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THE NONCOMMUTATIVE GEOMETRY OF INNER FORMS OF p-ADIC SPECIAL LINEAR GROUPS

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Abstract. Let G be any reductive p-adic group. We conjecture that every Bernstein component in the space of irreducible smooth G-representations can be described as a "twisted extended quotient" of the associated Bernstein torus by the asssociated finite group. We also pose some conjectures about L-packets and about the structure of the Schwartz algebra of G in these noncommutative geometric terms. Ultimately, our conjectures aim to reduce the classification of irreducible representations to that of supercuspidal representations, and similarly for the local Langlands correspondence. These conjectures generalize earlier versions, which are only expected to hold for quasi-split groups.

We prove our conjectures for inner forms of general linear and special linear groups over local non-archimedean fields. This relies on our earlier study of Hecke algebras for types in these groups. We also make the relation with the local Langlands correspondence explicit.

Contents

Introduction

The aim of this paper is twofold. Firstly, we generalize our earlier conjecture [ABP, ABPS2] to all reductive p-adic groups, split or not. This is done in the introduction and the appendices. Secondly, we prove all these conjectures for the inner forms of general and special linear groups. In this respect the paper is a sequel to [ABPS3, ABPS4].

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Let F be a non-archimedean local field, and let G be a connected reductive algebraic group over F. We denote the space of (equivalence classes of) irreducible smooth complex G-representations by $\mathrm{Irr}(G)$, and its subset of supercuspidal representations by $\mathrm{Irr}_{\mathrm{cusp}}(G)$. For a Levi subgroup L (of a parabolic subgroup) of G we put $W(G,L) = N_G(L)/L$. Let $\mathcal{L} = \mathcal{L}(G)$ be a set of representatives for the conjugacy classes of Levi subgroups of G. The roughest form of our conjectures asserts that there exists a bijection

$$\mathrm{Irr}(G) \longleftrightarrow \bigsqcup\nolimits_{L \in \mathcal{L}} (\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(G,L))_\natural . \tag{1}$$

The right hand side is a twisted extended quotient, see Appendix B. It means that in the ordinary quotient $\mathrm{Irr}_{\mathrm{cusp}}(L)/W(G,L)$ we replace every point $\omega \in \mathrm{Irr}_{\mathrm{cusp}}(L)$ by the set of irreducible representations of the twisted group algebra $\mathbb{C}[W(G,L)_\omega, \natural(\omega)]$ determined by the 2-cocycle $\natural(\omega)$. In general the map (1) is not canonical, but the non-canonicity is limited. This already shows an important aspect of our work: relating, in a new and very precise way, the classification of irreducible smooth representations to that of supercuspidal representations.

To formulate our conjectures more accurately, we need the Bernstein decomposition of the category of smooth G-representations. Let L be a Levi subgroup of G and $\omega \in \mathrm{Irr}_{\mathrm{cusp}}(L)$. The inertial equivalence class $\mathfrak{s} = [L,\omega]_G$ determines a subset $\mathrm{Irr}^{\mathfrak{s}}(G) \subset \mathrm{Irr}(G)$. Bernstein also attached to (L,ω) a torus $T_{\mathfrak{s}} = \mathrm{Irr}^{[L,\omega]_L}(L)$ and a finite group $W_{\mathfrak{s}}$, the stabilizer of $T_{\mathfrak{s}}$ in $W(G,L)$. We note that $T_{\mathfrak{s}}$ is isomorphic to the quotient of the group of unramified characters $X_{\mathrm{nr}}(L)$ by a finite subgroup $X_{\mathrm{nr}}(L,\omega)$.

Let $\mathrm{Irr}_{\mathrm{temp}}(G)$ be the set of irreducible tempered G-representations (still considered up to isomorphism), and write

$$\mathrm{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G) = \mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}_{\mathrm{temp}}(G).$$

Let $T_{\mathfrak{s},\mathrm{un}}$ be the set of unitary representations in $T_{\mathfrak{s}}$. It is a real compact subtorus and

$$T_{\mathfrak{s}} = T_{\mathfrak{s},\mathrm{un}} \times \mathrm{Hom}_{\mathbb{Z}}(X^*(T_{\mathfrak{s}}), \mathbb{R}_{>0}). \tag{2}$$

Then $T_{\mathfrak{s},\mathrm{un}} \cong X_{\mathrm{unr}}(L)/X_{\mathrm{nr}}(L,\omega)$, where $X_{\mathrm{unr}}(L)$ is the group of unitary unramified characters of L.

Conjecture 1. [Bijection with extended quotients]
There exist a family of 2-cocycles $\natural$ *and a bijection*

$$\mathrm{Irr}^{\mathfrak{s}}(G) \longleftrightarrow (T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_\natural \tag{3}$$

such that:

- *It restricts to a bijection* $\mathrm{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G) \longleftrightarrow (T_{\mathfrak{s},\mathrm{un}}/\!/W_{\mathfrak{s}})_\natural$*, and* (3) *is determined by this restriction.*
- *Suppose that* $\pi \in \mathrm{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G)$ *is mapped to* $[t,\rho] \in (T_{\mathfrak{s},\mathrm{un}}/\!/W_{\mathfrak{s}})_\natural$*. Then* $W_{\mathfrak{s}}t \in T_{\mathfrak{s},\mathrm{un}}/W_{\mathfrak{s}}$ *is the unitary part of the cuspidal support of* π *(an element of* $T_{\mathfrak{s}}/W_{\mathfrak{s}}$*), with respect to the polar decomposition* (2).

It is already clear how (3) should be determined by its restriction to tempered representations. Namely, by some kind of analytic continuation, as in [ABPS1]. Together with the second bullet this already determines the $T_{\mathfrak{s}}$-coordinates of (3).

Let $t \in T_{\mathfrak{s}}$. Define $W_{\mathfrak{s},t}$ as the isotropy group of t in $W_{\mathfrak{s}}$, with respect to the canonical action of $W(G,L)$ on $\mathrm{Irr}(L)$. It is a semi-direct product

$$W_{\mathfrak{s},t} = W(R_{\mathfrak{s},t}) \rtimes \mathfrak{R}_{\mathfrak{s},t},$$

where $W(R_{\mathfrak{s},t})$ is the Weyl group of a root system $R_{\mathfrak{s},t}$ attached by Harish-Chandra to t by means of zeros of the μ-function [Wal, V.2], and $\mathfrak{R}_{\mathfrak{s},t}$ is the associated R-group in $W_{\mathfrak{s},t}$, see [Sil1] and [ABPS1, §1]. The group $W_{\mathfrak{s},t}$ may be viewed as the "Weyl group" of a (possibly disconnected) group $\check{G}_{\mathfrak{s},t}$ with connected component $\check{G}^0_{\mathfrak{s},t}$ the complex Lie group with Weyl group $W(R_{\mathfrak{s},t})$. The Springer correspondence for $\check{G}^0_{\mathfrak{s},t}$ has been extended to the group $\check{G}_{\mathfrak{s},t}$ in [ABPS4, §4]. For a given irreducible representation ρ of $W_{\mathfrak{s},t}$, we will call the pair attached to it by this extended Springer correspondence the *Springer parameter of* ρ.

Conjecture 2. [L-packets]
Assume that a local Langlands correspondence exists for $\mathrm{Irr}^{\mathfrak{s}}(G)$.

- *The 2-cocycle* $\natural(t) : W_{\mathfrak{s},t} \times W_{\mathfrak{s},t} \to \mathbb{C}^\times$ *factors through* $W_{\mathfrak{s},t}/W(R_{\mathfrak{s},t}) \times W_{\mathfrak{s},t}/W(R_{\mathfrak{s},t})$.
- *The bijection* (3) *is canonical up to permutations within L-packets in* $\mathrm{Irr}^{\mathfrak{s}}(G)$.
- *Two G-representations with images $[t,\rho]$ and $[t',\rho']$ belong to the same L-packet if and only if there is a $w \in W_{\mathfrak{s}}$ such that $wt' = t$ and the $W(R_{\mathfrak{s},t})$-representations ρ and $w\cdot\rho'$ have Springer parameters with the same unipotent class (in the complex reductive group with maximal torus $T_{\mathfrak{s}}$, root system $R_{\mathfrak{s},t}$ and "Weyl group" $W_{\mathfrak{s},t}$).*

The first bullet ensures that $\mathbb{C}[W_{\mathfrak{s},t}, \natural(t)]$ contains $\mathbb{C}[W(R_{\mathfrak{s},t})]$, which is necessary for ρ to be a linear (i.e. not projective) representation of $W(R_{\mathfrak{s},t})$. Although $\rho|_{W(R_{\mathfrak{s},t})}$ may be reducible, all its irreducible constituents are $W_{\mathfrak{s},t}$-conjugate. Therefore the unipotent parts of their Springer parameters are in the same conjugacy class in the indicated complex reductive group.

This conjecture implies that the intersection of $\mathrm{Irr}^{\mathfrak{s}}(G)$ with the L-packet of $[t,\rho]$ is in bijection with a set of projective representations of a certain subgroup of $W_{\mathfrak{s},t}/W(R_{\mathfrak{s},t})$. This can be compared with the conjectures about R-groups and L-packets in [Art] and [ABPS1, §5].

We expect that the 2-cocycles $\natural(t)$ are trivial whenever G is split. However, $\natural(t)$ does not always represent the neutral element of $H^2(W_{\mathfrak{s},t}, \mathbb{C}^\times)$. In [ABPS4, Example 5.6] we worked out a Bernstein component for $G = GL_5(D)_{\mathrm{der}}$ (with D a 4-dimensional noncommutative division algebra over F), for which $\natural(t)$ is not trivial.

Recall that the Harish-Chandra–Schwartz algebra $\mathcal{S}(G)$ has $\mathrm{Irr}_{\mathrm{temp}}(G)$ as its space of irreducible representations. Let $\mathcal{S}(G)^{\mathfrak{s}} \subset \mathcal{S}(G)$ be the ideal corresponding to $\mathrm{Irr}^{\mathfrak{s}}(G)$.

Conjecture 3. [Schwartz algebras]
There exist a projective representation $V_{\mathfrak{s}}$ of $X_{\mathrm{nr}}(L,\omega) \rtimes W_{\mathfrak{s}}$ and a homomorphism of topological algebras

$$\zeta^{\mathfrak{s}}_G : (C^\infty(X_{\mathrm{unr}}(L)) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mathfrak{s}}))^{X_{\mathrm{nr}}(L,\omega)} \rtimes W_{\mathfrak{s}} \longrightarrow \mathcal{S}(G)^{\mathfrak{s}}$$

such that:

- *There are canonical bijections*

$$(T_{\mathfrak s,\mathrm{un}}/\!/W_{\mathfrak s})_\natural \longleftrightarrow (X_{\mathrm{unr}}(L)/\!/X_{\mathrm{nr}}(L,\omega)\rtimes W_{\mathfrak s})_\natural \longleftrightarrow \mathrm{Irr}\big((C^\infty(X_{\mathrm{unr}}(L))\otimes \mathrm{End}_{\mathbb C}(V_{\mathfrak s}))^{X_{\mathrm{nr}}(L,\omega)}\rtimes W_{\mathfrak s}\big). \tag{4}$$

- *The morphism $\zeta_G^{\mathfrak s}$ is spectrum preserving with respect to filtrations (see Appendix A).*
- *The map* $\mathrm{Irr}^{\mathfrak s}_{\mathrm{temp}}(G) = \mathrm{Irr}(\mathcal S(G)^{\mathfrak s}) \to (T_{\mathfrak s,\mathrm{un}}/\!/W_{\mathfrak s})_\natural$ *induced by $\zeta_G^{\mathfrak s}$ and* (4) *equals* (1).

The first bullet is true by the general principle Lemma B.3, if the family of 2-cocycles $\natural$ satisfies a mild condition. The shape of the domain of $\zeta_G^{\mathfrak s}$ is motivated by the Fourier transform of $\mathcal S(G)^{\mathfrak s}$, see [Mis, Wal]. Indeed, we expect that the relation between these two algebras is that certain parameters $q \in \mathbb R_{>1}$ associated to $\mathcal S(G)^{\mathfrak s}$ are changed to 1. The second bullet is a generalization of results known for Schwartz completions of affine Hecke algebras [Sol2]. This part of the conjecture replaces the affine Hecke algebras appearing in earlier versions [ABP]. The new version is more flexible because it avoids the use of asymptotic Hecke algebras.

We have already verified Conjectures 1 and 2 for principal series representations of split reductive p-adic groups in [ABPS5]. That situation is simpler than the general case, because all the 2-cocycles are trivial. Conjecture 3 also holds for those representations, that is a consequence of [ABPS5, Theorems 11.2 and 15.1] and Lemma 6.5.

Our conjectures interact with the local Langlands correspondence (LLC), mainly because elements of $(T_{\mathfrak s}/\!/W_{\mathfrak s})_\natural$ are rather close to Langlands parameters. This was used in [ABPS5, §16] to establish the LLC for for principal series representations of split reductive p-adic groups (except for a few cases in which the residual characteristic of F is bad for G).

If one accepts that a local Langlands correspondence exists for all supercuspidal representations of Levi subgroups of G, then (1) can be transferred to a similar bijection for Langlands parameters (which in examples is easier to prove that (1) itself). In this way our conjectures help to reduce the local Langlands correspondence for G to that for supercuspidal representations of its Levi subgroups. Conjecture 3 can be regarded as an outline to prove a part of the LLC for G.

Let D be a central simple F-algebra with $\dim_F(D) = d^2$. Then $G := \mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_{md}(F)$ and $G^\sharp := \mathrm{GL}_m(D)_{\mathrm{der}}$ is an inner form of $\mathrm{SL}_{md}(F)$. As announced, we will prove the above conjectures for G and $G^\sharp$. This relies heavily on our earlier paper [ABPS4], whose main results we recall in Section 1.

Most of the work for the conjectures for G was already done in [ABPS4], the remainder is contained in Theorems 4.3 and 5.3 and Lemmas 5.2 and 6.5.

Our strategy for $G^\sharp$ is based on restriction of a Bernstein component $\mathrm{Irr}^{\mathfrak s}(G)$ to $G^\sharp$, as in [ABPS4, §2.2]. This yields a finite number of Bernstein components for $G^\sharp$. The relation between the Bernstein tori for G and those for $G^\sharp$ can be formulated nicely in terms of extended quotients, see Section 2 and the first part of Section 4. In Section 3 we use the Hecke algebras for Bernstein components of $G^\sharp$ (as computed in [ABPS4]), as well as Lusztig's asymptotic Hecke algebras, to establish geometric equivalences (see Appendix A) between the appropriate algebras. These are used to prove Conjecture 1 for $G^\sharp$ in Theorem 4.4 and Lemma 4.5.

Then we invoke the LLC for $G^\sharp$, known from [HiSa] and [ABPS3], to compare the L-packets with these twisted extended quotients. Conjecture 2 for $G^\sharp$ is established in Lemma 5.5.

Finally, in Section 6 we turn to the Schwartz algebras for G and $G^\sharp$. The Hecke algebras for Bernstein components of G and $G^\sharp$ are closely related to affine Hecke algebras. Likewise, the Schwartz algebras for Bernstein components of G and $G^\sharp$ turn out to be isomorphic to some algebras derived from Schwartz completions of affine Hecke algebras [DeOp]. The link is established in Subsection 6.1, by comparing the Fourier transforms of these algebras. Then we apply some techniques [Sol2] for affine Hecke algebras and their Schwartz completions to prove Conjecture 3 for $G^\sharp$ (Corollary 6.7).

1. Preliminaries

We start with some generalities, to fix the notations. Then we recall the main results of [ABPS4].

Let G be a connected reductive group over a local non-archimedean field F of residual characteristic p. All our representations are tacitly assumed to be smooth and over the complex numbers. We write $\mathrm{Rep}(G)$ for the category of such G-representations and $\mathrm{Irr}(G)$ for the collection of isomorphism classes of irreducible representations therein.

Let P be a parabolic subgroup of G with Levi factor L. The "Weyl" group of L is $W(G,L) = N_G(L)/L$. It acts on equivalence classes of L-representations π by

$$(w \cdot \pi)(g) = \pi(\bar{w} g \bar{w}^{-1}),$$

where $\bar{w} \in N_G(L)$ is a chosen representative for $w \in W(G,L)$. We write

$$W_\pi = \{w \in W(G,L) \mid w \cdot \pi \cong \pi\}.$$

Let ω be an irreducible supercuspidal L-representation. The inertial equivalence class $\mathfrak{s} = [L,\omega]_G$ gives rise to a category of smooth G-representations $\mathrm{Rep}^{\mathfrak{s}}(G)$ and a subset $\mathrm{Irr}^{\mathfrak{s}}(G) \subset \mathrm{Irr}(G)$. Write $X_{\mathrm{nr}}(L)$ for the group of unramified characters $L \to \mathbb{C}^\times$. Then $\mathrm{Irr}^{\mathfrak{s}}(G)$ consists of all irreducible irreducible constituents of the parabolically induced representations $I_P^G(\omega \otimes \chi)$ with $\chi \in X_{\mathrm{nr}}(L)$. We note that I_P^G always means normalized, smooth parabolic induction from L via P to G.

The set $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ with $\mathfrak{s}_L = [L,\omega]_L$ can be described explicitly, namely by

$$X_{\mathrm{nr}}(L,\omega) = \{\chi \in X_{\mathrm{nr}}(L) : \omega \otimes \chi \cong \omega\}, \tag{5}$$

$$\mathrm{Irr}^{\mathfrak{s}_L}(L) = \{\omega \otimes \chi : \chi \in X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)\}. \tag{6}$$

Several objects are attached to the Bernstein component $\mathrm{Irr}^{\mathfrak{s}}(G)$ of $\mathrm{Irr}(G)$ [BeDe]. Firstly, there is the torus

$$T_{\mathfrak{s}} := X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega),$$

which is homeomorphic to $\mathrm{Irr}^{\mathfrak{s}_L}(L)$. Secondly, we have the groups

$$\begin{aligned} N_G(\mathfrak{s}_L) =& \{g \in N_G(L) \mid g \cdot \omega \in \mathrm{Irr}^{\mathfrak{s}_L}(L)\} \\ =& \{g \in N_G(L) \mid g \cdot [L,\omega]_L = [L,\omega]_L\}, \\ W_{\mathfrak{s}} :=& \{w \in W(G,L) \mid w \cdot \omega \in \mathrm{Irr}^{\mathfrak{s}_L}(L)\} = N_G(\mathfrak{s}_L)/L. \end{aligned}$$

Of course $T_\mathfrak{s}$ and $W_\mathfrak{s}$ are only determined up to isomorphism by $\mathfrak{s}$, actually they depend on $\mathfrak{s}_L$. To cope with this, we tacitly assume that $\mathfrak{s}_L$ is known when talking about $\mathfrak{s}$.

The choice of $\omega \in \mathrm{Irr}^{\mathfrak{s}_L}(L)$ fixes a bijection $T_\mathfrak{s} \to \mathrm{Irr}^{\mathfrak{s}_L}(L)$, and via this bijection the action of $W_\mathfrak{s}$ on $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ is transferred to $T_\mathfrak{s}$. The finite group $W_\mathfrak{s}$ can be thought of as the "Weyl group" of $\mathfrak{s}$, although in general it is not generated by reflections.

Let $C_c^\infty(G)$ be the vector space of compactly supported locally constant functions $G \to \mathbb{C}$. The choice of a Haar measure on G determines a convolution product $*$ on $C_c^\infty(G)$. The algebra $(C_c^\infty(G), *)$ is known as the Hecke algebra $\mathcal{H}(G)$. There is an equivalence between $\mathrm{Rep}(G)$ and the category $\mathrm{Mod}(\mathcal{H}(G))$ of $\mathcal{H}(G)$-modules V such that $\mathcal{H}(G) \cdot V = V$. We denote the collection of inertial equivalence classes for G by $\mathfrak{B}(G)$. The Bernstein decomposition

$$\mathrm{Rep}(G) = \prod\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}^\mathfrak{s}(G)$$

induces a factorization in two-sided ideals

$$\mathcal{H}(G) = \prod\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^\mathfrak{s}.$$

From now on we discuss things that are specific for $G = \mathrm{GL}_m(D)$, where D is a central simple F-algebra. We write $\dim_F(D) = d^2$. Every Levi subgroup L of G is isomorphic to $\prod_j \mathrm{GL}_{\tilde m_j}(D)$ for some $\tilde m_j \in \mathbb{N}$ with $\sum_j \tilde m_j = m$. Hence every irreducible L-representation ω can be written as $\otimes_j \tilde\omega_j$ with $\tilde\omega_j \in \mathrm{Irr}(\mathrm{GL}_{\tilde m_j}(D))$. Then ω is supercuspidal if and only if every $\tilde\omega_j$ is so. As above, we assume that this is the case. Replacing (L, ω) by an inertially equivalent pair allows us to make the following simplifying assumptions:

Condition 1.1.

- *if* $\tilde m_i = \tilde m_j$ *and* $[\mathrm{GL}_{\tilde m_j}(D), \tilde\omega_i]_{\mathrm{GL}_{\tilde m_j}(D)} = [\mathrm{GL}_{\tilde m_j}(D), \tilde\omega_j]_{\mathrm{GL}_{\tilde m_j}(D)}$, *then* $\tilde\omega_i = \tilde\omega_j$*;*
- $\omega = \prod_i \omega_i^{\otimes e_i}$, *such that* ω_i *and* ω_j *are not inertially equivalent if* $i \neq j$*;*
- $L = \prod_i L_i^{e_i} = \prod_i \mathrm{GL}_{m_i}(D)^{e_i}$, *embedded diagonally in* $\mathrm{GL}_m(D)$ *such that factors* L_i *with the same* (m_i, e_i) *are in subsequent positions;*
- *as representatives for the elements of* $W(G, L)$ *we take permutation matrices;*
- P *is the parabolic subgroup of* G *generated by* L *and the upper triangular matrices;*
- *if* $m_i = m_j, e_i = e_j$ *and* ω_i *is isomorphic to* $\omega_j \otimes \gamma$ *for some character* γ *of* $\mathrm{GL}_{m_i}(D)$, *then* $\omega_i = \omega_j \otimes \gamma\chi$ *for some* $\chi \in X_{\mathrm{nr}}(\mathrm{GL}_{m_i}(D))$.

Most of the time we will not need the conditions for stating the results, but they are useful in many proofs. Under Conditions 1.1 we define

$$M_i = Z_G\big(\prod\nolimits_{j \neq i} L_j^{e_j}\big) = \mathrm{GL}_{m_i e_i}(D), \tag{7}$$

then $\prod_i M_i$ is a Levi subgroup of G containing L. For $\mathfrak{s} = [L, \omega]_G$ we have

$$W_\mathfrak{s} = N_{\prod_i M_i}(L)/L = \prod\nolimits_i N_{M_i}(L_i^{e_i})/L_i^{e_i} \cong \prod\nolimits_i S_{e_i}, \tag{8}$$

a direct product of symmetric groups. Writing $\mathfrak{s}_i = [L_i, \omega_i]_{L_i}$, the torus associated to $\mathfrak{s}$ becomes

$$T_{\mathfrak{s}} = \prod\nolimits_i (T_{\mathfrak{s}_i})^{e_i} = \prod\nolimits_i T_i, \tag{9}$$

$$T_{\mathfrak{s}_i} = X_{\mathrm{nr}}(L_i)/X_{\mathrm{nr}}(L_i, \omega_i). \tag{10}$$

By our choice of representatives for $W(G, L)$, $\omega_i^{\otimes e_i}$ is stable under $N_{M_i}(L_i^{e_i})/L_i^{e_i} \cong S_{e_i}$. If $R_i \subset X^*(\prod_i T_i)$ denotes the coroot system of $(M_i, L_i^{e_i})$, we can identify S_{e_i} with $W(R_i)$. The action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ is just permuting coordinates in the standard way and

$$W_{\mathfrak{s}} = W_{\omega}. \tag{11}$$

The reduced norm map $D \to F$ gives rise to a group homomorphism $\mathrm{Nrd} : G \to F^\times$. We denote its kernel by $G^\sharp$, so $G^\sharp$ is also the derived group of G. For subgroups $H \subset G$ we write

$$H^\sharp = H \cap G^\sharp.$$

In [ABPS4] we determined the shape of the Hecke algebras associated to types for $G^\sharp$, starting with those for G. As an intermediate step, we did this for the group $G^\sharp Z(G)$, where $Z(G) \cong F^\times$ denotes the centre of G. The advantage is that the comparison between $G^\sharp$ and $G^\sharp Z(G)$ is easy, while $G^\sharp Z(G) \subset G$ can be treated as an extension of finite index. In fact it is a subgroup of finite if p does not divide md. In case p does divide md, the quotient $G/G^\sharp Z(G)$ is compact and similar techniques can be applied.

For an inertial equivalence class $\mathfrak{s} = [L, \omega]_G$ we define $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp)$ as the set of irreducible $G^\sharp$-representations that are subquotients of $\mathrm{Res}^G_{G^\sharp}(\pi)$ for some $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G)$, and $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$ as the collection of $G^\sharp$-representations all whose irreducible subquotients lie in $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp)$. We want to investigate the category $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$. It is a product of finitely many Bernstein blocks for $G^\sharp$ (see [ABPS4]):

$$\mathrm{Rep}^{\mathfrak{s}}(G^\sharp) = \prod\nolimits_{\mathfrak{t}^\sharp \prec \mathfrak{s}} \mathrm{Rep}^{\mathfrak{t}^\sharp}(G^\sharp). \tag{12}$$

We note that the Bernstein components $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ which are subordinate to one $\mathfrak{s}$ (i.e., such that $\mathfrak{t}^\sharp \prec \mathfrak{s}$) form precisely one class of L-indistinguishable components: every L-packet for $G^\sharp$ which intersects one of them intersects them all.

Analogously we define $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp Z(G))$, and we obtain

$$\mathrm{Rep}^{\mathfrak{s}}(G^\sharp Z(G)) = \prod\nolimits_{\mathfrak{t} \prec \mathfrak{s}} \mathrm{Rep}^{\mathfrak{t}}(G^\sharp Z(G)),$$

where the $\mathfrak{t}$ are inertial equivalence classes for $G^\sharp Z(G)$.

The restriction of $\mathfrak{t}$ to $G^\sharp$ is a single inertial equivalence class $\mathfrak{t}^\sharp$, and by [ABPS4, (43)]:

$$T_{\mathfrak{t}^\sharp} = T_{\mathfrak{t}}/X_{\mathrm{nr}}(\mathrm{Nrd}(Z(G))). \tag{13}$$

For $\pi \in \mathrm{Irr}(G)$ we put

$$X^G(\pi) := \{\gamma \in \mathrm{Irr}(G/G^\sharp) : \gamma \otimes \pi \cong \pi\}.$$

The same notation will be used for representations of parabolic subgroups of G which admit a central character. For every $\gamma \in X^L(\pi)$ there exists a nonzero intertwining operator

$$I(\gamma, \pi) \in \mathrm{Hom}_G(\pi \otimes \gamma, \pi) = \mathrm{Hom}_G(\pi, \pi \otimes \gamma^{-1}), \tag{14}$$

which is unique up to a scalar. As $G^\sharp \subset \ker(\gamma)$, $I(\gamma, \pi)$ can also be considered as an element of $\mathrm{End}_{G^\sharp}(\pi)$. As such, these operators determine a 2-cocycle κ_π by

$$I(\gamma,\pi) \circ I(\gamma',\pi) = \kappa_\pi(\gamma,\gamma') I(\gamma\gamma',\pi). \tag{15}$$

By [HiSa, Lemma 2.4] they span the $G^\sharp$-intertwining algebra of π:

$$\mathrm{End}_{G^\sharp}(\mathrm{Res}^G_{G^\sharp}\pi) \cong \mathbb{C}[X^G(\pi), \kappa_\pi], \tag{16}$$

where the right hand side denotes the twisted group algebra of $X^G(\pi)$. Furthermore by [HiSa, Corollary 2.10]

$$\mathrm{Res}^G_{G^\sharp}\pi \cong \bigoplus_{\rho \in \mathrm{Irr}(\mathbb{C}[X^G(\pi),\kappa_\pi])} \mathrm{Hom}_{\mathbb{C}[X^G(\pi),\kappa_\pi]}(\rho,\pi) \otimes \rho \tag{17}$$

as representations of $G^\sharp \times X^G(\pi)$.

The analogous groups for $\mathfrak{s} = [L,\omega]_G$ and $\mathfrak{s}_L = [L,\omega]_L$ are

$$X^L(\mathfrak{s}) := \{\gamma \in \mathrm{Irr}(L/L^\sharp Z(G)) : \gamma \otimes \omega \in [L,\omega]_L\},$$
$$X^G(\mathfrak{s}) := \{\gamma \in \mathrm{Irr}(G/G^\sharp Z(G)) : \gamma \otimes I_P^G(\omega) \in [L,\omega]_G\}.$$

The role of the group $W_\mathfrak{s}$ for $\mathrm{Rep}(G^\sharp)^\mathfrak{s}$ is played by

$$W_\mathfrak{s}^\sharp := \{w \in W(G,L) \mid \exists \gamma \in \mathrm{Irr}(L/L^\sharp Z(G)) \text{ such that } w(\gamma \otimes \omega) \in [L,\omega]_L\}$$

By [ABPS4, Lemma 2.3]

$$W_\mathfrak{s}^\sharp = W_\mathfrak{s} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp \text{ , where } \mathfrak{R}_\mathfrak{s}^\sharp = W_\mathfrak{s}^\sharp \cap N_G(P \cap \prod\nolimits_i M_i)/L. \tag{18}$$

while [ABPS4, Lemma 2.4.d] says that

$$X^G(\mathfrak{s})/X^L(\mathfrak{s}) \cong \mathfrak{R}_\mathfrak{s}^\sharp. \tag{19}$$

Now we collect some notations which are needed specifically to state the final results of [ABPS4].

From [SéSt1] we know that there exists a simple type (K,λ) for $[L,\omega]_M$, and in [SéSt2] it was shown to admit a G-cover (K_G,λ_G). We denote the associated central idempotent of $\mathcal{H}(K)$ by e_λ, and similarly for other irreducible representations. Then $V_\lambda = e_\lambda V_\omega$.

For the restriction process we need an idempotent that is invariant under $X^G(\mathfrak{s})$. To that end we replace λ_G by the sum of the representations $\gamma \otimes \lambda_G$ with $\gamma \in X^G(\mathfrak{s})$, which we call μ_G. Of course

$$V_\mu := \sum\nolimits_{\gamma \in X^G(\mathfrak{s})} e_{\gamma\otimes\lambda} V_\omega$$

is reducible as a representation of K.

In [ABPS4, (128)] we defined a finite dimensional subspace $V_\mu \subset V_\omega$ which is stable under the operators $I(\gamma,\omega)$ with $\gamma \in X^L(\mathfrak{s})$. In [ABPS4, (91)] we constructed an idempotent $e_{\mu_G} \in \mathcal{H}(G)$ which is supported on a compact open subgroup $K_G \subset G$. It follows from the work of Sécherre and Stevens [SéSt2] that $e_{\mu_G}\mathcal{H}(G)e_{\mu_G}$ is Morita equivalent with $\mathcal{H}(G)^\mathfrak{s}$. By [Séc] and [ABPS4, Proposition 3.15] there is an affine Hecke algebra $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ such that

$$e_{\mu_G}\mathcal{H}(G)e_{\mu_G} \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes_\mathbb{C} \mathbb{C}\mathfrak{R}_\mathfrak{s}^\sharp). \tag{20}$$

The groups $X^G(\mathfrak{s})$ and $X_{\rm nr}(G)$ act on $e_{\mu_G}\mathcal{H}(G)e_{\mu_G}$ by pointwise multiplication of functions $G \to \mathbb{C}$ with characters of G. However, for technical reasons we use the action

$$\alpha_\gamma(f)(g) = \gamma^{-1}(g)f(g) \qquad f \in \mathcal{H}(G), \gamma \in \mathrm{Irr}(G/G^\sharp), g \in G. \tag{21}$$

The action on the right hand side of (20) preserves the tensor factors, and on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}\mathfrak{R}_{\mathfrak{s}}^\sharp)$ it is the natural action of $X^G(\mathfrak{s})/X^L(\mathfrak{s}) \cong \mathfrak{R}_{\mathfrak{s}}^\sharp$.

Although e_{μ_G} looks like the idempotent of a type, it is not clear whether it is one, because the associated K_G-representation is reducible and no more suitable compact subgroup of G is in sight. Let $e_{\mu_{G^\sharp}}$ (respectively $e_{\mu_{G^\sharp Z(G)}}$) be the restriction of $e_{\mu_G} : G \to \mathbb{C}$ to $G^\sharp$ (resp. $G^\sharp Z(G)$). We normalize the Haar measure on $G^\sharp$ (resp. $G^\sharp Z(G)$) such that it becomes an idempotent in $\mathcal{H}(G^\sharp)$ (resp. $\mathcal{H}(G^\sharp Z(G))$).

In [ABPS4, Lemma 3.3] we can constructed a certain finite set $[L/H_\lambda]$, consisting of representatives for a normal subgroup $H_\lambda \subset L$. Consider the elements

$$\begin{aligned} e^\sharp_{\lambda_G} &:= \sum\nolimits_{a \in [L/H_\lambda]} a e_{\mu_G} a^{-1} \in \mathcal{H}(G), \\ e^\sharp_{\lambda_{G^\sharp Z(G)}} &:= \sum\nolimits_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp Z(G)}} a^{-1} \in \mathcal{H}(G^\sharp Z(G)), \\ e^\sharp_{\lambda_{G^\sharp}} &:= \sum\nolimits_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp}} a^{-1} \in \mathcal{H}(G^\sharp). \end{aligned} \tag{22}$$

It follows from [ABPS4, Lemma 3.12] that they are again idempotent. Notice that $e^\sharp_{\lambda_G}$ detects the same category of G-representations as e_{μ_G}, namely $\mathrm{Rep}^{\mathfrak{s}}(G)$. In the proof of [ABPS4, Proposition 3.15] we established that (20) extends to an isomorphism

$$e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G} \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes_{\mathbb{C}} \mathbb{C}\mathfrak{R}_{\mathfrak{s}}^\sharp) \otimes M_{[L:H_\lambda]}(\mathbb{C}). \tag{23}$$

Theorem 1.2. [ABPS4, Theorem 4.13]

(a) *$\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ is Morita equivalent with its subalgebra*

$$e^\sharp_{\lambda_{G^\sharp Z(G)}}\mathcal{H}(G^\sharp Z(G))e^\sharp_{\lambda_{G^\sharp Z(G)}} = \bigoplus\nolimits_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp Z(G)}} a^{-1}\mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1}$$

(b) *Each of the algebras $a e_{\mu_{G^\sharp Z(G)}} a^{-1}\mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1}$ is isomorphic to*

$$\big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp. \tag{24}$$

(c) *Under these isomorphisms the action of $X_{\rm nr}(G)$ on $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ becomes the action of $X_{\rm nr}(L/L^\sharp) \cong X_{\rm nr}(G)$ on (24) via translations on $T_{\mathfrak{s}}$.*

To describe the Hecke algebras for $G^\sharp$ in similar terms, let $T_{\mathfrak{s}}^\sharp$ be the restriction of $T_{\mathfrak{s}}$ to $L^\sharp$, that is,

$$T_{\mathfrak{s}}^\sharp := T_{\mathfrak{s}}/X_{\rm nr}(G) = T_{\mathfrak{s}}/X_{\rm nr}(L/L^\sharp) \cong X_{\rm nr}(L^\sharp)/X_{\rm nr}(L,\omega), \tag{25}$$

where $X_{\rm nr}(L/L^\sharp)$ denotes the group of unramified characters of L which are trivial on $L^\sharp$. With this torus we build an affine Hecke algebra $\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}})$.

Theorem 1.3. [ABPS4, Theorem 4.15]

(a) *$\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ is Morita equivalent with*

$$e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}} = \bigoplus\nolimits_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp}} a^{-1}\mathcal{H}(G^\sharp) a e_{\mu_{G^\sharp}} a^{-1}$$

(*b*) *Each of the algebras* $a e_{\mu_{G^\sharp}} a^{-1} \mathcal{H}(G^\sharp) a e_{\mu_{G^\sharp}} a^{-1}$ *is isomorphic to*

$$\big(\mathcal{H}(T_{\mathfrak s}^\sharp, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu) \big)^{X^L (\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp .$$

Let us describe the above actions of the group $X^G (\mathfrak s)$ explicitly. The action on

$$a e_{\mu_{G^\sharp Z(G)}} a^{-1} \mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1} \cong \mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu). \tag{26}$$

does not depend on $a \in [L / H_\lambda]$ because

$$\gamma \cdot (a f a^{-1}) = a (\gamma \cdot f) a^{-1} \qquad f \in \mathcal{H}(G).$$

For another way to view $X^G (\mathfrak s)$, we start with

$$\mathrm{Stab}(\mathfrak s) := \{ (w,\gamma) \in N_G (L) / L \times \mathrm{Irr}(L / L^\sharp Z(G)) \mid w (\gamma \otimes \omega) \in [L,\omega]_L \} .$$

The normal subgroup $W_{\mathfrak s}$ has a complement:

$$\mathrm{Stab}(\mathfrak s) = \mathrm{Stab}(\mathfrak s, P \cap \prod\nolimits_i M_i) \ltimes W_{\mathfrak s} := \mathrm{Stab}(\mathfrak s)^+ \ltimes W_{\mathfrak s}$$

$$\mathrm{Stab}(\mathfrak s)^+ := \{ (w,\gamma) \in N_G (P \cap \prod\nolimits_i M_i) / L \times \mathrm{Irr}(L / L^\sharp Z(G)) \mid w (\gamma \otimes \omega) \in [L,\omega]_L \}$$

By [ABPS4, Lemma 2.4.a] projection of $\mathrm{Stab}(\mathfrak s)$ on the second coordinate gives an isomorphism

$$X^G (\mathfrak s) \cong \mathrm{Stab}(\mathfrak s) / W_{\mathfrak s} \cong \mathrm{Stab}(\mathfrak s)^+ \tag{27}$$

In particular

$$\mathrm{Stab}(\mathfrak s)^+ / X^L (\mathfrak s) \cong \mathfrak{R}_{\mathfrak s}^\sharp . \tag{28}$$

This yields an action α of $\mathrm{Stab}(\mathfrak s)^+$ on (26). As in [ABPS4, (159)–(161)] we choose $\chi_\gamma \in X_{\mathrm{nr}}(L)^{W_{\mathfrak s}}$ for $(w,\gamma) \in \mathrm{Stab}(\mathfrak s)^+$, such that

$$w(\omega) \otimes \gamma \cong \omega \otimes \chi_\gamma . \tag{29}$$

Notice that χ_γ is unique up to $X_{\mathrm{nr}}(L,\omega)$. Furthermore we choose an invertible

$$J(\gamma, \omega \otimes \chi_\gamma^{-1}) \in \mathrm{Hom}_L (\omega \otimes \chi_\gamma^{-1}, w^{-1}(\omega) \otimes \gamma^{-1}). \tag{30}$$

This generalizes (14) in the sense that

$$J(\gamma, \omega \otimes \chi_\gamma^{-1}) = I(\gamma,\omega) \quad \text{if} \quad \gamma \in X^L (\omega) \text{ and } \chi_\gamma = 1.$$

We may assume that

$$\chi_\gamma = \gamma \quad \text{and} \quad J(\gamma, \omega \otimes \chi_\gamma^{-1}) = \mathrm{id}_{V_\omega} \quad \text{if} \quad \gamma \in X_{\mathrm{nr}}(L / L^\sharp Z(G)). \tag{31}$$

By definition [ABPS4, (119)] the algebra $\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$ has a $\mathbb C$-basis $\{ \theta_x [w] : x \in X^* (T_{\mathfrak s}), w \in W_{\mathfrak s} \}$ such that

- the span of the θ_x is identified with the algebra $\mathcal{O}(T_{\mathfrak s})$ of regular functions on $T_{\mathfrak s}$;
- the span of the $[w]$ is the finite dimensional Iwahori–Hecke algebra $\mathcal{H}(W_{\mathfrak s}, q_{\mathfrak s})$;
- the multiplication between these two subalgebras is given by

$$f[s] - [s](s \cdot f) = (q_{\mathfrak s}(s) - 1)(f - (s \cdot f))(1 - \theta_{-\alpha})^{-1} \qquad f \in \mathcal{O}(T_{\mathfrak s}), \tag{32}$$

  for a simple reflection $s = s_\alpha$;
- the parameter function $q_{\mathfrak s}$ is given explicitly in [Séc].

Thus $\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$ is a tensor product of affine Hecke algebras of type GL_e, but written in such a way that the torus $T_{\mathfrak s}$ appears canonically in it (i.e. independent of the choice of a base point of $T_{\mathfrak s}$).

Theorem 1.4. [ABPS4, Lemmas 3.5 and 4.11]

(*a*) *The action of* $\mathrm{Stab}(\mathfrak{s})^+$ *on* $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)$ *in Theorem 1.2 preserves both tensor factors. On* $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ *it is given by*

$$\alpha_{(w,\gamma)}(\theta_x[v]) = \chi_\gamma^{-1}(x)\theta_{w(x)}[wvw^{-1}] \qquad x \in X^*(T_\mathfrak{s}), v \in W_\mathfrak{s},$$

and on $\mathrm{End}_\mathbb{C}(V_\mu)$ *by*

$$\alpha_{(w,\gamma)}(h) = J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}.$$

(*b*) *The subgroup of elements that act trivially is*

$$X^L(\omega, V_\mu) = \{\gamma \in X^L(\omega) \mid I(\gamma, \omega)|_{V_\mu} \in \mathbb{C}^\times \mathrm{id}_{V_\mu}\}.$$

Its cardinality equals $[L : H_\lambda]$.

(*c*) *Part (a) and Theorem 1.2.c also describe the action of* $\mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(G)$ *on* $\mathcal{H}(T_\mathfrak{s}^\sharp, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)$ *in Theorem 1.3. The subgroup of elements that act trivially on this algebra is*

$$X^L(\omega, V_\mu) X_{\mathrm{nr}}(G) = X^L(\omega, V_\mu) X_{\mathrm{nr}}(L/L^\sharp).$$

Let us compare Theorem 1.3 with the situation for $L^\sharp$, which is simpler.

Theorem 1.5. (*a*) *There exist idempotents* $e_L^\mathfrak{s} \in \mathcal{H}(L), e_{L^\sharp}^\mathfrak{s} \in \mathcal{H}(L^\sharp)$ *such that* $\mathcal{H}(L^\sharp)^\mathfrak{s}$ *is Morita equivalent with*

$$e_{L^\sharp}^\mathfrak{s} \mathcal{H}(L^\sharp) e_{L^\sharp}^\mathfrak{s} \cong e_L^\mathfrak{s} \mathcal{H}(L)^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)} e_L^\mathfrak{s} \cong \big(\mathcal{O}(T_\mathfrak{s}^\sharp) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}[L/H_\lambda])\big)^{X^L(\mathfrak{s})}$$
$$\cong \bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal{O}(T_\mathfrak{s}^\sharp) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})/X^L(\omega, V_\mu)}.$$

(*b*) *Under the equivalences from part (a) and Theorem 1.3, the normalized parabolic induction functor*

$$I_{P^\sharp}^{G^\sharp} : \mathrm{Rep}^{\mathfrak{s}_L}(L^\sharp) \to \mathrm{Rep}^\mathfrak{s}(G^\sharp)$$

corresponds to induction from the last algebra in part (a) to

$$\bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal{H}(T_\mathfrak{s}^\sharp, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})/X^L(\omega, V_\mu)} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp.$$

Proof. Part (a) is a consequence of [ABPS4, (169)] and [ABPS4, Lemma 4.8], which shows that $\mathrm{End}_\mathbb{C}(\mathbb{C}[L/H_\lambda]) \cong \mathbb{C}[L/H_\lambda]$.

The analogue of part (b) for L and G says that

$$I_P^G : \mathrm{Rep}^{\mathfrak{s}_L}(L) \to \mathrm{Rep}^\mathfrak{s}(G)$$

corresponds to induction from

$$e_L^\mathfrak{s} \mathcal{H}(L) e_L^\mathfrak{s} \cong \mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}[L/H_\lambda]) \quad \text{to}$$
$$e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}\mathfrak{R}_\mathfrak{s}^\sharp).$$

To see that it is true, we reduce with [ABPS4, Theorem 4.5] to the algebras

$$e_{\lambda_L} \mathcal{H}(L) e_{\lambda_L} \cong \mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\lambda),$$
$$e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\lambda).$$

Then we are in the situation where (K_G, λ_G) is a cover of a type (K_L, λ_L), and the statement about the induction functors follows from [ABPS4, (126)] and [BuKu, Corollary 8.4].

We note that here, for a given algebra homomorphism $\phi : A \to B$, we must use induction in the version $\mathrm{Ind}_A^B(M) = \mathrm{Hom}_A(B, M)$. However, in all the cases we encounter B is free of finite rank as a module over A and it is endowed with a canonical anti-involution

$$f \mapsto [f^\vee : g \mapsto \overline{f(g^{-1})}].$$

Hence we may identify $\mathrm{Hom}_A(B, M) \cong B^* \otimes_A M \cong B \otimes_A M$.

Now we have shown the desired claim for I_P^G. Since $G^\sharp/P^\sharp \cong G/P$, $I_P^G = I_{P^\sharp}^{G^\sharp}$ on $\mathrm{Rep}^{\mathfrak{s}_L}(L)$. The functor $\mathrm{Res}^L_{L^\sharp}$ corresponds to $\mathrm{Res}^{e^{\lambda_L}\mathcal{H}(L)e^{\lambda_L}}_{e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}}$, and $\mathrm{Res}^G_{G^\sharp}$ to restriction from $e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}$ to $e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}}$, which is the algebra appearing in the statement.

This proves part (b) on $\mathrm{Res}^L_{L^\sharp}(\mathrm{Rep}^{\mathfrak{s}_L}(L))$. Since every irreducible $L^\sharp$-representation appears as a summand of an L-representation, this implies the statement on the whole of $\mathrm{Rep}^{\mathfrak{s}_L}(L^\sharp)$. $\square$

2. Bernstein tori

We will determine the Bernstein tori for $G^\sharp Z(G)$ and $G^\sharp$, in terms of those for G. From [ABPS4, (169)] we get a Morita equivalence

$$\mathcal{H}(L^\sharp Z(G))^{\mathfrak{s}_L} \sim_M \big(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda])\big)^{X^L(\mathfrak{s})}. \tag{33}$$

The group $X^L(\mathfrak{s})$ acts on $T_{\mathfrak{s}} = \mathrm{Irr}^{\mathfrak{s}_L}(L)$ by $\pi \mapsto \pi \otimes \gamma$. By [ABPS4, Proposition 2.1] $\mathrm{Res}^L_{L^\sharp}(\omega)$ and $\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi)$ with $\chi \in X_{\mathrm{nr}}(L)$ have a common irreducible subquotient if and only if there is a $\gamma \in X^L(\mathfrak{s})$ such that $\omega \otimes \chi \cong \omega \otimes \chi_\gamma$. Like in (30) we choose a nonzero

$$J(\gamma, \omega) \in \mathrm{Hom}_L(\omega, \omega \otimes \chi_\gamma \gamma^{-1}) = \mathrm{Hom}_L(\omega \otimes \gamma, \omega \otimes \chi_\gamma).$$

Then $J(\gamma, \omega) \in \mathrm{Hom}_{L^\sharp}(\omega, \omega \otimes \chi_\gamma)$ and for every irreducible subquotient $\sigma^\sharp$ of $\mathrm{Res}^L_{L^\sharp}(\omega)$

$$\gamma * (\sigma^\sharp \otimes \chi) : m \mapsto J(\gamma, \omega) \circ (\sigma^\sharp \otimes \chi)(m) \circ J(\gamma, \omega)^{-1} \tag{34}$$

is an irreducible subquotient representation of

$$\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi\chi_\gamma) = \mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi\chi_\gamma\gamma^{-1}).$$

This prompts us to consider

$$X^L(\mathfrak{s}, \sigma^\sharp) := \{\gamma \in X^L(\mathfrak{s}) \mid \gamma * \sigma^\sharp \cong \sigma^\sharp \otimes \chi_\gamma\}. \tag{35}$$

By [ABPS4, Lemma 4.14]

$$\sigma^\sharp \otimes \chi \cong \sigma^\sharp \text{ for all } \chi \in X_{\mathrm{nr}}(L, \omega). \tag{36}$$

Hence the group (35) is well-defined, that is, independent of the choice of the χ_γ. For $\gamma \in X^L(\omega)$ (34) reduces to $\sigma^\sharp \otimes \chi$, so $\gamma \in X^L(\mathfrak{s}, \sigma^\sharp)$. By (31) the same goes for $\gamma \in X_{\mathrm{nr}}(L/L^\sharp Z(G))$, so there is always an inclusion

$$X^L(\omega) X_{\mathrm{nr}}(L/L^\sharp Z(G)) \subset X^L(\mathfrak{s}, \sigma^\sharp). \tag{37}$$

We gathered enough tools to describe the Bernstein tori for $G^\sharp$ and $G^\sharp Z(G)$. Recall that $\mathfrak{s}_L = [L, \omega]_L, T_{\mathfrak{s}} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega)$ and that $T^\sharp_{\mathfrak{s}}$ denotes the "restriction" of $T_{\mathfrak{s}}$ to $L^\sharp$.

Proposition 2.1. *Let $\sigma^\sharp$ be an irreducible subquotient of $\mathrm{Res}^L_{L^\sharp}(\omega)$ and write $\mathfrak{t} = [L^\sharp Z(G), \sigma^\sharp]_{G^\sharp Z(G)}$ and $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$.*

(a) $X^L(\mathfrak{s}, \sigma^\sharp)$ depends only on $\mathfrak{s}_L$, not on the particular $\sigma^\sharp$.
(b) $X_{\mathrm{nr}}(L, \omega)\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}$ is a subgroup of $X_{\mathrm{nr}}(L)$ which contains $X_{\mathrm{nr}}(L/L^\sharp Z(G))$.
(c) $T_{\mathfrak{t}} \cong T_{\mathfrak{s}}/\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\} \cong X_{\mathrm{nr}}(L^\sharp Z(G))/X_{\mathrm{nr}}(L, \omega)\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}$.
(d) $T_{\mathfrak{t}^\sharp} \cong T^\sharp_{\mathfrak{s}}/\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\} \cong X_{\mathrm{nr}}(L^\sharp)/X_{\mathrm{nr}}(L, \omega)\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}$.

Proof. (a) By [ABPS4, Proposition 2.1] every two irreducible subquotients of $\mathrm{Res}^L_{L^\sharp}(\omega)$ are direct summands and are conjugate by an element of L. Given $\gamma \in X^L(\mathfrak{s})$, pick $m_\gamma \in L$ such that

$$\gamma * \sigma^\sharp \cong (\omega(m_\gamma)^{-1} \circ \sigma^\sharp \circ \omega(m_\gamma)) \otimes \chi_\gamma = (m_\gamma \cdot \sigma^\sharp) \otimes \chi_\gamma.$$

For any other irreducible summand $\tau = m_\tau \cdot \sigma^\sharp$ of $\mathrm{Res}^L_{L^\sharp}(\omega)$ we compute

$$\begin{aligned}
\gamma * \tau = \gamma * (m_\tau \cdot \sigma^\sharp) &= J(\gamma, \omega) \circ \omega(m_\tau)^{-1} \circ \sigma^\sharp \circ \omega(m_\tau) \circ J(\gamma, \omega)^{-1} \\
&= (\chi_\gamma \gamma^{-1} \otimes \omega)(m_\tau^{-1}) \circ J(\gamma, \omega) \circ \sigma^\sharp \circ J(\gamma, \omega)^{-1} \circ (\chi_\gamma \gamma^{-1} \otimes \omega)(m_\tau) \\
&\cong \omega(m_\tau^{-1}) \circ (m_\gamma \cdot \sigma^\sharp) \otimes \chi_\gamma \circ \omega(m_\tau) \\
&\cong (m_\tau m_\gamma \cdot \sigma^\sharp) \otimes \chi_\gamma.
\end{aligned}$$

As $L/L^\sharp$ is abelian, we find that $m_\tau m_\gamma \cdot \sigma^\sharp \cong m_\gamma m_\tau \cdot \sigma^\sharp$ and that

$$\gamma * \tau \cong (m_\gamma m_\tau \cdot \sigma^\sharp) \otimes \chi_\gamma = m_\gamma \tau \otimes \chi_\gamma.$$

Writing $L_\tau = \{m \in L \mid m \cdot \tau \cong \tau\}$, we deduce the following equivalences:

$$\gamma * \sigma^\sharp \cong \sigma^\sharp \otimes \chi_\gamma \Leftrightarrow m_\gamma \in L_{\sigma^\sharp} \Leftrightarrow m_\gamma \in m_\tau L_{\sigma^\sharp} m_\tau^{-1} = L_\tau \Leftrightarrow \gamma * \tau \cong \tau \otimes \chi_\gamma.$$

This means that $X^L(\mathfrak{s}, \sigma^\sharp) = X^L(\mathfrak{s}, \tau)$.
(b) By (31) and (37)

$$X_{\mathrm{nr}}(L/L^\sharp Z(G)) \subset \{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}.$$

In view of the uniqueness property of χ_γ the map

$$X^L(\mathfrak{s}) \to X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega) : \gamma \mapsto \chi_\gamma$$

is a group homomorphism with kernel $X^L(\omega)$. Hence the χ_γ form a subgroup of $X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega)$, isomorphic to $X^L(\mathfrak{s})/X^L(\omega)$.
(c) Consider the family of $L^\sharp Z(G)$-representations

$$\{\sigma^\sharp \otimes \chi \mid \chi \in X_{\mathrm{nr}}(L)\}.$$

We have to determine the χ for which $\sigma^\sharp \otimes \chi \cong \sigma^\sharp \in \mathrm{Irr}(L^\sharp Z(G))$. From [ABPS4, Lemma 4.14] we see that this includes all the elements of $X_{\mathrm{nr}}(L, \omega) X_{\mathrm{nr}}(L/L^\sharp Z(G))$. By [ABPS4, Proposition 2.1.b] and part (a), all the remaining γ come from $\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}$. This gives the first isomorphism, and the second follows with part (b).
(d) This is a consequence of part (c) and (13). ☐

Proposition 2.1 entails that for every inertial equivalence class

$$\mathfrak{t} = [L^\sharp Z(G), \sigma^\sharp]_{G^\sharp Z(G)} \prec \mathfrak{s} = [L, \omega]_G$$

the action (34) of $X^L(\mathfrak{s},\sigma^\sharp)$ leads to

$$T_\mathfrak{t} \cong T_\mathfrak{s}/X^L(\mathfrak{s},\sigma^\sharp).$$

However, some of the tori

$$T_\mathfrak{t} = T_{\mathfrak{t}_L} = \mathrm{Irr}^{[L^\sharp Z(G),\sigma^\sharp]_{L^\sharp Z(G)}}(L^\sharp Z(G))$$

associated to inequivalent $\sigma^\sharp \subset \mathrm{Res}^L_{L^\sharp}(\omega)$ can coincide as subsets of $\mathrm{Irr}(L^\sharp Z(G))$. This is caused by elements of $X^L(\mathfrak{s}) \setminus X^L(\mathfrak{s},\sigma^\sharp)$ via the action (34). With (36), (33) and (17) we can write

$$\mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp Z(G)) = \bigcup\nolimits_{\mathfrak{t}_L \prec \mathfrak{s}_L} T_{\mathfrak{t}_L} = \big(T_\mathfrak{s} \times \mathrm{Irr}(\mathbb{C}[X^L(\omega),\kappa_\omega])\big)/X^L(\mathfrak{s}), \tag{38}$$

where $(\omega\otimes\chi,\rho) \in T_\mathfrak{s} \times \mathrm{Irr}(\mathbb{C}[X^L(\omega),\kappa_\omega])$ corresponds to

$$\mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho,\omega\otimes\chi) \;\in\; \mathrm{Irr}(L^\sharp Z(G)).$$

With (13) we can deduce a similar expression for $L^\sharp$:

$$\begin{aligned} \mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp) = \bigcup\nolimits_{\mathfrak{t}^\sharp_L \prec \mathfrak{s}_L} T_{\mathfrak{t}^\sharp_L} &= \big(T^\sharp_\mathfrak{s} \times \mathrm{Irr}(\mathbb{C}[X^L(\omega),\kappa_\omega])\big)/X^L(\mathfrak{s}) \\ &= \big(T_\mathfrak{s} \times \mathrm{Irr}(\mathbb{C}[X^L(\omega),\kappa_\omega])\big)/X^L(\mathfrak{s})X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp). \end{aligned} \tag{39}$$

In the notation of (38) and (39) the action of $\gamma \in X^L(\mathfrak{s})$ becomes

$$\gamma\cdot(\omega\otimes\chi,\rho) = (\omega\otimes\chi\chi_\gamma,\phi_{\omega,\gamma}\rho), \tag{40}$$

where $\phi_{\omega,\gamma}$ is yet to be determined. Any $\gamma \in X^L(\omega)$ can be adjusted by an element of $X_{\mathrm{nr}}(L,\omega)$ to achieve $\chi_\gamma = 1$. Then (36) shows that $\phi_{\omega,\gamma}\rho \cong \rho$ for all $\gamma \in X^L(\omega)$.

Lemma 2.2. *For $\gamma \in X^L(\mathfrak{s})$, $\phi_{\omega,\gamma}\rho$ is ρ tensored with a character of $X^L(\omega)$, which we also call $\phi_{\omega,\gamma}$. Then*

$$X^L(\mathfrak{s}) \to \mathrm{Irr}(X^L(\omega)) : \gamma \mapsto \phi_{\omega,\gamma}$$

is a group homomorphism.

Proof. Let $N_{\gamma'}$ be a standard basis element of $\mathbb{C}[X^L(\omega),\kappa_\omega]$. In view of (34) $\phi_{\omega,\gamma}\rho$ is given by

$$N_{\gamma'} \mapsto J(\gamma,\omega)I(\gamma',\omega)J(\gamma,\omega)^{-1} \in \mathrm{Hom}_L(\omega\otimes\gamma'\chi_\gamma,\omega\otimes\chi_\gamma). \tag{41}$$

Since these are irreducible L-representations, there is a unique $\lambda \in \mathbb{C}^\times$ such that

$$\begin{aligned} &J(\gamma,\omega)I(\gamma',\omega)J(\gamma,\omega)^{-1} = \lambda^{-1}I(\gamma',\omega\otimes\chi_\gamma), \\ &(\phi_{\omega,\gamma}\rho)(N_{\gamma'}) = \rho(\lambda I(\gamma',\omega)) = \lambda\rho(N_{\gamma'}). \end{aligned}$$

Moreover the relation

$$I(\gamma'_1,\omega\otimes\chi_\gamma)I(\gamma'_2,\omega\otimes\chi_\gamma) = \kappa_{\omega\otimes\chi_\gamma}(\gamma'_1,\gamma'_2)I(\gamma'_1\gamma'_2,\omega\otimes\chi_\gamma) \tag{42}$$

also holds with $J(\gamma,\omega)I(\gamma'_i,\omega)J(\gamma,\omega)^{-1}$ instead of $I(\gamma'_i,\omega)$– a basic property of conjugation. It follows that $\gamma' \mapsto \lambda$ defines a character of $X^L(\omega)$ which implements the action $\rho \mapsto \phi_{\omega,\gamma}\rho$. As $\phi_{\omega,\gamma}$ comes from conjugation by $J(\gamma,\omega\otimes\chi)$ and by (42), $\gamma \mapsto \phi_{\omega,\gamma}$ is a group homomorphism. □

A straightforward check, using the above proof, shows that

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho,\omega\otimes\chi) & \to & \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\phi_\gamma\rho,\omega\otimes\chi\chi_\gamma) \\ f & \mapsto & J(\gamma,\omega\otimes\chi)\circ f \end{array} \tag{43}$$

is an isomorphism of $L^\sharp Z(G)$-representations.

3. Hecke algebras and spectrum preserving morphisms

We will show that the Hecke algebras obtained in Theorems 1.2 and 1.3 fit in the framework of spectrum preserving morphisms and geometric equivalence of finite type algebras, see Appendix A.

To apply this to the algebras from Section 1 we first exhibit an algebra that interpolates between

$$(\mathcal{H}(T_\mathfrak{s},W_\mathfrak{s},q_\mathfrak{s})\otimes\mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})}\rtimes\mathfrak{R}_\mathfrak{s}^\sharp$$

and $(\mathcal{O}(T_\mathfrak{s})\rtimes W_\mathfrak{s}\otimes\mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})}\rtimes\mathfrak{R}_\mathfrak{s}^\sharp$. Recall that Conditions 1.1 are in force and write

$$T_\mathfrak{s}=\prod\nolimits_i T_i,\ R_\mathfrak{s}=\bigsqcup\nolimits_i R_i,\ W_\mathfrak{s}=\prod\nolimits_i W(R_i)=\prod\nolimits_i S_{e_i}.$$

Let q_i be the restriction of $q_\mathfrak{s}: X^*(T_\mathfrak{s})\rtimes W_\mathfrak{s}\to\mathbb{R}_{>0}$ to $X^*(T_i)\rtimes W(R_i)$. Recall Lusztig's asymptotic Hecke algebra $J(X^*(T_i)\rtimes W(R_i))$ from [Lus2, Lus3]. We remark that, although in [Lus2] it is supposed that the underlying root datum is semisimple, this assumption is shown to be unneccesary in [Lus3]. This algebra is unital and of finite type over $\mathcal{O}(T_i)^{W(R_i)}$. It has a distinguished $\mathbb{C}$-basis $\{t_{xv}\mid x\in X^*(T_i),v\in W(R_i)\}$ and the t_x with $x\in X^*(T_i)^{W(R_i)}$ are central. We define

$$J(X^*(T_\mathfrak{s})\rtimes W_\mathfrak{s})=\bigotimes\nolimits_i J(X^*(T_i)\rtimes W(R_i)).$$

This is a unital finite type algebra over $\mathcal{O}(T_\mathfrak{s})^{W_\mathfrak{s}}$, in fact for several different $\mathcal{O}(T_\mathfrak{s})^{W_\mathfrak{s}}$-module structures.

Lemma 3.1. *The group* $\mathrm{Stab}(\mathfrak{s})^+$ *acts on* $J(X^*(T_\mathfrak{s})\rtimes W_\mathfrak{s})\otimes\mathrm{End}_\mathbb{C}(V_\mu)$ *and on* $\mathcal{O}(T_\mathfrak{s})\rtimes W_\mathfrak{s}\otimes\mathrm{End}_\mathbb{C}(V_\mu)$ *like the action on* $\mathcal{H}(T_\mathfrak{s},W_\mathfrak{s},q_\mathfrak{s})\otimes\mathrm{End}_\mathbb{C}(V_\mu)$ *in Theorem 1.4.*

Proof. For $w\in\mathfrak{R}_\mathfrak{s}^\sharp$ the group automorphism

$$xv\mapsto wxvw^{-1}\quad\text{of}\quad X^*(T_\mathfrak{s})\rtimes W_\mathfrak{s} \tag{44}$$

permutes the subgroups $X^*(T_i)\rtimes W(R_i)$ and preserves $q_\mathfrak{s}$. Thus (44) can be factorized as

$$\prod_j\omega_j\text{ with }\omega_j\in\mathrm{Aut}\big(\prod_{i:R_i=R_j,q_i=q_j}X^*(T_i)\rtimes W(R_i)\big)$$

The function $q_\mathfrak{s}$ takes the same value on all simple (affine) roots associated to the group for one j in (44), so the algebra

$$\bigotimes\nolimits_{i|R_i=R_j,q_i=q_j}J(X^*(T_i)\rtimes W(R_i)) \tag{45}$$

is of the kind considered in [Lus3, §1]. Then ω_j is an automorphism which fits in a group called Ω in [Lus3, §1.1], so it gives rise to an automorphism of the algebra (45). In this way the group $\mathfrak{R}_\mathfrak{s}^\sharp$ acts naturally on $J(X^*(T_\mathfrak{s})\rtimes W_\mathfrak{s})$.

Since $T_{\mathfrak s}^{W_{\mathfrak s}}$ is central in $T_{\mathfrak s} \rtimes W_{\mathfrak s}$, every $\chi \in T_{\mathfrak s}^{W_{\mathfrak s}}$ gives rise to an algebra automorphism of $J(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s})$:

$$t_{xv} \mapsto \chi(x) t_{xv} \qquad x \in X^*(T_{\mathfrak s}), v \in W_{\mathfrak s}. \tag{46}$$

Thus we can make $\mathrm{Stab}(\mathfrak s)^+$ act on $J(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s})$ by

$$(w,\gamma) \cdot t_{xv} = \chi_\gamma^{-1}(x) t_{wxvw^{-1}} \qquad x \in X^*(T_{\mathfrak s}), v \in W_{\mathfrak s}.$$

The action of $\mathrm{Stab}(\mathfrak s)^+$ on $\mathrm{End}_{\mathbb C}(V_\mu)$ may be copied to this setting, so we can define the following action on $J(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)$:

$$\alpha_{(w,\gamma)}(t_{xv} \otimes h) = \chi_\gamma^{-1}(x) t_{wxvw^{-1}} \otimes J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}.$$

Of course the above also works with the label function 1 instead of $q_{\mathfrak s}$. That yields a similar action of $\mathrm{Stab}(\mathfrak s)^+$ on $\mathcal O(T_{\mathfrak s}) \rtimes W_{\mathfrak s} \otimes \mathrm{End}_{\mathbb C}(V_\mu)$, namely

$$\alpha_{(w,\gamma)}(xv \otimes h) = \chi_\gamma^{-1}(x)\; wxvw^{-1} \otimes J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}, \tag{47}$$

where $xv \in X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}$. $\square$

Lusztig [Lus3, §1.4] defined injective algebra homomorphisms

$$\mathcal H(T_i, W(R_i), q_i) \xrightarrow{\phi_{i,q_i}} J(X^*(T_i) \rtimes W(R_i)) \xleftarrow{\phi_{i,1}} \mathcal O(T_i) \rtimes W(R_i) \tag{48}$$

with many nice properties. Among these, we record that

$$\phi_{i,q_i} \text{ and } \phi_1 \text{ are the identity on } \mathbb C[X^*(T_i)^{W(R_i)}] \cong \mathcal O(X_{\mathrm{nr}}(Z(M_i))). \tag{49}$$

There exist $\mathcal O(T_i)^{W(R_i)}$-module structures on $J(X^*(T_i) \rtimes W(R_i))$ for which the maps (48) are $\mathcal O(T_i)^{W(R_i)}$-linear, namely by letting $\mathcal O(T_i)^{W(R_i)}$ act via the map ϕ_{i,q_i} or via $\phi_{i,1}$. Taking tensor products over i in (48) and with the identity on $\mathrm{End}_{\mathbb C}(V_\mu)$ gives algebra homomorphisms

$$\begin{array}{rrcl} \phi_{q_{\mathfrak s}} : & \mathcal H(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu) & \to & J(X(T_{\mathfrak s}) \rtimes W_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu), \\ \phi_1 : & \mathcal O(T_{\mathfrak s}) \rtimes W_{\mathfrak s} \otimes \mathrm{End}_{\mathbb C}(V_\mu) & \to & J(X(T_{\mathfrak s}) \rtimes W_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu). \end{array} \tag{50}$$

The maps $\phi_{q_{\mathfrak s}}$ and ϕ_1 are $\mathcal O(T_{\mathfrak s})^{W_{\mathfrak s}}$-linear with respect to the appropriate module structure on $J(X(T_{\mathfrak s}) \rtimes W_{\mathfrak s})$.

Lemma 3.2. *The $\mathcal O(T_{\mathfrak s})^{W_{\mathfrak s}}$-algebra homomorphisms $\phi_{q_{\mathfrak s}}$ and ϕ_1 from* (50) *are spectrum preserving with respect to filtrations.*

Proof. It suffices to consider the map $\phi_{q_{\mathfrak s}}$, for the same reasoning will apply to ϕ_1. Our argument is a generalization of [BaNi, Theorem 10], which proves the analogous statements for $J(X^*(T_i) \rtimes W(R_i))$. Recall the function

$$a : X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s} \to \mathbb Z_{\geq 0} \tag{51}$$

from [Lus3, §1.3]. For fixed $n \in \mathbb Z_{\geq 0}$, the subspace of $J(X^*(T_i) \rtimes W(R_i))$ spanned by the t_{xv} with $a(xv) = n$ is a two-sided ideal, let us call it $J^{i,n}$. Then

$$J(X^*(T_i) \rtimes W(R_i)) = \bigoplus\nolimits_{n \geq 0} J^{i,n}$$

and the sum is finite by [Lus1, §7]. Moreover

$$\mathcal H^{i,n} := \phi_{q_i,i}^{-1}\big(\bigoplus\nolimits_{k \geq n} J^{i,k}\big)$$

is a two-sided ideal of $\mathcal{H}(T_i, W(R_i), q_i)$. According to [Lus2, Corollary 3.6] the morphism of $\mathcal{O}(T_i)^{W(R_i)}$-algebras

$$\mathcal{H}^{i,n}/\mathcal{H}^{i,n+1} \to J^{i,n} \text{ induced by } \phi_{q_i,i}$$

is spectrum preserving. For any irreducible $J^{i,n}$-module M_J^i the $\mathcal{H}^{i,n}$-module $\phi_{q_i,i}^*(M_J^i)$ has a distinguished quotient $M_{\mathcal{H}}^i$, which is an irreducible $\mathcal{H}^{i,n}/\mathcal{H}^{i,n+1}$-module.

Let $\mathbf{n}$ be a vector with coordinates $n_i \in \mathbb{Z}_{\geq 0}$ and put $|\mathbf{n}| = \sum_i n_i$. We write $\mathbf{n} \leq \mathbf{n}'$ if $n_i \leq n_i'$ for all i. We define the two-sided ideals

$$\begin{array}{lclcl} J^{\mathbf{n}} & = & \bigotimes_i J^{i,n_i} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) & \subset & J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu), \\ \mathcal{H}^{\mathbf{n}} & = & \bigotimes_i \mathcal{H}^{i,n_i} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) & \subset & \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu), \\ \mathcal{H}^{\mathbf{n}+} & = & \sum_{\mathbf{n}' \geq \mathbf{n}, |\mathbf{n}'| = |\mathbf{n}|+1} \mathcal{H}^{\mathbf{n}'}. & & \end{array}$$

It follows from the above that the morphism of $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$-algebras

$$\bigotimes_i (\mathcal{H}^{i,n_i}/\mathcal{H}^{i,n_i+1}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \cong \mathcal{H}^{\mathbf{n}}/\mathcal{H}^{\mathbf{n}+} \to J^{\mathbf{n}} \tag{52}$$

induced by $\phi_{q_{\mathfrak{s}}}$ is spectrum preserving, and that every irreducible $J^{\mathbf{n}}$-module M_J has a distinguished quotient $M_{\mathcal{H}}$ which is an irreducible $\mathcal{H}^{\mathbf{n}}/\mathcal{H}^{\mathbf{n}+}$-module.

Next we define, for $n \in \mathbb{Z}_{\geq 0}$:

$$J^n := \bigoplus_{|\mathbf{n}|=n} J^{\mathbf{n}}, \; \mathcal{H}^n := \bigoplus_{|\mathbf{n}|=n} \mathcal{H}^{\mathbf{n}}.$$

The aforementioned properties of the map (52) are also valid for

$$\mathcal{H}^n/\mathcal{H}^{n+1} \to J^n, \tag{53}$$

which shows that $\phi_{q_{\mathfrak{s}}}$ is spectrum preserving with respect to the filtrations $(\mathcal{H}^n)_{n\geq 0}$ and $(\oplus_{m \geq n} J^m)_{n \geq 0}$. $\square$

It follows from [Lus3, §1] that the maps (50) are $\mathfrak{R}_{\mathfrak{s}}^\sharp$-equivariant for the actions α defined in Lemma 3.1. Now

$$\begin{array}{l} \big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp, \\ \big(J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp, \\ \big(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \end{array} \tag{54}$$

are unital finite type $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$-algebras, and $\phi_{q_{\mathfrak{s}}}$ and ϕ_1 provide morphisms between them.

Theorem 3.3. (*a*) *The above morphisms between the $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$-algebras* (54) *are spectrum preserving with respect to filtrations, in the sense of* (112).

(*b*) *The same holds for the three algebras of* (54) *with $T_{\mathfrak{s}}^\sharp$ instead of $T_{\mathfrak{s}}$.*

Proof. (a) We use the notations from the proof of Lemma 3.2. Since (44) gives rise to an automorphism of the algebra $J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}, q_{\mathfrak{s}})$,

$$a(wxvw^{-1}) = a(xv) \text{ for all } x \in X^*(T_{\mathfrak{s}}), v \in W_{\mathfrak{s}}, w \in \mathfrak{R}_{\mathfrak{s}}^\sharp.$$

Hence J^n and $\mathcal{H}^n$ are stable under the respective actions α and (51) is $\mathrm{Stab}(\mathfrak{s})^+$-equivariant. Consider the restriction of M_J to $(J^n)^{X^L(\mathfrak{s})}$. By Clifford theory (see [RaRa, Appendix]) its decomposition is governed by a twisted group algebra of the stabilizer of M_J in $X^L(\mathfrak{s})$. Since (53) is $X^L(\mathfrak{s})$-equivariant and $M_{\mathcal{H}}$ is a quotient of M_J, the decomposition of $M_{\mathcal{H}}$ as module over $(\mathcal{H}^n/\mathcal{H}^{n+1})^{X^L(\mathfrak{s})}$ is governed by the same twisted group algebra in the same way. Therefore (53) restricts to a spectrum preserving morphism of $\mathcal{O}(T_\mathfrak{s})^{W_\mathfrak{s} \times X^L(\mathfrak{s})}$-algebras

$$(\mathcal{H}^n/\mathcal{H}^{n+1})^{X^L(\mathfrak{s})} \to (J^n)^{X^L(\mathfrak{s})}.$$

Now a similar argument with Clifford theory for crossed product algebras shows that

$$(\mathcal{H}^n/\mathcal{H}^{n+1})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \to (J^n)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s}$$

is a spectrum preserving morphism of $\mathcal{O}(T_\mathfrak{s})^{\mathrm{Stab}(\mathfrak{s})}$-algebras. By definition [BaNi, §5], this means that the map

$$\phi'_{q_\mathfrak{s}} : \big(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \to \big(J(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \tag{55}$$

induced by $\phi_{q_\mathfrak{s}}$ is spectrum preserving with respect to filtrations.

The same reasoning is valid with $\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$ instead of $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ – it is simply the case $q_\mathfrak{s} = 1$ of the above.

(b) Recall that $T_\mathfrak{s} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)$. The torus $T_\mathfrak{s}/X_{\mathrm{nr}}(L/L^\sharp Z(G))$ can be identified with

$$X_{\mathrm{nr}}(L^\sharp Z(G))/X_{\mathrm{nr}}(L,\omega). \tag{56}$$

Since the elements of $X_{\mathrm{nr}}(L,\omega)$ are trivial on $Z(L) \supset Z(G)$ and $L^\sharp \cap Z(G) \cong \mathfrak{o}_F^\times$ is compact, (56) factors as

$$X_{\mathrm{nr}}(L^\sharp)/X_{\mathrm{nr}}(L,\omega) \times X_{\mathrm{nr}}(Z(G)) = T^\sharp_\mathfrak{s} \times X_{\mathrm{nr}}(Z(G)).$$

By Theorem 1.2 the action of $X_{\mathrm{nr}}(L/L^\sharp Z(G)) \subset X^L(\mathfrak{s})$ on the algebras (54) comes only from its action on the torus $T_\mathfrak{s}$. Hence these three algebras do not change if we replace $T_\mathfrak{s}$ by (56). Equivalently, we may replace $T_\mathfrak{s}$ by $T^\sharp_\mathfrak{s} \times X_{\mathrm{nr}}(Z(G))$. It follows that

$$\big(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \cong \big(\mathcal{O}(X_{\mathrm{nr}}(Z(G))) \otimes \mathcal{H}(T^\sharp_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s}.$$

The action of $\mathrm{Stab}(\mathfrak{s})^+$ fixes $\mathcal{O}(X_{\mathrm{nr}}(Z(G)))$ pointwise, so this equals

$$\big(\mathcal{H}(T^\sharp_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \otimes \mathcal{O}(X_{\mathrm{nr}}(Z(G))).$$

The other two algebras in (54) can be rewritten similarly. By (49) the morphisms $\phi_{q_\mathfrak{s}}$ and ϕ_1 fix the respective subalgebras $\mathcal{O}(X_{\mathrm{nr}}(Z(G)))$ pointwise. It follows that (56) decomposes as

$$\phi^\sharp_{q_\mathfrak{s}} \otimes \mathrm{id} : \big(\mathcal{H}(T^\sharp_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \otimes \mathcal{O}(X_{\mathrm{nr}}(Z(G))) \to \big(J(X^*(T^\sharp_\mathfrak{s}) \rtimes W_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \otimes \mathcal{O}(X_{\mathrm{nr}}(Z(G))),$$

and similarly for ϕ'_1. From part (a) we know that $\phi'_{q_\mathfrak{s}} = \phi^\sharp_{q_\mathfrak{s}} \otimes \mathrm{id}$ and $\phi'_1 = \phi^\sharp_1 \otimes \mathrm{id}$ are spectrum preserving with respect to filtrations. So $\phi^\sharp_{q_\mathfrak{s}}$ and

$$\phi^\sharp_1 : \big(\mathcal{O}(T^\sharp_\mathfrak{s}) \rtimes W_\mathfrak{s} \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \to \big(J(X^*(T^\sharp_\mathfrak{s}) \rtimes W_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s}$$

have that property as well. □

With Theorem 3.3 we can show that the Hecke algebras for $G^\sharp$ and for $G^\sharp Z(G)$ are geometrically equivalent (confer Appendix A) to much simpler algebras. Recall the subgroup $H_\lambda \subset L$ from [ABPS4, Lemma 3.3].

Theorem 3.4. (*a*) *The algebra $\mathcal{H}(G^\sharp Z(G))^{\mathfrak s}$ is geometrically equivalent with*

$$\bigoplus\nolimits_1^{[L:H_\lambda]} \big(\mathcal{O}(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes W_{\mathfrak s}^\sharp,$$

where the action of $w \in W_{\mathfrak s}^\sharp$ is $\alpha_{(w,\gamma)}$ for any $\gamma \in \mathrm{Irr}(L/L^\sharp Z(G))$ such that $(w,\gamma) \in \mathrm{Stab}(\mathfrak s)$.

(*b*) *The algebra $\mathcal{H}(G^\sharp)^{\mathfrak s}$ is geometrically equivalent with*

$$\bigoplus\nolimits_1^{[L:H_\lambda]} \big(\mathcal{O}(T_{\mathfrak s}^\sharp) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes W_{\mathfrak s}^\sharp,$$

with respect to the same action of $W_{\mathfrak s}^\sharp$.

Remark. In principle one could factorize the above algebras according to single Bernstein components for $G^\sharp Z(G)$ and $G^\sharp$. However, this would result in less clear formulas.

Proof. (a) Recall from Theorem 1.2 that $\mathcal{H}(G^\sharp Z(G))^{\mathfrak s}$ is Morita equivalent with

$$\bigoplus\nolimits_1^{[L:H_\lambda]} \big(\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp. \tag{57}$$

Consider the sequence of algebras

$$\begin{aligned} & \big(\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp \\ & \to \big(J(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp \\ & = \big(J(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp \\ & \leftarrow \big(\mathcal{O}(T_{\mathfrak s}) \rtimes W_{\mathfrak s} \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp. \end{aligned} \tag{58}$$

In Theorem 3.3.a we proved that the map between the first two lines is spectrum preserving with respect to filtrations. The equality sign does nothing on the level of $\mathbb C$-algebras, but we use it to change the $\mathcal{O}(T_{\mathfrak s})^{\mathrm{Stab}(\mathfrak s)}$-module structure, such that the map from

$$\big(\mathcal{O}(T_{\mathfrak s}) \rtimes W_{\mathfrak s} \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes \mathfrak{R}_{\mathfrak s}^\sharp \tag{59}$$

becomes $\mathcal{O}(T_{\mathfrak s})^{\mathrm{Stab}(\mathfrak s)}$-linear. By Theorem 3.3.a that map is also spectrum preserving with respect to filtrations.

Every single step in the above sequence is an instance of geometric equivalence defined in Appendix A, so $\mathcal{H}(G^\sharp Z(G))^{\mathfrak s}$ is geometrically equivalent with a direct sum of $[L : H_\lambda]$ copies of (59). Since $\chi_\gamma \in T_{\mathfrak s}$ in (47) is $W_{\mathfrak s}$-invariant, the actions of $X^L(\mathfrak s)$ and $W_{\mathfrak s}$ on $\mathcal{O}(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)$ commute. This observation and (18) allow us to identify (59) with

$$\Big(\big(\mathcal{O}(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes W_{\mathfrak s}\Big) \rtimes \mathfrak{R}_{\mathfrak s}^\sharp = \big(\mathcal{O}(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)\big)^{X^L(\mathfrak s)} \rtimes W_{\mathfrak s}^\sharp. \tag{60}$$

The description of the action of $W_{\mathfrak s}^\sharp$ can be derived from Theorem 1.4.
(b) This follows from Theorem 1.3 and the same proof as for part (a). □

4. Twisted extended quotients

Twisted extended quotients appear naturally in the description of the Bernstein tori for $L^\sharp Z(G)$ and $L^\sharp$.

Lemma 4.1. *Let $\mathfrak{s}_L = [L,\omega]_L$ and define a two-cocycle κ_ω by (15).*
(a) Equation (17) for L determines bijections

$$(T_\mathfrak{s} // X^L(\mathfrak{s}))_{\kappa_\omega} \to \mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp Z(G)),$$
$$(T_\mathfrak{s} // X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} = (T_\mathfrak{s} // X^L(\mathfrak{s}))_{\kappa_\omega} / X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp) \to \mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp).$$

(b) The induced maps

$$\mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp Z(G)) \to T_\mathfrak{s} / X^L(\mathfrak{s}) \quad \text{and} \quad \mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp) \to T_\mathfrak{s} / X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)$$

are independent of the choice of κ_ω.
(c) Let $T_{\mathfrak{s},\mathrm{un}}$ be the real subtorus of unitary representations in $T_\mathfrak{s}$. The subspace of tempered representations $\mathrm{Irr}^{\mathfrak{s}_L}_{\mathrm{temp}}(L^\sharp Z(G))$ corresponds to $(T_{\mathfrak{s},\mathrm{un}} // X^L(\mathfrak{s}))_{\kappa_\omega}$. Similarly $\mathrm{Irr}^{\mathfrak{s}_L}_{\mathrm{temp}}(L^\sharp)$ is obtained by restricting the second line of part (a) to $T_{\mathfrak{s},\mathrm{un}}$.

Proof. (a) Apart from the equality, this is a reformulation of the last page of Section 2. For the equality, we note that by (39) the action of

$$X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp) \cong X_{\mathrm{nr}}(L/L^\sharp) / (X^L(\mathfrak{s}) \cap X_{\mathrm{nr}}(L/L^\sharp))$$

on $(T_\mathfrak{s} // X^L(\mathfrak{s}))_{\kappa_\omega}$ is free. Hence the isotropy groups for the action of $X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)$ are the same as for $X^L(\mathfrak{s})$, and we can use the same 2-cocycle κ_ω to construct a twisted extended quotient.
(b) By (17) a different choice of κ_ω in part (a) would only lead to the choice of another irreducible summand of $\mathrm{Res}^G_{G^\sharp}(\pi)$ for $\pi \in T_\mathfrak{s}$, and similarly for $G^\sharp Z(G)$.
(c) Since ω is supercuspidal, the set of tempered representations in $T_\mathfrak{s} = \mathrm{Irr}^{\mathfrak{s}_L}(L)$ is $T_{\mathfrak{s},\mathrm{un}}$. In the decomposition (17), an irreducible representation of $L^\sharp$ or $L^\sharp Z(G)$ is tempered if and only if it is contained in a tempered L-representation $\omega \otimes \chi$. This proves the statement for L. The claim for $L^\sharp$ follows upon dividing out the free action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$. □

The subgroup $X^L(\omega, V_\mu)$ acts trivially on $\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)$, and for that reason it can be pulled out of the extended quotient from Lemma 4.1.

Lemma 4.2. *There are bijections*

$$(T_\mathfrak{s} // X^L(\mathfrak{s}))_{\kappa_\omega} \longleftrightarrow \big(T_\mathfrak{s} // X^L(\mathfrak{s}) / X^L(\omega, V_\mu)\big)_{\kappa_\omega} \times \mathrm{Irr}(X^L(\omega, V_\mu)),$$
$$(T_\mathfrak{s} // X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} \longleftrightarrow \big(T_\mathfrak{s} // X\big)_{\kappa_\omega} \times \mathrm{Irr}(X^L(\omega, V_\mu)),$$

where $X = X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp) / X^L(\omega, V_\mu)$. They fix the coordinates in $T_\mathfrak{s}$.

Proof. In Lemma 4.1 we saw that $(T_\mathfrak{s} // X^L(\mathfrak{s}))_{\kappa_\omega}$ is in bijection with $\mathrm{Irr}(\mathcal{H}(L^\sharp Z(G))^{\mathfrak{s}_L})$. By [ABPS4, (169)] $\mathcal{H}(L^\sharp Z(G))^{\mathfrak{s}_L}$ is Morita equivalent with $(\mathcal{H}(L)^{\mathfrak{s}_L})^{X^L(\mathfrak{s})}$ and with the subalgebra

$$\begin{aligned} (61) \quad e^\mathfrak{s}_L \mathcal{H}(L)^{X^L(\mathfrak{s})} e^\mathfrak{s}_L &\cong \big(\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(e^\mathfrak{s}_L V_\omega)\big)^{X^L(\mathfrak{s})} \\ &\cong \bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})/X_{\mathrm{nr}}(\omega, V_\mu)}. \end{aligned}$$

Here the $X^L(\mathfrak{s})$-action on the middle term comes from an isomorphism

$$\mathrm{End}_{\mathbb{C}}(e_L^{\mathfrak{s}} V_\omega) \cong \mathrm{End}_{\mathbb{C}}(V_\mu) \otimes \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}[L/H_\lambda]^*.$$

We recall that by [ABPS4, Lemma 3.5] there is a group isomorphism

$$L/H_\lambda \cong \mathrm{Irr}(X^L(\omega, V_\mu)). \tag{62}$$

The stabilizer of an irreducible representation $\mathbb{C}_\chi \otimes V_\mu$ of the right hand side of (61) is $X^L(\omega)/X_{\mathrm{nr}}(\omega, V_\mu)$. Comparing the spaces of irreducible representations of (61), we find that

$$\{\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega]) : \rho|_{X^L(\omega,V_\mu)} = \mathrm{triv}\}$$

corresponds bijectively to $\mathrm{Irr}(\mathrm{End}_{\mathbb{C}}(V_\mu)^{X^L(\mathfrak{s})/X^L(\omega,V_\mu)})$. It follows that every irreducible representation of $\mathbb{C}[X^L(\omega)/X^L(\omega, V_\mu), \kappa_\omega]$ appears in V_μ. This is equivalent to each irreducible representation of $\mathrm{End}_{\mathbb{C}}(V_\mu) \rtimes X^L(\omega)/X^L(\omega, V_\mu)$ having nonzero vectors fixed by $X^L(\omega)/X^L(\omega, V_\mu)$. Thus Lemma B.2 can be applied to $X^L(\omega)/X^L(\omega, V_\mu)$ acting on $\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, and it shows that the irreducible representations on the right hand side of (61) are in bijection with $(T_\mathfrak{s}/\!/X^L(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \times \mathrm{Irr}(X^L(\omega, V_\mu))$.

The second bijection follows by dividing out the free action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$, as in the proof of Lemma 4.1.a. □

It turns out that any Bernstein component for G can be described in a canonical way with an extended quotient. Before we prove that, we recall the parametrization of irreducible representations of $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$.

Let $\check{G}_\mathfrak{s}$ be the complex reductive group with root datum $(X^*(T_\mathfrak{s}), R_\mathfrak{s}, X_*(T_\mathfrak{s}), R_\mathfrak{s}^\vee)$, it is isomorphic to $\prod_i \mathrm{GL}_{e_i}(\mathbb{C})$, embedded in $\check{G} = \mathrm{GL}_{md}(\mathbb{C})$ as

$$\check{G}_\mathfrak{s} = Z_{\check{G}}(\check{L}) = Z_{\mathrm{GL}_{md}(\mathbb{C})}\Big(\prod_i \mathrm{GL}_{m_i d}(\mathbb{C})^{e_i}\Big).$$

Recall that a Kazhdan–Lusztig triple for $\check{G}_\mathfrak{s}$ consists of:

- a unipotent element $u = \prod_i u_i \in \check{G}_\mathfrak{s}$;
- a semisimple element $t_q \in \check{G}_\mathfrak{s}$ with $t_q u t_q^{-1} = u^{q_\mathfrak{s}} := \prod_i u_i^{q_i}$;
- a representation $\rho_q \in \mathrm{Irr}(\pi_0(Z_{\check{G}_\mathfrak{s}}(t_q, u)))$ which appears in the homology of variety of Borel subgroups of $\check{G}_\mathfrak{s}$ containing $\{t_q, u\}$.

Typically such a triple is considered up to $\check{G}_\mathfrak{s}$-conjugation, we denote its equivalence class by $[t_q, u, \rho_q]_{\check{G}_\mathfrak{s}}$. These equivalence classes parametrize $\mathrm{Irr}(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}))$ in a natural way, see [KaLu]. We denote that by

$$[t_q, u, \rho_q]_{\check{G}_\mathfrak{s}} \mapsto \pi(t_q, u, \rho_q). \tag{63}$$

Recall from [ABPS5, §7] that an affine Springer parameter for $\check{G}_\mathfrak{s}$ consists of:

- a unipotent element $u = \prod_i u_i \in \check{G}_\mathfrak{s}$;
- a semisimple element $t \in Z_{\check{G}_\mathfrak{s}}(u)$;
- a representation $\rho \in \mathrm{Irr}(\pi_0(Z_{\check{G}_\mathfrak{s}}(t, u)))$ which appears in the homology of variety of Borel subgroups of $\check{G}_\mathfrak{s}$ containing $\{t, u\}$.

Again such a triple is considered up to $\check{G}_\mathfrak{s}$-conjugacy, and then denoted $[t,u,\rho]_{\check{G}_\mathfrak{s}}$. Kato [Kat] established a natural bijection between such equivalence classes and $\mathrm{Irr}(\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s})$, say

$$[t,u,\rho]_{\check{G}_\mathfrak{s}} \mapsto \tau(t,u,\rho). \tag{64}$$

From [KaLu, §2.4] we get a canonical bijection between Kazhdan–Lusztig triples and affine Springer parameters:

$$[t_q,u,\rho_q]_{\check{G}_\mathfrak{s}} \longleftrightarrow [t,u,\rho]_{\check{G}_\mathfrak{s}}. \tag{65}$$

Basically it adjusts t_q in a minimal way so that it commutes with u, and then there is only one consistent way to modify ρ_q to ρ.

Via Lemma 3.2 the algebra homomorphisms (50) give riso to a bijection

$$\mathrm{Irr}(\mathcal{H}(T_\mathfrak{s},W_\mathfrak{s},q_\mathfrak{s})) \longleftrightarrow \mathrm{Irr}(\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s})). \tag{66}$$

We showed in [ABPS5, (90)] that (66) is none other than the composition of (65) with (64) and the inverse of (63):

$$\pi(t_q,u,\rho_q) \longleftrightarrow \tau(t,u,\rho). \tag{67}$$

Theorem 4.3. *The Morita equivalence $\mathcal{H}(G)^\mathfrak{s} \sim_M \mathcal{H}(T_\mathfrak{s},W_\mathfrak{s},q_\mathfrak{s})$ and* (66) *give rise to a bijection*

$$\mathrm{Irr}^\mathfrak{s}(G) \longleftrightarrow (T_\mathfrak{s}/\!/W_\mathfrak{s})_2 \tag{68}$$

with the following properties:

1. (68) *restricts to a bijection $\mathrm{Irr}^\mathfrak{s}_{\mathrm{temp}}(G) \longleftrightarrow (T_{\mathfrak{s},\mathrm{un}}/\!/W_\mathfrak{s})_2$.*
2. (68) *can be obtained from its restriction to tempered representations by analytic continuation, as in* [ABPS1].
3. *If $\pi \in \mathrm{Irr}^\mathfrak{s}_{\mathrm{temp}}(G)$ is mapped to $[t,\rho] \in (T_{\mathfrak{s},\mathrm{un}}/\!/W_\mathfrak{s})_2$ and has cuspidal support $W_\mathfrak{s}\sigma \in T_\mathfrak{s}/W_\mathfrak{s}$, then $W_\mathfrak{s}t$ is the unitary part of $W_\mathfrak{s}\sigma$, with respect to the polar decomposition*
$$T_\mathfrak{s} = T_{\mathfrak{s},\mathrm{un}} \times \mathrm{Hom}_\mathbb{Z}(X^*(T_\mathfrak{s}),\mathbb{R}_{>0}).$$
4. *In the notation of (3), suppose that the Springer parameter of $\rho \in \mathrm{Irr}(W_{\mathfrak{s},t})$ is a unipotent class $[u]$ which is distinguished in a Levi subgroup $\check{M} \subset Z_{\check{G}_\mathfrak{s}}(t)$. Then $\pi = I_{PM}^G(\delta)$, where $M \supset L$ is the unique standard Levi subgroup of G corresponding to $\check{M}$ and $\delta \in \mathrm{Irr}^{[L,\omega]_M}_{\mathrm{temp}}(M)$ is square-integrable modulo centre.*

Moreover (68) *is the unique bijection with the properties (1)–(4).*

Proof. The isomorphism (20) gives a bijection

$$\mathrm{Irr}^\mathfrak{s}(G) \longleftrightarrow \mathrm{Irr}(\mathcal{H}(T_\mathfrak{s},W_\mathfrak{s},q_\mathfrak{s})). \tag{69}$$

Via Lemmas 3.2 and B.1 the right hand side is in bijection with

$$\mathrm{Irr}(\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s} \cong (T_\mathfrak{s}/\!/W_\mathfrak{s})_2.$$

In this way we define the map (68).
(1) follows from [ABPS5, (92) and Proposition 9.3].
(2) Consider the bijection (66) and its formulation (67). Here the representations are tempered if and only if $t \in T_\mathfrak{s}$ is unitary. Thus (67) for tempered representations determines the bijection (66), by analytic continuation (in the parameters t and t_q) of the formula.

The relation between $\mathrm{Irr}^{\mathfrak s}(G)$ and $\mathrm{Irr}^{\mathfrak s}_{\mathrm{temp}}(G)$ is similar, see [ABPS1, Proposition 2.1]. Hence (69) is can also be deduced from its restriction to tempered representations, with the method from [ABPS1, §4].
(3) In [Séc, Théorème 4.6] a $\mathfrak s_L$-type (K_L, λ_L) is constructed, with

$$e_{\lambda_L} \mathcal H(L) e_{\lambda_L} \cong \mathcal O(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\lambda).$$

It [SéSt2] it is shown that it admits a cover (K_G, λ_G) with

$$e_{\lambda_G} \mathcal H(G) e_{\lambda_G} \cong \mathcal H(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\lambda),$$

see also [ABPS4, §4.1]. By (20) and (23) the inclusions

$$e_{\lambda_G} \mathcal H(G) e_{\lambda_G} \to e_{\mu_G} \mathcal H(G) e_{\mu_G} \to e^\sharp_{\lambda_G} \mathcal H(G) e^\sharp_{\lambda_G}$$

are Morita equivalences, of the simple form tensoring with a finite dimensional matrix algebra. This means that (69) comes from a cover of a $\mathfrak s_L$-type. With [BuKu, §7] this implies that (68) translates the cuspidal support of a $(\pi, V_\pi) \in \mathrm{Irr}^{\mathfrak s}(G)$ to the unique $W_{\mathfrak s} t_q \in T_{\mathfrak s} / W_{\mathfrak s}$ such that $e_{\lambda_G} V_\pi$ is a subquotient of $\mathrm{ind}_{\mathcal O(T_{\mathfrak s})}^{\mathcal H(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})} (\mathbb C_{t_q}) \otimes V_\lambda$. It follows from [ABPS5, (33) and Lemma 7.1] that the bijection (67) sends any tempered irreducible subquotient of $\mathrm{ind}_{\mathcal O(T_{\mathfrak s})}^{\mathcal H(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})} (\mathbb C_{t_q})$ to an irreducible $\mathcal O(T_{\mathfrak s}) \rtimes W_{\mathfrak s}$-representation with $\mathcal O(T_{\mathfrak s})$-weights $W_{\mathfrak s} (t_q \, |t_q|^{-1})$. The associated element of $(T_{\mathfrak s} /\!/ W_{\mathfrak s})_2$ is then $[t = t_q \, |t_q|^{-1}, \rho]$ with $\rho \in \mathrm{Irr}(W_{\mathfrak s, t})$.
(4) By (67) the $\mathcal H(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$-representation associated to $[t, \rho]$ is $\pi(t_q, u, \mathrm{triv})$. Then (t_q, u, triv) is also a Kazhdan–Lusztig triple for $\mathcal H(T_{\mathfrak s}, W_{\mathfrak s, M}, q_{\mathfrak s})$ and by [KaLu, §7.8]

$$\pi(t_q, u, \mathrm{triv}) = \mathrm{ind}_{\mathcal H^M}^{\mathcal H} \pi_M (t_q, u, \mathrm{triv}).$$

By [ABPS5, Proposition 9.3] (see also [KaLu, Theorem 8.3]) $\pi_M(t_q, u, \mathrm{triv})$ is essentially square-integrable and tempered, that is, square-integrable modulo centre.

Since (K_G, λ_G) is a a cover of a $\mathfrak s_L$-type (K_L, λ_L), there is a a $[L, \omega]_M$-type (K_M, λ_M) which covers (K_L, λ_L) and is covered by (K_G, λ_G). By [ABPS5, Proposition 16.1] $\pi_M(t_q, u, \mathrm{triv})$ corresponds to a M-representation δ which is square-integrable modulo centre. By [BuKu, Corollary 4.8] the bijection (69) respects parabolic induction, so $\pi(t_q, u, \mathrm{triv})$ corresponds to $I_{PM}^G(\delta)$.
Now we check that (68) is canonical in the specified sense. By (1) and (2) it suffices to do so for tempered representations. For $\pi \in \mathrm{Irr}^{\mathfrak s}_{\mathrm{temp}}(G)$, property (3) determines the $W_{\mathfrak s}$-orbit $W_{\mathfrak s} t$. Fix a t in this orbit. By a result of Harish-Chandra [Wal, Proposition III.4.1] there are a Levi subgroup $M \subset G$ containing L and a square-integrable (modulo centre) representation $\delta \in \mathrm{Irr}(M)$ such that π is a subquotient of $I_{PM}^G(\delta)$. Moreover (M, δ) is unique up to conjugation.

For $t \in T_{\mathfrak s, \mathrm{un}}$, $W_{\mathfrak s, t}$ is a product of symmetric groups S_e and $Z_{\check G_{\mathfrak s}}(t)$ is a product group $\mathrm{GL}_e(\mathbb C)$. Hence the Springer correspondence for $W_{\mathfrak s, t}$ is a bijection between $\mathrm{Irr}(W_{\mathfrak s, t})$ and unipotent classes in $Z_{\check G_{\mathfrak s}}(t)$. A general linear group $\mathrm{GL}_e(\mathbb C)$ has a unique distinguished unipotent class, so $\mathrm{Irr}(W_{\mathfrak s, t})$ is also in canonical bijection with the set of $W_{\mathfrak s, t}$-conjugacy classes of Levi subgroups $\check M \subset \check G_{\mathfrak s}$ containing $Z_{\check G_{\mathfrak s}}(t)$.

Viewed in this light, properties (3) and (4) entail that for every pair $(\check M, t)$ as above there is precisely one square-integrable modulo centre $\delta \in \mathrm{Irr}(M)$ such that $W_{\mathfrak s, t}$ is the unitary part of the cuspidal support of $I_{PM}^G(\delta)$. Thus (3) and (4) determine the (tempered) G-representation associated to $[t, \rho] \in (T_{\mathfrak s, \mathrm{un}} /\!/ W_{\mathfrak s})_2$. $\square$

As a result of the work in Section 3, twisted extended quotients can also be used to describe the spaces of irreducible representations of $G^\sharp Z(G)$ and $G^\sharp$. Let us extend κ_ω to a two-cocycle of $\mathrm{Stab}(\mathfrak{s})$, trivial on the normal subgroup $W_\mathfrak{s} \times X^L(\omega, V_\mu)$, by

$$J(\gamma,\omega)J(\gamma',\omega) = \kappa_\omega(\gamma,\gamma')J(\gamma\gamma',\omega) \qquad \gamma,\gamma' \in X^G(\mathfrak{s}). \tag{70}$$

Theorem 4.4. (*a*) *Lemmas B.1 and B.2 gives rise to bijections*

$$(T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s})/X^L(\omega,V_\mu))_{\kappa_\omega} \to \mathrm{Irr}\big((\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_\mathfrak{s}^\sharp\big),$$
$$(T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)/X^L(\omega,V_\mu))_{\kappa_\omega} \to \mathrm{Irr}\big((\mathcal{O}(T_\mathfrak{s}^\sharp) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_\mathfrak{s}^\sharp\big).$$

(*b*) *The geometric equivalences from 3.4 provide bijections*

$$(T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s}))_{\kappa_\omega} \to (T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s})/X^L(\omega,V_\mu))_{\kappa_\omega} \times \mathrm{Irr}(X^L(\omega,V_\mu)) \to \mathrm{Irr}^\mathfrak{s}(G^\sharp Z(G)),$$
$$(T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} \to (T_\mathfrak{s}/\!/S)_{\kappa_\omega} \times \mathrm{Irr}(X^L(\omega,V_\mu)) \to \mathrm{Irr}^\mathfrak{s}(G^\sharp),$$

where $S = \mathrm{Stab}(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)/X^L(\omega,V_\mu)$.

(*c*) *In part (b)* $\mathrm{Irr}^\mathfrak{s}_{\mathrm{temp}}(G^\sharp Z(G))$ *(respectively* $\mathrm{Irr}^\mathfrak{s}_{\mathrm{temp}}(G^\sharp)$*) corresponds to the same extended quotient, only with* $T_{\mathfrak{s},\mathrm{un}}$ *instead of* $T_\mathfrak{s}$.

Proof. In each of the three parts the second claim follows from the first upon dividing out the action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$, like in Lemma 4.1.a
(a) In the proof of Lemma 4.2 we exhibited a bijection

$$(T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s})/X^L(\omega,V_\mu))_{\kappa_\omega} \longleftrightarrow \mathrm{Irr}\big((\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})}\big).$$

With Lemma B.2 we deduce a Morita equivalence

$$(\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \sim_M (\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)) \rtimes (X^L(\mathfrak{s})/X^L(\omega,V_\mu)). \tag{71}$$

In the notation of (115) this means that $p := p_{X^L(\mathfrak{s})/X^L(\omega,V_\mu)}$ is a full idempotent in the right hand side of (71), that is, the two-sided ideal it generates is the entire algebra. Then p is also full in

$$(\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)) \rtimes (\mathrm{Stab}(\mathfrak{s})/X^L(\omega,V_\mu)), \tag{72}$$

which implies that (72) is Morita equivalent with

$$p\big((\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)) \rtimes (\mathrm{Stab}(\mathfrak{s})/X^L(\omega,V_\mu))\big)p \cong$$
$$(\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})/X^L(\omega,V_\mu)} \rtimes (\mathrm{Stab}(\mathfrak{s})/X^L(\mathfrak{s})).$$

As a direct consequence of (18), (27) and (28),

$$\mathrm{Stab}(\mathfrak{s})/X^L(\mathfrak{s}) \cong W_\mathfrak{s}^\sharp.$$

In this way we reach the algebra featuring in part (a). By the above Morita equivalence, its irreducible representations are in bijection with those of (72). Apply Lemma B.1.a to the latter algebra.
(b) All the morphisms in (58) are spectrum preserving with respect to filtrations. In combination with the other remarks in the proof of Theorem 3.4.a this gives a bijection

$$\mathrm{Irr}^\mathfrak{s}(G^\sharp Z(G)) \to \mathrm{Irr}\big((\mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_\mathfrak{s}^\sharp\big) \times [L/H_\lambda]. \tag{73}$$

By part (a) and (62) the right hand side of (73) is in bijection with

$$(T_\mathfrak{s}/\!/\mathrm{Stab}(\mathfrak{s})/X^L(\omega,V_\mu))_{\kappa_\omega} \times \mathrm{Irr}(X^L(\omega,V_\mu)). \tag{74}$$

Let $X^L(\mathfrak{s})$ act on

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}[L/H_\lambda]) \cong \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}[L/H_\lambda]^*$$

by extension of its action on $\mathcal{H}(L)$. Then we have an isomorphism

$$\begin{aligned} (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp \otimes \mathbb{C}[L/H_\lambda] \cong \\ (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp . \end{aligned} \tag{75}$$

We note that (74) is also the space of irreducible representations of (75). In the proof of Lemma 4.2 we encountered a bijection

$$(T_{\mathfrak{s}} // X^L(\mathfrak{s}))_{\kappa_\omega} \longleftrightarrow \mathrm{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \big).$$

It implies a Morita equivalence

$$(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \sim_M (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda])) \rtimes X^L(\mathfrak{s}).$$

Just as in the proof of part (a), this extends to

$$\begin{aligned} (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp \sim_M \\ (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda])) \rtimes \mathrm{Stab}(\mathfrak{s}). \end{aligned} \tag{76}$$

Finally we apply Lemma B.1.a to the right hand side and we combine it with (76), (75) and (73).
(c) The first bijection in part (b) obviously preserves the subspaces associated to $T_{\mathfrak{s},\mathrm{un}}$. We need to show that the second bijection sends them to $\mathrm{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G^\sharp Z(G))$. This is a property of the geometric equivalences in Theorem 3.4, as we will now check.

We may and will assume that ω is unitary, or equivalently that it is tempered. The Morita equivalence between $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ and (57) is induced by an idempotent $e^\sharp_{\lambda_{G^\sharp Z(G)}} \in \mathcal{H}(G^\sharp Z(G))$, see Theorem 1.2. Its construction (which starts around (20)) shows that eventually it comes from a central idempotent in the algebra of a profinite group, so it is a self-adjoint element. Hence, by [BHK, Theorem A] this Morita equivalence preserves temperedness. The notion of temperedness in [BHK] agrees with temperedness for representations of affine Hecke algebras (see page 36) because both are based on the Hilbert algebra structure and the canonical tracial states on these algebras.

The sequence of algebras (58) is derived from its counterpart for $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$. By Theorem 4.3 that one matches tempered representations with $(T_{\mathfrak{s},\mathrm{un}} // W_{\mathfrak{s}})_2$. By Clifford theory any irreducible representation π of

$$(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \tag{77}$$

is contained in a sum of irreducible representations $\tilde{\pi}$ of $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, which are all in the same $\mathrm{Stab}(\mathfrak{s})$-orbit. Temperedness of π depends only on the action of the subalgebra $\mathcal{O}(T_{\mathfrak{s}}) \cong \mathbb{C}[X^*(T_{\mathfrak{s}})]$, and in fact can already be detected on $\mathbb{C}[X]$ for any finite index sublattice $X \subset X^*(T_{\mathfrak{s}})$. The analogous statement for (77) holds as well, with $X = X^*(T_{\mathfrak{s}} / X^L(\mathfrak{s}))$, and it is stable under the action of $\mathrm{Stab}(\mathfrak{s})$. Consequently π is tempered if and only if $\tilde{\pi}$ is tempered.

These observations imply that the sequence of algebra homomorphisms (58) preserves temperedness of irreducible representations, and that it maps such representations of (77) to irreducible representations of (60) with $\mathcal{O}(T_{\mathfrak{s}}/X^L(\mathfrak{s}))$-weights in $T_{\mathfrak{s},\mathrm{un}}/X^L(\mathfrak{s})$.

Now we invoke this property for every $a \in L/H_\lambda \cong \mathrm{Irr}(X^L(\omega, V_\mu))$ and we deduce that the second map in part (b) has the required property with respect to temperedness. □

We work out what Theorem 4.4 says for a single Bernstein component of $G^\sharp$. Let $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$ be an inertial equivalence class for $G^\sharp$, with $\mathfrak{t}^\sharp \prec \mathfrak{s} = [L, \omega]_G$.

We abbreviate $\phi_{\omega, X^L(\mathfrak{s})} = \{\phi_\omega : \gamma \in X^L(\mathfrak{s})\}$. By (39) there is a unique $X^L(\mathfrak{s})$-orbit

$$\phi_{\omega, X^L(\mathfrak{s})}\, \rho \;\subset\; \mathrm{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega]) \tag{78}$$

such that $T_{\mathfrak{t}^\sharp} = (T^\sharp_{\mathfrak{s}} \times \phi_{\omega, X^L(\mathfrak{s})}\rho)/X^L(\mathfrak{s})$. Then $\phi_{\omega, X^L(\mathfrak{s})}\rho$ determines a unique summand $\mathbb{C}a$ of $\mathbb{C}[L/H_\lambda]$, namely the irreducible representation of $X^L(\omega, V_\mu)$ obtained by restricting ρ. Let $V_{\sigma^\sharp} \subset \mathbb{C}a \otimes_{\mathbb{C}} V_\mu$ be the subspace associated to $\phi_{\omega, X^L(\mathfrak{s})}\rho$, and let $\mathfrak{R}_{\mathfrak{t}^\sharp}$ be its stabilizer in $\mathfrak{R}^\sharp_{\mathfrak{s}}$. Then $\mathfrak{R}_{\mathfrak{t}^\sharp}$ is also the stabilizer of $\mathfrak{t}^\sharp$ in $\mathfrak{R}^\sharp_{\mathfrak{s}}$ and

$$W_{\mathfrak{t}^\sharp} = W_{\mathfrak{s}} \rtimes \mathfrak{R}_{\mathfrak{t}^\sharp}, \tag{79}$$

by [ABPS4, Lemma 2.3]. Via the formula (70) the operators $J(\gamma, \omega)|_{V_{\sigma^\sharp}}$ determine a 2-cocycle κ'_ω of the group

$$W' = \{(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}) : w \in W_{\mathfrak{t}^\sharp}\}. \tag{80}$$

Since (70) is 1 on $W_{\mathfrak{s}}$, so is κ'_ω. By (19) $W'/X^L(\mathfrak{s}) \cong W_{\mathfrak{t}^\sharp}$. As $V_{\sigma^\sharp}$ is associated to the single $X^L(\mathfrak{s})$-orbit (78), $\kappa'_\omega((w,\gamma),(w',\gamma'))$ depends only on (w, w'). Thus it determines a 2-cocycle $\kappa_{\sigma^\sharp}$ of $W_{\mathfrak{t}^\sharp}$, which factors through $\mathfrak{R}_{\mathfrak{t}^\sharp} \cong W_{\mathfrak{t}^\sharp}/W_{\mathfrak{s}}$.

Lemma 4.5. (*a*) *The bijections in Theorem 4.4 restrict to*

$$\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp) \longleftrightarrow (T_{\mathfrak{t}^\sharp}/\!/W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}},$$
$$\mathrm{Irr}^{\mathfrak{t}^\sharp}_{\mathrm{temp}}(G^\sharp) \longleftrightarrow (T_{\mathfrak{t}^\sharp,\mathrm{un}}/\!/W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}},$$

where $T_{\mathfrak{t}^\sharp,\mathrm{un}}$ denotes the space of unitary representations in $T_{\mathfrak{t}^\sharp}$.

(*b*) *Suppose $\pi \in \mathrm{Irr}^{\mathfrak{t}^\sharp}_{\mathrm{temp}}(G^\sharp)$ corresponds to $[t, \rho]$ and has cuspidal support $W_{\mathfrak{t}^\sharp}(\chi \otimes \sigma^\sharp) \in T_{\mathfrak{t}^\sharp}/W_{\mathfrak{t}^\sharp}$. Then $W_{\mathfrak{t}^\sharp} t$ is the unitary part of $\chi \otimes \sigma^\sharp$, with respect to the polar decomposition*

$$T_{\mathfrak{t}^\sharp} = T_{\mathfrak{t}^\sharp,\mathrm{un}} \times \mathrm{Hom}_{\mathbb{Z}}(X^*(T_{\mathfrak{t}^\sharp}), \mathbb{R}_{>0}).$$

Proof. (a) Recall that $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ consists of those irreducible representations that are contained in $I_{P^\sharp}^{G^\sharp}(\chi \otimes \sigma^\sharp)$ for some $\chi \otimes \sigma^\sharp \in T_{\mathfrak{t}^\sharp}$. In Theorem 1.5.b we translated $I_{P^\sharp}^{G^\sharp}$ to induction between two algebras. The first one, Morita equivalent with $\mathcal{H}(L^\sharp)^{\mathfrak{s}_L}$, was

$$\mathbb{C}[L/H_\lambda] \otimes \big(\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})}.$$

The second algebra, Morita equivalent with $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$, was

$$\mathbb{C}[L/H_\lambda] \otimes \big(\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

By Theorem 1.5 $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ is in bijection with the spaces of irreducible representations of the two Morita equivalent algebras

$$\begin{aligned}(81)\qquad &\big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}\mathfrak{R}_{\mathfrak{s}}^\sharp \cdot V_{\sigma^\sharp})\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp,\\ &\big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp})\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{t}^\sharp}.\end{aligned}$$

The constructions in Section 3 restrict to geometric equivalences between (81) and

$$\begin{aligned}(82)\qquad &(\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}\mathfrak{R}_{\mathfrak{s}}^\sharp \cdot V_{\sigma^\sharp}))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp,\\ &(\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{t}^\sharp}.\end{aligned}$$

By Proposition 2.1.d

$$(83)\qquad \mathrm{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}))^{X^L(\mathfrak{s})}\big) \cong T_{\mathfrak{t}^\sharp}.$$

As explained above with (80), the 2-cocycle κ_ω of $\mathrm{Stab}(\mathfrak{s})$ reduces to the 2-cocycle $\kappa_{\sigma^\sharp}$ for the action of $W_{\mathfrak{t}^\sharp}$ in (82). Now we apply Lemma B.1.a to (82) and we find the first bijection. To obtain the second bijection, we use Theorem 4.4.c.
(b) For the geometric equivalence between

$$\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}) \text{ and } \mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}) \rtimes W_{\mathfrak{s}}$$

the analogous claim about the cuspidal support is property (3) of Theorem 4.3. Clifford theory relates the irreducible representations of these algebras to those of (81) and (82), in a way already discussed after (77). This implies that the desired property of the cuspidal support persists to the geometric equivalence between (81) and (82), which underlies part (a). □

5. Relation with the local Langlands correspondence

We show how the local Langlands correspondence (LLC) for G and $G^\sharp$ can be reconstructed in terms of twisted extended quotients.

Let $\mathbf{W}_F$ be the Weil group of the local non-archimedean field F. Recall that the Langlands dual group of $G = \mathrm{GL}_m(D)$ is $\check{G} = \mathrm{GL}_{md}(\mathbb{C})$. A Langlands parameter for G is continuous group homomorphism $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \check{G}$ such that:

- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is a homomorphism of algebraic groups.
- $\phi(\mathbf{W}_F)$ consists of semisimple elements.
- ϕ is relevant for G: if $\check{L}$ is a Levi subgroup of $\check{G}$ which contains $\mathrm{im}(\phi)$ and is minimal for that property, then (the conjugacy class of) $\check{L}$ corresponds to (the conjugacy class of) a Levi subgroup of G.

We denote the collection of Langlands parameters for G, modulo conjugation by $\check{G}$, by $\Phi(G)$.

Every smooth character of G is of the form $\nu \circ \mathrm{Nrd}$, with ν a smooth character of $F^\times$. Via Artin reciprocity it determines a Langlands parameter (trivial on $\mathrm{SL}_2(\mathbb{C})$)

$$(84)\qquad \hat{\nu} : \mathbf{W}_F \to \mathbb{C}^\times \cong Z(\mathrm{GL}_{md}(\mathbb{C})).$$

For any $\phi \in \Phi(G)$, $\phi\hat{\nu}$ is a well-defined element of $\Phi(G)$ because the image of $\hat{\nu}$ is central in $\check{G}$.

Theorem 5.1. *The local Langlands correspondence for G is a canonical bijection*

$$\mathrm{rec}_{D,m} : \mathrm{Irr}(G) \to \Phi(G)$$

with the following properties:

(*a*) $\pi \in \mathrm{Irr}(G)$ *is tempered if and only if* $\mathrm{rec}_{D,m}(\pi)$ *is bounded, that is, if* $\mathrm{rec}_{D,m}(\pi)(\mathbf{W}_F)$ *is a bounded subset of* $\check{G}$.

(*b*) *The L-packet* $\Pi_\phi(G)$ *is the single representation* $\mathrm{rec}_{D,m}^{-1}(\pi)$.

(*c*) $\mathrm{rec}_{D,m}$ *is equivariant for the two actions of* $\mathrm{Irr}(G/G^\sharp)$*: on* $\mathrm{Irr}(G)$ *by twisting with smooth characters and on* $\Phi(G)$ *by multiplication with central Langlands parameters as in* (84).

Proof. For the bijection and part (a) see [HiSa, §11] and [ABPS3, §2]. Ultimately it relies on the Jacquet–Langlands correspondence from [DKV, Bad].
(b) This is a direct consequence of the bijectivity.
(c) Since $\mathrm{rec}_{D,m}$ is determined completely by its behaviour on essentially square integrable representations of Levi subgroups of G [ABPS3, (13)], it suffices to prove (c) for such representations. Via the Jacquet–Langlands correspondence the issue can be transferred to $\mathrm{Irr}(\mathrm{GL}_n(F))$ with $n \leq md$. For general linear groups (c) is a well-known property of the LLC, and in fact a starting point of the construction, confer [Hen, 1.2]. $\square$

For $\mathfrak{s} = [L, \omega]_G$ we define $\Phi(G)^{\mathfrak{s}}$ as the image of $\mathrm{Irr}^{\mathfrak{s}}(G)$ under the bijection $\mathrm{rec}_{D,m}$. Similarly we define $\Phi(L)^{\mathfrak{s}_L} \subset \Phi(L)$.

Lemma 5.2. *The LLC for G fits in a commutative diagram of canonical bijections*

$$\begin{array}{ccc} \mathrm{Irr}^{\mathfrak{s}}(G) & \xrightarrow{\mathrm{rec}_{D,m}} & \Phi(G)^{\mathfrak{s}} \\ \updownarrow & & \updownarrow \\ (T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2 & \longleftrightarrow & (\Phi(L)^{\mathfrak{s}_L}/\!/W_{\mathfrak{s}})_2 \end{array}$$

Here the bottom map comes from the LLC for $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ *and the left hand side comes from Theorem 4.3.*

(*a*) *Suppose that* $[\phi_L] \in \Phi(L)^{\mathfrak{s}_L}$ *and that* $\rho \in \mathrm{Irr}(W_{\mathfrak{s},\phi_L})$ *has as Springer parameter a unipotent class* $[u] \in Z_{\check{G}_{\mathfrak{s}}}(\phi_L)$*. Then there is a representative u such that the right hand side sends* $[\phi_L, \rho]$ *to a Langlands parameter* ϕ *with* $\phi|_{\mathbf{W}_F} = \phi_L|_{\mathbf{W}_F}$ *and* $\phi(1, (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})) = \phi_L(1, (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}))u$.

(*b*) *Conjecture 2.2 holds for* $\mathrm{Irr}^{\mathfrak{s}}(G)$.

Proof. Apart from the right hand side, the maps have already been established as bijective and canonical. So there is a unique, canonical way to complete the commutative diagram.
(a) To work out the map on the right hand side, it suffices to consider

$$L = \prod\nolimits_i L_i^{e_i} \quad \text{and} \quad \omega = \prod\nolimits_i \omega_i^{e_i}$$

such that (L_i, ω_i) is not isomorphic to (L_j, ω_j) for $i \neq j$. Let $\phi_i : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{m_i d}(\mathbb{C})$ be a Langlands parameter for ω_i. Then

$$\phi_L = \prod\nolimits_i \phi_i^{e_i} : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \prod\nolimits_i \mathrm{GL}_{m_i d}(\mathbb{C})^{e_i}$$

is a Langlands parameter for ω. We have $W_{\mathfrak{s},\phi_L} = \prod_i S_{e_i}$, where S_{e_i} is embedded in $N_{\mathrm{GL}_{e_i d_i m}(\mathbb{C})}(\mathrm{GL}_{m_i d}(\mathbb{C})^{e_i})$ as permutation matrices. The unipotent class

$$[u] = [\prod\nolimits_i u_i] \in \prod\nolimits_i \mathrm{GL}_{e_i m_i d}(\mathbb{C}) \subset Z_{\check{G}_{\mathfrak{s}}}(\phi_L)$$

is determined by the standard Levi subgroup in which it is distinguished, say

$$\check{M} = \prod\nolimits_{i,j} \mathrm{GL}_{b_{ij} m_i d}(\mathbb{C})^{c_{ij}} \quad \text{with} \quad \sum\nolimits_j c_{ij} b_{ij} = e_i .$$

Assume for the moment that ω is tempered. By Theorem 4.3 $[\omega, \rho] \in (T_{\mathfrak{s},\mathrm{un}} /\!/ W_{\mathfrak{s}})_2$ corresponds to $I_{PM}^G(\delta)$, where

$$\delta = \prod\nolimits_{i,j} \delta_{ij}^{c_{ij}} \in \mathrm{Irr}_{\mathrm{temp}}^{[L,\omega]_M}(M)$$

is the unique square-integrable modulo centre representation such that $W_{\mathfrak{s},M}\omega$ is the unitary part of the cuspidal support of δ. By construction [ABPS3, §2] the Langlands parameter ϕ of $I_{PM}^G(\delta)$ is the same as that of δ, namely $\phi = \prod_{i,j} \phi_{ij}^{c_{ij}}$ with $\phi_{ij}|_{\mathbf{W}_F} = \phi_i^{b_{ij}}|_{\mathbf{W}_F}$ and

$$\phi_{ij}(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = \phi_i(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))^{b_{ij}} u_{ij}$$

where u_{ij} is a distinguished unipotent element in $Z_{\mathrm{GL}_{b_{ij} m_i d}(\mathbb{C})}(\mathrm{GL}_{m_i d}(\mathbb{C})^{b_{ij}})$. Thus $\phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ is distinguished in $\check{M}$ and ϕ has the asserted shape.

The general case, where ω is not necessarily tempered, follows from the tempered case. The reason is that all the maps in the commutative diagram (a priori except the right hand side) can be obtained from their tempered parts by some kind of analytic continuation, as in [ABPS1] and Theorem 4.3.

(b) The first part holds by the definition of κ_ω (70) and the second part because our commutative diagram is canonical.

For the third part, by Theorem 5.1.b the elements of $(T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}})_2$ are in bijection with the L-packets in $\mathrm{Irr}^{\mathfrak{s}}(G)$. Two elements $[t, \rho]$ and $[t', \rho']$ are equal if and only if there is a $w \in W_{\mathfrak{s}}$ such that $wt' = t$ and $w \cdot \rho' = \rho$. We note also that for every $t \in T_{\mathfrak{s}}$ the group $W_{\mathfrak{s},t} = W(R_{\mathfrak{s},t})$ is product of symmetric groups. Hence all irreducible representations of $W_{\mathfrak{s},t}$ are parametrized by different unipotent classes in connected complex reductive group with maximal torus $T_{\mathfrak{s}}$ and root system $R_{\mathfrak{s},t}$. So the condition becomes that ρ and $w \cdot \rho'$ have the same unipotent class as Springer parameter. $\square$

Let $\mathrm{Irr}_{\mathrm{cusp}}(L)$ be the space of supercuspidal L-representations and let $\Phi(L)_{\mathrm{cusp}}$ be its image in $\Phi(L)$. The Weyl group

$$W(G, L) = N_G(L)/L \cong N_{\check{G}}(\check{L})/\check{L}$$

acts naturally on both sets.

Theorem 5.3. *Let $\mathcal{L}$ be a set of representatives for the conjugacy classes of Levi subgroups of G. The maps from Lemma 5.2 combine to a commutative diagram of canonical bijections*

$$\begin{array}{ccc} \mathrm{Irr}(G) & \xrightarrow{\ \mathrm{rec}_{D,m}\ } & \Phi(G) \\ \updownarrow & & \updownarrow \\ \bigsqcup_{L \in \mathcal{L}} \big(\mathrm{Irr}_{\mathrm{cusp}}(L) /\!/ W(G, L) \big)_2 & \longleftrightarrow & \bigsqcup_{L \in \mathcal{L}} \big(\Phi(L)_{\mathrm{cusp}} /\!/ W(G, L) \big)_2 \end{array}$$

Here the tempered representations correspond to the bounded Langlands parameters.

Proof. The action of $W(G,L)$ on L is simply by permuting some direct factors of L, and the same for $\check{L}$. Hence the canonical bijection $\mathrm{Irr}(L) \leftrightarrow \Phi(L)$ is $W(G,L)$-equivariant. The group $W_{\mathfrak s}$ is defined as the stabilizer in $W(G,L)$ of $T_{\mathfrak s} = \mathrm{Irr}^{\mathfrak s_L}(L)$, and by the above equivariance it is also the stabilizer $\Phi^{\mathfrak s_L}(L)$. Consequently

$$
\begin{aligned}
(\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(G,L))_2 &\cong \bigsqcup\nolimits_{\mathfrak s = [L,\omega]_G} (T_{\mathfrak s}/\!/W_{\mathfrak s})_2,\\
(\Phi(L)_{\mathrm{cusp}}/\!/W(G,L))_2 &\cong \bigsqcup\nolimits_{\mathfrak s = [L,\omega]_G} (\Phi^{\mathfrak s_L}(L)/\!/W_{\mathfrak s})_2.
\end{aligned}
$$

Now we simply take the union of the commutative diagrams of Lemma 5.2. The characterization of temperedness and boundedness comes from Theorems 5.1.a and 4.4.c. □

To formulate the LLC for $G^\sharp$, we need enhanced Langlands parameters. In fact these are already present in the LLC for G, but there the enhancement can be neglected without any problems.

Recall that a Langlands parameter for $G^\sharp = \mathrm{GL}_m(D)_{\mathrm{der}}$ is a homomorphism $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb C) \to \mathrm{PGL}_{md}(\mathbb C)$ subject to the same requirements as a Langlands parameter for G. The set of such parameters modulo conjugation by $\check{G}^\sharp = \mathrm{PGL}_{md}(\mathbb C)$ is denoted $\Phi(G^\sharp)$. We note that the simply connected cover $\mathrm{SL}_{md}(\mathbb C)$ of $\mathrm{PGL}_{md}(\mathbb C)$ also acts by conjugation on Langlands parameters for $G^\sharp$.

An enhancement of ϕ is an irreducible representation ρ of $\pi_0(Z_{\mathrm{SL}_{md}(\mathbb C)}(\phi))$. In order that (ϕ,ρ) is relevant for $G^\sharp$, an extra condition is needed. For this we have to regard D as part of the data of $G^\sharp$, in other words, we must consider not just the inner form $G^\sharp$ of $\mathrm{SL}_{md}(F)$, but even the inner twist determined by $(G^\sharp, D)$. The Hasse invariant of D gives a character χ_D of $Z(\mathrm{SL}_{md}(\mathbb C)) \cong \mathbb Z/md\mathbb Z$ with kernel $m\mathbb Z/md\mathbb Z$. Notice that, by Schur's lemma, every enhancement ρ of ϕ determines a character of $Z(\mathrm{SL}_{md}(\mathbb C))$. We define an enhanced Langlands parameter for $G^\sharp = \mathrm{GL}_m(D)_{\mathrm{der}}$ as a pair (ϕ,ρ) such that $\rho|_{Z(\mathrm{SL}_{md}(\mathbb C))} = \chi_D$. The collection of these, modulo conjugation by $\mathrm{SL}_{md}(\mathbb C)$, is denoted $\Phi_e(G^\sharp)$.

The LLC for $G^\sharp$ [ABPS3] is a bijection

$$
\Phi_e(G^\sharp) \longleftrightarrow \mathrm{Irr}(G^\sharp) : (\phi,\rho) \mapsto \pi(\phi,\rho). \tag{85}
$$

such that

- if ϕ lifts to a Langlands parameter $\tilde\phi$ for G, then $\pi(\phi,\rho)$ is a direct summand of $\mathrm{Res}^G_{G^\sharp}(\mathrm{rec}^{-1}_{D,m}(\tilde\phi))$,
- $\pi(\phi,\rho)$ is tempered if and only if ϕ is bounded,
- the L-packet

  $$\Pi_\phi(G^\sharp) = \{\pi(\phi,\rho) : \rho \in \mathrm{Irr}(\pi_0(Z_{\mathrm{SL}_{md}(\mathbb C)}(\phi))), \rho|_{Z(\mathrm{SL}_{md}(\mathbb C))} = \chi_D\}$$

  is canonically determined.

As $\mathrm{Irr}^{\mathfrak s}(G^\sharp)$ is defined in terms of restriction from $\mathrm{Irr}^{\mathfrak s}(G)$, it is a union of L-packets for $G^\sharp$. With the second property of (85), it canonically determines a set $\Phi_e(G^\sharp)^{\mathfrak s}$ of enhanced Langlands parameters for $G^\sharp$.

In the same way as for G, the LLC for a Levi subgroup $L^\sharp = L \cap G^\sharp$ follows from that for $L = \prod_i \mathrm{GL}_{m_i}(D)$. It involves enhancements from the action of

$$(\check{L}^\sharp)_{sc} = \mathrm{SL}_{md}(\mathbb C) \cap \prod\nolimits_i \mathrm{GL}_{m_i d}(\mathbb C).$$

Given $\mathfrak{s}_L = [L, \omega]_L$, $\mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp)$ is a union of L-packets for $L^\sharp$. Hence the corresponding set $\Phi_e(G^\sharp)^{\mathfrak{s}}$ of enhanced Langlands parameters is well-defined.

Lemma 5.4. *The LLC for $G^\sharp$ and the maps from Lemma 4.1, Theorem 4.4.b and Corollary 6.7 fit in the following commutative bijective diagram:*

$$
\begin{array}{ccc}
\mathrm{Irr}^{\mathfrak{s}}(G^\sharp) & \longleftrightarrow & \Phi_e(G^\sharp)^{\mathfrak{s}} \\
\updownarrow & & \updownarrow \\
(\mathrm{Irr}^{\mathfrak{s}_L}(L^\sharp) /\!/ W_{\mathfrak{s}}^\sharp)_{\kappa_\omega} & \longleftrightarrow & (\Phi_e(L^\sharp)^{\mathfrak{s}_L} /\!/ W_{\mathfrak{s}}^\sharp)_{\kappa_\omega} \\
\updownarrow & & \updownarrow \\
\big((T_{\mathfrak{s}} /\!/ X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} /\!/ W_{\mathfrak{s}}^\sharp \big)_{\kappa_\omega} & \longleftrightarrow & \big((\Phi(L)^{\mathfrak{s}_L} /\!/ X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} /\!/ W_{\mathfrak{s}}^\sharp \big)_{\kappa_\omega} \\
\updownarrow & & \updownarrow \\
\big(T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp) \big)_{\kappa_\omega} & \longleftrightarrow & \big(\Phi(L)^{\mathfrak{s}_L} /\!/ \mathrm{Stab}(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp) \big)_{\kappa_\omega} \\
\updownarrow & & \updownarrow \\
\big((T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}})_2 /\!/ \mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(L/L^\sharp) \big)_{\kappa_\omega} & \longleftrightarrow & \big((\Phi(L)^{\mathfrak{s}_L} /\!/ W_{\mathfrak{s}})_2 /\!/ \mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(L/L^\sharp) \big)_{\kappa_\omega}
\end{array}
$$

All these maps are canonical up to permutations within L-packets. In the last row the collection of L-packets is in bijection with $(T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}})_2 / \mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(L/L^\sharp)$ and with $(\Phi(L)^{\mathfrak{s}_L} /\!/ W_{\mathfrak{s}})_2 / \mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(L/L^\sharp)$.

Proof. The bijection between the first and the fourth set on the left hand side is given by Theorem 4.4.b. Then Corollary B.4 and (76) give bijections to the third and fifth sets on the left, as the 2-cocycle κ_ω is by construction (70) trivial on $W_{\mathfrak{s}}$. The bijection between the second and third sets on the left comes from Lemma 4.1.a. By Lemma 4.1.b it is canonical up to permutations within L-packets.

The LLC for L is equivariant for permutations of the direct factors of L and for twisting with characters of L (because the LLC for $\mathrm{GL}_m(D)$ is so). This gives the three lower horizontal bijections. Applying Corollary B.4 to the three lower terms on the right hand side gives bijections between them, and shows that the two lower squares in the diagram are canonical and commutative.

Similarly the LLC for $L^\sharp$ is equivariant for the action of $W_{\mathfrak{s}}^\sharp$, which leads to the second horizontal bijection. We define the upper two maps on the right hand side as the unique bijections that make the diagram commute. Since all the other maps in the upper two squares are canonical up to permutations within L-packets, so are the last two.

An L-packet for $G^\sharp$ consists of the irreducible $G^\sharp$-constituents of an irreducible G-representation. In view of Lemma 5.2, the collection of L-packets in $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp)$ is canonically in bijection with $(T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}})_2$. From (58) we can see how

$$\big((T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}})_2 /\!/ \mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(L/L^\sharp) \big)_{\kappa_\omega}$$

is constructed on the level of representations. We take an element $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G)$ and transform it to an irreducible representation of $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ by a geometric equivalence. Then we form the twisted extended quotient by $\mathrm{Stab}(\mathfrak{s})^+$, using Lemmas

B.1 and B.2, which corresponds to identifying π with π' if they have the same restriction to $G^\sharp Z(G)$, and decomposing π in irreducible $G^\sharp Z(G)$-subrepresentations. Finally we divide out the action of $X_{\rm nr}(L^\sharp Z(G)/L^\sharp)$, thus identifying the $G^\sharp Z(G)$-representations with the same restriction to $G^\sharp$. The implies the description of the L-packets in the lower left term of the commutative diagram, and hence also in the lower right term. $\square$

The bijection between the upper and the lower term on the right hand side of Lemma 5.4 can also be obtained as follows. First apply the recipe from Lemma 5.2 $(\Phi(L)^{\mathfrak s_L} // W_{\mathfrak s})_2$, then take the twisted extended quotient with respect to ${\rm Stab}(\mathfrak s)^+$, and finally divide out the free action of $X_{\rm nr}(L^\sharp Z(G)/L^\sharp)$ to reach $\Phi_e(G^\sharp)^{\mathfrak s_L}$.

Lemma 5.5. *Let $\mathfrak t^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$ be an inertial equivalence class subordinate to $\mathfrak s = [L,\omega]_G$. Lemma 4.5.a and the LLC for $G^\sharp$ and for $L^\sharp$ provide a commutative, bijective diagram*

$$\begin{array}{ccc} {\rm Irr}^{\mathfrak t^\sharp}(G^\sharp) & \longleftrightarrow & \Phi_e(G^\sharp)^{\mathfrak t^\sharp} \\ \updownarrow & & \updownarrow \\ (T_{\mathfrak t^\sharp} // W_{\mathfrak t^\sharp})_{\kappa_{\sigma^\sharp}} & \longleftrightarrow & (\Phi_e(L^\sharp)^{[L^\sharp,\sigma^\sharp]_{L^\sharp}} // W_{\mathfrak t^\sharp})_{\kappa_{\sigma^\sharp}} \end{array}$$

Two elements $[t,\rho],[t',\rho'] \in (T_{\mathfrak t^\sharp} // W_{\mathfrak t^\sharp})_{\kappa_{\sigma^\sharp}}$ are mapped to $G^\sharp$-representations in the same L-packet if and only if

- *$wt' = w$ for some $w \in W_{\mathfrak t^\sharp}$;*
- *the $W_{\mathfrak s,t}$-representations ρ and $w \cdot \rho'$ have Springer parameters with the same unipotent class, in the complex reductive group with maximal torus $T_{\mathfrak t^\sharp}$, root system $R_{\mathfrak t^\sharp,t}$ and Weyl group $W_{\mathfrak t^\sharp,t}$.*

Proof. The commutative diagram is obtained from Lemma 5.4, taking (39) into account. To see whether $[t,\rho]$ and $[t',\rho']$ belong to the same L-packet, Lemma 5.4 says that it suffices to look at their images in $(T_{\mathfrak s} // W_{\mathfrak s})_2 / {\rm Stab}(\mathfrak s)^+ X_{\rm nr}(L/L^\sharp)$.

Let $\tilde t \in T_{\mathfrak s}$ be a lift of t. Then $W_{\mathfrak t^\sharp,t}$ is the isotropy group of $X^L(\mathfrak s) X_{\rm nr}(L/L^\sharp)(\tilde\sigma^\sharp) \in T_{\mathfrak t^\sharp}$ in $W_{\mathfrak s}^\sharp$. Here $\sigma^\sharp$ is a projective representation of

$$(X^L(\mathfrak s) X_{\rm nr}(L/L^\sharp))_{\tilde t} = X^L(\omega).$$

With Lemma B.1 we get

$$\sigma^\sharp \rtimes \rho \in {\rm Irr}(\mathbb C[({\rm Stab}(\mathfrak s) X_{\rm nr}(L/L^\sharp)_{\tilde t}, \kappa_\omega]).$$

The intersection of $({\rm Stab}(\mathfrak s) X_{\rm nr}(L/L^\sharp)_{\tilde t}$ with $W_{\mathfrak s}$ is $W_{\mathfrak s,\tilde t} = W(R_{\mathfrak s,\tilde t})$. Since $W_{\mathfrak s}$ commutes with $X^L(\mathfrak s) X_{\rm nr}(L/L^\sharp)$, the restriction of $\sigma^\sharp \rtimes \rho$ to $W_{\mathfrak s,\tilde t}$ is $\dim(\sigma^\sharp)$ times $\rho|_{W_{\mathfrak s,\tilde t}}$. We want to show that

$$R_{\mathfrak t^\sharp,t} = R_{\mathfrak s,\tilde t}, \tag{86}$$

although in general $W_{\mathfrak t^\sharp,t}$ is strictly larger than $W_{\mathfrak s,\tilde t}$. Both root systems are defined in terms of zeros of Harish-Chandra μ-functions associated to roots $\alpha \in R_{\mathfrak s}$. The function μ_α (for G) is defined via intertwining operators betweeen G-representations, see [Wal, §IV.3 and §V.2]. These remain well-defined as intertwining operators between $G^\sharp$-representations, which implies that μ_α factors through $T_{\mathfrak s} \to T_{\mathfrak t^\sharp}$ and in this way gives the function μ_α for $G^\sharp$. By [Sil2, Theorem 1.6] all zeros of μ_α are fixed points of the reflection $s_\alpha \in W_{\mathfrak s}$. Hence $\mu_\alpha(t) \neq 0$ if $s_\alpha(\tilde t) \neq \tilde t$, proving (86).

It follows that $[t,\rho]$ maps to $[\tilde t, \rho|_{W(R_{\mathfrak t^\sharp,t})}$ in $(T_{\mathfrak s}//W_{\mathfrak s})_2/\mathrm{Stab}(\mathfrak s)^+ X_{\mathrm{nr}}(L/L^\sharp)$, and similarly for $[t',\rho']$. The $\mathrm{Stab}(\mathfrak s)^+ X_{\mathrm{nr}}(L/L^\sharp)$-orbits of $[\tilde t,\rho|_{W(R_{\mathfrak t^\sharp,t})}$ and $[\tilde t',\rho|_{W(R_{\mathfrak t^\sharp,t'})}$ are equal if and only if

$$\text{there is a } w \in W_{\mathfrak t^\sharp} \text{ such that } wt' = t \text{ and } (w\rho')|_{W(R_{\mathfrak t^\sharp,t})} = \rho|_{W(R_{\mathfrak t^\sharp,t})}.$$

By Lemma 5.2.b the last condition is equivalent to $w\rho'$ and ρ having the same unipotent class as Springer parameter. Because w is only determined up to $W_{\mathfrak t^\sharp t}$, these unipotent classes must be considered in the complex reductive group with maximal torus $T_{\mathfrak t^\sharp}$, root system $R_{\mathfrak t^\sharp,t}$ and Weyl group $W_{\mathfrak t^\sharp,t}$. □

As before, let $\mathcal L$ be a set of representatives for the conjugacy classes of Levi subgroups of G. Then $\{L^\sharp : L \in \mathcal L\}$ is a set of representatives for the conjugacy classes of Levi subgroups of $G^\sharp$.

Theorem 5.6. *The maps from Lemma 5.4 combine to a commutative diagram of bijections*

$$\begin{array}{ccc}
\mathrm{Irr}(G^\sharp) & \longleftrightarrow & \Phi_e(G^\sharp) \\
\updownarrow & & \updownarrow \\
\bigsqcup_{L\in\mathcal L} \big(\mathrm{Irr}_{\mathrm{cusp}}(L^\sharp)//W(G^\sharp,L^\sharp)\big)_\natural & \longleftrightarrow & \bigsqcup_{L\in\mathcal L} \big(\Phi(L^\sharp)_{\mathrm{cusp}}//W(G^\sharp,L^\sharp)\big)_\natural \\
\updownarrow & & \updownarrow \\
\bigsqcup_{L\in\mathcal L} \big(\mathrm{Irr}_{\mathrm{cusp}}(L)//\mathrm{Irr}(L/L^\sharp)W(G,L)\big)_\natural & \longleftrightarrow & \bigsqcup_{L\in\mathcal L} \big(\Phi(L)_{\mathrm{cusp}}//\mathrm{Irr}(L/L^\sharp)W(G,L)\big)_\natural
\end{array}$$

Here the family of 2-cocycles $\natural$ restricts to κ_ω on $\mathrm{Irr}^{[L,\omega]_L}(L)$. The tempered representations correspond to the bounded enhanced Langlands parameters and the entire diagram is canonical up to permutations within L-packets.

Proof. The upper square follows quickly from Lemma 5.4, in the same way as Theorem 5.3 followed from Lemma 5.2.

Recall from Lemma 4.1 that

$$\mathrm{Irr}^{\mathfrak s_L}(L^\sharp) \text{ is in bijection with } (T_{\mathfrak s}//X^L(\mathfrak s)X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega}.$$

Here $X^L(\mathfrak s)$ is the stabilizer of $\mathfrak s_L = [L,\omega]_L$ in $\mathrm{Irr}(L/L^\sharp Z(G))$. A character of $L/L^\sharp$ which is ramified on $Z(G)$ cannot stabilize $\mathfrak s_L$, so $X^L(\mathfrak s)X_{\mathrm{nr}}(L/L^\sharp)$ is the stabilizer of $\mathfrak s_L$ in $\mathrm{Irr}(L/L^\sharp)$. By Theorem 5.1 the LLC for L is bijective and $\mathrm{Irr}(L/L^\sharp)$-equivariant, so $X^L(\mathfrak s)X_{\mathrm{nr}}(L/L^\sharp)$ is also the stabilizer of $\Phi(L)^{\mathfrak s_L}$ in $\mathrm{Irr}(L/L^\sharp)$. This implies

$$\begin{aligned}
(\mathrm{Irr}_{\mathrm{cusp}}(L)//\mathrm{Irr}(L/L^\sharp))_\natural &\cong \bigsqcup\nolimits_{\mathfrak s_L=[L,\omega]_L} (\mathrm{Irr}^{\mathfrak s_L}(L)//X^L(\mathfrak s)X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} \\
&\cong \bigsqcup\nolimits_{\mathfrak s_L=[L,\omega]_L} \mathrm{Irr}^{\mathfrak s_L}(L^\sharp) = \mathrm{Irr}_{\mathrm{cusp}}(L^\sharp),
\end{aligned}$$

and similarly for Langlands parameters. These bijections are equivariant for permutations of the direct factors of L, so applying $(?//W(G,L))_{\kappa_\omega}$ to all of them produces a commutative square as in the theorem, but with lower row

$$\bigsqcup\nolimits_{L\in\mathcal L} \big((\mathrm{Irr}_{\mathrm{cusp}}(L) /\!/ \mathrm{Irr}(L/L^\sharp))_\natural /\!/ W(G,L) \big)_\natural \longleftrightarrow$$
$$\bigsqcup\nolimits_{L\in\mathcal L} \big((\Phi(L)_{\mathrm{cusp}} /\!/ \mathrm{Irr}(L/L^\sharp))_\natural /\!/ W(G,L) \big)_\natural .$$

We apply Corollary B.4 to get the row in the theorem. The canonicity of the thus obtained commutative diagram is a consequence of the analogous property in Lemma 5.4. The temperedness/boundedness correspondence follows from the properties of the local Langlands correspondences for $G, G^\sharp, L$ and $L^\sharp$. $\square$

6. Schwartz algebras

Harish–Chandra's Schwartz algebra $\mathcal S(G)$ is a completion of the Hecke algebra $\mathcal H(G)$. It is particularly useful for the harmonic analysis on G, see e.g. [Wal]. By definition a smooth G-representation is tempered if and only if it extends to a $\mathcal S(G)$-module.

On the other hand, for affine Hecke algebras like

$$\mathcal H(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}, q_{\mathfrak s}) \cong \mathcal H(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$$

a Schwartz completion was defined and studied in [DeOp]. In this section we will compare these two kinds of Schwartz algebras. We do so both for G and for its derived group $G^\sharp$. Throughout this section we assume that $\mathfrak s = [L,\omega]$ with $\omega \in \mathrm{Irr}(L)$ supercuspidal and unitary (and hence tempered).

First we need to recall the precise definition of $\mathcal S(G)$. Let d be a G–invariant metric on the (enlarged) Bruhat–Tits building $\mathcal B(G)$. Fix a special vertex x_0 of $\mathcal B(G)$. Now

$$d_{x_0} : g \mapsto d(g x_0, x_0)$$

is a locally constant function $G \to \mathbb R_{\geq 0}$. For any $N \in \mathbb N$ one defines a norm on $\mathcal H(G)$ by

$$\nu_N(f) := \big\| (1 + d_{x_0})^N f \big\|_2 = \Big(\int_G (1 + d_{x_0}(g))^{2N} |f(g)|^2 \mathrm d\mu(g) \Big)^{1/2}. \tag{87}$$

For any compact open subgroup $K \subset G$, ν_N becomes a norm on

$$\mathcal H(G,K) = e_K \mathcal H(G) e_K .$$

As in [Vig, §9] one defines $\mathcal S(G,K)$ as the completion of $\mathcal H(G,K)$ with respect to the family of norms $\{\nu_N : N \in \mathbb N\}$. Finally one puts

$$\mathcal S(G) = \bigcup\nolimits_K \mathcal S(G,K),$$

where the union runs over all compact open subgroups K. The definitions of $\mathcal S(G^\sharp)$ and $\mathcal S(G^\sharp Z(G))$ are analogous.

Given an inertial equivalence class $\mathfrak s$ for G, $\mathcal S(G)^{\mathfrak s}$ denotes the completion of $\mathcal H(G)^{\mathfrak s}$ in $\mathcal S(G)$. Equivalently, $\mathcal S(G)^{\mathfrak s}$ is the two-sided ideal of $\mathcal S(G)$ generated by $\mathcal H(G)^{\mathfrak s}$. The definition with ideals can clearly be applied to $G^\sharp$ and $G^\sharp Z(G)$. Hence the modules of $\mathcal S(H)^{\mathfrak s}$ are precisely the tempered representations in $\mathrm{Rep}^{\mathfrak s}(H)$, where $H \in \{G, G^\sharp, G^\sharp Z(G)\}$. Like in Bushnell–Kutzko theory, idempotents can be used to construct smaller, Morita equivalent subalgebras of $\mathcal S(H)^{\mathfrak s}$.

Lemma 6.1. *There are Morita equivalences*

$$\begin{aligned}
\mathcal{S}(G)^{\mathfrak{s}} &\sim_M e^\sharp_{\lambda_G}\mathcal{S}(G)e^\sharp_{\lambda_G} &&\cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}[L/H_\lambda])\otimes e_{\mu_G}\mathcal{S}(G)e_{\mu_G}\\
\mathcal{S}(G^\sharp Z(G))^{\mathfrak{s}} &\sim_M e^\sharp_{\lambda_{GZ(G)}}\mathcal{S}(G^\sharp Z(G))e^\sharp_{\lambda_{G^\sharp Z(G)}} &&\cong \bigoplus_{a\in[L/H_\lambda]} e_{\mu_{G^\sharp Z(G)}}\mathcal{S}(G^\sharp Z(G))e_{\mu_{G^\sharp Z(G)}}\\
\mathcal{S}(G^\sharp)^{\mathfrak{s}} &\sim_M e^\sharp_{\lambda_{G^\sharp}}\mathcal{S}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}} &&\cong \bigoplus_{a\in[L/H_\lambda]} e_{\mu_{G^\sharp}}\mathcal{S}(G^\sharp)e_{\mu_{G^\sharp}}.
\end{aligned}$$

Proof. Recall that the analogous Morita equivalences for Hecke algebras were already proven in [ABPS4], see (23) and Theorems 1.2 and 1.3. Let H denote any of the groups $G, G^\sharp, G^\sharp Z(G)$. With the bimodules $e^\sharp_{\lambda_H}\mathcal{S}(H)$ and $\mathcal{S}(H)e^\sharp_{\lambda_H}$ we calculate

$$\begin{aligned}
e^\sharp_{\lambda_H}\mathcal{S}(H)\otimes_{\mathcal{S}(H)^{\mathfrak{s}}}\mathcal{S}(H)e^\sharp_{\lambda_H} &= e^\sharp_{\lambda_H}\mathcal{S}(H)^{\mathfrak{s}}\otimes_{\mathcal{S}(H)^{\mathfrak{s}}}\mathcal{S}(H)^{\mathfrak{s}}e^\sharp_{\lambda_H}\\
&\cong e^\sharp_{\lambda_H}\mathcal{S}(H)^{\mathfrak{s}}e^\sharp_{\lambda_H} = e^\sharp_{\lambda_H}\mathcal{S}(H)e^\sharp_{\lambda_H},\\
\mathcal{S}(H)e^\sharp_{\lambda_H}\otimes_{e^\sharp_{\lambda_H}\mathcal{S}(H)e^\sharp_{\lambda_H}} e^\sharp_{\lambda_H}\mathcal{S}(H) &\cong \mathcal{S}(H)e^\sharp_{\lambda_H}\mathcal{S}(H)\\
&= \mathcal{S}(H)\mathcal{H}(H)^{\mathfrak{s}}\mathcal{S}(H) = \mathcal{S}(H)^{\mathfrak{s}}.
\end{aligned}$$

This means that these bimodules implement the desired Morita equivalences.

Recall the formulas (22) for the involved idempotents. The $ae_{\mu_H}a^{-1}$ with $a \in [L/H_\lambda]$ are mutually orthogonal. For $H = G$ they are conjugate, which leads to the isomorphism for $H = G$, see [ABPS4, Proposition 3.15]. For $H \in \{G^\sharp, G^\sharp Z(G)\}$ all the $ae_{\mu_H}a^{-1}$ live in different Bernstein components. The desired isomorphisms are consequence thereof, see Theorems 1.2.a and 1.3.a. □

6.1. Fourier transforms.

The comparison of the various Schwartz algebras will go via their Fourier transforms. To get to grips with them, we first work them out for G. The Plancherel isomorphism for G [Wal] provides a description of $\mathcal{S}(G)$ in terms of the space $\mathrm{Irr}_{\mathrm{temp}}(G)$ of irreducible tempered G-representations. As a consequence of this isomorphism, $\mathrm{Irr}_{\mathrm{temp}}(G)$ is the support of the Plancherel measure on $\mathrm{Irr}(G)$. This means that the tracial state $f \mapsto f(1)$ can be computed as an integral of $\mathrm{tr}(\pi(f))$ over $\mathrm{Irr}_{\mathrm{temp}}(G)$, endowed with the Plancherel measure.

Let $e_{\mathfrak{s}} \in \mathcal{H}(G)$ be an idempotent such that $\mathcal{H}(G)e_{\mathfrak{s}}\mathcal{H}(G) = \mathcal{H}(G)^{\mathfrak{s}}$. It gives rise to the following data.

- A finite set $\Delta_{G,\mathfrak{s}}$ of pairs (P,σ), where $P = MU$ is a standard parabolic subgroup of G and (σ, V_σ) is an irreducible square-integrable (modulo centre) representation of M. $\Delta_{G,\mathfrak{s}}$ contains one element for every such pair (P,σ) with $e_{\mathfrak{s}}I_P^G(V_\sigma) \neq 0$, considered up to G-conjugation and up to twists by unramified unitary characters.
- For every such pair a torus
$$T_{P,\sigma} = \{\sigma\otimes\chi \in \mathrm{Irr}(M) : \chi \in X_{\mathrm{unr}}(M)\},$$
where $X_{\mathrm{unr}}(M)$ denotes the group of unitary unramified characters of M. We identify $T_{P,\sigma}$ with $X_{\mathrm{unr}}(M)/X_{\mathrm{nr}}(M,\sigma)$ via $\chi \mapsto \sigma\otimes\chi$.
- For every $(P,\sigma) \in \Delta_{G,\mathfrak{s}}$ a finite group $W_{\mathfrak{s},\sigma}$, namely the stabilizer of $T_{P,\sigma}$ in $W_{\mathfrak{s}}$.
- For every $w \in W_{P,\sigma}$ an intertwining operator
$$I(w,\sigma\otimes\chi) \in \mathrm{Hom}_{G\times G}\big(\mathrm{End}_{\mathbb{C}}(I_P^G(\sigma\otimes\chi)), \mathrm{End}_{\mathbb{C}}(I_P^G(w(\sigma\otimes\chi)))\big).$$

- The Fréchet algebra $C^\infty(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}_{\mathbb{C}}(e_{\mathfrak{s}} I_P^G(V_\sigma))$.
- An action of $W_{P,\sigma} := W_{\mathfrak{s},\sigma} \ltimes X_{\mathrm{nr}}(M,\sigma)$ on this algebra by
$$(w \cdot f)(\chi) = I(w, w^{-1}(\sigma \otimes \chi)) f(w^{-1}(\chi)),$$
where $X_{\mathrm{nr}}(M,\sigma)$ acts by translations on $X_{\mathrm{unr}}(M)$.

Based on [Wal], it was checked in [Sol1, Theorem 2.9] that Harish-Chandra's Plancherel isomorphism restricts to an isomorphism of Fréchet algebras

$$\begin{array}{ccc} e_{\mathfrak{s}} \mathcal{S}(G) e_{\mathfrak{s}} & \longrightarrow & \bigoplus_{(P,\sigma) \in \Delta_{G,\mathfrak{s}}} \big(C^\infty(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}_{\mathbb{C}}(e_{\mathfrak{s}} I_P^G(V_\sigma)) \big)^{W_{P,\sigma}} \\ h & \mapsto & [(P,\sigma,\chi) \mapsto I_P^G(\sigma \otimes \chi)(h)]. \end{array} \tag{88}$$

Let us consider an affine Hecke algebra $\mathcal{H}(W,q)$ based on an (extended) affine Weyl group W and a parameter function q, as for example in [Opd]. It is assumed among others that $W = X \rtimes W_0$ where X is a lattice containing a root system with a finite Weyl group W_0.

We need a length function $\mathcal{N} : W \to \mathbb{R}_{\geq 0}$ which is "close" to the length function of the affine Coxeter group contained in W. There are many suitable choices. In the important case $W = \mathbb{Z}^m \rtimes S_m$ we can take

$$\mathcal{N}(x\sigma) = \|x\|_2 \,,$$

and the other cases we encounter can be derived from that. The algebra $\mathcal{H}(W,q)$ comes with a distinguished basis $\{N_w : w \in W\}$, where $N_w = [w] q(w)^{-1/2}$ in the notation of Section 1. For each $N \in \mathbb{N}$ one defines a norm on $\mathcal{H}(W,q)$ by

$$p_N \big(\sum_{w \in W} c_w N_w \big) = \sup_{w \in W} |c_w| (1 + \mathcal{N}(w))^N . \tag{89}$$

Then the Schwartz algebra $\mathcal{S}(W,q)$ is the completion of $\mathcal{H}(W,q)$ with respect to the family of norms $\{p_N : N \in \mathbb{N}\}$. On elementary grounds [OpSo, (130)] this family is equivalent with the family of norms

$$p'_N \big(\sum_{w \in W} c_w N_w \big) = \big(\sum_{w \in W} |c_w|^2 (1 + \mathcal{N}(w))^{2N} \big)^{1/2} .$$

Recall from (32) that $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ is defined to be the vector space $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathcal{H}(W_{\mathfrak{s}}, q_{\mathfrak{s}})$ with certain multiplication rules. The choice of a basepoint of $T_{\mathfrak{s}}$ determines an isomorphism

$$\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \cong \mathcal{H}(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}, q_{\mathfrak{s}}),$$

and we use that to transfer the norms p'_N to norms p''_N on $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$. The completion with respect to the latter family of norms is a Fréchet algebra $\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ which is isomorphic to $\mathcal{S}(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}, q_{\mathfrak{s}})$. The equivalence class of the norm p''_N does not depend on the choice of a basepoint of $T_{\mathfrak{s}}$ if we suppose that it belongs to the maximal compact subtorus of $T_{\mathfrak{s}}$. Hence $\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ is defined canonically.

An $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$-module is called tempered if it extends continuously to a $\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$-module. It is known from [Opd] that the space

$$\mathrm{Irr}_{\mathrm{temp}}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})) = \mathrm{Irr}(\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}))$$

is precisely the support of the Plancherel measure on $\mathrm{Irr}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}))$. Here the Plancherel measure comes from the standard trace on $\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$:

$$\tau_{\mathcal{H}}(\theta_x [v]) = \begin{cases} 1 & \text{if } x = 0, v = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In [DeOp] the Plancherel isomorphism for $\mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ was worked out. It is built with the following data.

- A collection $\Delta_{\mathcal{H},\mathfrak{s}}$ of pairs (M, δ), where $M \subset G$ is a standard Levi subgroup containing L and (δ, V_δ) is a square-integrable (modulo centre) representation of the parabolic subalgebra $\mathcal{H}^M \subset \mathcal{H} = \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$. $\Delta_{\mathcal{H},\mathfrak{s}}$ forms a set of representatives for such pairs up to $W_\mathfrak{s}$-conjugation and character twists.
- For every such pair a torus
$$T^M_{\mathfrak{s},\mathrm{un}} := \{t \in T_{\mathfrak{s},\mathrm{un}} : t(x) = 1 \text{ if } x \in \mathbb{Q}R(M, L) \cap X^*(T_\mathfrak{s})\},$$
a quotient of $X_{\mathrm{unr}}(M)/(X_{\mathrm{unr}}(M) \cap X_{\mathrm{nr}}(L, \omega))$.
- A finite group $W_{\mathfrak{s},\delta}$, the stabilizer of $\delta \otimes T^M_{\mathfrak{s},\mathrm{un}}$ in $W_\mathfrak{s}$, and a finite group
$$T^M_{\mathfrak{s},M} = \{t \in T^M_{\mathfrak{s},\mathrm{un}} : t(x) = 1 \text{ if } \alpha(x) = 1 \ \forall \alpha \in R(M, L)\}.$$
- For every $w \in W_{M,\delta} := W_{\mathfrak{s},\delta} \ltimes T^M_{\mathfrak{s},M}$ an intertwining operator
$$I(w, \delta \otimes t) \in \mathrm{Hom}_{\mathcal{H}\times\mathcal{H}^{op}}\big(\mathrm{End}_\mathbb{C}(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^M}(\delta \otimes t)), \mathrm{End}_\mathbb{C}(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^M}(w(\delta \otimes t)))\big).$$
- The Fréchet algebra
$$\bigoplus\nolimits_{(M,\delta)\in\Delta_{\mathcal{H},\mathfrak{s}}} C^\infty(T_{M,\delta}) \otimes \mathrm{End}_\mathbb{C}(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^M}(V_\delta)).$$
- An action of $W_{M,\delta}$ on this algebra by
$$(w \cdot f)(t) = I(w, w^{-1}(\delta \otimes t)) f(w^{-1}t),$$
where $T^M_{\mathfrak{s},M}$ acts on $T^M_{\mathfrak{s},\mathrm{un}}$ by translations and $W_{\mathfrak{s},\delta}$ by $w(\delta \otimes t) = \delta \otimes w(t)$.

With these notations, the main result of [DeOp] states that

$$\begin{array}{ccc} \mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) & \longrightarrow & \bigoplus_{(M,\delta)\in\Delta_{\mathcal{H},\mathfrak{s}}} \big(C^\infty(T_{M,\delta} \otimes \mathrm{End}_\mathbb{C}(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^M}(V_\delta))\big)^{W_{M,\delta}} \\ h & \mapsto & [(M, \delta, t) \mapsto \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^M}(\delta \otimes t)(h)] \end{array} \tag{90}$$

is an isomorphism of Fréchet algebras.

Recall the isomorphism
$$e_{\mu_G}\mathcal{H}(G)e_{\mu_G} \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s})$$
from (20). As tensoring with $\mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s})$ is a Morita equivalence, it is natural to call a module $V \otimes V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s}$ over (20) tempered if and only if $V \in \mathrm{Mod}(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}))$ is tempered.

Theorem 6.2. *The isomorphism* (20) *extends in a unique way to an isomorphism of Fréchet algebras*
$$e_{\mu_G}\mathcal{S}(G)e_{\mu_G} \cong \mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s}).$$

Proof. The unicity is clear from the continuity and the density of the subalgebras (20) in their Schwartz completions.

Since $e_{\mu_G}\mathcal{H}(G)e_{\mu_G}$ is Morita equivalent with $\mathcal{H}(G)^\mathfrak{s}$,

$$\begin{array}{ccc} \mathrm{Rep}^\mathfrak{s}(G) & \longrightarrow & \mathrm{Mod}(e_{\mu_G}\mathcal{H}(G)e_{\mu_G}) \\ V & \mapsto & e_{\mu_G}V \end{array} \tag{91}$$

is an equivalence of categories. According to [BHK, Theorem A] it restricts to a homeomorphism between the spaces of irreducible tempered representations on both sides, and it preserves the Plancherel measures (up to some normalization factor). The isomorphism (20) also preserves temperedness of irreducible representations, because it matches the tracial states on which the Plancherel measures of both algebras are based, namely $f \mapsto f(1)$ and $\tau_{\mathcal{H}} \otimes \mathrm{tr}_{V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}}}$. Consequently (91) and (20) induce a homeomorphism

$$
\begin{array}{ccc} \mathrm{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G) & \longrightarrow & \mathrm{Irr}_{\mathrm{temp}}\big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})\big) \\ V & \mapsto & e_{\mu_G} V. \end{array} \tag{92}
$$

From the Plancherel isomorphisms for $e_{\mu_G}\mathcal{S}(G)e_{\mu_G}$ (88) we see that $\Delta_{G,\mathfrak{s}}$ contains precisely one pair (P, σ) for every connected component of $\mathrm{Irr}_{\mathrm{temp}}(G)$. Similarly $\Delta_{\mathcal{H},\mathfrak{s}}$ is in bijection with the set of components $\mathrm{Irr}_{\mathrm{temp}}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}))$, and this does not change upon tensoring the algebra with $\mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})$. Hence we may choose $\Delta_{\mathcal{H},\mathfrak{s}}$ such that

$$
\{e_{\mu_G} I_P^G(V_\sigma) : (P, \sigma) \in \Delta_{G,\mathfrak{s}}\} = \{\mathrm{ind}_{\mathcal{H}^M}^{\mathcal{H}}(V_\delta) \otimes V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}} : (M, \delta) \in \Delta_{\mathcal{H},\mathfrak{s}}\}.
$$

Then (92) induces a bijection

$$
X_{\mathrm{unr}}(M)/X_{\mathrm{nr}}(M, \sigma) \cong T_{P,\sigma} \;\to\; T_{M,\delta} \cong T^M_{\mathfrak{s},\mathrm{un}}/T^M_{\mathfrak{s},M}.
$$

It follows that $W_{\mathfrak{s},\sigma} = W_{\mathfrak{s},\delta}$. Since $T^M_{\mathfrak{s},\mathrm{un}}$ is a quotient of $X_{\mathrm{unr}}(M)$, also

$$
T^M_{\mathfrak{s},M} \cong X_{\mathrm{nr}}(M, \sigma)/(X_{\mathrm{nr}}(M, \sigma) \cap X_{\mathrm{nr}}(L, \omega)).
$$

Consider a $k \in X_{\mathrm{nr}}(M, \sigma) \cap X_{\mathrm{nr}}(L, \omega)$. Then $k = 1$ in $X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega) \cong T_{\mathfrak{s}}$, so the $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$-modules

$$
\mathrm{ind}_{\mathcal{H}^M}^{\mathcal{H}}(\delta \otimes k \otimes \chi) \quad \text{and} \quad \mathrm{ind}_{\mathcal{H}^M}^{\mathcal{H}}(\delta \otimes \chi)
$$

are the same for all $\chi \in X_{\mathrm{nr}}(M)$. Hence

$$
e_{\mu_G} I_P^G(\sigma \otimes k \otimes \chi) = e_{\mu_G} I_P^G(\sigma \otimes \chi)
$$

for all $\chi \in X_{\mathrm{nr}}(M)$ and

$$
I(k, \sigma \otimes \chi)|_{e_{\mu_G} I_P^G(\sigma \otimes \chi)} \in \mathbb{C}\,\mathrm{id}_{e_{\mu_G} I_P^G(\sigma \otimes \chi)}.
$$

Therefore the action of $W_{P,\sigma}$ on $C^\infty(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}_{\mathbb{C}}(e_{\mu_G} I_P^G(V_\sigma))$ is built from an action of $X_{\mathrm{nr}}(M) \cap X_{\mathrm{nr}}(L, \omega)$ on $X_{\mathrm{unr}}(M)$ and an action of the quotient

$$
W_{P,\sigma}/(X_{\mathrm{nr}}(M) \cap X_{\mathrm{nr}}(L, \omega)) \cong W_{M,\delta}
$$

on the $X_{\mathrm{nr}}(M) \cap X_{\mathrm{nr}}(L, \omega)$-invariant elements. Now (88) becomes an isomorphism

$$
e_{\mu_G}\mathcal{S}(G)e_{\mu_G} \to \bigoplus\nolimits_{(P,\sigma) \in \Delta_{G,\mathfrak{s}}} \big(C^\infty(T^M_{\mathfrak{s},un}) \otimes \mathrm{End}_{\mathbb{C}}(e_{\mu_G} I_P^G(V_\sigma))\big)^{W_{P,\sigma}}. \tag{93}
$$

Comparing this with (90) tensored with $\mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})$, we see that the Fourier transforms correspond via (92). Thus we obtain an isomorphism of topological algebras

$$
e_{\mu_G}\mathcal{S}(G)e_{\mu_G} \to \mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})
$$

which extends (20). $\square$

Both sides of Theorem 6.2 are defined as the completion of a subspace with respect to a family of (semi-)norms, namely the norms ν_N on $e_{\mu_G}\mathcal{H}(G)e_{\mu_G}$ and the norms $p_N \otimes \|\cdot\|_{\mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})}$ on $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})$, where $\|\cdot\|_{\mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}})}$ denotes any norm on this algebra. Hence these families of norms are equivalent under the isomorphisms (20) and (93).

In fact one can also prove Theorem 6.2 comparing these norms directly, generalising [DeOp, §10]. However, that would involve many tedious computations. We feel that the above proof is conceptually clearer.

Proposition 6.3.
(a) $\mathcal{S}(G^\sharp Z(G))$ *is Morita equivalent with* $\bigoplus_{a \in [L/H_\lambda]} (e_{\mu_G} \mathcal{S}(G) e_{\mu_G})^{X^G(\mathfrak{s})}$.
(b) There are isomorphisms of Fréchet algebras

$$\begin{aligned}(e_{\mu_G} \mathcal{S}(G) e_{\mu_G})^{X^G(\mathfrak{s})} &\cong (\mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes \mathbb{C}\mathfrak{R}_\mathfrak{s}^\sharp))^{X^G(\mathfrak{s})} \\ &\cong (\mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp.\end{aligned}$$

Proof. (a) By Lemma 6.1 it suffices to show that

$$e_{\mu_{G^\sharp Z(G)}} \mathcal{S}(G^\sharp Z(G)) e_{\mu_{G^\sharp Z(G)}} \cong (e_{\mu_G} \mathcal{S}(G) e_{\mu_G})^{X^G(\mathfrak{s})}. \tag{94}$$

The corresponding statement for Hecke algebras is the isomorphism

$$(e_{\mu_G} \mathcal{H}(G) e_{\mu_G})^{X^G(\mathfrak{s})} \to e_{\mu_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e_{\mu_{G^\sharp Z(G)}} \tag{95}$$

from [ABPS4, Lemma 4.9 and Corollary 4.10]. The underlying map is simply the restriction of functions $f : G \to \mathbb{C}$ to $G^\sharp Z(G)$. The norms defining the Schwartz completions are ν_N (87) and

$$\nu'_N(f) = \int_{G^\sharp Z(G)} (1 + d_{x_0}(g))^{2N} |f(g)|^2 \mathrm{d}\mu'(g).$$

To see that these norms are compatible, we consider the l-th congruence subgroup $C_l \subset G$. We write $C'_l = C_l \cap G^\sharp Z(G)$. For l sufficiently large, it was shown in the proof of [ABPS4, Lemma 3.10] that

$$\mathcal{H}(G^\sharp Z(G), C'_l)^\mathfrak{s} \cong (\mathcal{H}(G, C_l)^\mathfrak{s})^{X^G(\mathfrak{s})}. \tag{96}$$

Hence we can normalize the Haar measures on G and $G^\sharp Z(G)$ such that

$$\nu_N(f) = \nu'_N(f) \text{ for all } f \in (e_{\mu_G} \mathcal{H}(G) e_{\mu_G})^{X^G(\mathfrak{s})} \subset (\mathcal{H}(G, C_l)^\mathfrak{s})^{X^G(\mathfrak{s})}.$$

Then (95) extends continuously to an isomorphism of the Schwartz completions, namely (94).
(b) The isomorphism from Theorem 6.2 is $X^G(\mathfrak{s})$-equivariant by definition of the action of $X^G(\mathfrak{s})$ on

$$\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu) \otimes \mathrm{End}_\mathbb{C}(\mathbb{C}\mathfrak{R}_\mathfrak{s}^\sharp).$$

Its restriction to $X^G(\mathfrak{s})$-invariant elements gives the first statement of part (b). The $X^G(\mathfrak{s})$-action on the above algebra preserves the tensor factors and is the natural action on $\mathrm{End}_\mathbb{C}(\mathbb{C}\mathfrak{R}_\mathfrak{s}^\sharp)$. Knowing that, a standard argument, as in [ABPS4, Lemma 3.7], proves the second claim of part (b). □

To obtain a version of Proposition 6.3 for $G^\sharp$, we need to involve the action α of $X_{\mathrm{nr}}(G)$ on $\mathcal{H}(G)$ from (21). However, this action does not extend to $\mathcal{S}(G)$, for example because a twist of a tempered representation by a non-unitary character is no longer tempered. Fortunately, the action of the subgroup $X_{\mathrm{unr}}(G)$ of unitary unramified characters does extend continuously to $\mathcal{S}(G)$, for it preserves the norms that define the Schwartz completion.

In Theorem 1.2 we saw that $X_{\rm nr}(G)$ acts on $\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)$ via translations on $T_{\mathfrak s}$. In terms of basis elements this becomes

$$\alpha_\gamma(\theta_x[v] \otimes h) = \gamma^{-1}(x)\theta_x[v] \otimes h, \text{ for } x \in X^*(T_{\mathfrak s}), v \in W_{\mathfrak s}, h \in \mathrm{End}_{\mathbb C}(V_\mu).$$

Clearly this action stabilizes $\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$, and the action of the subgroup $X_{\rm unr}(G)$ on that subalgebra preserves the norms ν_N defining the Schwartz completion. Hence the action of $X_{\rm unr}(G)$ extends continuously to $\mathcal{S}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu)$, and it is still given by translations on $T_{\mathfrak s}$.

Theorem 6.4.
(a) $\mathcal{S}(G^\sharp)$ *is Morita equivalent with* $\bigoplus_{a \in [L/H_\lambda]} (e_{\mu_G} \mathcal{S}(G) e_{\mu_G})^{X^G(\mathfrak s) X_{\rm unr}(G)}$.
(b) There are isomorphisms of Fréchet algebras

$$\begin{aligned} (e_{\mu_G}\mathcal{S}(G)e_{\mu_G})^{X^G(\mathfrak s)X_{\rm unr}(G)} &\cong (\mathcal{S}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu \otimes \mathbb{C}\mathfrak{R}^\sharp_{\mathfrak s}))^{X^G(\mathfrak s)X_{\rm unr}(G)} \\ &\cong (\mathcal{S}(T^\sharp_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu))^{X^L(\mathfrak s)} \rtimes \mathfrak{R}^\sharp_{\mathfrak s}. \end{aligned}$$

Proof. (a) By Lemma 6.1 it suffices to show that

$$e_{\mu_{G^\sharp}} \mathcal{S}(G^\sharp) e_{\mu_{G^\sharp}} \cong (e_{\mu_G}\mathcal{S}(G)e_{\mu_G})^{X^G(\mathfrak s)X_{\rm unr}(G)}. \tag{97}$$

From [ABPS4, Theorem 3.17 and Lemma 4.9] we know that

$$e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}} \cong (e_{\mu_G}\mathcal{H}(G)e_{\mu_G})^{X^G(\mathfrak s)X_{\rm nr}(G)}. \tag{98}$$

Since $X_{\rm unr}(G)$ is Zariski-dense in $X_{\rm nr}(G)$, we may just as well replace $X_{\rm nr}(G)$ by $X_{\rm unr}(G)$ on the right hand side of (98). Next we compare the relevant Schwartz norms in the same way as in the proof of Proposition 6.3.a. To that end we need to know that

$$\mathcal{H}(G^\sharp, C^\sharp_l)^{\mathfrak s} \cong (\mathcal{H}(G, C_l)^{\mathfrak s})^{X^G(\mathfrak s)X_{\rm nr}(G)} = (\mathcal{H}(G, C_l)^{\mathfrak s})^{X^G(\mathfrak s)X_{\rm nr}(G)},$$

which follows from (96) and and the proof of [ABPS4, Theorem 3.17]. These considerations lead to an isomorphism between the Fréchet algebras (97).
(b) This follows from Theorem 6.2 and Proposition 6.3.b. □

6.2. Spectrum preserving morphisms.

In [Sol2] an algebra homomorphism

$$\zeta_0 : C^\infty(T_{\mathfrak s,\rm un}) \rtimes W_{\mathfrak s} \to \mathcal{S}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \tag{99}$$

with many nice properties was constructed. Notice that the left hand side is the $q_s = 1$ version of the right hand side. We will generalize this to the Schwartz algebras from Theorem 6.4.

In [Sol2, Lemma 5.3.2] it was shown that there exist filtrations on $C^\infty(T_{\mathfrak s,\rm un}) \rtimes W_{\mathfrak s}$ and on $\mathcal{S}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$ which are respected by ζ_0, and with respect to which ζ_0 is spectrum preserving. The choice of such filtrations determines a bijection

$$\mathrm{Irr}(\zeta_0) : \mathrm{Irr}(\mathcal{S}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})) \to \mathrm{Irr}(C^\infty(T_{\mathfrak s,\rm un}) \rtimes W_{\mathfrak s}).$$

However, different filtrations can produce different bijections.

Lemma 6.5. *There exist filtrations on* $\mathcal{S}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$ *and* $C^\infty(T_{\mathfrak s,\rm un}) \rtimes W_{\mathfrak s}$ *with respect to which* $\mathrm{Irr}(\zeta_0)$ *equals* (67).

Proof. Lusztig's a-function (51) is also defined on $\mathrm{Irr}(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}))$, by

$$a(\pi) = \max\{n : \pi(\mathcal{H}^n) \neq 0\},$$

where $\mathcal{H}^n$ is as in (53). According to [Lus3, Theorem 4.8.c] this can also be described as

$$a(\pi(t_q, u, \rho_q)) = \dim_\mathbb{C}(\mathcal{B}^u), \tag{100}$$

where $\mathcal{B}^u$ denotes the variety of Borel subgroups of $\check{G}_\mathfrak{s}$ that contain u. Define

$$I_n := \{h \in \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) : \pi(t_q, u, \rho_q) = 0 \text{ if } \dim_\mathbb{C}(\mathcal{B}^u) < n\}. \tag{101}$$

Then $I_n \supset \mathcal{H}^n$ and

$$\mathrm{Irr}(I_n / I_{n+1}) = \{\pi(t_q, u, \rho_q) : \dim_\mathbb{C}(\mathcal{B}^u) = n\}.$$

In [ABPS5, Proposition 9.3] it was characterized when $\pi(t_q, u, \rho_q)$ is tempered, namely when the associated element $t \in G$ lies in a compact subgroup. The collection of Kazhdan–Lusztig triples (t_q, u, ρ_q) with u and ρ_q fixed can be written as a union of cosets of complex tori, see [ABPS1, §3]. With (100) it follows that the set of tempered irreducible $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$-representations of a-weight n is dense in the space $\{\pi \in \mathrm{Irr}(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})) : a(\pi) = n\}$ (endowed with the Jacobson topology). In particular

$$I_n = \{h \in \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) : \pi(h) = 0 \text{ if } \pi \in \mathrm{Irr}(\mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})), a(\pi) < n\}. \tag{102}$$

Similarly we put

$$\begin{aligned} J_n &:= \{f \in \mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s} : \tau(t, u, \rho)(f) = 0 \text{ if } \dim_\mathbb{C}(\mathcal{B}^u) < n\} \\ &= \{f \in \mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s} : \tau(t, u, \rho)(f) = 0 \text{ if } \dim_\mathbb{C}(\mathcal{B}^u) < n \text{ and } t \in \check{G}_\mathfrak{s} \text{ is compact}\}. \end{aligned} \tag{103}$$

Then $\mathrm{Irr}(J_n / J_{n+1}) = \{\tau(t, u, \rho) : \dim_\mathbb{C}(\mathcal{B}^u) = n\}$. Let

$$\mathcal{S}I_n \subset \mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \quad \text{and} \quad \mathcal{S}J_n \subset C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_\mathfrak{s}$$

be the two-sided ideals generated by I_n and J_n. These ideals can also be described by the conditions in (102) and the second line of (103). We claim that

$$\zeta_0(\mathcal{S}J_n) \subset \mathcal{S}I_n. \tag{104}$$

Let (M, δ) be as in (90). By construction

$$\zeta_0^*(\mathrm{ind}_{\mathcal{H}^M}^{\mathcal{H}}(\delta)) = \mathrm{ind}_{\mathcal{O}(T_\mathfrak{s}) \rtimes W_M}^{\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}}(\zeta_{0,M}^* \delta),$$

where $\zeta_{0,M}^* \delta$ is obtained from δ by modifying the parameters q of the algebra $\mathcal{H}^M$. More concretely:

- $\zeta_{0,M}^* \delta|_{\mathcal{O}(T_\mathfrak{s})}$ arises by replacing each $\mathcal{O}(T_\mathfrak{s})$-weight t_q of δ by $t = t_q \, |t_q|^{-1}$;
- $\zeta_{0,M}^* \delta|_{\mathbb{C}[W_{\mathfrak{s},M}]}$ comes from the realization of V_δ as a quotient of $H_*(\mathcal{B}_{\check{M}}^{u,t})$.

If $\delta = \pi_M(t_q, u, \rho_{q,M})$, then it is a quotient of

$$\mathrm{Hom}_{\pi_0(Z_{\check{M}_\mathfrak{s}}(t_q, u))}(\rho_{q,M}, H_*(\mathcal{B}_{\check{M}_\mathfrak{s}}^{t_q, u}, \mathbb{C})).$$

By [ABPS5, Lemma 8.3] its structure as a $\mathbb{C}[W_{\mathfrak{s},M}]$-module is

$$\tau_M(t, u, \rho) \oplus (\text{terms } \tau_M(t, u', \rho') \text{ with } u < u').$$

Here $u' > u$ means that $\overline{\mathcal{O}_{u'}} \supsetneq \mathcal{O}_u$, where $\mathcal{O}_{u'}$ denotes the $\check{M}_{\mathfrak{s}}$-conjugacy class of u'. This condition implies

$$\dim Z_{\check{M}_{\mathfrak{s}}}(u') < \dim Z_{\check{M}_{\mathfrak{s}}}(u) \quad \text{and} \quad \dim \mathcal{B}^{u'}_{\check{M}_{\mathfrak{s}}} < \dim \mathcal{B}^{u}_{\check{M}_{\mathfrak{s}}}.$$

The summands of $\mathrm{ind}_{\mathcal{H}^M}^{\mathcal{H}} \pi(t_q, u, \rho_{q,M})$ are of the form $\pi(t_q, u, \rho_q)$ where $\rho_q \in \mathrm{Irr}(\pi_0(Z_{\check{G}_{\mathfrak{s}}}(t_q, u)))$ contains $\rho_{q,M}$. It follows that

$$\zeta_0^*(\pi(t_q, u, \rho_q)) = \tau(t, u, \rho) \oplus (\text{terms } \tau(t, u', \rho') \text{ with } \dim \mathcal{B}^{u'} < \dim \mathcal{B}^u). \tag{105}$$

Let $h \in \mathcal{S}J_n \subset C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_{\mathfrak{s}}$ and suppose that $\dim \mathcal{B}^u < n$. Then (105) shows that

$$\pi(t_q, u, \rho_q)(\zeta_0(h)) = \zeta_0^*(\pi(t_q, u, \rho_q))(h) = 0.$$

Hence $\zeta_0(h) \in \mathcal{S}I_n \subset \mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$, which proves the claim (104). Assume now that $\dim \mathcal{B}^u = n$ and consider the algebra homomorphism

$$\mathcal{S}J_n / \mathcal{S}J_{n+1} \to \mathcal{S}I_n / \mathcal{S}I_{n+1}$$

induced by ζ_0. Then (105) shows that $\mathrm{Irr}(\zeta_0)(\pi(t_q, u, \rho_q)) = \tau(t, u, \rho)$. □

To extend ζ_0 to the setting of this paper, we must check that it is $\mathrm{Stab}(\mathfrak{s})^+$-equivariant. As ζ_0 is not even unique, we will rather check that we control the construction so that it becomes equivariant. In [Sol2] more general algebra homomorphisms

$$\zeta_0 \otimes \mathrm{id}_{\mathbb{C}[\Gamma]} : (C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_{\mathfrak{s}}) \rtimes \Gamma \to \mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \rtimes \Gamma \tag{106}$$

are constructed. Here Γ is a finite group of particular automorphisms of $\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$, namely those coming from automorphisms of the Dynkin diagram of $R_{\mathfrak{s}}$ that preserve $q_{\mathfrak{s}} : R_{\mathfrak{s}} \to \mathbb{R}_{>0}$. But in fact the setup of [Sol2] is even more general. By [Sol2, Theorem 4.4.2.e and Lemma 4.2.3.a]

$$\zeta_0 \text{ is } T_{\mathfrak{s},\mathrm{un}}^{W_0}\text{-equivariant,} \tag{107}$$

for the action induced by translations on $T_{\mathfrak{s},\mathrm{un}}$. So in (106) we may take for Γ any finite group consisting of diagram automorphisms and translations by subgroups of $T_{\mathfrak{s},\mathrm{un}}$. In particular we can take $\Gamma = \mathrm{Stab}(\mathfrak{s})^+$ with the actions described in Theorem 1.4 and (47). In this way (106) implies that ζ_0 is $\mathrm{Stab}(\mathfrak{s})^+$-equivariant. Then

$$\zeta_0 \otimes \mathrm{id}_{\mathrm{End}_{\mathbb{C}}(V_\mu)} : (C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \to \mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$$

is also equivariant and induces

$$\zeta^{\mathfrak{s}}_{G^\sharp Z(G)} := \bigoplus_{a \in [L/H_\lambda]} \zeta_0 \otimes \mathrm{id} : \bigoplus_{a \in [L/H_\lambda]} \big((C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \to$$

$$\bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal{S}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}. \tag{108}$$

Proposition 6.6. *$\zeta^{\mathfrak{s}}_{G^\sharp Z(G)}$ is spectrum preserving with respect to filtrations. There exist filtrations on the above algebras such that*

$$\mathrm{Irr}(\zeta^{\mathfrak{s}}_{G^\sharp Z(G)}) : \mathrm{Irr}(\mathcal{S}(G^\sharp Z(G))^{\mathfrak{s}}) \to (T_{\mathfrak{s},\mathrm{un}} /\!/ \mathrm{Stab}(\mathfrak{s}))_{\kappa_\omega}$$

equals the inverse of the map from Theorem 4.4.

Proof. First we check that the action of $\mathrm{Stab}(\mathfrak{s})^+$ on $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ preserves the a-weights of irreducible representations. Recall that $\pi(t_q, u, \rho_q)$ is a quotient of the standard module $H_*(\mathcal{B}^{t_q,u}, \mathbb{C}) \otimes V_\mu$. By construction the weights for the action of the subalgebra $\mathcal{O}(T_\mathfrak{s})$ on $H_*(\mathcal{B}^{t_q,u}, \mathbb{C})$ are precisely the $w(t_q)$ with $w \in W_\mathfrak{s}$. Translation by $\chi_\gamma \in T_{\mathfrak{s},\mathrm{un}}^{W_0}$ induces an automorphism of $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ which is the identity on $\mathcal{H}(W_0, q_\mathfrak{s})$. This implies that

$$
\begin{aligned}
&\chi_\gamma^*(H_*(\mathcal{B}^{t_q,u}, \mathbb{C})) = H_*(\mathcal{B}^{\chi_\gamma t_q,u}, \mathbb{C}),\\
&\chi_\gamma^*(\pi(t_q, u, \rho_q)) = \pi(\chi_\gamma t_q, u, \rho_q).
\end{aligned} \tag{109}
$$

With [ABPS5, (66)] we obtain

$$
\alpha^*_{(w,\gamma)}(\pi(t_q, u, \rho_q)) = \pi(\chi_\gamma^{-1} w t_q w^{-1}, wuw^{-1}, \rho_q \circ \mathrm{Ad}_w^{-1}).
$$

Here conjugation with w takes place in $\check{G}_\mathfrak{s}$, which is possible because $W(G,L) \cong W(\check{G}_\mathfrak{s}, \check{L})$. As $\dim \mathcal{B}^{wuw^{-1}} = \dim \mathcal{B}^u$,

$$
a(\alpha^*_{(w,\gamma)}\pi) = a(\pi) \quad \text{for all} \quad \pi \in \mathrm{Irr}(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})).
$$

Hence $\mathrm{Stab}(\mathfrak{s})^+$ stabilizes the ideals $I_n \subset \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ and $\mathcal{S}I_n \subset \mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$. Similarly, it stabilizes the ideals $J_n \subset \mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$ and $\mathcal{S}J_n \subset C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_\mathfrak{s}$. This enables us to define ideals in the algebras from (108):

$$
\begin{aligned}
I'_n &:= \bigoplus\nolimits_{a \in [L/H_\lambda]} (\mathcal{S}I_n \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp,\\
J'_n &:= \bigoplus\nolimits_{a \in [L/H_\lambda]} (\mathcal{S}J_n \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp.
\end{aligned}
$$

By (104) $\zeta^\mathfrak{s}_{G^\sharp Z(G)}(J'_n) \subset I'_n$. The way to obtain $\mathrm{Irr}(I'_n / I'_{n+1})$ from $\mathrm{Irr}(\mathcal{S}I_n / \mathcal{S}I_{n+1})$ is the same as from $\mathrm{Irr}(J'_n / J'_{n+1})$ from $\mathrm{Irr}(\mathcal{S}J_n / \mathcal{S}J_{n+1})$, and described by Clifford theory. From the proofs of Lemma B.2 and Theorem 4.4 we see that irreducible representations of

$$
\bigoplus\nolimits_{a \in [L/H_\lambda]} (\mathcal{S}(T_\mathfrak{s}^\sharp, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp.
$$

can be parametrized by triples (a, π, σ) with $a \in [L/H_\lambda], \pi \in \mathrm{Irr}(\mathcal{S}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}))$ and $\sigma \in \mathrm{Irr}(\mathbb{C}[\mathrm{Stab}(\mathfrak{s})^+_\pi, \kappa_\pi])$. Similarly

$$
\mathrm{Irr}\big(\bigoplus\nolimits_{a \in [L/H_\lambda]} (C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_\mathfrak{s} \otimes \mathrm{End}_\mathbb{C}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp\big)
$$

can be parametrized by triples (a, π, σ) with $\pi \in \mathrm{Irr}(C^\infty(T_{\mathfrak{s},\mathrm{un}}) \rtimes W_\mathfrak{s})$. Like in Lemma B.1 we denote the associated representation by $(a, \pi \rtimes \sigma)$. Then (105) shows that

$$
\begin{aligned}
&\zeta^\mathfrak{s}_{G^\sharp Z(G)}(a, \pi(t_q, u, \rho_q) \rtimes \sigma) =\\
&\qquad (a, \tau(t, u, \rho) \rtimes \sigma) \oplus (\text{terms } (a, \tau(t, u', \rho') \rtimes \sigma') \text{ with } \dim \mathcal{B}^{u'} < \dim \mathcal{B}^u).
\end{aligned}
$$

As in the proof of Lemma 6.5, this implies that $\zeta^\mathfrak{s}_{G^\sharp Z(G)}$ is spectrum preserving with respect to the filtrations $(I'_n)_{n \geq 0}$ and $(J'_n)_{n \geq 0}$, and that

$$
\mathrm{Irr}(\zeta^\mathfrak{s}_{G^\sharp Z(G)})((a, \pi(t_q, u, \rho_q) \rtimes \sigma) = (a, \tau(t, u, \rho) \rtimes \sigma). \tag{110}
$$

The map in Theorem 4.4.b is based on Theorem 3.4.a, in particular on (58). This in turn relies on Lemma 3.2 and the associated bijection

$$
\mathrm{Irr}(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)) \longleftrightarrow \mathrm{Irr}(\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s} \otimes \mathrm{End}_\mathbb{C}(V_\mu)).
$$

By (67) the latter can be identified with

$$\pi(t_q, u, \rho_q) \otimes V_\mu \longleftrightarrow \tau(t, u, \rho) \otimes V_\mu .$$

Consequently (110) is the inverse of the map in Theorem 4.4.b. □

Corollary 6.7. *$\zeta^{\mathfrak s}_{G^\sharp Z(G)}$ restricts to a map*

$$\zeta^{\mathfrak s}_{G^\sharp} : \bigoplus_{a \in [L/H_\lambda]} \zeta_0 \otimes \mathrm{id} : \bigoplus_{a \in [L/H_\lambda]} \big((C^\infty (T^\sharp_{\mathfrak s, \mathrm{un}}) \rtimes W_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu) \big)^{X^L(\mathfrak s)} \rtimes \mathfrak R^\sharp_{\mathfrak s} \to \bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal S (T^\sharp_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu) \big)^{X^L(\mathfrak s)} \rtimes \mathfrak R^\sharp_{\mathfrak s} .$$

With respect to the filtrations coming from those in Proposition 6.6, $\zeta^{\mathfrak s}_{G^\sharp}$ is spectrum preserving and

$$\mathrm{Irr}(\zeta^{\mathfrak s}_{G^\sharp}) : \mathrm{Irr}(\mathcal S (G)^{\mathfrak s}) \to (T_{\mathfrak s,\mathrm{un}} /\!/ \mathrm{Stab}(\mathfrak s) X_{\mathrm{unr}}(L/L^\sharp))_{\kappa_\omega}$$

equals the inverse of the map in Theorem 4.4.b.

Proof. It follows from (107) that $\zeta^{\mathfrak s}_{G^\sharp Z(G)}$ is equivariant for the action of $X_{\mathrm{unr}}(G) \cong X_{\mathrm{unr}}(L/L^\sharp)$ on $T_{\mathfrak s,\mathrm{un}}$ by translations. Hence we can restrict $\zeta^{\mathfrak s}_{G^\sharp Z(G)}$ to the subspaces of $X_{\mathrm{unr}}(G)$-invariant elements on both sides, which gives $\zeta^{\mathfrak s}_{G^\sharp}$. By (109) and (100) the a-weights of irreducible representations of $\mathcal H (T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s})$ or $\mathcal O (T_{\mathfrak s}) \rtimes W_{\mathfrak s}$ are invariant under the action of $X_{\mathrm{unr}}(G)$. Therefore the ideals $\mathcal S I_n$ and $\mathcal S J_n$ are stabilized by $X_{\mathrm{unr}}(G)$. This enables us to take $X_{\mathrm{unr}}(G)$-invariants in the entire proof of Proposition 6.6, which leads to the desired conclusions. □

Appendix A. Geometric equivalences

Let X be a complex affine variety and let $k = \mathcal O (X)$ be its coordinate algebra. Equivalently, k is a unital algebra over $\mathbb C$ which is commutative, finitely generated, and nilpotent-free. A k algebra is an algebra A over $\mathbb C$ which is a k-module (with an evident compatibility between the algebra structure of A and the k-module structure of A). For A a k-algebra, let $\mathrm{Prim}(A)$ denote its primitive ideal spectrum, that is, the set of primitive ideals of A.

In this appendix, we will consider only k-algebras A that satisfy the following property: the map

$$\mathrm{Irr}(A) \to \mathrm{Prim}(A) : (\pi, V_\pi) \mapsto \ker(\pi) \tag{111}$$

is a bijection. For example, this is the case if A is of finite type (that is, if A is finitely generated as a k-module), or more generally if every irreducible A-representation has (at most) countable dimension. For such k-algebras, we will introduce a weakening of Morita equivalence called *geometric equivalence.*

The new equivalence relation preserves the primitive ideal spectrum and the periodic cyclic homology. However, it permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence.

Spectrum preserving morphisms of k-algebras

Assume that B has the same property. By definition, a $\mathbb C$-algebra homomorphism $\phi \colon A \to B$ is *spectrum preserving* if

- for every primitive ideal I of B, the ideal $\phi^{-1}(I)$ is contained in a unique primitive ideal of A, say $\phi^*(I)$;

- the resulting map $\phi^*\colon \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ is bijective.

More generally, $\phi\colon A \to B$ is called *spectrum preserving with respect to filtrations* if there exist chains of two-sided ideals

$$(0) = I_0 \subset I_1 \cdots \subset I_n = A \quad \text{and} \quad (0) = J_0 \subset J_1 \cdots \subset J_n = B \tag{112}$$

such that, for every i, $\phi(I_i) \subset J_i$ and the induced map $\phi\colon I_k/I_{i-1} \to J_i/J_{i-1}$ is spectrum preserving.

These data determine a bijection $\mathrm{Prim}(B) \to \mathrm{Prim}(A)$ which, however, need not be continuous.

Algebraic variation of k-structure

Denote the centre of a k-algebra A by $\mathrm{Z}(A)$. If A is a $\mathbb{C}$-algebra, $A[t, t^{-1}]$ is the $\mathbb{C}$-algebra of Laurent polynomials in the indeterminate t with coefficients in A. Note that $\mathrm{Z}(A[t, t^{-1}]) = \mathrm{Z}(A)[t, t^{-1}]$.

Let A be a unital $\mathbb{C}$-algebra, and let $\Psi\colon k \to A[t, t^{-1}]$ be a unital morphism of $\mathbb{C}$-algebras. For $z \in \mathbb{C}^\times$, let $\mathrm{ev}(z)$ denotes the "evaluation at z" map:

$$\mathrm{ev}(z)\colon \quad \begin{array}{ccc} A[t, t^{-1}] & \to & A \\ \sum a_j t^j & \mapsto & \sum a_j z^j \end{array} .$$

Consider the composition $\mathrm{ev}(z) \circ \Psi\colon k \to \mathrm{Z}(A)$, and denote the unital k-algebra so obtained by A_z. The underlying $\mathbb{C}$-algebra of A_z is A. Assume that for all $z \in \mathbb{C}^\times$, A_z is a finite type k-algebra. Then for $z, z' \in \mathbb{C}^\times$, we will say that $A_{z'}$ is obtained from A_z by an *algebraic variation of k-structure*.

Definition A.1. *With k-fixed, geometric equivalence for k-algebras (such that* (111) *is a bijection) is the equivalence relation generated by the two elementary moves:*

- *spectrum preserving morphisms with respect to filtrations,*
- *algebraic variation of k-structure.*

Thus two k-algebras A, B as above are geometrically equivalent if there exists a finite sequence

$$A = A_0, A_1, \ldots, A_r = B$$

with each A_j a k-algebra such that (111) is bijective and for $j = 0, 1, \ldots, r-1$ one of the following three possibilities is valid:

1. A_{j+1} is obtained from A_j by an algebraic variation of k-structure,
2. there is a spectrum preserving morphism with respect to filtrations $A_j \to A_{j+1}$,
3. there is a spectrum preserving morphism with respect to filtrations $A_{j+1} \to A_j$.

To give a geometric equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spectra and an isomorphism of periodic cyclic homology are determined:

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B) \qquad \text{and} \qquad \mathrm{HP}_*(A) \simeq \mathrm{HP}_*(B).$$

Proposition A.2. *If two unital k-algebras (such that the corresponding maps* (111) *are bijective) are Morita equivalent, then they are geometrically equivalent.*

Proof. Two unital k-algebras A, B are Morita equivalent if there is an equivalence of categories

$$\text{(unital left } A\text{-modules)} \cong \text{(unital left } B\text{-modules)}.$$

Any such equivalence of categories is implemeted by a *Morita context*, i.e. by a pair of unital bimodules $({}_A V_B, {}_B W_A)$ together with given isomorphisms of bimodules

$$\alpha\colon V \otimes_B W \to A, \qquad \beta\colon W \otimes_A V \to B,$$

which are "associative" in the following sense. Writing

$$\alpha(v \otimes w) = vw \quad \text{and} \quad \beta(w \otimes v) = wv,$$

one requires that

$$(vw)v' = v(wv') \quad \text{and} \quad (wv)w' = w(vw') \qquad \text{for all } v, v' \in V,\ w, w' \in W.$$

The *linking algebra* is defined as

$$\mathrm{M}_{2\times 2}({}_A V_B, {}_B W_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}.$$

Then the map (111) corresponding to $\mathrm{M}_{2\times 2}({}_A V_B, {}_B W_A)$ is a bijection. The inclusions

$$\begin{array}{ccc} A & \hookrightarrow & \mathrm{M}_{2\times 2}({}_A V_B, {}_B W_A) \\ a & \mapsto & \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \hookrightarrow & \mathrm{M}_{2\times 2}({}_A V_B, {}_B W_A) \\ b & \mapsto & \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right) \end{array}$$

are spectrum preserving morphisms of k-algebras. Hence A and B are geometrically equivalent. $\square$

Appendix B. Extended quotients

Let Γ be a group acting on a topological space X. In [ABPS5, §2] we studied various extended quotients of X by Γ. In this paper we need the most general version, the twisted extended quotients.

Let $\natural$ be a given function which assigns to each $x \in X$ a 2-cocycle

$$\natural(x) : \Gamma_x \times \Gamma_x \to \mathbb{C}^\times, \text{ where } \Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

It is assumed that $\natural(\gamma x)$ and $\gamma_* \natural(x)$ define the same class in $H^2(\Gamma_x, \mathbb{C}^\times)$, where $\gamma_* : \Gamma_x \to \Gamma_{\gamma x}$ sends α to $\gamma \alpha \gamma^{-1}$. Define

$$\widetilde{X}_\natural := \{(x, \rho) : x \in X, \rho \in \mathrm{Irr}\, \mathbb{C}[\Gamma_x, \natural(x)]\}.$$

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} : \mathbb{C}[\Gamma_x, \natural(x)] \to \mathbb{C}[\Gamma_{\gamma x}, \natural(\gamma x)]$$

such that:

- $\phi_{\gamma,x}$ is inner if $\gamma x = x$;
- $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma, x \in X$.

We call these maps connecting homomorphisms, because they are reminiscent of a connection on a vector bundle. Then we can define a Γ-action on $\widetilde{X}_\natural$ by

$$\gamma \cdot (x, \rho) = (\gamma x, \rho \circ \phi_{\gamma,x}^{-1}).$$

We form the *twisted extended quotient*

$$(X/\!/\Gamma)_\natural := \widetilde{X}_\natural / \Gamma.$$

We note that this reduces to the extended quotient of the second kind $(X//\Gamma)_2$ from [ABPS5, §2] if $\natural(x)$ is trivial for all $x \in X$ and $\phi_{\gamma,x}$ is conjugation by γ.

Such twisted extended quotients typically arise in the following situation. Let A be a $\mathbb{C}$-algebra such that all irreducible A-modules have countable dimension over $\mathbb{C}$. Let Γ be a group acting on A by automorphisms and form the crossed product $A \rtimes \Gamma$.

Let $X = \mathrm{Irr}(A)$. Now Γ acts on $\mathrm{Irr}(A)$ and we get $\natural$ as follows. Given $x \in \mathrm{Irr}(A)$ choose an irreducible representation (π_x, V_x) whose isomorphism class is x. For each $\gamma \in \Gamma$ consider π_x twisted by γ:

$$\gamma \cdot \pi_x : a \mapsto \pi_x(\gamma^{-1} a \gamma).$$

Then $\gamma \cdot x$ is defined as the isomorphism class of $\gamma \cdot \pi_x$. Since $\gamma \cdot \pi_x$ is equivalent to $\pi_{\gamma x}$, there exists a nonzero intertwining operator

$$T_{\gamma,x} \in \mathrm{Hom}_A(\gamma \cdot \pi_x, \pi_{\gamma x}). \tag{113}$$

By Schur's lemma (which is applicable because $\dim V_x$ is countable) $T_{\gamma,x}$ is unique up to scalars, but in general there is no preferred choice. For $\gamma, \gamma' \in \Gamma_x$ there exists a unique $c \in \mathbb{C}^\times$ such that

$$T_{\gamma,x} \circ T_{\gamma',x} = c T_{\gamma\gamma',x}.$$

We define the 2-cocycle by

$$\natural(x)(\gamma, \gamma') = c.$$

Let $N_{\gamma,x}$ with $\gamma \in \Gamma_x$ be the standard basis of $\mathbb{C}[\Gamma_x, \natural(x)]$. The algebra homomorphism $\phi_{\gamma,x}$ is essentially conjugation by $T_{\gamma,x}$, but we must be careful if some of the T_γ coincide. The precise definition is

$$\phi_{\gamma,x}(N_{\gamma',x}) = \lambda N_{\gamma\gamma'\gamma^{-1},\gamma x} \quad \text{if} \quad T_{\gamma,x} T_{\gamma',x} T_{\gamma,x}^{-1} = \lambda T_{\gamma\gamma'\gamma^{-1},\gamma x}, \lambda \in \mathbb{C}^\times. \tag{114}$$

Notice that (114) does not depend on the choice of $T_{\gamma,x}$.

Suppose that Γ_x is finite and $(\tau, V_\tau) \in \mathrm{Irr}(\mathbb{C}[\Gamma_x, \natural(x)])$. Then $V_x \otimes V_\tau^*$ is an irreducible $A \rtimes \Gamma_x$-module, in a way which depends on the choice of intertwining operators $T_{\gamma,x}$.

Lemma B.1. [ABPS5, Lemma 2.3]
Let A and Γ be as above and assume that the action of Γ on $\mathrm{Irr}(A)$ has finite isotropy groups.

(*a*) *There is a bijection*

$$\begin{array}{ccc} (\mathrm{Irr}(A)//\Gamma)_\natural & \longleftrightarrow & \mathrm{Irr}(A \rtimes \Gamma) \\ (\pi_x, \tau) & \mapsto & \pi_x \rtimes \tau := \mathrm{Ind}_{A \rtimes \Gamma_x}^{A \rtimes \Gamma}(V_x \otimes V_\tau^*). \end{array}$$

(*b*) *If all irreducible A-modules are one-dimensional, then part (a) becomes a natural bijection*

$$(\mathrm{Irr}(A)//\Gamma)_2 \longleftrightarrow \mathrm{Irr}(A \rtimes \Gamma).$$

Via the following result twisted extended quotients also arise from algebras of invariants.

Lemma B.2. *Let Γ be a finite group acting on a $\mathbb{C}$-algebra A. There is a bijection*

$$\begin{array}{ccc} \{V \in \mathrm{Irr}(A \rtimes \Gamma) : V^\Gamma \neq 0\} & \longleftrightarrow & \mathrm{Irr}(A^\Gamma) \\ V & \mapsto & V^\Gamma. \end{array}$$

If all elements of Irr(A) *have countable dimension, it becomes*

$$\begin{array}{ccc} \{(\pi_x, \tau) \in (\mathrm{Irr}(A)/\!/\Gamma)_\natural : \mathrm{Hom}_{\Gamma_x}(V_\tau, V_x) \neq 0\} & \longleftrightarrow & \mathrm{Irr}(A^\Gamma) \\ (\pi_x, \tau) & \mapsto & \mathrm{Hom}_{\Gamma_x}(V_\tau, V_x). \end{array}$$

Proof. Consider the idempotent

$$p_\Gamma = |\Gamma|^{-1} \sum\nolimits_{\gamma \in \Gamma} \gamma \in \mathbb{C}[\Gamma]. \tag{115}$$

It is well-known and easily shown that

$$A^\Gamma \cong p_\Gamma (A \rtimes \Gamma) p_\Gamma$$

and that the right hand side is Morita equivalent with the two-sided ideal

$$I = (A \rtimes \Gamma) p_\Gamma (A \rtimes \Gamma) \subset A \rtimes \Gamma.$$

The Morita equivalence sends a module V over the latter algebra to

$$p_\Gamma (A \rtimes \Gamma) \otimes_{(A \rtimes \Gamma) p_\Gamma (A \rtimes \Gamma)} V = V^\Gamma.$$

As I is a two-sided ideal,

$$\mathrm{Irr}(I) = \{V \in \mathrm{Irr}(A \rtimes \Gamma) : I \cdot V \neq 0\} = \{V \in \mathrm{Irr}(A \rtimes \Gamma) : p_\Gamma V = V^\Gamma \neq 0\}$$

This gives the first bijection. From Lemma B.1.a we know that every such V is of the form $\pi_x \rtimes \tau$. With Frobenius reciprocity we calculate

$$(\pi_x \rtimes \tau)^\Gamma = \left(\mathrm{Ind}_{A \rtimes \Gamma_x}^{A \rtimes \Gamma} (V_x \otimes V_\tau^*)\right)^\Gamma \cong (V_x \otimes V_\tau^*)^{\Gamma_x} = \mathrm{Hom}_{\Gamma_x}(V_\tau, V_x).$$

Now Lemma B.1.a and the first bijection give the second. □

Let A be a commutative $\mathbb{C}$-algebra all whose irreducible representations are of countable dimension over $\mathbb{C}$. Then Irr(A) consists of characters of A and is a T_1-space. Typical examples are $A = C_0(X)$ (with X locally compact Hausdorff), $A = C^\infty(X)$ (with X a smooth manifold) and $A = \mathcal{O}(X)$ (with X an algebraic variety).

As a kind of converse to Lemmas B.1 and B.2, we show that every twisted extended quotient of Irr(A) appears as the space of irreducible representations of some algebras. With small modifications, the argument also works for smooth manifolds and algebraic varieties.

Let Γ be a group acting on A by algebra automorphisms, such that Γ_x is finite for every $x \in \mathrm{Irr}(A)$. Recall that every 2-cocycle $\natural$ of Γ arises from a projective Γ-representation (μ, V_μ) by

$$\mu(\gamma)\mu(\gamma') = \natural(\gamma, \gamma')\mu(\gamma, \gamma').$$

Let Γ act on $A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ by

$$\gamma \cdot (a \otimes h) = \gamma(a) \otimes \mu(\gamma) h \mu(\gamma)^{-1}.$$

Lemma B.3. *There are bijections*

$$\begin{array}{lcl} \mathrm{Irr}((A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes \Gamma) & \longleftrightarrow & (\mathrm{Irr}(A)/\!/\Gamma)_\natural, \\ \mathrm{Irr}((A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^\Gamma) & \longleftrightarrow & \{[x, \rho] \in (X/\!/\Gamma)_\natural : \rho \text{ appears in } V_\mu\}. \end{array}$$

Proof. We can identify Irr$(A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))$ with $\{\mathbb{C}_x \otimes V_\mu : x \in \mathrm{Irr}(A)\}$. It follows directly from (113) that we can take $T_{\gamma,x} = \mu(\gamma)$ for all $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(A)$. Thus the first bijection is an instance of Lemma B.1.a.

Let $x \in \mathrm{Irr}(A)$ and $(\tau, V_\tau) \in \mathrm{Irr}(\mathbb{C}[\Gamma_x, \natural])$. Then

$$\mathrm{Hom}_{\Gamma_x}(\tau, \mathbb{C}_x \otimes V_\mu) = \mathrm{Hom}_{\Gamma_x}(\tau, V_\mu),$$

and this is nonzero if and only if τ appears in V_μ. Now an application of Lemma B.2 proves the second bijection. □

Corollary B.4. *In the above setting, suppose that $\Gamma = \Gamma_1 \rtimes \Gamma_2$ is a semidirect product. Then there is a canonical bijection*

$$(\mathrm{Irr}(A)//\Gamma)_\natural \longleftrightarrow ((\mathrm{Irr}(A)//\Gamma_1)_\natural//\Gamma_2)_\natural .$$

Proof. The bijection is obtained from Lemma B.3 and

$$(A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes \Gamma = ((A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes \Gamma_1) \rtimes \Gamma_2$$

It is canonical because the same 2-cocycle is used on both sides. □

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