Affine Weyl groups and Langlands duality

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AFFINE WEYL GROUPS AND LANGLANDS DUALITY

Abstract. Let G be a compact connected semisimple Lie group. We show that, as well as the duality between K-theory and K-homology, there is also a Langlands duality in the Baum-Connes correspondence for the (extended) affine Weyl group attached to G.

1. Introduction

In this paper we examine the Baum-Connes correspondence in the context of (extended) affine Weyl groups associated with a compact connected semisimple Lie group G. The extended affine Weyl group W_a' of a Lie group G can be realised as a group of affine isometries of the Lie algebra of a maximal torus of G. We denote the maximal torus by T and its Lie algebra by $\mathfrak{t}$. The action of W_a' on the Lie algebra $\mathfrak{t}$ provides a universal example for proper actions of W_a' and hence the domain of the assembly map is the equivariant K-homology group $K_*^{W_a'}(\mathfrak{t})$. The group W_a' is the semidirect product of a lattice Γ of translations of $\mathfrak{t}$ by the Weyl group W of G, which acts linearly on $\mathfrak{t}$. The quotient of $\mathfrak{t}$ by the translation action of Γ recovers the torus T and hence the left hand side of the assembly map can be identified as

$$K_*^{W_a'}(\mathfrak{t}) \cong K_W^*(C_0(\mathfrak{t}) \rtimes \Gamma) \cong K_*^W(T)$$

see below.

On the other hand, the right hand side of the assembly map is the K-theory of the algebra

$$C_r^* W_a' \cong C_r^* \Gamma \rtimes W \cong C(\widehat{\Gamma}) \rtimes W.$$

Here $\widehat{\Gamma}$ denotes the Pontryagin dual of the lattice Γ, which is a torus of the same dimension as T. One might therefore be tempted to think that the Baum-Connes correspondence in this case is an isomorphism between the W-equivariant K-homology and K-theory of the torus T. While such an isomorphism very often exists, this is not the Baum-Connes correspondence. Indeed although $\widehat{\Gamma}$ is a torus of the same dimension as T, there is in general no W-equivariant identification of the two tori.

An example (studied in section ??) is given by the Lie group SU(3) whose extended affine Weyl group (which in this case is its affine Weyl group) is the $(3,3,3)$-triangle group acting on the plane. The maximal torus T can be realised as a hexagon X with opposing sides identified and the Weyl group W (which is the dihedral group D_3) acts by reflecting in the three diagonals of X. By contrast we show that the dual torus $\widehat{\Gamma}$ can be realised as a different hexagon $X^\vee$ with opposing sides identified. The new hexagon should be viewed as the dual hexagon of X and the group W now acts by reflections in the bisectors of the edges of $X^\vee$. The corresponding action on the plane is by an index 3 extension of the triangle group, obtained by adjoining an order 3 rotation. Hence the dual picture has C_3 isotropy (as well as C_2 and D_3 isotropy), while the undualised picture does not. The tori cannot therefore be W-equivariantly identified in this example.

Given that the left- and right-hand sides of the Baum-Connes correspondence look so different in this example the isomorphism might almost appear coincidental. This 'coincidence' however can be explained by a duality between the tori T and $\widehat{\Gamma}$ which, as we will show, yields a duality in K-theory. This is in addition to the Poincaré duality from K-theory to K-homology and Fourier-Pontryagin duality from $C_r^*(\Gamma)$ to $C(\widehat{\Gamma})$. The torus $\widehat{\Gamma}$ is the T-dual of the torus T, which means at the level of Lie groups that $\widehat{\Gamma}$ is the maximal torus $T^\vee$ of the Langlands dual $G^\vee$ of G. We show that the identification of $\widehat{\Gamma}$ with $T^\vee$ is W-equivariant and thus the action of W on $\widehat{\Gamma}$ corresponds to the action of the extended affine Weyl group of $G^\vee$ on the Lie algebra $\mathfrak{t}^\vee$ (which is canonically identified with the dual space $\mathfrak{t}^*$). We show that the duality between T and $T^\vee$ yields a natural isomorphism from the W-equivariant K-homology of T to the W-equivariant K-theory of $T^\vee$. We thus obtain the following commutative diagram,

$$\begin{array}{ccc} KK^*_{W_a'}(C_0(\mathfrak{t}),\mathbb{C}) & \xrightarrow{\mu} & KK(\mathbb{C}, C_r^* W_a') \\ \downarrow\cong & & \uparrow\cong \\ KK^*_W(C(T),\mathbb{C}) & \xrightarrow{\cong} & KK^*_W(\mathbb{C}, C(T^\vee)) \end{array}$$

where μ is the Baum-Connes assembly map.

We obtain the bottom isomorphism as the composition of the Poincaré duality isomorphism from $KK^*_W(C(T),\mathbb{C})$ to $KK^*_W(\mathbb{C}, C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$ with a 'Langlands' isomorphism from $KK^*_W(\mathbb{C}, C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$ to $KK^*_W(\mathbb{C}, C(T^\vee))$, thereby giving an independent proof of the Baum-Connes correspondence in this context. Here $\mathcal{C}\ell(\mathfrak{t})$ denotes the complex Clifford algebra of $\mathfrak{t}$.

The duality between G and $G^\vee$ is further amplified by the following theorem.

Theorem 1.1. *Let G be a compact connected semisimple Lie group and $G^\vee$ its Langlands dual. Let $W_a'(G)$, $W_a'(G^\vee)$ denote the extended affine Weyl groups of G and $G^\vee$ respectively. Then there is a natural isomorphism*

$$K_*(C_r^*(W_a'(G))) \cong K_*(C_r^*(W_a'(G^\vee))).$$

Hence in particular, if G is of adjoint type then for $W_a(G)$ the affine Weyl group of G we have

$$K_*(C_r^*(W_a'(G))) \cong K_*(C_r^*(W_a(G))).$$

In the above example the Langlands dual of SU(3) is PSU(3), and the index 3 extension of the triangle group is the extended affine Weyl group of PSU(3).

Insert diagram here!!

We note that the last isomorphism does not hold in general in the case of a group of non adjoint type. We consider the example of $\mathrm{SU}(4)/\{\pm I\}$ in section ??.

2. Langlands duality

2.1. **Complex reductive groups.** Let $\mathbf{H}$ be a connected complex reductive group, with maximal torus $\mathbf{S}$. This determines a root datum

$$R(\mathbf{H},\mathbf{S}) := (\mathbf{X}^*(\mathbf{S}), R, \mathbf{X}_*(\mathbf{S}), R^\vee)$$

Here R and $R^\vee$ are the sets of roots and coroots of $\mathbf{H}$, while

$$\mathbf{X}^*(\mathbf{S}) := \mathrm{Hom}(\mathbf{S}, \mathbb{C}^\times) \quad \textbf{and} \quad \mathbf{X}_*(\mathbf{S}) := \mathrm{Hom}(\mathbb{C}^\times, \mathbf{S}) \tag{1}$$

are its character and co-character lattices.

The root datum implicitly includes the pairing $\mathbf{X}^*(\mathbf{S}) \times \mathbf{X}_*(\mathbf{S}) \to \mathbb{Z}$ and the bijection $R \to R^\vee$, $\alpha \mapsto h_\alpha$ between roots and coroots. Root data classify complex reductive Lie groups, in the sense that two such groups are isomorphic if and only if their root data are isomorphic (in the obvious sense) [SGA], [Ste].

A root datum determines the reductive group $\mathbf{H}$ up to isomorphism. Interchanging the roles of roots and coroots and of the character and co-character lattices results in a new root datum:

$$R(\mathbf{H},\mathbf{S})^\vee := (\mathbf{X}_*(\mathbf{S}), R^\vee, \mathbf{X}^*(\mathbf{S}), R)$$

The Langlands dual group of $\mathbf{H}$ is the complex reductive group $\mathbf{H}^\vee$ (unique up to isomorphism) determined by the dual root datum $R(\mathbf{H},\mathbf{S})^\vee$. A root datum also implies a choice of maximal torus $\mathbf{S} \subset \mathbf{H}$ via the canonical isomorphism $\mathbf{S} \simeq \mathrm{Hom}(\mathbf{X}^*(\mathbf{S}), \mathbb{C}^\times)$, and likewise a natural choice of maximal torus for the Langlands dual group $\mathbf{H}^\vee : \mathbf{S}^\vee := \mathrm{Hom}(\mathbf{X}_*(\mathbf{S}), \mathbb{C}^\times) \subset \mathbf{H}^\vee$.

In particular, we have the equation

$$\mathbf{X}^*(\mathbf{S}^\vee) = \mathbf{X}_*(\mathbf{S}) \tag{2}$$

2.2. **Compact semisimple groups.** Let now G be a compact connected semisimple Lie group, with maximal torus T. We recall that a compact connected Lie group is semisimple if and only if it has finite centre [B, p.285]. The classical examples are the compact real forms

$$\mathrm{SU}_n,\ \mathrm{SO}_{2n+1},\ \mathrm{Sp}_{2n},\ \mathrm{SO}_{2n},\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

The passage from G to its Langlands dual $G^\vee$ is via the *complexification* $G_\mathbb{C}$ of G. The correspondence

$$G \mapsto G_\mathbb{C}$$

is *bijective*, see [D, 27.17.11], and we have $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. Then pass to the Langlands dual $(G_\mathbb{C})^\vee$ of the complex group $G_\mathbb{C}$. Finally, define $G^\vee$ to be a maximal compact subgroup of $(G_\mathbb{C})^\vee$.

The dual group $G^\vee$ is determined up to isomorphism by the condition

$$(G^\vee)_\mathbb{C} = (G_\mathbb{C})^\vee.$$

This determines $G^\vee$ up to $(G_\mathbb{C})^\vee$-conjugacy.

Let T be a maximal torus in G. Then $\mathbf{S} := T_\mathbb{C}$ is a maximal torus in $\mathbf{H} := G_\mathbb{C}$, and so the dual torus $\mathbf{S}^\vee$ is well-defined in the dual group $\mathbf{H}^\vee$. Then $T^\vee$ is determined by the condition

$$(T^\vee)_\mathbb{C} = \mathbf{S}^\vee.$$

By definition, the torus $T^\vee$ is the T-dual of T . Corresponding to (2), we have the T-duality equation

$$X^*(T^\vee) = X_*(T) \tag{3}$$

where we agree that $X^*(T)$ shall mean the group of morphisms from the Lie group T to the Lie group $\mathbf{U} = \{z \in \mathbb{C} : |z| = 1\}$, and $X_*(T)$ shall mean the group of morphisms from the Lie group $\mathbf{U}$ to the Lie group T.

2.3. **Example:** $\mathrm{SU}_3(\mathbb{C})$**.** Let $G = \mathrm{SU}_3(\mathbb{C})$. Then $G^\vee = \mathrm{PSU}_3(\mathbb{C})$ and we have

$$T = \{(z_1, z_2, z_3) : z_j \in \mathbf{U}, z_1z_2z_3 = 1\}$$

$$T^\vee = \{(z_1 : z_2 : z_3) : z_j \in \mathbf{U}, z_1z_2z_3 = 1\}$$

the latter being in homogeneous coordinates. The map

$$T \to T^\vee, \quad (z_1, z_2, z_3) \mapsto (z_1 : z_2 : z_3)$$

is a 3-fold cover: the pre-image of $(z_1 : z_2 : z_3)$ is the set

$$\{\eta z_1, \eta\ z_2, \eta z_3) : \eta \in \mathbf{U}, \eta^3 = 1\}.$$

The Lie group G and its dual $G^\vee$ admit a common Weyl group

$$W = N(T)/T = N(T^\vee)/T^\vee$$

in this case the symmetric group $\mathfrak{S}_3$. Note that, in general, T and $T^\vee$ are *not* isomorphic as W-spaces. With $G = \mathrm{SU}_3(\mathbb{C})$, T admits three W-fixed points, namely

$$\{(1, 1, 1), (\omega, \omega, \omega), (\omega^2, \omega^2, \omega^2) : \omega = \exp(2\pi i/3)\}$$

whereas the unique W-fixed point in $T^\vee$ is the identity $I \in T^\vee$.

2.4. **The nodal group.** Let G be as above, T a maximal torus in G and define the *nodal group*

$$\Gamma(T) := \ker(\exp : \mathfrak{t} \to T)$$

Lemma 2.1. *We have a W-equivariant isomorphism*

$$X_*(T) \simeq \Gamma(T)$$

Proof. The group $X_*(T)$ is the group of morphisms from the Lie group $\mathbf{U}$ to the Lie group T. Given $f \in X_*(T)$, we have

$$\Gamma(f) : \Gamma(\mathbf{U}) \to \Gamma(T).$$

We identify $\Gamma(\mathbf{U})$ with the subgroup $2\pi i\mathbb{Z}$ of $L(\mathbf{U}) = i\mathbb{R}$. We then have the isomorphism

$$X_*(T) \simeq \Gamma(T), \qquad f \mapsto \Gamma(f)(2\pi i)$$

as in [B, p.307]. □

Lemma 2.2. *If A is a locally compact abelian topological group, let $\widehat{A}$ denote its Pontryagin dual. Then we have a W-equivariant isomorphism*

$$\widehat{\Gamma(T)} \simeq T^\vee.$$

Proof. We have, by Lemma (2.1) and the T-duality equation (3),

$$\Gamma(T) \simeq X_*(T) = X^*(T^\vee) = \widehat{T^\vee}.$$

Now apply Pontryagin duality. □

The groups $\Gamma(T)$ and $T^\vee$ are *in duality* in the sense of locally compact abelian topological groups.

2.5. **A table of Langlands dual groups.** The *connection index* is a numerical invariant denoted f in [B, VI, p.240]. The connection indices are listed in [B, VI, Plates I–X, p.265–292]. The connection index is a useful invariant, thanks to the following property:

$$|\pi_1(G)| \cdot |\mathcal{Z}(G)| = f$$

see [B, IX, p.320]. For example, we have

$$\pi_1 \mathrm{SO}_{2n+1} = \mathbb{Z}/2\mathbb{Z}, \qquad \mathcal{Z}(\mathrm{SO}_{2n+1}) = 1, \qquad f = 2$$

Here is a table of Langlands duals and connection indices for compact connected semisimple groups:

G	$G^\vee$	f
$A_n = \mathrm{SU}_{n+1}$	PSU_{n+1}	$n+1$
$B_n = \mathrm{SO}_{2n+1}$	Sp_{2n}	2
$C_n = \mathrm{Sp}_{2n}$	SO_{2n+1}	2
$D_n = \mathrm{SO}_{2n}$	SO_{2n}	4
E_6	E_6	3
E_7	E_7	2
E_8	E_8	1
F_4	F_4	1
G_2	G_2	1

In this table, the simply-connected form of E_6 (resp. E_7) corresponds to the adjoint form of E_6 (resp. E_7).

3. Affine Weyl groups

There is a vital distinction between the affine Weyl group W_a and the extended affine Weyl group W'_a. The quotient W'_a/W_a is a finite abelian group which dominates the discussion.

Our reference at this point is [B, IX, p.309–327]. Let $\mathfrak{t}$ denote the Lie algebra of T, and let $\exp : \mathfrak{t} \to T$ denote the exponential map. The map $\exp : \mathfrak{t} \to T$ is a morphism of Lie groups, surjective with discrete kernel [B, p.282]. The kernel of exp is the *nodal group* $\Gamma(T)$.

The inclusion $\iota : T \to G$ induces the homomorphism $\pi_1(\iota) : \pi_1(T) \to \pi_1(G)$. Now $f(G,T)$ will denote the composite of the canonical isomorphism from $\Gamma(T)$ to $\pi_1(T)$ and the homomorphism $\pi_1(\iota)$:

$$f(G,T) : \Gamma(T) \simeq \pi_1(T) \to \pi_1(G).$$

Denote by $N(G,T)$ the kernel of $f(G,T)$. We have a short exact sequence

$$0 \to N(G,T) \to \Gamma(T) \to \pi_1(G) \to 0 \tag{4}$$

see [B, p.315].

Denote by $N_G(T)$ the normalizer of T in G. Let W denote the Weyl group $N_G(T)/T$. The *affine Weyl group* is

$$W_a = N(G,T) \rtimes W$$

and the *extended affine Weyl group* is

$$W'_a = \Gamma(T) \rtimes W$$

The subgroup W_a of W'_a is normal.

If $\mathfrak{t} - \mathfrak{t}_r$ denotes the union of the singular hyperplanes in $\mathfrak{t}$, then the *alcoves* of $\mathfrak{t}$ are the connected component of $\mathfrak{t}_r$.

The group W_a operates simply-transitively on the set of alcoves. Let A be an alcove. Then $\overline{A}$ is a fundamental domain for the operation of W_a on $\mathfrak{t}$.

Let H_A be the stabilizer of A in W'_a. Then H_A is a finite abelian group which can be identified naturally with $\pi_1(G)$, see [B, IX, p.326]. The extended affine Weyl group W'_a is the semi-direct product

$$W'_a = W_a \rtimes H_A.$$

View $\mathfrak{t}$ as an additive group, and form the Euclidean group $\mathfrak{t} \rtimes O(\mathfrak{t})$. We have $W'_a \subset \mathfrak{t} \rtimes O(\mathfrak{t})$ and so W'_a acts as affine transformations of $\mathfrak{t}$. Now H_A leaves $\overline{A}$ invariant, so H_A acts as affine transformations of $\overline{A}$. Let $v_0, v_1, \ldots, v_n$ be the vertices of the simplex $\overline{A}$. We will use barycentric coordinates, so that

$$x = \sum_{i=0}^{n} t_i v_i$$

with $x \in \overline{A}$. The barycentre x_0 has coordinates $t_j = 1$, $0 \leq j \leq n$ and so is H_A-fixed. Then $\overline{A}$ is equivariantly contractible to x_0:

$$r_t(x) := tx_0 + (1-t)x \tag{5}$$

with $0 \leq t \leq 1$. This is an affine H_A-equivariant retract from $\overline{A}$ to x_0.

The exponential map $\overline{A} \to T$ and the canonical injection $T \to G$ induce a homeomorphism

$$\overline{A}/H_A \to T/W$$

see [B, p.326]. It follows that T/W is a contractible space.

Example. In the special case of SU_3, the vector space $\mathfrak{t}$ is the Euclidean plane $\mathbb{R}^2$. The singular hyperplanes tessellate $\mathbb{R}^2$ into equilateral triangles. The interior of each equilateral triangle is an alcove. Barycentric subdivision refines this tessellation into isosceles triangles. The extended affine Weyl group W'_a acts simply transitively on the set of these isosceles triangles, but the closure $\overline{\Delta}$ of one such triangle is not a fundamental domain (in the strict sense) for the action of W'_a. The corresponding quotient space is [B, IX. §5.2]:

$$\mathfrak{t}/W'_a \simeq \overline{A}/H_A.$$

The abelian group H_A is the cyclic group $\mathbb{Z}/3\mathbb{Z}$ which acts on $\overline{A}$ by rotation about the barycentre of $\overline{A}$ through $2\pi/3$.

4. C^*-algebras

Theorem 4.1. *We have*

$$C^*(W'_a) \simeq C(T^\vee) \rtimes W$$

Proof. By Lemma (2.2), the spectrum of the commutative C^*-algebra $C^*(\Gamma(T))$ is homeomorphic to the compact Hausdorff space $T^\vee$, and we have the *Gelfand* isomorphism [Sp, p.67]:

$$C^*(\Gamma(T)) \simeq C(T^\vee)$$

Related to this, we have, by the Mackey machine:

$$C^*(W'_a) = C^*(\Gamma(T) \rtimes W) \tag{6}$$

$$\simeq C(T^\vee) \rtimes W \tag{7}$$

by Lemma (2.2). □

Lemma 4.2. *Let $\widetilde{G}$ denote the universal cover (simply connected covering group) of G and let $\widetilde{T}$ denote a maximal torus in $\widetilde{G}$. We have*

$$N(T) = \Gamma(\widetilde{T}) = N(\widetilde{T})$$

Proof. Let $\pi_1 = \pi_1(G)$. According to [B, p.291], we have

$$G = \widetilde{G}/\pi_1, \qquad T = \widetilde{T}/\pi_1$$

and π_1 is central in G.

Consider the adjoint representation

$$\mathrm{Ad}_{\widetilde{G}} : \widetilde{G} \to \mathrm{Aut}(\mathfrak{g})$$

Since π_1 is a central subgroup, this representation descends to the adjoint representation of G:

$$\mathrm{Ad}_G : G \to \mathrm{Aut}(\mathfrak{g})$$

Since the roots are the nonzero weights in the adjoint representation, it follows that

$$R(\widetilde{G}, \widetilde{T}) = R(G, T).$$

Now $N(\widetilde{T})$ is the subgroup of $\mathfrak{t}$ generated by the nodal vectors

$$\{K_\alpha : \alpha \in R(\widetilde{G}, \widetilde{T})\}$$

and $N(T)$ is the subgroup of $\mathfrak{t}$ generated by the nodal vectors

$$\{K_\alpha : \alpha \in R(G, T)\}$$

see [B, p.314]. It follows that

$$N(T) = N(\widetilde{T}).$$

By (4) we infer that

$$N(\widetilde{T}) = \Gamma(\widetilde{T}).$$

□

Theorem 4.3. *We have*

$$C^*(W_a) \simeq C(T^\vee_{adj}) \rtimes W$$

where $T^\vee_{adj}$ is dual to $\widetilde{T}$, i.e. $T^\vee_{adj}$ is a maximal torus in the adjoint *form of G.*

Proof. By Lemma (4.2), we have

$$\begin{aligned} W_a(G) &= N(T) \rtimes W \\ &= \Gamma(\widetilde{T}) \rtimes W \\ &= N(\widetilde{T}) \rtimes W \end{aligned}$$

so that we have

$$W_a(G) = W'_a(\widetilde{G}) = W_a(\widetilde{G}).$$

Then we infer that

$$\begin{aligned} C^*(W_a(G)) &= C^*(W'_a(\widetilde{G})) \\ &= C(T^\vee_{adj}) \rtimes W \end{aligned}$$

□

5. “KK-Langlands”

In this section we will establish the isomorphism from $KK^*_W(C(T),\mathbb{C})$ to $KK^*_W(\mathbb{C},C(T^\vee))$. Poincaré duality (see Kasparov [K] section 4) yields an isomorphism from $KK^*_W(C(T),\mathbb{C})$ to $KK^*_W(\mathbb{C},C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$. We will show that $C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t})$ is KK-equivalent to $C(T^\vee)$ hence obtaining the required isomorphism from $KK^*_W(\mathbb{C},C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$ to $KK^*_W(\mathbb{C},C(T^\vee))$.

5.1. **Construction of the Dirac class.** We begin with the construction of a W-equivariant Dirac class $[D]$ in the KK-group $KK_W(C(T),C(T^\vee)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*))$. Let $L^2_c(\mathfrak{t})$ denote the space of compactly supported square-integrable functions on $\mathfrak{t}$, where $\mathfrak{t}$ has the Haar measure normalised so that $T=\Gamma\backslash\mathfrak{t}$ has mass 1. This is a Γ-space in the obvious way. We equip the space $L^2_c(\mathfrak{t})$ with the $\mathbb{C}[\Gamma]$-valued inner product

$$\langle u,v\rangle=\sum_{\gamma\in\Gamma}\int_{\mathfrak{t}}\overline{u(x)}(\gamma\cdot v)(x)\,dx[\gamma]$$

and right $\mathbb{C}[\Gamma]$-module structure defined by

$$v[\gamma]=\gamma^{-1}\cdot v.$$

Completing $L^2_c(\mathfrak{t})$ we obtain a $C^*_r(\Gamma)$-Hilbert module $\overline{L^2_c(\mathfrak{t})}$, which we view as a $C(T^\vee)$-Hilbert module via the isomorphism $C^*_r(\Gamma)\cong C(T^\vee)$. In this form the inner product is given explicitly by

$$\langle u,v\rangle(\exp(\eta))=\sum_{\gamma\in\Gamma}\int_{\mathfrak{t}}\overline{u(x)}(\gamma\cdot v)(x)\,dx\,e^{2\pi i\langle\eta,\gamma\rangle}$$

where $\eta\in t^*$. We note that the inner product can be expressed in a more symmetrical form, at the cost of selecting a fundamental domain X for the action of Γ on $\mathfrak{t}$. The integral over $\mathfrak{t}$ can be expressed as the sum over $\delta\in\Gamma$ of the integrals over translates of X. This gives

$$\int_{\mathfrak{t}}\overline{u(x)}(\gamma\cdot v)(x)\,dx=\sum_{\delta\in\Gamma}\int_X\overline{\delta\cdot u(x)}((\delta+\gamma)\cdot v)(x)\,dx$$

and changing variables to $\gamma'=\delta+\gamma$ we obtain the formula

$$\langle u,v\rangle(\exp(\eta))=\sum_{\gamma',\delta\in\Gamma}\int_X\overline{(\delta\cdot u)(x)e^{2\pi i\langle\eta,\delta\rangle}}(\gamma'\cdot v)(x)\,e^{2\pi i\langle\eta,\gamma'\rangle}\,dx. \tag{8}$$

Now, having constructed $\overline{L^2_c(\mathfrak{t})}$, we form the tensor product Hilbert module $\mathcal{E}=\overline{L^2_c(\mathfrak{t})}\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*)$ over $C(T^\vee)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*)$.

The Weyl group W acts on $\mathfrak{t}$ and hence on $L^2_c(\mathfrak{t})$, giving rise to an action of W on the completion $\overline{L^2_c(\mathfrak{t})}$. Dually W acts on $\mathfrak{t}^*$ and hence on the Clifford algebra $\mathcal{C}\ell(\mathfrak{t}^*)$. We equip $\mathcal{E}$ with the diagonal action of W on the two factors. It is easy to verify that this makes $\mathcal{E}$ into a W-equivariant Hilbert module.

Next we define a representation ρ of $C(T)$ as adjointable operators on $\mathcal{E}$. Viewing a function f in $C(T)$ as a Γ-periodic function $\tilde{f}$ on $\mathfrak{t}$, we simply define $(\rho(f)(v \otimes a))(x) = \tilde{f}(x)v(x) \otimes a$. The action of W on $\mathfrak{t}$ is (tautologically) compatible with the action on T, hence this representation is W-equivariant.

Finally we define the operator D on $\mathcal{E}$. This is an unbounded operator defined (using Einstein summation convention) by

$$D(v \otimes a) = \frac{\partial}{\partial x^j} v \otimes \varepsilon^j a$$

where $\{\varepsilon^j\}$ denotes the dual basis of $\mathfrak{t}^*$ corresponding to the basis $\{\mathbf{e}_j = \frac{\partial}{\partial x^j}\}$ of $\mathfrak{t}$.

Self-adjointness of D follows easily by the usual Stokes' Theorem argument, along with the observation that $\frac{\partial}{\partial x^j}(\gamma \cdot v) = \gamma \cdot \frac{\partial}{\partial x^j} v$. For f in $C(T)$ differentiable, it is immediate that $[D, \rho(f)]$ extends to a bounded operator on $\mathcal{E}$.

Proposition 5.1. *The operator D on $\mathcal{E}$ is given by a field over $T^\vee$ of operators on the Hilbert module $L^2(X) \widehat{\otimes} \mathcal{C}\ell(\mathfrak{t}^*)$, which have discrete spectrum (with finite multiplicities), namely $\{\pm 2\pi|\chi + \eta| : \chi \in \Gamma^\vee\}$ at the point* $\exp(\eta)$ *in $T^\vee$. Hence D is regular and has compact resolvent.*

Proof. Evaluation at a point $\exp(\eta) \in T^\vee$ gives a map $C(T^\vee) \widehat{\otimes} \mathcal{C}\ell(\mathfrak{t}^*) \to \mathcal{C}\ell(\mathfrak{t}^*)$, and we let $\mathcal{E}_\eta = \mathcal{E} \widehat{\otimes}_{C(T^\vee) \widehat{\otimes} \mathcal{C}\ell(\mathfrak{t}^*)} \mathcal{C}\ell(\mathfrak{t}^*)$ be the corresponding Hilbert module. Using Equation (8), the map $\phi_\eta : \mathcal{E}_\eta \to L^2(X) \otimes \mathcal{C}\ell(\mathfrak{t}^*)$ defined by

$$\phi_\eta((v \otimes a) \otimes 1) = \sum_{\gamma \in \Gamma} v(-\gamma + x) e^{2\pi i \langle \eta, \gamma \rangle} \otimes a$$

preserves the inner product and hence is an isomorphism. We are interested in the localisation of the operator D given by transferring $D \otimes 1$ on $\mathcal{E}_\eta$ onto $L^2(X) \otimes \mathcal{C}\ell(\mathfrak{t}^*)$. We will denote the operator on $L^2(X) \otimes \mathcal{C}\ell(\mathfrak{t}^*)$ by D_η. Like D, the operator D_η is a differential operator satisfying the local formula $D_\eta = \frac{\partial}{\partial x^j} \otimes \varepsilon^j$, however we must also determine the boundary conditions. These are given by considering the image of the smooth functions on $\mathfrak{t}$ under the map ϕ_η: for v a smooth function we have

$$\begin{aligned} \phi_\eta((v \otimes a) \otimes 1)(\delta + x) &= \sum_{\gamma \in \Gamma} v(-\gamma + \delta + x) e^{2\pi i \langle \eta, \gamma \rangle} \otimes a \\ &= \sum_{\gamma' \in \Gamma} v(-\gamma' + x) e^{2\pi i \langle \eta, \delta + \gamma' \rangle} \otimes a \end{aligned}$$

so $\phi_\eta((v \otimes a) \otimes 1)(\delta + x) = e^{2\pi i \langle \eta, \delta \rangle} \phi_\eta((v \otimes a) \otimes 1)(x)$. We note that the boundary conditions vary with the point η, however this variation is $\Gamma^\vee$-periodic, so in fact they depend only on $\exp(\eta)$.

Fixing η, the space $L^2(X)$ has an orthonormal basis consisting of functions of the form $e^{2\pi i \langle \chi + \eta, x \rangle}$ where χ ranges over the dual lattice $\widehat{T} = \Gamma^\vee$. These functions satisfy the boundary conditions. Applying the operator D_η and then pulling the coordinates $\chi_j + \eta_j$ through the tensor we obtain

$$D_\eta(e^{2\pi i\langle\chi+\eta,x\rangle} \otimes b) = \frac{\partial}{\partial x^j} e^{2\pi i\langle\chi+\eta,x\rangle} \otimes \varepsilon^j b = 2\pi i e^{2\pi i\langle\chi+\eta,x\rangle} \otimes (\chi+\eta) b$$

whence the operator D_η is $2\pi i \otimes (\chi+\eta)$ on the corresponding subspace. The submodules

$$E_{\chi,\pm} = \{e^{2\pi i\langle\chi+\eta,x\rangle} \otimes (i(\chi+\eta) \pm |\chi+\eta|)a : a \in \mathcal{C}\ell(\mathfrak{t}^*)\}$$

are eigenspaces with eigenvalue $\pm 2\pi|\chi+\eta|$. For each η the set of χ such that $|\chi+\eta|$ takes a given value is finite, hence the eigenvalues have finite multiplicity.

It now follows easily that D is regular and has compact resolvent. □

To show that the triple $(\mathcal{E}, \rho, D)$ is an unbounded W-equivariant Kasparov triple, thereby defining an element of $KK_W(C(T), C(T^\vee)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*))$, it remains to show that the Dirac operator D is W-equivariant. This follows from a more general statement, Proposition 5.2 below, which shows in particular that $\frac{\partial}{\partial x^j} \otimes \varepsilon^j$ in $\mathfrak{t} \otimes \mathfrak{t}^*$ is invariant under the natural action of $\mathrm{GL}(\mathfrak{t})$ and hence that it is W-invariant. The Proposition moreover shows an invariance result for elements of $\mathcal{C}\ell(\mathfrak{t}) \otimes \mathcal{C}\ell(\mathfrak{t}^*)$ which we will make use of later to carry out a W-equivariant restriction from the Clifford algebras to spinors.

Consider the abstract setup of a finite dimensional vector space V. The natural action of $\mathrm{GL}(V)$ on V, induces a diagonal action on $V \otimes V^*$.

If V is equipped with a non-degenerate symmetric bilinear form g then we can form the Clifford algebra $\mathcal{C}\ell(V)$. The subgroup $\mathrm{O}(g)$ of $\mathrm{GL}(V)$, consisting of those elements preserving g, acts naturally on $\mathcal{C}\ell(V)$. The bilinear form additionally gives an isomorphism from V to V^* and hence induces a bilinear form g^* on V^*, allowing us to form the Clifford algebra $\mathcal{C}\ell(V^*)$. Clearly the dual action of $\mathrm{O}(g)$ on V^* preserves g^* hence there is a diagonal action of $\mathrm{O}(g)$ on $\mathcal{C}\ell(V)\widehat{\otimes}\mathcal{C}\ell(V^*)$ which we identify with $\mathcal{C}\ell(V \times V^*)$.

We say that an element a of $\mathcal{C}\ell(V \times V^*)$ is *symmetric* if there exists a g-orthonormal[1] basis $\{\mathbf{e}_j : j = 1, \dots, n\}$ with dual basis $\{\varepsilon^j : j = 1, \dots, n\}$ such that a can be written as $p(\mathbf{e}_1\varepsilon^1, \dots, \mathbf{e}_n\varepsilon^n)$ where $p(x_1, \dots, x_n)$ is a symmetric polynomial.

Proposition 5.2. *For any basis $\{\mathbf{e}_j\}$ of V with dual basis $\{\varepsilon^j\}$ for V^*, the Einstein sum $\mathbf{e}_j \otimes \varepsilon^j$ in $V \otimes V^*$ is $\mathrm{GL}(V)$-invariant.*

Suppose moreover that V is equipped with a non-degenerate symmetric bilinear form g and that the underlying field has characteristic zero. Then every symmetric element of $\mathcal{C}\ell(V)\widehat{\otimes}\mathcal{C}\ell(V^*) \cong \mathcal{C}\ell(V \times V^*)$ is $\mathrm{O}(g)$-invariant.

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[1] We say that $\{\mathbf{e}_j\}$ is g-orthonormal if $g_{jk} = \pm\delta_{jk}$ for each j, k.

Proof. Identifying $V \otimes V^*$ with endomorphisms of V in the natural way, the action of $\mathrm{GL}(V)$ is the action by conjugation and $\mathbf{e}_j \otimes \varepsilon^j$ is the identity which is invariant under conjugation.

For the second part, over a field of characteristic zero the symmetric polynomials are generated by power sum symmetric polynomials $p(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k$, so it suffices to consider

$$\begin{aligned} p(\mathbf{e}_1\varepsilon^1, \dots, \mathbf{e}_n\varepsilon^n) &= (\mathbf{e}_1\varepsilon^1)^k + \cdots + (\mathbf{e}_n\varepsilon^n)^k \\ &= (-1)^{k(k-1)/2}\Big((\mathbf{e}_1)^k(\varepsilon^1)^k + \cdots + (\mathbf{e}_n)^k(\varepsilon^n)^k\Big). \end{aligned}$$

When k is even, writing $(\mathbf{e}_j)^k = (\mathbf{e}_j^2)^{k/2} = (g_{jj})^{k/2}$ and similarly $(\varepsilon^j)^k = (g^{jj})^{k/2}$, we see that each term $(\mathbf{e}_j)^k(\varepsilon^j)^k$ is 1 since $g_{jj} = g^{jj} = \pm 1$ for an orthonormal basis. Thus $p(\mathbf{e}_1\varepsilon^1, \dots, \mathbf{e}_n\varepsilon^n) = n(-1)^{k(k-1)/2}$ which is invariant.

Similarly when k is odd we get $(\mathbf{e}_j)^k(\varepsilon^j)^k = \mathbf{e}_j\varepsilon^j$ so

$$p(\mathbf{e}_1\varepsilon^1, \dots, \mathbf{e}_n\varepsilon^n) = (-1)^{k(k-1)/2}(\mathbf{e}_1\varepsilon^1 + \cdots + \mathbf{e}_n\varepsilon^n).$$

As the sum $\mathbf{e}_j \otimes \varepsilon^j$ in $V \otimes V^*$ is invariant under $\mathrm{GL}(V)$, it is in particular invariant under $\mathrm{O}(g)$, and hence the sum $\mathbf{e}_j\varepsilon^j$ is $\mathrm{O}(g)$-invariant in the Clifford algebra. □

5.2. **The Kasparov product.** In the previous section we constructed a class $(\mathcal{E}, \rho, D)$ in $KK_W(C(T), C(T^\vee)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*))$. Replacing the group G with its Langlands dual this construction produces an element $(\mathcal{E}^\vee, \rho^\vee, D^\vee)$ in $KK_W(C(T^\vee), C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$. Taking the Kasparov product with $(\mathcal{E}^\vee, \rho^\vee, D^\vee)$ gives a map

$$KK_W(\mathbb{C}, C(T^\vee)) \to KK_W(\mathbb{C}, C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$$

which we will prove is an isomorphism. To construct the inverse, first we take the Kasparov product with $(\mathcal{E}, \rho, D)$ over $C(T)$ which gives a map

$$KK_W(\mathbb{C}, C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t})) \to KK_W(\mathbb{C}, C(T^\vee)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$$

and then use a W-equivariant Morita equivalence to get a map

$$KK_W(\mathbb{C}, C(T^\vee)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}^*)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t})) \to KK_W(\mathbb{C}, C(T^\vee)).$$

The composition of these three maps gives a map

$$KK_W(\mathbb{C}, C(T^\vee)) \to KK_W(\mathbb{C}, C(T^\vee)) \tag{9}$$

which we will show is the identity.

Consider the projection $P = \prod_j \frac{1}{2}(1 + i\mathbf{e}_j\varepsilon^j)$ in the Clifford algebra $\mathcal{C}\ell(\mathfrak{t}^*)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}) \cong \mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$. By Proposition 5.2 this is W-invariant. The corner algebra $P\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})P$ is $\mathbb{C}P$, and we will identify this with $\mathbb{C}$. Now let $\mathcal{S} = \mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})P$, which is a finite dimensional Hilbert space, with inner product given by $\langle aP, bP\rangle = Pa^*bP$. This is naturally equipped with a representation of $\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$ as adjointable operators, indeed $\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$ is identified with the algebra of compact operators on $\mathcal{S}$. Thus $\mathcal{S}$ gives a W-equivariant graded Morita equivalence from $\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$ to $\mathbb{C}$.

Our inverse to the element $(\mathcal{E}^\vee, \rho^\vee, D^\vee)$ is thus given by the $\mathcal{S}$-spinor reduction of $(\mathcal{E}, \rho, D)$, viz.

$$((\mathcal{E} \mathbin{\widehat{\otimes}} \mathcal{C}\ell(\mathfrak{t})) \mathbin{\widehat{\otimes}}_{\mathcal{C}\ell(\mathfrak{t}^*) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(\mathfrak{t})} \mathcal{S}, \rho \otimes 1 \otimes 1, D \otimes 1 \otimes 1).$$

The module $(\mathcal{E} \mathbin{\widehat{\otimes}} \mathcal{C}\ell(\mathfrak{t})) \mathbin{\widehat{\otimes}}_{\mathcal{C}\ell(\mathfrak{t}^*) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(\mathfrak{t})} \mathcal{S}$ simplifies as $\overline{L^2_c(\mathfrak{t})} \mathbin{\widehat{\otimes}} \mathcal{S}$, with the representation of $C(T)$ acting on $\overline{L^2_c(\mathfrak{t})}$, and with the Dirac operator now acting on spinor fields. By a slight abuse of notation we will denote the representation and operator on $\overline{L^2_c(\mathfrak{t})} \mathbin{\widehat{\otimes}} \mathcal{S}$ by $\rho \otimes 1$ and $D \otimes 1$ respectively.

We will now proceed to compute the Kasparov product

$$(\mathcal{E}^\vee, \rho^\vee, D^\vee) \otimes_{C(T) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(\mathfrak{t})} (\overline{L^2_c(\mathfrak{t})} \mathbin{\widehat{\otimes}} \mathcal{S}, \rho \otimes 1, D \otimes 1)$$

corresponding to the map (9).

The first step is to consider the Hilbert module for this Kasparov product which is the following $C(T^\vee)$-Hilbert module:

$$\mathcal{E}^\vee \mathbin{\widehat{\otimes}}_{C(T) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(\mathfrak{t})} (\overline{L^2_c(\mathfrak{t})} \mathbin{\widehat{\otimes}} \mathcal{S}) \cong \overline{L^2_c(\mathfrak{t}^*)} \mathbin{\widehat{\otimes}}_{C(T)} \overline{L^2_c(\mathfrak{t})} \mathbin{\widehat{\otimes}} \mathcal{S}.$$

We remark that the group W acts diagonally on all three factors of this tensor product, and indeed that the action on $\mathcal{S}$ is diagonal in terms of the $\mathfrak{t}^*$ and $\mathfrak{t}$ parts.

We will define an isomorphism ϕ from $\overline{L^2_c(\mathfrak{t}^*)} \mathbin{\widehat{\otimes}}_{C(T)} \overline{L^2_c(\mathfrak{t})}$ to the Hilbert module $L^2(\mathfrak{t}^*) \mathbin{\widehat{\otimes}} C(T^\vee)$.

Firstly we construct a map μ from $\overline{L^2_c(\mathfrak{t})}$ to the module of continuous functions $w : \mathfrak{t}^* \to L^2(T)$ such that $w(x, \zeta + \chi) = w(x, \zeta) e^{2\pi i \langle \chi, x \rangle}$, where $\zeta \in \mathfrak{t}^*$, $x \in \mathfrak{t}$ and w is Γ-periodic in x. We define

$$\mu(v)(x, \zeta) = \sum_{\gamma \in \Gamma} v(-\gamma + x) e^{2\pi i \langle \zeta, \gamma - x \rangle}, \tag{10}$$

which we note is Γ-periodic in x. By Equation 8, μ is isometric and a standard density argument (considering v supported on a single fundamental domain) shows that it is surjective.

We now take the Fourier coefficients of $\mu(v)$ defined by

$$\widehat{\mu(v)}_\chi(\zeta) = \int_T \mu(v)(x, \zeta) e^{2\pi i \langle \chi, x \rangle} \, dx$$

for $\chi \in \Gamma^\vee$. Where there is no risk of confusion we will abbreviate $\widehat{\mu(v)}_\chi(\zeta)$ as $\widehat{\mu(v)}_\chi$. We now define $\phi : \overline{L^2_c(\mathfrak{t}^*)} \mathbin{\widehat{\otimes}}_{C(T)} \overline{L^2_c(\mathfrak{t})} \to L^2(\mathfrak{t}^*) \mathbin{\widehat{\otimes}} C(T^\vee)$ by

$$\phi(u \otimes v)(\eta, \exp(\zeta)) = \sum_{\chi \in \Gamma^\vee} u(-\chi + \eta + \zeta) \widehat{\mu(v)}_\chi(\zeta).$$

Note that $\mu(v)$ is square-integrable in x, hence the Fourier coefficients are square-summable. If u is supported on a single fundamental domain for the action of $\Gamma^\vee$ then (fixing ζ) its translates $u(-\chi+\eta+\zeta)$ are pairwise orthogonal functions with the same norm, whence the sum $\sum_{\chi\in\Gamma^\vee} u(-\chi+\eta+\zeta)\widehat{\mu(v)}_\chi(\zeta)$ is square-integrable in ζ. Hence splitting the support of a general compactly supported u into finitely many translates of a fundamental domain we see that for each ζ the function $\eta \mapsto \phi(u\otimes v)(\eta, \exp(\zeta))$ is in $L^2(\mathfrak{t}^*)$.

By continuity of the left regular representation $(\chi-\eta)\cdot u$ varies continuously with ζ. Since $\mu(v)$ is continuous in ζ it follows (using square-summability of the Fourier coefficients) that $\phi(u\otimes v)$ is a continuous function from $\mathfrak{t}^*$ to $L^2(\mathfrak{t}^*)$.

Expanding the definition of $\widehat{\mu(v)}$ and noting that $e^{2\pi i\langle\chi,-\gamma\rangle}=1$ we have:

$$\phi(u\otimes v)(\eta,\exp(\zeta)) = \sum_{\chi\in\Gamma^\vee} u(-\chi+\eta+\zeta)\int_T \sum_{\gamma\in\Gamma} v(-\gamma+x)e^{2\pi i\langle\chi-\zeta,-\gamma+x\rangle}\,dx.$$

We note that the summation over χ ensures that this function is $\Gamma^\vee$-periodic in the ζ variable and thus depends only on the point $\exp(\zeta)\in T^\vee$, not the chosen representative ζ in $\mathfrak{t}^*$.

Lemma 5.3. *The map ϕ preserves the inner product and gives an isomorphism of Hilbert modules*

$$\phi : \overline{L^2_c(\mathfrak{t}^*)}\,\widehat{\otimes}_{C(T)}\overline{L^2_c(\mathfrak{t})} \to L^2(\mathfrak{t}^*)\,\widehat{\otimes}\, C(T^\vee).$$

Proof. We need to compute

$$\langle u\otimes v, u'\otimes v'\rangle = \langle v, \langle u, u'\rangle v'\rangle.$$

Let $w=\mu(v), w'=\mu(v')$. The function $\langle u,u'\rangle(x)$ is Γ-periodic, hence using Equation 8 to compute the inner product in $\overline{L^2_c(\mathfrak{t})}$ we have

$$\begin{aligned}\langle v, \langle u,u'\rangle v'\rangle(\exp(\zeta)) &= \int_T \overline{w(x,\zeta)}\,\big(\langle u,u'\rangle(x)\,w'(x,\zeta)\big)\,dx\\ &= \int_T \overline{w(x,\zeta)}\sum_{\chi\in\Gamma^\vee}\int_{\mathfrak{t}^*}\overline{u(\eta)}u'(-\chi+\eta)\,d\eta\, e^{2\pi i\langle\chi,x\rangle}\,w'(x,\zeta)\,dx.\end{aligned}$$

We can rearrange this as

$$\sum_{\chi\in\Gamma^\vee}\int_{\mathfrak{t}^*}\overline{u(\eta)}u'(-\chi+\eta)\,d\eta\int_T\overline{w(x,\zeta)}\,e^{2\pi i\langle\chi,x\rangle}\,w'(x,\zeta)\,dx$$

since the sum over χ is finite and the inner integral is independent of x.

The Fourier coefficients of $e^{2\pi i\langle\chi,x\rangle}\,w'(x,\zeta)$ are the Fourier coefficient of w' shifted by χ hence by Parseval's identity the integral over T is $\sum_\psi \overline{\widehat{w}_\psi}\widehat{w}'_{\chi+\psi}$.

Thus

$$\begin{aligned}\langle v, \langle u, u'\rangle v'\rangle &= \sum_{\chi} \int_{\mathfrak{t}^*} \overline{u(\eta)} u'(-\chi+\eta) d\eta \sum_{\psi} \overline{\widehat{w}_\psi} \widehat{w}'_{\chi+\psi} \\ &= \sum_{\chi} \sum_{\psi} \int_{\mathfrak{t}^*} \overline{u(\eta)} u'(-\chi+\eta) d\eta\, \overline{\widehat{w}_\psi} \widehat{w}'_{\chi+\psi}\end{aligned}$$

By invariance of the measure we can change variables, replacing η by $\eta' = \eta + \psi$ to obtain

$$\begin{aligned}&\sum_{\chi} \sum_{\psi} \int_{\mathfrak{t}^*} \overline{u(-\psi+\eta')} u'(-\chi-\psi+\eta') d\eta'\, \overline{\widehat{w}_\psi} \widehat{w}'_{\chi+\psi} \\ &= \sum_{\chi} \int_{\mathfrak{t}^*} \sum_{\psi} \overline{u(-\psi+\eta')} u'(-\chi-\psi+\eta')\, \overline{\widehat{w}_\psi} \widehat{w}'_{\chi+\psi} d\eta' \\ &= \int_{\mathfrak{t}^*} \sum_{\psi} \overline{u(-\psi+\eta')} \sum_{\chi} u'(-\chi-\psi+\eta')\, \overline{\widehat{w}_\psi} \widehat{w}'_{\chi+\psi} d\eta'\end{aligned}$$

noting that the sum over χ is a finite sum. Now substitute $\chi' = \chi + \psi$ to obtain

$$\begin{aligned}&\int_{\mathfrak{t}^*} \sum_{\psi} \overline{u(-\psi+\eta')} \sum_{\chi'} u'(-\chi'+\eta')\, \overline{\widehat{w}_\psi} \widehat{w}'_{\chi'} d\eta' \\ &= \int_{\mathfrak{t}^*} \sum_{\psi} \overline{u(-\psi+\eta') \widehat{w}_\psi} \sum_{\chi'} u'(-\chi'+\eta')\, \widehat{w}'_{\chi'} d\eta'.\end{aligned}$$

Finally we make a further change of variables in the integral to obtain

$$\int_{\mathfrak{t}^*} \sum_{\psi} \overline{u(-\psi+\eta''+\zeta) \widehat{w}_\psi} \sum_{\chi'} u'(-\chi'+\eta''+\zeta)\, \widehat{w}'_{\chi'} d\eta'' = \langle \phi(u \otimes v), \phi(u' \otimes v')\rangle(\exp(\zeta)).$$

To show surjectivity of ϕ, let z be a continuous function from $T^\vee$ to $L^2(\mathfrak{t}^*)$, and without loss of generality assume there exists a compact subset K of $\mathfrak{t}^*$ such that $z(\exp(\zeta))$ is supported in K for all ζ. We will show that z is in the image of ϕ.

Changing variables, let $z'(\zeta, \eta) = z(\exp(\zeta))(\eta - \zeta)$, which we note is $\Gamma^\vee$-periodic for the diagonal action of $\Gamma^\vee$. To establish that z is in the image of ϕ it suffices to show that z' can be approximated, as a continuous function from $\mathfrak{t}^*$ to $L^2(\mathfrak{t}^*)$, by (sums of) elements of the form

$$\sum_{\psi} u(-\psi+\eta) \widehat{w}_\psi(\zeta).$$

The diagonal $\Gamma^\vee$-periodicity allows us to restrict η to a fundamental domain $X^\vee$ for $\Gamma^\vee$ without loss of information. This allows us to think of z' as a compactly supported function from $\mathfrak{t}^*$ to $L^2(X^\vee)$ which can thus be approximated by (sums of) products $u(\eta)f(\zeta)$ with $u \in L^2(X^\vee)$ and $f \in C_c(\mathfrak{t}^*)$.

Now take $w : \mathfrak{t}^* \to L^2(T)$ to be the function with (pointwise) Fourier coefficients $\widehat{w}_\psi(\zeta) = f(\zeta - \psi)$. We note that these coefficients are diagonally periodic, hence the function w satisfies the condition $w(x, \zeta + \chi) = w(x, \zeta)e^{2\pi i\langle\chi, x\rangle}$. Thus there exists $v \in L^2_c(\mathfrak{t})$ such that $w = \mu(v)$.

Extending u to be zero outside $X^\vee$, we note that restricting

$$\sum_\psi u(-\psi + \eta)\widehat{w}_\psi(\zeta)$$

to η in $X^\vee$ we recover $u(\eta)\widehat{w}_0(\zeta) = u(\eta)f(\zeta)$ as required. □

The algebra $C(T^\vee)$ is represented on $\overline{L^2_c(\mathfrak{t}^*)}\widehat{\otimes}_{C(T)}\overline{L^2_c(\mathfrak{t})}$ by pointwise multiplication on the first factor, viewing elements f in $C(T^\vee)$ as periodic functions $\tilde{f}$ on $\mathfrak{t}^*$. Applying ϕ we have

$$\begin{aligned}\phi(\tilde{f}u \otimes v)(\eta, \exp(\zeta)) &= \sum_{\chi \in \Gamma^\vee} \tilde{f}(-\chi + \eta + \zeta)u(-\chi + \eta + \zeta)\widehat{\mu(v)}_\chi(\zeta)\\ &= \tilde{f}(\eta + \zeta)\phi(u \otimes v)(\eta, \exp(\zeta)).\end{aligned}$$

The representation of $C(T^\vee)$ on $\overline{L^2_c(\mathfrak{t}^*)}\widehat{\otimes}_{C(T)}\overline{L^2_c(\mathfrak{t})}$ thus corresponds to the representation σ of $C(T^\vee)$ on $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)$ defined by pointwise multiplication with $\tilde{f}(\eta + \zeta)$. Note that for $\tilde{f}$ of the form $\tilde{f}(\xi) = e^{2\pi i\langle\xi, \gamma\rangle}$, for $\gamma \in \Gamma$ we have $\tilde{f}(\eta + \zeta) = \tilde{f}(\eta)\tilde{f}(\zeta)$ so the representation σ is 'diagonal' on such functions.

We will now define an unbounded operator on the Hilbert module of spinor-valued functions $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\widehat{\otimes}\mathcal{S}$. We define the operator $\widehat{D}$ by the formula

$$\widehat{D}(u \otimes g \otimes s)(\eta, \exp(\zeta)) = -2\pi i\eta_j u(\eta) \otimes g(\exp(\zeta)) \otimes \varepsilon^j s$$

where η_j denotes the jth coordinate of η, and we are using Einstein summation convention. We can alternatively write $\eta_j = \langle\eta, \mathbf{e}_j\rangle$, hence W-invariance of $\widehat{D}$ follows from the W-invariance of $\mathbf{e}_j \otimes \varepsilon^j$ (Proposition 5.2).

Recall that $D \otimes 1_S$ is an unbounded operator on $\overline{L^2_c(\mathfrak{t})}\widehat{\otimes}\mathcal{S}$. Identifying the Hilbert module $\mathcal{E}^\vee\widehat{\otimes}_{C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t})}(\overline{L^2_c(\mathfrak{t})}\widehat{\otimes}\mathcal{S})$ with $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\widehat{\otimes}\mathcal{S}$, the operator $\widehat{D}$ is an unbounded connection for $D \otimes 1_S$ in the following sense.

Lemma 5.4. *The bounded operator $\widehat{D}(1+\widehat{D}^2)^{-1/2}$ is a $D(1+D^2)^{-1/2}\otimes 1_S$-connection.*

Proof. We begin by fixing a fundamental domain $X^\vee$ for the action of $\Gamma^\vee$ on $\mathfrak{t}^*$. For each point $\eta \in \mathfrak{t}^*$ choose an element $\chi(\eta)$ of the lattice $\Gamma^\vee$ such that η lies in the translate $\chi(\eta) + X^\vee$ of the fundamental domain. On the complement of a set of measure zero, $\chi(\eta)$ is uniquely defined.

We define a function $c : \mathfrak{t}^* \to \mathfrak{t}^*$ by

$$c(\eta) = \frac{-2\pi i\eta}{\sqrt{1 + 4\pi^2|\eta|^2}}.$$

We think of $c(\eta)$ as the standard normalisation of the multiplication operator $-2\pi i\eta$ in the Clifford algebra $\mathcal{C}\ell(\mathfrak{t}^*)$.

Consider the multiplication operator M on $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\otimes S$ defined by

$$M(z)(\exp(\zeta))(\eta) = c(\chi(\eta+\zeta)-\zeta)z(\exp(\zeta))(\eta).$$

Note $\chi(\eta+\zeta)-\zeta$ is periodic in the ζ variable as required. We remark that although $\chi(\eta+\zeta)$ is not a continuous function, it is continuous almost everywhere, which suffices for $M(z)$ to be continuous as a function from $T^\vee$ to $L^2(\mathfrak{t}^*)\widehat{\otimes}S$.

Let $\Phi = \phi\otimes 1_S$ denote the isomorphism from $\overline{L^2_c(\mathfrak{t}^*)}\widehat{\otimes}_{C(T)}\overline{L^2_c(\mathfrak{t})}\widehat{\otimes}S$ to $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\widehat{\otimes}S$. We will show that M is a connection for the normalised operator $F = D(1+D^2)^{-1/2}\otimes 1$, identifying the above modules using Φ.

To compute $\Phi(u\otimes F(v\otimes s))$ we begin by considering the Fourier coefficients $(\widehat{\mu}\otimes 1_S)(D(v\otimes s))$. We have

$$(\widehat{\mu}\otimes 1_S)(D(v\otimes s))_\psi(\zeta) = \int_T \sum_{\gamma\in\Gamma} D(v\otimes s)(-\gamma+x)e^{2\pi i\langle\psi-\zeta,-\gamma+x\rangle}\,dx$$

Let D_T denote the Dirac operator on the torus, given by the same local formula as D. Then the integral over T of

$$\sum_{\gamma\in\Gamma} D((v\otimes s)(-\gamma+x)e^{2\pi i\langle\psi-\zeta,-\gamma+x\rangle}) = D_T\Big(\sum_{\gamma\in\Gamma}(v\otimes s)(-\gamma+x)e^{2\pi i\langle\psi-\zeta,-\gamma+x\rangle}\Big)$$

is zero by Stokes' theorem. Thus

$$\begin{aligned}(\widehat{\mu}\otimes 1_S)(D(v\otimes s))_\psi(\zeta) &= -\int_T \sum_{\gamma\in\Gamma} v(-\gamma+x)D(e^{2\pi i\langle\psi-\zeta,-\gamma+x\rangle}\otimes s)\,dx\\ &= -2\pi i\int_T \sum_{\gamma\in\Gamma} v(-\gamma+x)e^{2\pi i\langle\psi-\zeta,-\gamma+x\rangle}\otimes(\psi-\zeta)s\,dx.\end{aligned}$$

Now normalising the operator we have

$$(\widehat{\mu}\otimes 1_S)(F(v\otimes s))_\psi(\zeta) = c(\psi-\zeta)(\widehat{\mu}\otimes 1_S)(v\otimes s).$$

Take u supported in $X^\vee$. Then the sum defining Φ has only a single non-zero term and we obtain

$$\begin{aligned}\Phi(u\otimes F(v\otimes s))(\exp(\zeta))(\eta) &= \sum_\psi u(-\psi+\eta+\zeta)c(\psi-\zeta)(\widehat{\mu}\otimes 1_S)(v\otimes s)_\psi(\zeta)\\ &= u(-\chi(\eta+\zeta)+\eta+\zeta)c(\chi(\eta+\zeta)-\zeta)(\widehat{\mu}\otimes 1_S)(v\otimes s)_{\chi(\eta+\zeta)}(\zeta)\\ &= c(\chi(\eta+\zeta)-\zeta)\Phi(u\otimes v\otimes s)(\exp(\zeta))(\eta)\end{aligned}$$

hence

$$\Phi(u\otimes F(v\otimes s)) = M(\Phi(u\otimes v\otimes s)).$$

Recall that the $C(T)$ module structure on $\overline{L^2_c(\mathfrak{t}^*)}$ is defined by $(u\cdot e^{-2\pi i\langle\psi,x\rangle})(\eta) = u(-\psi+\eta)$ hence for u' supported on a translate $\psi + X^\vee$ of the fundamental domain we have

$$u' = u \cdot e^{-2\pi i\langle\psi,x\rangle}$$

for some u supported on $X^\vee$. Now

$$\begin{aligned}\Phi(u' \otimes F(v\otimes s)) &- M\Phi(u'\otimes v\otimes s)\\ &= \Phi(u\otimes e^{-2\pi i\langle\psi,x\rangle}F(v\otimes s)) - M\Phi(u\otimes e^{-2\pi i\langle\psi,x\rangle}v\otimes s)\\ &= \Phi(u\otimes[e^{-2\pi i\langle\psi,x\rangle},F](v\otimes s)).\end{aligned}$$

Hence the map $v\mapsto \Phi(u'\otimes F(v\otimes s)) - M\Phi(u'\otimes v\otimes s)$ is a compact operator. Writing an arbitrary u as a sum of functions each supported on a single translate of the fundamental domain, we again obtain a compact operator and hence M is a connection as claimed.

Now we will compare the operator M with the claimed connection $\widehat{D}(1+\widehat{D}^2)^{-1/2}$. Note that the operator $\widehat{D}(1+\widehat{D}^2)^{-1/2}$ on $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\widehat{\otimes}\mathcal{S}$ is multiplication by $c(\eta)$. Since $\eta+\zeta-\chi(\eta+\zeta)$ is in the fundamental domain, the distance from $\chi(\eta+\zeta)$ to $\eta+\zeta$ is bounded. Hence the difference between $\chi(\eta+\zeta)-\zeta$ and η is also bounded. It follows that the function

$$f(\eta,\zeta) = c(\chi(\eta+\zeta)-\zeta) - c(\eta)$$

tends to zero as η goes to infinity. Hence we may approximate f by a function f' which is compactly supported in η and periodic in ζ.

A specimen compact operator from $\overline{L^2_c(\mathfrak{t}^*)}\widehat{\otimes}\mathcal{S}$ to $L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\widehat{\otimes}\mathcal{S}$ is given by

$$v\otimes s\mapsto z\langle\widehat{w},\widehat{\mu(v)}\rangle\otimes as$$

where $z\in L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)$, $\widehat{w}$ is a diagonally $\Gamma^\vee$ invariant element of $C(\mathfrak{t}^*,\ell^2(\Gamma^\vee))$ and a is an element of the Clifford algebra. Introducing coordinates we have

$$(z\langle\widehat{w},\widehat{\mu}(v)\rangle)(\exp(\zeta))(\eta) = \sum_{\psi\in\Gamma^\vee} z(\exp(\zeta))(\eta)\widehat{w}_\psi(\zeta)\widehat{\mu(v)}_\psi(\zeta).$$

On the other hand

$$\begin{aligned}f'(\eta,\zeta)\Phi(u\otimes v\otimes s) &= \sum_{\psi\in\Gamma^\vee} f'(\eta,\zeta)(u(-\psi+\eta+\zeta)\widehat{\mu(v)}_\psi(\zeta)\otimes s)\\ &= \sum_{\psi\in\Gamma^\vee} z'_j(\zeta,\eta,\psi)\widehat{\mu(v)}_\psi(\zeta)\otimes\varepsilon^j s\end{aligned}$$

where $z'_j(\zeta,\eta,\psi) = f'(\eta,\zeta)_j u(-\psi+\eta+\zeta)$.

Fixing ζ, the function $z'_j(\zeta,-,-)$ is square summable/integrable in the other two variables, hence z'_j can be thought of as an equivariant function from $\mathfrak{t}^*$ to $L^2(\mathfrak{t}^*)\widehat{\otimes}\ell^2(\Gamma^\vee)$ (where the action on the tensor product is on the second factor). Hence z'_j can be approximated by a sum of elementary tensor valued functions on $\mathfrak{t}^*$, $z_j \otimes \widehat{w}$, with z_j periodic and $\widehat{w}$ equivariant. The operators

$$v \otimes s \mapsto z_j \langle \widehat{w}, \widehat{\mu(v)} \rangle \otimes \varepsilon^j s$$

are compact, from which we deduce that

$$v \otimes s \mapsto f' \Phi(u \otimes v \otimes s)$$

defines a compact operator. The same therefore holds for f, thus $\widehat{D}(1 + \widehat{D}^2)^{-1/2}$ is also a connection for F as required. □

We will show that the Kasparov product is represented by the unbounded triple

$$(L^2(\mathfrak{t}^*) \widehat{\otimes} C(T^\vee) \widehat{\otimes} \mathcal{S}, \sigma \otimes 1, Q)$$

where the operator Q is, using summation convention,

$$Q = D^\vee \otimes 1 + \widehat{D} = \frac{\partial}{\partial \eta_j} \otimes 1 \otimes \mathbf{e}_j - 2\pi i \eta_j \otimes 1 \otimes \varepsilon^j.$$

We remark that, without summing j, k, we have

$$[\frac{\partial}{\partial \eta_j} \otimes 1 \otimes \mathbf{e}_j, -2\pi i \eta_k \otimes 1 \otimes \varepsilon^k] = [\frac{\partial}{\partial \eta_j}, -2\pi i \eta_k] \otimes 1 \otimes \mathbf{e}_j \varepsilon^k$$

which is $-2\pi i \otimes 1 \otimes \mathbf{e}_j \varepsilon^j$ when $j = k$ and vanishes for $j \neq k$. Hence in particular the commutator $[D^\vee \otimes 1, \widehat{D}]$ is a bounded operator. We will now show that the commutator of the bounded forms of these operators is compact.

Lemma 5.5. *Let $F_1 = D^\vee(1 + (D^\vee)^2)^{-1/2}$ and let $\widehat{F}_2 = \widehat{D}(1 + \widehat{D}^2)^{-1/2}$. Viewing $F_1 \otimes 1$ as an operator on $L^2(\mathfrak{t}^*) \widehat{\otimes} C(T^\vee) \widehat{\otimes} \mathcal{S}$ (using the isomorphism ϕ) the commutator $[F_1 \otimes 1, \widehat{F}_2]$ is compact.*

Proof. Using the identity

$$a^{-1/2} = \frac{1}{\pi} \int_0^\infty \frac{1}{t^{1/2}(t + a)} dt$$

we have

$$(1 + \widehat{D}^2)^{-1/2} = \frac{1}{\pi} \int_0^\infty \frac{1}{t^{1/2}(t + 1 + \widehat{D}^2)} dt$$

with the integral norm-convergent in the space of compact operators. The commutator $[F_1 \otimes 1, \widehat{F}_2]$ is therefore given by the integral

$$\frac{1}{\pi} \int_0^\infty \frac{1}{t^{1/2}} \Big[F_1 \otimes 1, \frac{\widehat{D}}{t + 1 + \widehat{D}^2} \Big] dt$$

and we will show that again the integrand is compact and the integral converges in norm.

The operator $\frac{\widehat{D}}{t+1+\widehat{D}^2}$ is the operator of multiplication by the Clifford algebra valued function

$$\frac{-2\pi i \eta_j}{t + 1 + 4\pi^2 |\eta|^2} \varepsilon^j$$

where as usual we observe Einstein's summation convention. In the Clifford algebra $\mathbf{e}_k$ gradedly commutes with ε^j for all j, k and hence $[F_1 \otimes 1, \varepsilon^j] = 0$ for all j. Hence

$$\Big[F_1 \otimes 1, \frac{\widehat{D}}{t+1+\widehat{D}^2}\Big] = \Big[F_1 \otimes 1, \frac{-2\pi i \eta_j}{t+1+4\pi^2|\eta|^2}\Big]\varepsilon^j.$$

As an operator on $L^2(\mathfrak{t}^*) \otimes \mathcal{S}$ the operator $F_1 \otimes 1$ is simply the bounded form of the Dirac operator on $\mathfrak{t}^*$. In particular it is pseudolocal, hence the commutator of $F_1 \otimes 1$ with the function $\frac{-2\pi i \eta_j}{t+1+4\pi^2|\eta|^2}$ is compact. We have thus shown that the integrand is a compact operator for all t.

We now consider the norm of

$$\Big[F_1 \otimes 1, \frac{\widehat{D}}{t+1+\widehat{D}^2}\Big] = [F_1 \otimes 1, \widehat{D}]\frac{1}{t+1+\widehat{D}^2} - \widehat{D}\Big[F_1 \otimes 1, \frac{1}{t+1+\widehat{D}^2}\Big]$$

The operator $[F_1 \otimes 1, \widehat{D}]$ is bounded. This is most easily seen using the inverse Fourier transform isomorphism from $L^2(\mathfrak{t}^*)$ to $L^2(\mathfrak{t})$. Under this identification the operator $\widehat{D}$ becomes the Dirac operator on $\mathfrak{t}$ while $F_1 \otimes 1$ becomes the operator of Clifford multiplication by

$$\frac{-ix^j}{1+|x|^2}\mathbf{e}_j.$$

Since the derivatives of this function are bounded it follows that the commutator $[F_1 \otimes 1, \widehat{D}]$ is a bounded operator. The operator $[F_1 \otimes 1, \widehat{D}]\frac{1}{t+1+\widehat{D}^2}$ is therefore bounded with norm asymptotically t^{-1}. We now consider the second term

$$\begin{aligned}\widehat{D}\Big[F_1 \otimes 1, \frac{1}{t+1+\widehat{D}^2}\Big] &= \frac{\widehat{D}}{t+1+\widehat{D}^2}[t+1+\widehat{D}^2, F_1 \otimes 1]\frac{1}{t+1+\widehat{D}^2}\\ &= \frac{\widehat{D}}{t+1+\widehat{D}^2}[\widehat{D}^2, F_1 \otimes 1]\frac{1}{t+1+\widehat{D}^2}\\ &= \frac{\widehat{D}}{t+1+\widehat{D}^2}(\widehat{D}[\widehat{D}, F_1 \otimes 1] - [\widehat{D}, F_1 \otimes 1]\widehat{D})\frac{1}{t+1+\widehat{D}^2}.\end{aligned}$$

We have already noted that $[\widehat{D}, F_1 \otimes 1]$ is bounded. The first term of this expansion is thus the product of $\frac{\widehat{D}^2}{t+1+\widehat{D}^2}[\widehat{D}, F_1 \otimes 1]$ which has bounded norm with the operator $\frac{1}{t+1+\widehat{D}^2}$ which has norm $\frac{1}{t+1}$.

The second term is

$$\frac{\widehat{D}}{t+1+\widehat{D}^2}[\widehat{D}, F_1 \otimes 1]\frac{\widehat{D}}{t+1+\widehat{D}^2}.$$

Since the function $\frac{x}{t+1+x^2}$ has maximum value $\frac{1}{2\sqrt{t+1}}$ the operator $\frac{\widehat{D}}{t+1+\widehat{D}^2}$ has norm at most $\frac{1}{2\sqrt{t+1}}$. The norm of the product is again asymptotically t^{-1} as t goes to infinity.

To conclude, the operator $\left[F_1 \otimes 1, \frac{\widehat{D}}{t+1+\widehat{D}^2}\right]$ has norm asymptotically t^{-1} and hence the integral

$$\frac{1}{\pi}\int_0^\infty \frac{1}{t^{1/2}}\left[F_1 \otimes 1, \frac{\widehat{D}}{t+1+\widehat{D}^2}\right]dt$$

converges in norm. Since the integrand is compact it follows that $[F_1 \otimes 1, \widehat{F}_2]$ is a compact operator. □

Theorem 5.6. *The triple* $(L^2(\mathfrak{t}^*)\widehat{\otimes}C(T^\vee)\widehat{\otimes}\mathcal{S}, \sigma \otimes 1, Q)$ *is an unbounded Kasparov triple representing the identity in* $KK_W(C(T^\vee), C(T^\vee))$.

Proof. Since the operator Q is 1 on the $C(T^\vee)$ factor we can consider it as an operator on the Hilbert space $L^2(\mathfrak{t}^*)\widehat{\otimes}\mathcal{S}$. In the following argument we will *not* use summation convention. We consider the following operators on $L^2(\mathfrak{t}^*)\widehat{\otimes}\mathcal{S}$:

$$\begin{aligned} p_j &= \frac{\partial}{\partial \eta_j} \otimes \mathbf{e}_j \\ x_j &= -2\pi i \eta_j \otimes \varepsilon^j \\ q_j &= \frac{1}{2}(1 - 1 \otimes i\mathbf{e}_j\varepsilon^j) \\ A_j &= \frac{1}{2\sqrt{\pi}}(x_j + p_j) \end{aligned}$$

Since A_j anti-commutes with $1 \otimes i\mathbf{e}_j\varepsilon^j$ we have $q_jA_j = A_j(1-q_j)$, hence we can think of A_j as an off-diagonal matrix with respect to q_j. We write A_j as $a_j + a_j^*$ where $a_j = q_jA_j = A_j(1-q_j)$ and hence $a_j^* = A_jq_j = (1-q_j)A_j$. We think of a_j^* and a_j as creation and annihilation operators respectively and we define a number operator $N_j = a_j^*a_j$. The involution $i\varepsilon^j$ intertwines q_j with $1-q_j$. We define A_j', N_j' to be the conjugates of A_j, N_j respectively by $i\varepsilon_j$. Note that

$$A_j' = \frac{1}{2\sqrt{\pi}}(x_j - p_j)$$

and hence

$$A_j^2 = (A_j')^2 + 2\frac{1}{4\pi}[x_j, p_j] = (A_j')^2 - 1 \otimes i\mathbf{e}_j\varepsilon^j.$$

We have $N_j' = A_j'(1-q_j)A_j' = q_j(A_j')^2$. Thus

$$a_ja_j^* = q_jA_j^2q_j = q_jA_j^2 = q_j(A_j')^2 - q_j(1 \otimes i\mathbf{e}_j\varepsilon^j) = N_j' + q_j.$$

Hence the spectrum of $a_ja_j^*$ (viewed as an operator on the range of q_j) is the spectrum of N_j' shifted by 1. However N_j' is conjugate to $N_j = a_j^*a_j$ so

we conclude that
$$\mathrm{Sp}(a_j a_j^*) = \mathrm{Sp}(a_j^* a_j) + 1.$$
But $\mathrm{Sp}(a_j a_j^*)\backslash\{0\} = \mathrm{Sp}(a_j^* a_j)\backslash\{0\}$ so we conclude that $\mathrm{Sp}(a_j^* a_j) = \{0, 1, 2, \dots\}$ while $\mathrm{Sp}(a_j a_j^*) = \{1, 2, \dots\}$.

Now since the operators A_j pairwise gradedly commute we have
$$Q^2 = 4\pi \sum_j A_j^2 = 4\pi \sum_j a_j^* a_j + a_j a_j^*$$
and noting that the summands commute we see that Q^2 has discrete spectrum. To show that $(1+Q^2)^{-1}$ is compact, and hence $(L^2(\mathfrak{t}^*) \widehat{\otimes} C(T^\vee) \widehat{\otimes} \mathcal{S}, \sigma \otimes 1, Q)$ is an unbounded Kasparov triple, it remains to verify that $\ker Q$ is finite dimensional (and hence that all eigenspaces are finite dimensional). We have
$$\ker Q = \ker Q^2 = \bigcap_j \ker A_j^2 = \bigcap_j \ker A_j.$$

Multiplying the differential equation $(x_j + p_j)f = 0$ by $-\mathbf{e}_j \exp(\pi\eta_j^2 \otimes i\mathbf{e}_j\varepsilon^j)$ we see that the kernel of A_j is the space of solutions of the differential equation
$$\frac{\partial}{\partial \eta_j}(\exp(\pi\eta_j^2 \otimes i\mathbf{e}_j\varepsilon^j) f) = 0$$
whence for f in the kernel we have
$$f(\eta_1, \dots, \eta_n) = \exp(-\pi\eta_j^2 \otimes i\mathbf{e}_j\varepsilon^j) f(\eta_1, \dots, \eta_{j-1}, 0, \eta_{j+1}, \dots, \eta_n).$$
Since the solutions must be square integrable the values of f must lie in the $+1$ eigenspace of the involution $i\mathbf{e}_j\varepsilon^j$, that is, the range of the projection $1-q_j$. On this subspace the operator $\exp(-\pi\eta_j^2 \otimes i\mathbf{e}_j\varepsilon^j)$ reduces to $e^{-\pi\eta_j^2}(1-q_j)$. Since the kernel of Q is the intersection of the kernels of the operators A_j an element of the kernel must have the form
$$f(\eta) = e^{-\pi|\eta|^2} \prod_j (1 - q_j) f(0)$$
so the kernel is 1-dimensional. Indeed the product $\prod_j (1 - q_j)$ is the projection P used to define the space of spinors $\mathcal{S} = \mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})P$, and hence $\prod_j (1 - q_j) f(0)$ lies in the 1-dimensional space $P\mathcal{S} = P\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})P$.

We now define a family of representations σ_t of $C(T^\vee)$ on $L^2(\mathfrak{t}^*) \widehat{\otimes} C(T^\vee)$ by
$$\sigma_t(f)(u \otimes g) = \tilde{f}(t\eta + \zeta)(u \otimes g)$$
where as usual $\tilde{f}$ is the pull-back of f to $\mathfrak{t}^*$. We thus obtain a homotopy from the above triple (where $\sigma = \sigma_1$) to
$$(L^2(\mathfrak{t}^*) \widehat{\otimes} C(T^\vee) \widehat{\otimes} \mathcal{S}, \sigma_0 \otimes 1, Q).$$

This decouples the operator, which is constant in the $T^\vee$-variable, from the representation which is now the identity representation of $C(T^\vee)$ on itself.

The space decomposes as $\ker Q \oplus \ker Q^\perp$ and as Q commutes with the representation, this also respects the decomposition, hence the triple decomposes as a direct sum. The $\ker Q^\perp$ summand is 'degenerate' in the sense that the operator is invertible, since Q has discrete spectrum, and commutes with the representation. Hence the latter summand is the zero element of $KK_W(C(T^\vee), C(T^\vee))$.

We have already observed that $\ker Q$ is a 1-dimensional subspace. This is evenly graded since $P\mathcal{S}$ lies in the even part of the Clifford algebra. Hence as the representation $\sigma_0 \otimes 1$ is the identity representation of $C(T^\vee)$ on itself we conclude that $(\ker Q, (\sigma_0 \otimes 1)|_{\ker Q}, 0)$ is the identity element in KK-theory. □

Theorem 5.7. *The Kasparov product*

$$(\mathcal{E}^\vee, \rho^\vee, D^\vee) \otimes_{C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t})} (\overline{L^2_c(\mathfrak{t})}\widehat{\otimes}\mathcal{S}, \rho\otimes 1, D\otimes 1)$$

is the identity in $KK_W(C(T^\vee), C(T^\vee))$.

Proof. By Theorem 5.6 it suffices to show that the Kasparov product is represented by the Kasparov triple

$$(L^2(\mathfrak{t}^*)\widehat{\otimes} C(T^\vee)\widehat{\otimes}\mathcal{S}, \sigma\otimes 1, Q).$$

Let $F, F_1, F_2, \widehat{F}_2$ denote the bounded forms of the operators $Q, D^\vee, D, \widehat{D}$ respectively, obtained by applying the function $x(1+x^2)^{-1/2}$. Note that $Q, \widehat{D}, F, \widehat{F}_2$ are operators on $L^2(\mathfrak{t}^*)\widehat{\otimes} C(T^\vee)\widehat{\otimes}\mathcal{S}$, and using the isomorphism ϕ we will also view $D^\vee\otimes 1$ and $F_1\otimes 1$ as operators on this Hilbert module.

We will prove that the bounded operator F is a Kasparov product for the bounded operators F_1, F_2. Certainly the triple $(L^2(\mathfrak{t}^*)\widehat{\otimes} C(T^\vee)\widehat{\otimes}\mathcal{S}, \sigma\otimes 1, F)$ is a bounded Kasparov triple, since the operator Q defines an unbounded triple. We must thus show that F is an F_2-connection, and that $[F_1\otimes 1, F]$ is positive modulo compact operators.

We define two operators M, N on $L^2(\mathfrak{t}^*)\widehat{\otimes} C(T^\vee)\widehat{\otimes}\mathcal{S}$ by

$$M = \frac{1}{\sqrt{1+Q^2}}\sqrt{1+(D^\vee)^2\otimes 1}$$
$$N = \frac{1}{\sqrt{1+Q^2}}\sqrt{1+\widehat{D}^2}$$

and note that

$$F = M(F_1\otimes 1) + N\widehat{F}_2.$$

We remark that

$$\begin{aligned} MM^* + NN^* &= \frac{1}{\sqrt{1+Q^2}}(2+(D^\vee)^2\otimes 1 + \widehat{D}^2)\frac{1}{\sqrt{1+Q^2}} \\ &= 1 + \frac{1}{\sqrt{1+Q^2}}(1-[D^\vee\otimes 1, \widehat{D}])\frac{1}{\sqrt{1+Q^2}} \end{aligned}$$

which is a compact perturbation of 1, since the commutator $[D^\vee \otimes 1, \widehat{D}]$ is bounded. In particular, M, N are bounded operators.

We will show that M is positive modulo compact operators for which is suffices to show that the commutator $[(1+Q^2)^{-1/4}, \sqrt{1+(D^\vee)^2 \otimes 1}]$ is compact.

For brevity letting $A = (1+Q^2)^{-1/4}, B = (1+(D^\vee)^2 \otimes 1)^{-1/2}$, we examine the commutator

$$[A, B(1+(D^\vee)^2 \otimes 1)] = [A,B](1+(D^\vee)^2 \otimes 1) + B[A, 1+(D^\vee)^2 \otimes 1].$$

We begin by examining the second commutator which we can expand as

$$[A, (D^\vee)^2 \otimes 1] = [A, D^\vee \otimes 1](D^\vee \otimes 1) + (D^\vee \otimes 1)[A, D^\vee \otimes 1].$$

Using the identity

$$a^{-1/4} = \frac{\sin(\pi/4)}{\pi} \int_0^\infty \frac{1}{t^{1/4}(t+a)} dt$$

we have

$$A = \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{1}{t^{1/4}(t+1+Q^2)} dt$$

with the integral norm-convergent in the space of compact operators.

Thus

$$\begin{aligned}[A, D^\vee \otimes 1](D^\vee \otimes 1) &= \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{1}{t^{1/4}} \Big[\frac{1}{t+1+Q^2}, D^\vee \otimes 1\Big](D^\vee \otimes 1) dt \\ &= \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{1}{t^{1/4}} \frac{1}{(t+1+Q^2)} \Big[D^\vee \otimes 1, t+1+Q^2\Big] \frac{1}{t+1+Q^2} (D^\vee \otimes 1) dt\end{aligned}$$

The commutator in the integrand is

$$\Big[D^\vee \otimes 1, Q^2\Big] = (D^\vee \otimes 1 - \widehat{D})[D^\vee \otimes 1, \widehat{D}] - [D^\vee \otimes 1, \widehat{D}](D^\vee \otimes 1 - \widehat{D})$$

The integrand thus has two terms. We consider the first term which is

$$\frac{1}{t^{1/4}} \frac{1}{(t+1+Q^2)} (D^\vee \otimes 1 - \widehat{D})[D^\vee \otimes 1, \widehat{D}] \frac{1}{t+1+Q^2} (D^\vee \otimes 1)$$

Note that $Q-i$ is invertible (as Q is self-adjoint) and $(Q-i)^{-1}(D^\vee \otimes 1)$ and $(Q-i)^{-1}\widehat{D}$ are (or extend to) bounded operators: to see this we note that

$$(Q-i)^{-1}Q^2(Q+i)^{-1} = (Q-i)^{-1}((D^\vee \otimes 1)^2 + \widehat{D}^2 + [D^\vee \otimes 1, \widehat{D}])(Q+i)^{-1}$$

is bounded with norm 1, and again use boundedness of the commutator $[D^\vee \otimes 1, \widehat{D}]$.

It follows that the norms of $\frac{1}{t+1+Q^2}(D^\vee \otimes 1)$ and $\frac{1}{t+1+Q^2}\widehat{D}$ are each bounded above by the norm of $\frac{Q-i}{t+1+Q^2}$. Since the function $\frac{x}{t+1+x^2}$ has maximum value $\frac{1}{2\sqrt{t+1}}$ the operator $\frac{Q-i}{(t+1+Q^2)}$ has norm at most $\frac{1}{2\sqrt{t+1}} + \frac{1}{t+1}$.

The norm of the first term of the integrand thus is asymptotically $t^{-5/4}$ as $t \to \infty$ and hence we see that this part of the integral converges in norm. The other term has a similar norm estimate and thus the integral converges, proving that the commutator $[A, D^\vee \otimes 1](D^\vee \otimes 1)$ is compact (as the integrands are compact operators). We remark that removing the factor of $D^\vee \otimes 1$, the operator $[A, D^\vee \otimes 1]$ is also compact, indeed the norm of the integrand is then asymptotically $t^{-7/4}$.

To see that $[A, (D^\vee)^2 \otimes 1]$ is compact we now simply note that $(D^\vee \otimes 1)[A, D^\vee \otimes 1]$ is the adjoint of $-[A, D^\vee \otimes 1](D^\vee \otimes 1)$.

We remark that since $Q^2 = (D^\vee)^2 \otimes 1 + \widehat{D}^2 + [D^\vee \otimes 1, \widehat{D}]$ commutes exactly with A, and $[D^\vee \otimes 1, \widehat{D}]$ is bounded, it follows that the commutator $[A, \widehat{D}^2]$ is also compact.

We will now prove compactness of $[A, B](1 + (D^\vee)^2 \otimes 1)$. We express B as an integral

$$B = \frac{1}{\pi} \int_0^\infty \frac{1}{t^{1/2}(t + 1 + (D^\vee)^2 \otimes 1)} dt$$

giving us

$$[A, B](1 + (D^\vee)^2 \otimes 1) = \frac{1}{\pi} \int_0^\infty \frac{1}{t^{1/2}} \Big[A, \frac{1}{t + 1 + (D^\vee)^2 \otimes 1}\Big](1 + (D^\vee)^2 \otimes 1) dt$$

$$= \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{1}{t^{1/2}} \frac{1}{(t + 1 + (D^\vee)^2 \otimes 1)} \Big[(D^\vee)^2 \otimes 1, A\Big] \frac{1 + (D^\vee)^2 \otimes 1}{t + 1 + (D^\vee)^2 \otimes 1} dt$$

The three factors are respectively a bounded operator of norm $1/(t+1)$, a compact operator (from the previous calculation) and a bounded operator of norm 1. Hence the integral again converges and we conclude that this term is also compact as required.

To summarise we have shown that the commutator $[A, B(1+(D^\vee)^2 \otimes 1)] = [(1 + Q^2)^{-1/4}, \sqrt{1 + (D^\vee)^2 \otimes 1}]$ is compact and hence M is equal to the positive operator

$$M_+ := (1 + Q^2)^{-1/4} \sqrt{1 + (D^\vee)^2 \otimes 1}(1 + Q^2)^{-1/4}$$

modulo compact operators. In particular the difference between M and M^* is also compact.

By writing the commutator $[A, \sqrt{1 + \widehat{D}^2}]$ as

$$[A, C(1 + \widehat{D}^2)] = [A, C](1 + \widehat{D}^2) + C[A, 1 + \widehat{D}^2]$$

where $C = (1 + \widehat{D}^2)^{-1/2}$, we can show that $[A, \sqrt{1 + \widehat{D}^2}]$ is compact so N is also positive modulo compact operators: we have already remarked that $[A, \widehat{D}^2]$ is compact, and expressing C as an integral (as we did for B) one can show that $[A, C](1 + \widehat{D}^2)$ is also compact.

We will now show that M is a connection for the zero operator, i.e. $M(k \otimes 1)$ and $(k \otimes 1)M$ are compact operators for all $k \in \mathfrak{K}(\mathcal{E}^\vee)$. We note that for

$k_0 = (1+(D^\vee)^2)^{-1/2}$ we have

$$M(k_0 \otimes 1) = (1+Q^2)^{-1/2}(1+(D^\vee)^2 \otimes 1)^{1/2}(1+(D^\vee)^2 \otimes 1)^{-1/2} = (1+Q^2)^{-1/2}$$

which is compact. The operator $(k_0 \otimes 1)M$ is $(M(k_0 \otimes 1))^*$ modulo compact operators (since $M = M^*$ mod compacts), hence $(k_0 \otimes 1)M$ is also a compact operator. Since $k_0 = (1+(D^\vee)^2)^{-1/2}$ is a strictly positive element of $\mathfrak{K}(\mathcal{E}^\vee)$ it follows that $M(k \otimes 1)$ and $(k \otimes 1)M$ are compact for all $k \in \mathfrak{K}(\mathcal{E}^\vee)$ as required. Hence M is a 0-connection.

As M, N are positive modulo compacts, we have

$$M^2 + N^2 = MM^* + NN^* = 1$$

modulo compacts. Since M is a 0-connection it follows that N is a 1-connection. Thus as $\widehat{F}_2$ is an F_2-connection it follows that $F = M(F_1 \otimes 1) + N\widehat{F}_2$ is also an F_2-connection.

It remains to show that

$$[F_1 \otimes 1, F] = [F_1 \otimes 1, M(F_1 \otimes 1) + N\widehat{F}_2]$$

is positive modulo compact operators. We note first that $[M, F_1 \otimes 1]$ is compact. To see this we write

$$M(F_1 \otimes 1) = A^2(D^\vee \otimes 1) = (D^\vee \otimes 1)A^2 = (F_1 \otimes 1)M^* = (F_1 \otimes 1)M$$

modulo compact operators. Since $M^2 + N^2 = 1$ modulo compacts it follows that N also commutes with $F_1 \otimes 1$ modulo compacts.

We can thus write

$$[F_1 \otimes 1, F] = 2(F_1 \otimes 1)M(F_1 \otimes 1) + N[F_1 \otimes 1, \widehat{F}_2]$$

modulo compacts. The first term is positive modulo compacts while $[F_1 \otimes 1, \widehat{F}_2]$ is a compact operator by Lemma 5.5. This completes the proof. ☐

5.3. **The KK-equivalence.** To complete our proof of the isomorphism

$$KK_W(\mathbb{C}, C(T^\vee)) \cong KK_W(\mathbb{C}, C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t}))$$

we prove the following.

Theorem 5.8. *The class $(\mathcal{E}^\vee, \rho^\vee, D^\vee)$ gives a KK-equivalence from $C(T^\vee)$ to $C(T)\widehat{\otimes}\mathcal{C}\ell(\mathfrak{t})$.*

Proof. For brevity of notation we will write $[D]$ for the class of $(\mathcal{E}, \rho, D)$ and $[D^\vee]$ for the class of $(\mathcal{E}^\vee, \rho^\vee, D^\vee)$. We will write $[\mathcal{S}]$ for the KK-class corresponding to the Morita equivalence $\mathcal{S} = \mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})P$ from $\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$ to $\mathbb{C}$, and $[\mathcal{S}^*]$ for its inverse, which is given by the module $\mathcal{S}^* = P\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$.

We showed in the previous section (Theorem 5.7) that $[D^\vee] \otimes [D] \otimes [\mathcal{S}] = 1_{C(T^\vee)}$.

On the other hand, replacing the group G by its Langlands dual $G^\vee$, Theorem 5.7 shows that $[D] \otimes [D^\vee] \otimes [\mathcal{S}] = 1_{C(T)}$, hence $[D] \otimes [D^\vee] = 1_{C(T)} \otimes [\mathcal{S}^*]$. We note that $\mathcal{S}$ and $\mathcal{S}^*$ remain unchanged (up to isomorphism) by the Langlands duality.

We conclude that the product of $1_{C(T)} \otimes [\mathcal{S}]$ with $[D] \otimes [D^\vee]$ gives $1_{C(T)}$ hence $[D^\vee]$ has a left inverse $(1_{C(T)} \otimes [\mathcal{S}]) \otimes [D]$ as well as right inverse $[D] \otimes [\mathcal{S}]$. Thus in fact these two elements of $KK_W(C(T) \widehat{\otimes} \mathcal{C}\ell(\mathfrak{t}), C(T^\vee))$ are equal and $[D^\vee]$ is invertible as required. □

Corollary 5.9. *The composition of Poincaré duality with the Langlands KK-equivalence $(\mathcal{E}^\vee, \rho^\vee, D^\vee)$ gives an isomorphism from $KK_W(C(T), \mathbb{C})$ to $KK_W(\mathbb{C}, C(T^\vee))$.*

6. Notes and computations

Theorem 6.1. *As well as the duality between K-theory and K-homology, there is also a Langlands duality in the Baum-Connes correspondence for the extended affine Weyl group W_a', namely*

$$K_*^W(T) \simeq K_W^*(T^\vee) \tag{11}$$

where W is the finite Weyl group.

Proof. The real vector space $\mathfrak{t}$ is a contractible space on which W_a and W_a' act properly, and serves as universal example for the (extended) affine Weyl group:

$$\mathfrak{t} = \underline{E}W_a = \underline{E}W_a'$$

The LHS of (11) relies on the fact that the lattice $\Gamma(T)$ acts freely on the Lie algebra $\mathfrak{t}$. Then we have — needs more work at this point —

$$\begin{aligned} K_*^{W_a'}(\mathfrak{t}) &\simeq K_*^W(\mathfrak{t}/\Gamma(T)) \\ &\simeq K_*^W(T) \end{aligned}$$

For the RHS of (11) we have

$$\begin{aligned} K_*(C^*(W_a')) &\simeq C(T^\vee) \rtimes W \\ &\simeq K_W^*(T^\vee) \end{aligned}$$

by Theorem (4.1) and Eqn.(12). □

6.1. **The extended quotient.** Let $K_W^j(T)$ denote the classical topological equivariant K-theory [A, 2.3] for the Weyl group W acting on the compact torus T.

By the Green-Julg theorem [Black, Theorem 11.7.1], we have

$$K_j(C(T^\vee) \rtimes W) \simeq K_W^j(T^\vee) \tag{12}$$

Applying the equivariant Chern character for discrete groups [BC] gives a map

$$\mathrm{ch}_W : K_W^j(T^\vee) \to \bigoplus_l H^{j+2l}(T^\vee//W; \mathbb{C}) \tag{13}$$

which becomes an isomorphism when $K^j_W(T^\vee)$ is tensored with $\mathbb{C}$. In fact, it is enough to tensor by $\mathbb{Q}(\zeta)$ where ζ is a primitive dth root of unity, where d is the order of W.

In the formula (13) for the equivariant Chern character, $T^\vee//W$ denotes the extended quotient of $T^\vee$ by W. This, for any W-space X, is defined as follows. Set

$$\widetilde{X} := \{(w,t) \in W \times X : wt = t\}.$$

Then $\widetilde{X} \subset W \times X$. The group W acts on $\widetilde{X}$:

$$W \times \widetilde{X} \to \widetilde{X}, \qquad \alpha(w,t) = (\alpha w \alpha^{-1}, \alpha t)$$

with $(w,t) \in \widetilde{X}, \alpha \in W$. The extended quotient is defined by

$$X//W := \widetilde{X}/W.$$

The extended quotient $X//W$ is the ordinary quotient for the action of W on $\widetilde{X}$.

6.2. **The extended affine Weyl group attached to** PSU_3**.** Let $G = \mathrm{PSU}_3$, then $G^\vee = \mathrm{SU}_3$. Then $\pi_1(G) = \mathbb{Z}/3\mathbb{Z}$ and so W_a has index 3 in W'_a. The dual torus $T^\vee$ is the standard maximal torus in SU_3. The finite Weyl group is the symmetric group $\mathfrak{S}_3$. Computing the extended quotient $T//W$ we find

$$T^\vee//W = T^\vee/W \sqcup (T^\vee)^{s_1}/Z(s_1) \sqcup (T^\vee)^{s_1 s_2}/Z(s_1 s_2)$$

Now

– $T^\vee/W$ is contractible as in §2

– the second term is a copy of $\mathbf{U}$

– the third term is the set $\{I, \omega I, \omega^2 I\}$ with ω a cube root of unity

By Theorem (4.1), we have (modulo torsion)

$$K_0\, C^*(W'_a) = \mathbb{Z}^5, \qquad K_1\, C^*(W'_a) = \mathbb{Z}$$

We obtain isomorphisms after tensoring over $\mathbb{Z}$ with $\mathbb{Q}(e^{2\pi i/6})$.

6.3. **The affine Weyl group attached to** PSU_3**.** We stay with $G = \mathrm{PSU}_3$, $G^\vee = \mathrm{SU}_3$. For W_a we need the maximal torus in the adjoint form of SU_3, namely PSU_3. Denote by S the standard maximal torus in PSU_3. In homogeneous coordinates, we have

$$S = \{(z_1 : z_2 : z_3) : z_j \in \mathbf{U}\}$$

We obtain

$$S//W = S/W \sqcup S^{s_1}/Z(s_1) \sqcup S^{s_1 s_2}/Z(s_1 s_2)$$

Now

– S/W is contractible as in §2

– the second term is a copy of $\mathbf{U}$

– the third term is the set $(1:1:1) \sqcup \{1 : \omega : \omega^2\} \sqcup \{1 : \omega^2 : \omega\}$ with ω a cube root of unity

By Theorem (4.3), we have (modulo torsion)

$$K_0\, C^*(W_a) = \mathbb{Z}^5, \qquad K_1\, C^*(W_a) = \mathbb{Z}$$

6.4. **The affine Weyl group attached to** SU_3**.** Let $G = \mathrm{SU}_3$. The group SU_3 is simply connected and $W_a = W'_a$. We have $G^\vee = \mathrm{PSU}_3$ and $T^\vee = S$ in the above notation. Modulo torsion, we have

$$K_0\, C^*(W_a) = K_0\, C^*(W'_a) = \mathbb{Z}^5, \qquad K_1\, C^*(W_a) = K_1\, C^*(W'_a) = \mathbb{Z}$$

6.5. **The affine Weyl group attached to** G_2**.** The exceptional Lie group G_2 has connection index 1, and so its compact real form is unique: it is simply connected and of adjoint type. The maximal torus has dimension 2, and the Weyl group W is the dihedral group of order 12.

The extended quotient $T//W$ may be computed directly. The extended quotient has 8 connected components:

$$T//W = \mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup T/W$$

where $\mathbb{I}$ is the closed unit interval. For the ordinary quotient, we have

$$T/W \simeq \overline{A}$$

which is a contractible space. The space $\mathbb{I}$ is contractible. So we have a homotopy equivalence

$$T//W \sim 8 \text{ isolated points}$$

so we have (modulo torsion)

$$K_0 = \mathbb{Z}^8, \qquad K_1 = 0$$

References

[A] M.F. ATIYAH, *K-Theory* (Benjamin, New York 1967).

[BC] P. BAUM and A. CONNES, 'The Chern character for discrete groups', A Fête of Topology, Academic Press, New York, 1988, 163 – 232.

[Black] B. BLACKADAR, *K-Theory for operator algebras* (Cambridge University Press, 2002.)

[B] N. BOURBAKI, *Lie groups and Lie algebras* (chapters 4-6, Springer 2002; chapters 7-9, Springer 2008).

[Sp] N. BOURBAKI, Théories Spectrales, Springer, 2007.

[D] J. DIEUDONNÉ, Éléments d'analyse, volume 5, Bordas, Paris, 1975.

[K] G. KASPAROV, Equivariant KK-theory and the Novikov conjecture Invent. math. 91,147-201 (1988).