \langle a, x \rangle \cong W_p \). By Corollary 5.1.4 either \( [a, a^x] = 1 \) for all \( i \in \{1, \ldots, p\} \) or \( [x, x^a] = 1 \) for all \( i \in \{1, \ldots, p\} \). Without loss of generality assume that \( [a, a^x] = 1 \) for all \( i = 1, \ldots, p \). An analogous argument can be used in the case that \( [x, x^a] = 1 \).

As \( [a, a^x] = 1 \), either \( \text{supp}(a^x_i) = \text{supp}(a_{i\sigma}) \) for each \( i = 1, \ldots, p \) and some \( p \)-cycle \( \sigma \in \text{Sym}(p) \), or \( |\text{supp}(a^x_i) \cap \text{supp}(a_{j\sigma})| = 1 \) for all \( i, j \). If the former case holds, then \( A_x = p \cdot Y_{\sigma} \). If the latter case holds, then in particular \( |\text{supp}(a^x_i) \cap \text{supp}(a_{i\sigma})| = 1 \). It follows that in the disjoint cycle decomposition of \( x \), there exists a cycle whose support contains two elements of \( \{1, \ldots, p\} \). By taking appropriate powers of \( a \) and \( x \), we deduce that \( \text{ord}(a^ix^j) < p^2 \) for some \( 1 \leq i, j \leq p - 1 \) and hence as \( \langle a, x \rangle \cong W_p \), it follows that \( \text{ord}(a^ix^j) = p \). Hence

\[
1 = (a^i) \cdot (a^ix^{x-j}) \cdot (a^ix^{-2j}) \cdots (a^ix^{(1-p)j}).
\]

Thus as the \( a^x \)-group pairwise commute, they must form an elementary abelian subgroup of order at most \( p^{p-1} \). Moreover, as \( x \) acts on this subgroup via conjugation, it would follow that \( |\langle a, x \rangle| \leq p^p < p^{p+1} = |W_p| \) and hence \( \langle a, x \rangle \ncong W_p \). We conclude that the latter of the two cases cannot hold, and hence \( A_x = p \cdot Y_{\sigma} \). Consequently \( \text{supp}(a^x_i) = \text{supp}(a_{i\sigma}) \) for each \( i \in \{1, \ldots, p\} \).
Define $b_i := \min \text{supp}(\chi_i)$ and write $x$ as a $p \times p$ array as

\[
\begin{align*}
(b_1, b_1\chi_1, b_1\chi_1^2, \ldots, b_1\chi_1^{p-1}) \\
(b_2, b_2\chi_2, b_2\chi_2^2, \ldots, b_2\chi_2^{p-1}) \\
\vdots \\
(b_p, b_p\chi_p, b_p\chi_p^2, \ldots, b_p\chi_p^{p-1})
\end{align*}
\] (5.2.5)

As $[a, a^x] = 1$ for all $i \in \{1, \ldots, p\}$, we see that $\alpha_i^x \in \langle \alpha_{i\sigma} \rangle$ for each $i \in \{1, \ldots, p\}$ and hence the columns of (5.2.5) correspond to powers of the $\alpha_i$. Consequently, $A_a^x = \text{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_1, \ldots, c_p \in \mathbb{N}$ with the sum of the $c_i$'s equal to $p$. In particular $X = 1$ is a root of the representer polynomial $f_{A_a^x}(X) \in \mathbb{Z}_p[X]$.

Define $\Gamma := \langle a^x \mid j = 1, \ldots, p \rangle$. Thus $\Gamma$ is an elementary abelian $p$-group of rank $p$. Moreover $\alpha_i^{x^{-1}} = \alpha_{i\sigma}^{e_i^{-1}}$ for some non-zero $e_1, \ldots, e_p \in \mathbb{Z}_p$. Define $a_i$ for $i = 1, \ldots, p$ recursively by $a_1 := \min \{i \mid c_i \neq 0\}$ and

$$a_i := \min \left\{ \ell \mid \sum_{j=1}^\ell j \cdot c_j > \sum_{j=1}^{i-1} a_j \right\}$$

for $i > 1$.

To understand the correspondence between the $a_i$, $c_i$ and $e_i$, we again consider (5.2.5). We see that the $a_i$ give the number of rows descended in (5.2.5) to go from the entry in the first row of a column to its image under $a$, while each $c_j$ gives the number of times that $j$ rows are descended within a column. Thus $\sum_{i=0}^{p-1} i \cdot c_i = \sum_{i=1}^{p} a_i$. Meanwhile, the $e_i$ give the exponent to which $\alpha_1$ is raised under the image of successive powers of $x$, and hence these are precisely the inverses of the $a_j$.

Consequently, as multisets we have that

$$\{a_i \mid i = 1, \ldots, p\} = \{e_i^{-1} \mid i = 1, \ldots, p\}.$$ 

Appealing to Lemma 5.2.1 we see that, as $\Gamma$ has rank $p$, then $a_1 + \cdots + a_p \not\equiv 0 \pmod{p}$. Consequently

$$f'_{A_a^x}(1) = \sum_{i=0}^{p-1} i \cdot c_i = \sum_{i=1}^{p} a_i \not\equiv 0 \pmod{p},$$

and the representer polynomial of $A_a^x$ has a simple root at $X = 1$ as required.

Conversely, assume that the conditions in (ii) hold. As $A_{x_{\sigma}} = p \cdot Y_{\sigma}$, it follows that supp$(\alpha_i^x) = \text{supp}(\alpha_{i\sigma})$. Moreover, as $[a, a^x] = 1$ we have $\alpha_i^x \in \langle \alpha_{i\sigma} \rangle$. Defining $\Gamma := \langle a^x \mid j = 1, \ldots, p \rangle$ we deduce that $\Gamma$ is an elementary abelian $p$-group upon which $\langle x \rangle$ acts, and $\Gamma \cap \langle x \rangle = \{1\}$. Thus it suffices to prove that rank($\Gamma$) = $p$. However, an analogous argument to that used above involving the $a_i$, $c_i$ and $e_i$ shows that this is equivalent to proving that $f'_{A_a^x}(1) \not\equiv 0 \pmod{p}$, which holds as $f_{A_a^x}(X)$ has a simple root at $X = 1$.

We note that all of the conditions in Proposition 5.2.2 are necessary. Indeed, the necessity of
the structure of $A_x$ and $A_x^x$ was evident in the proof of the proposition. However, if \( p = 5 \) and

\[
x = (1, 6, 11, 16, 21)(2, 7, 14, 18, 25)(3, 9, 15, 17, 23)
\]

\[
(4, 8, 12, 20, 24)(5, 10, 13, 19, 22),
\]

then the matrices $A_x$ and $A_x^x$ are given by

\[
A_x = 5 \cdot \pi \quad \text{and} \quad A_x^x = \text{circ}(0, 2, 1, 1, 1)
\]

both of which satisfy the conditions in the proposition. However, in this case $[a, a^x] \neq 1$. Indeed, $\langle a, x \rangle$ is a group of order 3,888,000,000 and exponent 150.

### 5.2.2 A Worked Example

To illustrate what is happening in the proofs of Lemma 5.2.1 and Proposition 5.2.2, we give an explicit example. Indeed, let $p = 5$ and consider $G = \text{Sym}(25)$, having standard 5-element $a = (1, 2, 3, 4, 5) \cdots (21, 22, 23, 24, 25) \in G$. Using the computer algebra system MAGMA we may take a random element, $x$, of the $G$-conjugacy class of $a$ that satisfies $\langle a, x \rangle \cong W_5$, namely

\[
x = (1, 6, 11, 16, 21)(2, 9, 12, 19, 22)(3, 7, 13, 17, 23)
\]

\[
(4, 10, 14, 20, 24)(5, 8, 15, 18, 25).
\]

We now consider the methodology from the proofs of Lemma 5.2.1 and Proposition 5.2.2 in this specific case.

**Lemma 5.2.1**

We have that

\[
\alpha_1^x = (6, 9, 7, 10, 8) = \alpha_3, \quad \alpha_1^2 = (11, 12, 13, 14, 15) = \alpha_5,
\]

\[
\alpha_1^3 = (16, 19, 17, 20, 18) = \alpha_4, \quad \alpha_1^4 = (21, 22, 23, 24, 25) = \alpha_5,
\]

\[
\alpha_1^5 = (1, 2, 3, 4, 5) = \alpha_1.
\]

Thus - using the notation from Lemma 5.2.1 - $\sigma = (1, 2, 3, 4, 5)$ and $(e_1, e_2, e_3, e_4, e_5) = (1, 3, 1, 3, 1)$. Conjugating $a$ by successive powers of $x$ gives

\[
a^x = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 = a_1^{e_1} \cdot a_2^{e_2} \cdot a_3^{e_3} \cdot a_4^{e_4} \cdot a_5^{e_5},
\]

\[
a^{x^2} = a_1^2 \cdot a_2^3 \cdot a_3^4 \cdot a_4^5 = a_1^{e_1^2} \cdot a_2^{e_2^2} \cdot a_3^{e_3^2} \cdot a_4^{e_4^2} \cdot a_5^{e_5^2},
\]

\[
a^{x^3} = a_1^3 \cdot a_2^4 \cdot a_3^5 = a_1^{e_1^3} \cdot a_2^{e_2^3} \cdot a_3^{e_3^3} \cdot a_4^{e_4^3} \cdot a_5^{e_5^3},
\]

\[
a^{x^4} = a_1^4 \cdot a_2^5 \cdot a_3 = a_1^{e_1^4} \cdot a_2^{e_2^4} \cdot a_3^{e_3^4} \cdot a_4^{e_4^4} \cdot a_5^{e_5^4},
\]

\[
a^{x^5} = a_1^5 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 = a_1^{e_1^5} \cdot a_2^{e_2^5} \cdot a_3^{e_3^5} \cdot a_4^{e_4^5} \cdot a_5^{e_5^5}.
\]

Hence we have confirmed the formula $a^{x^j} = \prod_{j=1}^p a_1^{e_1 \sigma^j} \cdot a_2^{e_2 \sigma^j} \cdot a_3^{e_3 \sigma^j} \cdot a_4^{e_4 \sigma^j} \cdot a_5^{e_5 \sigma^j}$ from the proof of the lemma. Consequently, a non-trivial relation between that $a^{x^j}$ will give a non-trivial solution to $Ax \equiv 0 \pmod p$. 
where the matrix $A = (A_{\ell,j})$ has the form

$$A = \begin{pmatrix}
    e_5^{-1}e_1 & e_4^{-1}e_1 & e_3^{-1}e_1 & e_2^{-1}e_1 & e_1^{-1}e_1 \\
    e_1^{-1}e_2 & e_5^{-1}e_2 & e_4^{-1}e_2 & e_3^{-1}e_2 & e_2^{-1}e_2 \\
    e_2^{-1}e_3 & e_1^{-1}e_3 & e_5^{-1}e_3 & e_4^{-1}e_3 & e_3^{-1}e_3 \\
    e_3^{-1}e_4 & e_2^{-1}e_4 & e_1^{-1}e_4 & e_5^{-1}e_4 & e_4^{-1}e_4 \\
    e_4^{-1}e_5 & e_3^{-1}e_5 & e_2^{-1}e_5 & e_1^{-1}e_5 & e_5^{-1}e_5
\end{pmatrix}.$$  

This has a non-trivial solution precisely when the matrix

$$B = \begin{pmatrix}
    e_5^{-1} & e_4^{-1} & e_3^{-1} & e_2^{-1} & e_1^{-1} \\
    e_1^{-1} & e_5^{-1} & e_4^{-1} & e_3^{-1} & e_2^{-1} \\
    e_2^{-1} & e_1^{-1} & e_5^{-1} & e_4^{-1} & e_3^{-1} \\
    e_3^{-1} & e_2^{-1} & e_1^{-1} & e_5^{-1} & e_4^{-1} \\
    e_4^{-1} & e_3^{-1} & e_2^{-1} & e_1^{-1} & e_5^{-1}
\end{pmatrix} = \text{circ}(e_5^{-1}, e_4^{-1}, e_3^{-1}, e_2^{-1}, e_1^{-1}) = \text{circ}(1, 2, 1, 2, 1)$$

has zero determinant (as the $e_i \in \mathbb{Z}_p$ are invertible). However, by Lemma 5.1.2 we have that

$$\det(B) \equiv e_5^{-1} + e_4^{-1} + e_3^{-1} + e_2^{-1} + e_1^{-1} \equiv 1 + 2 + 1 + 2 + 1 \equiv 2 \pmod{5}.$$  

Thus there are no non-trivial solutions to $Ax \equiv 0 \pmod{5}$ and hence $\langle a^x \rangle_{j = 0, \ldots, 4}$ has rank 5 - as can easily be verified.

**Proposition 5.2.2**

By the work above we have that $[a, a^x] = 1$ and that $\langle a^x \rangle_{j = 0, \ldots, 4}$ is an elementary abelian 5-group of rank 5. Moreover

$$A_x = \begin{pmatrix}
    0 & 5 & 0 & 0 & 0 \\
    0 & 0 & 5 & 0 & 0 \\
    0 & 0 & 0 & 5 & 0 \\
    0 & 0 & 0 & 0 & 5 \\
    5 & 0 & 0 & 0 & 0
\end{pmatrix} = 5 \cdot Y_{(1,2,3,4,5)} \quad \text{and}$$

$$A_{a}^x = \begin{pmatrix}
    0 & 3 & 2 & 0 & 0 \\
    0 & 0 & 3 & 2 & 0 \\
    0 & 0 & 0 & 3 & 2 \\
    2 & 0 & 0 & 0 & 3 \\
    3 & 2 & 0 & 0 & 0
\end{pmatrix} = \text{circ}(0, 3, 2, 0, 0).$$

We define $(c_1, c_2, c_3, c_4) = (3, 2, 0, 0), a_1 = \min\{i | c_i \neq 0\}$ and

$$a_i = \min \left\{ \ell \mid \sum_{j=1}^{\ell} j \cdot c_j > \sum_{j=1}^{i-1} a_j \right\}$$
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for $i = 2, \ldots, 5$. Thus $(a_1, a_2, a_3, a_4, a_5) = (1, 1, 1, 2, 2)$.

Finally, considering the representer polynomial of $A_5^r$ we have that

$$f'_{A_5^r}(1) \equiv (3 + 4X)|_{X=1} \equiv 2 \equiv \sum_{i=1}^{5} a_i \not\equiv 0 \pmod{5}$$

and hence $X = 1$ is a simple root of $f_{A_5^r}(X) \in \mathbb{Z}_5[X]$ as required.

We note that by considering $x$ as a $5 \times 5$ array of the form

$$x = \begin{pmatrix} 1, & 6, & 11, & 16, & 21 \\ 2, & 9, & 12, & 19, & 22 \\ 3, & 7, & 13, & 17, & 23 \\ 4, & 10, & 14, & 20, & 24 \\ 5, & 8, & 15, & 18, & 25 \end{pmatrix},$$

the image of each entry in the first column of $x$ under the action of $a$ is in the same column, and one row below the original entry. Similarly, for columns 2, 3, 4 and 5 the action of $a$ corresponds to “descending” by $2, 1, 2$ and $1$ rows respectively. Thus the image of 1 under $(ax)^5$ will be in the first column of $x$ in (5.2.6), but will have descended $2 \equiv 1 + 2 + 1 + 2 \equiv \sum_{i=1}^{5} a_i \pmod{5}$ rows.

5.3 An Equivalent Formulation

Our characterisation of $\langle a, x \rangle$ in terms of the matrices $A_x$ and $A_x^r$ given in Section 5.2 depended on the representer polynomial of a given circulant matrix. However, we may also consider the determinant of the circulant matrix. This characterisation relies on the following well known result regarding the determinant of a circulant matrix.

**Theorem 5.3.1.** [KST12, Theorem 6] Let $C = \text{circ}(c_0, c_1, \ldots, c_{n-1})$ be a circulant matrix with complex coefficients. Then

$$\det(C) = \prod_{j=0}^{n-1} \left( \sum_{i=0}^{n-1} c_i \omega^j \right),$$

where $\omega = \exp(2\pi i/n)$.

We now describe this approach using determinants.

**Proposition 5.3.2.** Let $p \geq 3$ be a prime and let $c_1, \ldots, c_{p-1}$ be non-negative integers such that $\sum_{i=1}^{p-1} c_i = p$. Let $C = \text{circ}(0, c_1, \ldots, c_{p-1})$, and let $f_C(X) = \sum_{i=1}^{p-1} c_i X^i \in \mathbb{Z}_p[X]$ be the representer polynomial of $C$. Then $|\det(C)|_p$ is divisible by $p^2$. Furthermore $|\det(C)|_p$ is divisible by $p^3$ if and only if $f_C(X) = 0$ or 1 is a root of $f_C(X)$ of multiplicity at least 2.

Before proceeding to prove Proposition 5.3.2 we first give three general results that we will require in the proof.

**Lemma 5.3.3.** Let $p \geq 3$ be a prime and $\omega = \exp(2\pi i/p)$ be a primitive $p^{th}$ root of unity. Then $\mathbb{Q} \cap \mathbb{Z}[\omega] = \mathbb{Z}$. In particular, $1/p^i \not\in \mathbb{Z}[\omega]$ for $i \geq 1$. 

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Proof. We note that $S = \{1, \omega, \ldots, \omega^{p-2}\}$ is a basis for both $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ and $\mathbb{Z}[\omega]$ over $\mathbb{Z}$. Suppose that $\alpha \in (\mathbb{Q} \cap \mathbb{Z}[\omega]) \setminus \mathbb{Z}$. Then $\alpha$ can be written as a $\mathbb{Z}$-linear combination of elements of $S$, thus giving two distinct ways of writing $\alpha$ as a $\mathbb{Q}$-linear combination of elements of $S$ in $\mathbb{Q}[\omega]$. This contradiction gives the result.

Lemma 5.3.4. Let $p \geq 3$ be a prime and let $\omega = \exp(2\pi i/p)$. Then

(i) $\prod_{j=1}^{p-1} (1 - \omega^{2j}) = p$; and

(ii) $\prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) = p$.

Proof. Set $h(z) = z^p - 1 = \prod_{j=0}^{p-1}(z - \omega^{2j})$. Differentiating $h(z)$ with respect to $z$ gives

$$p z^{p-1} = h'(z) = \prod_{j=1}^{p-1} (z - \omega^{2j}) + (z - 1) \sum_{j=1}^{p-1} \prod_{i=1 \atop i \neq j}^{p-1} (z - \omega^{2i}).$$

It follows that $p = h'(1) = \prod_{j=1}^{p-1} (1 - \omega^{2j})$ as required to prove (i).

Part (ii) then follows immediately since

$$\prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) = \prod_{j=1}^{p-1} \omega^j (1 - \omega^{(p-2)j})$$

$$= \omega^{p(p-1)/2} \prod_{j=1}^{p-1} (1 - \omega^{(p-2)j})$$

$$= \prod_{j=1}^{p-1} (1 - \omega^{(p-2)j}) \prod_{j=1}^{p-1} (1 - \omega^{2j}) = p.$$

Lemma 5.3.5. Let $p \geq 3$ be a prime. Then

$$1 + \sum_{i=2}^{p-1} \prod_{j=2}^{i} (1 - j^{-1}) \equiv 0 \pmod{p}. \quad (5.3.7)$$

Proof. As the elements $(1 - j^{-1})$ for $j = 2, \ldots, p - 1$ are the non-zero, non-identity elements of $\mathbb{Z}_p$, it follows that $\prod_{j=2}^{p-1} (1 - j^{-1}) \equiv p - 1 \pmod{p}$. Thus (5.3.7) is equivalent to proving that $\sum_{i=2}^{p-1} \prod_{j=2}^{i} (1 - j^{-1}) \equiv 0 \pmod{p}$. We will actually prove that

$$1 + \prod_{j=i+1}^{p-i} (1 - j^{-1}) \equiv 0 \pmod{p} \quad (5.3.8)$$

for all $i = 2, \ldots, (p - 1)/2$. This is sufficient to prove the result, since

$$\prod_{j=2}^{i} (1 - j^{-1}) + \prod_{j=2}^{p-i} (1 - j^{-1}) = \prod_{j=2}^{i} (1 - j^{-1}) \left(1 + \prod_{j=i+1}^{p-i} (1 - j^{-1})\right)$$

$$\equiv 0 \pmod{p},$$

from which the result follows immediately.
We have that (5.3.8) holds by induction on $\ell = (p+1)/2 - i$. When $\ell = 1$, then $i = (p-1)/2$ and so (5.3.8) becomes $1 + (1 - ((p+1)/2)^{-1}) \equiv 0 \pmod{p}$, which clearly holds since $((p+1)/2)^{-1} = 2 \pmod{p}$. Assume that (5.3.8) holds for $\ell = (p+1)/2 - k < (p+1)/2$. It follows that

$$1 + \prod_{j=k}^{p-k+1} (1 - j^{-1}) \equiv 1 + (1 - k^{-1})(1 - (p - k + 1)^{-1}) \prod_{j=k+1}^{p-k} (1 - j^{-1})$$

$$\equiv 1 - (1 - k^{-1})(1 - (p - k + 1)^{-1})$$

$$\equiv k^{-1} + (p - k + 1)^{-1} - k^{-1}(p - k + 1)^{-1}$$

$$\equiv k^{-1}(p - k + 1)^{-1}(p - k + 1 + k - 1)$$

$$\equiv 0 \pmod{p}.$$

Thus by induction (5.3.8) holds for all $i = 2, \ldots, (p - 1)/2$, as required to complete the proof.

We are now in a position to prove Proposition 5.3.2.

**Proof of Proposition 5.3.2** If $c_i = p$ for some $i$, then $C = p \cdot \pi^i$. Thus the result clearly holds since $|\det(C)|_p = p^p$ so is divisible by $p^3$, whilst $f_C(X) = 0 \in \mathbb{Z}_p[X]$. Conversely, if $f_C(X) = 0 \in \mathbb{Z}_p[X]$, then some $c_i$ must equal $p$, and hence $|\det(C)|_p = p^q$. Consequently, we may restrict to the case that $f_C(X) \neq 0$ and hence that $c_i < p$ for each $i = 1, \ldots, p - 1$.

Consider the general form of the determinant of $C$ and define

$$q_j = c_1\omega^j + c_2\omega^{2j} + \cdots + c_{p-2}\omega^{(p-2)j} - (c_1 + \cdots + c_{p-2})\omega^{(p-1)j}$$

$$= c_1(\omega^j - \omega^{(p-1)j}) + c_2(\omega^{2j} - \omega^{(p-1)j}) + \cdots + c_{p-2}(\omega^{(p-2)j} - \omega^{(p-1)j}),$$

where $\omega = \exp(2\pi i/p)$. As $c_1 + \cdots + c_{p-1} = p$, applying Theorem 5.3.1 to $C = \text{circ}(0, c_1, \ldots, c_{p-1})$ we have that

$$\det(C) = \prod_{j=0}^{p-1} \left( c_1\omega^j + c_2\omega^{2j} + \cdots + c_{p-2}\omega^{(p-2)j} \right.\right.$$

$$+ (p - c_1 - \cdots - c_{p-2})\omega^{(p-1)j}\left.\right)$$

$$= p \cdot \prod_{j=1}^{p-1} \left( c_1\omega^j + c_2\omega^{2j} + \cdots + c_{p-2}\omega^{(p-2)j} \right.\right.$$

$$+ (p - c_1 - \cdots - c_{p-2})\omega^{(p-1)j}\left.\right)$$

$$= p \cdot \prod_{j=1}^{p-1} (q_j + p \cdot \omega^{(p-1)j}).$$
We now extract the terms in this expansion with a factor of $p$, $p^2$ and at least $p^3$. Define

$$
\alpha(C) = \sum_{i=1}^{p-1} \sum_{k=1}^{p-1} \omega^{-i-k} \prod_{j=1 \atop j \neq i}^{p-1} (q_j + p \cdot \omega^{(p-1)j}),
$$

$$
\beta(C) = \sum_{i=1}^{p-1} \omega^{-i} \prod_{j=1 \atop j \neq i}^{p-1} q_j,
$$

$$
\gamma(C) = \prod_{j=1}^{p-1} q_j.
$$

It follows that

$$
\det(C) = p^3 \cdot \alpha(C) + p^2 \cdot \beta(C) + p \cdot \gamma(C) \quad (5.3.9)
$$

We now consider $\beta(C)$ and $\gamma(C)$ in turn.

**Claim 1:** $\beta(C) = p \cdot \beta'(C)$ for some $\beta'(C) \in \mathbb{Z}[\omega]$.

**Proof of Claim 1:** Setting

$$
\beta_j(C) = \frac{q_j}{w^j - w^{(p-1)j}},
$$

we see that

$$
\beta_j(C) = c_1 + c_2(1 + \omega^j + \omega^{2j} \cdots + \omega^{(p-1)j} + \omega^j) + c_3(1 + \omega^2j + \omega^{3j} \cdots + \omega^{(p-1)j} + \omega^j + \omega^{3j}) + c_5(1 + \omega^{2j} + \omega^{4j}) + \cdots + c_{p-2}(1 + \omega^{2j} + \omega^{4j} \cdots + \omega^{(p-3)j}).
$$

Consider the formulation of $\beta(C)$ as:

$$
\beta(C) = \sum_{i=1}^{p-1} \omega^{-i} \prod_{j=1 \atop j \neq i}^{p-1} \left(c_1(\omega^j - \omega^{(p-1)j}) + \cdots + c_{p-2}(\omega^{(p-2)j} - \omega^{(p-1)j})\right)
$$

$$
= \sum_{i=1}^{p-1} \omega^{-i} \prod_{j=1 \atop j \neq i}^{p-1} (\omega^j - \omega^{(p-1)j}) \beta_j(C)
$$

$$
= \sum_{i=1}^{p-1} \omega^{-i} \prod_{j=1 \atop j \neq i}^{p-1} (\omega^j - \omega^{(p-1)j}) \prod_{j=1 \atop j \neq i}^{p-1} \beta_j(C). \quad (5.3.10)
$$

If $k \equiv -(j+1)^{-1} \pmod{p}$, then $2k(j+1) + 1 \equiv -1 \pmod{p}$. Consequently

$$
\delta_j := \omega + \omega^{2(j+1)+1} + \cdots + \omega^{2k(j+1)+1} + \cdots + \omega^{-1}
$$

has $1 - (j+1)^{-1} \pmod{p}$ terms. As a consequence, we have the relation

$$
\omega^{p-j} - \omega^j = (\omega^{p-j-1} - \omega^{j+1})(\omega + \omega^{2j+3} + \cdots + \omega^{2k(j+1)+1} + \cdots + \omega^{-1})
$$

$$
= (\omega^{p-j-1} - \omega^{j+1})\delta_j \quad (5.3.11)
$$
for $j = 1, \ldots, p - 2$.

Let $i \in \{1, \ldots, p - 2\}$. By (5.3.11) we have that

$$\omega^{p-1} - \omega = \delta_1 \cdots \delta_{p-i-1}(\omega^i - \omega^{p-i}).$$  

(5.3.12)

Applying (5.3.12) to (5.3.10) we obtained

$$\beta(C) = \omega \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-2} \beta_j(C) + \sum_{i=1}^{p-2} \omega^{-i} \prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-1} \beta_j(C)$$

$$= \omega \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-2} \beta_j(C) + \sum_{i=1}^{p-2} \omega^{-i} \delta_1 \cdots \delta_{p-i-1} \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-1} \beta_j(C)$$

$$= \Gamma(C) \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}),$$

(5.3.13)

where

$$\Gamma(C) := \omega \prod_{j=1}^{p-2} \beta_j(C) + \sum_{i=1}^{p-2} \omega^{-i} \delta_1 \cdots \delta_{p-i-1} \prod_{j=1}^{p-1} \beta_j(C).$$

Define $\Gamma(C)(X) \in \mathbb{Z}_p[X]$ to be the polynomial obtained by replacing every occurrence of $\omega$ in $\Gamma(C)$ by the indeterminate $X$. Similarly define $\beta_j(C)(X), \delta_j(X) \in \mathbb{Z}_p[X]$. Since $\beta_j(C)(1) = \beta_i(C)(1)$ for all $i, j = 1, \ldots, p - 1$, we may denote this common value by $\xi$. Moreover, our observation above asserts that

$$\delta_j(1) \equiv 1 - (j + 1)^{-1} \pmod{p}.$$  

It follows that

$$\Gamma(C)(1) \equiv \xi^{p-2} + \sum_{i=1}^{p-2} (1 - 2^{-1}) \cdots (1 - (p - i)^{-1}) \xi^{p-2}$$

$$\equiv \xi^{p-2} \left( 1 + \sum_{i=1}^{p-2} \prod_{j=2}^{p-i} (1 - j^{-1}) \right)$$

$$\equiv \xi^{p-2} \left( 1 + \sum_{i=2}^{p-1} \prod_{j=2}^{i} (1 - j^{-1}) \right)$$

$$\equiv 0 \pmod{p}$$

where the last equivalence arises from Lemma 5.3.5. Consequently, $(X - 1)$ is a factor of $\Gamma(C)(X) \in \mathbb{Z}_p[X]$ and hence evaluating $\Gamma(C)(X)$ at $X = \omega$ yields

$$\Gamma(C) = (\omega - 1)F(\omega) + pG(\omega)$$

for some $F(\omega), G(\omega) \in \mathbb{Z}[\omega]$. Substituting into (5.3.13) gives

$$\beta(C) = \left( \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}) \right) ((\omega - 1)F(\omega) + pG(\omega)).$$
Finally, as
\[ \omega - 1 = (\omega^{p-1} - \omega)(\omega^2 + \omega^4 + \cdots + \omega^{p-1}) \]
we may appeal to Lemma 5.3.4(ii) to get
\[ \beta(C) = \left( \prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) \right) (\omega^2 + \omega^4 + \cdots + \omega^{p-1}) F(\omega) \]
\[ + p \left( \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}) \right) G(\omega) \]
\[ = p(\omega^2 + \omega^4 + \cdots + \omega^{p-1}) F(\omega) + p \left( \prod_{j=1}^{p-2} (\omega^j - \omega^{(p-1)j}) \right) G(\omega) \]
\[ = p\beta'(C) \]
as required.

Now we consider \( \gamma(C) \).

**Claim 2:** \( \gamma(C) = p \cdot \gamma'(C) \) for some \( \gamma'(C) \in \mathbb{Z} \).

**Proof of Claim 2:** We may factorise \( \gamma(C) \) as
\[ \gamma(C) = \prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) \left( c_1 + c_2(1 + \omega^{2j} + \cdots + \omega^{(p-1)j}) + c_3(1 + \omega^{2j}) \right) \]
\[ + c_4(1 + \omega^{2j} + \cdots + \omega^{(p-1)j}) + \cdots + c_p(1 + \omega^{2j} + \cdots + \omega^{(p-1)j}) \]
\[ = \prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) \beta_j(C) \]
\[ = \prod_{j=1}^{p-1} (\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-1} \beta_j(C) \]
\[ = p \cdot \prod_{j=1}^{p-1} \beta_j(C) \quad (5.3.14) \]
where the last equality in (5.3.14) follows from Lemma 5.3.4(ii). If we consider \( \prod_{j=1}^{p-1} \beta_j(C) \), then by taking \( s_i \) to be the coefficient of \( \omega^i \) in \( \beta_j(C) \) we see that the \( s_i \) are non-negative integers and
\[ s_0 = c_1 + \cdots + c_p = p - c_{p-1} \] is non-zero. Defining \( S = \text{circ}(s_0, s_1, \ldots, s_{p-1}) \), we see that \( \det(S) \in \mathbb{Z} \) and so
\[ \prod_{j=1}^{p-1} \beta_j(C) = \frac{1}{s_0 + s_1 + \cdots + s_{p-1}} \det(S) \in \mathbb{Q} \cap \mathbb{Z}[\omega]. \]
Appealing to Lemma 5.3.3, we conclude that \( \prod_{j=1}^{p-1} \beta_j(C) \in \mathbb{Z} \) as required to prove Claim 2.

It follows from (5.3.9) that
\[ \det(C) = p^2 \left( p\alpha(C) + p\beta'(C) + \gamma'(C) \right). \]
Since \( p\alpha(C) + p\beta(C) + \gamma'(C) = \det(C)/p^2 \in \mathbb{Q} \cap \mathbb{Z}[\omega] = \mathbb{Z} \), we conclude that \( \det(C) \) is divisible by \( p^2 \). To complete the proof of the proposition, we make one final claim:

**Claim 3:** \( \gamma'(C) = p \cdot \gamma''(C) \) for some \( \gamma''(C) \in \mathbb{Z}[\omega] \) if and only if \( X = 1 \) is a root of \( f_C(X) \in \mathbb{Z}_p[X] \) of multiplicity at least 2.

**Proof of Claim 3:** First note that \( X = 1 \) is a root of \( f_C(X) \) of multiplicity at least 2 if and only if \( X = 1 \) is a root of the derivative \( f'_C(X) \) of \( f_C(X) \). This occurs precisely when \( \sum_{i=1}^{p-1} i \cdot c_i \equiv 0 \pmod{p} \). Moreover

\[
\sum_{i=1}^{p-1} i \cdot c_i = \sum_{i=1}^{p-2} i \cdot c_i + (p-1) \left( p - \sum_{i=1}^{p-2} c_i \right) \\
\equiv \sum_{i=1}^{p-2} (i + 1) \cdot c_i \pmod{p} \\
= \sum_{i=1}^{(p-3)/2} \left( 2i \cdot c_{2i-1} + (2i + 1) \cdot c_{2i} \right) + (p-1) \cdot c_{p-2} \pmod{p}. 
\]

Thus it suffices to prove that \( \gamma'(C) \) has the desired factorisation precisely when

\[
\sum_{i=1}^{(p-3)/2} \left( 2i \cdot c_{2i-1} + (2i + 1) \cdot c_{2i} \right) + (p-1) \cdot c_{p-2} \equiv 0 \pmod{p}. 
\]

In fact we will prove that

\[
\gamma'(C) \equiv \left( \sum_{i=1}^{(p-3)/2} \left( i \cdot c_{2i-1} + \left( i + \frac{p+1}{2} \right) c_{2i} \right) + \frac{p-1}{2} \cdot c_{p-2} \right)^{p-1} \pmod{p} \tag{5.3.15} 
\]

We note that

\[
\gamma'(C) = \prod_{j=1}^{p-1} \left( c_1 + c_2 (1 + \omega^{2j}) + \cdots + \omega^{(p-1)j} + \omega^j \right) + c_3 (1 + \omega^{2j}) \\
+ c_4 (1 + \omega^{2j}) + \cdots + \omega^{(p-1)j} + \omega^j + \omega^{3j} \\
+ c_5 (1 + \omega^{2j} + \omega^{3j}) + \cdots + c_{p-2} (1 + \omega^{2j} + \cdots + \omega^{(p-3)j}) \\
= \prod_{j=1}^{p-1} \left( (c_1 + c_3 + \cdots + c_{p-2} + c_2 + c_4 + \cdots + c_{p-3}) \\
+ (c_3 + \cdots + c_{p-2} + c_2 + c_4 + \cdots + c_{p-3}) \omega^{2j} \\
+ (c_5 + \cdots + c_{p-2} + c_2 + c_4 + \cdots + c_{p-3}) \omega^{4j} \\
+ \cdots + (c_2 + c_4 + \cdots + c_{p-3}) \omega^{(p-1)j} + (c_2 + c_4 + \cdots + c_{p-3}) \omega^j \\
+ (c_4 + c_6 + \cdots + c_{p-3}) \omega^{3j} + \cdots + c_{p-3} \omega^{(p-4)j} \right). \tag{5.3.16} 
\]

Now let \( C' \) be the circulant matrix \( C' = \text{circ}(c'_0, \ldots, c'_{p-1}) \) where \( c'_i \) is the coefficient of \( \omega^j \) in the \( j \)th factor of (5.3.16) and set \( \lambda := \sum_{i=0}^{p-1} c'_i \). Since \( \gamma'(C) = \prod_{j=1}^{p-1} \sum_{i=0}^{p-1} c'_i \omega^{ij} \), we see that \( \det(C') = \lambda \cdot \gamma'(C) \). An alternative formulation of \( \lambda \) is given by

\[
\lambda = \left( \sum_{i=1}^{(p-3)/2} i \cdot c_{2i-1} + \left( i + \frac{p+1}{2} \right) c_{2i} \right) + \frac{p-1}{2} \cdot c_{p-2}. \tag{5.3.17} 
\]
There are now two possible cases to consider. First suppose that \( \lambda \) is divisible by \( p \). Then an analogous argument to that used for the matrix \( C \lambda \) can be used to show that \( \det(C') \) is divisible by \( p^2 \) (as the coefficient of \( \omega^{(p-2)j} \) is zero, and so the matrix is of the required form). Since \( \lambda < p^2 \), it follows that \( \gamma'(C) \) is divisible by \( p \), and hence Claim 3 holds.

Now suppose that \( \lambda \) is not divisible by \( p \). Then \( 5.3.15 \) becomes \( \gamma'(C) \equiv 1 \pmod{p} \) by Fermat’s Little Theorem. Combining our formulation of \( \det(C') \) and Lemma 5.1.2 we have that \( \lambda \equiv \det(C') \equiv \lambda \cdot \gamma'(C) \pmod{p} \), and hence as \( \lambda \not\equiv 0 \pmod{p} \) it follows that \( \gamma'(C) \equiv 1 \pmod{p} \) as required. This completes the proof of Claim 3.

To complete the proof of Proposition 5.3.2 we recall that we have already reduced to the case that \( f_C(X) \neq 0 \in \mathbb{Z}_p[X] \). Substituting the results of Claims 1 and 2 into \( 5.3.3 \) we obtain

\[
\det(C) = p^3(\alpha(C) + \beta'(C)) + p^2\gamma'(C).
\]

If \( X = 1 \) is a root of \( f_C(X) \in \mathbb{Z}_p[X] \) of multiplicity at least 2, then by Claim 3 we obtain

\[
\det(C) = p^3(\alpha(C) + \beta'(C)) + \gamma''(C),
\]

and hence as \( \gamma''(C) \) is divisible by \( p \) we have \( \det(C) \equiv 0 \pmod{p^3} \). Conversely, if \( X = 1 \) is not a root of multiplicity at least 2 of \( f_C(X) \), then Claim 3 asserts that there is no \( \gamma''(C) \in \mathbb{Z}[\omega] \) such that \( \gamma'(C) = p \cdot \gamma''(C) \). Thus \( \det(C) = \alpha(C) + \beta'(C) + p^2\gamma'(C) \) and as \( \gamma'(C) \in \mathbb{Z} \), appealing to Lemma 5.3.3 we obtain that \( \det(C) = p^3m + p^2n \) where \( p \) and \( n \) are coprime. Thus \( |\det(C)|_p = p^2 \).

\[\square\]

Theorem 5.0.1 now follows immediately from Propositions 5.2.2 and 5.3.2.

### 5.4 The \( n = p^3 \) Case

To consider the case that \( G = \text{Sym}(p^3) \), we follow a similar approach to that used in Section 5.2. We first develop the theoretical results in Subsections 5.4.1 and 5.4.2 before giving a worked example for the case that \( p = 3 \) in Subsection 5.4.3.

#### 5.4.1 General Theory

In Section 5.3 we saw that the multiplicity of \( X = 1 \) as a root of the representer polynomial of given circulant matrices was important. We begin by noting that the multiplicity is preserved under cyclic shifts of our circulant matrix.

**Lemma 5.4.1.** Let \( p \) be a prime and \( C := \text{circ}(c_0, c_1, \ldots, c_{p-1}) \) be an integer circulant matrix. If \( X = 1 \) is a root of multiplicity \( i \) of \( f_C(X) \in \mathbb{Z}_p[X] \), then it is a root of multiplicity \( i \) of \( f_{\pi^jC}(X) \in \mathbb{Z}_p[X] \) for each \( j = 0, \ldots, p - 1 \).

**Proof.** Define \( C_j = \pi^{-j}C \) for each \( j = 0, \ldots, p - 1 \). We shall prove that \( X = 1 \) is a root of multiplicity \( i \) of the representer polynomial of each \( C_j \). By assumption the result holds for \( j = 0 \) and hence assume it holds for \( j = k \). We will prove the result holds for \( C_{k-1} \). Indeed

\[
f_{C_k}(X) = c_k + c_{k+1}X + \cdots + c_{p-1}X^{p-k-1} + c_0X^{p-k} + \cdots + c_{k-1}X^{p-1} = (X - 1)^k g(X)
\]

where \( g(X) \) is a polynomial of degree less than \( p \). Since \( X = 1 \) is a root of \( f_{C_k}(X) \), then

\[
(X - 1)^k g(X) = 0.
\]

This implies that \( g(X) \) is divisible by \( (X - 1)^k \), and hence \( X = 1 \) is a root of multiplicity \( k \) of \( f_{C_{k-1}}(X) \). Therefore, the multiplicity is preserved under cyclic shifts of our circulant matrix.

\[\square\]
for some \( g(X) \in \mathbb{Z}_p[X] \) with \( g(1) \not\equiv 0 \pmod{p} \). Thus

\[
\begin{align*}
f_{C_{k-1}}(X) &= c_{k-1} + c_K X + \cdots + c_{p-1}X^{p-k} + c_0X^{p+1-k} + \cdots + c_{k-2}X^{p-1} \\
&= f_{C_k}(X) \cdot X + c_{k-1}(1 - X^p) \\
&= (X - 1)^i g(X) \cdot X + c_{k-1}(1 - X)^p.
\end{align*}
\]

Hence \( X = 1 \) is a root of multiplicity \( i \) of \( f_{C_{k-1}}(X) \), and thus by induction of \( f_{C_j}(X) \) for each \( j = 0, \ldots, p - 1 \).

We may use Lemma 5.4.1 to consider quotients of \( W_p \).

**Lemma 5.4.2.** Let \( N \) be a non-trivial normal subgroup of \( W_p \) and define \( Q := W_p/N \). Then \( \text{Exp}(Q) \leq p \).

**Proof.** Denote the base group of \( W_p \) by \( \Gamma \), so that \( W_p = \Gamma \rtimes C_p \) where \( C_p = \langle x \rangle \) - a cyclic group of order \( p \). Assume for sake of contradiction that \( \text{Exp}(Q) = p^2 \), and let \( q \in Q \) satisfy \( \text{ord}(q) = p^2 \).

We consider elements of \( W_p \) to be ordered pairs of the form \((\gamma, c)\) for \( \gamma \in \Gamma \) and \( c \in C_p \). Thus there exists \( \gamma \in \Gamma \) such that \( q = (\gamma, x^i)N \) for some \( i = 1, \ldots, p - 1 \). Thus

\[
q^p = (\gamma^{x^{-i}} \gamma^{x^{-2i}} \cdots \gamma^{x^{(1-p)i}}, 1)N = (\gamma \gamma^{x^2} \cdots \gamma^{x^{p-1}}, 1)N. \tag{5.4.18}
\]

Moreover, \( \gamma \gamma^{x^2} \cdots \gamma^{x^{p-1}} \not\in N \) as \( \text{ord}(q) = p^2 \).

If \( N \not\leq \Gamma \), then there exists \( \gamma_0 \in \Gamma \) such that \( (\gamma_0, x^{-i}) \in N \). Consequently

\[
q = (\gamma, x^i)(\gamma_0, x^{-i})N = (\gamma \gamma^{x^{-i}}, 1)N
\]

contradicting the fact that \( q \) has order \( p^2 \). Thus \( N \leq \Gamma \).

If \( \Gamma = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \cdots \times \langle \alpha_p \rangle \) where \( \alpha_i^p = \alpha_{i-1} \) for \( i = 2, \ldots, p \) and \( \alpha_1^p = \alpha_p \), then from (5.4.18) we may deduce that

\[
1 \neq \gamma \gamma^{x^2} \cdots \gamma^{x^{p-1}} = (\alpha_1 \alpha_2 \cdots \alpha_p)^j \tag{5.4.19}
\]

for some \( j = 1, \ldots, p - 1 \). Moreover, for any \( \delta \in \Gamma \) we have that

\[
\delta \delta^{x^2} \cdots \delta^{x^{p-1}} \in \langle \alpha_1 \alpha_2 \cdots \alpha_p \rangle.
\]

If there exists \( \delta \in N \) such that \( \delta \delta^{x^2} \cdots \delta^{x^{p-1}} \neq 1 \), then \( q^p \) is trivial in \( Q \). Thus every \( \delta \in N \) must satisfy \( \delta \delta^{x^2} \cdots \delta^{x^{p-1}} = 1 \). Consequently, every \( \delta \in N \) has the form

\[
\delta = \alpha_1^{c_1} \alpha_2^{c_2} \cdots \alpha_p^{c_p} \tag{5.4.20}
\]

with \( c_1 + \cdots + c_p \equiv 0 \pmod{p} \).

Let \( \delta \in N\setminus\{1\} \). Thus \( \delta \) has the form (5.4.20). Set \( C = \text{circ}(c_1, c_2, \ldots, c_p) \) and let \( f_C(X) \in \mathbb{Z}_p[X] \) be the corresponding representer polynomial. As \( c_1 + \cdots + c_p \equiv 0 \pmod{p} \) we have that \( X = 1 \) is a root of \( f_C(X) \). Assume that this root has multiplicity \( i \). Thus \( X = 1 \) is a root of \( f_{C_{i-1}}(X) \) but is not a root of \( f_{C_i}(X) \). Hence set

\[
\lambda := f_{C_i}(1) = c_{i+1} \cdot \frac{i!}{0!} + c_{i+2} \cdot \frac{(i + 1)!}{1!} + \cdots + c_p \cdot \frac{(p - 1)!}{(p - 1 - i)!} \in \mathbb{Z}_p^*.
\]
Define $d_j$ for $j = 1, \ldots, p$ by
\[ d_{j+1} := \begin{cases} 
\frac{j!}{(j - i)!} & \text{if } j = i, \ldots, p - 1; \text{ and} \\
0 & \text{otherwise},
\end{cases} \]
so that $\lambda = \sum_{j=1}^{p} c_j d_j$.

Defining $C_j = \pi^{1-j}C$ for $j = 1, \ldots, p$ we have that $f^i_{C_1}(1) = f^i_{C_2}(1) = \lambda$. In fact $f^i_{C_k}(1) = \lambda$ for all $k = 1, \ldots, p$. Indeed, by Lemma 5.4.1 we have that $f^{i-1}_{C_k}(1) = 0$ for each $k$. Moreover by iterating the calculations in the proof of the lemma we obtain
\[ f_{C_k}(X) = f_{C_1}(X) \cdot X^{p-k+1} + (X - 1)^p g_k(X) \]
for some $g_k(X) \in \mathbb{Z}_p[X]$. Consequently, $f^{i-1}_{C_k}(1) = f^i_{C_1}(1) = \lambda$. We conclude that
\[
\begin{bmatrix}
c_1 & c_2 & \cdots & c_{p-1} & c_p \\
c_p & c_1 & \cdots & c_{p-2} & c_{p-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
c_3 & c_4 & \cdots & c_1 & c_2 \\
c_2 & c_3 & \cdots & c_p & c_1
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{p-1} \\
d_p
\end{bmatrix}
= \begin{bmatrix}
\lambda \\
\lambda \\
\vdots \\
\lambda
\end{bmatrix}.
\]
As $\delta$ has the form given in (5.4.20) we see that
\[
(\alpha_1 \alpha_2 \cdots \alpha_p)^\lambda = \prod_{i=1}^{p} (\delta^{x_i-1})^{d_i} \in N.
\]
Thus by (5.4.18) and (5.4.19), $q^p = N$, contradicting the fact that $\text{ord}(q) = p^2$. Thus the result holds.

A presentation of $W_p$ was given by Drozd and Skuratovskii.

**Theorem 5.4.3.** [DrozdSkuratovskii2008] The wreath product $C_p \wr C_p$ has a presentation given by
\[ C_p \wr C_p = \langle a, x \mid a^p = 1, x^p = 1, [a, a^{x^i}] = 1 \text{ for } 1 \leq k \leq (p - 1)/2 \rangle. \]

Combining Corollary 5.1.4, Lemma 5.4.2 and Theorem 5.4.3 we obtain the following result.

**Corollary 5.4.4.** Let $G$ be a group and let $a, x \in G$ be elements of order $p$. Then $\langle a, x \rangle \cong W_p$ if and only if $\text{ord}(ax) = p^2$ and either

(i) $[a, a^{x^i}] = 1$ for $i = 1, \ldots, (p - 1)/2$; or

(ii) $[x, a^{x^i}] = 1$ for $i = 1, \ldots, (p - 1)/2$.

**Proof.** Assume that $\langle a, x \rangle \cong W_p$. By Corollary 5.1.4 the desired commutator relations hold. Without loss assume that $[a, a^{x^i}] = 1$ for $i = 1, \ldots, (p - 1)/2$. Thus $a$ is in the base group of $W_p$. Moreover, the base group is defined by $\langle a^{x^i} \mid i = 0, \ldots, p - 1 \rangle$ and hence $(ax)^p = a \cdot a^{x^{p-1}} \cdots a^x \neq 1$. Thus $\text{ord}(ax) = p^2$.

Conversely, assume that the commutator relations hold. By Theorem 5.4.3 we have that $\langle a, x \rangle$ is isomorphic to a quotient of $W_p$. However, as $\text{ord}(ax) = p^2$ this quotient must be $W_p$ by Lemma 5.4.2.

To generalise the results of Sections 5.2 and 5.3 we begin by considering conjugate elements of full support in Sym($p^3$). Indeed, let $G := \text{Sym}(p^3)$, $a \in G$ be the standard $p$-element of $G$, $\mathcal{X} = a^G$ and let $x \in \mathcal{X}$. In such a situation the matrices $A_x$ and $A_a^x$ are both $p^2 \times p^2$ matrices. We make the following definition.

**Definition 5.4.5.** Let $r > 1$ be an integer and let $M$ be a $pr \times pr$ matrix. The block sum matrix of $M$, denoted $BS(M)$, is the $p \times p$ matrix given by

$$(BS(M))_{i,j} := \sum_{v,w=0}^{r-1} M_{i+pv,j+pw}.$$ 

Given a $pr \times pr$ matrix $M$, we see that $BS(M)$ is the matrix obtained by partitioning $M$ into $p \times p$ blocks and then forming the formal sum of these blocks. Of particular interest is the case when $M$ can be partitioned into circulant blocks. In this case we see that $BS(M)$ is a circulant matrix.

We shall also consider a variant on the representer polynomial.

**Definition 5.4.6.** Let $M$ be a $pr \times pr$ matrix having $p \times p$ circulant blocks. We define the reduced representer polynomial of $M$ to be the polynomial

$$g_M(X) := g_0 + g_1 X + \cdots + g_{r-1} X^{r-1}$$

where

$$g_i := \sum_{j=0}^{r-1} \sum_{k=1}^{p} M_{jp+1, (j+i)p+k}$$

and $M_{jp+1, (j+i)p+k}$ is replaced by $M_{jp+1, (j+i-r)p+k}$ if $j + i \geq r$.

The reduced representer polynomial of the matrix $M$ in some senses represents the matrix formed by replacing each $p \times p$ circulant block of $M$ by its row/column sum.

### 5.4.2 The Results

Before giving a characterisation of which elements of $\mathcal{X}$ generate $W_p$ with $a$, we make the following simple observation. If $\langle a^x \mid i = 0, \ldots, p-1 \rangle$ is an abelian group that is closed under conjugation by $x$, then

$$(ax)^p = a \cdot a^{x^{p-1}} \cdot a^{x^{p-2}} \cdots a^x.$$ 

In particular $(ax)^p$ is invariant under conjugation by both $a$ and $x$ and hence by $\langle a, x \rangle$.

We are now in a position to prove Theorem 5.0.2.

**Proof of Theorem 5.0.2** Assume that $\langle a, x \rangle \cong W_p$ and denote the base group of $W_p$ by $\Gamma$. By Corollary 5.1.4 we either have $\Gamma = \langle a^x \mid i = 0, \ldots, p-1 \rangle$ or $\Gamma = \langle x^a \mid i = 0, \ldots, p-1 \rangle$. Without loss of generality assume the former case holds. We will prove that the given conditions hold (an analogous argument may be used to show that in the latter case the conditions hold with the roles
of \(a\) and \(x\) interchanged). By assumption the commutator relations hold and thus it remains to prove the conditions on \(A_x\) and \(A^e_a\).

For distinct \(i,j \in \{1, \ldots, p^2\}\) such that \(\text{supp}(\alpha_i^x) \cap \text{supp}(\alpha_j) = \emptyset\), we have that \(|\text{supp}(\alpha_i^x) \cap \text{supp}(\alpha_j)| = 1 \text{ or } p\). In the latter case the commutator relations assert that \(\alpha_i^x \in \langle \alpha_j \rangle\) and this contributes an entry of \(p\) in the \((i,j)\) position of \(A_x\) and zeros in all other entries of the \(i\)th row and \(j\)th column. In the former case, we see that there must be distinct \(j_1 = j, j_2, \ldots, j_p \in \{1, \ldots, p^2\}\) such that \(|\text{supp}(\alpha_i^x) \cap \text{supp}(\alpha_{j_l})| = 1\) for all \(l = 1, \ldots, p\). However, the commutator relations then imply that there exist distinct \(i_1, i_2, \ldots, i_p \in \{1, \ldots, p^2\}\) such that \(|\text{supp}(\alpha_{i_1}^x) \cap \text{supp}(\alpha_{i_j})| = 1\) for all \(k, \ell \in \{1, \ldots, p\}\).

If every entry of \(A_x\) is equal to 0 or \(p\), then we may decompose \(a = a_1 \cdots a_p\) and \(x = x_1 \cdots x_p\) with each pair \((a_i, x_i)\) sitting inside a copy of \(\text{Sym}(p^2)\). Our assumptions on \(a\) and \(x\) together with the commutator relations assert that \(\langle a_i, x_i \rangle\) is isomorphic to a proper quotient of \(W_p\) for each \(i = 1, \ldots, p\), and hence the same is true of \(\langle a, x \rangle\). We conclude that there is a block of \(A_x\) in which every entry is equal to one and up to a suitable renumbering of the \(a_i\), the matrix \(A_x\) has the required form.

Since \(A_x\) has the given form, it follows that \(|\text{supp}(\alpha_i) \cap \text{supp}(\chi_j)| = 0 \text{ or } 1\) for all \(i, j \in \{1, \ldots, p^2\}\). Thus we see that \(w \in \text{supp}(\chi_w)\) for \(w \in \{1, \ldots, p^2\}\). Setting \(q := p^2 - p\), we consider \(x\) as

\[
\begin{align*}
x &= \begin{pmatrix}
1, \chi_1, \ldots, 1 \cdot \chi_1^{p-1} \\
2, 2 \cdot \chi_2, \ldots, 2 \cdot \chi_2^{p-1} \\
\vdots \\
p, p \cdot \chi_p, \ldots, p \cdot \chi_p^{p-1}
\end{pmatrix} \\
&= \begin{pmatrix}
q + 1, (q + 1) \cdot \chi_{q+1}, \ldots, (q + 1) \cdot \chi_{q+1}^{p-1} \\
q + 2, (q + 2) \cdot \chi_{q+2}, \ldots, (q + 2) \cdot \chi_{q+2}^{p-1} \\
\vdots \\
p^2, (p^2) \cdot \chi_p^p, \ldots, (p^2) \cdot \chi_p^{p-1}
\end{pmatrix}
\end{align*}
\]

(5.4.21)

In the subsequent work, we shall refer to the columns of \(x\) according to (5.4.21). By this, we mean that the first column of \(x\) is equal to \(\{1, 2, \ldots, p\}\), the second column is equal to \(\{1 \cdot \chi_1, 2 \cdot \chi_2, \ldots, p \cdot \chi_p\}\) and so on. We will also refer to the permutation \(\sigma_i\) corresponding to column \(i\). This will be the permutation defined on column \(i\) by a cyclic permuting of the entries. Thus \(\sigma_1 = (1, 2, \ldots, p)\), \(\sigma_2 = (1 \cdot \chi_1, 2 \cdot \chi_2, \ldots, p \cdot \chi_p)\), and so on.

We consider the structure of \(A^e_a\) via the structure of \(A_x\). First suppose that \(\text{supp}(\alpha_i^x) = \text{supp}(\alpha_j)\) for some \(i = 1, \ldots, p\), \(j = 1, \ldots, p^2\) and \(\ell = 1, \ldots, p\). As noted above it follows that \(\alpha_i^x \in \langle \alpha_j \rangle\), and hence the column of \(x\) in (5.4.21) corresponding to \(\alpha_i^x\) will contribute a power of \(\pi\), say \(\pi^r\), to one diagonal block of \(A^e_a\). Indeed, this is the only way that a diagonal block of \(A^e_a\) can be constructed. Moreover, as there is at least one block of \(A_x\) consisting of constant 1s, there exists some \(q \in \{1, \ldots, p\}\) such that none of the \(q^{th}\), \((q + p)^{th}\), \(\ldots\), \((q + (p^2 - p))^{th}\) columns of \(x\) are equal to the support of some \(\alpha_j\). For each such set of columns, the commutator relations then ensure that the combination of the columns add circulant blocks to the matrix \(A^e_a\) corresponding to the way that these columns are mapped onto one another by \(a\). It follows that the matrix \(A^e_a\) is a block matrix with \(p \times p\) circulant blocks. Finally, as there exists at least one set of \(p\) columns as defined above, we may use these together with the commutator relations to see that the diagonal blocks of \(A^e_a\) must be equal.
It remains to prove that the block sum matrix or the reduced representer polynomial of $A^x_\sigma$ has the desired form. To do this we consider $y := (ax)^p$. The commutator relations ensure that the columns of $x$ in (5.4.21) are permuted under the action of $a$ and hence the $i$th column of $x$ in (5.4.21) is mapped to the $(pr + i)$th column of $x$ under $y$ for some value of $r$. First assume that each column of $x$ is invariant under $y$. It follows from the commutator relations that $y = \sigma_1^i \sigma_2^i \cdots \sigma_{pp}^i$ for some $i_1, \ldots, i_p \in \{0, \ldots, p - 1\}$. However, as $y$ is invariant under conjugation by $\langle a, x \rangle$ we may conclude that the $i_j$ are all equal.

We now construct polynomials $f_j(x) \in \mathbb{Z}_p[X]$ for $j = 1, \ldots, p$ recursively. To do this initially set $f_j(X) = 0$ - the zero polynomial. Now consider $((j - 1)p + 1) \cdot ax$. Redefine $f_j(X)$ to be $f_j(X) := f_j(X) + X^r$ where $r$ is the number of rows descended in (5.4.21) to go from the entry $((j - 1)p + 1)$ to the entry $((j - 1)p + 1) \cdot ax$. Repeat this with $((j - 1)p + 1)$ replaced by $((j - 1)p + 1) \cdot (ax)^k$ for $k = 1, \ldots, p - 1$. Thus $f_j(X)$ encodes the circulant nature of the action of $a$ on $x$ in the columns of (5.4.21) corresponding to the orbit of the $((j - 1)p + 1)$th column under the action of $\langle ax \rangle$. We see that the contribution of these columns to $BS(A^x_\sigma)$ is precisely the $p \times p$ circulant matrix for which $f_j(x)$ is the representer polynomial. Moreover, a similar analysis to that used in Section 5.2 shows that $X = 1$ is a simple root of $f_j(x) \in \mathbb{Z}_p[X]$ precisely when $i_{(j-1)p+1} \neq 0 \mod p$ since

$$f_j'(1) \equiv \begin{cases} 0 & \text{if } i_{(j-1)p+1} \equiv 0 \mod p; \\ i_{(j-1)p+1}^{-1} & \text{otherwise}. \end{cases} \tag{5.4.22}$$

Considering the representer polynomial of the block sum matrix $BS(A^x_\sigma)$ we see that

$$f_{BS(A^x_\sigma)}(X) = \sum_{j=1}^p f_j(X).$$

Since the $i_j$ are all equal, we see that $f'_{BS(A^x_\sigma)}(1) \equiv 0 \mod p$ and $X = 1$ is a root of multiplicity at least two of the representer polynomial of $BS(A^x_\sigma)$.

To obtain the final conclusion we note that as the $i_j$ are all equal, we have that $f_1'(1) = f_2'(1) = \cdots = f_p'(1)$. Combining this with (5.4.22) we see that $y = 1$ precisely when $f_i'(1) = pk$ for some $k \in \mathbb{N} \setminus \{0\}$ and for all $i = 1, \ldots, p$. This occurs precisely when

$$f'_{BS(A^x_\sigma)}(1) = \sum_{j=1}^p f_j'(1) = kp^2.$$

Thus $X = 1$ is a root of $f'_{BS(A^x_\sigma)}(X) \in \mathbb{Z}_p[X]$ precisely when $y = 1$. Hence as $\langle a, x \rangle \cong W_p$ we conclude that $X = 1$ is not a root of $f'_{BS(A^x_\sigma)}(X) \in \mathbb{Z}_p[X]$. Hence if the columns of $x$ in (5.4.21) are invariant under the action of $y$, then case (c1) holds.

Now assume that the columns of $x$ are not fixed by $y$. We shall prove that case (c2) must be true. Define block $j$ of $x$ to be the set of columns $jp + 1, \ldots, jp + p$ of $x$ in (5.4.21) for $j = 0, \ldots, p - 1$ and consider the matrix $A^x_j$. Each entry of $A^x_j$ corresponds to the image of a column of $x$ under $a$. There are two possibilities: either the column $i$ is left invariant under the action of $a$, or it is mapped onto the column $pr + i$ for some $1 \leq r \leq p - 1$. This corresponds to the column being mapped from block $j$ to block $j + r \mod p$. The former case occurs precisely when a power of
π is added to a diagonal block of $A^*_p$, whilst in the latter case, a power of π is added to an off diagonal block of $A^*_p$ which is $r$ blocks to the right of the leading diagonal. We see that the partial representer polynomial $g_{A^*_p}(X)$ of $A^*_p$ is of the form

$$g_{A^*_p}(X) = g_0 + g_1 X^1 + \cdots + g_{p-1} X^{p-1},$$

where for each $r = 0, \ldots, p-1$ the coefficient $g_r$ is the number of columns $i$ of $x$ in (5.4.21) that are mapped to the column $i + pr$. Equivalently,

$$g_r = \sum_{j=0}^{p-1} \left\{ i \mid \begin{array}{l}
 \text{column } i \text{ is in block } j \text{ and is mapped to a column in block } j + r \pmod{p} \\
 \text{under the action of } a
\end{array} \right\}.$$

Consider the action of $y$ on the column $i$ of $x$. This corresponds to applying $ax$ to the column $p$ times. It follows that the action of $y$ is determined by the $p$ successive actions of $a$. Since each column of $x$ has an orbit of size $p^2$ under the action of $ax$, we see that each of the actions of $a$ on the columns of $x$ occurs $p$ times in the action of $y$ on the columns of $x$. Thus to obtain the number of blocks that column $i$ has passed through in reaching its image under the action of $y$, we need to sum the number of blocks passed through for each successive action of $a$. It follows that the total number of blocks passed through by all columns of $x$ under the action of $y$ is

$$p \cdot \sum_{r=0}^{p-1} r \cdot g_r = p \cdot g'_{A^*_p}(1). \quad (5.4.23)$$

Since $y$ is invariant under the action of $x$, we see that if column $i$ in block $j$ is mapped to column $i + pr$ in block $j + r \pmod{p}$ under the action of $y$, then every column of block $j$ will be mapped to the corresponding column in block $j + r \pmod{p}$ under the action of $y$. Hence $y$ simply permutes the blocks of $x$.

Suppose that block $j$ of $x$ passes through $p \cdot k_j + \ell_j$ blocks under the action of $y$ for some $k_j, \ell_j \in \{0, \ldots, p-1\}$. Since $y$ is invariant under the action of $a$, we deduce that the $k_j$ are all equal to some common value say $k$. It follows that the total number of blocks passed through by all columns of $x$ under the action of $y$ is

$$\sum_{j=0}^{p-1} p \cdot (p \cdot k + \ell_j) = p \left( p^2 k + \sum_{j=0}^{p-1} \ell_j \right). \quad (5.4.24)$$

Combining (5.4.23) and (5.4.24) we obtain

$$g'_{A^*_p}(1) \equiv \sum_{j=0}^{p-1} \ell_j \pmod{p^2}.$$ 

Finally, we note that $\sum_{j=1}^{p-1} \ell_j \equiv 0 \pmod{p^2}$ precisely when $\ell_j = 0$ for each $j = 0, \ldots, p-1$. Hence $g'_{A^*_p}(1) \equiv 0 \pmod{p^2}$ precisely when the columns of $x$ in (5.4.21) are fixed by $y$. Since by assumption the columns of $x$ are permuted by $y$, we conclude that case (c2) holds.

Conversely assume that the given conditions hold (again the argument for when $a$ and $x$ are interchanged is analogous). By the above we see that $(ax)^p \neq 1$. Combining this with the commutator relations and Corollary 5.4.4 we obtain that $\langle a, x \rangle \cong W_p$, as required.
As in Section 5.2 we note that the conditions given in Theorem 5.0.2 are all required. As before this is evident in the proof of the theorem for the conditions on the matrices $A_x$ and $A_x^c$. To see that the commutator relations are all necessary, we note that if $G := \text{Sym}(125)$ and $a$ is the standard 5-element of $G$, then we may take

\[ x = (1, 26, 51, 76, 101)(2, 27, 56, 79, 106)(3, 28, 61, 77, 111)(4, 29, 66, 80, 116) \\
(5, 30, 71, 78, 121)(6, 41, 54, 91, 117)(7, 42, 59, 94, 122)(8, 43, 64, 92, 102) \\
(9, 44, 69, 95, 107)(10, 45, 74, 93, 112)(11, 49, 72, 83, 110)(12, 50, 52, 81, 115) \\
(13, 46, 57, 84, 120)(14, 47, 62, 82, 125)(15, 48, 67, 85, 105)(16, 36, 68, 90, 118) \\
(17, 37, 73, 88, 123)(18, 38, 53, 86, 103)(19, 39, 58, 89, 108)(20, 40, 63, 87, 113) \\
(21, 32, 70, 100, 119)(22, 33, 75, 98, 124)(23, 34, 55, 96, 104)(24, 35, 60, 99, 109) \\
(25, 31, 65, 97, 114). \]

The reader may check that $A_x$ and $A_x^c$ have the required form and $[a, a^x] = 1$. However, $[a, a^{x^2}] \neq 1$.

### 5.4.3 A Worked Example

As in Section 5.2 we follow the technical steps of the proof of Theorem 5.0.2 in the case of a couple of specific examples. Indeed, consider $G := \text{Sym}(27)$ and the following $G$-conjugates of the standard 3-element $a$ of $G$,

\[ x_1 = (1, 10, 20)(4, 14, 22)(7, 18, 27) \]
\[ x_2 = (2, 13, 23)(5, 17, 25)(8, 12, 21) \]  
\[ (3, 16, 26)(6, 11, 19)(9, 15, 24) \quad (5.4.25) \]

and

\[ x_2 = (1, 10, 22)(4, 14, 19)(7, 12, 25) \]
\[ (2, 13, 23)(5, 17, 20)(8, 15, 26) \quad (5.4.26) \]

\[ (3, 16, 24)(6, 11, 21)(9, 18, 27) \]

It is easily seen that $\langle a, x_1 \rangle \cong \langle a, x_2 \rangle \cong W_3$. We shall show that for the pair $(a, x_1)$ conditions (i), (ii) and (iii) (c1) of Theorem 5.0.2 hold, whilst for the pair $(a, x_2)$ conditions (i), (ii) and (iii) (c2) hold.

As in the proof of Theorem 5.0.2 we consider the columns of $x_1$ in (5.4.25) and note that columns
2, 5 and 8 ensure that the matrix $A_{x_1}$ has a block of constant ones. Indeed

\[
A_{x_1} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

whilst

\[
A_{x_1}^a = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Consider the representation of $x_1$ in (5.4.25) and the element

\[
y_1 := (ax_1)^3 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 13, 16)(11, 14, 17)
\]

\[
\]

Denoting the permutation corresponding to the $i^{th}$ column of $x_1$ by $\sigma_i$ we see that

\[
y_1 = \sigma_1^{i_1} \cdot \sigma_2^{i_2} \cdot \sigma_3^{i_3} \cdot \sigma_4^{i_4} \cdot \sigma_5^{i_5} \cdot \sigma_6^{i_6} \cdot \sigma_7^{i_7} \cdot \sigma_8^{i_8} \cdot \sigma_9^{i_9},
\]

where $i_1 = i_2 = \cdots = i_9 = 1$.

We now define the polynomials $f_j(X) \in Z_3[X]$ as described in the proof of the theorem. First set $f_1(X) := 0 \in Z_3[X]$ and consider the orbit of 1 under $ax_1$. Since $1^{ax_1} = 13$ and 13 is in the second row of $x_1$ in (5.4.25), we have “descended” by 1 row. Hence we redefine $f_1(X) := f_1(X) + X^1 = X^1$. As $13^{ax_1} = 22$ and 22 is in the first row of $x_1$ we have cyclically descended a further 2 rows, and so we set $f_1(X) := f_1(X) + X^2 = X + X^2$. Finally, $22^{ax_1} = 2$ and 2 is in the second row of $x_1$. Thus we have descended by a further row, and our final polynomial is

\[
f_1(X) := f_1(X) + X^1 + X^2 = 2X + X^2 \in Z_3[X].
\]

By considering the images of the first column of $x_1$ under the action of $(ax_1)^1$, $(ax_1)^2$ and $(ax_1)^3$
we see that we have accounted for an entry of 1 in each of the circled entries of

\[
A_{a}^{x_1} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\] (5.4.27)

Hence the contribution to the block sum matrix \(BS(A_{a}^{x_1})\) from these columns is

\[
\begin{pmatrix}
0 & 2 & 1 \\
1 & 0 & 2 \\
2 & 1 & 0 \\
\end{pmatrix} = \text{circ}(0, 2, 1).
\]

This is the matrix for which \(f_1(X)\) is the representer polynomial.

Similarly \(f_2(X) = 2X + X^2 \in \mathbb{Z}_3[X]\), which accounts for additional entries of 1 in (5.4.27). This gives

\[
A_{a}^{x_1} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\] (5.4.28)

Finally, \(f_3(X) = 2X + X^2 \in \mathbb{Z}_3[X]\) which contributes a 1 to the remaining entries of (5.4.28). We note that in general, it is not the case that the \(f_i(X) \in \mathbb{Z}_p[X]\) are all equal.

The final step of the proof of Theorem 5.0.2 considers the representer polynomials \(f_1(X), f_2(X), f_3(X) \in \mathbb{Z}_3[X]\) and \(f_{BS(A_{a}^{x_1})}(X) \in \mathbb{Z}_9[X]\). We see that \(f'_j(1) = 1 = i_j^{-1} \in \mathbb{Z}_3\) for all \(j\), as given in (5.4.22). Moreover, considering \(f'_j(X) \in \mathbb{Z}_9(X)\) we have that \(f'_j(1) = 4\) for all \(j\). Consequently

\[
f'_{BS(A_{a}^{x_1})}(1) = \sum_{j=1}^{3} f'_j(1) = 3 \neq 0 \in \mathbb{Z}_9
\]

and case (c1) holds.
(a, x_2):

As with x_1, we see that columns 2, 5 and 8 of x_2 in (5.4.26) ensure that A_{x_2} has the desired form, namely

\[
A_{x_2} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

whilst

\[
A_a^{x_2} = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0
\end{pmatrix}
\]

In this case we see that

\[
y_2 := (ax_2)^3 = (1, 5, 8)(2, 6, 9)(3, 4, 7)(10, 17, 15)(11, 18, 13)
\]

\[
(12, 16, 14)(19, 25, 24)(20, 26, 22)(21, 27, 23)
\]

and so clearly y_2 does not fix the columns of x_2 in (5.4.26). We thus consider the blocks of columns of x_2 in (5.4.26), where block 0 is given by the first three columns, block 1 by the middle three columns and block 2 by the final three columns. Denoting the columns of x_2 by c_1, ..., c_9 we see that the action of a on the columns of x_2 is given by c_2 \mapsto c_5, c_5 \mapsto c_8, c_8 \mapsto c_2 \text{ and } c_i \mapsto c_i \text{ for } i = 1, 3, 4, 6, 7, 9. It follows that there are six columns whose image under a stays in the same block, meaning that g_0 = 6. The remaining three columns are mapped from block i to block i + 1 (mod 3) for some i = 0, 1, 2. Hence g_1 = 3 and g_2 = 0.

Considering the reduced representor polynomial of A_a^{x_2} we have that

\[
g_{A_a^{x_2}}(X) = \sum_{j=0}^{2} g_j X^i = 6 + 3X.
\]

Moreover we see that the columns of x_2 in (5.4.26) are permuted under the action of ax_2 in the following cyclic manner

\[
c_1 \mapsto c_2 \mapsto c_6 \mapsto c_4 \mapsto c_5 \mapsto c_9 \mapsto c_7 \mapsto c_8 \mapsto c_3 \mapsto c_1.
\]
It follows that column \( i \) is mapped under \( y_2 \) to column \( i + 3 \) (or \( i - 6 \) if \( i + 3 > 9 \)) and in doing so passes through a single block of \( x_2 \). Thus using the notation of \((5.4.24)\) we have that \( k_j = 0 \) and \( \ell_j = 1 \) for all \( j = 0, 1, 2 \).

Consequently the total number of blocks passed through by all columns of \( x_2 \) under the action of \( y_2 \) is

\[
9 = 3 \cdot g'_{A^x_2}(1)
\]

and

\[
g'_{A^x_2}(1) = 3 \equiv \sum_{j=0}^{2} \ell_j \pmod{9}.
\]

In particular \( g'_{A^x_2}(1) \neq 0 \pmod{9} \) and case (c2) holds.

### 5.5 The General Case

We now consider the most general setting that \( G := \text{Sym}(n) \) for some \( n \geq p^2 \) and let

\[
a = (1, 2, \ldots, p)(p + 1, p + 2, \ldots, 2p) \cdots (p(r - 1) + 1, p(r - 1) + 2, \ldots, pr) \in G
\]

for some \( r \geq p \). As previously we set \( \mathcal{X} := a^G \) and consider \( x \in \mathcal{X} \). Denote the \( p \)-cycles forming \( a \) (respectively \( x \)) by \( \alpha_1, \ldots, \alpha_r \) (respectively \( \chi_1, \ldots, \chi_r \)). We note that if \( \langle a, x \rangle \cong W_p \), then \( |\text{supp}(\alpha_i) \cap \text{supp}(\chi_j)| = 0, 1 \) or \( p \) for each \( i, j \in \{1, \ldots, r\} \).

Before considering the most general situation, we look at the case that \( \text{supp}(a) = \text{supp}(x) \) and prove Theorem 5.0.3.

**Proof of Theorem 5.0.3**: Assume that \( \langle a, x \rangle \cong W_p \). Without loss of generality assume that \( a \) is in the base group of \( W_p \). We will show that the given conditions hold true (if \( x \) is in the base group, then an analogous argument may be used to show that the conditions hold with the roles of \( a \) and \( x \) interchanged). The commutator relations follow immediately.

First consider the case that \( \text{supp}(\alpha_i) = \text{supp}(\chi_j) \) for some \( i, j \in \{1, \ldots, r\} \). This results in matrix entries \((A_x)_{i,i} = (A^x)_{j,j} = p\). By a suitable renumbering of the \( \alpha_i \) and \( \chi_j \), we may assume that such a situation arises for pairs \((\alpha_i, \chi_i)\) with \( i = 1, \ldots, m \) for some \( m \). This gives the diagonal block \( D_1 \) of \( A_x \) and \( E_1 \) of \( A^x \). It follows that for all other \( \alpha_i \) we have \( \text{supp}(\alpha_i^\ell) \cap \text{supp}(\alpha_i) = \emptyset \).

If \( \alpha_i^\ell \in \langle \alpha_i \rangle \) for \( \ell = 1, \ldots, p \) and for some \( i, \ell \), then we may consider these \( \alpha_i \) and the \( \chi_j \) that share their common support as lying inside a copy of \( \text{Sym}(p^2) \). Thus we may apply the results of Section 5.2 to obtain the general structures of \( D_2 \) and \( E_2 \). In all other cases, we use a similar approach to find a subset \( I \subseteq \{1, \ldots, r\} \) of size \( p^2 \) such that \( \prod_{i \in I} \alpha_i \) and \( \prod_{i \in I} \chi_i \) are conjugate elements of full support inside a copy of \( \text{Sym}(p^3) \). The results of Section 5.4 then give the structures of \( D_3 \) and \( E_3 \). The final condition on one of the blocks of \( E_2 \) or \( E_3 \) then follows from the fact that \( ax \) has order \( p^2 \) and the results of Sections 5.2 and 5.4.

Conversely, assume that the conditions in Theorem 5.0.3 hold (again the argument for when \( a \) and \( x \) are interchanged is analogous). By Corollary 5.4.4 it remains to check that \( ax \) has order \( p^2 \).
5.5. THE GENERAL CASE

Suppose that $D_1$ is an $m_1 \times m_1$ matrix, $D_2$ is a $pm_2 \times pm_2$ matrix and $D_3$ is a $p^2m_3 \times p^2m_3$ matrix for some $m_1$, $m_2$ and $m_3$. Define $a_1 = \alpha_1 \cdots \alpha_{m_1}$,

\[a_{2,i} = \alpha_{m_1+p(i-1)+1} \cdots \alpha_{m_1+p(i-1)+p}\quad\text{for } i = 1, \ldots, m_2, \quad\text{and}\]
\[a_{3,j} = \alpha_{m_1+pm_2+p^2(j-1)+1} \cdots \alpha_{m_1+pm_2+p^2(j-1)+p^2}\quad\text{for } j = 1, \ldots, m_3.

Define $x_1, x_{2,1}, \ldots, x_{2,m_2}, x_{3,1}, \ldots, x_{3,m_3}$ analogously. We see that the pair $(a_1, x_1)$ corresponds to the blocks $D_1$ and $E_1$ of $A_x$ and $A^T_a$, each pair $(a_{2,i}, x_{2,i})$ corresponds to a block of $D_2$ and its associated block in $E_2$, whilst every pair $(a_{3,j}, x_{3,j})$ corresponds to blocks of $D_3$ and $E_3$. Moreover, since for each pair $(a_s, x_s)$ we have $\text{supp}(a_s) = \text{supp}(x_s)$ and $[a_s, a_s^x] = 1$, it suffices to check that for one such pair, the element $a_s x_s \in \text{Sym}(\text{supp}(a_s))$ has order $p^2$. However, this follows immediately from condition [IV] and the results of Sections 5.2 and 5.4. Thus $\langle a, x \rangle \cong W_p$. 

In the general setting, we note that by the preceding arguments we must have that $|\text{supp}(a) \cap \text{supp}(x)|$ is divisible by $p$. Thus the above result will still hold if $n = pr + s$ for some $s < p$. However this is not the case in general, since if we no longer require $\text{supp}(a) = \text{supp}(x)$, the condition that $a$ and $x$ are $G$-conjugate becomes weaker. Indeed, if we consider $G = \text{Sym}(p^2)$, we have that $W_p$ is embedded into $G$ using the generators $a = (1, 2, \ldots, p)$ and

\[x = (1, p + 1, \ldots, p(p - 1) + 1)(2, p + 2, \ldots, p(p - 1) + 2) \cdots (p, 2p, \ldots, p^2).
\]

However, we may also use $x$ and $y$ to generate $W_p$, where

\[y = (1, 2, \ldots, p)(p + 1, p + 2, \ldots, 2p) \cdots (p(p - 2) + 1, p(p - 2) + 2, \ldots, p(p - 1)).\]

If we now relax our assumption that $\text{supp}(a) = \text{supp}(x)$ and move into $G := \text{Sym}(p(p + 1))$, we see that taking $x \in G$ as above and

\[\bar{y} = y \cdot (p^2 + 1, p^2 + 2, \ldots, p(p + 1)),\]

we have that $\langle x, \bar{y} \rangle \cong W_p$. However, it is clear that the generation of $W_p$ results from the elements $x, y \in \text{Sym}(p^2) \hookrightarrow G$. It follows that although $x$ and $\bar{y}$ are $G$-conjugate, this conjugation is in some sense artificial. Consequently, producing a theorem such as Theorem 5.0.3 would not be realistic in the most general setting.
Chapter 6

Cuspidal Characters of Finite Groups

Let $G$ be a finite group of Lie type. In Chapters 2 and 3 we saw how the building, $\Delta$, of $G$ is a rich structure which admits the construction of modules for $G$ over finite fields from modules defined on smaller subgroups known as parabolic subgroups. We then mirrored the structure of a building in the form of minimal parabolic systems, with appropriate groups being known as parabolic subgroups. Each parabolic subgroup, $P$, contained in $\Delta$ admits an amiable decomposition as $P = U \cdot L$, with unipotent radical $U$ and Levi complement $L$. For an arbitrary finite group, we saw how the $p$-core of a subgroup can in some senses replicate the role of the unipotent radical. However, in general there will be no complement to the $p$-core of a parabolic subgroup. The subsequent simplicial complex of parabolic subgroups then permitted the construction of modules over finite fields in a comparable manner to the motivating case of groups of Lie type.

Stepping away from the building site of construction work, we may return to the familiar setting of characters of finite groups defined over the complex numbers. Although a great knowledge of such characters already exists, there is a natural desire to gain a deeper understanding of them. One way to gain such insight is to classify the known characters of a given group, $G$, dependant on whether they satisfy certain pre-described properties. An example of such a property for a group of Lie type is that of being a cuspidal character. This relies on the truncation of a character to each parabolic subgroup $P$ of $G$, which in turn depends on the unipotent radical of $P$.

In the case that $G$ is a sporadic simple group, the complex characters of $G$ are described in full in [CCN`09]. Using the $p$-core of each parabolic subgroup of $G$, we may consider a notion of cuspidal characters in this more general setting. Since such a notion depends on the $p$-core, we obtain $p$-cuspidal characters for each prime divisor $p$ of $G$. These $p$-cuspidal characters are the primary focus of this chapter.

This chapter is arranged as follows. We begin in Section 6.1 by formally defining an $X$-parabolic system for a group $G$ with respect to a given subgroup $X$. Such systems are a natural generalisation of the $p$-minimal parabolic systems introduced in Chapter 2. This is followed by a brief exposition of cuspidal characters of groups of Lie type. These motivate the subsequent definition of cuspidal characters of finite groups with respect to $X$-parabolic systems, whose properties are then investigated.

The subsequent four sections provide a full survey of the $p$-cuspidal characters of the sporadic
simple groups, with 2-, 3- and 5-cuspidal characters being considered in Sections 6.2, 6.3 and 6.4 respectively. The remaining \( p \)-cuspidal characters for \( p > 5 \) are studied in Section 6.5. With the exception of the baby monster, \( \mathbb{B} \), for the prime \( p = 2 \) we fully determine the \( p \)-cuspidal characters of each sporadic group in these sections. When \( G = \mathbb{B} \) and \( p = 2 \) we make the assumption that \( Co_2 \) has a unique 22-dimensional representation over \( GF(2) \). With this assumption, our calculations leave us with two potential 2-cuspidal characters, but at the time of writing we have been unable to ascertain for definite whether or not they are indeed 2-cuspidal. For ease of reference, a summary of the 2-minimal parabolic systems for each of the sporadic simple groups together with their associated 2-cuspidal characters is given in Tables 6.1 and 6.2. A complete list of \( p \)-cuspidal characters for the sporadic groups in the case that \( p > 2 \) is then given in Table 6.3. In each table, for a given group \( G \) the character \( \chi_i \) will equal the corresponding character of \( G \) given in the character table in [CCN `09].

The chapter concludes in Section 6.6 with a brief discussion on possible geometries arising from the \( p \)-cuspidal characters summarised in Tables 6.1-6.3.

### 6.1 Parabolic Systems and Cuspidal Characters

#### 6.1.1 Parabolic Systems

Let \( G \) be a finite group and let \( X \) be a subgroup of \( G \). Let \( I \) be an index set with \( |I| = n \geq 0 \).

**Definition 6.1.1.** An \( X \)-parabolic system, \( \mathcal{X} \), of \( G \) of rank \( n \) is a set of pairs of subgroups of \( G \), \((P_J, Q_J)\), indexed by subsets \( J \) of \( I \) such that

(i) for each \( J \subseteq I \), \( X \subseteq P_J, Q_J \subseteq P_J; \)

(ii) for \( K \subseteq J \subseteq I \), \( Q_J \subseteq Q_K; \)

(iii) \( P_I = G \) and \( Q_I = 1 \); and

(iv) \( X = P_\varnothing \).

We shall write \( \mathcal{X} = \{(P_J, Q_J)|J \subseteq I\} \) and note that by part (ii) of Definition 6.1.1 that all \( Q_J \) are subgroups of \( Q_\varnothing \) and that \( Q_\varnothing \leq P_\varnothing = X \). We allow the possibility that \((P_J, Q_J) = (P_K, Q_K)\) with \( J \neq K \), but this will not occur in most of the cases that follow.

For ease of notation, if our index set is \( I = \{1, 2, \ldots, n\} \) for some \( n \geq 1 \), then given a subset \( \{i_1, i_2, \ldots, i_r\} \subseteq I \) with \( i_j < i_{j+1} \) for all \( j \), we denote the subgroups \( P_{\{i_1, i_2, \ldots, i_r\}} \) and \( Q_{\{i_1, i_2, \ldots, i_r\}} \) by \( P_{i_1 i_2 \ldots i_r} \) and \( Q_{i_1 i_2 \ldots i_r} \), respectively.

**Example 6.1.2.** Let \( G = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle \) be the dihedral group of order 8, \( X = \langle r \rangle \) and let \( I = \{1, 2\} \). Then defining

\[
P_{12} = G, \quad P_1 = G, \quad P_2 = G, \quad P_\varnothing = X; \quad \text{and}
\]

\[
Q_{12} = 1, \quad Q_1 = \langle r^2 \rangle, \quad Q_2 = X, \quad Q_\varnothing = X,
\]

we have that \( \mathcal{X} = \{(P_J, Q_J)|J \subseteq I\} \) is an \( X \)-parabolic system of \( G \) of rank 2.
<table>
<thead>
<tr>
<th>Family</th>
<th>Group</th>
<th>2-Minimal Parabolic System</th>
<th>Rank</th>
<th>Geometric/Non-Geometric</th>
<th>2-Cuspidal Characters (character degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathieu Groups</td>
<td>$M_{11}$</td>
<td>{P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_2 \sim 3^4.\text{SD}<em>{16}} \quad {P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_3 \sim \text{Alt}(6).2} \quad {P_2 \sim 3^4.\text{SD}</em>{16}, P_3 \sim \text{Alt}(6).2}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>$M_{12}$</td>
<td>{P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_2 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3)}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>$M_{22}$</td>
<td>{P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_2 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3)}</td>
<td>2</td>
<td>Geometric</td>
<td>\chi_3 (10), \chi_4 (10)</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>{P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_2 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_3 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>\chi_3 (45), \chi_4 (45)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_{24}$</td>
<td>{P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_2 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_3 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>\chi_3 (45), \chi_4 (45)</td>
</tr>
<tr>
<td>HS</td>
<td>{P_1 \sim 2^{16}, \text{Sym}(5), P_2 \sim 4^{2^7}, \text{Sym}(3)}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>$J_3$</td>
<td>{P_1 \sim 2^{+^{16}}, \text{Sym}(3), P_2 \sim 2^{+^{16}}, L_2(4)}</td>
<td>2</td>
<td>Geometric</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>Co$_1$</td>
<td>{P_1 \sim 2^{4^6}, \text{Sym}(5), P_2 \sim 2^{4^6}, \text{Sym}(3), P_3 \sim 2^{4^6}, \text{Sym}(3), P_4 \sim 2^{4^6}, \text{Sym}(3)}</td>
<td>4</td>
<td>Geometric</td>
<td>\chi_2 (276), \chi_3 (37674), \chi_{11} (94875)</td>
<td></td>
</tr>
<tr>
<td>Co$_2$</td>
<td>{P_1 \sim [2^{15}], \text{Sym}(5), P_2 \sim [2^{17}], \text{Sym}(3), P_3 \sim [2^{17}], \text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>\chi_3 (253), \chi_{10} (9625), \chi_{11} (9625), \chi_{12} (10395), \chi_{13} (10395), \chi_{16} (31625), \chi_{33} (2390985), \chi_{32} (239085)</td>
<td></td>
</tr>
<tr>
<td>Co$_3$</td>
<td>{P_1 \sim 2^{+^{2^{17}}}, \text{Sym}(3), P_2 \sim 2^{+^{2^{17}}}, \text{Sym}(3), P_3 \sim 2^{+^{2^{17}}}, \text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>Leech Lattice and Conway Groups</td>
<td>$M_{cl}$</td>
<td>{P_1 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_2 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3), P_3 \sim 2^{+^{2^{2^2}}}, \text{Sym}(3)}</td>
<td>3</td>
<td>Non-Geometric</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>$S_{uz}$</td>
<td>{P_1 \sim 2^{+^{2^{16}}}, L_2(4), P_2 \sim 2^{+^{2^{16}}}, \text{Sym}(3), P_3 \sim 2^{+^{2^{16}}}, \text{Sym}(3)}</td>
<td>3</td>
<td>Geometric</td>
<td>None</td>
</tr>
</tbody>
</table>

Table 6.1: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Mathieu, Leech Lattice and Conway sporadic simple groups.
### Table 6.2: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Monster group, its subgroups and the pariah sporadic simple groups.

<table>
<thead>
<tr>
<th>Family</th>
<th>Group</th>
<th>p-Minimal Parabolic System</th>
<th>Geometric/Non-Geometric</th>
<th>Rank</th>
<th>Character Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monster</td>
<td>2A</td>
<td>( P_2 \sim Z_2 \times A_5 \times SL(2,5) \times SL(2,5) )</td>
<td>Geometric</td>
<td>4</td>
<td>( x_0 ) (1407)</td>
</tr>
<tr>
<td>and Subgroups</td>
<td>2B</td>
<td>( P_2 \sim Z_2 \times A_5 \times SL(2,5) \times SL(2,5) )</td>
<td>Geometric</td>
<td>4</td>
<td>( x_0 ) (1407)</td>
</tr>
<tr>
<td>2-Pariahs</td>
<td>3</td>
<td>( P_2 \sim Z_2 \times A_5 \times SL(2,5) \times SL(2,5) )</td>
<td>Geometric</td>
<td>4</td>
<td>( x_0 ) (1407)</td>
</tr>
</tbody>
</table>

### Table 6.3: The p-cuspidal characters and associated p-minimal parabolic systems of the sporadic simple groups in the case that \( p \neq 2 \).

<table>
<thead>
<tr>
<th>Family</th>
<th>Group</th>
<th>p-Minimal Parabolic System</th>
<th>Geometric/Non-Geometric</th>
<th>Rank</th>
<th>Character Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conway Groups</td>
<td>2A</td>
<td>( P_2 \sim Z_2 \times A_5 \times SL(2,5) \times SL(2,5) )</td>
<td>Geometric</td>
<td>4</td>
<td>( x_0 ) (1407)</td>
</tr>
<tr>
<td>and Subgroups</td>
<td>2B</td>
<td>( P_2 \sim Z_2 \times A_5 \times SL(2,5) \times SL(2,5) )</td>
<td>Geometric</td>
<td>4</td>
<td>( x_0 ) (1407)</td>
</tr>
<tr>
<td>2-Pariahs</td>
<td>3</td>
<td>( P_2 \sim Z_2 \times A_5 \times SL(2,5) \times SL(2,5) )</td>
<td>Geometric</td>
<td>4</td>
<td>( x_0 ) (1407)</td>
</tr>
</tbody>
</table>
Given an $X$-parabolic system $\mathfrak{X} = \{(P_J,Q_J) | J \subseteq I \}$ of $G$ and $J \subseteq I$, we set $\overline{P}_J := P_J/Q_J$. Furthermore, for any subgroup $Q_J \leq Y \leq P_J$, we use the standard bar notation $\overline{Y} := Y/Q_J$. We may use $\mathfrak{X}$ to form an $\mathfrak{X}$-parabolic system, $\overline{\mathfrak{X}}_J$, of rank $|J|$ for $\overline{P}_J$ given by

$$\overline{\mathfrak{X}}_J = \{(\overline{P}_J \cap \overline{P}_K, \overline{Q}_K) | (P_K, Q_K) \in \mathfrak{X}, K, J \subseteq I \}.$$ 

For each prime $p$, taking $S = \text{Syl}_p(G)$ and $B := N_G(S)$, we may define a $B$-parabolic system for $G$. Indeed, we recall from Chapter 2 that a subgroup $P$ of $G$ is called $p$-minimal (with respect to $B$) if $B < P$ and $B$ is contained in a unique maximal subgroup of $P$. We note that our definition of $p$-minimality is slightly more general than that given in [RS84], where the subgroup $P$ was also required to have non-trivial $p$-core, $O_p(P)$. Defining

$$\mathcal{M}(G,B) = \{P | P \text{ is a } p \text{-minimal subgroup of } G \text{ (with respect to } B) \},$$

then a set

$$\mathcal{M}_0 = \{P_i | P_i \in \mathcal{M}(G,B), i \in I \}$$

is called a minimal parabolic system of characteristic $p$ for $G$ or a $p$-minimal parabolic system of $G$, if $G = \langle P_j \rangle$ for $i \in I$ and $G \neq \langle P_j \rangle$ for any $i \in I$. The rank of $\mathcal{M}_0$ is $|I|$. We call $\mathcal{M}_0$ a geometric $p$-minimal parabolic system if for all $J, K \subseteq I$ we have $P_{J \cap K} = P_J \cap P_K$. Otherwise $\mathcal{M}_0$ is called non-geometric.

The promised $B$-parabolic system, $\mathfrak{X} = \{(P_J,Q_J) | J \subseteq I \}$, is given by defining

$$P_J = \begin{cases} \langle P_j | j \in J \rangle & \text{if } \emptyset \neq J \subseteq I; \text{ and} \\ B & \text{if } J = \emptyset \end{cases}$$

and $Q_J = O_p(P_J)$ for all $J \subseteq I$. It follows that every group $G$ has a $p$-minimal parabolic system, and hence a $B$-parabolic system for each prime $p$ (although this is not the case if the additional condition on non-trivial $p$-cores is imposed). The minimal parabolic systems for the sporadic groups (in the case that $S$ is non-cyclic) are described in [RS84].

### 6.1.2 Cuspidal Characters

Given a finite simple group, $G$, of Lie type, having a Borel subgroup $X$, let $P_J$ be a standard parabolic subgroup of $G$ and let $U_J$ be the associated unipotent radical. For an irreducible character $\chi$ of $G$, we recall from [Car93] that the truncation, $T_{P_J/U_J}(\chi)$, of $\chi$ is the character of $P_J$ defined by

$$(T_{P_J/U_J}(\chi))(p) = \frac{1}{|U_J|} \sum_{u \in U_J} \chi(up)$$

for $p \in P_J$. The character $\chi$ is then called cuspidal if for all standard parabolic subgroups $P_J \neq G$, the truncation $T_{P_J/U_J}(\chi)$ is zero. Equivalently, a character $\chi$ is cuspidal if $(\chi|_{U_J}, 1|_{U_J}) = 0$ for all $U_J \neq X$ (see [Car93 Proposition 9.1.1]).
Cuspidal characters play an important role in determining the irreducible characters of finite simple groups of Lie type, and an exposition of them may be found in [Car93, Chapter 9]. It would be of interest to form an analogous definition of cuspidal characters for the sporadic simple groups, and see whether any interesting properties arise. To do this, we note that for a finite simple group, $G$, of Lie type as described above, taking $Q_J$ to be the unipotent radical of $P_J$ we obtain an $X$-parabolic system of $G$ given by $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$. We may use this observation to define a form of cuspidal characters in a more general setting. Indeed, let $G$ be an arbitrary finite group and denote by $\text{Irr}_{\mathbb{C}} G$ the set of all irreducible complex characters of $G$.

**Definition 6.1.3.** Let $\mathfrak{X}$ be an $X$-parabolic system of $G$ where $X \subseteq G$, and let $\chi \in \text{Irr}(G)$. Then $\chi$ is called $\mathfrak{X}$-cuspidal if for all $(P_J, Q_J) \in \mathfrak{X}$ with $Q_J \neq 1$ we have

$$\sum_{g \in Q_J} \chi(g) = 0. \quad (6.1.1)$$

The condition (6.1.1) will be known as the *cuspidal condition on* $Q_J$ and is equivalent to $(\chi_{Q_J}, 1_{Q_J}) = 0$.\footnote{In an abuse of terminology, we will also sometimes refer to the cuspidal relation holding for $P_J$ when (6.1.1) occurs.} Clearly, when the index set $I = \emptyset$, we have $G = P_\emptyset = X$ and $Q_\emptyset = 1$, and hence every irreducible character is vacuously $\mathfrak{X}$-cuspidal. When $\mathfrak{X}$ is a $B$-parabolic system associated to a $p$-minimal parabolic system of $G$, then any $\mathfrak{X}$-cuspidal character will also be called a $p$-cuspidal character of $G$.

We recall the notion of the intertwining number of two modules.

**Definition 6.1.4.** [Kar94] Let $F$ be a field of characteristic 0, with algebraic closure $\overline{F}$, let $G$ be a finite group and let $V$ and $W$ be $\overline{F}G$-modules. The intertwining number, denoted $i(V, W)$, is defined by

$$i(V, W) := \dim_F \text{Hom}_{\overline{F}G}(V, W)$$

The intertwining number of modules will be of importance due to its connection with the inner product of the associated characters.

**Theorem 6.1.5.** [Kar94, Chapter 3, Theorem 1.1] Let $F$ be an arbitrary field of characteristic 0 and let $\lambda$ and $\mu$ be arbitrary characters of $G$ afforded by $\overline{F}G$-modules $V$ and $W$ respectively. Then

$$(\lambda, \mu) = i(V, W).$$

We may use Theorem 6.1.5 to prove an analogue of Proposition 9.1.3 of [Car93], the proof of which is almost identical to that used in Carter’s Proposition.

**Theorem 6.1.6.** Let $\mathfrak{X}$ be an $X$-parabolic system of $G$ and $\chi \in \text{Irr}(G)$. Then there exists $(P_J, Q_J) \in \mathfrak{X}$ and an $\mathfrak{X}_J$-cuspidal character $\psi$ of $\overline{P}_J = P_J/Q_J$ such that $(\chi, \psi^{G}) \neq 0$.

**Proof.** Let $\mathcal{S} = \{J \subseteq I | (\chi_{Q_J}, 1_{Q_J}) \neq 0\}$. Note that $\mathcal{S} \neq \emptyset$ as $Q_I = 1$. Let $J$ be a minimal element of $\mathcal{S}$ and let $V$ be an irreducible $\mathbb{C}G$-module that affords $\chi$. Define

$$V' = \{v \in V | v \cdot u = v \text{ for all } u \in Q_J\}.$$

$\chi|_{Q_J}$ is the restriction of $\chi$ to $Q_J$. For all $u \in Q_J$, we have $\chi|_{Q_J}(u) = 0$ if and only if $u \neq 1_{Q_J}$. Indeed, if $u = 1_{Q_J}$, then $\chi |_{Q_J}(u) = \chi(u) = 1$, and if $u \neq 1_{Q_J}$, then $\chi |_{Q_J}(u) = \chi|_{Q_J}(u)$ by Proposition 9.1.3 of [Car93]. Therefore, $\chi|_{Q_J}$ is the restriction of $\chi$ to $Q_J$.
By Theorem 6.1.5, as \((\chi_{Q_1}, 1_{Q_1}) \neq 0\), there exists a non-zero \(CQ_J\)-homomorphism from the trivial \(CQ_J\)-module to \(V\), and hence \(V'\) is non-empty.

Clearly \(V'\) is a linear subspace of \(V\), and given \(g \in P_J\) and \(u \in Q_J\) we have that

\[(vg)u = vgug^{-1}g = vg,
\]
as \(Q_J \leq P_J\). Hence \(V'\) is a \(CP_J\)-module.

Consider \(V'\) as a \(C\overline{P}_J\)-module, having associated character \(\phi = \sum_i \phi_i\) (with the \(\phi_i\) irreducible \(C\overline{P}_J\)-characters). So \(V'\) affords \(\phi_{P_J} = \sum_i (\phi_i)_{P_J}\) and \(V\) affords the character \(\chi_{P_J}\). Now \(V'\) is a \(CP_J\)-submodule of \(V\), hence each \((\phi_i)_{P_J}\) is a component of \(\chi_{P_J}\). Consequently

\[((\phi_i)^G, \chi) = ((\phi_i)_{P_J}, \chi_{P_J}) \neq 0,
\]
and \(\chi\) is a component of \((\phi_i)^G\). Thus it remains to prove that \(\phi_i\) is a cuspidal character.

If \(\phi_i\) is not cuspidal, then \((\phi_i)_{Q_K}, 1_{Q_K}) \neq 0\) for some \(K \subseteq J\). It follows that \(\dim_{C} \text{Hom}_{Q_K}(1, V) \neq 0\) and hence \((\chi_{Q_K}, 1_{Q_K}) \neq 0\). Hence \(K \in \mathcal{S}\), contradicting the minimality of \(J\). Thus the result holds true. \(\square\)

We illustrate this behaviour with an example.

**Example 6.1.7.** \(G = \text{Alt}(7)\) has a rank 2 characteristic 2 minimal parabolic system \(\{P_1, P_2\} \leq \mathcal{M}(G, B)\) with \(B \cong \text{Dih}(8)\), \(P_1 \cong \text{Dih}(8) : C_3\) and \(P_2 \cong \text{Sym}(4)\). Take \(B = \langle(1,2)(3,4), (1,3)(5,6)\rangle, P_1 = \langle B, (5,6,7)\rangle\) and \(P_2 = \langle B, (1,2,5)(3,4,6)\rangle\). Let \(\mathfrak{X}\) be the \(B\)-parabolic system given by \(\{(P_1, Q_J) | J \subseteq \{1, 2\}\}\) where \(Q_{12} = B\). So \(Q_{12} = B\), \(Q_1 = \langle(1,2)(3,4), (1,3)(2,4)\rangle, Q_2 = \langle(1,3)(5,6), (2,4)(5,6)\rangle\) and \(Q_{\{1,2\}} = 1\). It follows using [CCN+09] that for \(i = 1, 2\) we have

\[
\sum_{g \in Q_i} \chi(g) = \chi(1) + 3\chi(2A) \neq 0
\]
for any \(\chi \in \text{Irr}(G)\). Thus \(G\) has no \(\mathfrak{X}\)-cuspidal characters.

For \(\overline{P}_i = P_i/Q_i \cong \text{Sym}(3)\) we have that \(Q_{12} \subseteq \overline{P}_{12} \cong C_2\). It follows that there is one \(\overline{P}_i\)-cuspidal character, namely the sign character. We denote this character by \(\phi_i\) and also think of it as a \(P_i\) character. Using the notation from [CCN+09], calculations show that the constituent characters of \(\phi_1^G\) are \(\chi_3, \chi_4, \chi_7, \chi_9\). Meanwhile the constituent characters of \(\phi_2^G\) are \(\chi_3, \chi_4, \chi_5, \chi_7, \chi_8, \chi_9\).

For \(\overline{P}_{12} = 1\) we note that trivial character is \(\overline{P}_{12}\)-cuspidal, and it lifts to the trivial character \(1_B\). We have that the constituent characters of \(1_B^G\) are \(\chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9\). Thus we observe - not very surprisingly - that Proposition 9.1.5 of [Car93] does not extend to our more general situation, since \((\phi_1^G, \phi_2^G) = 5\), whilst \(\phi_1^G \neq \phi_2^G\).

Although by definition, to determine if a character is cuspidal we must check the cuspidal condition for every subgroup in the \(X\)-parabolic system, we shall shortly see that this is not actually necessary. First we give the analogue of Car93 Proposition 9.1.2.

**Proposition 6.1.8.** Let \(G\) be a group, \(X \leq G\) and \(\chi \in \text{Irr}(G)\). If \(\mathfrak{X}\) is an \(X\)-parabolic system of \(G\) of rank \(n\) having underlying indexing set \(I\), then the following are equivalent:
(i) $\chi$ is a $X$-cuspidal character of $G$.

(ii) $(\chi_{Q_J}, 1_{Q_J}) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(iii) $(\chi, 1^G_{Q_J}) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(iv) $\sum_{q \in Q_J} \chi(qg) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$ and all $g \in G$.

(v) $\sum_{q \in Q_J} \chi(qg) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$ and all $g \in G$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\chi$ is a $X$-cuspidal character of $G$. Thus for each pair of subgroups $(P_J, Q_J)$ for $J \subseteq I$ we have that either $Q_J = 1$ or

$$\sum_{q \in Q_J} \chi(q) = 0.$$ 

In particular

$$\sum_{q \in Q_J} \chi(q) 1_{Q_J}(q) = 0$$

and hence $(\chi_{Q_J}, 1_{Q_J}) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(ii) $\Rightarrow$ (iv) Let $Q_J$ be such that $Q_J \neq 1$ (if no such $Q_J$ exists, the result is vacuously true). Let $\rho$ be an irreducible representation corresponding to $\chi$, let $\rho'$ be an irreducible constituent of $\rho|_{Q_J}$ and let $d$ denote the degree of $\rho'$. The module corresponding to $\rho'$ has basis $\{e_1, \ldots, e_d\}$ and hence we may define coefficient functions $\rho'_{ij}$ for $i, j = 1, \ldots, d$ by

$$e_i g = \sum_{j=1}^d \rho'_{ij}(g) e_j.$$ 

By the orthogonality relations for the coefficient functions (as given in [Car93, Section 6.1]), it follows that

$$(\rho'_{ij}, (1_{Q_J})_{11}) = 0$$

for all $i, j = 1, \ldots, d$, as $1_{Q_J}$ is not an irreducible constituent of $\chi_{Q_J}$. Thus

$$\sum_{q \in Q_J} \rho'_{ij}(q) = 0$$

for all $i, j$ and hence

$$\sum_{q \in Q_J} \rho'(q) = 0$$

for all $i, j$. Since this holds for all irreducible components $\rho'$ of $\chi_{Q_J}$ we deduce that

$$\sum_{q \in Q_J} \rho(q) = 0.$$ 

(6.1.2)

Now let $g \in G$ be given. Multiplying (6.1.2) on the right by $\rho(g)$ gives

$$\sum_{q \in Q_J} \rho(qg) = \left( \sum_{q \in Q_J} \rho(q) \right) \rho(g) = 0.$$
Consequently, taking traces we obtain
\[
\sum_{q \in Q_J} \chi(qg) = 0.
\]

**(iv) \implies (i)** Taking \( g = 1 \) we see that \( \chi \) is a \( \mathfrak{X} \)-cuspidal character of \( G \).

**(ii) \implies (v) \implies (i)** This follows analogously by multiplying on the left by \( \rho(g) \) in (6.1.2).

**(ii) \implies (iii)** This follows by Frobenius reciprocity.

Proposition 6.1.8 infers that we only have to check that the cuspidal condition holds for certain “maximal” subgroups of a parabolic system to ascertain that a character is cuspidal.

**Corollary 6.1.9.** Let \( G \) be a group, \( X \subseteq G \), \( I = \{1, \ldots, n\} \) and let \( \mathfrak{X} = \{(P_J, Q_J)|J \subseteq I\} \) be an \( X \)-parabolic system of \( G \). Define
\[
\mathfrak{I} := \{J \subseteq I|Q_J \neq 1 \text{ and if } J \not\subseteq K \subseteq I, \text{ then } Q_K = 1\}.
\]
Then \( \chi \in \text{Irr}(G) \) is cuspidal precisely when
\[
\sum_{q \in Q_J} \chi(q) = 0
\]
for all \( Q_J \) such that \( J \in \mathfrak{I} \).

**Proof.** The condition is clearly necessary. To see that it is sufficient, let \( J' \subseteq I \) be such that \( Q_{J'} \neq 1 \). We shall show that
\[
\sum_{q \in Q_{J'}} \chi(q) = 0.
\]
Since \( Q_{J'} \neq 1 \), we see that \( J' \not\subseteq I \) and there exists some \( J \in \mathfrak{I} \) such that \( J' \subseteq J \). Consequently \( Q_J \subseteq Q_{J'} \). By assumption
\[
\sum_{q \in Q_J} \chi(q) = 0,
\]
and so \( (\chi|Q_J, 1_{Q_J}) = 0 \). The proof of Proposition 6.1.8 asserts that
\[
\sum_{q \in Q_J} \chi(qg) = 0
\]
for all \( g \in G \).

Let \( T \) denote a right transversal of \( Q_J \) in \( Q_{J'} \). Then
\[
\sum_{q \in Q_{J'}} \chi(q) = \sum_{t \in T} \left( \sum_{q \in Q_J} \chi(qt) \right) = 0
\]
as required. \( \square \)

The final result that we will use in classifying the \( p \)-cuspidal characters of the sporadic simple groups concerns irreducible characters of odd degree.
Lemma 6.1.10. Let $G$ be a finite group and $p$ an odd prime such that $|G| = p^a m$ for some $a \geq 1$ with $(p, m) = 1$. Assume that $G$ has a $p$-minimal parabolic system containing a parabolic subgroup with non-trivial $p$-core. If for each $G$-conjugacy class, $C$, of elements of order $p^b$ for $b \leq a$ and all $g \in C$ we have that

$$\langle g \rangle \cap \{g \in G \mid \text{ord}(g) = p^b\} \subseteq C, \quad (6.1.3)$$

then every $p$-cuspidal character of $G$ has even degree.

Proof. Assume that condition (6.1.3) holds for all non-trivial powers of $p$. Then a non-trivial $p$-core, $Q$, of a parabolic subgroup will intersect every conjugacy class of $p$-elements in a set of even order. Thus if the degree of $\chi \in \text{Irr}(G)$ is odd, then the same is true of

$$\sum_{g \in Q} \chi(g),$$

and hence $\chi$ is not a $p$-cuspidal character of $G$. \qed

6.2 2-Cuspidal Characters

We now work systematically through the sporadic simple groups, determining for each group $G$ and each 2-minimal parabolic system of $G$, which characters $\chi \in \text{Irr}(G)$ are 2-cuspidal. A summary of our results is given in Tables 6.1 and 6.2. Throughout, the notation $\chi_i \in \text{Irr}(G)$ is the same as that used in [CCN+09]. We shall also use the standard notation from [CCN+09] for the conjugacy classes of $G$.

6.2.1 The Mathieu Groups

$M_{11}$

There are three 2-minimal parabolic subgroups of $M_{11}$, namely

$$P_1 \sim 2_2^{1+2}.\text{Sym}(3), \quad P_2 \sim 3^3.\text{SD}_{16}, \quad \text{and} \quad P_3 \sim \text{Alt}(6).2,$$

(where $SD_{16}$ is the semidihedral group of order 16) and these give rise to three 2-minimal parabolic systems, each of rank 2. Since $O_2(P_2) = O_2(P_3) = 1$, we must consider a Sylow 2-subgroup of $M_{11}$. Such a subgroup will intersect the $M_{11}$-conjugacy classes $1A, 2A, 4A, 8A$ and $8B$ in $1, 5, 6, 2$ and $2$ elements respectively. It follows that the cuspidal relation for a Sylow 2-subgroup holds for $\chi_3, \chi_4 \in \text{Irr}(M_{11})$ (both of degree 10). Consequently, $\chi_3$ and $\chi_4$ are 2-cuspidal characters of the minimal parabolic system $\{P_2, P_3\}$. Finally, as $O_2(P_1)$ contains 1, 1 and 6 elements from the classes $1A, 2A$ and $4A$ respectively and

$$\chi_i(1A) + \chi_i(2A) + 6 \cdot \chi_i(4A) = 8$$

for $i = 3, 4$, we see that the minimal parabolic systems containing $P_1$ admit no 2-cuspidal characters.
6.2. 2-CUSPIDAL CHARACTERS

$M_{12}$

There are no 2-cuspidal characters for the unique 2-minimal parabolic system of $M_{12}$ given by

$$\{P_1 \sim 4^{2+2} \cdot \text{Sym}(3), P_2 \sim 2^{1+4} \cdot \text{Sym}(3)\}.$$ 

To see this, we observe that $O_2(P_1)$ intersects the $M_{12}$-conjugacy classes $1A, 2A, 2B, 4A$ and $4B$ in 1, 4, 15, 6 and 6 elements respectively, whilst $O_2(P_2)$ intersects these classes in 1, 12, 7, 6 and 6 elements respectively. Consequently the only character satisfying the cuspidal relation for $O_2(P_1)$ is $\chi_{13}$ (of degree 120). However

$$\chi_{13}(1A) + 12 \cdot \chi_{13}(2A) + 7 \cdot \chi_{13}(2B) + 6 \cdot \chi_{13}(4A) + 6 \cdot \chi_{13}(4B) = 64.$$ 

$M_{22}$

There is a unique 2-minimal parabolic system for $M_{22}$, namely

$$\{P_1 \sim 2^{4+2} \cdot \text{Sym}(3), P_2 \sim 2^4 \cdot \text{Sym}(5)\}.$$ 

The 2-cores $O_2(P_1)$ and $O_2(P_2)$ intersect the $M_{22}$-conjugacy classes $1A, 2A, 4A$ and $4B$ in 1, 27, 12, 24 and 1, 15, 0 and 0 elements respectively. The only elements $\chi \in \text{Irr}(M_{22})$ satisfying

$$\chi(1A) + 27 \cdot \chi(2A) + 12 \cdot \chi(4A) + 24 \cdot \chi(4B) = \chi(1A) + 15 \cdot \chi(2A) = 0$$

are the two characters of degree 45, $\chi_3$ and $\chi_4$.

$M_{23}$

The group $M_{23}$ has seven conjugacy classes of 2-minimal parabolic subgroups, six of which feature in 2-minimal parabolic systems of $M_{23}$. Using the notation of [RS84], these subgroups are

$$P_1 \sim 2^{4+2} \cdot \text{Sym}(3), \ P_2 \sim 2^{4+2} \cdot \text{Sym}(3), \ P_3 \sim 2^{4+2} \cdot \text{Sym}(3)$$

$$P_4 \sim 2^{4+2} \cdot \text{Sym}(3), \ P_6 \sim 2^4 \cdot \text{Sym}(5), \ P_7 \sim 2^4 \cdot \text{Sym}(5).$$

The 2-minimal parabolic systems are given by $\{P_1, P_3, P_7\}, \{P_3, P_4, P_7\}, \{P_2, P_3, P_7\}, \{P_1, P_6, P_7\}, \{P_2, P_6, P_7\}, \{P_3, P_6, P_7\}$ and $\{P_4, P_6, P_7\}$.

Considering the maximal 2-parabolic subgroups of these systems, we see that the maximal 2-parabolic subgroups involving $P_6$ and $P_7$ have trivial 2-cores. Thus we need to check sub-maximal parabolics in order to apply Corollary 6.1.9. The 2-cores $O_2(P_i)$ for $i = 1, 2, 3, 4$ intersect the $M_{23}$-classes $1A, 2A$ and $4A$ in 1, 27 and 36 elements respectively. The remaining sub-maximal parabolics, namely $P_6$, $P_7$, $P_{13}$ and $P_{34}$, have rank 4 elementary abelian 2-groups for their 2-cores. Thus the non-trivial elements of their 2-cores lie in the $M_{23}$-class $2A$. For each sub-maximal parabolic, the only irreducible $M_{23}$-characters satisfying the cuspidal relation are $\chi_3$ and $\chi_4$. Hence for each of the 2-minimal parabolic systems, the characters $\chi_3$ and $\chi_4$ of degree 45 are 2-cuspidal characters.
<table>
<thead>
<tr>
<th>Parabolic Subgroup</th>
<th>2-core</th>
<th>Order of intersection with $M_{24}$-class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{12}$</td>
<td>$2^6$</td>
<td>1 45 18 0 0</td>
</tr>
<tr>
<td>$P_{13}$</td>
<td>$2^{6+2}$</td>
<td>1 57 54 72 72</td>
</tr>
<tr>
<td>$P_{23}$</td>
<td>$2^{4+3}$</td>
<td>1 29 42 56 0</td>
</tr>
</tbody>
</table>

Table 6.4: The 2-cores of maximal parabolic subgroups of $M_{24}$.

The Mathieu group $M_{24}$ has a unique 2-minimal parabolic system given by

$$\{P_1 \sim 2^{6+3}.\text{Sym}(3), P_2 \sim 2^{6+3}.\text{Sym}(3), P_3 \sim 2^{6+3}.\text{Sym}(3)\}.$$

The maximal parabolic subgroups $P_{12}$, $P_{13}$ and $P_{23}$ all have non-trivial 2-cores, and their intersections with the $M_{24}$-conjugacy classes are summarised in Table 6.4. It follows that the characters $\chi_3, \chi_4, \chi_{12}, \chi_{13}, \chi_{15}$ and $\chi_{16}$ satisfy the cuspidal relation for $P_{12}$, as do $\chi_3, \chi_4, \chi_5, \chi_6, \chi_{12}, \chi_{13}, \chi_{15}$ and $\chi_{16}$ for $P_{13}$ and $\chi_3, \chi_4, \chi_5, \chi_6$ and $\chi_8$ for $P_{23}$. We conclude that $\chi_3$ and $\chi_4$ - both of degree 45 - are the only 2-cuspidal characters of $M_{24}$.

### 6.2.2 The Leech Lattice and Conway Groups

#### $HS$

The Higman-Sims group has a unique 2-minimal parabolic system of the form

$$\{P_1 \sim 4.2^4.\text{Sym}(5), P_2 \sim 4^3.2^2.\text{Sym}(3)\}.$$

Considering $O_2(P_1)$, we see that it intersects the $HS$-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$ and $4C$ in 1, 31, 0, 2, 30 and 0 elements respectively. Consequently, there are no 2-cuspidal characters of $HS$, as the cuspidal relation does not hold for $P_1$.

#### $J_2$

There is a unique 2-minimal parabolic system of $J_2$ given by

$$\{P_1 \sim 2^{2+4}.3\text{Sym}(3), P_2 \sim 2^{1+4}.L_2(4)\}.$$

The intersections of $O_2(P_1)$ with the $J_2$-classes $1A$, $2A$, $2B$ and $4A$ have orders 1, 3, 24 and 36 respectively. Consequently, the cuspidal relation on $P_1$ holds for the irreducible characters $\chi_4, \chi_5, \chi_{14}$ and $\chi_{15}$. Meanwhile, the 2-core $O_2(P_2)$ intersects the given $J_2$-classes in 1, 11, 0 and 20 elements respectively, meaning that the cuspidal relation holds on $P_2$ for the characters $\chi_8, \chi_9$ and $\chi_{18}$. We conclude that $J_2$ admits no 2-cuspidal characters.

#### $Co_1$

The largest Conway group, $Co_1$, admits a unique 2-minimal parabolic system, having rank 4. Its minimal parabolic subgroups are given by $P_i \sim [2^{20}].\text{Sym}(3)$ for $i = 1, \ldots, 4$ and the corresponding 2-maximal parabolic subgroups have the form $P_{123} \sim 2^{2+12+3}.(\text{Sym}(3) \times L_3(2))$,
Corresponding Characters

\[
\chi_1 (276), \chi_4 (1771), \chi_5 (8855), \chi_8 (37674), \chi_{11} (94875), \chi_{13} (345345), \\
\chi_{15} (483000), \chi_{21} (1434510), \chi_{23} (1771000), \chi_{27} (2464749), \chi_{28} (2464749)
\]

Table 6.5: The intersections of the 2-cores of the maximal 2-parabolic subgroups of \(Co_1\) with the \(Co_1\)-conjugacy classes.

<table>
<thead>
<tr>
<th>Parabolic Subgroup</th>
<th>Corresponding Characters (character degrees)</th>
</tr>
</thead>
</table>
| \(P_{123}\)        | \(\chi_2 (276), \chi_4 (1771), \chi_5 (8855), \chi_8 (37674), \chi_{11} (94875), \chi_{13} (345345), \)
|                    | \(\chi_{15} (483000), \chi_{21} (1434510), \chi_{23} (1771000), \chi_{27} (2464749), \chi_{28} (2464749)\) |
| \(P_{124}\)        | \(\chi_2 (276), \chi_8 (37674), \chi_{11} (94875), \chi_{21} (1434510), \chi_{27} (2464749), \chi_{28} (2464749)\) |
| \(P_{134}\)        | \(\chi_2 (276), \chi_4 (1771), \chi_5 (8855), \chi_8 (37674), \chi_{11} (94875), \chi_{13} (345345), \)
|                    | \(\chi_{15} (483000), \chi_{21} (1434510), \chi_{23} (1771000), \chi_{27} (2464749), \chi_{28} (2464749)\) |
| \(P_{234}\)        | \(\chi_2 (276), \chi_4 (1771), \chi_8 (37674), \chi_9 (44275), \chi_{11} (94875), \chi_{13} (345345)\) |

Table 6.6: The elements of \(\text{Irr}(Co_1)\) satisfying the cuspidal relation for each maximal 2-parabolic subgroup of \(Co_1\).

\[P_{124} \sim 2^{11+12} \cdot (\text{Sym}(3) \times 3S_4(2)), \quad P_{134} \sim 2^{11} \cdot M_{24} \quad \text{and} \quad P_{234} \sim 2^{1+8+6} \cdot L_4(2).\]

The orders of the intersections of the 2-cores of the maximal parabolic subgroups with the \(Co_1\)-conjugacy classes are given in Table 6.5. A summary of the elements of \(\text{Irr}(Co_1)\) which satisfy the cuspidal relation for each of the maximal parabolics is given in Table 6.6. We conclude that \(Co_1\) admits three 2-cuspidal characters, namely \(\chi_2, \chi_8\) and \(\chi_{11}\).

\(Co_2\)

The group \(Co_2\) has a 2-minimal parabolic system of the form \(\{P_1, P_2, P_3\}\), where \(P_i \sim [2^{15}] \cdot \text{Sym}(5)\) and \(P_i \sim [2^{17}] \cdot \text{Sym}(3)\) for \(i = 2, 3\). This system has maximal parabolic subgroups \(P_{12} \sim 2^{4+10} \cdot (\text{Sym}(3) \times \text{Sym}(5))\), \(P_{13} \sim 2^{10} \cdot M_{22} 2\) and \(P_{23} \sim 2^{1+8+6} \cdot L_3(2)\). The orders of the intersections of the 2-cores of these maximal parabolics with the relevant \(Co_2\)-conjugacy classes are given in Table 6.7. It follows that the cuspidal relation holds on \(P_{12}\) for the characters \(\chi_2, \chi_3, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{21}, \chi_{22}, \chi_{23}, \chi_{31}, \chi_{32}\) and \(\chi_{37}\); it holds on \(P_{13}\) for \(\chi_3, \chi_5, \chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{21}, \chi_{22}, \chi_{23}, \chi_{25}, \chi_{31}, \chi_{32}, \chi_{36}, \chi_{37}\) and \(\chi_{40}\). Consequently, \(Co_2\) admits eight 2-cuspidal characters namely \(\chi_3\) (of degree 253), \(\chi_{10} (9625), \chi_{11} (9625), \chi_{12} (10395), \chi_{13} (10395), \chi_{16} (31625), \chi_{31} (239085)\) and \(\chi_{32} (239085)\).

<table>
<thead>
<tr>
<th>2-core</th>
<th>Order of intersection with (Co_2)-conjugacy class</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1A)</td>
</tr>
<tr>
<td>(O_2(P_{12}))</td>
<td>1</td>
</tr>
<tr>
<td>(O_2(P_{13}))</td>
<td>1</td>
</tr>
<tr>
<td>(O_2(P_{23}))</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6.7: The 2-cores of the maximal parabolic subgroups of \(Co_2\).
Co3

There is a unique 2-minimal parabolic system of Co3 given by

\[ \{P_1 \cong 2^{4+4+1}.\text{Sym}(3), P_2 \cong 2^{4+4+1}.\text{Sym}(3), P_3 \cong 2^{4+4+1}.\text{Sym}(3)\}, \]

which has maximal parabolic subgroups \( P_{12} \cong 2^{2+6}.3(\text{Sym}(3) \times \text{Sym}(3)), P_{13} \cong 4\cdot 2L(3)(2) \) and \( P_{23} \cong 4.2^4.\text{Sp}_4(2) \). We have that \( O_2(P_{23}) \) intersects the Co3-conjugacy classes 1A, 2A, 2B, 4A and 4B in 1, 31, 0, 2 and 30 elements respectively, and hence the cuspidal condition does not hold for \( P_{23} \). Thus Co3 admits no 2-cuspidal characters.

McL

The McLaughlin group has a multitude of 2-minimal parabolic systems comprising of the 2-minimal parabolic subgroups \( P_i \) and \( P_i^\sigma \) for \( i = 1, \ldots, 5 \), where \( \sigma \) is the non-trivial outer automorphism of McL (as seen in Example 2.5.6). Here \( P_3^\sigma = P_3, P_4^\sigma = P_4, P_i \cong 2^{4+2}.\text{Sym}(3) \) for \( i = 1, 2, 3, 4 \) and 5 \( P_5 \cong 2^4.\text{Sym}(5) \). These subgroups give rise to the minimal parabolic systems

\[ \{P_1, P_3, P_5^\sigma\}, \{P_2, P_5, P_5^\sigma\}, \{P_2, P_3, P_5^\sigma\}, \{P_1, P_4, P_5^\sigma\}, \{P_1, P_4, P_5\}. \]

Since the 2-cores of \( P_3 \) and \( P_5^\sigma \) are elementary abelian of rank 4, we see that any minimal parabolic system containing either of these minimal parabolics will not admit a 2-cuspidal character. Conversely, any 2-minimal parabolic system not containing these subgroups will contain the parabolic subgroup \( P_{11} := \langle P_1, P_1^\sigma \rangle \). Since \( O_2(P_{11}) \) intersects the McL-conjugacy classes 1A, 2A and 4A in 1, 19 and 12 elements respectively, we see that the cuspidal relation does not hold for \( P_{11} \) and hence none of the 2-minimal parabolic systems of McL admit a 2-cuspidal character.

Suz

The group Suz has a unique 2-minimal parabolic system, which has rank 3. Its minimal parabolic subgroups satisfy \( P_1 \cong 2^{4+6+1}.L_2(4), P_2 \cong 2^{4+6+2}.(3 \times L_2(2)) \) and \( P_3 \cong 2^{6+4+2}.(3 \times L_2(2)) \). The maximal parabolic subgroups are given by \( P_{12} \cong 2^{1+6}.\text{U}_4(2), P_{13} \cong 2^{2+8}.(\text{Sym}(3) \times L_2(4)) \) and \( P_{23} \cong 2^{4+6}.\text{Sp}_4(2)' \). The 2-core \( O_2(P_{12}) \) intersects the Suz-conjugacy classes 1A, 2A and 4A in 1, 55 and 72 elements respectively. We deduce that there are no 2-cuspidal characters of Suz.

6.2.3 The Monster Group and its Subgroups

He

There are four 2-minimal parabolic subgroups of He given by \( P_1 \cong P_1 \cong 2^{6+3}.\text{Sym}(3) \) and \( P_2 \cong P_3 \cong 2^{6+3}.\text{Sym}(3) \). These give rise to the 2-minimal parabolic systems \( \{P_1, P_2, P_3\} \) and \( \{P_1, P_3, P_4\} \). Considering the maximal parabolic subgroups \( P_{14} \) and \( P_{13} \cong P_{24} \) we see that \( O_2(P_{14}) \) intersects the He-conjugacy classes 1A, 2A, 2B, 4A, 4B and 4C in 1, 18, 45, 0, 0 and 0 elements respectively,
whilst $O_2(P_{13})$ intersects the respective conjugacy classes in $1$, $42$, $29$, $0$, $56$ and $0$ elements. It follows that the cuspidal relation for $P_{14}$ holds for the characters $\chi_7, \chi_8 \in \text{Irr}(He)$, whilst for $P_{13} \cong P_{24}$ the cuspidal relation holds for $\chi_4, \chi_5 \in \text{Irr}(He)$. Since each 2-minimal parabolic system contains $P_{14}$ and either $P_{13}$ or $P_{24}$, we conclude that there are no 2-cuspidal characters of $He$.

$HN$

There is a unique 2-minimal parabolic system of $HN$ given by

$$\{P_1 \sim 2^{1+8}.\text{Alt}(5) \wr \mathbb{Z}_2, P_2 \sim 2^{2+3+6+2}_3\text{Sym}(3)\}.$$  

We consider the minimal parabolic subgroup $P_1$, whose character table is given in [Har76 TABLE IV], and we adopt the notation given in [Har76] for the $P_1$-conjugacy classes. We have that each $P_1$-class is either contained in, or is disjoint from $O_2(P_1)$. It follows that $O_2(P_1) = 1_1 \cup 2_1 \cup 2_2 \cup 2_3 \cup 4_1$.

Considering the centralizer orders of $1_1$, $2_1$, $2_2$, $2_3$ and $4_1$ in $P_1$ and the orders of the centralizers of 2-elements in $HN$, we see that the $P_1$-classes $2_1$ and $2_2$ are contained in the $HN$-class $2B$, the $P_1$-class $2_3$ lies in either $HN$-class $2A$ or $2B$, and the $P_1$-class $4_1$ is contained in the $HN$-class $4A$. It follows that $|O_2(P_1) \cap 2A| = 0$ or $120$, $|O_2(P_1) \cap 2B| = 151$ or $271$ and $|O_2(P_1) \cap 4A| = 240$. It follows that the cuspidal relation on $P_1$ does not hold for any $\chi \in \text{Irr}(HN)$, and hence there are no 2-cuspidal characters of $HN$.

$Th$

The 2-minimal parabolic system

$$\{P_1 \sim 2^{1+8}.\text{Alt}(9), P_2 \sim 2^{5+6+2+1}\cdot\text{Sym}(3)\}$$

of $Th$ is unique. Considering fusion within the maximal subgroup $2^5.L_5(2) > P_2$ we find that $O_2(P_2)$ intersects the $Th$-conjugacy classes $1A, 2A, 4A, 4B, 8A$ and $8B$ in $1, 687, 656, 7104, 4864$ and $3072$ elements respectively. It follows that the cuspidal relation holds on $P_2$ for $\chi_2, \chi_6 \in \text{Irr}(Th)$. Considering the normal subgroups of a Sylow 2-subgroup of $2^5.L_5(2)$ having order $2^9$ and exponent $4$, we see that for each such subgroup the only element of $\text{Irr}(Th)$ for which the cuspidal relation holds is $\chi_2$ of degree $248$. Thus $\chi_2$ is the unique 2-cuspidal character of $Th$.

$Fi_{22}$

There is a unique 2-minimal parabolic system of $Fi_{22}$ given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{16}].\text{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{14}].\text{Sym}(5)$. This system has maximal parabolic subgroups $P_{12} \sim 2^{9+4+2}.(\text{Sym}(3) \times \text{Sym}(3))$, $P_{13} \sim 2^{2+8}.U_4(2)2$ and $P_{23} \sim 2^{10}.M_{22}$. The 2-core $O_2(P_{13})$ intersects the $Fi_{22}$-conjugacy classes $1A, 2A, 2B, 2C, 4A, 4B, 4C, 4D$ and $4E$ in $1, 2, 271, 270, 480, 0, 0, 0$ and $0$ elements respectively. We see that the cuspidal relation does not hold on $P_{13}$ and hence $Fi_{22}$ admits no 2-cuspidal characters.
The group $Fi_{23}$ has eight 2-minimal parabolic subgroups, seven of which feature in 2-minimal parabolic systems. Using the notation of [RSS4] these have the form $P_i \sim [2^{17}] \cdot \text{Sym}(3)$ for $i = 1, \ldots, 5$ and $P_6 \sim [2^{15}] \cdot \text{Sym}(5)$ for $i = 7, 8$. These give rise to the geometric 2-minimal parabolic systems $\{P_1, P_3, P_5, P_8\}$ and $\{P_1, P_4, P_6, P_8\}$ and the non-geometric systems $\{P_1, P_2, P_5, P_8\}$ and $\{P_1, P_7, P_8\}$. The maximal parabolic subgroups of these systems are

\[
P_{125} \sim 2^{10+4} \cdot \text{Sym}(3) \times \text{Alt}(7),
\]
\[
P_{128} = P_{138} = P_{148} \sim 2^2 \times 2^{1+8}(3 \times U_4(2))_2,
\]
\[
P_{135} \sim [2^{14}] \cdot (\text{Sym}(3) \times L_3(2)),
\]
\[
P_{145} \sim [2^{14}] \cdot (\text{Sym}(3) \times L_3(2)),
\]
\[
P_{158} \sim 2F_{22},
\]
\[
P_{18} \sim [2^{11}] \cdot U_4(2)_2,
\]
\[
P_{258} = P_{358} = P_{458} \sim 2^{11} \cdot M_{23},
\]
\[
P_{78} \sim 2^{11} \cdot M_{21}.
\]

It is easy to check that the cuspidal relation does not hold for $P_{158}$, and hence the three 2-minimal parabolic systems of rank 4 do not admit any 2-cuspidal characters.

Finally, we consider the maximal parabolic subgroup $P_{18} < P_{158}$. We see that the 2-core $O_2(P_{18})$ intersects the $Fi_{23}$-conjugacy classes $1A, 2A, 2B, 2C, 4A, 4B, 4C$ and $4D$ in $1, 3, 273, 811, 0, 960, 0$ and $0$ elements respectively. Consequently, the cuspidal relation does not hold for $P_{18}$, and hence there are no 2-cuspidal characters of $Fi_{23}$.

$Fi'_{24}$

There is a unique 2-minimal parabolic system of $Fi'_{24}$, which has rank 4. The maximal parabolic subgroups are $P_a \sim 2^{1+12} \cdot (3U_4(3) \cdot 2)$, $P_b \sim 2^{3+12} \cdot (\text{Sym}(3) \times Sp_4(2)^r)$, $P_c \sim 2^{8+6+3} \cdot (L_3(2) \times \text{Sym}(3))$ and $P_d \sim 2^{11} \cdot M_{24}$. Since $O_2(P_d)$ is elementary abelian, we consider the minimum value that each $\chi \in \text{Irr}(Fi'_{24})$ takes on elements of order 2. We immediately deduce that the only possible 2-cuspidal character of $Fi'_{24}$ is $\chi_2$ of degree 8671. For $\chi_2$ to be 2-cuspidal, we would require an integer solution to

\[
\chi_2(1A) + j \cdot \chi_2(2A) + (2^{11} - j - 1) \cdot \chi_2(2B) = 0.
\]

Since no such solution exists, we conclude that there are no 2-cuspidal characters of $Fi'_{24}$.

B

The baby monster has five conjugacy classes of 2-minimal parabolic subgroups having representatives $P_i \sim [2^{40}] \cdot \text{Sym}(3)$ for $i = 1, \ldots, 4$ and $P_5 \sim [2^{38}] \cdot \text{Sym}(5)$. These give rise to a unique 2-minimal parabolic system $\{P_1, P_2, P_3, P_5\}$. The maximal parabolic subgroups of this system are given by $P_{123} \sim 2^{9+16+6+4} \cdot L_4(2)$, $P_{125} \sim 2^{3+32} \cdot (L_3(2) \times \text{Sym}(5))$, $P_{135} \sim 2^{2+10+20} \cdot (\text{Sym}(3) \times M_{22}^2)$ and $P_{235} \sim 2^{12+22} \cdot C_2$. All of these maximal parabolic subgroups are 2-radical. Indeed, from [Yos05] we observe that all 2-parabolic subgroups generated by $P_1, \ldots, P_5$ are 2-radical with the exception of $P_3, P_4$ and $P_{34}$. The reader can find further information regarding the structure of the 2-radical parabolic subgroups in [Yos05].
2.2. 2-CUSPIDAL CHARACTERS

| Element, $m_i$ | $|m_i^G|\,$ | $\text{Stab}_{Co_2}(m_i)$ | $[\text{Stab}_{Co_2}(m_i)]$ |
|---------------|----------------|-----------------------------|-----------------------------|
| $m_1$         | 2,049,300      | $(2^2 \times 2^4)\cdot \text{Sym}(8)$ | $20,643,840 = 2^{10} \cdot 3^2 \cdot 5.7$ |
| $m_2$         | 1,619,200      | $U_4(3) : \text{Dih}(8)$ | $26,127,260 = 2^{10} \cdot 3^6 \cdot 5.7$ |
| $m_3$         | 476,928        | $HS : 2$                     | $88,704,000 = 2^{10} \cdot 3^2 \cdot 5^3 \cdot 7.11$ |
| $m_4$         | 46,575         | $2^{10} : M_{22} : 2$        | $908,328,960 = 2^{18} \cdot 3^2 \cdot 5.7.11$ |
| $m_5$         | 2,300          | $U_6(2) : 2$                 | $18,393,661,440 = 2^{16} \cdot 3^6 \cdot 5.7.11$ |

Table 6.8: The orbits of the 22-dimensional $GF(2)Co_2$-module, $M$.

| Conjugacy class, $*$ | $|C_E(*)|$ | Conjugacy class, $*$ | $|C_E(*)|$ |
|----------------------|------------|----------------------|------------|
| $2A$                 | $2^{38} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ | $4D$ | $2^{28} \cdot 3^3 \cdot 5.7$ |
| $2B$                 | $2^{41} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ | $4E$ | $2^{25} \cdot 3^4 \cdot 5.7$ |
| $2C$                 | $2^{27} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ | $4F$ | $2^{29} \cdot 3^2 \cdot 5$ |
| $2D$                 | $2^{38} \cdot 3^5 \cdot 5 \cdot 7$ | $4G$ | $2^{27} \cdot 3^2 \cdot 5$ |
| $4A$                 | $2^{32} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | $4H$ | $2^{25} \cdot 3^2$ |
| $4B$                 | $2^{32} \cdot 3^6 \cdot 5 \cdot 7$ | $4I$ | $2^{14} \cdot 3^3 \cdot 5 \cdot 13$ |
| $4C$                 | $2^{30} \cdot 3^3 \cdot 7$ | $4J$ | $2^{23} \cdot 3.5$ |

Table 6.9: The orders of centralizers of elements of order 2 and 4 in $E$.

Noting that $P_{235} \cong C_E(2B)$, we focus on the 2-core $Q := O_2(P_{235}) \cong 2^{4+22}$ and we shall assume that $Co_2$ admits a unique class of 22-dimensional modules over $GF(2)$, namely that given in $[ABL]$, having representative $M$. Consider the orbits of $M$ under $Co_2$. There are five such orbits, represented by elements $m_1, \ldots, m_5$. Details of orbits and stabilizers of the $m_i$ under $Co_2$ are given in Table 6.8 whilst the orders of the centralizers of elements of order 2 and 4 in $E$ are given in Table 6.9. As an aside, we note that the stabilizers of $m_2, m_3, m_4$ and $m_5$ are all maximal in $Co_2$.

Let $z$ denote the central involution of $Z(Q)$. Thus each $m_i$ gives rise to a subgroup of $Q$ of order 4. Hence as each element of $E$ of order 4 is conjugate to its cube, we see that each $m_i$ either gives rise to two $Co_2$-orbits of involutions or two $Co_2$-orbits of elements of order 4, each having size as given in Table 6.8. Since $Q$ contains 4,196,351 elements of order 2 and 4,192,256 elements of order 4, we conclude that $m_1$, $m_4$ and $m_5$ correspond to orbits of involutions, whilst $m_2$ and $m_3$ correspond to orbits of elements of order 4. Comparing the stabilizer orders in Table 6.8 with the centralizer orders in Table 6.9 we conclude that both orbits arising from $m_2$ contain elements of the $E$-class 4B, whilst both orbits arising from $m_3$ contain elements of the $E$-class 4A.

Let $n_i$ denote a non-central involution in the subgroup of $Q$ that is the pre-image of $m_i$ in $M$ for each $i = 1, 4, 5$. Since $Q/Z(Q)$ is elementary abelian and $|Z(Q)| = 2$ we see that $[Q : C_Q(z, n_i)] = 2$, and hence $n_i$ and $z \cdot n_i$ are $P_{235}$-conjugate. Thus the $P_{235}$-orbits of $n_1$, $n_4$ and $n_5$ have respective orders 409,8600, 93,150 and 4600. Since $|\text{Stab}_{Co_2}(m_4)| = 2^{18} \cdot 3^2 \cdot 5 \cdot 7.11$, we have that $|\text{Stab}_{P_{235}}(n_4)| = 2^{40} \cdot 3^2 \cdot 5 \cdot 7.11$, and hence elements of the orbit of $n_4$ lie in the $E$-class 2B. This leaves the elements $n_1$ and $n_5$ for which $|\text{Stab}_{P_{235}}(n_1)| = 2^{38} \cdot 3^2 \cdot 5 \cdot 7$ and $|\text{Stab}_{P_{235}}(n_5)| = 2^{38} \cdot 3^6 \cdot 5 \cdot 7.11$.

Let $x$ be an element of the $E$-class 2A. By the ATLAS we have that there are five orbits of $x^E$ under the action of $C_E(x) \sim 2.2.E_6(2) : 2$, as summarised in Table 6.10. Let $x, y \in 2A_E$ be such that $xy = z \in 2B$. Thus $\langle x, y \rangle \leq Q$. From Table 6.10 we see that $[C_E(x) : C_E(x) \cap C_E(xy)] = 3,968,055$.
and hence
\[ |C_{BE}(x) \cap Q| = |C_{BE}(x) \cap C_{BE}(xy)| = 77,148,607,752,437,760 = 2^{38} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11. \]

It follows that \( n_5 \) is an element of the \( BE \)-class 2A, and that the orbit of \( n_5 \) under \( P_{235} \) accounts for all elements of class 2A in \( Q \).

Finally, consider the \( P_{235} \)-orbit of \( n_1 \). Consideration of the centralizer order and the preceding paragraph allows us to deduce that \( n_1 \) is in \( 2BE \) or \( 2D \). If \( n_1 \in 2BE \), then there would exist an element of \( 2BE \) admitting a \( P_{235} \)-orbit of size 4098600. However, the sizes of such orbits are given in [RW04, Theorem 1] and none have this order. Hence \( n_1 \in 2D \). We conclude that \( Q = O_2(P_{235}) \) contains 1, 4600, 93151, 4098600, 953856 and 3238800 elements of the \( BE \)-classes 1A, 2A, 2B, 2D, 4A and 4B respectively, and thus the cuspidal relation on \( P_{235} \) holds for the characters \( \chi_{10} \) and \( \chi_{11} \) of respective degrees 1, 407, 126, 890 and 3, 214, 743, 741.

At the time of writing, we have been unable to ascertain if the cuspidal relations for \( P_{123}, P_{125} \) or \( P_{135} \) hold for either \( \chi_{10} \) or \( \chi_{11} \), and hence whether \( \chi_{10} \) and \( \chi_{11} \) are 2-cuspidal characters of \( B \).

\[ \mathcal{M} \]

The monster group has a unique 2-minimal parabolic system, \( \{P_1, P_2, P_3, P_4, P_5\} \), where \( P_i \sim [2^{45}]L_2(2) \) for \( i = 1, \ldots, 5 \). The maximal parabolic subgroups are given by \( P_{1234} \sim 2^{5+5+16+10}L_5(2), P_{1235} \sim 2^{4+1+2+8+8+12+4}(L_4(2) \times \text{Sym}(3)), P_{1245} \sim 2^{3+36}(L_3(2) \times 3S_4(2)), P_{1345} \sim 2^{2+11+22}(\text{Sym}(3) \times M_{24}) \) and \( P_{2345} \sim 2^{1+21}.Co_1 \).

We observe that there is no \( \chi \in \text{Irr}(\mathcal{M}) \) that satisfies the cuspidal relation for \( O_2(P_{2345}) \). Indeed, let \( z \) be an involution of \( \mathcal{M} \) in class 2B and let \( \Lambda \) be the leach lattice as defined in [Asc94]. Moreover, let \( \Lambda_i \) be the set of all vectors in \( \Lambda \) of type \( i \) defined as
\[ \Lambda_i := \{v \in \Lambda | (v, v)/16 = i\}. \]

Then calculations in [Asc94] show that
\[ |\Lambda_2| = 196,560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13, \]
\[ |\Lambda_3| = 2^{12}(2^{12} - 1) = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13, \text{ and} \]
\[ |\Lambda_4| = 398,034,000 = 2^4 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13. \]

Let \( G := -1 \) - the automorphism group of \( \Lambda \), and let \( \widetilde{G} := G/\langle \varepsilon_X \rangle \) (where \( \varepsilon_X \) is the scalar map defined on \( \Lambda \) by \(-1\)). Thus \( \widetilde{G} \) is equal to \( Co_1 \). Since \( \varepsilon_X \) acts trivially on \( \widetilde{\Lambda} := \Lambda/2\Lambda \), and \( G \) acts transitively on the set \( \Lambda_2 \) ([Asc94, Lemma 22.12(1)]) and on each of the sets \( \Lambda_3, \Lambda_4 \) ([Asc94, Lemma 22.14(1)]), it follows that \( \widetilde{G} \) acts transitively on the sets \( \widetilde{\Lambda}_2, \widetilde{\Lambda}_3 \) and \( \widetilde{\Lambda}_4 \) (where \( \widetilde{\Lambda}_i \) is the image of \( \Lambda_i \) in \( \widetilde{\Lambda} \)).
Without loss, we may assume that $z$ is the central involution of the extra-special group $O_2(P_{2345})$ and we define $P_{2345} := P_{2345}/\langle z \rangle$. By considering the action of $G$ on $\tilde{A}$ we have that
\[
\begin{align*}
|\text{Stab}_G(\tilde{\lambda}_2)| &= 2^{17} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_G(\tilde{\lambda}_3)| &= 2^9 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_G(\tilde{\lambda}_4)| &= 2^{17} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23
\end{align*}
\]
(where $\lambda_i \in \Lambda_i$). It follows that
\[
\begin{align*}
|\text{Stab}_{P_{2345}}(\tilde{\lambda}_2)| &= 2^{41} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_{P_{2345}}(\tilde{\lambda}_3)| &= 2^{33} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|\text{Stab}_{P_{2345}}(\tilde{\lambda}_4)| &= 2^{41} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.
\end{align*}
\] (6.2.4)

The question remains, how does the element $\lambda_i$ lift to the extra-special group $2^{1+24} = O_2(P_{2345})$? Once this is established, we may then use the centralizer orders
\[
\begin{align*}
|C_{\tilde{\Gamma}}(2A)| &= 2^{42} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47, \\
|C_{\tilde{\Gamma}}(2B)| &= 2^{46} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23, \\
|C_{\tilde{\Gamma}}(4A)| &= 2^{34} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\
|C_{\tilde{\Gamma}}(4B)| &= 2^{27} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17, \\
|C_{\tilde{\Gamma}}(4C)| &= 2^{34} \cdot 3^4 \cdot 5^7, \text{ and} \\
|C_{\tilde{\Gamma}}(4D)| &= 2^{27} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13
\end{align*}
\] (6.2.5)
to determine the fusion within $O_2(P_{2345})$. Indeed, we have that $C_{\tilde{\Gamma}_{P_{2345}}} \sim 2^{24}.C_{\tilde{\Gamma}3} \sim 2^{24}.Co_3$ and $C_{\tilde{\Gamma}_{P_{2345}}} \sim 2^{24}.(2^{11} : M_{24})$.

There are two possible ways in which a $\tilde{\lambda}_i$ can lift into $2^{1+24}$, namely to an abelian subgroup of order 4 of the form $\langle \lambda_i, z \rangle$ having exponent 2 or 4. The former case occurs when $\text{ord}(\lambda_i) = 2$, whilst the latter case occurs when $\text{ord}(\lambda_i) = 4$ and hence $\lambda_i^2 = z$. Since $(4A)^3 = 4A$, $(4B)^3 = 4B$, $(4C)^3 = 4C$ and $(4D)^3 = 4D$, we may use the centralizer and stabilizer orders from (6.2.5) and (6.2.4) to see that the only possible elements of order 4 in $P_{2345}$ must lie in the $M$-conjugacy class $4A$. Since the exponent of $2^{1+24}$ is 4, we conclude that the elements of the orbit $\Gamma_{\tilde{\lambda}_3}^G$ lift to cyclic groups of order 4 containing 1, $z$ and two elements from the $M$-class $4A$. This means that
\[
|O_2(P_{2345}) \cap 4A| = 2 \cdot |\tilde{\Lambda}_3| = |\Lambda_3| = 2^{12}(2^{12} - 1) = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 16,773,120.
\]

Next we consider the lifts of $\tilde{\lambda}_2$ and $\tilde{\lambda}_4$. We see that these must lift to the elementary abelian subgroups $\langle \lambda_2, z \rangle$ and $\langle \lambda_4, z \rangle$, respectively. Since there are only two $M$-classes of involutions, we have that $(\lambda_2 z)^{g_2} = \lambda_2$ and $(\lambda_4 z)^{g_4} = \lambda_4$ for some $g_2, g_4 \in M$. To determine which element lifts to class $2A$ and which element lifts to $2B$, we note that by (MS02) Lemma 4.4 for $x \neq z$ an involution of $P_{2345}$, either $C_{P_{2345}}(x) \sim 2^{1+23}.Co_2$ (if $x$ is not 2-central) or $C_{P_{2345}}(x) \sim 2^{1+23}.(2^{11} : M_{24})$ if $x$ is 2-central. Here
\[
\begin{align*}
|2^{1+23}.Co_2| &= 2^{42} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 \text{ and} \\
|2^{1+23}.(2^{11} : M_{24})| &= 2^{45} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23.
\end{align*}
\] (6.2.6)
Combining (6.2.6) with (6.2.4) and the fact that the 2-central elements of $P_{2345}$ lie in $2B$, we have that $\widetilde{\lambda}_2$ lifts to an elementary abelian subgroup generated by $z$ and an involution of the $\mathcal{M}$-class $2A$, whilst $\widetilde{\lambda}_4$ lifts to a subgroup generated by $z$ and an element of $2B$. Since $z$ also lies in $2B$, we conclude that

\[
|O_2(P_{2345}) \cap 2A| = 2 \cdot |\lambda_2| = |\lambda_2| = 196,560,
\]

\[
|O_2(P_{2345}) \cap 2B| = 2 \cdot |\lambda_4| + 1 = |\lambda_4|/24 + 1 = 16,584,751 \quad \text{and}
\]

\[
|O_2(P_{2345}) \cap 4A| = 2 \cdot |\lambda_3| = |\lambda_3| = 16,773,120.
\]

As there is no $\chi \in \text{Irr}(\mathcal{M})$ satisfying

\[
\chi(1A) + 196560 \cdot \chi(2A) + 16584751 \cdot \chi(2B) + 16773120 \cdot \chi(4A) = 0,
\]

it follows that there are no 2-cuspidal characters of $\mathcal{M}$.

### 6.2.4 The Pariahs

#### $J_1$

The normalizer of a Sylow 2-subgroup of $J_1$ is maximal. Thus as the cuspidal relation does not hold for such a Sylow subgroup, $J_1$ has no 2-cuspidal characters.

#### $O'N$

The group $O'N$ admits a unique 2-minimal parabolic system of the form

\[
\{P_1 \sim 4^3 \cdot 2^2 \cdot \text{Sym}(3), P_2 \sim 4 \cdot L_3(4)2\}.
\]

The generators of $O_2(P_2)$ are elements of the $O'N$-conjugacy class $4A$. Thus we see that the cuspidal relation does not hold on $P_2$ for any $\chi \in \text{Irr}(O'N)$. Hence $O'N$ admits no 2-cuspidal characters.

#### $J_3$

There is a unique 2-minimal parabolic system of $J_3$, given by

\[
\{P_1 \sim 2^{2+4} \cdot (3 \times \text{Sym}(3)), P_2 \sim 2^{1+4} \cdot L_2(4)\}.
\]

Considering the 2-core $O_2(P_2)$, it contains 1, 11 and 20 elements from the $J_3$-conjugacy classes $1A$, $2A$ and $4A$ respectively. It follows that $J_3$ does not admit any 2-cuspidal characters.

#### $Ru$

There are three 2-minimal parabolic subgroups of $Ru$ given by $P_1 \sim 2^{5+6} \cdot \text{Sym}(5)$ and $P_i \sim 2^{5+6+2} \cdot \text{Sym}(3)$ for $i = 2, 3$. Since $P_3 \leq P_1$, we obtain a unique 2-minimal parabolic system, namely $\{P_1, P_2\}$. Considering the 2-cores of $P_1$ and $P_2$, we see that $O_2(P_1)$ intersects the $Ru$-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$, $4C$, $4D$, $8A$, $8B$ and $8C$ in $1, 271, 0, 512, 64, 240, 960, 0, 0$ and $0$ elements respectively, whilst $O_2(P_2)$ intersects the given $Ru$-classes in respectively $1, 367, 192, 608, 448, 1296, 1440, 1536, 768$ and $1536$ elements. It follows that the cuspidal relation holds on $P_1$ for $\chi_2, \chi_3 \in \text{Irr}(Ru)$ and on $P_2$ for $\chi_2, \chi_3, \chi_4 \in \text{Irr}(Ru)$. We conclude that the two characters $\chi_2$ and $\chi_3$ of degree 378 are the only 2-cuspidal characters of $Ru$. 
There is a unique 2-minimal parabolic system of $J_4$ given by \( \{P_1, P_2, P_3\} \) where \( P_i \sim [2^{20}] \, \text{Sym}(3) \) for \( i = 1, 2 \) and \( P_3 \sim [2^{18}] \, \text{Sym}(5) \). The maximal 2-parabolic subgroups of this system are given by \( P_{12} \sim [2^{18}] \, L_3(2) \), \( P_{13} \sim [2^{17}] \, (\text{Sym}(3) \times \text{Sym}(5)) \) and \( P_{23} \sim 2^{4+13}M_{222} \). By considering centralizer orders, powering up classes and conjugation of representatives of certain $P_{12}$-, $P_{13}$- and $P_{23}$-conjugacy classes by random elements in $J_4$, we may determine the fusion of $O_2(P_{12})$-, $O_2(P_{13})$- and $O_2(P_{23})$-classes in $J_4$. We detail the orders of the intersections of the 2-cores of the maximal parabolic subgroups with the $J_4$-conjugacy classes in Table 6.11. Consequently, the cuspidal relation holds on $P_{12}$ for the characters $\chi_2$, $\chi_3$, $\chi_4$ and $\chi_5$, it holds on $P_{13}$ for $\chi_2$, $\chi_3$, $\chi_4$, $\chi_5$, $\chi_6$, $\chi_7$, $\chi_9$, $\chi_{10}$, $\chi_{12}$ and $\chi_{13}$, whilst the cuspidal relation holds on $P_{23}$ for the irreducible characters $\chi_2$ and $\chi_3$. We conclude that $J_4$ admits two 2-cuspidal characters, $\chi_2$ and $\chi_3$, both of degree 1333.

There are six 2-minimal parabolic subgroups of $L_y$, three of which feature in the two 2-minimal parabolic systems of $L_y$. These are $P_1 \sim [2^7] \, \text{Sym}(3)$, $P_2 \sim [2^5] \, \text{Sym}(5)$, $P_3 \sim 2 \, \text{Sym}(9)$ and they give rise to the systems \( \{P_1, P_3\} \) and \( \{P_1, P_3\} \). Since \( |O_2(P_1)| = 2^7 \), \( |O_2(P_2)| = 2^5 \) and \( |O_2(P_3)| = 2 \), it is easy to see that the cuspidal relation does not hold for any of the 2-minimal parabolic subgroups of $L_y$, and hence there are no 2-cuspidal characters of $L_y$.

### 6.3 3-Cuspidal Characters

We now describe the 3-cuspidal characters for each of the sporadic groups.

#### 6.3.1 The Mathieu Groups

$M_{11}$

The normalizer of a Sylow 3-subgroup of $M_{11}$ is the maximal subgroup $M_9 : 2$ of $M_{11}$. Consequently, we see that $M_{11}$ admits two 3-cuspidal characters, $\chi_6$ and $\chi_7$, both of degree 16.
CHAPTER 6. CUSPIDAL CHARACTERS OF FINITE GROUPS

$M_{12}$

The group $M_{12}$ has a unique 3-minimal parabolic system

$$\{P_1 \sim 3^2GL_2(3), P_2 \sim 3^2GL_2(3)\}.$$  

The non-trivial elements of the 3-cores $O_3(P_1)$ and $O_3(P_2)$ lie in the $M_{12}$-conjugacy class 3A. It follows that the 3-cuspidal characters of $M_{12}$ are $\chi_4$ and $\chi_5$, both of which have degree 16.

$M_{22}$

There is a unique 3-minimal parabolic system of $M_{22}$, namely

$$\{P_1 \cong M_{10}, P_2 \cong L_3(4)\}.$$  

Since $O_3(P_i) = 1$ for $i = 1, 2$, an element $\chi \in \text{Irr}(M_{22})$ will be 3-cuspidal if and only if the cuspidal relation holds for a Sylow 3-subgroup. Since this is never the case, $M_{22}$ has no 3-cuspidal characters.

$M_{23}$

The 3-cores of the two 3-minimal parabolic subgroups comprising the unique 3-minimal parabolic system

$$\{P_1 \cong M_{11}, P_2 \sim L_3(4) : 2\}$$

of $M_{23}$ are both trivial. Since the cuspidal relation does not hold for a Sylow 3-subgroup, we conclude that $M_{23}$ admits no 3-cuspidal characters.

$M_{24}$

There are three 3-minimal parabolic subgroups of $M_{24}$ given by $P_1 \sim 3 \cdot \text{Sym}(6)$, $P_2 \sim 2^63^{1+}2^{1+}2^{1+}2^{1+}$ and $P_3 \sim M_{12} : 2$. These give rise to two 3-minimal parabolic systems of $M_{24}$, namely $\{P_1, P_3\}$ and $\{P_2, P_3\}$. It is easy to observe that there is no $\chi \in \text{Irr}(M_{24})$ satisfying the cuspidal relation for a Sylow 3-subgroup of $M_{24}$. It follows that neither 3-minimal parabolic system of $M_{24}$ admits a 3-cuspidal character.

6.3.2 The Leech Lattice and Conway Groups

$HS$

As a Sylow 3-subgroup of $HS$ has order 9, it is easy to see that the cuspidal relation will not hold for such a subgroup, and hence $HS$ admits no 3-cuspidal characters.

$J_2$

There is a unique 3-minimal parabolic system of $J_2$ given by

$$\{P_1 \sim 3 \cdot \text{Alt}(6)2, P_2 \cong U_3(3)\}.$$  

The non-trivial elements of $O_3(P_1)$ are contained in the $J_2$-conjugacy class 3A. We immediately see that $J_2$ has no 3-cuspidal characters.
6.3. 3-CUSPIDAL CHARACTERS

$Co_1$

There is a solitary 3-minimal parabolic system of $Co_1$, which has rank 3. Its maximal parabolic subgroups are $P_{12} \sim 3^{1+4}.Sp_4(3) \cdot 2$, $P_{13} \sim 3^{3+4}.GL_2(3)^2$ and $P_{23} \sim 3^6.2M_{12}$.

The $Co_1$-fusion of the 3-core of $P_{23}$ is described in [Cur80]. We see that $O_3(P_{23})$ intersects the $Co_1$-conjugacy classes $1A$, $3A$, $3B$, $3C$ and $3D$ in 1, 24, 264, 440 and 0 elements respectively. It follows that the cuspidal relation does not hold for $P_{23}$ and hence $Co_1$ admits no 3-cuspidal characters.

$Co_2$

The unique 3-minimal parabolic system of $Co_2$ has the form

$$\{P_1 \sim 3^{1+4}.2^{1+4}\Sym(5), P_2 \sim 3^4.L_2(9) \Dih(8)\}.$$ 

Since $O_3(P_2)$ intersects the respective $Co_2$-conjugacy classes $1A$, $3A$ and $3B$ in 1, 20 and 60 elements, we see that there are no 3-cuspidal characters of $Co_2$.

$Co_3$

There are two 3-minimal parabolic subgroups of $Co_3$ and they form the unique 3-minimal parabolic system

$$\{P_1 \sim 3^{1+4}.4\Sym(6), P_2 \sim 3^5.(M_{11} \times 2)\}.$$ 

The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the $Co_3$-classes $1A$, $3A$, $3B$ and $3C$ in 1, 2, 240 and 0 and 1, 110, 132 and 0 elements respectively. It follows that the cuspidal relation holds on $P_1$ for $\chi_6, \chi_7 \in \Irr(Co_3)$ and on $P_2$ for $\chi_{10}, \chi_{11} \in \Irr(Co_3)$. Consequently, there are no 3-cuspidal characters of $Co_3$.

$McL$

The McLaughlin group has a unique 3-minimal parabolic system given by

$$\{P_1 \sim 3^4.M_{10}, P_2 \sim 3^{1+4}.2\Sym(5)\}.$$ 

The 3-core $O_3(P_1)$ intersects the $McL$-conjugacy classes $1A$, $3A$ and $3B$ in 1, 20 and 60 elements respectively. It follows that no $\chi \in \Irr(McL)$ satisfies the cuspidal relation for $P_1$ and hence $McL$ has no 3-cuspidal characters.

$Suz$

The unique 3-minimal parabolic system

$$\{P_1 \sim 3^5.M_{11}, P_2 \sim 3^{2+4}.2(Alt(4) \times 2^2)2\}$$

of $Suz$ does not admit any 3-cuspidal characters. To see this, we note that the 3-core $O_3(P_1)$ intersects the $Suz$-classes $1A$, $3A$, $3B$ and $3C$ in 1, 22, 220 and 0 elements respectively, meaning that the cuspidal relation does not hold on $P_1$ for any $\chi \in \Irr(Suz)$. 
6.3.3 The Monster Group and its Subgroups

$He$

Since a Sylow 3-subgroup of $He$ has order 27 and exponent 3, we easily observe that there are no 3-cuspidal characters of $He$.

$HN$

The unique 3-minimal parabolic system of $HN$ is

$$\{P_1 \sim 3^{1+4}.2\text{Sym}(5), P_2 \sim 3^4.2(\text{Alt}(4) \times \text{Alt}(4))4\}.$$  

The character table of the subgroup $M = 3^{1+4}.SL_2(5) \leq HN$ is given as [Har76, TABLE II]. Since $M \leq P_1$, it follows that $O_3(M) = O_3(P_1)$. Moreover, every $M$-conjugacy class is either contained in, or disjoint from $O_3(M)$. Using the notation from [Har76], we have that

$$O_3(M) = 1 \cup 3_1 \cup 3_1^2 \cup 3_2 \cup 3_3.$$  

Considering centralizer orders in $M$ and $HN$, we see that $3_1$ and $3_1^2$ are contained in the $HN$-class $3B$, whilst $3_2$ and $3_3$ lie in either $3A$ or $3B$. It follows that $|O_3(P_1) \cap 3A| = |O_3(M) \cap 3A| = 0, 120$ or $240$ and $|O_3(P_1) \cap 3B| = |O_3(M) \cap 3B| = 2, 122$ or $242$. For each of these possibilities, we see that the cuspidal relation would not hold for $P_1$, and hence there are no 3-cuspidal characters of $HN$.

$Th$

The Thompson group has a single 3-minimal parabolic system. It is of rank 2 and has the form

$$\{P_1 \sim 3^{(1+2)+4+2}.GL_2(3), P_2 \sim 3^{(2+3)+4}.GL_2(3)\}.$$  

The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the respective $Th$-conjugacy classes $1A, 3A, 3B, 3C, 9A, 9B$ and $9C$ in 1, 270, 2186, 4104, 2106, 5184 and 5832 elements and 1, 756, 2672, 4590, 648, 5184 and 5832 elements respectively. It follows that for both $P_1$ and $P_2$ there is a unique element of $\text{Irr}(Th)$ satisfying the cuspidal relation, namely $\chi_2$ of degree 248. We conclude that $\chi_2$ is the unique 3-cuspidal character of $Th$.

$Fi_{22}$

There is a unique 3-minimal parabolic system of $Fi_{22}$, which has rank 3 and minimal parabolic subgroups of the form $P_i \sim [3^8].2.PGL_2(3)$ for $i = 1, 2, 3$. The maximal parabolic subgroups of this system are

$$P_{12} \cong P_{13} \sim 3^{4+2}.L_3(3) \quad \text{and} \quad P_{23} \sim 3^{1+6}.2^2.SL_2(3).\text{Sym}(4).$$  

The maximal parabolics $P_{12}$ and $P_{13}$ are maximal subgroups of the maximal subgroups of $Fi_{22}$ which are isomorphic to $O_T(3)$, and hence their 3-cores can be easily computed. Meanwhile,
Table 6.12: The orders of the intersections of the $Fi_{22}$-conjugacy classes with the 3-cores of maximal 3-parabolic subgroups.

| Conjugacy Class, $C$ | $|O_3(P_{12}) \cap C|$ | 1A | 3A | 3B | 3C | 3D |
|----------------------|-------------------------|----|----|----|----|----|
| $O_3(P_{12}) \cap C$ | 1                       | 0  | 260| 234| 234|    |
| $O_3(P_{13}) \cap C$ | 1                       | 0  | 260| 234| 234|    |
| $O_3(P_{23}) \cap C$ | 1                       | 72 | 386| 576| 1152|   |

by [Asc97] (39.6), the 3-core $O_3(P_{23})$ is isomorphic to the Fitting subgroup of the normalizer in $Fi_{22}$ of an element of the $Fi_{22}$-class $3B$. The orders of the intersections of the 3-cores of these maximal parabolics with the $Fi_{22}$-conjugacy classes is summarised in Table 6.12.

We see that the cuspidal relation holds on $P_{12}$ and $P_{13}$ for the characters $\chi_2, \chi_5 \in \text{Irr}(Fi_{22})$, and it holds on $P_{23}$ for $\chi_2$. Consequently, $\chi_2$ (of degree 78) is the unique 3-cuspidal character of $Fi_{22}$.

$Fi_{23}$

The group $Fi_{23}$ has a unique 3-minimal parabolic system given by

$$\{ P_1 \sim [3^{12}] \cdot 2^2 PGL_2(3), P_2 \sim [3^{12}] \cdot 2^2 PGL_2(3), P_3 \sim [3^9, 2L_2(3), Sym(3)) \}.$$ 

The corresponding 3-maximal parabolic subgroups are

$$P_{12} \sim 3^{3+7} \cdot GL_3(3), \quad P_{13} \sim 3^{1+8} : 2^1 \cdot 3^1 \cdot 2 \cdot \text{Sym}(4) \quad \text{and} \quad P_{23} \sim D_4(3) \cdot \text{Sym}(3).$$

Using the information on $Fi_{23}$-fusion within $O_3(P_{13})$ given in [Wil87, Table 2], and the fact that the non-trivial elements of $Z(3^{1+8})$ lie in the $Fi_{23}$-class $3B$, we see that $O_3(P_{13})$ intersects the $Fi_{23}$-classes $3A, 3B, 3C$ and $3D$ in 864, 1538, 3456 and 13824 elements respectively. (We note that the above fusion can also be calculated within $P_{23}$. Indeed, there are nine $P_{23}$-classes of elements of order 3, say $3a, \ldots, 3i$, having centralizer orders in $P_{23}$ of 408146688, 37791360, 37791360, 12737088, 2834352, 944784, 314928, 78732 and 17496 respectively. It follows that the $P_{23}$-classes satisfy the following inclusions; $3c, 3d \subseteq 3A$, $3a, 3e \subseteq 3B$, $3b, 3g \subseteq 3C$ and $3f, 3h, 3i \subseteq 3D$). Consequently, the cuspidal relation does not hold on $P_{13}$ for any $\chi \in \text{Irr}(Fi_{23})$, and $Fi_{23}$ admits no 3-cuspidal characters.

$Fi'_{24}$

As with $Fi_{23}$ we see that there is a unique 3-minimal parabolic system of $Fi'_{24}$, namely

$$\{ P_1 \sim [3^{15}] \cdot 2^2 PGL_2(3), P_2 \sim [3^{15}] \cdot 2^2 PGL_2(3), P_3 \sim [3^{15}] \cdot 2 \cdot \text{Sym}(5) \},$$

having maximal parabolics

$$P_{12} \sim 3^{3+7+3} \cdot L_3(3) \cdot 2, \quad P_{13} \sim 3^{2+4+8} \cdot (SL_2(3) \times \text{Alt}(5)) \cdot 2 \quad \text{and} \quad P_{23} \sim 3^{1+10} \cdot U_5(2) \cdot 2.$$ 

Consider the extra-special 3-core, $O_3(P_{23}) \cong 3^{1+10}_+$. Since there is a unique irreducible 10-dimensional $GF(3)^e U_5(2)$-module, $M$, we can explicitly determine the sizes of the orbits of elements of this module. These are summarised in Table 6.13. We see that there are at most four classes of non-central elements of $3^{1+10}_+$. 

Table 6.13: The non-trivial orbits of the unique irreducible 10-dimensional $GF(3)U_5(2)$-module.

| Element, $m$ | $\text{Stab}_{U_5(2)}(m)$ | $\text{Stab}_{U_5}(m)$ | $|m|^{U_5(2)}$ |
|-------------|----------------|----------------|----------------|
| $m_1$       | $3^3.\text{Alt}(5)$ | 4,860          | 2,816          |
| $m_2$       | 3.\text{Alt}(6)    | 1,080          | 12,672         |
| $m_3$       | $2^{1+6}.3^{1+2}.3$ | 10,368         | 1,320          |
| $m_4$       | $3^3.\text{Alt}(4)$ | 324            | 42,240         |

Let $z \in Z(3^{1+10}_+) \setminus \{1\}$ and let $x \in 3^{1+10}_+ \setminus Z(3^{1+10}_+)$. Thus $x$ represents a non-zero vector in $M$. Then $x$, $x^2$, $zx$, $zx^2$, $z^2x$ and $z^2x^2$ are all $F'_{24}$-conjugate. Indeed, from the ATLAS we have that for any 3-element $w \in F'_{24}$, the elements $w$ and $w^2$ are $F'_{24}$-conjugate. Suppose that $g \in F'_{24}$ is such that $x^g = x^2$. Then $(zx)^g = z^{x^2}$ as $z$ is central. Thus $zx$ and $z^2x$ lie in a common $F'_{24}$, and are joined by $(zx)^2 = z^2x^2$ and $(zx^2)^2 = z^2x$. Finally, as $zx$ and $z^2x$ are $F'_{24}$-conjugate, there exists $h \in F'_{24}$ satisfying $z^2x = (zx)^h = z^{hx}$ and hence $zx = x^h$. We conclude that the orbits of $m_1$, $m_2$, $m_3$ and $m_4$ give rise to orbits of $3^{1+10}_+ Z(3^{1+10}_+)$ of respective sizes 8448, 38016, 3960 and 126720.

Label the orbit of $3^{1+10}_+$ arising from $m_i$ by $M_i$ for $i = 1, \ldots, 4$. Considering the orders of stabilizers given in Table 6.13 together with the centralizer orders of elements of order 3 in $F'_{24}$ given in the ATLAS, we deduce that $M_1, M_2, M_3 \leq 3A \cup 3B \cup 3C$, whilst elements in $M_4$ could form a subset of $3A, 3B, 3C, 3D$ or $3E$. Since $N(3B) \cong P_{23}$, it follows that the non-trivial central elements of $3^{1+10}_+$ lie in the $F'_{24}$-class $3B$.

The menagerie of information obtained above results in 135 different possibilities for the fusion of 3-elements within $O_3(P_{23}) \cong 3^{1+10}_+$. Feeding each possibility into MAGMA and allowing it to roam over all 108 complex characters of $F'_{24}$, we see that the cuspidal relation never holds for $P_{23}$, and hence $F'_{24}$ admits no 3-cuspidal characters.

Aside 6.3.1. We note that the $F'_{24}$-fusion within the 3-core $O_3(P_{23})$ has previously been studied by Wilson. Indeed, in [Wil87, Section 2.2] Wilson calculates that $O_3(P_{23})$ contains 3960, 8450, 38016 and 126720 elements from the respective $F'_{24}$-classes $3A$, $3B$, $3C$ and $3D$. However, these calculations are based heavily on an unpublished preprint, and we have been unable to verify them.

\[ \mathbb{B} \]

There is a unique 3-minimal parabolic system of $\mathbb{B}$, which has rank 3. The maximal parabolic subgroups of this system are $P_{12} \sim 3^{3+7}.GL_2(3)$, $P_{13} \sim 3^{2+3+6}.GL_2(3)^2$ and $P_{23} \sim 3^{1+8}.2^{1+6}.PSp_4(3).2$. Considering the minimum value that each $\chi \in \text{Irr}(\mathbb{B})$ takes on 3-elements, we see that the only possible 3-cuspidal character of $\mathbb{B}$ is $\chi_2$ of degree 4371. However, as $\mathbb{B}$ satisfies the condition of Lemma 6.1.10, we see that $\chi_2$ cannot be 3-cuspidal, and hence $\mathbb{B}$ admits no 3-cuspidal characters.

\[ \mathbb{M} \]

The monster group, $\mathbb{M}$, has a unique 3-minimal parabolic system, which has rank 3. Its maximal parabolic subgroups are given by $P_{12} \sim 3^{3+8+6}.L_3(3)$, $P_{13} \sim 3^{2+5+10}.(GL_2(3) \times M_{11})$ and $P_{23} \sim 3^{1+12}.2.Suz.2$. 

By considering the 3-core $O_3(P_{23})$, we see that it has exponent 3. Moreover, appealing to [IM97, Lemma 1.5] we see that
\[ |O_3(P_{23}) \cap 3A| = 196,560 \quad \text{and} \quad |O_3(P_{23}) \cap 3B| = 1,397,762. \]
It immediately follows that $\mathbb{M}$ admits no 3-cuspidal characters.

### 6.3.4 The Pariahs

#### $J_1$

The cuspidal relation does not hold for a Sylow 3-subgroup of $J_1$. Hence there are no 3-cuspidal characters for the unique 3-minimal parabolic system, $\{J_1\}$, of $J_1$.

#### $O'N$

A Sylow 3-subgroup of $O'N$ has order 81. Moreover, since there is a unique $O'N$-conjugacy class of non-trivial 3-elements, we see that the cuspidal relation does not hold for a Sylow 3-subgroup of $O'N$. Consequently, $O'N$ admits no 3-cuspidal characters.

#### $J_3$

The normalizer of a Sylow 3-subgroup of $J_3$ is maximal. Moreover, such a Sylow subgroup intersects the $J_3$-conjugacy classes 1A, 3A, 3B, 9A, 9B and 9C in 1, 18, 8, 72, 72 and 72 elements respectively. Checking the cuspidal relation for each $\chi \in \text{Irr}(J_3)$ for a Sylow 3-subgroup, we see that there are no 3-cuspidal characters of $J_3$.

#### $Ru$

Since a Sylow 3-subgroup of $Ru$ has order 27 and there is a unique $Ru$-conjugacy class of non-trivial 3-elements, it is easy to see that $Ru$ admits no 3-cuspidal characters.

#### $J_4$

The non-trivial elements of a Sylow 3-subgroup of $J_4$ lie in the $J_4$-class 3A. Since such a subgroup has order 27, it is easy to see that the cuspidal relation does not hold for a Sylow 3-subgroup of $J_4$. Hence, $J_4$ has no 3-cuspidal characters.

#### $Ly$

There is a unique 3-minimal parabolic system of $Ly$ given by
\[ \{P_1 \sim 3^{2+4}.\text{Sym}(5), P_2 \sim 3^5.(M_{11} \times 2)\}. \]
Considering the minimum value that each $\chi \in \text{Irr}(Ly)$ takes on elements of order 3, we see that the only possible candidates for 3-cuspidal characters are $\chi_7$ and $\chi_8$ of degree 120064. However, these characters take strictly negative values on all 3-elements, and hence the cuspidal relation
cannot hold for them for both $O_3(P_2)$ and a Sylow 3-subgroup of $L_y$. We conclude that there are no 3 cuspidal characters of $L_y$.

6.4 5-Cuspidal Characters

The groups $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, $Suz$, $He$, $Fi_{22}$, $Fi_{23}$, $Fi'_{24}$, $J_1$, $O'N$, $J_3$, $Ru$ and $J_4$ have Sylow 5 subgroups of exponent 5 and of order at most $5^3$. Thus, considering the minimal values that an irreducible character takes on elements of order 5 together with the character degree, we see that none of these groups admit a 5 cuspidal character. We now consider the remaining eleven sporadic groups in turn.

$HS$

A Sylow 5 subgroup of $HS$ intersects the $HS$ conjugacy classes 1A, 5A, 5B and 5C in 1, 4, 40 and 80 elements respectively. It follows immediately that $HS$ has no 5 cuspidal characters.

$J_2$

The normalizer in $J_2$ of a Sylow 5 subgroup, $S$, is maximal and $S$ intersects each of the $J_2$ conjugacy classes 5A, 5B, 5C and 5D in 6 elements. From this we deduce that $J_2$ has a unique 5 cuspidal character given by $\chi_6$ of degree 36.

$Co_1$

There is a unique 5 minimal parabolic system of $Co_1$ given by

$$\{P_1 \sim 5^3.(4 \times \text{Alt}(5)) \cdot 2, P_2 \sim 5^{1+2}.GL_2(5)\}.$$  

Considering the minimum value that each $\chi \in \text{Irr}(Co_1)$ takes on elements of order 5, and the order of the 5 cores of the minimal parabolic subgroups, we see that there are no 5 cuspidal characters of $Co_1$.

$Co_2$, $Co_3$, $McL$

If $G \in \{Co_2, Co_3, McL\}$, then a Sylow 5 subgroup of $G$ intersects the $G$ conjugacy classes 1A, 5A and 5B in 1, 4 and 120 elements respectively. It follows that the cuspidal relation does not hold for a Sylow 5 subgroup for any $\chi \in \text{Irr}(G)$ and hence $G$ has no 5 cuspidal characters.

$HN$

There is a unique 5 minimal parabolic system of $HN$ given by

$$\{P_1 \sim 5^{1+4}.(2^{1+4}.5.4), P_2 \sim 5^{2+1+2}.4.\text{Alt}(5)\}.$$  

The character table of $P_1$ is given as [Har76 Table III], whilst a partial character table of $P_2$ featuring the conjugacy classes of elements of order 2 and classes contained in $O_5(P_2)$ - is given in

...
### 6.4. CUSPIDAL CHARACTERS

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**Table 6.14:** A partial character table of the 5-minimal parabolic subgroup \( P_2 \sim 5^{2+1+2} \cdot 4 \cdot \text{Alt}(5) \) of \( HN \) (where \( w = \exp(2\pi i/5) \)).
Table 6.14. By considering the restriction of $\chi_2 \in \text{Irr}(HN)$ to $P_1$ and $P_2$, we may calculate the $HN$-fusion within $O_5(P_1)$ and $O_5(P_2)$. We see that $O_5(P_1)$ intersects the $HN$-classes $1A$, $5A$, $5B$, $5C$, $5D$ and $5E$ in 1, 400, 324, 800, 800 and 800 elements respectively, whilst $O_5(P_2)$ intersects the given classes in 1, 0, 624, 650, 650, and 1200 elements respectively. Consequently, the cuspidal relation holds on $P_1$ for $\chi_4 \in \text{Irr}(HN)$ and on $P_2$ for $\chi_4, \chi_5 \in \text{Irr}(HN)$. We conclude that $\chi_4$ (of degree 760) is the unique 5-cuspidal character of $HN$.

Th

The normalizer of a Sylow 5-subgroup of $Th$ is a maximal subgroup. Thus a character $\chi \in \text{Irr}(Th)$ will be 5-cuspidal precisely when the cuspidal condition holds for the Sylow subgroup. Since $Th$ has a unique conjugacy class of elements of order 5, we observe that there is a unique 5-cuspidal character of $Th$, namely $\chi_2$ of degree 248.

$B$

There is a unique 5-minimal parabolic system of $B$ given by

$$\{P_1 \sim 5^{1+4}.2^{1+4}\text{Sym}(5)2, P_2 \sim 5^{2+1+2}\text{GL}_2(5)\}.$$ Considering the minimum value that each $\chi \in \text{Irr}(B)$ takes on $B$-classes of 5-elements, we see that the only possible 5-cuspidal character of $B$ is $\chi_2$. However, $\deg(\chi_2) = 4371$, and $B$ satisfies the conditions of Lemma 6.1.10. Thus there are no 5-cuspidal characters of $B$.

$M$

The monster has a unique 5-minimal parabolic system, namely

$$\{P_1 \sim 5^{1+6}.2(J_2 \times 2)2, P_2 \sim 5^{2+2+4}\text{Sym}(3)\text{GL}_2(5)\}.$$ Using a similar approach to that used for the baby monster, with the 5-core $O_5(P_1)$, we deduce that there are no 5-cuspidal characters of $M$.

$Ly$

From [RSS04], we see that there is a unique 5-minimal parabolic system of $Ly$, having rank 3. Its minimal parabolic subgroups are $P_1 \sim 5^{1+4}.4\text{PGL}_2(5)$, $P_2 \sim 5^{3+2}.4\text{PGL}_2(5)$ and $P_3 \sim 5^{1+4}.4\text{PGL}_2(5)$. The maximal 5-parabolic subgroups are given by

$$P_{12} \sim 5^3.\text{SL}_3(5), \quad P_{13} \sim 5^{1+4}.2\text{Alt}(6)4 \quad \text{and} \quad P_{23} \cong G_2(5).$$ By considering the elementary abelian subgroup $O_5(P_{12}) \cong 5^3$, we see that the only possible 5-cuspidal characters of $Ly$ are $\chi_2$ and $\chi_3$ (both of degree 2480), and that for these characters to be 5-cuspidal, we must have that the non-trivial elements of $O_5(P_{12})$ are contained in the $Ly$-conjugacy class $5A$. Defining $S$ to be our given Sylow 5-subgroup of $Ly$, we see that $S$ has a unique normal elementary abelian 5-subgroup of order $5^3$. Considering this within the maximal subgroup
6.5 $p$-Cuspidal Characters ($p > 5$)

In the case that $p > 5$, most sporadic groups with order divisible by $p$ have a cyclic Sylow $p$-subgroup of order $p$. The exceptions are $(Co_1, p = 7)$, $(He, p = 7)$, $(Th, p = 7)$, $(Fi'_{24}, p = 7)$, $(E, p = 7)$, $(HS, p = 7, 11, 13)$, $(O'N, p = 7)$ and $(J_4, p = 11)$.

The normalizer of a Sylow 7-subgroup, $S$, of $He$ is maximal in $He$, and hence there is a unique 7-minimal parabolic system given by $\{He\}$. Consequently, an element $\chi \in \text{Irr}(He)$ will be 7-cuspidal precisely when the cuspidal condition holds for $S$. We have that $S$ contains 1, 42, 42, 132, 63 and 63 elements from the $He$-classes 1A, 7A, 7B, 7C, 7D and 7E respectively. It follows that the 7-cuspidal characters of $He$ are $\chi_2$ and $\chi_3$ of degree 51.

In all other cases, since the sporadic group in question contains no elements of order $p^a$ for $a > 1$, we may consider the minimum value that each irreducible character takes on elements of order $p$, to conclude that there are no $p$-cuspidal characters.

We conclude by considering the $p$-cuspidal characters arising from sporadic groups having a cyclic Sylow $p$-subgroup of order $p$. Since such a subgroup will necessarily be the $p$-core of its normalizer, it is easy to see that the cuspidal relation must hold for the Sylow subgroup. Moreover, a character will be cuspidal precisely when this is true. This gives an additional four cuspidal characters for the sporadic groups; the 10-dimensional characters $\chi_2, \chi_3, \chi_4 \in \text{Irr}(M_{11})$ are 11-cuspidal and the 22-dimensional character $\chi_2 \in \text{Irr}(M_{23})$ is 23-cuspidal.

6.6 Possible Connections with Geometries

In Section 6.1 we saw how geometries could be used to classify characters. Indeed, the definition of $p$-cuspidal characters is entirely dependant on a given $p$-minimal parabolic geometry. Having classified the $p$-cuspidal characters of the sporadic groups\footnote{with the exception of the 2-cuspidal characters of $E$}, we may now turn the situation on its head and ask the following question.

**Question 6.6.1.** Let $G$ be a group, $p$ a prime divisor of $|G|$ and suppose that $\chi$ is a $p$-cuspidal character of $G$. Are there any “nice” geometries arising from $\chi$?

For the tuples $(G, p)$ given by $(M_{11}, 11)$ and $(M_{23}, 23)$, it is unlikely that the $p$-cuspidal characters of $G$ will give rise to any interesting geometries. Indeed, in both cases a Sylow $p$-subgroup of $G$
is cyclic of order $p$, and the resulting $p$-cuspidal characters have degree $p - 1$. Similarly, when $(G, p)$ is $(M_{11}, 3)$ or $(Th, 5)$, there is a unique class of elements of order $p$, and a Sylow $p$-subgroup has exponent $p$. Hence it is improbable that the resulting $p$-cuspidal characters lead to interesting geometries. Finally, as the maximal 3-parabolic subgroups of $M_{12}$ are both normalizers of elementary abelian subgroups of rank 2 generated by elements of the $M_{12}$-class $3A$, we shall not examine these 3-cuspidal characters any further. Thus we consider the remaining $p$-cuspidal characters given in Tables 6.1-6.3 with the caveat that - at the time of writing - our subsequent comments are purely speculative and serve more as a survey of literature that may be related to the remaining $p$-cuspidal characters.

In the early 1990s, Margolin looked at a geometry for the Mathieu group $M_{24}$ [Mar93]. Margolin’s interest stemmed from the two 1333-dimensional irreducible $GF(2)J_4$-representations. Since $2^{11} : M_{24}$ is a maximal subgroup of $J_4$, Margolin considered the restriction of these representations to $2^{11} : M_{24}$, namely as a faithful 1288-dimensional representation and a 45-dimensional representation having kernel $2^{11}$. Hence Margolin sought to find a simple explanation for this 45-dimensional representation and this resulted in the construction of a geometry. We note that both 1333-dimensional $J_4$-characters are 2-cuspidal, as are both of the resulting irreducible $M_{24}$-characters of degree 45, along with their irreducible restrictions to $M_{22}$ and $M_{23}$.

Considering the 3-cuspidal character of $Th$, we note that $Th$ is a subgroup of the exceptional group $E_8(q)$ when $q = 3$ (see for example [Wil09]). The lowest character degree of $E_8(3)$ is 248. Thus it would be worth considering whether this subgroup inclusion is related to the fact that this character of lowest degree is 3-cuspidal, and if so, is there an amiable geometry arising from this character?

The existence of the geometry constructed by Margolin and the subgroup inclusion of $Th$ in $E_8(3)$ may be completely unrelated to the existence of the aforementioned $p$-cuspidal characters. Indeed, there do not appear to exist publications on geometries related to the other $p$-cuspidal characters described in Sections 6.2-6.5. However, it would be worth investigating these two situations further. Moreover, it would be worth considering the $p$-cuspidal characters from Tables 6.1-6.3 from a general geometric viewpoint, and thus looking at answering Question 6.6.1. However, such consideration is far beyond the scope of this thesis.
Chapter 7

Conclusions and Future Work

We conclude the thesis by briefly summarising our findings and suggesting prospective areas of further study for the interested reader. For ease of reading, we break this down under the four main chapter headings.

7.1 Homology of Presheaves of Abelian Groups

In Chapter 3, our main contribution to the subject area was the calculation of the zero-homology groups of universal panel-irreducible presheaves of the symmetric group Sym(6) and the the Mathieu groups $M_{11}$ and $M_{22}$ together with calculating the irreducible quotients of the zero-homology groups of such presheaves for $M_{12}$, $M_{23}$ and $M_{24}$. These presheaves were all defined over the finite field, $GF(2)$. Further work would allow for the undetermined homology groups to be explicitly constructed. However, describing their structure would be a complex affair.

The main aim in calculating these homology groups/quotients of these homology groups was to try to ascertain the answer to Question 3.9.1, namely for a given group $G$, a 2-minimal parabolic system $S := \{P_i \mid i = 1, \ldots, n\}$ for $G$ such that $P_i \sim O_2(P_i).\text{Sym}(3)$ for all $i = 1, \ldots, n$ and an irreducible $GF(2)G$-module $V$, does there exist a universal panel-irreducible presheaf, $\lambda$, for $G$ defined on $S$ over $GF(2)$ such that $V$ is a quotient of $H_0(\lambda)$? We saw in Section 3.8 that the groups Sym(6), $M_{12}$ and $M_{24}$ satisfied the given hypothesis and in each case and for every 2-minimal parabolic system $S$ of the given group, every irreducible module over $GF(2)$ was indeed a quotient of the zero-homology group of some universal panel-irreducible presheaf defined on $S$ over $GF(2)$. Future research into this area would be of interest, to try and establish the answer to Question 3.9.1.

Further areas that we did not consider in Chapter 3 but that are large areas of potential future research include presheaves defined over fields other than $GF(2)$, and presheaves defined on the order complex of simplicial complexes other than those defined by minimal parabolic systems. It is likely that such areas would be of great interest and may well shed further light on the presheaves defined on 2-minimal parabolic systems over $GF(2)$ that we have examined.

A final area of possible research in this area would be to try and develop the theoretical-side of the subject further. During our work, we have focused predominantly on the computational-side of things. However, we believe that further theoretical results, either expanding the results of
Section 3.6 or taking a completely new approach to the subject, would be beneficial in developing the field of study further.

### 7.2 $\pi$-Product Graphs in Symmetric Groups

The main result of Chapter 4 was Theorem 4.0.1. This concerned the case that $G := \text{Sym}(n)$, $t = (1,2)\cdots(2m-1,2m) \in G$ and $X$ is the $G$-conjugacy class of $t$. The result fully classified when the $\{4\}$-product graph $P_{\{4\}}(G,X)$ was connected, and in the case that it was connected, the diameter was shown to equal 2. The result followed by a combinatorial analysis of the possible connected components of related $x$-graphs, and was constructive in nature. Indeed, given an element $x \in X$ such that the order of $tx$ does not equal 4, then by following the proof of Theorem 4.0.1 the reader may construct an element $y \in X$ such that $d(t,y) = d(y,x) = 1$.

The graph $P_{\{4\}}(G,X)$ was chosen as two elements $x,y \in X$ are neighbours in $P_{\{4\}}(G,X)$ precisely when $\langle x,y \rangle \cong \text{Dih}(8)$. A natural generalisation of this is to consider when the graph $P_{\{2^a\}}(G,X)$ is connected for $a > 2$, where $x,y \in X$ are neighbouring vertices precisely when $\langle x,y \rangle \cong \text{Dih}(2^{a+1})$. The connectivity of such graphs was determined in Theorem 4.0.2, where crude upper and lower bounds were also obtained for the diameter in the case of connected graphs.

Explicit values of the diameters of the connected graphs $P_{\{8\}}(G,X)$ were obtained in Theorem 4.0.3 in the case that the support of $t$ has order 8. However, we did not investigate the diameters of the connected graphs from Theorem 4.0.2 any further in the case that $a > 3$. This is an area that would be worth contemplating in the future, and a more general approach to that used in proving Theorem 4.0.1 might be worth considering in such future work.

The final two results of Chapter 4, namely Theorems 4.0.4 and 4.0.5 were in a similar vein to Theorem 4.0.2. However, all three of these theorems concerned $\pi$-product graphs where $\pi = \{\ell\}$ and $\ell$ is closely related to $|\text{supp}(t)|$. We did not consider the situation when such a relation did not hold. Since such a greater generality is in a similar vein to the greater generality of Theorem 4.0.1 it may be worth looking at the possibility of generalising Theorems 4.0.2, 4.0.4 and 4.0.5 in the future.

### 7.3 Conjugate $p$-elements of Full Support that Generate the Wreath Product $C_p \wr C_p$

Motivated by the graph-theoretic approach of Chapter 4, we considered possible generalisations of the $x$-graph in Chapter 5 resulting in matrices $A^x_a$ and $A^x_z$ being defined in Definition 5.1.1. These matrices encapsulated some of the information that a generalised $x$-graph would contain, whilst losing other information. However, they proved to have adequate versatility for us to determine when two conjugate elements of order $p$ having full support in a given symmetric group generate the wreath product $C_p \wr C_p$ of two cyclic groups of order $p$. In the case of $x$-graphs, the work of Bates, Bundy, Perkins and Rowley [BBPR03b] illustrated that $x$-graphs may be applied to obtain a number of different results. The matrices $A^x_a$ and $A^x_z$ lack the majority of this versatility and
were defined specifically for our chosen line of research. However, it may be worth considering if they have further applications. It is unlikely that the results of Chapter 5 can in themselves be extended.

7.4 Cuspidal Characters of Finite Groups

The final chapter of the thesis considered cuspidal characters of finite groups. The idea was to consider an analogue to cuspidal characters of groups of Lie type for an arbitrary finite group. Many of the results presented in [Car93] proved to have analogues in the finite setting and this enabled \( p \)-cuspidal characters of finite groups to be considered. Moreover, we showed in Corollary 6.1.9 that it suffices to check the cuspidal relation for a certain subset of parabolic subgroups.

The main body of Chapter 6 was devoted to considering the \( p \)-cuspidal characters of each sporadic simple group. All \( p \)-cuspidal characters were determined with the exception of the 2-cuspidal characters of the baby monster, \( B \). This is obviously an area that would warrant further work, as would consideration of possible geometries associated to the other \( p \)-cuspidal characters of the sporadic groups - a topic briefly mentioned in Section 6.6.

The approach used throughout Chapter 6 involved \( p \)-minimal parabolic systems of groups. As we saw, these were specific cases of \( X \)-parabolic systems as defined in Definition 6.1.1. We did not investigate these systems in greater generality. Further work could involve considering such systems and contemplating if other forms of cuspidal characters could be defined for finite groups using other \( X \)-parabolic systems.
Appendix A

Vertex Term Calculations for Universal Panel-Irreducible Presheaves of $M_{23}$ and $M_{24}$ over $GF(2)$

In Chapter 3 we calculated the irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic systems of $M_{23}$ and $M_{24}$ over $GF(2)$. These minimal parabolic systems all have rank 3, and thus the associated simplicial complexes have chamber, panel and vertex terms. In this appendix, we give a full examination of the vertex terms of these presheaves.

A.1 Vertex Term Calculations for Universal Panel-Irreducible Presheaves of $M_{23}$ over $GF(2)$

As seen in Chapter 3 there are seven 2-minimal parabolic systems of $M_{23}$, each of rank 3. We denote these by $S_i := \{P_i, P_6, P_7\}$ for $i = 1, 2, 3, 4$, $S_5 := \{P_1, P_3, P_7\}$, $S_6 := \{P_2, P_3, P_7\}$ and $S_7 := \{P_3, P_4, P_7\}$. Here we are adopting the notation of [RS84, p77], and so $P_i \sim 2^2.\text{Sym}(3)$ for $i = 1, 2, 3, 4$ and $P_j \sim 2^4.\text{Sym}(5)$ for $j = 6, 7$. The diagrams of the geometries arising from $S_1$, $S_2$ and $S_4$ are given in Figure A.1 whilst the diagrams arising from $S_3$, $S_5$, $S_6$ and $S_7$ are given in Figures A.2, A.3, A.4 and A.5 respectively.

The minimal parabolic subgroups $P_i$ each admit two classes of irreducible $GF(2)P_i$-modules, for $i = 1, \ldots, 4$, denoted by $1P_i$ and $2P_i$. For $j = 6, 7$ there are three classes of irreducible $GF(2)P_j$-modules, namely $1P_j$, $4P_j$ and $4P_j$. Continuing the notation of Chapter 3 we denote the presheaf defined on the minimal parabolic system $\{P_i, P_j, P_k\}$ having a *dim-dimensional irreducible $GF(2)P_*$-module at panels of type $P_*$ by $\lambda^{(ijk)}_{\text{dim} \cdot \text{dim}, \text{dim}}$ (using the bar notation to differentiate between classes of 4-dimensional irreducibles in the case of $P_6$ and $P_7$). The vertex terms of $\lambda^{(ijk)}_{\text{dim} \cdot \text{dim}, \text{dim}, \text{dim}}$ are themselves zero-homology groups of presheaves defined on rank two 2-minimal parabolic systems of $P_{ij}$, $P_{jk}$ and $P_{ik}$. We denote by $\lambda^{(ij)}_{\text{idim} \cdot \text{idim}}$ the universal panel-irreducible presheaf of $P_{ij}$ having a 1-dimensional irreducible module at its chambers and satisfying
Figure A.1: The diagram of the geometries arising from the 2-minimal parabolic systems $S_i$ of $M_{23}$ for $i = 1, 2, 4$.

Figure A.2: The diagram of the geometry arising from the 2-minimal parabolic system $S_3$ of $M_{23}$.

Figure A.3: The diagram of the geometry arising from the 2-minimal parabolic system $S_5$ of $M_{23}$.
Figure A.4: The diagram of the geometry arising from the 2-minimal parabolic system \( S_6 \) of \( M_{23} \).

Figure A.5: The diagram of the geometry arising from the 2-minimal parabolic system \( S_7 \) of \( M_{23} \).
Irreducible $GF(2) P$-Modules

<table>
<thead>
<tr>
<th>Parabolic Subgroup, $P$</th>
<th>Shape of $P$</th>
<th>$\text{Irreducible GF(2) } P$-Modules</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{13}$</td>
<td>$2^{i} L_{2}(2)$</td>
<td>$1_{P_{13}}, 3_{P_{13}}, 3_{P_{13}}, 8_{P_{13}}$</td>
</tr>
<tr>
<td>$P_{16}$</td>
<td>$2^{i} \text{ Alt}(7)$</td>
<td>$1_{P_{16}}, 4_{P_{16}}, 7_{P_{16}}, 6_{P_{16}}, 14_{P_{16}}, 20_{P_{16}}$</td>
</tr>
<tr>
<td>$P_{17}$</td>
<td>$(3 \times \text{ Alt}(5)): 2$</td>
<td>$1_{P_{17}}, 2_{P_{17}}, 4^{b}<em>{P</em>{17}}, 4^{b}<em>{P</em>{17}}, 4^{b}<em>{P</em>{17}}, 4^{d}<em>{P</em>{17}}, 8_{P_{17}}$</td>
</tr>
<tr>
<td>$P_{23}$</td>
<td>$2^{i} \text{ Alt}(7)$</td>
<td>$1_{P_{23}}, 4_{P_{23}}, 7_{P_{23}}, 6_{P_{23}}, 14_{P_{23}}, 20_{P_{23}}$</td>
</tr>
<tr>
<td>$P_{26}$</td>
<td>$2^{i} \text{ Alt}(7)$</td>
<td>$1_{P_{26}}, 4_{P_{26}}, 7_{P_{26}}, 6_{P_{26}}, 14_{P_{26}}, 20_{P_{26}}$</td>
</tr>
<tr>
<td>$P_{27}$</td>
<td>$(3 \times \text{ Alt}(5)): 2$</td>
<td>$1_{P_{27}}, 2_{P_{27}}, 4^{b}<em>{P</em>{27}}, 4^{b}<em>{P</em>{27}}, 4^{d}<em>{P</em>{27}}, 8_{P_{27}}$</td>
</tr>
<tr>
<td>$P_{34}$</td>
<td>$2^{i} L_{2}(2)$</td>
<td>$1_{P_{34}}, 3_{P_{34}}, 3_{P_{34}}, 3_{P_{34}}, 8_{P_{34}}$</td>
</tr>
<tr>
<td>$P_{36}$</td>
<td>$\text{ Alt}(7)$</td>
<td>$1_{P_{36}}, 4_{P_{36}}, 7_{P_{36}}, 6_{P_{36}}, 14_{P_{36}}, 20_{P_{36}}$</td>
</tr>
<tr>
<td>$P_{37}$</td>
<td>$M_{22}$</td>
<td>$1_{P_{37}}, 10_{P_{37}}, 10_{P_{37}}, 10_{P_{37}}, 34_{P_{37}}, 98_{P_{37}}, 140_{P_{37}}$</td>
</tr>
<tr>
<td>$P_{46}$</td>
<td>$2^{i} \text{ Alt}(7)$</td>
<td>$1_{P_{46}}, 4_{P_{46}}, 7_{P_{46}}, 6_{P_{46}}, 14_{P_{46}}, 20_{P_{46}}$</td>
</tr>
<tr>
<td>$P_{47}$</td>
<td>$(3 \times \text{ Alt}(5)): 2$</td>
<td>$1_{P_{47}}, 2_{P_{47}}, 4^{b}<em>{P</em>{47}}, 4^{b}<em>{P</em>{47}}, 4^{d}<em>{P</em>{47}}, 8_{P_{47}}$</td>
</tr>
<tr>
<td>$P_{67}$</td>
<td>$M_{21}, 2$</td>
<td>$1_{P_{67}}, 9_{P_{67}}, 9_{P_{67}}, 16_{P_{67}}, 64_{P_{67}}$</td>
</tr>
</tbody>
</table>

Table A.1: The irreducible modules over $GF(2)$ of the 2-parabolic subgroups of $M_{23}$ of rank 2.

$(\lambda_{\dim,k_{\dim}}^{(ij)})_{P_{i}} := (\lambda_{\dim,k_{\dim}}^{(ij)})_{P_{i}}$ and $(\lambda_{\dim,k_{\dim}}^{(ij)})_{P_{j}} := (\lambda_{\dim,k_{\dim}}^{(ij)})_{P_{j}}$. Similarly for $P_{ik}$ and $P_{jk}$. Consequently $(\lambda_{\dim,k_{\dim}}^{(ij)})_{P_{i}} := H_{0}(\lambda_{\dim,k_{\dim}}^{(ij)})$. It follows that we must consider the universal panel-irreducible presheaves of $P_{13}, P_{16}, P_{17}, P_{23}, P_{26}, P_{27}, P_{34}, P_{36}, P_{37}, P_{46}, P_{47}$ and $P_{67}$. A summary of the irreducible modules over $GF(2)$ admitted by each of these parabolic subgroups is given in Table A.1. We note that $P_{16} = P_{23} = P_{26} = P_{36} = P_{46}$ and $P_{17} = P_{27} = P_{47}$. However, we will subsequently be considering presheaves for each group $P_{ij}$ defined on the minimal parabolic system $\{P_{1}, P_{j}\}$. Thus, we have listed each of these labellings of the groups as separate entries in Table A.1.

There are four universal panel-irreducible presheaves for each parabolic subgroup not containing $P_{6}$ or $P_{7}$, six universal panel-irreducible presheaves for each parabolic subgroup containing either $P_{6}$ or $P_{7}$, and nine universal panel-irreducible presheaves for $P_{67}$. To calculate the zero-homology groups of each of these presheaves, we use the methods described in Chapter 3, namely considering possible quotients of induced modules, straightforward geometric spanning arguments and looking at which irreducible modules admit certain presheaves. In the case of presheaves defined on $P_{23}$, we also use Ronan’s Duality Theorem to obtain the dimension of $H_{0}(\lambda_{23}^{(23)})$. With the dimension determined, the exact structure of the homology group may then be obtained by considering the possible quotients of induced modules. Details of the universal panel-irreducible presheaves of the parabolic subgroups $P_{13}, P_{16}, P_{17}, P_{23}, P_{26}, P_{27}, P_{34}, P_{36}, P_{37}, P_{46}, P_{47}$ and $P_{67}$ are given in Tables A.2 and A.13. Using these tables and the observation above that $(\lambda_{\dim,k_{\dim}}^{(ij)})_{P_{ij}} := H_{0}(\lambda_{\dim,k_{\dim}}^{(ij)})$, the reader can easily determine the vertex terms of the universal panel-irreducible presheaves of $M_{23}$.

---

1. We note that - due to the construction of the universal presheaf - this definition is independent of the values of $k$ and $k_{\dim}$.

2. As $P_{57} \cong M_{22}$, a full discussion of the zero-homology groups of presheaves for $P_{57}$ is given in Chapter 3. In that discussion $P_{3}$ and $P_{7}$ are labelled as $P_{1}$ and $P_{2}$ respectively.
### A.1. Vertex Term Calculations for Presheaves of \( M_{23} \)

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
<th>Notes about ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{1,1}^{(13)} )</td>
<td>1</td>
<td>1</td>
<td>( (\lambda_{1,1}^{(13)}) \ast \cong \lambda_{2,2}^{(13)} )</td>
</tr>
<tr>
<td>( \lambda_{1,2}^{(13)} )</td>
<td>3</td>
<td>3</td>
<td>( (\lambda_{1,2}^{(13)}) \ast \cong \lambda_{2,1}^{(13)} )</td>
</tr>
<tr>
<td>( \lambda_{2,1}^{(13)} )</td>
<td>3</td>
<td>3</td>
<td>( (\lambda_{2,1}^{(13)}) \ast \cong \lambda_{1,2}^{(13)} )</td>
</tr>
<tr>
<td>( \lambda_{2,2}^{(13)} )</td>
<td>8</td>
<td>8</td>
<td>( (\lambda_{2,2}^{(13)}) \ast \cong \lambda_{1,1}^{(13)} )</td>
</tr>
</tbody>
</table>

Table A.2: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_1, P_3\} \) of the parabolic subgroup \( P_{13} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
<th>Notes about ( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{1,1}^{(16)} )</td>
<td>1</td>
<td>1</td>
<td>( \frac{463}{6} )</td>
</tr>
<tr>
<td>( \lambda_{1,4}^{(16)} )</td>
<td>6</td>
<td>( 6/\left(\frac{463}{6}\right) )</td>
<td>( \frac{4}{3} )</td>
</tr>
<tr>
<td>( \lambda_{1,7}^{(16)} )</td>
<td>4</td>
<td>( \frac{4}{3} )</td>
<td>( \frac{6}{1} )</td>
</tr>
<tr>
<td>( \lambda_{1,16}^{(16)} )</td>
<td>6</td>
<td>( \frac{6}{1} )</td>
<td>( \frac{14}{14/\left(14 \oplus 1/20/1\right)} )</td>
</tr>
<tr>
<td>( \lambda_{2,16}^{(16)} )</td>
<td>14</td>
<td>( 14/(14 \oplus 1/20/1) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( \lambda_{2,7}^{(16)} )</td>
<td>4, 20</td>
<td>( 4 \oplus 20 )</td>
<td>The submodule 1/20/1 is uniserial.</td>
</tr>
</tbody>
</table>

Table A.3: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_1, P_6\} \) of the parabolic subgroup \( P_{16} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{1,1}^{(17)} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_{1,4}^{(17)} )</td>
<td>( 4^a )</td>
<td>( 4^a )</td>
</tr>
<tr>
<td>( \lambda_{1,7}^{(17)} )</td>
<td>( 4^b )</td>
<td>( 4^b )</td>
</tr>
<tr>
<td>( \lambda_{1,16}^{(17)} )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \lambda_{2,16}^{(17)} )</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>( \lambda_{2,7}^{(17)} )</td>
<td>( 4^c, 4^d )</td>
<td>( 4^c \oplus 4^d )</td>
</tr>
</tbody>
</table>

Table A.4: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_1, P_7\} \) of the parabolic subgroup \( P_{17} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
<th>Notes about ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{1,1}^{(23)} )</td>
<td>1</td>
<td>1</td>
<td>( (\lambda_{1,1}^{(23)}) \ast \cong \lambda_{2,2}^{(23)} )</td>
</tr>
<tr>
<td>( \lambda_{1,2}^{(23)} )</td>
<td>6, 14</td>
<td>( 6 \oplus 14 )</td>
<td>( (\lambda_{1,2}^{(23)}) \ast \cong \lambda_{2,1}^{(23)} )</td>
</tr>
<tr>
<td>( \lambda_{2,1}^{(23)} )</td>
<td>4, 4</td>
<td>( 4/6 \oplus 4/6 )</td>
<td>( (\lambda_{2,1}^{(23)}) \ast \cong \lambda_{1,2}^{(23)} )</td>
</tr>
<tr>
<td>( \lambda_{2,2}^{(23)} )</td>
<td>14, 20</td>
<td>( 20/1/14/1/20 \oplus (14/1 \oplus 20/1)/14 )</td>
<td>( (\lambda_{2,2}^{(23)}) \ast \cong \lambda_{1,1}^{(23)} )</td>
</tr>
</tbody>
</table>

Table A.5: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_2, P_3\} \) of the parabolic subgroup \( P_{23} \) of \( M_{23} \).
APPENDIX A. VERTEX TERM CALCULATIONS

### Table A.6: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_2, R_6\} \) of the parabolic subgroup \( P_{26} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
<th>Notes about ( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(26)}_{1,1} )</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \lambda^{(26)}_{1,4} )</td>
<td>6</td>
<td>( 6/(4\oplus 1/6) )</td>
<td></td>
</tr>
<tr>
<td>( \lambda^{(26)}_{1,7} )</td>
<td>None</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( \lambda^{(26)}_{2,1} )</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>( \lambda^{(26)}_{2,4} )</td>
<td>14</td>
<td>( 14/(14 \oplus 1/20/1) )</td>
<td>The submodule 1/20/1 is uniserial.</td>
</tr>
<tr>
<td>( \lambda^{(26)}_{2,7} )</td>
<td>4, ( \mathfrak{4} ), 20</td>
<td>( 4 \oplus \mathfrak{4} \oplus 20/1 )</td>
<td></td>
</tr>
</tbody>
</table>

### Table A.7: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_2, R_1\} \) of the parabolic subgroup \( P_{27} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(27)}_{1,1} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda^{(27)}_{1,2} )</td>
<td>4(^a)</td>
<td>4(^a)</td>
</tr>
<tr>
<td>( \lambda^{(27)}_{1,7} )</td>
<td>4(^c)</td>
<td>4(^c)</td>
</tr>
<tr>
<td>( \lambda^{(27)}_{2,1} )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \lambda^{(27)}_{2,4} )</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>( \lambda^{(27)}_{2,7} )</td>
<td>4(^b), 4(^d)</td>
<td>( 4(^b\oplus 4(^d))</td>
</tr>
</tbody>
</table>

### Table A.8: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_3, R_1\} \) of the parabolic subgroup \( P_{34} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
<th>Notes about ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(34)}_{1,1} )</td>
<td>1</td>
<td>1</td>
<td>( \lambda^{(34)}<em>{1,1} ) * ( \lambda^{(34)}</em>{2,2} )</td>
</tr>
<tr>
<td>( \lambda^{(34)}_{1,2} )</td>
<td>( \mathfrak{3} )</td>
<td>( \mathfrak{3} )</td>
<td>( \lambda^{(34)}<em>{1,2} ) * ( \lambda^{(34)}</em>{2,1} )</td>
</tr>
<tr>
<td>( \lambda^{(34)}_{2,1} )</td>
<td>3</td>
<td>3</td>
<td>( \lambda^{(34)}<em>{2,1} ) * ( \lambda^{(34)}</em>{1,2} )</td>
</tr>
<tr>
<td>( \lambda^{(34)}_{2,2} )</td>
<td>8</td>
<td>8</td>
<td>( \lambda^{(34)}<em>{2,2} ) * ( \lambda^{(34)}</em>{1,1} )</td>
</tr>
</tbody>
</table>

### Table A.9: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_3, R_6\} \) of the parabolic subgroup \( P_{36} \) of \( M_{23} \).

<table>
<thead>
<tr>
<th>Presheaf, ( \lambda )</th>
<th>Irreducible quotients of ( H_0(\lambda) )</th>
<th>( H_0(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(36)}_{1,1} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda^{(36)}_{1,4} )</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>( \lambda^{(36)}_{1,7} )</td>
<td>4, ( \mathfrak{4} )</td>
<td>( 4 \oplus \mathfrak{4} )</td>
</tr>
<tr>
<td>( \lambda^{(36)}_{2,1} )</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>( \lambda^{(36)}_{2,4} )</td>
<td>6, 14</td>
<td>( 6 \oplus 14/1 )</td>
</tr>
<tr>
<td>( \lambda^{(36)}_{2,7} )</td>
<td>20</td>
<td>( 20/1 )</td>
</tr>
</tbody>
</table>

Table A.6: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \( \{P_2, R_6\} \) of the parabolic subgroup \( P_{26} \) of \( M_{23} \).
### A.1. Vertex Term Calculations for Presheaves of $M_{23}$

<table>
<thead>
<tr>
<th>Presheaf, $\lambda$</th>
<th>Irreducible quotients of $H_0(\lambda)$</th>
<th>$H_0(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,1}^{(37)}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{1,4}^{(37)}$</td>
<td>10</td>
<td>$10/10/1$</td>
</tr>
<tr>
<td>$\lambda_{1,7}^{(37)}$</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>$\lambda_{2,1}^{(37)}$</td>
<td>$\Pi_0$</td>
<td>$\Pi_0/1$</td>
</tr>
<tr>
<td>$\lambda_{2,4}^{(37)}$</td>
<td>98</td>
<td>$98/1/\left(\frac{34}{1/10/10/10} \oplus 10/10/1\right)/34/1$</td>
</tr>
<tr>
<td>$\lambda_{2,7}^{(37)}$</td>
<td>140</td>
<td>$140/(1 \oplus 1/34/1/10/34)$</td>
</tr>
</tbody>
</table>

Table A.10: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_7\}$ of the parabolic subgroup $P_{37}$ of $M_{23}$.

<table>
<thead>
<tr>
<th>Presheaf, $\lambda$</th>
<th>Irreducible quotients of $H_0(\lambda)$</th>
<th>$H_0(\lambda)$</th>
<th>Notes about $H_0(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,1}^{(46)}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{1,4}^{(46)}$</td>
<td>6</td>
<td>$6/\left(\frac{467}{6}\right)$</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{1,7,4}^{(46)}$</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{2,1}^{(46)}$</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{2,4}^{(46)}$</td>
<td>14</td>
<td>$14/(14 \oplus 1/20/1)$</td>
<td>The submodule $1/20/1$ is uniserial.</td>
</tr>
<tr>
<td>$\lambda_{2,7}^{(46)}$</td>
<td>4, 20</td>
<td>$4 \oplus 20$</td>
<td></td>
</tr>
</tbody>
</table>

Table A.11: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_4, P_8\}$ of the parabolic subgroup $P_{46}$ of $M_{23}$.

<table>
<thead>
<tr>
<th>Presheaf, $\lambda$</th>
<th>Irreducible quotients of $H_0(\lambda)$</th>
<th>$H_0(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,1}^{(47)}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{1,4}^{(47)}$</td>
<td>$4^a$</td>
<td>$4^a$</td>
</tr>
<tr>
<td>$\lambda_{1,7,4}^{(47)}$</td>
<td>$4^b$</td>
<td>$4^b$</td>
</tr>
<tr>
<td>$\lambda_{2,1}^{(47)}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_{2,4}^{(47)}$</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$\lambda_{2,7}^{(47)}$</td>
<td>$4^c, 4^d$</td>
<td>$4^c \oplus 4^d$</td>
</tr>
</tbody>
</table>

Table A.12: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_4, P_7\}$ of the parabolic subgroup $P_{47}$ of $M_{23}$. 
APPENDIX A. VERTEX TERM CALCULATIONS

<table>
<thead>
<tr>
<th>Presheaf, $\lambda$</th>
<th>Irreducible quotients of $H_0(\lambda)$</th>
<th>$H_0(\lambda)$</th>
<th>Notes about $H_0(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{(67)}^{(1)}$</td>
<td>1</td>
<td>1</td>
<td>The module 9/1/1 is uniserial.</td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(4)}$</td>
<td>9</td>
<td>9/1/1</td>
<td>The module (\bar{9}/1/1) is uniserial.</td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(7)}$</td>
<td>None</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(4,1)}$</td>
<td>(\bar{9})</td>
<td>(\bar{9}/1/1)</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(4,4)}$</td>
<td>64</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(7,3)}$</td>
<td>None</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(7,4)}$</td>
<td>None</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(7,7)}$</td>
<td>None</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{(67)}^{(7,7)}$</td>
<td>16</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Table A.13: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system \(\{P_6, P_7\}\) of the parabolic subgroup $P_{67}$ of $M_{23}$.

Figure A.6: The diagram of the geometry arising from the unique 2-minimal parabolic system of $M_{24}$.

A.2 Vertex Term Calculations for Universal Panel-Irreducible Presheaves of $M_{24}$ over $GF(2)$

There is a unique 2-minimal parabolic system of $M_{24}$ given by $S := \{P_1, P_3, P_4\}$, where each $P_i$ has shape $2^{6+3}.\text{Sym}(3)$. The maximal parabolic subgroups are given by $P_{13} \sim 2^6.3\text{Sym}(6)$, $P_{14} \sim 2^{6+2}.(\text{Sym}(3) \times \text{Sym}(3))$ and $P_{34} \sim 2^{4+3}.L_3(2)$. Calculations show that the incidence graph of a residue of flag of cotype $\{1, 3\}$ of the corresponding geometry has 1-diameter and 3-diameter both equal to 8 and girth equal to 10. The corresponding graph of a residue of flag of cotype $\{1, 4\}$ has 1- and 4-diameter equal to 2 and girth equal to 4, whilst that of a residue of flag of cotype $\{3, 4\}$ has 3- and 4-diameter equal to 3 and girth equal to 6. The diagram of the geometry arising from $S$ is given in Figure A.6.

Each minimal parabolic subgroup $P_i$ admits two classes of irreducible $GF(2)P_i$-modules, $1_{P_i}$ and $2_{P_i}$. Meanwhile, the maximal parabolic subgroups $P_{13}$, $P_{14}$ and $P_{34}$ admit seven, four and four classes of irreducible modules over $GF(2)$ respectively, given by $1_{P_{13}}$, $4_{P_{13}}$, $\bar{4}_{P_{13}}$, $6_{P_{13}}$, $\bar{6}_{P_{13}}$, $16_{P_{13}}$.
Table A.14: The vertex terms of the universal panel-irreducible presheaves for $M_{24}$ over $GF(2)$.

<table>
<thead>
<tr>
<th>Presheaf, $(\lambda_{i,j,k})_{P13}$</th>
<th>Vertex Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,1,1}$</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{1,1,2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{1,2,1}$</td>
<td>$4/1 \oplus \overline{6}$</td>
</tr>
<tr>
<td>$\lambda_{1,2,2}$</td>
<td>$4/1 \oplus \overline{6}$</td>
</tr>
<tr>
<td>$\lambda_{2,1,1}$</td>
<td>$4/1 \oplus 6$</td>
</tr>
<tr>
<td>$\lambda_{2,1,2}$</td>
<td>$4/1 \oplus 6$</td>
</tr>
<tr>
<td>$\lambda_{2,2,1}$</td>
<td>$16 \oplus \frac{18}{6} \oplus \overline{6}$</td>
</tr>
<tr>
<td>$\lambda_{2,2,2}$</td>
<td>$16 \oplus \frac{18}{6} \oplus \overline{6}$</td>
</tr>
</tbody>
</table>

and $18_{P13}, 1_{P14}, 2_{P14}, \overline{7}_{P14}$ and $4_{P14}$ and $1_{P34}, 3_{P34}, \overline{7}_{P34}$ and $8_{P34}$.

We proceed by defining $\lambda^{(i,j)}_{a_{i,j}}$ to be the universal panel-irreducible presheaf defined on the maximal parabolic subgroup $P_{ij}$ having a 1-dimensional chamber term, and a- and b-dimensional terms at panels of type $P_i$ and $P_j$ respectively. Thus $H_0(\lambda^{(i,j)}_{a_{i,j}}) \cong P_{ij}$ for each $(i, j) \in \{(1, 3), (1, 4), (3, 4)\}$.

Since $[P_i : B] = 3$ for $i = 1, 3, 4$, we note that the duality relations $\left(\lambda^{(i,j)}_{a_{i,j}}\right)^* \cong \lambda^{(i,j)}_{a_{i,j}}$ and \left(\lambda^{(i,j)}_{a_{i,j}}\right)^* \cong \lambda^{(i,j)}_{a_{i,j}}$ hold true.

In particular, combining Ronan’s Duality Theorem with the Euler characteristic of $\lambda^{(i,j)}_{a_{i,j}}$ we see that

$$\dim(H_0(\lambda^{(i,j)}_{a_{i,j}})) = \chi(\lambda^{(i,j)}_{a_{i,j}}) + 1 = [P_{ij} : P_i] + 1. \quad \text{(A.2.1)}$$

**Presheaves defined on $P_{13}$**

The irreducible $GF(2)P_{13}$-modules $1_{P13}, 4_{P13}, \overline{7}_{P13}, 6_{P13}, \overline{6}_{P13}, 16_{P13}$ and $18_{P13}$ each admit a unique universal panel-irreducible presheaf, these respectively being $\lambda^{(13)}_{1,3,1}, \lambda^{(13)}_{1,3,2}, \lambda^{(13)}_{1,3,3}, \lambda^{(13)}_{1,3,4}, \lambda^{(13)}_{1,3,5}, \lambda^{(13)}_{1,3,6}, \lambda^{(13)}_{1,3,7}$ and $\lambda^{(13)}_{1,3,8}$. By considering possible quotients of the induced modules $(1_{P1})^{P1}_{P1}, (2_{P1})^{P1}_{P1}, (1_{P3})^{P3}_{P3}$ and $(2_{P3})^{P3}_{P3}$ in conjunction with (A.2.1) we see that

$$H_0(\lambda^{(13)}_{1,3,3}) \cong \overline{4}/1 \oplus \overline{6}, \quad H_0(\lambda^{(13)}_{1,3,4}) \cong 4/1 \oplus 6 \quad \text{and} \quad H_0(\lambda^{(13)}_{1,3,8}) \cong 16 \oplus \frac{18}{6} \oplus \overline{6}.$$ 

**Presheaves defined on $P_{14}$ and $P_{34}$**

Since $[P_{14} : P_1] = [P_{14} : P_3] = 3$ and $[P_{34} : P_3] = [P_{34} : P_4] = 7$, the homology groups $H_0(\lambda^{(14)}_{a_{1,4}}, a_{1,4})$ and $H_0(\lambda^{(34)}_{a_{3,4}}, a_{3,4})$ for $a, b \in \{1, 2\}$ may be directly calculated and all turn out to be irreducible. We have that

$$H_0(\lambda^{(14)}_{1,2,4}) \cong \overline{2}_{P_{14}}, \quad H_0(\lambda^{(14)}_{1,2,4}) \cong 2_{P_{14}}, \quad H_0(\lambda^{(14)}_{1,2,4}) \cong 4_{P_{14}}, \quad H_0(\lambda^{(34)}_{3,4,2}) \cong 3_{P_{34}}, \quad H_0(\lambda^{(34)}_{3,4,2}) \cong \overline{3}_{P_{34}}$$

and $H_0(\lambda^{(34)}_{3,4,2}) \cong 8_{P_{34}}$.

**Presheaves defined on $M_{24}$**

As in Chapter 3, we denote the presheaf for $M_{24}$ having a 1-dimensional chamber term and $i$-, $j$- and $k$-dimensional panel terms at panels of type $P_1, P_3$ and $P_4$ by $\lambda_{i,j,k}$. Utilising the calculations above, we detail the vertex terms of these presheaves in Table A.14.
Appendix B

$x$-Graphs Related to $\pi$-Product Graphs in Symmetric Groups

We now present the various $x$-graphs that were omitted for the sake of brevity in Chapter 4.

Figure B.1: The $x$-graphs $G_{y_{ij}}^{t_i,t_j}$ and $G_{x_{i}x_{j}}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 0$ and $q$ is even.
Figure B.2: The x-graphs $G_{y_{ij}}$ and $G_{x_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 0$ and $q$ is odd.

Figure B.3: The x-graphs $G_{y_{ij}}$ and $G_{x_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 1$ and $q$ is even.
Figure B.4: The $x$-graphs $G_{y_{ij}}^t$ and $G_{x_{ij}}^y$ from Lemma 4.2.6(i) in the case that $r = 1$ and $q$ is odd.
Figure B.5: The $x$-graphs $G^{t_{ij},t}_{y_{ij}}$ and $G^{y_{ij}}_{x_{i},x_{j}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and $q, r$ are both even.
Figure B.6: The $x$-graphs $G_{yi,j}^{t_{i,j}}$ and $G_{x_i,x_j}^{y_{i,j}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and $q$ is even and $r$ is odd.
Figure B.7: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_i,x_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and $q$ is odd and $r$ is even.
Figure B.8: The $x$-graphs $G_{y_{ij}}^{t,t_{ij}}$ and $G_{y_{ij}}^{y_{ij},x_{ij}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and $q, r$ are both odd.
Figure B.9: The $x$-graphs $G_{y_{i}}^{t_{i}}$ and $G_{x_{i}}^{y_{i}}$ from Lemma 4.2.6(ii) in the case that $q$ is even.

Figure B.10: The $x$-graphs $G_{y_{i}}^{t_{i}}$ and $G_{x_{i}}^{y_{i}}$ from Lemma 4.2.6(ii) in the case that $q$ is odd.
Figure B.11: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{i,x_j}}^{y_{ij}}$ from Lemma 4.2.6(iii) in the case that $q = 2$.

Figure B.12: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{i,x_j}}^{y_{ij}}$ from Lemma 4.2.6(iii) in the case that $q \geq 3$ is even.
Figure B.13: The $x$-graphs $G_{g_{ij}}^{t_{ij}}$ and $G_{x_{i}x_{j}}^{y_{ij}}$ from Lemma 4.2.6(iii) in the case that $q \geq 3$ is odd.

Figure B.14: The $x$-graphs $G_{g_{ij}}^{t_{ij}}$ and $G_{x_{i}x_{j}}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q = r = 1$. 
Figure B.15: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{ij}}^{u_{ij}}$ from Lemma 4.2.6(iv) in the case that $q = 1$ and $r > 1$ is even.

Figure B.16: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{ij}}^{u_{ij}}$ from Lemma 4.2.6(iv) in the case that $q = 1$ and $r > 1$ is odd.
Figure B.17: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{i},x_{j}}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q > 1$ and $q, r$ are both even.
Figure B.18: The x-graphs $G_{uv}^t$ and $G_{uv}^y$ from Lemma 4.2.6(iv) in the case that $q > 1$ is even and $r$ is odd.
Figure B.19: The $x$-graphs $G_{y_{i,j}}^{t_{i,j}}$ and $G_{x_{i,j}}^{y_{i,j}}$ from Lemma [4.2.6(iv)] in the case that $q > 1$ is odd and $r$ is even.
Figure B.20: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{i,x_j}}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q > 1$ and $q,r$ are both odd.

Figure B.21: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{i,x_j}}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q = r = 1$. 
Figure B.22: The $x$-graphs $G_{yi,j}^{t_i t_j}$ and $G_{xi,x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > 1$ is even and $r = 1$.

Figure B.23: The $x$-graphs $G_{yi,j}^{t_i t_j}$ and $G_{xi,x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > 1$ is odd and $r = 1$. 
Figure B.24: The $x$-graphs $G_{ij}^{\ell_{ij}}$ and $G_{x,j}^{\ell_{ij}}$ from Lemma 4.2.6(v) in the case that $q = r > 1.$
Figure B.25: The $x$-graphs $G_{y_{ij}}^{t_{ij}}$ and $G_{x_{ij}}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > r > 1$ and $q$ and $r$ have the same parity.
$G_{y_{ij}}^{t_{i} t_{j}}$:

1. $\{1, 2\}$, $\{2(q-r)-1, 2(q-r)\}$
2. $\{q-r-2, q-r-1\}$, $\{q-r+2, q-r+3\}$, $\{q-r, q-r+1\}$
3. $\{2(q-r)+1, 2(q-r)+2\}$, $\{2q+1, 2q+2\}$
4. $\{2q-3, 2q-2\}$, $\{v-3, v-2\}$
5. $\{2q-1, 2q\}$, $\{v-1, v\}$, $\{v+1\}$, $\{v+2\}$

$G_{x_i x_j}^{y_{ij}}$:

1. $\{1, 2(q-r)\}$, $\{2(q-r) +1, 2q+1\}$
2. $\{2, 2(q-r) -1\}$, $\{3, 2(q-r) -2\}$
3. $\{q-r-1, q-r+2\}$, $\{q-r, q-r+1\}$
4. $\{2(q-r)+2, 2q+2\}$, $\{2(q-r)+3, 2q+3\}$
5. $\{2q-2, v-2\}$, $\{2q-1, v-1\}$
6. $\{2q\}$, $\{v+1, v+2\}$, $\{v\}$

Figure B.26: The $x$-graphs $G_{y_{ij}}^{t_{i} t_{j}}$ and $G_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > r > 1$ and $q$ and $r$ have different parities.
Figure B.27: The $x$-graphs $G_{y_{ij}}^{t_{lj}t_{lk}}$ and $G_{x_{t_j,x_k}}^{y_{ij}}$ from Lemma 4.2.7(i) in the case that $q = r = s = 1$.

Figure B.28: The $x$-graphs $G_{y_{ij}}^{t_{lj}t_{lk}}$ and $G_{x_{t_j,x_k}}^{y_{ij}}$ from Lemma 4.2.7(i) in the case that $q, r$ are both even and $s = 1$. 
Figure B.29: The $x$-graphs $G^t_{yijk}$ and $G^y_{xixjxk}$ from Lemma 4.2.7(i) in the case that $q$ is even, $r$ is odd and $s = 1$. 
Figure B.30: The $x$-graphs $G_{yijk}^{t_it_jt_k}$ and $G_{xixjxk}^{yijk}$ from Lemma 4.2.7(i) in the case that $q$ is odd, $r$ is even and $s = 1.$
Figure B.31: The $x$-graphs $G_{y_{ij}k}^{t_{i}t_{j}t_{k}}$ and $G_{x_{i}x_{j}x_{k}}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q, r$ are both odd and $s = 1.$
Figure B.32: The $x$-graphs $G_{yijk}^{t_1 t_2 t_3}$ and $G_{xijk}^{yijk}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ are all even.
Figure B.33: The $x$-graphs $\mathcal{G}_{ik}^{t,i,t,k}$ and $\mathcal{G}_{ijk}^{t,i,t,k}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ with $q, r$ even and $s$ odd.
Figure B.34: The $x$-graphs $G_{t_{ij}tk}^{\ell}$ and $G_{x_{xj}xk}^{q}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ with $q, s$ both even and $r$ odd.
Figure B.35: The $x$-graphs $G^t_{ijk}$ and $G^y_{ijk}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ with $q$ even and $r, s$ both odd.
Figure B.36: The x-graphs $\mathcal{G}^{t_{ijk}}_{y_{ijk}}$ and $\mathcal{G}^{y_{ijk}}_{x_{ijk}x_{jk}}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ with $q$ odd and $r, s$ both even.
Figure B.37: The $x$-graphs $g_{y_{ijk}}^{t_{ijk}k}$ and $g_{x_{ij}x_{jk}}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ with $q, s$ both odd and $r$ even.
Figure B.38: The $x$-graphs $\mathcal{G}^{u_i u_j u_k}_{y_{i,j,k}}$ and $\mathcal{G}^{y_{i,j,k}}_{x_i x_j x_k}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ with $q, r$ both odd and $s$ even.
Figure B.39: The $x$-graphs $G_{y_{ijk}}^{t_{ij}, t_k}$ and $G_{x_{ijk}}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geq r \geq s > 1$ all odd.
Figure B.40: The $x$-graphs $G_{y_{ijk}}^{t_{ij}t_{jk}}$ and $G_{x_{i}x_{j}x_{k}}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q = r = 1$.

Figure B.41: The $x$-graphs $G_{y_{ijk}}^{t_{ij}t_{jk}}$ and $G_{x_{i}x_{j}x_{k}}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q = 1$ and $r > 1$ is even.
Figure B.42: The $x$-graphs $G_{ijk}$ and $G_{xixjxk}$ from Lemma 4.2.7(ii) in the case that $q = 1$ and $r > 1$ is odd.

Figure B.43: The $x$-graphs $G_{ijk}$ and $G_{xixjxk}$ from Lemma 4.2.7(ii) in the case that $q = 2$ and $r > 1$ is even.
Figure B.44: The $x$-graphs $\mathcal{G}_{y_{ijk}}^{t_i,t_j,t_k}$ and $\mathcal{G}_{x_i,x_j,x_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q = 2$ and $r > 1$ is odd.
Figure B.45: The $x$-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_{i,j,k}}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q > 2$, $r > 1$ and $q$ and $r$ are both even.
Figure B.46: The $x$-graphs $G_{y_{ijk}}^{t_i,t_j,t_k}$ and $G_{x_{i},x_{j},x_{k}}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q > 2$ is even and $r > 1$ is odd.
Figure B.47: The $x$-graphs $G_{y_ijk}^{l_j}$ and $G_{x, x, x_k}^{y_ijk}$ from Lemma 4.2.7(ii) in the case that $q > 2$ is odd and $r > 1$ is even.
Figure B.48: The $x$-graphs $G_{ij,k}^{t_i,t_j,t_k}$ and $G_{x_i,x_j,x_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q > 2$, $r > 1$ and $q$ and $r$ are both odd.
Figure B.49: The $x$-graphs $G_y$ and $G'_y$ from Lemma 4.2.8 in the case that $G_x$ contains a cycle of two black vertices and $m = 2$.

Figure B.50: The $x$-graphs $G_y$ and $G'_y$ from Lemma 4.2.8 in the case that $G_x$ contains a cycle of two black vertices and $m = 3$. 
Appendix B. X-Graphs from Chapter 4

Figure B.51: The $x$-graphs $G_y$ and $G^y_x$ from Lemma 4.2.8 in the case that $G_x$ contains a cycle of three black vertices and $m = 3$.

Figure B.52: The $x$-graphs $G^I_{y,l}$ and $G^y_{x,l}$ from (4.2.1) when $u = 1$.

Figure B.53: The $x$-graphs $G^I_{y,l}$ and $G^y_{x,l}$ from (4.2.1) when $u > 1$ is even.
Figure B.54: The $x$-graphs $G_{yi}^{ti}$ and $G_{yi}^{yi}$ from (4.2.1) when $u > 1$ is odd.
Figure B.55: The $x$-graphs $G^l_{yi}$ and $G^r_{yi}$ from (4.2.2).

Figure B.56: The $x$-graphs $G^l_{yi}$ and $G^r_{yi}$ from (4.2.3) with $u = 1$.

Figure B.57: The $x$-graphs $G^l_{yi}$ and $G^r_{yi}$ from (4.2.3) with $u > 1$ even.
Figure B.58: The $x$-graphs $G_{yi}^1$ and $G_{zi}^y$ from \(4.2.3\) with $u > 1$ odd.

Figure B.59: The $x$-graph $G_{yi}^1$ from \(4.3.5\).

Figure B.60: The $x$-graph $G_{yi}^1$ from \(4.3.6\).

Figure B.61: The $x$-graph $G_{yi}^1$ from \(4.3.7\).
Figure B.62: The $x$-graphs $G_{y_1}^{z_1}$ and $G_{y_2}^{z_2}$ from (4.3.11) and (4.3.12) respectively.

Figure B.63: The $x$-graphs $G_{y_i+1}^{z_i+1}$ and $G_{y_i+1}^{z_i+1}$ from (4.3.10) and (4.3.12) respectively.
Figure B.64: The $x$-graphs $G_y^z$, and $G_z^y$ from (4.3.13) and (4.3.14) respectively.

Figure B.65: The $x$-graphs $G_y^z$, and $G_z^y$ from (4.3.13) and (4.3.14) respectively.
Figure B.66: The $x$-graphs $G_{x2}^{y_1}$ and $G_{y2}^{z_2}$ from (4.3.15) and (4.3.16) respectively.

Figure B.67: The $x$-graphs $G_{z_{i+1}}^{y_i}$ and $G_{y_{i+1}}^{z_{i+1}}$ from (4.3.15) and (4.3.16) respectively.
Figure B.68: The $x$-graphs $G_{z_1}$ and $G_{y_1}^{z_1}$ from (4.3.17).

Figure B.69: The $x$-graphs $G_{z_1}$ and $G_{y_1}^{z_1}$ from (4.3.18).

Figure B.70: The $x$-graphs $G_{z_1}$ and $G_{y_1}^{z_1}$ from (4.3.19).
**APPENDIX B. X-GRAPHS FROM CHAPTER 4**

Figure B.71: The $x$-graphs $G_{z_1}$ and $G_{y_1}^z$ from (4.3.21).

Figure B.72: The $x$-graph $G_{y_1}$ from (4.3.22).

Figure B.73: The $x$-graph $G_{y_1}$ from (4.3.23).

Figure B.74: The $x$-graphs $G_{z_1}$ and $G_{y_1}^z$ from (4.3.24).

Figure B.75: The $x$-graphs $G_{y_1}^{y_1}$ and $G_{y_2}^{z_2}$ from (4.3.25).
Figure B.76: The $x$-graphs $G_{zi+1}^y$ and $G_{yi+1}^{zi}$ from (4.3.25).

Figure B.77: The $x$-graphs $G_{z2}^y$ and $G_{y2}^z$ from (4.3.26).
Figure B.78: The $x$-graphs $G_{z_{i+1}}^{y_i}$ and $G_{y_{i+1}}^{z_{i+1}}$ from (4.3.26).
Figure B.79: The x-graphs $G_{y_1}^{y_2}$ and $G_{z_2}^{z_2}$ from (4.3.27).

Figure B.80: The x-graphs $G_{z_1+1}^{y_i}$ and $G_{y_i+1}^{z_i+1}$ from (4.3.27).

Figure B.81: The x-graphs $G_{z_2}^{y_1}$ and $G_{y_2}^{z_2}$ from (4.3.28).
\( \mathcal{G}_{y_{i+1}} \):

- \( x_{i+1,1} \)
- \( x_{1,1}, x_{1,2} \)
- \( x_{2,1}, x_{2,2} \)

\( \mathcal{G}_{z_{i+1}} \):

- \( x_{i,1}, x_{i,2} \)
- \( w_{i+1,1}, w_{i+1,2} \)
- \( w_{m,1}, w_{m,2} \)

\( \mathcal{G}_{y_{i+1}} \):

- \( x_{i+1,2} \)
- \( x_{i,1}, x_{i+1,1} \)
- \( x_{1,2}, x_{2,1} \)
- \( x_{i-2}, x_{i,1} \)

\( \mathcal{G}_{z_{i+1}} \):

- \( x_{i,2}, w_{i+1,1} \)
- \( w_{i+1,2}, w_{i+2,1} \)
- \( w_{m-1,2}, w_{m,1} \)

Figure B.82: The \( x \)-graphs \( \mathcal{G}_{y_{i+1}} \) and \( \mathcal{G}_{z_{i+1}} \) from (4.3.28).

\( \mathcal{G}_{z_{1}} \):

- \( 1, 2 \)
- \( 3, 4 \)
- \( 2m - 1, 2m \)
- \( 2m + 1 \)

\( \mathcal{G}_{y_{1}} \):

- \( 1 \)
- \( 2, 3 \)
- \( 4, 5 \)
- \( 2m, 2m + 1 \)

Figure B.83: The \( x \)-graphs \( \mathcal{G}_{z_{1}} \) and \( \mathcal{G}_{y_{1}} \) from (4.3.29).

\( \mathcal{G}_{z_{1}} \):

- \( 1, 2 \)
- \( 3, 4 \)
- \( 2m - 1, 2m \)
- \( 2m + 1 \)

\( \mathcal{G}_{y_{1}} \):

- \( 1 \)
- \( 2m, 2m + 1 \)
- \( 2m - 2, 2m - 1 \)
- \( 2, 3 \)

Figure B.84: The \( x \)-graphs \( \mathcal{G}_{z_{1}} \) and \( \mathcal{G}_{y_{1}} \) from (4.3.30).
Appendix C

MAGMA Code

The following code may be used in the computer algebra system MAGMA ([BC06], [CPB08] and [CP08]) to generate the matrix $A_x$ for conjugate $p$-elements $a$ and $x$ in $\text{Sym}(n)$ having full support (see Chapter 5 for full details). Here $p$ is a prime number.

```magma
function CreateMatrix(a,x)
    Deca:=CycleDecomposition(a);
    n:=#Deca-#Fix(a);
    p:=Order(a);
    Ax:=Matrix(RationalField(),n,n,
                [<i,j,#(Deca[i]^x meet Deca[j])>:i,j in [1..n]]);
    return Ax;
end function;

For a given matrix $A$, we can check if it is circulant using the code

```magma
function IsCirculant(A)
    n:=#Rows(A);
    I:=Matrix(RationalField(),n,n,[<i,i+1,1>:i in [1..n-1]])
        +Matrix(RationalField(),n,n,[<n,1,1>]);
    B:=ZeroMatrix(RationalField(),n,n);
    for i in {1..n} do
        B:=B+A[1,i]*I^(i-1);
    end for;
    if A eq B then return true; else return false; end if;
end function;
```

For a given $p$-elements $a, x \in \text{Sym}(n)$, we may check if $\langle a, x \rangle \cong C_p \wr C_p$ using the code

```magma
function IsIsomorphicWreath(a,x)
    p:=Order(a);
    if Order(a) ne Order(x) then return false; else
        Cp:=CyclicGroup(p);
    end if;
end function;
```
Wp:=WreathProduct(Cp,Cp);
if IsIsomorphic(sub<Parent(a)|a,x>,Wp) then return true;
else return false; end if;
end if;
end function;

We may also create the circulant matrix \( \text{circ}(y_0, \ldots, y_{p-1}) \) from the vector \( y = (y_0, \ldots, y_{p-1}) \) over the finite field \( GF(p) \)

function CreateCirculant(y)
p:=#y;
P:=Matrix(GF(p),p,p, [<i,i+1,1>: i in [1..p-1]])
    +Matrix(GF(p),p,p, [<p,1,1>]);
C:=y[1]*P^p;
for i in {2..p} do
    C:=C+y[i]*P^(i-1);
end for;
return C;
end function;

or over the real numbers.

function CreateRealCirculant(y)
p:=#y;
P:=Matrix(RealField(),p,p, [<i,i+1,1>: i in [1..p-1]])
    +Matrix(RealField(),p,p, [<p,1,1>]);
C:=y[1]*P^p;
for i in {2..p} do
    C:=C+y[i]*P^(i-1);
end for;
return C;
end function;
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<table>
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<tr>
<th>Notation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$(\cdot,\cdot)$</td>
<td>the inner product of complex characters.</td>
</tr>
<tr>
<td>$[2^n]$</td>
<td>a group of order $2^n$</td>
</tr>
<tr>
<td>$[p^n]$</td>
<td>a group of order $p^n$</td>
</tr>
<tr>
<td>$:= $</td>
<td>is defined to be equal to.</td>
</tr>
<tr>
<td>$</td>
<td>\det(B)</td>
</tr>
<tr>
<td>$d_r$</td>
<td>the $r$-th boundary map of the chain complex of a presheaf.</td>
</tr>
<tr>
<td>$</td>
<td>G</td>
</tr>
<tr>
<td>$[g,h]$</td>
<td>the commutator of $g$ with $h$.</td>
</tr>
<tr>
<td>$[G:H]$</td>
<td>the index of a subgroup $H$ in a group $G$.</td>
</tr>
<tr>
<td>$</td>
<td>S_p(G)</td>
</tr>
<tr>
<td>$0_{kG}$</td>
<td>the zero $kG$-module.</td>
</tr>
<tr>
<td>$1_Q$</td>
<td>the trivial character of the subgroup $Q$.</td>
</tr>
<tr>
<td>$2^{1+2r}, 2^{1+2r}_-$</td>
<td>the extra-special 2-groups of order $2^{1+2r}$.</td>
</tr>
<tr>
<td>$\varphi_{\sigma \tau}$</td>
<td>the connecting homomorphism from $F_{\sigma}$ to $F_{\tau}$ in the presheaf/coefficient system $F$.</td>
</tr>
<tr>
<td>$\chi_H^G, \chi_H^G, \chi^G, \chi^G$</td>
<td>the induction of the $H$-character $\chi$ to the overgroup $G$.</td>
</tr>
<tr>
<td>$\psi</td>
<td>_H, \psi_H$</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>a presheaf or coefficient system (of abelian groups).</td>
</tr>
<tr>
<td>$\mathcal{F}^\ast$</td>
<td>the dual of the presheaf $\mathcal{F}$.</td>
</tr>
<tr>
<td>$\mathcal{F}_{\sigma}$</td>
<td>the term of the coefficient system or presheaf $\mathcal{F}$ at $\sigma$.</td>
</tr>
<tr>
<td>$g^G$</td>
<td>the $G$-conjugacy class of $g$.</td>
</tr>
<tr>
<td>$g^h$</td>
<td>the conjugate of $g$ by $h$.</td>
</tr>
</tbody>
</table>
\( G_\sigma \) the stabilizer in \( G \) of \( \sigma \).

\( G \cong X \) the group \( G \) has the shape \( X \).

\( H \leq G, H < G, H \trianglelefteq G \) \( H \) is respectively a subgroup, proper subgroup or normal subgroup of \( G \).

\( H \circ K \) the central product of the subgroups \( H \) and \( K \).

\( K \wr H \) the wreath product of \( K \) by \( H \).

\( K \wr \Gamma H \) the wreath product of \( K \) by \( H \) with respect to \( \Gamma \).

\( M | S \) the restriction of \( M \) to \( S \).

\( N_1/N_2/\cdots/N_n \) the module having successive composition factors \( N_1, N_2, \ldots, N_n \).

\( N.H, NH \) an extension of \( H \) by \( N \).

\( N : H \) a split extension of \( H \) by \( N \).

\( N \rtimes H \) the semidirect product of \( N \) with \( H \).

\( N \leq M \) \( N \) is a submodule of \( M \).

\( \alpha_i \) the \( i \)-th \( p \)-cycle in the decomposition of the standard \( p \)-element of \( \text{Sym}(n) \).

\( \Gamma_F \) the residue geometry of the flag \( F \).

\( \delta_i \) the coboundary map from \( C^i(\Delta, \mathcal{F}) \to C^{i+1}(\Delta, \mathcal{F}) \).

\( \Delta(\mathcal{P}) \) the set of flags of subspaces of the vector space \( V \) corresponding to \( \mathcal{P} \).

\( \Delta_i(x) \) the disc of elements distance \( i \) from \( x \) in a given graph.

\( \sigma^G \) the \( G \)-orbit of \( \sigma \).

\( \Phi(G) \) the Frattini subgroup of the group \( G \).

\( \chi(\mathcal{F}) \) the Euler characteristic of the presheaf \( \mathcal{F} \).

\( \chi_i \) the \( i \)-th \( p \)-cycle in the decomposition of an arbitrary \( p \)-element of \( \text{Sym}(n) \).

\( A^a, A_x^a \) the matrices generalising \( x \)-graphs to \( p \)-elements of \( \text{Sym}(n) \).

\( Ab \) the category of abelian groups.

\( A_p(G) \) the poset of all non-trivial elementary abelian \( p \)-subgroups of the group \( G \).

\( B_{r-1}(\Delta, \mathcal{F}) \) the space of \( r \)-boundaries of the presheaf \( \mathcal{F} \) defined on \( \Delta \).

\( B_{r-1}(\mathcal{F}) \) the space of \( r \)-boundaries of the presheaf \( \mathcal{F} \).
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS(M)</td>
<td>the block sum matrix of the matrix M</td>
</tr>
<tr>
<td>C(G,X)</td>
<td>the commuting graph of the group G with vertex set X</td>
</tr>
<tr>
<td>C_G(g)</td>
<td>the centralizer in G of g.</td>
</tr>
<tr>
<td>char k</td>
<td>the characteristic of the field k.</td>
</tr>
<tr>
<td>C^i(Δ, F)</td>
<td>the i-th co-chain space of the presheaf F defined on Δ.</td>
</tr>
<tr>
<td>C^i(F)</td>
<td>the i-th co-chain space of the presheaf F</td>
</tr>
<tr>
<td>circ(c_0, c_1, ..., c_n-1)</td>
<td>the n x n circulant matrix having first row (c_0, c_1, ..., c_n-1)</td>
</tr>
<tr>
<td>C_r</td>
<td>the cyclic group of order r.</td>
</tr>
<tr>
<td>C_r(Δ, F)</td>
<td>the r-th chain space of the presheaf F defined on Δ.</td>
</tr>
<tr>
<td>C_r(F)</td>
<td>the r-th chain space of the presheaf F</td>
</tr>
<tr>
<td>d(·, ·)</td>
<td>the standard distance metric on a graph</td>
</tr>
<tr>
<td>det(B)</td>
<td>the determinant of the matrix B</td>
</tr>
<tr>
<td>Dih(2n)</td>
<td>the dihedral group of order 2n</td>
</tr>
<tr>
<td>F(G,X)</td>
<td>the local fusion graph of the group G having vertex set X</td>
</tr>
<tr>
<td>f_C(X)</td>
<td>the representer polynomial of the circulant matrix C</td>
</tr>
<tr>
<td>fix(g)</td>
<td>the fixed-point set of the element g ∈ Sym(n).</td>
</tr>
<tr>
<td>Fun(Γ, K)</td>
<td>the set of all functions from Γ to K</td>
</tr>
<tr>
<td>F_V</td>
<td>the fixed-point presheaf of V.</td>
</tr>
<tr>
<td>GF(p^n)</td>
<td>the Galois field of order p^n</td>
</tr>
<tr>
<td>g_M(X)</td>
<td>the reduced representer polynomial of the matrix M</td>
</tr>
<tr>
<td>G_x</td>
<td>the x-graph corresponding to x</td>
</tr>
<tr>
<td>G_y</td>
<td>the x-graph corresponding to x with respect to y</td>
</tr>
<tr>
<td>H^i(Δ, F)</td>
<td>the i-th cohomology group of the presheaf F defined on Δ.</td>
</tr>
<tr>
<td>H^i(F)</td>
<td>the i-th cohomology group of the presheaf F</td>
</tr>
<tr>
<td>H_r(Δ, F)</td>
<td>the r-th homology group of the presheaf F defined on Δ.</td>
</tr>
<tr>
<td>H_r(F)</td>
<td>the r-th homology group of the presheaf F</td>
</tr>
<tr>
<td>i(V,W)</td>
<td>the intertwining number of the modules V and W</td>
</tr>
</tbody>
</table>
$i_{kG}$, $i_G$ the unique irreducible $kG$-module of dimension $i$.

Ind $L^R_S$, $L^G_S$ the induction of the $S$-module $L$ to the overring $R$.

Irr($G$) the set of irreducible complex characters of $G$.

$K_V$ the constant presheaf of $V$.

$\mathcal{M}(G, B)$ the set of all minimal parabolic subgroups of $G$ with respect to $B$.

$m_{\sigma}$ an element of a presheaf morphism $m$.

$N_G(H)$ the normalizer in $G$ of the subgroup $H$.

$O_p(G)$ the $p$-core of the group $G$.

ord($g$) the order of the element $g$.

$\mathcal{P}(V)$ the projective geometry of $V$.

$p_{1+2r}$, $p_{-1+2r}$ the extra-special $p$-groups of order $p^{1+2r}$.

$p^a$ the elementary abelian $p$-group $C_p^a$.

$\mathcal{P}_{\pi}(G, X)$ the $\pi$-product graph associated to the group $G$ having vertex set $X$.

$p^{r_1+\cdots+r_n}$ a $p$-group having elementary abelian sections of ranks $r_1, \ldots, r_n$.

soc($V$) the socle of the module $V$.

$S_p(G)$ the poset of all non-trivial $p$-subgroups of the group $G$.

Stab($\sigma$) the stabilizer of $\sigma$.

supp($g$) the support of the element $g \in \text{Sym}(n)$.

supp($x_1, x_2, \ldots, x_r$) the union of the supports of the elements $x_1, \ldots, x_r \in \text{Sym}(n)$.

$Syl_p(G)$ the set of Sylow $p$-subgroups of $G$.

$T_{P_J/\mathcal{U}_J}(\chi)$ the truncation of the character $\chi$ with respect to the parabolic subgroup $P_J$.

$Y_{\sigma}$ the permutation matrix corresponding to $\sigma \in \text{Sym}(n)$.

$Z(G)$ the centre of the group $G$.

$Z_r(\Delta, \mathcal{F})$ the space of $r$-cycles of a presheaf $\mathcal{F}$ defined on $\Delta$.

$Z_r(\mathcal{F})$ the space of $r$-cycles of a presheaf $\mathcal{F}$.
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