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STRUCTURED MAPPING PROBLEMS FOR MATRICES ASSOCIATED WITH SCALAR PRODUCTS PART I: LIE AND JORDAN ALGEBRAS*

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Abstract. Given a class of structured matrices $\mathbb{S}$, we identify pairs of vectors x, b for which there exists a matrix $A \in \mathbb{S}$ such that $Ax = b$, and also characterize the set of all matrices $A \in \mathbb{S}$ mapping x to b. The structured classes we consider are the Lie and Jordan algebras associated with orthosymmetric scalar products. These include (skew-)symmetric, (skew-)Hamiltonian, pseudo (skew-)Hermitian, persymmetric and perskew-symmetric matrices. Structured mappings with extremal properties are also investigated. In particular, structured mappings of minimal rank are identified and shown to be unique when rank one is achieved. The structured mapping of minimal Frobenius norm is always unique and explicit formulas for it and its norm are obtained. Finally the set of all structured mappings of minimal 2-norm is characterized. Our results generalize and unify existing work, answer a number of open questions, and provide useful tools for structured backward error investigations.

Key words. Lie algebra, Jordan algebra, scalar product, bilinear form, sesquilinear form, orthosymmetric, adjoint, structured matrix, backward error, Hamiltonian, skew-Hamiltonian, Hermitian, complex symmetric, skew-symmetric, persymmetric, perskew-symmetric, minimal rank, minimal Frobenius norm, minimal 2-norm.

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1. Introduction. The problem of finding all matrices A that map a given nonzero vector $x \in \mathbb{K}^n$ to a given vector $b \in \mathbb{K}^m$, where $\mathbb{K}$ is a fixed field, can be solved using elementary means [10]. Trenkler [21] recently revisited this problem, giving a solution using generalized inverses:

$$(1.1) \qquad A = bx^\dagger + Z(I_n - xx^\dagger)$$

where I_n is the $n \times n$ identity matrix, $Z \in \mathbb{K}^{m \times n}$ is arbitrary and $x^\dagger$ is any generalized inverse of x. In this work we restrict the permissible transformations to a class of structured matrices $\mathbb{S} \subset \mathbb{K}^{n \times n}$ and consider the following *structured mapping problems:*

Existence: For which vectors x, b does there exist some $A \in \mathbb{S}$ such that $Ax = b$?

Characterization: Determine the set $\mathcal{S} = \{ A \in \mathbb{S} : Ax = b \}$ of all structured mappings taking x to b.

We present a complete, unified solution for these two problems when $\mathbb{S}$ is the Lie or Jordan algebra associated with an orthosymmetric scalar product. These $\mathbb{S}$ include, for example, symmetric and skew-symmetric, Hermitian, pseudo-Hermitian and skew-Hermitian, Hamiltonian, persymmetric and perskew-symmetric matrices. We will assume that $x \neq 0$ throughout, since both problems have trivial solutions if $x = 0$.

Answers to some particular instances of these structured mapping problems can be found in the literature. Liu and Leaker [9, Lem. 1] show that for $x, b \in \mathbb{R}^n$, x can be

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mapped to b by a real skew-symmetric matrix if and only if x and b are orthogonal. Khatri and Mitra [8] and later Sun [17] address the existence and characterization problems for the matrix equation $AX = B$, where X, B are matrices and the unknown A is Hermitian; the skew-Hermitian and complex symmetric cases are covered in [19]. Restricting the results of [8], [17] and [19] to the case when X and B are vectors yields one among the many representations of the set $\mathcal{S}$ identified in this paper. Structured mapping problems for double structures, for structures that do not arise in the context of a scalar product, and for some specific nonlinear structures have also been investigated (see [5], [6], [15], [18] and [22] for examples).

One of our motivations for studying these problems stems from the analysis of structured backward errors in the solutions to structured linear systems and structured eigenproblems [7], [19], [20]. Recall that a backward error of an approximate solution y to a linear system $Ax = b$ is a measure of the smallest perturbation E such that $(A + E)y = b$. When A is in some linearly structured class $\mathbb{S}$ one may want to require E to have the same structure; the structured backward error is then a measure of the smallest structured perturbation E such that $Ey = r := b - Ay$. Hence solving the structured mapping problem is the first step towards obtaining explicit expressions for structured backward errors.

For any linear matrix structure $\mathbb{S}$ it is possible to obtain a characterization of the structured mapping set $\mathcal{S}$ using the Kronecker product approach of [4], which we briefly outline here. The equation $Ax = b$ is first rewritten as $(x^T \otimes I_n) \operatorname{vec}(A) = b$, where $\otimes$ denotes the Kronecker product and vec is the operator that stacks the columns of a matrix into one long vector. The linear nature of the matrix structure is then encoded by $\operatorname{vec}(A) = \Pi_{\mathbb{S}} p$, where $\Pi_{\mathbb{S}}$ is an $n^2 \times m$ pattern matrix giving (in essence) a basis for the structured class $\mathbb{S}$, and p is an m-dimensional vector of parameters ($m = \dim \mathbb{S} \le n^2$). Hence

$$(1.2) \qquad \mathcal{S} = \{\, A \in \mathbb{K}^{n \times n} : \ (x^T \otimes I_n) \Pi_{\mathbb{S}} p = b, \ \operatorname{vec}(A) = \Pi_{\mathbb{S}} p \,\}.$$

Note that there may be no solution to the system $(x^T \otimes I_n)\Pi_{\mathbb{S}} p = b$ if $(x^T \otimes I_n)\Pi_{\mathbb{S}}$ is rank deficient or if the system is overdetermined ($n > m$). When they exist, solutions can be obtained from the singular value decomposition of $(x^T \otimes I_n)\Pi_{\mathbb{S}}$. In particular, if the system is underdetermined and consistent, and if the pattern matrix $\Pi_{\mathbb{S}}$ is chosen so that $\|p\|_2 = \|A\|_F$ for all $A \in \mathbb{S}$ (i.e., $\Pi_{\mathbb{S}}$ contains an orthonormal basis for $\mathbb{S}$ in the Frobenius inner product), then the solution $A \in \mathbb{S}$ with minimal Frobenius norm is given in terms of the pseudo-inverse by $p = \big((x^T \otimes I_n)\Pi_{\mathbb{S}}\big)^+ b$. As a result a computable expression for the structured backward error is obtained:

$$(1.3) \ \ \eta_F(y) = \min\{\, \|E\|_F : (A + E)y = b, \ E \in \mathbb{S} \,\} = \big\|\big((y^T \otimes I_n)\Pi_{\mathbb{S}}\big)^+ (b - Ay)\big\|_2.$$

There are several disadvantages associated with this Kronecker product approach. The existence of structured solutions to $Ax = b$ may not be easy to check. In addition, the set $\mathcal{S}$ of all structured mappings is given only implicitly by (1.2). Also, amongst all solutions in $\mathcal{S}$, it is difficult to distinguish ones with special properties, other than the one of minimal Frobenius norm. The structured backward error expression in (1.3) is expensive to evaluate and difficult to compare with its unstructured counterpart $\|b - Ay\|_2$.

By contrast, the approach presented in this paper gives easy-to-check conditions for the existence problem and an *explicit* solution for the characterization problem when $\mathbb{S}$ is the Lie or Jordan algebra of a scalar product. The set $\mathcal{S}$ is rewritten as

$$(1.4) \qquad\qquad \mathcal{S} = B + \{\, A \in \mathbb{S} : \ Ax = 0 \,\},$$

where B is any particular solution of the nonhomogeneous mapping problem. We provide a set of possible particular solutions B for a given class $\mathbb{S}$ and given vectors x and b, thus giving multiple ways of representing $\mathcal{S}$. This enables one to more easily identify structured mappings with minimal rank or minimal Frobenius norm and to readily derive bounds for the ratio between the structured and unstructured backward errors. A multiplicative representation, by contrast with the additive representation in (1.4), is used to characterize the set of all minimal 2-norm structured mappings in $\mathcal{S}$. From this characterization, minimal 2-norm mappings of minimal rank and minimal 2-norm mappings of minimal Frobenius norm can be identified.

To give an idea of the scope of the paper, we give here an illustration of what is obtained by applying our general results to a particular structure $\mathbb{S}$, in this case the Lie algebra of complex skew-symmetric matrices. For given $x, b \in \mathbb{C}^n$ our results imply that

$$\mathcal{S} := \{\, A \in \mathbb{C}^{n \times n} : \ Ax = b, \ A^T = -A \,\} \ \text{ is nonempty} \quad \Longleftrightarrow \quad x^T b = 0\,,$$

and that

$$(1.5) \quad \mathcal{S} \ = \ \{\, bw^T - wb^T + (I - vx^T)L(I - xv^T) : \ L \in \mathbb{C}^{n \times n}, \ L^T = -L \,\}\,,$$

where $w, v \in \mathbb{C}^n$ are any fixed but arbitrary vectors chosen such that $w^T x = v^T x = 1$. All mappings in $\mathcal{S}$ of the form $bw^T - wb^T$ (corresponding to setting $L = 0$ in (1.5)) have minimal rank two, and the choice $w = \bar{x}/\|x\|_2^2$, $L = 0$ gives the unique mapping A_{opt} of minimal Frobenius norm:

$$(1.6) \quad A_{\mathrm{opt}} = (b\bar{x}^T - \bar{x}b^T)/\|x\|_2^2\,, \quad \|A_{\mathrm{opt}}\|_F = \min_{A \in \mathcal{S}} \|A\|_F = \sqrt{2}\|b\|_2/\|x\|_2\,.$$

The set $\mathcal{M} := \{\, A \in \mathcal{S} : \ \|A\|_2 = \min_{B \in \mathcal{S}} \|B\|_2 \,\}$ of all minimal 2-norm mappings can be characterized by

$$\mathcal{M} = \frac{\|b\|_2}{\|x\|_2} \left\{ U^T \mathrm{diag}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, S \right) U : \ S \in \mathbb{C}^{(n-2) \times (n-2)}, \ S^T = -S, \ \|S\|_2 \leq 1 \right\},$$

where $U^*[\,x \ \bar{b}\,] = [\,\|x\|_2 e_1 \ \|b\|_2 e_2\,]$, i.e., U is the unitary factor of the QR factorization of $[\,x \ \bar{b}\,]$ with R forced to have positive entries. For this structure $\mathbb{S}$ it turns out that $A_{\mathrm{opt}} \in \mathcal{M}$, so A_{opt} is simultaneously a mapping of minimal rank, minimal Frobenius norm and minimal 2-norm. As a consequence of (1.6) an explicit formula for the structured backward error in (1.3) for this class $\mathbb{S}$ is given for the Frobenius norm by

$$\eta_F(y) = \sqrt{2}\, \frac{\|Ay - b\|_2}{\|y\|_2}\,,$$

which is immediately seen to differ from its unstructured counterpart by a factor of only $\sqrt{2}$. For the 2-norm the structured and unstructured backward errors are equal.

In summary, the results here generalize and unify existing work, answer a number of open questions, and provide useful tools for the investigation of structured backward errors. After some preliminaries in section 2, a complete solution to the existence and characterization problems is presented in sections 3 and 4. In section 5 we identify structured mappings of minimal rank, minimal Frobenius norm and minimal 2-norm, and investigate their uniqueness. Some technical proofs are delegated to an appendix.

2. Preliminaries.

2.1. Scalar products. A *bilinear form* on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) is a map $(x, y) \mapsto \langle x, y \rangle$ from $\mathbb{K}^n \times \mathbb{K}^n$ to $\mathbb{K}$, which is linear in each argument. If $\mathbb{K} = \mathbb{C}$, the map $(x, y) \mapsto \langle x, y \rangle$ is a *sesquilinear form* if it is conjugate linear in the first argument and linear in the second. To a bilinear form on $\mathbb{K}^n$ is associated a unique $M \in \mathbb{K}^{n \times n}$ such that $\langle x, y \rangle = x^T M y$ for all $x, y \in \mathbb{K}^n$; if the form is sesquilinear, $\langle x, y \rangle = x^* M y$ for all $x, y \in \mathbb{C}^n$, where the superscript $*$ denotes the conjugate transpose. The form is said to be *nondegenerate* when M is nonsingular.

A bilinear form is symmetric if $\langle x, y \rangle = \langle y, x \rangle$ or, equivalently, if $M^T = M$, and skew-symmetric if $\langle x, y \rangle = -\langle y, x \rangle$ or, equivalently, if $M^T = -M$. A sesquilinear form is Hermitian if $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and skew-Hermitian if $\langle x, y \rangle = -\overline{\langle y, x \rangle}$. The matrices associated with such forms are Hermitian and skew-Hermitian, respectively.

We will use the term *scalar product* to mean a nondegenerate bilinear or sesquilinear form on $\mathbb{K}^n$. When we have more than one scalar product under consideration, we will denote $\langle x, y \rangle$ by $\langle x, y \rangle_{\mathrm{M}}$, using the matrix M defining the form as a subscript to distinguish the forms under discussion.

2.2. Adjoints. The *adjoint* of A with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$, denoted by $A^\star$, is uniquely defined by the property $\langle Ax, y \rangle_{\mathrm{M}} = \langle x, A^\star y \rangle_{\mathrm{M}}$ for all $x, y \in \mathbb{K}^n$. It can be shown that the adjoint is given explicitly by

$$A^\star = \begin{cases} M^{-1} A^T M & \text{for bilinear forms,} \\ M^{-1} A^* M & \text{for sesquilinear forms.} \end{cases}$$

The following properties of adjoint, all analogous to properties of transpose (or conjugate transpose) follow easily, and hold for all scalar products.

LEMMA 2.1. $(A + B)^\star = A^\star + B^\star$, $(AB)^\star = B^\star A^\star$, $(A^{-1})^\star = (A^\star)^{-1}$ *and*

$$(\alpha A)^\star = \begin{cases} \alpha A^\star & \text{for bilinear forms,} \\ \overline{\alpha} A^\star & \text{for sesquilinear forms.} \end{cases}$$

The involutory property $(A^\star)^\star = A$ does not hold for all scalar products; this issue is discussed in section 2.4.

2.3. Lie and Jordan algebras. Associated with $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is a Lie algebra $\mathbb{L}$ and a Jordan algebra $\mathbb{J}$, defined by

$$\mathbb{L} := \left\{ A \in \mathbb{K}^{n \times n} : \langle Ax, y \rangle_{\mathrm{M}} = -\langle x, Ay \rangle_{\mathrm{M}} \ \forall x, y \in \mathbb{K}^n \right\} = \left\{ A \in \mathbb{K}^{n \times n} : A^\star = -A \right\},$$
$$\mathbb{J} := \left\{ A \in \mathbb{K}^{n \times n} : \langle Ax, y \rangle_{\mathrm{M}} = \langle x, Ay \rangle_{\mathrm{M}} \ \forall x, y \in \mathbb{K}^n \right\} = \left\{ A \in \mathbb{K}^{n \times n} : A^\star = A \right\}.$$

All the structured matrices considered in this paper belong to one of these two classes. Note that $\mathbb{L}$ and $\mathbb{J}$ are linear subspaces of $\mathbb{K}^{n \times n}$. Table 2.1 shows a sample of well-known structured matrices in some $\mathbb{L}$ or $\mathbb{J}$ associated with a scalar product.

2.4. Orthosymmetric and unitary scalar products. Scalar products for which vector orthogonality is a symmetric relation, i.e.,

$$\langle x, y \rangle_{\mathrm{M}} = 0 \iff \langle y, x \rangle_{\mathrm{M}} = 0, \ \forall x, y \in \mathbb{K}^n,$$

will be referred to as *orthosymmetric scalar products* [12], [13]. One can show that $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is orthosymmetric if and only if it satisfies any one (and hence all) of the following equivalent properties:

TABLE 2.1
Structured matrices associated with some orthosymmetric scalar products.

$$R = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \text{ with } p + q = n.$$

Space	M	Adjoint $A^\star$	Jordan Algebra $\mathbb{J} = \{A : A^\star = A\}$	Lie Algebra $\mathbb{L} = \{A : A^\star = -A\}$
Symmetric bilinear forms				
$\mathbb{R}^n$	I	A^T	Symmetrics	Skew-symmetrics
$\mathbb{C}^n$	I	A^T	Complex symmetrics	Complex skew-symmetrics
$\mathbb{R}^n$	$\Sigma_{p,q}$	$\Sigma_{p,q} A^T \Sigma_{p,q}$	Pseudosymmetrics	Pseudoskew-symmetrics
$\mathbb{C}^n$	$\Sigma_{p,q}$	$\Sigma_{p,q} A^T \Sigma_{p,q}$	Complex pseudosymm.	Complex pseudoskew-symm.
$\mathbb{R}^n$	R	$R A^T R$	Persymmetrics	Perskew-symmetrics
Skew-symmetric bilinear forms				
$\mathbb{R}^{2n}$	J	$-J A^T J$	Skew-Hamiltonians	Hamiltonians
$\mathbb{C}^{2n}$	J	$-J A^T J$	Complex J-skew-symm.	Complex J-symmetrics
Hermitian sesquilinear forms				
$\mathbb{C}^n$	I	A^*	Hermitian	Skew-Hermitian
$\mathbb{C}^n$	$\Sigma_{p,q}$	$\Sigma_{p,q} A^* \Sigma_{p,q}$	Pseudo-Hermitian	Pseudoskew-Hermitian
Skew-Hermitian sesquilinear forms				
$\mathbb{C}^{2n}$	J	$-J A^* J$	J-skew-Hermitian	J-Hermitian

1. The adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is involutory, i.e., $(A^\star)^\star = A, \forall A \in \mathbb{K}^{n \times n}$.
2. $M = \alpha M^T$ with $\alpha = \pm 1$ for bilinear forms; $M = \alpha M^*$ with $\alpha \in \mathbb{C}, |\alpha| = 1$ for sesquilinear forms.
3. $\mathbb{K}^{n \times n} = \mathbb{L} \oplus \mathbb{J}$.

See [12, Thm. A.4] or [13, Thm. 1.6] for a proof of this equivalence along with a list of additional equivalent properties. The second property says that orthosymmetric bilinear forms are always either symmetric or skew-symmetric. On the other hand, an orthosymmetric sesquilinear form $\langle x, y \rangle_{\mathrm{M}} = x^* M y$, where $M = \alpha M^*, |\alpha| = 1, \alpha \in \mathbb{C}$ is always closely tied to a Hermitian form: defining the Hermitian matrix $H = \bar{\alpha}^{1/2} M$ gives $\langle x, y \rangle_{\mathrm{H}} = \bar{\alpha}^{1/2} \langle x, y \rangle_{\mathrm{M}}$, for all $x, y \in \mathbb{C}^n$. Consequently, the Jordan algebra of $\langle \cdot, \cdot \rangle_{\mathrm{H}}$ is identical to the Jordan algebra of $\langle \cdot, \cdot \rangle_{\mathrm{M}}$:

$$\langle Ax, y \rangle_{\mathrm{H}} = \langle x, Ay \rangle_{\mathrm{H}} \iff \bar{\alpha}^{1/2} \langle Ax, y \rangle_{\mathrm{M}} = \bar{\alpha}^{1/2} \langle x, Ay \rangle_{\mathrm{M}} \iff \langle Ax, y \rangle_{\mathrm{M}} = \langle x, Ay \rangle_{\mathrm{M}}.$$

Similarly, the Lie algebras of $\langle \cdot, \cdot \rangle_{\mathrm{H}}$ and $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ are also identical. Thus a result established for Hermitian sesquilinear forms immediately translates into a corresponding result for orthosymmetric sesquilinear forms. Up to a scalar multiple, then, there are really only three distinct types of orthosymmetric scalar products: symmetric and skew-symmetric bilinear, and Hermitian sesquilinear. We will, however, continue to include separately stated results (without separate proofs) for skew-Hermitian forms for convenience, as this is a commonly occurring special case.

The results in this paper hold only for orthosymmetric scalar products, which as just mentioned, are those for which the useful and simplifying property $(A^\star)^\star = A$

holds for all matrices [12], [13]. For these scalar products, adjoints of rank-one matrices will often be needed,

$$(2.1) \qquad \begin{aligned} (yz^T M)^\star &= zy^T M^T & \text{for orthosymmetric bilinear forms,} \\ (yz^* M)^\star &= zy^* M^* & \text{for orthosymmetric sesquilinear forms.} \end{aligned}$$

Some of our results will require the extra property that the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is also *unitary*, that is, βM is unitary for some $\beta > 0$ [12]. One can show that in unitary scalar products, "the stars commute", i.e, $(A^*)^\star = (A^\star)^*$ for all $A \in \mathbb{K}^{n \times n}$ and that for every unitarily invariant norm $\| \cdot \|$, $\|A^\star\| = \|A\|$ for all $A \in \mathbb{K}^{n \times n}$ [13, Thm. 1.8]. Finally, note that important classes of structured matrices arise in the context of scalar products that are both orthosymmetric and unitary, as witnessed by the entries in Table 2.1 (for all of which $\alpha = \pm 1$ and $\beta = 1$). The results in this paper are not confined to just the examples in the table, however.

2.5. Projections. Projections that map x to the zero vector form a key part in our solution to the structured mapping problems.

Since the matrix M of the scalar product is nonsingular, given a nonzero $x \in \mathbb{K}^n$ one can always construct many $w \in \mathbb{K}^n$ such that $\langle w, x \rangle_{\mathrm{M}} = 1$. For example, when x is nonisotropic (i.e $\langle x, x \rangle_{\mathrm{M}} \neq 0$), $w = x/\langle x, x \rangle_{\mathrm{M}}$ will work for bilinear forms, and $w = x/\overline{\langle x, x \rangle}_{\mathrm{M}}$ can be used for sesquilinear forms. If x is isotropic (i.e $\langle x, x \rangle_{\mathrm{M}} = 0$), choose k so that $x_k \neq 0$; then $w = M^{-T} e_k / x_k$ will have the desired property for bilinear forms, and $w = M^{-*} e_k / \overline{x}_k$ will work for sesquilinear forms.

With w chosen so that $\langle w, x \rangle_{\mathrm{M}} = 1$, it is easy to show that for bilinear forms, $xw^T M$ is idempotent and hence a projection with range span$\{x\}$. Replacing ^{T} by * gives a similar result for sesquilinear forms. The complementary projections P_w defined by

$$(2.2) \qquad P_w := \begin{cases} I - xw^T M, & \langle w, x \rangle_{\mathrm{M}} = 1 & \text{for bilinear forms,} \\ I - xw^* M, & \langle w, x \rangle_{\mathrm{M}} = 1 & \text{for sesquilinear forms,} \end{cases}$$

have kernel span$\{x\}$, and in particular map x to the zero vector.

3. The existence problem. Throughout the rest of the paper we assume that $x, b \in \mathbb{K}^n$ with $x \neq 0$, but that any b is allowed unless otherwise stated. For a scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ we will denote by $\mathbb{S}$ the corresponding Jordan algebra $\mathbb{J}$ or Lie algebra $\mathbb{L}$ and will write

$$(3.1) \quad \mathcal{S} := \{ A \in \mathbb{S} : Ax = b \}, \ \mathcal{J} := \{ A \in \mathbb{J} : Ax = b \}, \ \mathcal{L} = \{ A \in \mathbb{L} : Ax = b \},$$

for the associated structured mapping sets. Note that $\mathcal{S} = \mathcal{J}$ when $\mathbb{S} = \mathbb{J}$ and $\mathcal{S} = \mathcal{L}$ when $\mathbb{S} = \mathbb{L}$.

As a preliminary step towards solving the existence problem, we show that the projections given in (2.2) can be used to construct maps that send x to b.

LEMMA 3.1. *Let* $x \neq 0$*, and let* $w \in \mathbb{K}^n$ *be chosen so that* $\langle w, x \rangle_{\mathrm{M}} = 1$ *with* $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ *orthosymmetric. Then* $^\pm B_w x = b$*, where* $^\pm B_w$ *is defined by*

$$(3.2) \qquad {}^\pm B_w := \begin{cases} bw^T M \pm (bw^T M)^\star P_w & \text{for bilinear forms,} \\ bw^* M \pm (bw^* M)^\star P_w & \text{for sesquilinear forms.} \end{cases}$$

Note that ^+B_w *and* ^-B_w *have rank at most two.*

Proof. Since $P_w x = 0$ and $\langle w, x \rangle_{\mathrm{M}} = 1$, we immediately conclude $^\pm B_w x = b$. Next, by (2.1) we see that in the bilinear case, $(bw^T M)^\star P_w = wb^T M^T P_w$ which is a

rank-one matrix, and hence ^+B_w, ^-B_w are the sum of two matrices of rank one. The proof in the sesquilinear case is similar. □

Thus, since ^+B_w, ^-B_w are always solutions to the unstructured mapping problem, they should be consistent with (1.1), which captures all solutions. Now $\langle w, x \rangle_{\mathrm{M}} = 1$ implies $w^T M x = 1$ in the bilinear case. Since any row vector u^T with the property $u^T x = 1$ is a generalized inverse $x^\dagger$ for the map $x : \mathbb{R} \to \mathbb{R}^n$, we can take $x^\dagger$ to be $w^T M$. Rewriting (3.2) for the bilinear case we get

$$(3.3) \qquad {}^{\pm}B_w = bx^\dagger \pm w b^T M^T (I - x x^\dagger)$$

which is of the form given by Trenkler in (1.1) with $Z = \pm w b^T M^T$. The argument for the sesquilinear case is similar, with the role of $x^\dagger$ being played by $w^* M$. It is worth observing that once the parameter w^T is chosen, both $x^\dagger$ and Z in (3.3) are determined, and thus we are confining our attention to a constrained subset of the maps given by (1.1).

We have still to determine when a structured mapping exists, and the role ^+B_w, ^-B_w play in such a mapping. The next theorem characterizes pairs of vectors x, b for which there exists $A \in \mathbb{L}$ or $A \in \mathbb{J}$ such that $Ax = b$. When a structured mapping exists, we show that either ^-B_w or ^+B_w will be in the Lie or Jordan algebra, thus yielding a constructive proof of existence.

THEOREM 3.2 (Existence for $\mathbb{L}$, $\mathbb{J}$). *Let* $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ *be an orthosymmetric scalar product. Then for any given pair of vectors* $x, b \in \mathbb{K}^n$ *with* $x \neq 0$ *and associated structured mapping sets* $\mathcal{J}$ *and* $\mathcal{L}$ *in* (3.1),

$$(3.4) \qquad \mathcal{J} \neq \emptyset \iff \langle b, x \rangle_{\mathrm{M}} = \langle x, b \rangle_{\mathrm{M}},$$
$$(3.5) \qquad \mathcal{L} \neq \emptyset \iff \langle b, x \rangle_{\mathrm{M}} = -\langle x, b \rangle_{\mathrm{M}}.$$

In particular, when $\langle b, x \rangle_{\mathrm{M}} = \langle x, b \rangle_{\mathrm{M}}$ *then* $^+B_w \in \mathcal{J}$ *and when* $\langle b, x \rangle_{\mathrm{M}} = -\langle x, b \rangle_{\mathrm{M}}$, $^-B_w \in \mathcal{L}$.

Proof. ($\Rightarrow$) Since $Ax = b$, in all cases we have

$$A \in \mathbb{J} \Rightarrow \langle b, x \rangle_{\mathrm{M}} = \langle Ax, x \rangle_{\mathrm{M}} = \langle x, Ax \rangle_{\mathrm{M}} = \langle x, b \rangle_{\mathrm{M}}.$$
$$A \in \mathbb{L} \Rightarrow \langle b, x \rangle_{\mathrm{M}} = \langle Ax, x \rangle_{\mathrm{M}} = \langle x, -Ax \rangle_{\mathrm{M}} = -\langle x, b \rangle_{\mathrm{M}}.$$

($\Leftarrow$) By Lemma 3.1 we know that ^+B_w, ^-B_w as defined in (3.2) map x to b. It suffices to prove that when $\langle b, x \rangle_M = \langle x, b \rangle_M$, $^+B_w \in \mathbb{J}$ and when $\langle b, x \rangle_M = -\langle x, b \rangle_M$, $^-B_w \in \mathbb{L}$. Using Lemma 2.1 and the expressions for the adjoints given in (2.1), we have for bilinear forms,

$$\begin{aligned} {}^{\pm}B_w &= b w^T M \pm (b w^T M)^\star (I - x w^T M) \\ &= b w^T M \pm (b w^T M)^\star \mp w(b^T M^T x) w^T M \\ &= b w^T M \pm (b w^T M)^\star \mp \langle x, b \rangle_{\mathrm{M}} w w^T M, \end{aligned}$$

and on the other hand,

$$\begin{aligned} {}^{\pm}B_w^\star &= (b w^T M)^\star \pm (I - w x^T M^T) b w^T M \\ &= (b w^T M)^\star \pm b w^T M \mp w(x^T M^T b) w^T M \\ &= (b w^T M)^\star \pm b w^T M \mp \langle b, x \rangle_{\mathrm{M}} w w^T M. \end{aligned}$$

If $\langle b, x \rangle_M = \langle x, b \rangle_M$ then clearly $^+B_w = {}^+B_w^\star$ so that $^+B_w \in \mathbb{J}$. If $\langle b, x \rangle_M = -\langle x, b \rangle_M$ then $^-B_w^\star = (b w^T M)^\star - b w^T M - \langle x, b \rangle_M w w^T M = -{}^-B_w$ so that $^-B_w \in \mathbb{L}$. The proof in the sesquilinear case is similar. □

The existence conditions in (3.4)–(3.5) are made more explicit in Corollary 3.3 for the main types of orthosymmetric scalar products. Observe that sometimes a condition on $\langle b, x \rangle_{\text{M}}$ is needed, while in other cases a structured mapping exists with no restrictions at all on x and b.

COROLLARY 3.3. *Let* $\langle \cdot, \cdot \rangle_{\text{M}}$ *be any orthosymmetric scalar product for which*

$$M = \begin{cases} \pm M^T & \text{for bilinear forms,} \\ \pm M^* & \text{for sesquilinear forms.} \end{cases}$$

Then for any given pair of vectors x, $b \in \mathbb{K}^n$ *with* $x \neq 0$, *let* $\mathcal{S}$ *denote either* $\mathcal{J}$ *or* $\mathcal{L}$ *as in* (3.1). *Then* $\mathcal{S} \neq \emptyset$ *if and only if the conditions given in the following table hold:*

Scalar Product	$\mathcal{J} \neq \emptyset$	$\mathcal{L} \neq \emptyset$
symmetric bilinear	*always*	$\langle b, x \rangle_{\text{M}} = 0$
skew-symmetric bilinear	$\langle b, x \rangle_{\text{M}} = 0$	*always*
Hermitian sesquilinear	$\langle b, x \rangle_{\text{M}} \in \mathbb{R}$	$\langle b, x \rangle_{\text{M}} \in i\mathbb{R}$
skew-Hermitian sesquilinear	$\langle b, x \rangle_{\text{M}} \in i\mathbb{R}$	$\langle b, x \rangle_{\text{M}} \in \mathbb{R}$

Proof. The conditions in the table follow from Theorem 3.2 and the definition of symmetric and skew-symmetric bilinear forms, and Hermitian and skew-Hermitian sesquilinear forms. □

Theorem 3.2 and Corollary 3.3 unify and generalize existence results in [9] for real skew-symmetric matrices, in [8] and [17] for symmetric and Hermitian matrices, in [19] for complex symmetric and skew-Hermitian structures, and in [16, Lem. 5.1] for real persymmetric matrices, which are particular instances of Lie and Jordan algebras associated with different bilinear and sesquilinear forms on $\mathbb{R}^n$ and $\mathbb{C}^n$ (see Table 2.1).

4. The characterization problem. We turn now to the task of determining the set of *all* matrices that map x to b and belong to a Lie or Jordan algebra.

LEMMA 4.1. *Let* $\mathbb{S}$ *denote the Lie or Jordan algebra of any orthosymmetric scalar product. Then*

(a) $A \in \mathbb{S} \Rightarrow Q^\star A Q \in \mathbb{S}$ *for all* Q; *that is,* $\star$-*congruence preserves* $\mathbb{L}$ *and* $\mathbb{J}$ *structures.*

(b) $\{P_w^\star S P_w : S \in \mathbb{S}\} \subseteq \{A \in \mathbb{S} : Ax = 0\}$, *where* P_w *is any particular one of the projection matrices defined in* (2.2).

(c) *For any* $w \in \mathbb{K}^n$ *such that* $\langle w, x \rangle_{\text{M}} = 1$, $A \in \mathbb{S}$, $Ax = 0 \Longrightarrow A = P_w^\star A P_w$.

Proof. (a) This is a direct consequence of adjoint being involutory in orthosymmetric scalar products.

(b) Follows immediately from the fact that $P_w x = 0$, together with (a).

(c) For any bilinear form, $A \in \mathbb{S} \Longrightarrow A = \pm A^\star = \pm M^{-1} A^T M \Longrightarrow MA = \pm A^T M \Longrightarrow x^T MA = \pm x^T A^T M = \pm (Ax)^T M$. But $Ax = 0$. Hence $x^T MA = 0$. From (2.2), we have $P_w = I - x w^T M$. Hence $A P_w = A - A x w^T M = A$, since $Ax = 0$. Using (2.1) and $x^T MA = 0$ we now obtain

$$P_w^\star A P_w = P_w^\star A = (I - w x^T M^T) A = A,$$

since for orthosymmetric bilinear forms $M^T = \pm M$. The proof for sesquilinear forms follows along the same lines. □

The complete solution to the *homogeneous mapping problem* can now be described.

THEOREM 4.2 (Characterization for $\mathbb{J}$ and $\mathbb{L}$: homogeneous case). *Let $\mathbb{S}$ denote the Lie or Jordan algebra of any orthosymmetric scalar product space. Given $x \in \mathbb{K}^n$ with $x \neq 0$, and $w \in \mathbb{K}^n$ such that $\langle w, x \rangle_{\mathrm{M}} = 1$,*

$$\{A \in \mathbb{S} : Ax = 0\} = \{P_w^{\star} S P_w : S \in \mathbb{S}\}$$

where P_w is defined in (2.2).

Proof. Follows immediately by combining (b) and (c) of Lemma 4.1. $\square$

COROLLARY 4.3. *If $v, w \in \mathbb{K}^n$, with $\langle v, x \rangle_{\mathrm{M}} = \langle w, x \rangle_{\mathrm{M}} = 1$, then*

$$\{P_v^{\star} S P_v : S \in \mathbb{S}\} = \{P_w^{\star} S P_w : S \in \mathbb{S}\}.$$

Thus we have several representations of the set of solutions to the homogeneous mapping problem. Now if $A, B \in \mathbb{S}$ are such that $Ax = Bx = b$, then $(A - B)x = 0$. By Theorem 4.2, $A - B = P_w^{\star} S P_w$, or equivalently, $A = B + P_w^{\star} S P_w$ for some $S \in \mathbb{S}$. Hence,

$$(4.1) \qquad \{A \in \mathbb{S} : Ax = b\} = B + \{A \in \mathbb{S} : Ax = 0\},$$

where B is any particular solution of the non-homogeneous mapping problem. By combining Theorems 3.2 and 4.2 together with (4.1) we now have the complete solution of the characterization part of the mapping problem for $\mathbb{J}$ and $\mathbb{L}$.

THEOREM 4.4 (Characterization for $\mathbb{J}$ and $\mathbb{L}$: non-homogeneous case). *Let $\mathbb{J}$ and $\mathbb{L}$ be the Jordan and Lie algebras of any orthosymmetric scalar product on $\mathbb{K}^n$. Let $x, b \in \mathbb{K}^n$ with $x \neq 0$ and let $\mathcal{J}$ and $\mathcal{L}$ be the structured mapping sets as in (3.1). Choose any $v, w \in \mathbb{K}^n$ such that $\langle v, x \rangle_{\mathrm{M}} = \langle w, x \rangle_{\mathrm{M}} = 1$, and use v and w to define P_v, $^{\pm}B_w$ as in (2.2) and (3.2) respectively. Consider the following sets:*

$$\mathcal{J}_+ = \{^+B_w + P_v^{\star} S P_v : S \in \mathbb{J}\}, \qquad \mathcal{L}_- = \{^-B_w + P_v^{\star} L P_v : L \in \mathbb{L}\}.$$

Then

$$\mathcal{J} = \begin{cases} \mathcal{J}_+ & \text{if } \langle x, b \rangle_{\mathrm{M}} = \langle b, x \rangle_{\mathrm{M}}, \\ \emptyset & \text{otherwise}, \end{cases} \qquad \mathcal{L} = \begin{cases} \mathcal{L}_- & \text{if } \langle x, b \rangle_{\mathrm{M}} = -\langle b, x \rangle_{\mathrm{M}}, \\ \emptyset & \text{otherwise}. \end{cases}$$

A more general problem for Hermitian, and real symmetric matrices in particular, was considered by Sun [17, Lem. 1.4]. For given matrices $X, B \in \mathbb{K}^{n \times \ell}$, Sun gave a characterization of the set

$$\mathcal{H} = \{A \in \mathbb{K}^{n \times n} : A^* = A \text{ and } AX = B\}$$

in terms of the pseudo-inverse X^+ of X, and the complementary orthogonal projections $\Pi_X = XX^+$, and $\Pi_{X^{\perp}} = I - \Pi_X$. He proved that $\mathcal{H} \neq \emptyset$ if and only if two conditions are satisfied: $B\Pi_{X^*} = B$ and $\Pi_X BX^+$ is Hermitian. In this case $\mathcal{H}$ can be expressed as

$$(4.2) \qquad \mathcal{H} = \{BX^+ + (BX^+)^* \Pi_{X^{\perp}} + \Pi_{X^{\perp}} S \Pi_{X^{\perp}} : \ S^* = S, \ S \in \mathbb{K}^{n \times n}\}.$$

When $\ell = 1$, writing X, B as x, b respectively, we get $\Pi_x = xx^*/(x^*x)$, and $x^+ = x^*/(x^*x)$. Since $\Pi_{x^*} = 1$, the conditions for $\mathcal{H}$ to be nonempty reduce to requiring that

$\Pi_x b x^+$ be Hermitian. A simple calculation shows that this happens if and only if $x^* b$ is real, which is in agreement with the condition in Corollary 3.3. Sun's characterization of $\mathcal{H}$ becomes

$$\mathcal{H} = \left\{ \frac{bx^*}{x^*x} + \frac{x^*b}{x^*x} \Pi_x + \Pi_x S \Pi_x, \ S^* = S \right\},$$

which corresponds to $\mathcal{J}_+$ in Theorem 4.4 with $M = I$ and the special choice $v = w = x/(x^*x)$. This choice of w corresponds to using an orthogonal projection in the representation for $\mathcal{J}_+$, since P_v is now $I - xx^*/(x^*x)$. Thus Sun's characterization is one among many given by Theorem 4.4.

A similar analysis of the real symmetric case shows that the results of Corollary 3.3 and Theorem 4.4 are compatible with Sun's solution for the case $\ell = 1$, and due to the freedom in the choice of v and w, gives a more flexible description of the set of real symmetric matrices mapping x to b.

5. Structured mappings with extremal properties. Let $\mathcal{J}$ and $\mathcal{L}$ be the sets of all structured solutions to the mapping problem as in (3.1). We now show how to find matrices in $\mathcal{J}$ or $\mathcal{L}$ with the extremal properties of minimal rank, minimal Frobenius norm, or minimal 2-norm, and investigate their uniqueness.

5.1. Structured mappings of minimal rank. In what follows, we assume $b \neq 0$.

THEOREM 5.1 (Rank-one structured mappings). *Let $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ be an orthosymmetric scalar product, and let $\mathcal{S}$ denote either $\mathcal{J}$ or $\mathcal{L}$ as in (3.1). Assume $b \neq 0$. A necessary condition for the existence of a rank-one matrix in $\mathcal{S}$ is $\langle b, x \rangle_{\mathrm{M}} \neq 0$. Whenever this rank-one matrix exists, it is unique and given by*

$$A = \begin{cases} bb^T M / \langle b, x \rangle_{\mathrm{M}} & \text{for bilinear forms,} \\ bb^* M / \langle b, x \rangle_{\mathrm{M}} & \text{for sesquilinear forms.} \end{cases}$$

Proof. Consider any rank-one matrix $A = uv^T$ such that $Ax = b$ with $b \neq 0$. Since $b \in \text{range}(A)$, u is a multiple of b, so without loss of generality we can take $u = b$.

Now suppose the orthosymmetric scalar product is bilinear, so $M = \pm M^T$. Since M is nonsingular, there exists $z \in \mathbb{K}^n$ such that $v^T = z^T M$, and so $A = uv^T = bz^T M$. For $A \in \mathbb{S}$ we have $A^\star = \epsilon A$ with $\epsilon = \pm 1$. Hence by (2.1) we have $\pm z b^T M = \epsilon b z^T M$ and so $z b^T = \pm \epsilon b z^T$. Thus $z = \mu b$ and $A = \mu b b^T M$ with μ a scalar. But $Ax = b \Rightarrow \mu b (b^T M x) = b \Rightarrow \mu \langle b, x \rangle_{\mathrm{M}} = 1$, thus forcing $\langle b, x \rangle_{\mathrm{M}}$ to be nonzero, and uniquely determining A by $A = bb^T M / \langle b, x \rangle_{\mathrm{M}}$. Similar reasoning applies for the sesquilinear case, leading to the formula $A = bb^* M / \langle b, x \rangle_{\mathrm{M}}$. ☐

COROLLARY 5.2. *Let $b \neq 0$. If $\langle b, x \rangle_{\mathrm{M}} \neq 0$, then either $\mathcal{S}$ is empty or there is a unique $A \in \mathcal{S}$ with $\text{rank}(A) = 1$.*

Proof. When $\mathcal{S} \neq \emptyset$, we know from Theorem 3.2 that $^+B_w \in \mathcal{J}$ and $^-B_w \in \mathcal{L}$ for any w such that $\langle w, x \rangle_{\mathrm{M}} = 1$. Since $\langle b, x \rangle_{\mathrm{M}} \neq 0$, choose $w = w_*$, where

$$(5.1) \qquad\qquad w_* := \begin{cases} b/\langle b, x \rangle_{\mathrm{M}} & \text{for bilinear forms,} \\ b/\overline{\langle b, x \rangle}_{\mathrm{M}} & \text{for sesquilinear forms} \end{cases}$$

so that $\langle w_*, x \rangle_{\mathrm{M}} = 1$. Substituting this choice of w into the formulas for ^+B_w, ^-B_w given in (3.2) yields the unique rank-one mapping specified in Theorem 5.1. ☐

TABLE 5.1
Rank-one structured mappings when $\langle b, x \rangle_M \neq 0$ with w_* as in (5.1).

Scalar Product	$\mathcal{J}$	$\mathcal{L}$
symmetric bilinear	$^{+}B_{w_*}$	$\mathcal{L}$ is empty
skew-symmetric bilinear	$\mathcal{J}$ is empty	$^{-}B_{w_*}$
Hermitian sesquilinear	$^{+}B_{w_*}$ if $0 \neq \langle b, x \rangle_M \in \mathbb{R}$. Otherwise $\mathcal{J}$ is empty.	$^{-}B_{w_*}$ if $0 \neq \langle b, x \rangle_M \in i\mathbb{R}$. Otherwise $\mathcal{L}$ is empty.
skew-Hermitian sesquilinear	$^{+}B_{w_*}$ if $0 \neq \langle b, x \rangle_M \in i\mathbb{R}$. Otherwise $\mathcal{J}$ is empty.	$^{-}B_{w_*}$ if $0 \neq \langle b, x \rangle_M \in \mathbb{R}$. Otherwise $\mathcal{L}$ is empty.

Particular cases of Corollary 5.2 are summarized in Table 5.1. The extra conditions in the sesquilinear cases come from the results in Corollary 3.3.

For nonzero b we have seen that the condition $\langle b, x \rangle_M \neq 0$, while necessary for the existence of structured rank-one mappings, is precisely the condition that precludes the existence of *any* structured mappings in certain cases (see Theorem 3.2). On the other hand, Theorem 3.2 also shows that structured mapping sets $\mathcal{S}$ are never empty when the condition $\langle b, x \rangle_M = 0$ is met. We turn to the question of determining what the minimal achievable rank is in this case.

THEOREM 5.3 (Rank-two structured mappings). *Let $\langle \cdot, \cdot \rangle_M$ be an orthosymmetric scalar product, and let $\mathcal{S}$ denote either $\mathcal{J}$ or $\mathcal{L}$ as in (3.1). Consider any nonzero x, $b \in \mathbb{K}^n$. If $\langle b, x \rangle_M = 0$ then*

$$\min_{A \in \mathcal{S}} \operatorname{rank}(A) = 2.$$

There are always infinitely many matrices in $\mathcal{S}$ attaining this minimal rank. Among these are $^{-}B_w \in \mathcal{L}$ and $^{+}B_w \in \mathcal{J}$, where $^{-}B_w$, $^{+}B_w$ are given by (3.2), with any choice of $w \in \mathbb{K}^n$ such that $\langle w, x \rangle_M = 1$.

Proof. If $\langle b, x \rangle_M = 0$, then by Theorem 5.1, the minimum possible rank for matrices in $\mathcal{S}$ is 2. We know $^{+}B_w$, $^{-}B_w$ map x to b for all $w \in \mathbb{K}^n$ such that $\langle w, x \rangle_M = 1$, and from Theorem 3.2 it follows that $^{+}B_w \in \mathbb{J}$ and $^{-}B_w \in \mathbb{L}$ for all such w. Since $^{+}B_w$, $^{-}B_w$ are at most rank two, and since they cannot be rank one, they are structured mappings of rank two.   □

5.2. Structured mappings of minimal Frobenius norm. Another important special property is minimal norm, since this is directly related to structured backward errors for linear systems and eigenvalue problems [19], [20] as well as to the derivation of quasi-Newton methods [3]. We first consider minimal Frobenius norm; the minimal 2-norm case will be treated in the next section. For real symmetric or Hermitian matrices, it is well known [2], [3] that minimal Frobenius norm is achieved by

$$A_{\text{opt}} = \frac{bx^* + xb^*}{x^*x} - \frac{(b^*x)}{(x^*x)^2}xx^*.$$

We show how to generalize this result to all Lie and Jordan algebras associated with scalar products that are both orthosymmetric and unitary. To prove the uniqueness of the structured mapping of minimal Frobenius norm, we will need the next two lemmas.

LEMMA 5.4. *In any real or complex inner product space, the associated norm* $\|\cdot\|$ *is strictly convex on independent vectors, that is,*

$$\|tu + (1-t)v\| \; < \; t\|u\| + (1-t)\|v\|, \quad 0 < t < 1,$$

for any linearly independent u and v.

Proof. The Cauchy-Schwarz inequality implies that $\langle u,v \rangle + \langle v,u \rangle < 2\|u\|\|v\|$ for linearly independent u,v. A straightforward calculation then establishes the result. ☐

LEMMA 5.5. *For $b \neq 0$, the Frobenius norm is strictly convex on $\mathcal{S}$ ($\mathcal{S} = \mathcal{J}, \mathcal{L}$).*

Proof. Assuming $b \neq 0$, distinct $A, B \in \mathcal{S}$ are linearly independent. Since the Frobenius norm arises from the inner product $\langle A,B \rangle = \mathrm{tr}(A^*B)$, the result is immediate from Lemma 5.4. ☐

As Lemma 3.1 and Theorem 3.2 show, whenever the set of structured mappings $\mathcal{S}$ is non-empty, we can construct a parametrized set of structured maps ^+B_w or ^-B_w that take x to b. The next theorem shows how this latitude in the choice of the parameter $w \in \mathbb{K}^n$, $\langle w,x \rangle_{\mathrm{M}} = 1$, can be exploited to identify the unique map of minimal Frobenius norm.

THEOREM 5.6 (Minimal Frobenius norm structured mapping). *Let $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ be a scalar product that is both orthosymmetric and unitary. Let $\mathcal{S}$ denote either $\mathcal{J}$ or $\mathcal{L}$ as in (3.1). If $\mathcal{S} \neq \emptyset$, the problem*

$$\min_{A \in \mathcal{S}} \|A\|_F$$

has a unique solution given by

$$(5.2) \qquad A_{\mathrm{opt}} = \frac{bx^*}{x^*x} + \epsilon \left(\frac{bx^*}{x^*x} \right)^{\star} \left(I - \frac{xx^*}{x^*x} \right), \quad \epsilon = \begin{cases} 1 & \text{if } \mathcal{S} = \mathcal{J}, \\ -1 & \text{if } \mathcal{S} = \mathcal{L}. \end{cases}$$

Moreover,

$$(5.3) \qquad \|A_{\mathrm{opt}}\|_F^2 = 2\frac{\|b\|_2^2}{\|x\|_2^2} - \beta^2 \frac{|\langle b,x \rangle_{\mathrm{M}}|^2}{\|x\|_2^4},$$

where $\beta > 0$ is such that βM is unitary.

Proof. Since $\mathcal{S} \neq \emptyset$, we know from Theorem 3.2 that $^+B_w \in \mathcal{J}$ and $^-B_w \in \mathcal{L}$ for any $w \in \mathbb{K}^n$ such that $\langle w,x \rangle_{\mathrm{M}} = 1$. Choose $w = w_0$, where

$$(5.4) \qquad\qquad w_0 = \begin{cases} M^{-T}\overline{x}/(x^*x) & \text{for bilinear forms,} \\ M^{-*}x/(x^*x) & \text{for sesquilinear forms.} \end{cases}$$

Then $\langle w_0, x \rangle_{\mathrm{M}} = 1$, and the expressions for the structured maps $^{\pm}B_{w_0}$ in (3.2) and the projection P_{w_0} in (2.2) become

$$(5.5) \qquad\qquad {}^{\pm}B_{w_0} = \frac{bx^*}{x^*x} \pm \left(\frac{bx^*}{x^*x} \right)^{\star} P_{w_0}, \qquad P_{w_0} = I - \frac{xx^*}{x^*x}.$$

For brevity, let A_0 denote $^{\pm}B_{w_0}$, and let P_0 denote the orthogonal projection P_{w_0}. We now show that A_0 is the unique map of minimal Frobenius norm in $\mathcal{S}$.

Complete $\{x/\|x\|_2\}$ to an orthonormal basis $\{x/\|x\|_2, u_2, \ldots, u_n\}$ with respect to the standard inner product on $\mathbb{K}^n$. We first observe that for all $A \in \mathcal{S}$,

$$\|Ax\|_F = \|b\|_F = \|A_0x\|_F.$$

The Characterization Theorem 4.4 with $v = w = w_0$ tells us that any $A \in \mathcal{S}$ can be written as $A = A_0 + P_0^\star S P_0$ for some $S \in \mathbb{S}$. Premultiplying u_i by A and taking norm yields

$$(5.6) \quad \|Au_i\|_2^2 = \|A_0 u_i\|_2^2 + \|P_0^\star S P_0 u_i\|_2^2 + 2 \operatorname{Re}\left((A_0 u_i)^* P_0^\star S P_0 u_i\right), \quad 2 \le i \le n.$$

When $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is unitary, the last term on the right hand side of (5.6) always vanishes. To see this, first consider the case when the form is bilinear. Since the stars commute in a unitary scalar product and $x^* u_i = 0$, $i = 2 : n$, we have

$$(A_0 u_i)^* = \pm u_i^* \left(\left(\frac{bx^*}{x^*x} \right)^\star \right)^* = \pm u_i^* \left(\frac{xb^*}{x^*x} \right)^\star = \pm \left(\frac{u_i^* M^{-1} \bar{b}}{x^*x} \right) x^T M =: \alpha_i x^T M$$

and,

$$(A_0 u_i)^* P_0^\star S P_0 u_i = \alpha_i x^T M \left(M^{-1} \left(I - \frac{\bar{x}x^T}{x^*x} \right) M \right) S u_i = \alpha_i (x^T - x^T) M S u_i = 0.$$

Similarly, for sesquilinear forms, $(A_0 u_i)^* = \alpha_i x^* M$ with $\alpha_i = \pm (u_i^* M^{-1} b)/(x^*x)$ and

$$(A_0 u_i)^* P_0^\star S P_0 u_i = \alpha_i x^* M \left((M^{-1} \left(I - \frac{xx^*}{x^*x} \right) M \right) S u_i = \alpha_i (x^* - x^*) M S u_i = 0.$$

Therefore from (5.6), $\|Au_i\|_2 \ge \|A_0 u_i\|_2$, $2 \le i \le n$. Recall that the Frobenius norm is unitarily invariant; since $\{ x/\|x\|_2, u_2, \dots, u_n \}$ forms an orthonormal basis for $\mathbb{K}^n$,

$$\|A\|_F^2 = \frac{\|Ax\|_2^2}{\|x\|_2^2} + \sum_{i=2}^n \|Au_i\|_2^2 \ge \frac{\|A_0 x\|_2^2}{\|x\|_2^2} + \sum_{i=2}^n \|A_0 u_i\|_2^2 = \|A_0\|_F^2, \quad \forall A \in \mathcal{S},$$

showing that A_0 has minimal Frobenius norm.

It is well-known that strictly convex functions have at most one minimizer [1, p.4]. Therefore Lemma 5.5 implies that A_0 is unique for $b \neq 0$. When $b = 0$, $A_0 \equiv 0$ is clearly unique. Thus A_{opt}, the unique structured map of minimal Frobenius norm is A_0, defined by (5.2).

Finally, for the Frobenius norm of A_{opt} we have

$$(5.7) \quad \|A_{\mathrm{opt}}\|_F^2 = \left(\|bx^*\|_F^2 + \|(bx^*)^\star P_0\|_F^2 + 2\epsilon \operatorname{Re}\left(\operatorname{tr}[xb^*(bx^*)^\star P_0] \right) \right)/\|x\|_2^4.$$

Now $P_0 x = 0$ implies

$$(5.8) \qquad\qquad \operatorname{tr}[xb^*(bx^*)^\star P_0] = \operatorname{tr}[P_0 xb^*(bx^*)^\star] = \operatorname{tr}(0) = 0.$$

Since P_0 is an orthogonal projection, $P_0^2 = P_0 = P_0^*$. Hence

$$\|(bx^*)^\star P_0\|_F^2 = \operatorname{tr}\left[(bx^*)^\star P_0 \left((bx^*)^\star \right)^* \right] = \|(bx^*)^\star\|_F^2 - \|(bx^*)^\star x\|_2^2/(x^*x)$$

$$(5.9) \qquad\qquad\qquad = \|(bx^*)\|_F^2 - x^*(bb^*)^\star x.$$

For the last equality we have used the fact that $\|X^\star\|_F = \|X\|_F$ for any unitary scalar product[1]. Recall that βM is unitary for some $\beta > 0$. Thus $M^{-1} = \beta^2 M^*$ and

$$(5.10) \quad x^*(bb^*)^\star x = \beta^2 \begin{cases} x^* M^* \bar{b} b^T M x & \text{(bilinear forms)} \\ x^* M^* b b^* M x & \text{(sesquilinear forms)} \end{cases} = \beta^2 |\langle b, x \rangle_{\mathrm{M}}|^2.$$

Now combining (5.8)–(5.10) into (5.7) gives the desired formula for $\|A_{\mathrm{opt}}\|_F^2$. $\qquad \square$

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[1]Surprisingly, this property characterizes unitary scalar products [13].

5.3. Structured mappings of minimal 2-norm. From $Ax = b$ it is clear that $\|b\|_2/\|x\|_2$ is always a lower bound for $\|A\|_2$. For a large class of scalar products Theorem 5.6 also yields an upper bound:

(5.11) $$\frac{\|b\|_2}{\|x\|_2} \leq \min_{A \in \mathcal{S}} \|A\|_2 \leq \min_{A \in \mathcal{S}} \|A\|_F \leq \sqrt{2}\frac{\|b\|_2}{\|x\|_2},$$

where $\mathcal{S}$ denotes either $\mathcal{J}$ or $\mathcal{L}$ as in (3.1). In this section we show that the lower bound is actually attained in any Lie or Jordan algebra of a scalar product that is both orthosymmetric and unitary.

Unlike the structured mapping of minimal Frobenius norm, mappings of minimal 2-norm in $\mathcal{S}$ are almost never unique. For example, consider the Jordan algebra of $n \times n$ symmetric matrices with $n \geq 3$, and take $x = e_1$ and $b = e_2$ to be the first and second column of the identity matrix, respectively. Then all matrices of the form $A = \text{diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, S\right)$ with S symmetric and $\|S\|_2 \leq 1$ satisfy $A^T = A$, $Ax = b$, and have $\|A\|_2 = \|b\|_2/\|x\|_2 = 1$. Indeed, this formula captures *all* symmetric matrices mapping e_1 to e_2 that have minimal 2-norm. We develop similar characterizations of the sets of structured mappings of minimal 2-norm for large classes of Lie and Jordan algebras by reducing to the characterization problem for the following special structures:

(5.12) $$\begin{aligned} \text{Sym}(n, \mathbb{K}) &= \{A \in \mathbb{K}^{n \times n} : A^T = A\}, \\ \text{Skew}(n, \mathbb{K}) &= \{A \in \mathbb{K}^{n \times n} : A^T = -A\}, \\ \text{Herm}(n, \mathbb{C}) &= \{A \in \mathbb{C}^{n \times n} : A^* = A\}. \end{aligned}$$

We will use the simplified notation $\text{Sym}(\mathbb{K})$, etc., when the size of the matrices is clear from the context. The technical details for these special structures can be found in Appendix A. Recall that for nonzero $\mu \in \mathbb{K}$,

$$\text{sign}(\mu) := \overline{\mu}/|\mu|.$$

THEOREM 5.7 (Minimal 2-norm structured mappings: general case). *Let $\mathbb{S}_n$ be the Lie algebra $\mathbb{L}$ or Jordan algebra $\mathbb{J}$ of a scalar product $\langle \cdot, \cdot \rangle_M$ on $\mathbb{K}^n$ that is both orthosymmetric and unitary, so that $M \cdot \mathbb{S}_n$ is either $\text{Sym}(n, \mathbb{K})$, $\text{Skew}(n, \mathbb{K})$, or $\gamma\text{Herm}(n, \mathbb{C})$ for some $|\gamma| = 1$. Also let $x, b \in \mathbb{K}^n \setminus \{0\}$ be vectors such that $\mathcal{S} = \{A \in \mathbb{S}_n : Ax = b\}$ is nonempty. Then*

$$\min_{A \in \mathcal{S}} \|A\|_2 = \frac{\|b\|_2}{\|x\|_2}$$

Furthermore, with $\mathcal{M} := \{A \in \mathcal{S} : \|A\|_2 = \|b\|_2/\|x\|_2\}$, there exists a unitary matrix U such that

(5.13) $$\mathcal{M} = \left\{\frac{\|b\|_2}{\|x\|_2}U^\star(\beta M)^{-1}\begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}U : S \in \widetilde{\mathbb{S}}_{n-r}, \|S\|_2 \leq 1\right\},$$

where $\beta > 0$ is a real constant such that βM is unitary and $\star$ denotes the adjoint of the scalar product $\langle \cdot, \cdot \rangle_M$. The number r, the structured class $\widetilde{\mathbb{S}}_{n-r}$, and $R \in \mathbb{K}^{r \times r}$ are given in each case by:

(i) *$M \cdot \mathbb{S}_n = \text{Sym}(n, \mathbb{K})$: $r = 1$ ($r = 2$) if x and $\overline{Mb}$ are linearly dependent (independent), $\widetilde{\mathbb{S}}_{n-r} = \text{Sym}(n - r, \mathbb{K})$, and*

$$R = \begin{cases} \text{sign}(\mu) & \text{if } \beta\overline{Mb} = \mu x \text{ for some } \mu \in \mathbb{K}, \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{otherwise}, \end{cases}$$

(ii) $M \cdot \mathbb{S}_n = \mathrm{Skew}(n, \mathbb{K})$: $r = 2$, $\widetilde{\mathbb{S}}_{n-r} = \mathrm{Skew}(n - 2, \mathbb{K})$, and $R = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$,

(iii) $M \cdot \mathbb{S}_n = \gamma \mathrm{Herm}(n, \mathbb{C})$ for some $|\gamma| = 1$: $r = 1$ ($r = 2$) if x and Mb are linearly dependent (independent), $\widetilde{\mathbb{S}}_{n-r} = \gamma \mathrm{Herm}(n - r, \mathbb{C})$, and

$$
R = \begin{cases} \gamma \, \mathrm{sign}(\mu) & \text{if } \gamma^{-1}\beta M b = \mu x \text{ for some } \mu \in \mathbb{R}, \\ \gamma \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] & \text{otherwise.} \end{cases}
$$

The matrix U can be taken as the product of at most two unitary Householder reflectors; when $\mathbb{K} = \mathbb{R}$, U is real orthogonal.

Proof. For orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ it is shown in [12, Thm. 8.4] that left multiplication by the matrix M defining the scalar product is a bijection from $\mathbb{K}^{n \times n}$ to $\mathbb{K}^{n \times n}$ that maps $\mathbb{L}$ and $\mathbb{J}$ to $\mathrm{Skew}(\mathbb{K})$ and $\mathrm{Sym}(\mathbb{K})$ for bilinear forms, and to unit scalar multiples of $\mathrm{Herm}(\mathbb{C})$ for sesquilinear forms. Furthermore βM being unitary implies that the map $\mathbb{S} \longmapsto \beta M \cdot \mathbb{S}$ is a 2-norm-preserving bijection. For bilinear forms, the equivalence of the equations $Ax = b$ and $\widetilde{A}x := (\beta M A)x = (\beta M b) =: \widetilde{b}$ thus reduces the structured mapping problem for $\mathcal{S} = \{A \in \mathbb{S}_n : Ax = b\}$ in a 2-norm-preserving way to the structured mapping problem for finding $\widetilde{A}$ in $\mathrm{Skew}(n, \mathbb{K})$ or $\mathrm{Sym}(n, \mathbb{K})$ such that $\widetilde{A}x = \widetilde{b}$. Similarly for sesquilinear forms, the equivalence of $Ax = b$ and $\widetilde{A}x := (\gamma^{-1}\beta M A)x = (\gamma^{-1}\beta M b) =: \widetilde{b}$ gives a 2-norm-preserving reduction of the structured mapping problem for $\mathcal{S}$ to that of finding $\widetilde{A}$ in $\mathrm{Herm}(n, \mathbb{C})$ such that $\widetilde{A}x = \widetilde{b}$.

The value of $\min_{A \in \mathcal{S}} \|A\|_2$ and the formula for $\mathcal{M}$ in (5.13) now follow by applying Theorem A.2 to the minimal 2-norm structured mapping problem for $\widetilde{A}x = \widetilde{b}$, and then using the correspondence between $\widetilde{A}$ and A.   □

Note the structure of formula (5.13), and how it automatically produces matrices in $\mathbb{S}_n$. In all cases $\frac{\|b\|_2}{\|x\|_2}(\beta M)^{-1} \mathrm{diag}(R, S)$ is in $\mathbb{S}_n$, since the scalar product is orthosymmetric and $\|b\|_2/\|x\|_2$ and β are real. Lemma 4.1(a) shows that $\star$-congruence preserves $\mathbb{L}$ and $\mathbb{J}$ structure, so $U^{\star} \frac{\|b\|_2}{\|x\|_2}(\beta M)^{-1} \mathrm{diag}(R, S)U$ is again in $\mathbb{S}_n$.

5.4. Comparison of the various "minimal" structured mappings. We conclude section 5 by exploring the relationships between the three types of extremal mappings—minimal rank, Frobenius norm, and 2-norm—under the assumption that the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is both unitary and orthosymmetric.

In general the minimal Frobenius norm solution A_{opt} differs from the minimal rank solution. The latter is usually rank one whereas A_{opt} is generally rank two. From (5.2) we see that A_{opt} is rank one if and only if $M^{-1}\bar{x} \in \mathrm{span}\{b\}$ for bilinear forms or $M^{-1}x \in \mathrm{span}\{b\}$ for sesquilinear forms.

For structured mappings of minimal 2-norm, the following corollary of Theorem 5.7 singles out the unique matrix of minimal rank as well as the one of minimal Frobenius norm.

COROLLARY 5.8. *Under the hypotheses of Theorem 5.7, let $\mathcal{M}$ denote the set of all minimal 2-norm mappings in $\mathcal{S} = \{A \in \mathbb{S} : Ax = b\}$. Assume further that x, b are vectors such that $\mathcal{S}$ is nonempty. Consider the particular mapping*

$$
(5.14) \qquad A_2 := \frac{\|b\|_2}{\|x\|_2} U^{\star} (\beta M)^{-1} \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} U \in \mathcal{M},
$$

obtained by setting S equal to 0 in (5.13). Then A_2 is the unique solution of both the minimal rank problem $\min_{A \in \mathcal{M}} \mathrm{rank}(A)$ and the minimal Frobenius norm problem $\min_{A \in \mathcal{M}} \|A\|_F$. Moreover, either

(1) A_2 has rank one and $\|A_2\|_F = \|b\|_2/\|x\|_2$, or

(2) A_2 has rank two and $\|A_2\|_F = \sqrt{2}\,\|b\|_2/\|x\|_2$.

Case (1) *occurs when x and $\overline{Mb}$ (x and Mb) are linearly dependent and the scalar product is bilinear (sesquilinear). (Note that if $M \cdot \mathbb{S} = \mathrm{Skew}(n, \mathbb{K})$, then this linear dependence implies that $\mathcal{S}$ is empty.) Otherwise case* (2) *holds.*

Are there any conditions under which there is a structured mapping in $\mathcal{S}$ that simultaneously has all three extremal properties? The next result provides a complete answer to this question.

THEOREM 5.9. *Let $\mathbb{S}$ be the Lie or Jordan algebra of a scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ that is both unitary and orthosymmetric. Assume that $x, b \in \mathbb{K}^n \setminus \{0\}$ are vectors such that $\mathcal{S} = \{A \in \mathbb{S} : Ax = b\}$ is nonempty and A_2 is the matrix defined (5.14). Then the unique minimal Frobenius norm mapping $A_{\mathrm{opt}} \in \mathcal{S}$ has both minimal 2-norm and minimal rank in $\mathcal{S}$ if and only if the pair of vectors (x, b) satisfies either property* (a) *or property* (b) *below.*

(a) *$M^{-1}\bar{x} \in \mathrm{span}\{b\}$ for bilinear forms or $M^{-1}x \in \mathrm{span}\{b\}$ for sesquilinear forms. In this case*

$$A_{opt} = A_2 = \begin{cases} bb^T M/\langle b, x \rangle_{\mathrm{M}} & \text{for bilinear forms,} \\ bb^* M/\langle b, x \rangle_{\mathrm{M}} & \text{for sesquilinear forms} \end{cases}$$

is the unique rank-one mapping in $\mathcal{S}$.

(b) *$\langle b, x \rangle_{\mathrm{M}} = 0$. In this case*

$$A_{\mathrm{opt}} = A_2 = \frac{bx^*}{x^*x} + \epsilon \left(\frac{bx^*}{x^*x} \right)^{\star}, \quad \epsilon = \begin{cases} 1 & \text{if } \mathbb{S} = \mathbb{J}, \\ -1 & \text{if } \mathbb{S} = \mathbb{L} \end{cases}$$

is the unique rank-two mapping in $\mathcal{M} = \{A \in \mathcal{S} : \|A\|_2 = \min_{B \in \mathcal{S}} \|B\|_2\}$.

Proof. ($\Rightarrow$) A_{opt} having minimal 2-norm in $\mathcal{S}$ means that $A_{\mathrm{opt}} \in \mathcal{M}$, with minimal Frobenius norm in $\mathcal{M}$; thus $A_{\mathrm{opt}} = A_2$ by Corollary 5.8. But A_2 is either rank one or rank two. A_2 with rank one means A_{opt} has rank one, and therefore property (a) holds by the remarks preceding the corollary. On the other hand A_2 with rank two implies $\|A_{\mathrm{opt}}\|_F = \|A_2\|_F = \sqrt{2}\,\|b\|_2/\|x\|_2$, which by (5.3) implies that property (b) holds.

($\Leftarrow$) Property (a) implies that A_{opt} is rank one by the remarks preceding the corollary. But property (a) is equivalent to the linear dependence of x and $\overline{Mb}$ (x and Mb) for bilinear (sesquilinear) forms, which are precisely the conditions in Corollary 5.8 which guarantee that A_2 is rank one. The uniqueness of rank-one mappings in $\mathcal{S}$ from Theorem 5.1 now implies that $A_{\mathrm{opt}} = A_2$ has all three minimality properties.

Property (b) implies that $\|A_{\mathrm{opt}}\|_F = \sqrt{2}\,\|b\|_2/\|x\|_2$ by (5.3), and that the minimal rank in $\mathcal{S}$ is two by Theorem 5.3. By Corollary 5.8 we know that $\|A_2\|_F \leq \sqrt{2}\,\|b\|_2/\|x\|_2$, so the uniqueness of minimal Frobenius norm mappings implies that $A_{\mathrm{opt}} = A_2$. This map has minimal rank two by case (2) of Corollary 5.8. ☐

6. Concluding remarks. In this paper we have presented complete, unified, and explicit solutions of the existence and characterization problems for structured mappings coming from Lie and Jordan algebras associated with orthosymmetric scalar products. In addition, in the set $\{A \in \mathbb{S} : Ax = b\}$ we have identified and characterized the structured mappings of minimal rank, minimal Frobenius norm, and minimal 2-norm. These results have already found application in the analysis of structured condition numbers and backward errors [7], [20], and constitute the first step towards characterizing the set $\{A \in \mathbb{S} : AX = B\}$, where X and B are given matrices.

In part II of this paper [14] we consider the same structured mapping problems for a third class of structured matrices $\mathbb{S}$ associated with a scalar product: the automorphism group $\mathbb{G}$ defined by

$$\mathbb{G} = \left\{ A \in \mathbb{K}^{n \times n} : \langle Ax, Ay \rangle_{\mathrm{M}} = \langle x, y \rangle_{\mathrm{M}} \ \forall x, y \in \mathbb{K}^n \right\} = \left\{ A \in \mathbb{K}^{n \times n} : A^{\star} = A^{-1} \right\}.$$

Unlike the corresponding Lie and Jordan algebras, the group $\mathbb{G}$ is a *nonlinear* subset of $\mathbb{K}^{n \times n}$; hence different (and somewhat more elaborate) techniques are needed to solve the structured mapping problems for $\mathbb{G}$. There are, however, some ways in which the results for groups $\mathbb{G}$ are actually simpler than the ones developed in this paper for $\mathbb{L}$ and $\mathbb{J}$. Consider, for example, the solution of the existence problem given in [14]: for any orthosymmetric scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$, there exists $A \in \mathbb{G}$ such that $Ax = b$ if and only if $\langle x, x \rangle_{\mathrm{M}} = \langle b, b \rangle_{\mathrm{M}}$. A clean, unified, and simply stated result. Examples of groups $\mathbb{G}$ covered by this theorem include the orthogonal, symplectic, and pseudo-unitary groups. Two types of characterization of the set $\{ A \in \mathbb{G} : Ax = b \}$ are also given in [14], both of which are expected to be useful in structured backward error investigations.

Appendix A. Structured mappings of minimal 2-norm for symmetric, skew-symmetric, and Hermitian structures.

Our goal in this appendix is to characterize the complete set of all minimal 2-norm mappings for each of the five key structures in (5.12). For example, for real symmetric matrices it is already well known that $A = (\|b\|_2/\|x\|_2)H$, where H is a Householder reflector mapping $x/\|x\|_2$ to $b/\|b\|_2$, provides a minimal 2-norm solution. However, the set of *all* minimal 2-norm symmetric matrices taking x to b has not previously been explicitly described.

First we consider the 2×2 case for a special type of (x, b) vector pair and for symmetric and Hermitian structures.

LEMMA A.1. *Let* $\mathbb{S}$ *be either* $\mathrm{Sym}(2, \mathbb{K})$ *or* $\mathrm{Herm}(2, \mathbb{C})$ *and*

$$\mathcal{S} = \left\{ A \in \mathbb{S} : \ A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \right\},$$

where $\alpha, \beta \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) \neq 0$ *and* $\beta \neq 0$ *when* $\mathbb{S} = \mathrm{Sym}(2, \mathbb{C})$, *and* $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ *otherwise. Then*

$$\min_{A \in \mathcal{S}} \|A\|_2 = 1,$$

with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *being the* unique *matrix in* $\mathcal{S}$ *of minimal 2-norm.*

Proof. Note that from (5.11) any $A \in \mathcal{S}$ satisfies $\|A\|_2 \geq 1$, and since $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{S}$ has unit 2-norm we have $\min_{A \in \mathcal{S}} \|A\|_2 = 1$. The rest of the proof consists of showing that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the unique minimizer of the 2-norm for $\mathcal{S}$.

We start by parameterizing $\mathcal{S}$ using (4.1):

$$\mathcal{S} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \left\{ A \in \mathbb{S} : \ A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\},$$

where $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a particular mapping in $\mathcal{S}$. Any $A \in \mathrm{Sym}(2, \mathbb{K})$ has the form $\begin{bmatrix} a & z \\ z & c \end{bmatrix}$ with $a, c, z \in \mathbb{K}$, so $A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $\begin{bmatrix} a & z \\ z & c \end{bmatrix} = z \begin{bmatrix} -\beta/\alpha & 1 \\ 1 & -\alpha/\beta \end{bmatrix}$. Similarly any $A \in \mathrm{Herm}(2, \mathbb{C})$ has the form $\begin{bmatrix} a & z \\ \bar{z} & c \end{bmatrix}$ with $a, c \in \mathbb{R}$; then $\alpha, \beta \in \mathbb{R}$ together with $A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

implies that $z \in \mathbb{R}$, and so A can once again be expressed in the form $z\begin{bmatrix} -\beta/\alpha & 1 \\ 1 & -\alpha/\beta \end{bmatrix}$.
Hence writing

$$
P(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + z \begin{bmatrix} -\frac{\beta}{\alpha} & 1 \\ 1 & -\frac{\alpha}{\beta} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\beta}{\alpha}z & z \\ z & -1 - \frac{\alpha}{\beta}z \end{bmatrix},
$$

we have $\mathcal{S} = \{P(z) : z \in \mathbb{K}\}$ if $\mathbb{S} = \mathrm{Sym}(2, \mathbb{K})$, and $\mathcal{S} = \{P(z) : z \in \mathbb{R}\}$ when
$\mathbb{S} = \mathrm{Herm}(2, \mathbb{C})$ and $\alpha, \beta \in \mathbb{R}$.

We can now calculate the 2-norm of $P(z)$ by computing the largest eigenvalue of

$$
P^*P(z) = \begin{bmatrix} 1 - \left(\frac{\beta}{\alpha}\bar{z} + \frac{\beta}{\alpha}z\right) + \left(1 + \frac{\beta^2}{|\alpha|^2}\right)|z|^2 & (z - \bar{z}) - \gamma|z|^2 \\ (\bar{z} - z) - \bar{\gamma}|z|^2 & 1 + \left(\frac{\alpha}{\beta}\bar{z} + \frac{\alpha}{\beta}z\right) + \left(1 + \frac{|\alpha|^2}{\beta^2}\right)|z|^2 \end{bmatrix},
$$

where $\gamma := (\alpha/\beta) + (\beta/\overline{\alpha})$. Much calculation and simplification yields

$$
\mathrm{tr}\, P^*P(z) = 2 + 2q(z), \qquad \det P^*P(z) = 1 + 2q(z) - 2\left(1 + \mathrm{Re}\, \frac{\alpha^2}{|\alpha|^2}\right)|z|^2,
$$

where $q(z) := \mathrm{Re}\left[\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)z\right] + |\gamma|^2|z|^2/2 \in \mathbb{R}$. Since the characteristic polynomial of
$P^*P(z)$ is $\lambda^2 - \mathrm{tr}\, P^*P(z)\lambda + \det P^*P(z)$, we get

$$
\lambda_\pm(z) = \frac{1}{2}\left(\mathrm{tr}\, P^*P(z) \pm \sqrt{\left(\mathrm{tr}\, P^*P(z)\right)^2 - 4\det P^*P(z)}\right)
$$

$$
= 1 + q(z) \pm \sqrt{q(z)^2 + 2\left(1 + \mathrm{Re}\, \frac{\alpha^2}{|\alpha|^2}\right)|z|^2}.
$$

Since $q(z) \in \mathbb{R}$, clearly the largest eigenvalue of $P^*P(z)$ is $\lambda_+(z)$. But the hypothesis
$\mathrm{Re}(\alpha) \neq 0$ means that $\mathrm{Re}\, \frac{\alpha^2}{|\alpha|^2} > -1$, so the second term under the square root
is strictly bigger than 0 for all nonzero z. Hence $\lambda_+(z)$ satisfies $\lambda_+(0) = 1$ and
$\lambda_+(z) > 1$ for all nonzero z. Thus $z = 0$ is the unique minimizer of $\lambda_+(z)$, and hence
$P(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the unique minimizer of the 2-norm for $\mathcal{S}$. □

We are now in a position to give a complete description of the set of all minimal
2-norm structured mappings for symmetric, skew-symmetric and Hermitian matrices.

THEOREM A.2 (Minimal 2-norm structured mappings: special cases). *Let* $\mathbb{S}_n$ *be
either* $\mathrm{Sym}(n, \mathbb{K})$, $\mathrm{Skew}(n, \mathbb{K})$, *or* $\mathrm{Herm}(n, \mathbb{C})$, *and let* $x, b \in \mathbb{K}^n \setminus \{0\}$ *be vectors such
that* $\mathcal{S} = \{A \in \mathbb{S}_n : Ax = b\}$ *is nonempty. Then*

$$
\min_{A \in \mathcal{S}} \|A\|_2 = \frac{\|b\|_2}{\|x\|_2}.
$$

Furthermore with $\mathcal{M} := \left\{A \in \mathcal{S} : \|A\|_2 = \|b\|_2/\|x\|_2\right\}$, *there exists an* $n \times n$ *unitary
matrix* U *such that*

$$
\mathcal{M} = \left\{\frac{\|b\|_2}{\|x\|_2}U^\star \mathrm{diag}(R, S)U : S \in \mathbb{S}_{n-r}, \|S\|_2 \leq 1\right\},
$$

where the adjoint $\star$, *the number* r, *and* $R \in \mathbb{S}_r$ *are given by:*
  (i) $\mathbb{S}_n = \mathrm{Sym}(n, \mathbb{K})$: $\star = T$ *and* $r = 1$ $(r = 2)$ *if* x *and* $\bar{b}$ *are linearly dependent
     (independent), with*

$$
R = \begin{cases} \mathrm{sign}(\mu) & \text{if } \bar{b} = \mu x \text{ for some } \mu \in \mathbb{K}, \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{otherwise}, \end{cases}
$$

(ii) $\mathbb{S}_n = \mathrm{Skew}(n, \mathbb{K})$: $\star = T$ *and* $r = 2$, *with* $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

(iii) $\mathbb{S}_n = \mathrm{Herm}(n, \mathbb{C})$: $\star = *$ *and* $r = 1$ $(r = 2)$ *if* x *and* b *are linearly dependent* (*independent*), *with*

$$R = \begin{cases} \mathrm{sign}(\mu) & \text{if } b = \mu x \text{ for some } \mu \in \mathbb{R}, \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{otherwise.} \end{cases}$$

The matrix U can be taken as the product of at most two unitary[2] Householder reflectors; when $\mathbb{K} = \mathbb{R}$, U is real orthogonal.

Proof. For any $A \in \mathbb{S}_n$ such that $Ax = b$, observe that the matrix $B = \frac{\|x\|_2}{\|b\|_2} A$ is also in $\mathbb{S}_n$ and maps $\frac{x}{\|x\|_2}$ to $\frac{b}{\|b\|_2}$; also note that $\overline{b} = \mu x$ (resp., $b = \mu x$) in the theorem is equivalent to $\frac{\overline{b}}{\|b\|_2} = \mathrm{sign}(\overline{\mu}) \frac{x}{\|x\|_2}$ $\left(\text{resp., } \frac{b}{\|b\|_2} = \mathrm{sign}(\overline{\mu}) \frac{x}{\|x\|_2}\right)$. Thus it suffices to prove the theorem for $x, b \in \mathbb{K}^n$ with $\|x\|_2 = \|b\|_2 = 1$ and the condition $\overline{b} = \mu x$ (resp., $b = \mu x$) replaced by $\overline{b} = \mathrm{sign}(\overline{\mu}) x$ $\left(\text{resp., } b = \mathrm{sign}(\overline{\mu}) x\right)$; we make these assumptions throughout the rest of the argument.

The proof proceeds by first developing a unitary U and accompanying R for each of the five structures in (5.12). Then these U and R matrices are used to build explicit families of matrices in the structured mapping set $\mathcal{S}$ that realize the lower bound in (5.11), and thus are of minimal 2-norm. Finally we show that for each structure these families account for all of $\mathcal{M}$.

We begin by constructing for each case of the theorem a unitary matrix U such that

$$(\mathrm{A}.1) \qquad Ux = \begin{bmatrix} y \\ 0 \end{bmatrix}, \quad (U^\star)^{-1} b = \begin{bmatrix} c \\ 0 \end{bmatrix},$$

with $y, c \in \mathbb{K}^r$ satisfying $Ry = c$, where $R \in \mathbb{S}_r$ is as defined in the theorem.

(i) First, suppose that $\mathbb{S}_n = \mathrm{Sym}(n, \mathbb{K})$. If $\overline{b} = \mathrm{sign}(\overline{\mu}) x$ for some $\mu \in \mathbb{K}$, then let U be the unitary Householder reflector mapping x to e_1, so that $y = 1$. Then $(U^\star)^{-1} b = \overline{U} b = \mathrm{sign}(\mu) e_1$, so $c = \mathrm{sign}(\mu)$. Clearly with $R := \mathrm{sign}(\mu) \in \mathbb{S}_1$ we have $Ry = c$.

When x and $\overline{b}$ are linearly independent then U can be taken as the product of two unitary Householder reflectors, $U = H_2 H_1$. The first reflector H_1 takes $x + \overline{b}$ to $\pm \| x + \overline{b} \|_2 e_1$; with $H_1 x = \begin{bmatrix} \alpha \\ v \end{bmatrix}$ and $H_1 \overline{b} = \begin{bmatrix} \gamma \\ w \end{bmatrix}$ we see that $w = -v$ with $v \neq 0$ because of the linear independence of x and $\overline{b}$, and $\alpha + \gamma = \pm \| x + \overline{b} \|_2 \in \mathbb{R} \setminus \{0\}$. Then $\|x\|_2 = \| \overline{b} \|_2 \Rightarrow \|H_1 x\|_2 = \|H_1 \overline{b}\|_2 \Rightarrow |\alpha| = |\gamma|$, which together with $\alpha + \gamma \in \mathbb{R}$ implies that $\gamma = \overline{\alpha}$, and hence $H_1 \overline{b} = \begin{bmatrix} \overline{\alpha} \\ -v \end{bmatrix}$. Note also that $2 \mathrm{Re}\, \alpha = \alpha + \overline{\alpha} = \pm \| x + \overline{b} \|_2 \neq 0$. For the second reflector pick $H_2 = \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{H}_2 \end{bmatrix}$ so that $\widetilde{H}_2 v = \beta e_1$ with $\beta = \pm \|v\|_2 \neq 0$. Hence

$$(\mathrm{A}.2) \qquad U \begin{bmatrix} x & \overline{b} \end{bmatrix} = \begin{bmatrix} \alpha & \overline{\alpha} \\ \beta & -\beta \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \mathrm{Re}\, \alpha \neq 0,\ 0 \neq \beta \in \mathbb{R},$$

and therefore $y = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $c = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$ satisfy $Ry = c$ with $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{S}_2$. Note that U can be taken to be real orthogonal when $\mathbb{K} = \mathbb{R}$.

---

[2]but not necessarily Hermitian, see [11].

(ii) For $\mathbb{S}_n = \mathrm{Skew}(n, \mathbb{K})$, Theorem 3.2 says that $\mathcal{S}$ is nonempty if and only if $b^T x = 0$. In this situation U can be taken as the product of two unitary Householder reflectors, $U = H_2 H_1$. The first reflector H_1 is defined to take x to e_1; then $H_1 \bar{b} = \left[\begin{smallmatrix} \alpha \\ v \end{smallmatrix}\right]$ for some $v \in \mathbb{K}^{n-1}$. The fact that $b^T x = 0$ implies $\alpha = 0$, since $b^T x = [\,\bar{\alpha} \quad v^*\,] H_1 H_1^* e_1 = \bar{\alpha}$. For the second reflector pick $H_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \tilde{H}_2 \end{smallmatrix}\right]$ so that $\tilde{H}_2 v = e_1 \in \mathbb{K}^{n-1}$. Then $U\,[\,x \quad \bar{b}\,] = [\,e_1 \quad e_2\,] \in \mathbb{K}^{n \times 2}$, giving $y = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $c = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ in $\mathbb{K}^2$ satisfying $Ry = c$ for $R = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathbb{S}_2$. Note once again that U can be taken to be real orthogonal when $\mathbb{K} = \mathbb{R}$.

(iii) Finally suppose that $\mathbb{S}_n = \mathrm{Herm}(n, \mathbb{C})$. Theorem 3.2 says that $\mathcal{S}$ is nonempty if and only if $b^* x \in \mathbb{R}$. If x and b are linearly dependent, then $b = \mathrm{sign}(\bar{\mu})x$ for some $\mu \in \mathbb{C}$, and $b^* x \in \mathbb{R}$ implies that $\mu \in \mathbb{R}$. In this case U can be taken as the unitary Householder reflector mapping x to e_1 so that $(U^\star)^{-1} b = Ub = \mathrm{sign}(\mu)e_1$ since μ is real. Hence $[\,y \quad c\,] = [\,1 \quad \mathrm{sign}(\mu)\,]$ and $Ry = c$ with $R = \mathrm{sign}(\mu) \in \mathbb{S}_1$.

On the other hand if x and b are linearly independent, then U can be taken as the product of two unitary Householder reflectors $U = H_2 H_1$ in a manner analogous to that described above in (i) for $\mathrm{Sym}(n, \mathbb{K})$; the only difference is that H_1 now takes $x + b$ to $\pm \|x + b\|_2 e_1$. In this case (A.2) holds with $\bar{b}$ replaced by b. Also $b^* x = (H_1 b)^* (H_1 x) = \alpha^2 - v^* v \in \mathbb{R}$ so that $\alpha^2 \in \mathbb{R}$. This together with $\mathrm{Re}\,\alpha \neq 0$ implies that $\alpha \in \mathbb{R}$. Hence we have $y = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ and $c = \left[\begin{smallmatrix} \alpha \\ -\beta \end{smallmatrix}\right]$ with $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, satisfying $Ry = c$ with $R = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \mathbb{S}_2$.

Using the unitary U and $R \in \mathbb{S}_r$ constructed above for each $\mathbb{S}_n$, we can now show that the lower bound $1 = \|b\|_2 / \|x\|_2 \leq \min_{A \in \mathcal{S}} \|A\|_2$ from (5.11) is actually attained by a whole family of $A \in \mathcal{S}$. For any $S \in \mathbb{S}_{n-r}$ with $\|S\|_2 \leq 1$, consider $A = U^\star \mathrm{diag}(R, S)U$. Then $A \in \mathbb{S}_n$, since $\mathbb{S}_n$ is preserved by any $\star$-congruence (see Lemma 4.1(a)) and $\mathrm{diag}(R, S) \in \mathbb{S}_n$. Also $Ax = b$ because of the properties of U in (A.1), and $\|A\|_2 = \|\mathrm{diag}(R, S)\|_2 = \|R\|_2 = 1$. Thus

$$\left\{ U^\star \mathrm{diag}(R, S)U : \; S \in \mathbb{S}_{n-r}, \; \|S\|_2 \leq 1 \right\} \subseteq \mathcal{M}.$$

Finally, we complete the characterization of $\mathcal{M}$ by showing that this containment is actually an equality. Consider an arbitrary $A \in \mathcal{M}$. Then $Ax = b \Rightarrow ((U^\star)^{-1} A U^{-1})(Ux) = (U^\star)^{-1} b$, so the matrix $B := (U^\star)^{-1} A U^{-1} = (U^{-1})^\star A U^{-1}$ is in $\mathbb{S}_n$ and maps the vector $Ux = \left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right]$ to $(U^\star)^{-1} b = \left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right]$. Let $B_{11} \in \mathbb{S}_r$ be the leading principal $r \times r$ submatrix of B, so $\|B_{11}\|_2 \leq \|B\|_2 = \|A\|_2 = 1$. The form of the two vectors $\left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right]$ implies that B_{11} maps y to c; since $\|y\|_2 = \|c\|_2 = 1$ we have $\|B_{11}\|_2 \geq 1$, and hence $\|B_{11}\|_2 = 1$. Using Lemma A.1 we can now show that $B_{11} = R$ in all cases.

(i) Suppose $\mathbb{S}_n = \mathrm{Sym}(n, \mathbb{K})$ and $B_{11} \in \mathbb{S}_r$. Then $\bar{b} = \mathrm{sign}(\bar{\mu})x$ for some $\mu \in \mathbb{K}$ implies $[\,y \; c\,] = [\,1 \; \mathrm{sign}(\mu)\,]$, so $B_{11}y = c$ implies $B_{11} = \mathrm{sign}(\mu) = R$. On the other hand if $\bar{b}$ and x are linearly independent, then $[\,y \; c\,] = \left[\begin{smallmatrix} \alpha & \alpha \\ \beta & -\beta \end{smallmatrix}\right]$ with $\mathrm{Re}(\alpha) \neq 0$ and $0 \neq \beta \in \mathbb{R}$. Since $B_{11}y = c$ with $\|B_{11}\|_2 = 1$, Lemma A.1 implies that $B_{11} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] = R$.

(ii) $B_{11} \in \mathrm{Skew}(2, \mathbb{K})$ must have the form $\left[\begin{smallmatrix} 0 & -\sigma \\ \sigma & 0 \end{smallmatrix}\right]$ for some $\sigma \in \mathbb{K}$. So $B_{11}y = c$ with $y = e_1$ and $c = e_2$ implies $\sigma = 1$, and hence $B_{11} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] = R$.

(iii) Finally consider $\mathbb{S}_n = \mathrm{Herm}(n, \mathbb{C})$ and $B_{11} \in \mathbb{S}_r$. If $b = \mathrm{sign}(\mu)x$ for some $\mu \in \mathbb{R}$ then $[\,y \; c\,] = [\,1 \; \mathrm{sign}(\mu)\,]$, so $B_{11}y = c$ implies $B_{11} = \mathrm{sign}(\mu) = R$. If b and x are linearly independent, then $[\,y \; c\,] = \left[\begin{smallmatrix} \alpha & \alpha \\ \beta & -\beta \end{smallmatrix}\right]$ with α, β both real and nonzero. Since $B_{11}y = c$ with $\|B_{11}\|_2 = 1$, Lemma A.1 implies that $B_{11} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] = R$.

The condition $\|B\|_2 = 1$ now forces the rest of the first r columns of B to be all zeroes; since $B \in \mathbb{S}_n$, the rest of the first r rows of B must also be all zeroes. Thus B has the form $B = \mathrm{diag}(R, S)$, with $S \in \mathbb{S}_{n-r}$. Finally, $\|B\|_2 = 1$ and $\|R\|_2 = 1$ implies that $\|S\|_2 \leq 1$. Thus we have $B := (U^\star)^{-1} A U^{-1} = \mathrm{diag}(R, S)$, so $A = U^\star \mathrm{diag}(R, S) U$ and hence $\mathcal{M} \subseteq \left\{ U^\star \mathrm{diag}(R, S) U : S \in \mathbb{S}_{n-r}, \|S\|_2 \leq 1 \right\}$. $\qquad\square$

Note that when $\mathbb{S}_n$ is the class of real symmetric matrices and if $x, y \in \mathbb{R}^n$ are linearly independent then choosing $S = I_n$ in Theorem A.2 yields the Householder reflector $I - 2U^T e_2 e_2^T U$ mapping x to b.

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