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# Adjoint and Coadjoint Orbits of the Special Euclidean Group

*Preliminary Version*

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**Abstract**

We give a geometric description of the adjoint and coadjoint orbits of the special Euclidean group. We implement the method of little subgroups as introduced by Rawnsley in 1975 and the method of types by Burgoyne and Cushman in 1977 to classify these orbits completely. The orbits are diffeomorphic to affine flag manifolds, whose definition and geometry we also explore. Since coadjoint orbits are naturally symplectic, such manifolds provide us with interesting examples of symplectic homogeneous spaces. As discovered by Cushman and van der Kallen in 2006, we identify a bijection between the coadjoint and adjoint orbits of the Euclidean group. Furthermore, we show that orbits corresponding under this bijection are homotopy equivalent. Whether the bijection for other groups, and especially the Poincaré group, preserves homotopy type of orbits remains an open question.

## 1 Introduction

For any compact or semisimple Lie group, the adjoint and coadjoint representations are equivalent (in both cases because there is an invariant quadratic form on the Lie algebra). However, for general Lie groups this is not true. In a recent paper [4], Cushman and van der Kallen showed that, for real affine orthogonal groups, where the actions are not isomorphic, there is nonetheless a surprising 1–1 correspondence between the adjoint and coadjoint orbits and this correspondence respects certain moduli of the orbits. These groups include both the Euclidean and Poincaré groups. In this paper we consider the geometry behind this correspondence for the (special) Euclidean group $SE(n)$.

Given a matrix Lie group $H \subset GL(V)$ one is concerned with describing the coadjoint orbits of the affine group $G := H \ltimes V$. An answer to this question was found by Rawnsley in [1]; the coadjoint orbits are in a one-to-one correspondence with certain fibre bundles defined by what is referred to in the literature as a "little subgroup". Cushman and van der Kallen used the method of types in [4] to classify the (co)adjoint orbits (building on work done in [2]) of affine orthogonal groups, in particular the Poincaré group and in [5] the odd real symplectic group. In this paper we will use the methods illustrated in these works to explicitly describe the geometry of the (co)adjoint orbits for the special Euclidean group $SE(n) = SO(n) \ltimes \mathbb{R}^n$. This example provides us with an intriguing collection of affine flag manifolds and exhibits a canonical symplectic structure on them. As remarked in [4] we establish a bijection between coadjoint and adjoint orbits preserving the modulus. Moreover we show that this orbit correspondence preserves homotopy type.

We begin by reviewing the situation in some generality before restricting attention to $SE(n)$. Our discussion here follows that given in [1]. We may write $\mathfrak{g} = \mathfrak{h} \times V$ and $\mathfrak{g}^* = \mathfrak{h}^* \times V^*$. Let $\Omega \in \mathfrak{h}^*$ and $\zeta \in V^*$ so that $(\Omega, \zeta) \in \mathfrak{g}^*$ and $(a, v)$ belong to $G$ where $a \in H$ and $v \in V$. The coadjoint action of $G$ on $\mathfrak{g}^*$ is given by,

$$\mathrm{Coad}_{(a,v)}(\Omega, \zeta) = \left(\mathrm{Coad}_a \Omega + v \odot a^{-*}\zeta, a^{-*}\zeta\right). \tag{1.1}$$

Here $a^*$ denotes the adjoint of $a$ and $a^{-*}$ the adjoint of $a^{-1}$. Following [3] we use the notation $v \odot \zeta$, where $v \in V$ and $\zeta \in V^*$ to denote the element in $\mathfrak{h}^*$ defined by satisfying,

$$\langle v \odot \zeta, A \rangle = \langle \zeta, Av \rangle, \ \forall A \in \mathfrak{h}.$$

The map $\mu : V \times V^* \to \mathfrak{h}$ sending $(v, \zeta)$ to $v \odot \zeta$ is the momentum map for the action of $H$ on the cotangent bundle $T^*V \cong V \times V^*$. Let $G_{\Omega,\zeta}$ be the isotropy subgroup fixing $(\Omega, \zeta)$ under the coadjoint action of $G$. For $(a, v)$ to belong to $G_{\Omega,\zeta}$ we clearly require $a^{-*}\zeta = \zeta$. Define $H_\zeta$ to be the subgroup consisting of all $a \in H$ satisfying $a^{-*}\zeta = \zeta$. The group $H_\zeta$ is often referred to as the *little subgroup*. The Lie algebra $\mathfrak{h}_\zeta$ of $H_\zeta$ consists of all $A \in \mathfrak{h}$ such that $A^{-*}\zeta = 0$. Given $(a, v) \in G_{\Omega,\zeta}$ we must have,

$$\mathrm{Coad}_a \Omega + v \odot \zeta = \Omega. \tag{1.2}$$

Let $v \in V$ be arbitrary and consider $v \odot \zeta$. Since $\langle v \odot \zeta, A \rangle = \langle \zeta, Av \rangle$ for all $A$ we also have, $\langle v \odot \zeta, A \rangle = \langle A^{-*}\zeta, v \rangle$. If $A \in \mathfrak{h}_\zeta$ it follows that $\langle v \odot \zeta, A \rangle = 0$ and hence that $v \odot \zeta$ belongs to the annihilator $\mathfrak{h}_\zeta^\circ$ of $\mathfrak{h}_\zeta$. In fact it can be shown that

$$\mathfrak{h}_\zeta^\circ = \{ v \odot \zeta \,|\, v \in V \}. \tag{1.3}$$

We may therefore restrict (1.2) to $\mathfrak{h}_\zeta$ and hence project away the term $v \odot \zeta$ to get,

$$\begin{aligned} (\mathrm{Coad}_a \Omega)\,|_{\mathfrak{h}_\zeta} &= \Omega|_{\mathfrak{h}_\zeta}, \\ \implies \mathrm{Coad}_a \left( \Omega|_{\mathfrak{h}_\zeta} \right) &= \Omega|_{\mathfrak{h}_\zeta}. \end{aligned}$$

We abbreviate $\Omega|_{\mathfrak{h}_\zeta}$ to $\overline{\Omega}$. For the above expression to hold we need $a \in H_{\overline{\Omega},\zeta}$ where now $H_{\overline{\Omega},\zeta}$ is the subgroup of $H_\zeta$ such that $\mathrm{Coad}_a \overline{\Omega} = \overline{\Omega}$. Conversely let $a \in H_{\overline{\Omega},\zeta}$. Then $\mathrm{Coad}_a \overline{\Omega} - \overline{\Omega} = 0$, so then $\mathrm{Coad}_a \Omega - \Omega \in \mathfrak{h}_\zeta^\bullet$. But then by (1.3) $\exists v \in V$ such that $\mathrm{Coad}_a \Omega - \Omega = v \times \zeta$. We therefore have a homomorphism $j : G_{\Omega,\zeta} \to H_{\overline{\Omega},\zeta}$ which sends $(a, v)$ to $a$. The kernel of $j$ is $(e, v)$ where $v$ must satisfy $v \odot \zeta = 0$. Hence we have an exact sequence,

$$0 \longrightarrow \{ v \,|\, v \odot \zeta = 0 \} \xrightarrow{i} G_{\Omega,\zeta} \xrightarrow{j} H_{\overline{H},\zeta} \longrightarrow 1 \tag{1.4}$$

where $i$ sends $v$ to $(e, v)$. Thus we have proved equation (1) of [1];

**Theorem 1.** *$G_{\Omega,\zeta}$ is an extension of $H_{\overline{\Omega},\zeta}$ by a vector space.*

As pointed out in [1], this extension is not usually split; however, in our example where $G = SE(n)$ it will be. Thus $G_{\Omega,\zeta}$ will be a semidirect product. We will see in section 3 how this isotropy group defines an affine flag manifold as an orbit.

# 2 Flags and affine flags

## 2.1 Definitions

A *flag* in $\mathbb{R}^n$ is defined to be a strictly ascending sequence of subspaces,

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_k = \mathbb{R}^n. \tag{2.1}$$

If we equip $\mathbb{R}^n$ with the standard metric, we may reinterpret the flag uniquely as being an ordered sequence of mutually orthogonal subspaces $V_1, \ldots V_k$, where $V_1 = E_1$ and $E_{i+1} = E_i \oplus V_{i+1}$;

$$0 \subset V_1 \subset V_1 \oplus V_2 \subset \ldots \subset \bigoplus_{i=1}^{k} V_i = \mathbb{R}^n. \tag{2.2}$$

Let $d_i$ denote the dimension of each $V_i$. Given $(d_1,...d_k)$ with $\Sigma_i d_i = n$ define the *flag manifold* of signature $(d_1,...,d_k)$, $\mathscr{F}(d_1,...,d_k)$ to be the manifold of all such flags as in (2.2). If each $d_i$ is equal to one (and hence $k=n$) we say that the flag is a *full flag* and call $\mathscr{F}(1,...,1)$ the manifold of full flags in $\mathbb{R}^n$. If the flag is not a full flag it is called a *partial flag*. The real projective space and Grassmannians are all examples of partial flag manifolds: $\mathbb{R}P^n = \mathscr{F}(1,n-1)$; $Gr(k;n-k) = \mathscr{F}(k;n-k)$.

An *oriented flag* is one where each subspace $E_i$ as in (2.1) is given an orientation. Note that this is equivalent to each $V_i$ acquiring an orientation. We denote the manifold of oriented flags with signature $(d_1,...,d_k)$ as $\widetilde{\mathscr{F}}(d_1,...d_k)$. We can also define a mixed flag to be one where only specific subspaces receive an orientation. We denote such a flag manifold as $\mathscr{F}(\tilde{d}_1,...,\tilde{d}_k)$, where the tilde above a given $d_i$ indicates that $V_i$ receives an orientation.

For our purposes it will also be useful to define the notion of a *Hermitian flag*. This is a flag where each $E_i$ is given a complex structure compatible with the metric on $\mathbb{R}^n$ (note therefore that each $d_i$ must be even). Note also that this is equivalent to each $V_i$ having a complex structure, that is an automorphism $J_i : V_i \to V_i$ satisfying $J_i^2 = -I$ and with $J_i$ oriented and orthogonal. We will denote the manifold of Hermitian flags by $\mathscr{HF}(d_1,..d_k)$. We will also need to consider mixed flags whereupon certain subspaces $E_i$ are given a complex structure, an orientation or nothing at all. We will write such a mixed flag manifold as $\mathscr{F}(d_1,\tilde{d}_2,...,{d_k}^{\mathbb{C}})$, where ${d_i}^{\mathbb{C}}$ indicates that $E_i$ has a complex structure and the tilde an orientation as before.

The group $SO(n)$ acts naturally on flags by sending each subspace $V_i$ to $a\cdot V_i$ for $a \in SO(n)$. If the flag is Hermitian then each complex structure $J_i$ defined on $V_i$ is sent to $a\circ J_i \circ a^{-1}$ on $a\cdot V_i$. We can therefore write the manifold of flags as a homogeneous $SO(n)$-space;

$$\mathscr{F}(d_1,...,\tilde{d}_2,...,d_3^{\mathbb{C}}) = \frac{SO(n)}{S\Big(O(d_1)\times\cdots\times SO(d_2)\times\cdots\times U(\frac{d_3}{2})\Big)} \tag{2.3}$$

The isotropy group of the flag requires some explanation. Clearly for $a$ to fix a flag it must leave invariant each subspace $V_i$. The restriction of $a$ to each subspace is orthogonal. Therefore $a$ must belong to the subgroup $S(O(d_1)\times\cdots\times O(d_k))$. If $V_i$ is oriented then the corresponding action of $a$ on $V_i$ must restrict to an element of $SO(d_i)$. Finally should $V_i$ possess a complex structure given by $J_i \in Aut(V_i)$ then the action of $a$ restricted to $V_i$ must satisfy $aJ_ia^{-1} = J_i$. If we identity $V_i$ with $\mathbb{C}^{d_i/2}$ (which we may using the complex structure $J_i$) then this condition is precisely that which says that $a$ acting on $\mathbb{C}^{d_i/2}$ should commute with multiplication by $i$. This is equivalent to saying that $a$ is a complex linear map on $\mathbb{C}^{d_i/2}$. Moreover the map is unitary since $a$ is orthogonal. Note that since $U(1)\cong SO(2)$, a complex structure on a plane is equivalent to a choice of orientation.

Given a flag $F$ we can displace each of its subspaces $E_i$ by a fixed vector $x\in\mathbb{R}^n$ to get an *affine flag* $F+x$. The bottom subspace $V_1+x$ has added significance since the flag $F+x$ is invariant under translations belonging to $V_1$. Given such an affine flag we refer to the space $V_1+x$ as the *flag pole*.

Given a flag manifold $\mathscr{F}$ we can define a tautological vector bundle $Taut\mathscr{F}$ by defining the fibre over each flag $F$ to be $\bigoplus_{i=2}^k V_i$, i.e. the sum of all subspaces but the first. The construction of this bundle is analogous to the tautological bundle over a projective space or grassmannian. Now consider the manifold of affine flags $\mathrm{Aff}\mathscr{F}(d_1;d_2,...,d_k)$. We can define a bijection between this manifold and $Taut\mathscr{F}(d_1;d_2,...,d_k)$ as follows; the affine flag $F+x$ is determined uniquely by $F$ and the flag pole $V_1+x$. However the flag pole $V_1+x$ is determined uniquely by its intersection with the orthogonal complement $V_1^\perp = \bigoplus_{i=2}^k V_i$. We may identify the flag $F+x$ uniquely to a particular $F$ and a point in $\bigoplus_{i=2}^k V_i$. This defines a unique point in $Taut\mathscr{F}(d_1;d_2,...,d_k)$. This bijection between the two spaces is clearly smooth. We have thus proved the following proposition;

**Proposition 2.1.** *The tautological bundle over a flag manifold, $Taut\mathscr{F}(d_1;d_2,...,d_k)$ is diffeomorphic to the affine flag manifold $\mathrm{Aff}\mathscr{F}(d_1;d_2,...,d_k)$.*

There is a transitive action of $SE(n)$ on $\mathrm{Aff}\mathscr{F}(d_1;d_2,...,d_k)$ defined by sending $F+x$ to $a\cdot F+(ax+v)$ where $(a,v)\in SO(n)\ltimes\mathbb{R}^n=SE(n)$. Affine flag manifolds are then homogeneous $SE(n)$-spaces with isotropy subgroup isomorphic to $H_{\mathscr{F}}\ltimes\mathbb{R}^{d_1}$ where $H_{\mathscr{F}}$ is the isotropy subgroup of $SO(n)$ fixing a flag in $\mathscr{F}(d_1,d_2,...,d_k)$ and $d_1$ is the dimension of the flag pole;

$$\mathrm{Aff}\mathscr{F}(d_1;d_2,...,d_k)=\frac{SE(n)}{H_{\mathscr{F}}\ltimes\mathbb{R}^{d_1}}. \tag{2.4}$$

## 2.2 Symplectic structure on Hermitian flags

It is possible to define a symplectic structure on certain flag manifolds in a canonical way. In order to do this we must first describe the tangent space to a flag manifold.

**Proposition 2.2.** *Given a flag $F=0\subset V_1\subset...\subset\bigoplus_{i=1}^s V_i=\mathbb{R}^n$ in $\mathscr{F}(d_1,...,d_s)$ we can identify the tangent space with a series of linear maps,*

$$T_F\mathscr{F}=\bigoplus_{i=1}^{s}\mathscr{L}(V_i,{E_i}^{\perp}), \tag{2.5}$$

*where $\mathscr{L}(V,W)$ is the set of linear maps $V\to W$ between vector spaces $V,W$.*

*Proof.* Let $F(t)$ be a curve in $\mathscr{F}$ so that $F(0)=F$ and $V_i(t)\in\mathscr{F}(d_i,n-d_i)$ the corresponding subspaces. Let $A_i(t)$ be a curve in $\mathscr{L}(\mathbb{R}^n,\mathbb{R}^{n-d_i})$ satisfying $\mathrm{Ker}A_i(t)=V_i(t)$ for all $t$. Let $\gamma_1(t)$ be an arbitrary curve in $\mathbb{R}^n$ such that $\gamma_1(t)\in V_1(t)$ for all $t$. Differentiating the identity $A_1(t)\gamma_1(t)=0$ at $t=0$ gives,

$$A_1(0)\gamma_1'(0)+A_1'(0)\gamma_1(0)=0.$$

The tangent vector is determined by $A_1'(t)$ and $\gamma_1(0)$ up to $\mathrm{Ker}A_1(0)=V_1$ and hence defines a class $\gamma_1(0)+V_1\in\mathbb{R}^n/V_1$ which we may identify with a unique $\widetilde{\gamma_1}(0)\in V_1^{\perp}$. The map sending $\gamma_1(0)$ to $\widetilde{\gamma_1}(0)$ is linear. Hence we have a linear map in $\mathscr{L}(V_1,V_1^{\perp})=\mathscr{L}(V_1,E_1^{\perp})$ determined uniquely by $V_1'(0)$. Fix an $i$ and suppose for induction that for $\forall\, j<i$, $V_j'(0)$ has been determined by a map in $\mathscr{L}(V_j,E_j^{\perp})$. Consider arbitrary curves $\gamma_1(t),...,\gamma_i(t)$ in $\mathbb{R}^n$ each satisfying $\gamma_j(t)\in V_j(t)$ for all $t$ and $j\le i$. Differentiating $A_i(t)\gamma_i(t)=0$ at $t=0$ gives as before,

$$A_i(0)\gamma_i'(0)+A_i'(0)\gamma_i(0)=0.$$

We may then suppose that $\gamma_i'(0)$ is in $V_i^{\perp}$ as before. However since the $V_i(t)$ are mutually orthogonal we additionally require that $\langle\gamma_i(t),\gamma_j(t)\rangle=0$ for all $j\le i$. Differentiating this at $t=0$ gives,

$$\langle\gamma_i'(0),\gamma_j(0)\rangle+\langle\gamma_i(0),\gamma_j'(0)\rangle=0.$$

This condition along with the fact that the $\gamma_j$s were arbitrary implies that the projections of $\gamma_i'(0)$ onto each $V_j$, $(j<i)$ are determined by $\gamma_i(0)$ and $\gamma_j'(0)$. The vector $\gamma_i'(0)$ therefore defines a class $\widetilde{\gamma}_i(0)$ in $V_i^{\perp}\bigcap_{j<i}V_j^{\perp}=E_i^{\perp}$. The map $\gamma_i(0)\to\widetilde{\gamma}_i(0)$ is linear. Therefore $V_i'(0)$ determines a unique map in $\mathscr{L}(V_i,E_i^{\perp})$. □

If the flag contains Hermitian subspaces then our tangent space needs to incorporate tangent vectors which arise from fixing each subspace of the flag but varying the complex structures. Let $V$ be a Hermitian subspace of the flag $F$ and $J:V\to V$ its complex structure. Define $\mathscr{C}(V)$ to be the manifold of oriented and orthogonal complex structures on $V$. Let $J(t)$ be a curve in $\mathscr{C}(V)$ such that $J(0)=J$. By differentiating the constraints $J(t)J^T(t)=I$ and $J(t)^2=-I$ we may identify $T_J\mathscr{C}(V)$ with the set $\{H\in\mathscr{L}(V,V):[H,J]=0,H+H^T=0\}$.

Now consider the flag manifold $\mathscr{F}=\mathscr{F}(d_1^{\mathbb{C}},...,d_{s-1}^{\mathbb{C}},d_s)$ where all subspaces $V_1,...,V_s$ are Hermitian with the optional exception of $V_s$. Since we may vary the subspaces $V_i$ and their complex structures independently

we may identify the tangent space with;

$$T_F\mathscr{F} = \bigoplus_{i=1}^{s} \left(\mathscr{L}(V_i, E_i^{\perp}) \oplus T_{J_i}\mathscr{C}(V_i)\right). \tag{2.6}$$

We can define a canonical symplectic form on $T_F\mathscr{F}$. Let $\mathbf{A} = \bigoplus_{1\leq i\leq s}(A_i, H_i)$ and $\mathbf{B} = \bigoplus_{1\leq i\leq s}(B_i, K_i)$ be tangent vectors in the sense of (2.6). That is $A_i, B_i \in \mathscr{L}(V_i, E_i^{\perp})$ and $H_i, K_i \in \{H \in \mathscr{L}(V_i, V_i) : [H, J_i] = 0, H + H^T = 0\}$. Define the following bilinear map;

$$\omega_F(\mathbf{A}, \mathbf{B}) = \frac{1}{2}\sum_{i=1}^{s}\left(Tr(A_i J_i B_i^T) + Tr(H_i J_i K_i^T)\right).$$

This form is clearly skew-symmetric. Furthermore it is non-degenerate since for $A_i \neq 0$, $Tr(A_i J_i (A_i J_i)^T) = Tr(A_i A_i^T) > 0$. Also notice that since $\mathscr{L}(V_s, E_s^{\perp})$ is trivial that we do not need a complex structure on $V_s$. Thus we have a symplectic form $\omega_F$ on $T_F\mathscr{F}$. This form varies smoothly with the fibres since each $J_i$ also varies smoothly. It follows that $\omega$ is a symplectic form on the flag $\mathscr{F}$.

Furthermore we can define a symplectic form on the flag manifolds $\mathrm{Aff}\mathscr{F}(\tilde{1}; d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s)$. That is, those affine flags with a one dimensional directed flag pole and the remaining subspaces Hermitian except for perhaps the last. Begin by observing that there is a fibre bundle structure $\mathrm{Aff}\mathscr{F}(\tilde{1}; d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s) \to \mathrm{Aff}\widetilde{\mathscr{F}}(1, n-1)$ with fibre $\mathscr{F}(d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s)$ defined by projecting the affine flag onto its flag pole. We have already defined a symplectic form $\omega_F$ on each fibre $\mathscr{F}(d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s)$. It suffices to show that the base space is symplectic with form $\omega_B$ to define a symplectic form $\omega_B \oplus \omega_F$ fibrewise on $\mathrm{Aff}\mathscr{F}((\tilde{1}; d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s)$. By Proposition 2.1 $\mathrm{Aff}\widetilde{\mathscr{F}}(1, n-1)$ is diffeomorphic to the tautological bundle over the manifold of directed lines in $\mathbb{R}^n$. This is known to be equal to the cotangent bundle of a sphere, $T^*S^{n-1}$. We may then take $\omega_B$ to be the canonical symplectic form on the cotangent bundle.

There is an alternative construction which is more geometric in flavour. Using a similar argument to before the tangent space to an affine directed line $l + x$ in $\mathrm{Aff}\widetilde{\mathscr{F}}(1, n-1)$ may be identified with the vector space $\mathrm{Aff}\mathscr{L}(l, l^{\perp})$ of affine linear maps of $l$ into $l^{\perp}$ ,i.e. affine linear maps $\mathbb{R} \to \mathbb{R}^{n-1}$. Such a map defines an affine line in $\mathbb{R}^{n-1}$. We identify such a line with a pair $[a, v]$ of vectors $a, v \in \mathbb{R}^{n-1}$ such that $a$ is a unit vector parallel to the line and $v$ a vector orthogonal to $a$ determining the translation of the line. Given two tangent vectors $[a, v]$ and $[b, w]$ identified in this way we may define the following symplectic form on $\mathrm{Aff}\mathscr{L}(l, l^{\perp})$;

$$\omega_F([a, v], [b, w]) = a^T w - b^T v.$$

Such a form $\omega_F$ can be shown to be symplectic and equivalent to the canonical form on $T^*S^{n-1}$ using our identification of $T^*S^{n-1}$ with $\mathrm{Aff}\widetilde{\mathscr{F}}(1, n-1)$. We have proved the following;

**Proposition 2.3.** *The flag manifolds $\mathscr{F}(d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s)$ and $\mathrm{Aff}\mathscr{F}(\tilde{1}; d_2^{\mathbb{C}}, ..., d_{s-1}^{\mathbb{C}}, d_s)$, (where $V_s$ may or may not be a Hermitian subspace) are symplectic.*

This result is important as it concurs with our later result that such flag manifolds occur as the coadjoint orbits of $SO(n)$ and $SE(n)$.

# 3 Orbits

## 3.1 Orbits of $SO(n)$

We begin by reviewing the orbits for $H = SO(n)$. Since the group is compact the adjoint and coadjoint representations are equivalent so it suffices to describe the adjoint orbits. Let $A \in \mathfrak{h} = \mathfrak{so}(\mathfrak{n})$, i.e. $A$ is a

real anti-symmetric matrix. $A$ is diagonalizable over $\mathbb{C}$ with purely imaginary eigenvalues which occur in pairs. That is if $i\rho$ is an eigenvalue ($\rho \in \mathbb{R}$) then so is $-i\rho$. To each non-zero eigenvalue pair $(i\rho, -i\rho)$ with multiplicity one there exists a unique invariant plane $\Pi_\rho \subset \mathbb{R}^n$ and a basis $x, y$ of orthogonal vectors that span $\Pi_\rho$ for which the action of $A$ on $\Pi_\rho$ is given by the $2 \times 2$ matrix,

$$\boxed{\rho} := \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}.$$

Assume the non-zero eigenvalues satisfy $\rho_1 \leq ... \leq \rho_r$. For $\rho \neq 0$ with multiplicity $m$ there are $m$ such invariant planes which are all orthogonal and for which the action on each plane is that of $\boxed{\rho}$.For all non-zero eigenvalue pairs $(i\rho_k, -i\rho_k)$ the invariant planes are orthogonal. Hence there exists a basis of orthogonal vectors in $\mathbb{R}^n$ such that $A$ is a matrix of the form,

$$\left(\begin{array}{c|ccc} 0 & & & \\ \hline & \boxed{\rho_1} & & \\ & & \ddots & \\ & & & \boxed{\rho_k} \end{array}\right) \tag{3.1}$$

Here 0 is a $k \times k$ matrix where $k$ is the dimension of $Ker\, A$. Therefore $A$ is conjugate to a matrix of the form above. The decomposition of $A$ into this form is referred to as a *type decomposition*. Borrowing notation from [2] we write this decomposition as

$$\Delta_A = \underbrace{\Delta_0 + ... + \Delta_0}_{k} + \Delta_{\rho_1} + ... + \Delta_{\rho_k}$$

The orbit through $A$ depends entirely on this type decomposition which is unique to $A$. The task of distinguishing all orbits now reduces to the task of classifying all such type decompositions for $A$. This is a fairly easy problem and is covered in [2]. We now claim that these orbits are flags.

**Proposition 3.1.** *There exists an $SO(n)$-equivariant diffeomorphism $\mathscr{F} : \mathcal{O}_{Ad}(A) \to \mathscr{F}$, where $\mathcal{O}_{Ad}(A)$ is the adjoint orbit through $A \in \mathfrak{so}(\mathfrak{n})$ and $\mathscr{F}$ a particular flag manifold in $\mathbb{R}^n$ determined by the type decomposition $\Delta_A$ of $A$.*

*Proof.* Let $K$ denote the kernel of $A$. For the non-zero eigenvalue pairs $(i\rho_k, -i\rho_k)$ consider the collection of invariant planes $\Pi_k$. For any $\lambda \in \mathbb{R}, \lambda \neq 0$ define $V_\lambda$ to be the sum of invariant planes $\Pi_i$ with $\rho_i = \lambda$. That is we group together the planes with a common eigenvalue to form a collection of even dimensional subspaces $V_1, ..., V_s$. The action of $A$ on each $V_i$ is given by $\boxed{\lambda_i} \oplus ... \oplus \boxed{\lambda_i}$ where the sum is taken over each $\Pi$ in $V_i$. We can define a complex structure $J_i$ on each $V_i$ by setting $J_i(v) = \frac{1}{\lambda_i}\left(\boxed{\lambda_i} \oplus ... \oplus \boxed{\lambda_i}\right)$. The action of $A$ on each $V_i$ is now determined by the complex structure $J_i$ and the corresponding eigenvalue $\lambda_i$. We define the following flag $\mathscr{F}(A)$,

$$\mathscr{F}(A) := 0 \subset K \subset K \oplus V_1 \subset ... \subset K \oplus \bigoplus_{i=1}^{s} V_i = \mathbb{R}^n.$$

Note that this flag is a Hermitian flag on all spaces except for the first subspace $K$. $\mathscr{F}(A) \in \mathscr{F}(k, d_1^{\mathbb{C}}, ..., d_s^{\mathbb{C}})$ where $d_i/2$ is the multiplicity of each eigenvalue $\lambda_i$. We aim to show that $\mathscr{F}(aAa^{-1}) = a \cdot \mathscr{F}(A)$. This is easy to see since the invariant subspaces for $aAa^{-1}$ are $a \cdot K, a \cdot V_1, ..., a \cdot V_s$ which are precisely the subspaces of the flag $a \cdot \mathscr{F}(A)$ and each complex structure $J_i$ on $a \cdot V_i$ is $aJ_ia^{-1}$. It follows that we have an $SO(n)$-equivariant map,

$$\mathscr{F} : \mathcal{O}_{\mathrm{Ad}}(A) \to \mathscr{F}(k, d_1^{\mathbb{C}}, ..., d_s^{\mathbb{C}}).$$

Injectivity of the map comes from the fact that the action of $A$ on $\mathbb{R}^n$ is determined uniquely by the eigenvalues $\lambda_1, ..., \lambda_s$, the kernel $K$ and the invariant subspaces together with their complex structures $(V_i, J_i)$.

Surjectivity follows from the fact that the action of $SO(n)$ on $\mathscr{F}(k, d_1^{\mathbb{C}}, ..., d_s^{\mathbb{C}})$ is transitive. Thus we have a $G$-equivariant bijection between smooth manifolds. Smoothness of $G = SO(n)$ implies that the map is smooth. □

Since $\mathscr{F}(k, d_1^{\mathbb{C}}, ..., d_k^{\mathbb{C}}) \cong \mathscr{F}(d_1^{\mathbb{C}}, ..., d_k^{\mathbb{C}}, k)$ we have from (2.3) that these orbits are symplectic. This is as expected since coadjoint orbits are naturally symplectic.

**Example:** For $SO(4)$ we list below the possible type decompositions for $A \in \mathfrak{so}(4)$ along with the corresponding isotropy subgroup and adjoint orbit through $A$.

| $\Delta_A$ | $\mathrm{Stab}_{\mathrm{Ad}}(A)$ | $\mathscr{O}_{\mathrm{Ad}}(A)$ | *Notes* |
|---|---|---|---|
| $4\Delta_0$ | $SO(4)$ | *Point* | . |
| $\Delta_\rho + \Delta_\kappa$ | $S(U(1) \times U(1))$ | $\mathscr{HF}(2,2)$ | $= \widetilde{Gr}(2;2), \rho \neq \kappa$ |
| $2\Delta_\rho$ | $U(2)$ | $\mathscr{HF}(4)$ | = Complex structures on $\mathbb{R}^4$. |
| $2\Delta_0 + \Delta_\rho$ | $S(O(2) \times U(1))$ | $\mathscr{F}(2, 2^{\mathbb{C}})$ | $= \widetilde{Gr}(2;2)$. |

(3.2)

### 3.2 Adjoint orbits of $SE(n)$

We now proceed to describe the adjoint orbits of $SE(n)$. There is a faithful representation of $SE(n)$ in $GL(n+1)$ defined by sending $(a, v)$ to the matrix $\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}$, where $a \in SO(n)$ and $v \in \mathbb{R}^n$. Using this representation we can write out the adjoint action of $SE(n)$ on a Lie algebra element $(A, X) \in \mathfrak{so}(\mathfrak{n}) \times V$;

$$\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}v \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} aAa^{-1} & -aAa^{-1}v + aX \\ 0 & 0 \end{pmatrix}.$$

Therefore we have,

$$Ad_{(a,v)}(A, X) = (Ad_a A,\ aX - (Ad_a A)v). \tag{3.3}$$

Hence the orbit is a bundle over the adjoint orbit $\mathscr{O}_{\mathrm{Ad}}(A)$ in $\mathfrak{so}(\mathfrak{n})$. We will separate the orbits into three cases.

**Case 1:** This is the trivial case of the orbit through the point $(0,0)$. The orbit is clearly that of a point. We define $A_1$ to be the set consisting solely of this point.

**Case 2:** We consider the orbit through a point belonging to the following set,

$$A_2 := \{(A, X) | X \in \mathrm{Im}A, A \neq 0\}.$$

Let $(A, X) \in A_2$ and $w$ a vector satisfying $Aw = X$. By acting on $(A, X)$ by $(I, w)$ we may wlog consider the orbit through a point $(A, 0)$. If we identify $\mathrm{Ad}_a A$ with $\mathscr{F}(\mathrm{Ad}_a A)$ then the orbit becomes a vector bundle with fibre $\mathrm{Im}(\mathrm{Ad}_a A)$ over $\mathscr{F}(\mathrm{Ad}_a A)$. Since $\mathscr{F}(\mathrm{Ad}_a A) = a \cdot \mathscr{F}(A)$ and $\mathrm{Im}(\mathrm{Ad}_a A) = a \cdot \mathrm{Im}A$, the orbit is the vector bundle over the flag manifold $\mathscr{F}_A$ determined by the type decomposition $\Delta_A$ with fibre equal to the subspace $\bigoplus_{i \geq 1} V_i$ for each flag. We recognise that this orbit is then the tautological bundle over $\mathscr{F}_A$ with $V_1 \cong \mathrm{Ker}A$. It follows from Proposition 2.1 that this orbit is the affine flag with signature determined by $\Delta_A$ and flag pole a subspace isomorphic to $\mathrm{Ker}A$. Note that if $A$ is non-singular then the orbit is the trivial vector bundle $\mathscr{F}_A \times \mathbb{R}^n$. This may be thought of as an affine flag in $\mathbb{R}^n$ where the flag pole is a point. In summary then we have the following proposition;

**Proposition 3.2.** *For any point $(A, X) \in A_2$ the orbit is an affine flag with signature and modulus determined by the type decomposition $\Delta_A$. The flag pole is isomorphic to the kernel of $A$. The orbit is a vector bundle over $\mathscr{O}_{Ad}(A)$ with fibre isomorphic to $\mathrm{Im}A$.*

Alternatively we can see this proposition by calculating the isotropy subgroup of $(A,0)$. Clearly $a$ must satisfy $\mathrm{Ad}_a A = A$. That is $a \in \mathrm{Stab}_{\mathrm{Ad}}(A)$. Additionally $v$ must belong to the kernel of $A$. Hence the isotropy subgroup is $\mathrm{Stab}_{\mathrm{Ad}}(A) \ltimes \mathrm{Ker}A$. Thanks to Proposition 3.1 $\mathrm{Stab}_{\mathrm{Ad}}(A)$ is isomorphic to the isotropy subgroup $H_{\mathscr{F}}$ of the flag $\mathscr{F}_A$. Proposition 3.2 now follows from (2.4).

**Case 3:** The final set to consider is,

$$A_3 = \{(A,X) | X \notin \mathrm{Im}A\}.$$

Since the image and kernel of $A$ are orthogonal complements of each other we may uniquely write $X$ as $X = X_k + X_k^\perp$ where $X_k \in \mathrm{Ker}A$ and $X_k^\perp \in \mathrm{Im}A$. Via the same method as in case 2 we may project away the part $X_k^\perp$ and hence wlog assume that $X$ is a non-zero element of $\mathrm{Ker}A$. We may also assume wlog that the kernel contains $e_1$ where $e_1, ..., e_n$ is the standard orthogonal basis on $\mathbb{R}^n$. By applying a suitable action we may further assume wlog that $X$ is of the form $|X|e_1$. Now to describe the isotropy group. Clearly $a$ must be in $\mathrm{Stab}_{\mathrm{Ad}}(A)$. We must then have that

$$aX - X = Av.$$

The left hand side of this expression belongs to $\mathrm{Ker}A$ (since $AaX = aAX = 0$) while the right hand side belongs to $\mathrm{Im}A$. Since these two sets have trivial intersection we must have that $aX = X$ and $v \in \mathrm{Ker}A$. Thus the isotropy group is

$$\left(SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Ad}}(A)\right) \ltimes \mathrm{Ker}A. \tag{3.4}$$

The group $\left(SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Ad}}(A)\right)$ is isomorphic to the group $\mathrm{Stab}_{\mathrm{Ad}}(\overline{A})$ where $\overline{A}$ indicates the restriction and projection of $A$ to $\langle e_2, ..., e_n\rangle$. We have established the following proposition which completes our classification of the adjoint orbits of $SE(n)$.

**Proposition 3.3.** *The orbit through a point $(A,X)$ in $A_3$ has isotropy group isomorphic to $Stab_{Ad}(\overline{A}) \ltimes KerA$. The modulus of the orbit is parametrised by $|X| \in \mathbb{R}^{>0}$ and the type decomposition $\Delta_{\overline{A}}$.*

**Example:** $SE(4)$. In this example we break the orbits down into the cases for $A_1$, $A_2$ and $A_3$. Trivially the orbit through $A_1$ is a point. Below we describe the orbits and isotropy groups for points through $A_2$. Recall from Proposition 3.2 that the orbit depends on the type decomposition $\Delta_A$.

| $\Delta_A$ | $\mathrm{Stab}_{\mathrm{Ad}}(A,0)$ | $Orbit$ |
|---|---|---|
| $\Delta_\rho + \Delta_\kappa$ | $S(U(1) \ltimes U(1))$ | $\mathbb{R}^4 \times \mathscr{HF}(2,2)$ |
| $2\Delta_\rho$ | $U(2)$ | $\mathbb{R}^4 \times \mathscr{HF}(4)$ |
| $2\Delta_0 + \Delta_\rho$ | $S(O(2) \times U(1)) \ltimes \mathbb{R}^2$ | $\mathrm{Aff}\mathscr{F}(2; 2^{\mathbb{C}})$ |

(3.5)

These orbits have 2, 1 and 1 moduli respectively. Now for orbits through $A_3$. The modulus of the orbit is determined by $\Delta_{\overline{A}}$ and $|X|$. Wlog suppose that $X = e_1$. This way $\Delta_{\overline{A}}$ may be understood as the type decomposition for an element $\overline{A} \in \mathfrak{so}(3)$. The orbits are classified below.

| $\Delta_{\overline{A}}$ | $\mathrm{Stab}_{\mathrm{Ad}}(\overline{A}) \ltimes \mathrm{Ker}A$ |
|---|---|
| $3\Delta_0$ | $SO(3) \ltimes \mathbb{R}^4$ |
| $\Delta_0 + \Delta_\rho$ | $S(O(1) \times U(1)) \ltimes \mathbb{R}^2$ |

(3.6)

These orbits have 1 and 2 moduli respectively. Observe that for $\Delta_{\overline{A}} = 3\Delta_0$ the orbit is that through $(0,X)$. It is clear from both the isotropy subgroup and (3.3) that for any $n$ the resulting orbit is a sphere $S^{n-1} \subset \mathbb{R}^n$.

### 3.3 Coadjoint orbits of $SE(n)$

Turning to the coadjoint orbits of $SE(n)$ we consider a point in the dual of the Lie algebra $(\Omega, \zeta) \in \mathfrak{so}(\mathfrak{n})^* \times \mathbb{R}^{n*} = \mathfrak{se}(\mathfrak{n})^*$ and let $(A,X) \in \mathfrak{se}(\mathfrak{n})$ be arbitrary. By definition we have,

$$\langle \mathrm{Coad}_{(a,v)}(\Omega,\zeta), (A,X)\rangle = \langle (\Omega,\zeta), \mathrm{Ad}_{(a,v)^{-1}}(A,X)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard pairing of $\mathfrak{se}(\mathfrak{n})^*$ with $\mathfrak{se}(\mathfrak{n})$. By using (3.3) we may rewrite the above as,

$$\begin{aligned}\langle \mathrm{Coad}_{(a,v)}(\Omega,\zeta),(A,X)\rangle &= \langle \Omega, \mathrm{Ad}_{a^{-1}}A\rangle + \langle \zeta, a^{-1}X\rangle + \langle \zeta, a^{-1}Av\rangle \\ &= \langle \mathrm{Coad}_a\Omega, A\rangle + \langle a^{-\star}\zeta, X\rangle + \langle \zeta, a^{-1}Av\rangle.\end{aligned}$$

We now introduce an isomorphism between dual spaces by using the inner products $\langle \zeta, X\rangle = \zeta^T X$ for elements $\zeta, X$ in $V$ and $\langle \Omega, A\rangle = \frac{1}{2}Tr(\Omega^T A)$ for $\Omega, A \in \mathfrak{so}(\mathfrak{n})$. In Section 1 we introduced the $\odot$ operator on $V \times V^*$. We shall do the same and rewrite the term $\langle \zeta, a^{-1}Av\rangle$ so that it is a pairing of elements in $\mathfrak{so}(\mathfrak{n})$ instead of $\mathbb{R}^n$.

$$\begin{aligned}\langle \zeta, a^{-1}Av\rangle &= Tr(\zeta^T a^{-1}Av) \\ &= \frac{1}{2}Tr(v\zeta^T a^{-1}A) + \frac{1}{2}Tr(v^T A^T a^{-T}\zeta) \\ &= \frac{1}{2}Tr\left((a\zeta v^T - v\zeta^T a^T)^T A\right).\end{aligned}$$

In this calculation we have used the cyclic property of the trace, $aa^T = I$ and $A + A^T = 0$. We rewrite $a\zeta v^T - v\zeta^T a^T$ as $a\zeta \wedge v$ where $w \wedge v$ is defined to equal $wv^T - vw^T$. It can be checked that $a\zeta \wedge v$ is antisymmetric and hence belongs to $\mathfrak{so}(\mathfrak{n})$. Therefore we have shown that $\langle \zeta, a^{-1}Av\rangle = \langle a\zeta \wedge v, A\rangle$. We have thus derived the following expression for the coadjoint action which is an analogue of the more general (1.1);

$$\mathrm{Coad}_{(a,v)}(\Omega,\zeta) = (\mathrm{Coad}_a\Omega + a\zeta \wedge v, a\zeta). \tag{3.7}$$

Observe that the orbits for $\zeta \neq 0$ are bundles over spheres $S^{n-1} \subset \mathbb{R}^n$. As in the adjoint case we will separate the orbits into three classes defined by what points they pass through. We partition $\mathfrak{se}(\mathfrak{n})^*$ into three disjoint sets, $C_1, C_2$ and $C_3$;

$$\begin{aligned}C_1 &= \{(0,0\} \\ C_2 &= \{(\Omega,0)|\Omega \neq 0\} \\ C_3 &= \{(\Omega,\zeta)|\zeta \neq 0\}\end{aligned}$$

**Case 1:** Trivially the orbit through $C_1$ is a point.

**Case 2:** From (3.7) it follows that the orbit through a point $(\Omega,0)$ in $C_2$ is $\mathcal{O}_{\mathrm{Coad}}(\Omega) \times \{0\}$. Since the coadjoint and adjoint representation are equivalent this orbit is diffeomorphic to $\mathcal{O}_{\mathrm{Ad}}(\Omega)$ in $\mathfrak{so}(\mathfrak{n})$ where we have identified $\Omega$ as an element in $\mathfrak{so}(\mathfrak{n})$. Consequently we have;

**Proposition 3.4.** *The orbit through a point* $(\Omega,0)$ *in* $C_2$ *is diffeomorphic to* $\mathcal{O}_{Ad}(\Omega)$ *and is hence defined by the type decomposition* $\Delta_\Omega$.

**Case 3:** We consider a point $(\Omega,\zeta) \in C_3$. We can apply an action of the group so that $\zeta$ is sent to $|\zeta|e_1$. Wlog then we assume that $\zeta = e_1$ and register that $|\zeta|$ is a modulus for the orbit. Since $\zeta = e_1$, the matrix corresponding to $\zeta \wedge v$ is given by,

$$\zeta \wedge v = \left(\begin{array}{c|c} 0 & v_2 \cdots v_n \\ \hline -v_2 & \\ \vdots & \mathbf{0} \\ -v_n & \end{array}\right). \tag{3.8}$$

We can therefore simplify the situation further by acting by a suitable $(I,v)$ so that $\Omega$ has all zeros in its first row and column. The remaining components for $\Omega$ are given by $\overline{\Omega}$, the projection and restriction to $\langle e_2,...,e_n\rangle$.

For $(a,v)$ to stabilize the point $(\Omega,e_1)$ we clearly require that $a \in SO(n)_{e_1}$, that is $ae_1 = e_1$. We are left with the following condition to hold,

$$\mathrm{Coad}_a\Omega - \Omega = v \wedge e_1.$$

Since we are identifying $\mathfrak{so}(\mathfrak{n})^*$ with $\mathfrak{so}(\mathfrak{n})$ we may replace $\mathrm{Coad}_a\Omega$ with $\mathrm{Ad}_a\Omega$ since the representations are equivalent. We must therefore satisfy $a\Omega a^{-1} - \Omega = v \wedge e_1$. Since we are assuming $\Omega$ has zeros in the first row and column it is easily seen that so must the left hand side of this expression. For $(a, v)$ to be in the isotropy subgroup for $(\Omega, e_1)$ is then equivalent to $a$ satisfying $\mathrm{Coad}_a\Omega = \Omega$ and $v$ such that $v \wedge e_1 = 0$. The isotropy subgroup is thus,

$$\left(SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Coad}}(\Omega)\right) \ltimes \mathbb{R} \cdot e_1 \tag{3.9}$$

This group is isomorphic to $\mathrm{Stab}_{\mathrm{Coad}}(\overline{\Omega})$, the isotropy subgroup for the coadjoint action of $SO(n-1)$ on $\mathfrak{so}(\mathfrak{n}-1)^*$. We collect this result in a proposition;

**Proposition 3.5.** *The coadjoint orbit through a point $(\Omega, \zeta)$ in $C_3$ is determined by $|\zeta| \in \mathbb{R}^{>0}$ and the type classification $\Delta_{\overline{\Omega}}$. The isotropy subgroup is isomorphic to the group,*

$$Stab_{Coad}(\overline{\Omega}) \ltimes \mathbb{R} \cdot \zeta.$$

From (2.3) we see that this group is the isotropy group for an affine flag with one dimensional directed flag pole and with the remaining subspaces determined by $\Delta_{\overline{\Omega}}$, (we have previously established from Proposition 2.3 that this is a symplectic manifold). Furthermore note that such an orbit fibres over $\tilde{\mathscr{F}}(1, n-1)$ with fibre equal to the flag manifold $\mathscr{F}_{\overline{\Omega}}$.

**Example:** $SE(4)$. Trivially the orbit through $C_1$ is a point. For points $(\Omega, 0) \in C_2$ we classify the orbits in the table below. Recall that they are defined by $\Delta_\Omega$ and are diffeemorphic to $\mathscr{O}_{\mathrm{Ad}}(A) \times \{0\}$.

| $\Delta_\Omega$ | $\mathscr{O}_{\mathrm{Coad}}(\Omega, 0)$ |
|---|---|
| $\Delta_\rho + \Delta_\kappa$ | $\mathscr{HF}(2,2) \times \{0\}$ |
| $2\Delta_\rho$ | $\mathscr{HF}(4) \times 0$ |
| $2\Delta_0 + \Delta_\rho$ | $\mathscr{F}(2, 2^{\mathbb{C}}) \times \{0\}$ |

(3.10)

These orbits have modulus 2, 1 and 1 respectively. For $(\Omega, \zeta) \in C_3$ recall that the modulus and orbit are determined by $\Delta_{\overline{\Omega}}$ and $|\zeta|$. Wlog suppose that $\zeta = e_1$. We classify the orbits and isotropy groups in the table below.

| $\Delta_{\overline{\Omega}}$ | $\mathrm{Stab}_{\mathrm{Coad}}(\overline{\Omega}) \ltimes \mathbb{R} \cdot \zeta$ | $\mathscr{O}_{\mathrm{Coad}}(\Omega, \zeta)$ |
|---|---|---|
| $3\Delta_0$ | $SO(3) \ltimes \mathbb{R}$ | $\mathrm{Aff}\mathscr{F}(1; n-1)$ |
| $\Delta_0 + \Delta_\rho$ | $S(O(1) \times U(1)) \ltimes \mathbb{R}$ | $\mathrm{Aff}\mathscr{F}(\tilde{1}; \tilde{1}, 2^{\mathbb{C}})$ |

(3.11)

These two orbits have modulus 1 and 2 respectively.

### 3.4 Orbit bijection

Compare (3.5) with (3.10) and (3.6) with (3.11). Notice that there is a bijection between the orbit types which preserves the modulus. Moreover we claim that this bijection is a homotopy equivalence between the orbits. That this bijection should be true for all $SE(n)$ is the main objective of this paper. To prove this we first need a well-known topological result; that a vector bundle is homotopic to the base space.

**Lemma.** *Given a vector bundle $p : E \to B$, the total space $E$ and the base space $B$ are homotopic.*

**Theorem 2.** *There exists a one-to-one correspondence between the coadjoint and adjoint orbit types of $SE(n)$ on $\mathfrak{se}(\mathfrak{n})^*$ and $\mathfrak{se}(\mathfrak{n})$ respectively. This correspondence preserves the modulus of each orbit. Furthermore two corresponding orbits are homotopic.*

*Proof.* Let $A_1, A_2, A_3, C_1, C_2, C_3$ be the sets defined before. Each orbit is contained in precisely one of these sets. For each set we have given an algebraic classification for the possible isotropy subgroups and moduli for an orbit passing through this set. It is this classification that defines what we mean when we say orbit type. We will prove that between each $A_i$ and $C_i$ there is a bijection of orbit types.

**Case 1:** For $A_1$ and $C_1$ the orbits are both a single point. Trivially these are homotopic.

**Case 2:** From Proposition 3.2 any point $(A, X)$ in $A_2$ has an orbit determined uniquely by the type decomposition $\Delta_A$. The orbit is a vector bundle over $\mathcal{O}_{\mathrm{Ad}}(A)$. By Proposition 3.4 the orbit through a point $(\Omega, 0)$ in $C_2$ is diffeomorphic to $\mathcal{O}_{\mathrm{Ad}}(\Omega)$ and determined uniquely by $\Delta_\Omega$. We can therefore identify the adjoint and coadjoint orbits with the same type decomposition, that is $\Delta_A = \Delta_\Omega$. Since the adjoint orbit is a vector bundle over the coadjoint orbit then from the lemma we have that the identified orbits are homotopic.

**Case 3:** Let $(A, X) \in A_3$.We may wlog instead consider the point $(A, \lambda e_1)$ where $\lambda > 0$. The isotropy subgroup for this point is given in (3.4) by $\left(SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Ad}}(A)\right) \ltimes \mathrm{Ker}A$ which we will denote by $G_{A,X}$. By Proposition 3.3 this orbit is determined by $\lambda$ and $\Delta_{\overline{A}}$ where $\overline{A}$ is the projection and restriction to $\langle e_2, ..., e_n \rangle$. Now consider a point in $C_3$. Similarly we may suppose the point is of the form $(\Omega, \mu e_1)$ where $\mu > 0$. From Proposition 3.5 and (3.9) the orbit is determined by $\mu$ and $\Delta_{\overline{\Omega}}$ and has isotropy group $G_{\Omega,\zeta} = \left(SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Coad}}(\Omega)\right) \ltimes \mathbb{R} \cdot e_1$. We may then identify the orbits which share the same modulus, that is if $(\lambda, \Delta_{\overline{A}}) = (\mu, \Delta_{\overline{\Omega}})$. Since the coadjoint and adjoint representations are equivalent we can identify the two groups $\mathrm{Stab}_{\mathrm{Coad}}(\overline{\Omega}) \equiv \mathrm{Stab}_{\mathrm{Ad}}(\overline{A})$. It can then be seen that for $\Delta_{\overline{A}} = \Delta_{\overline{\Omega}}$ we have,

$$SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Coad}}(\Omega) = SO(n)_{e_1} \cap \mathrm{Stab}_{\mathrm{Ad}}(A).$$

Therefore the isotropy subgroup $G_{\Omega,\zeta}$ is a subgroup of $G_{A,X}$. There is a natural projection map $\mathcal{O}_{\mathrm{Coad}}(\Omega, \zeta) \to \mathcal{O}_{\mathrm{Ad}}(A, X)$ defined by,

$$\mathcal{O}_{\mathrm{Coad}}(\Omega, \zeta) = \frac{SE(n)}{G_{\Omega,\zeta}} \to \frac{SE(n)}{G_{A,X}} = \mathcal{O}_{\mathrm{Ad}}(A, X).$$

This projection is defined coset-wise and is a principal bundle with fibre $G_{A,X}/G_{\Omega,\zeta}$ which is isomorphic to $\mathrm{Ker}A/\mathbb{R} \cdot e_1$, (where recall we have assumed $e_1 \in \mathrm{Ker}A$). The fibre is a vector space and so it follows from the lemma that this projection map between adjoint and coadjoint orbits is a homotopy equivalence. □

# 4 Further study

The methods outlined in this paper can, with slight modifications, be applied to any affine unitary or orthogonal group, definite and indefinite. Such groups yield a forthcoming description of the term $v \odot w$ in equation (1.1) analogous to our use of the term $v \wedge w$. In the case where $H = SU(n)$ the orbits are complex flag manifolds and the orbits of $G = SU(n) \ltimes \mathbb{C}^n$ are affine complex flags. For an indefinite group some care must be taken in handling timelike, null and spacelike vectors (that is vectors with positive, zero and negative length respectively). As with $SE(n)$ the orbits of the affine group are fibre bundles over a flag of directed lines with fibre equal to the coadjoint orbit of a little subgroup of $H$. It would be interesting to consider the orbits of the group $G = \sigma(H) \ltimes W$ where $\sigma : H \to GL(W)$ is an alternative representation of $H$ in a larger vector space $W$. Rawnsley considered this more general situation and showed that such orbits are still defined by a little subgroup and (1.4) still holds for an isotropy subgroup. Recent work by Mykytyuk [6] generalizes the situation to describing the coadjoint orbits of a Lie algbera $\mathfrak{g}$ containing an ideal $\mathfrak{n}$ and proves that the orbits are bundles with affine spaces for fibres. It would be interesting to see whether under such generality a result relating the adjoint and coadjoint orbits similar to Theorem 2 holds.

## References


[1] Rawnsley, J.H. *Representations of a semidirect product by quantization*. Math. Proc. Camb. Philos. Soc. **78**, 345-350 (1975)

[2] Burgoyne, N., Cushman, R. *Conjugacy classes in linear groups*. J. Algebra **44**, 339-362 (1977)

[3] Guillemin, V., Sternberg, S. *Symplectic techniques in physics*. Cambridge University Press, UK (1984)

[4] Cushman, R. van der Kallen, W. *Adjoint and coadjoint orbits of the Poincaré group*. Acta Appl. Math. **90**, 65-89 (2006)

[5] Bernatska, J., Petro, H. *Geometry and topology of coadjoint orbits of semisimple Lie groups*. Proceeding of 9-th International Conference on Geometry Integrability and Quantization Varna, 1-21 (2007)

[6] Mykytyuk, I.V. *Structure of the coadjoint orbits of Lie groups*. arXiv:1007.2446 [math.DG] (2010)

[7] Cushman, R. *Adjoint orbits of the odd real symplectic group*. Regular and Chaotic Dynamics **12.6**, 746-755 (2007)