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OVERSHOOTS AND UNDERSHOOTS OF LÉVY PROCESSES

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We obtain a new fluctuation identity for a general Lévy process giving a quintuple law describing the time of first passage, the time of the last maximum before first passage, the overshoot, the undershoot and the undershoot of the last maximum. With the help of this identity, we revisit the results of Klüppelberg et al. (2004) concerning asymptotic overshoot distribution of a particular class of Lévy processes with semi-heavy tails and refine some of their main conclusions. In particular we explain how different types of first passage contribute to the form of the asymptotic overshoot distribution established in the aforementioned paper. Applications in insurance mathematics are noted with emphasis on the case that the underlying Lévy process is spectrally one sided.

1. Lévy processes and ladder processes This paper concerns overshoots and undershoots of Lévy processes at first upwards passage of a constant boundary. We will therefore begin by introducing some necessary but standard notation.

In the sequel X will always denote a Lévy process defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is assumed to satisfy the usual assumptions of right continuity and completion. Its characteristic exponent will be given by $\Psi(\theta) := -\log E(e^{i\theta X_1})$ and its jump measure by Π_X. We will work with the probabilities $\{P_x : x \in \mathbb{R}\}$ such that $P_x(X_0 = x) = 1$ and $P_0 = P$. The probabilities $\{\widehat{P}_x : x \in \mathbb{R}\}$ will be defined in a similar sense for the dual process, $-X$.

Denote by $\{(L_t^{-1}, H_t) : t \geq 0\}$ and $\{(\widehat{L}_t^{-1}, \widehat{H}_t) : t \geq 0\}$ the (possibly killed) bivariate subordinators representing the ascending and descending ladder processes. Denote by $\kappa(\alpha, \beta)$ and $\widehat{\kappa}(\alpha, \beta)$ their joint Laplace exponents for $\alpha, \beta \geq 0$. For convenience we will write

$$\kappa(0, \beta) = q + \xi(\beta) = q + \mathrm{c}\beta + \int_{(0,\infty)} (1 - e^{-\beta x})\Pi_H(\mathrm{d}x),$$

where $q \geq 0$ is the killing rate of H so that $q > 0$ if and only if $\lim_{t\uparrow\infty} X_t = -\infty$, $\mathrm{c} \geq 0$ is the drift of H and Π_H is its jump measure. The quantity ξ is a true subordinator Laplace exponent. Similar notation will also be used for $\widehat{\kappa}(0, \theta)$ by replacing q, ξ, c and Π_H by $\widehat{q}$, $\widehat{\xi}$, $\widehat{\mathrm{c}}$ and $\Pi_{\widehat{H}}$. Note that when $q > 0$ we have $\widehat{q} = 0$.

Associated with the ascending and descending ladder processes are the bivariate renewal functions $\mathcal{U}$ and $\widehat{\mathcal{U}}$. The former is defined by

$$\mathcal{U}(\mathrm{d}x, \mathrm{d}s) = \int_0^\infty \mathrm{d}t \cdot P(H_t \in \mathrm{d}x, L_t^{-1} \in \mathrm{d}s)$$

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and taking double Laplace transforms shows that

$$\int_0^\infty \int_0^\infty e^{-\beta x - \alpha s} \mathcal{U}(\mathrm{d}x, \mathrm{d}s) = \frac{1}{\kappa(\alpha, \beta)} \text{ for } \alpha, \beta \geq 0 \tag{1}$$

with a similar definition and relation holding for $\widehat{\mathcal{U}}$. These bivariate renewal measures are essentially the Green's functions of the ascending and descending ladder processes. By $U(\mathrm{d}x)$ and $\widehat{U}(\mathrm{d}x)$ we will denote the marginal measures $\mathcal{U}(\mathrm{d}x, [0,\infty))$ and $\widehat{\mathcal{U}}(\mathrm{d}x, [0,\infty))$ respectively. Note that local time at the maximum is defined only up to a multiplicative constant. For this reason, the exponent κ can only be defined up to a multiplicative constant and hence the same is true of the measure $\mathcal{U}$ (and then obviously this argument applies to $\widehat{\mathcal{U}}$).

Let

$$\overline{X}_t := \sup_{u \leq t} X_u \text{ and } \underline{X}_t := \inf_{u \leq t} X_u.$$

The symbol $\mathbf{e}_q$ will always denote a random variable which is independent of X and distributed according to an exponential distribution with parameter $q > 0$. In addition, define for each $x \in \mathbb{R}$,

$$\tau_x^+ = \inf\{t > 0 : X_t > x\} \text{ and } \tau_x^- = \inf\{t > 0 : X_t < x\}.$$

2. Asymptotic overshoots Let us now move to the setting of Klüppelberg et al. (2004) and, in part, the motivation for this paper. For this it will be necessary to introduce some more notation.

For each $\alpha \geq 0$, $\mathcal{S}^{(\alpha)}$ will denote the class of non-lattice convolution equivalent distributions. That is to say distributions, F, with a non-lattice support on $[0,\infty)$ such that $\overline{F}(x) := 1 - F(x) > 0$ for all $x > 0$ satisfying

$$\lim_{u \uparrow \infty} \frac{\overline{F}(u - x)}{\overline{F}(u)} = e^{\alpha x} \text{ for each } x \in \mathbb{R} \text{ and} \tag{2}$$

$$\lim_{u \uparrow \infty} \frac{\overline{F^{*2}}(u)}{\overline{F}(u)} = 2M \text{ for some } M > 0.$$

There are several additional facts which follow from this definition. The constant M was identified as equal to $\int_{[0,\infty)} e^{\alpha x} F(\mathrm{d}x)$ (and hence the latter Laplace-Stieltjes transform is necessarily finite); see Chover et al. (1973), Cline (1987), Rogozin (2000) and Shimura and Watanabe (2004). The condition (2) implies that $F(\mathrm{d}x)/\overline{F}(x)$ converges in the weak sense to an exponential distribution with parameter α. It can also be shown that any measure Π which is tail equivalent to a distribution $F \in \mathcal{S}^{(\alpha)}$, that is to say $\overline{\Pi}(u) := \Pi(u,\infty) \sim \overline{F}(u)$ as $u \uparrow \infty$ for $F \in \mathcal{S}^{(\alpha)}$, also belongs to $\mathcal{S}^{(\alpha)}$; see Embrechts and Goldie (1982).

The following assumptions are included in the set-up in Klüppelberg et al (2004).

ASSUMPTION 1. *Fix $\alpha > 0$.*

(i) $X_0 = 0$, $\lim_{t \uparrow \infty} X_t = -\infty$ *almost surely and* $\mathrm{supp}\Pi \cap (0,\infty) \neq \emptyset$,

(ii) $\overline{\Pi}_H \in \mathcal{S}^{(\alpha)}$ *and*

(iii) $q + \xi(-\alpha) > 0$.

One of the main contributions of Klüppelberg et al. (2004) was the following result.

THEOREM 2. *Under Assumptions 1 we have*

$$\lim_{x\uparrow\infty} P(X_{\tau_x^+} - x > u | \tau_x^+ < \infty) = \overline{G}(u)$$

where

$$\overline{G}(u) = \frac{e^{-\alpha u}}{q}\left(q + \xi(-\alpha) + \int_{(u,\infty)} (e^{\alpha y} - e^{\alpha u})\Pi_H(\mathrm{d}y)\right). \tag{3}$$

In this paper we aim to recapture and explain in more detail the above result by proving stronger versions of asymptotic results concerning the overshoot and undershoot of both X and $\overline{X}$. Specifically we will show that the two components

$$\frac{e^{-\alpha u}}{q}(q + \xi(-\alpha)) \text{ and } \frac{e^{-\alpha u}}{q}\left(\int_{(u,\infty)} (e^{\alpha y} - e^{\alpha u})\Pi_H(\mathrm{d}y)\right)$$

in (3) are the consequence of two types of asymptotic overshoot; namely first passage occurring as a result of

- an arbitrarily large jump from a finite position after a finite time, or
- a finite jump from a finite distance relative to the barrier after an arbitrarily large time

respectively.

Our method appeals directly to a new fluctuation identity for a general Lévy process at first passage over a fixed level which specifies the quintuple law of

- the time of first passage relative to the time of the last maximum at first passage,
- the time of the last maximum at first passage,
- the overshoot at first passage,
- the undershoot at first passage and
- the undershoot of the the last maximum at first passage.

This quintuple law can be expressed entirely in terms of the quantities Π_X, $\mathcal{U}$ and $\widehat{\mathcal{U}}$.

Once this identity is established, it becomes a straightforward exercise to deal with the asymptotic behaviour of this quintuple law conditional on first passage occurring under Assumption 1. Indeed what will prevail in our analysis is the use of the facts that under this assumption, $U(\cdot,\infty)$ and $\overline{\Pi}^+_X(\cdot)$ both belong to $\mathcal{S}^{(\alpha)}$. These two facts can be deduced from the combined conclusions of Proposition 2.5, Lemma 3.5, Theorem 4.1 and Proposition 5.3 in Klüppelberg et al. (2004). Specifically it was proved that when Assumption 1 (i) and (iii) hold, then $U(\cdot,\infty)$, $\overline{\Pi}_H(\cdot)$ and $\overline{\Pi}^+_X(\cdot)$ are all in $\mathcal{S}^{(\alpha)}$ simultaneously or not at all. In the case they all belong to $\mathcal{S}^{(\alpha)}$,

$$U(u,\infty) \sim \frac{1}{(q+\xi(-\alpha))^2}\overline{\Pi}_H(u) \sim \frac{1}{(q+\xi(-\alpha))^2\widehat{\xi}(\alpha)}\overline{\Pi}^+_X(u) \qquad (4)$$

as u tends to infinity.

The outline of the remainder of the paper is as follows. In the next section we prove the new fluctuation identity for first passage of a general Lévy process over a fixed level. In Section 4 we consider the asymptotic joint laws of the space-time overshoot of X, the undershoot of X and the space-time undershoot of $\overline{X}$, all under Assumption 1. We conclude with some additional remarks, in particular with regard to applications in insurance mathematics.

3. A quintuple law for overshoots and undershoots The main purpose of this section is to prove the following quintuple law for space-time positions of overshoots and undershoots. We will use the notation

$$\overline{G}_t = \sup\{s \le t : X_s = \overline{X}_s\} \text{ and } \underline{G}_t = \sup\{s \le t : X_s = \underline{X}_s\}.$$

THEOREM 3. *Suppose that X is not a compound Poisson process. Then for a suitable choice of normalising constant of the local time at the maximum, for each $x>0$ we have on $u>0$, $v\ge y$, $y\in[0,x]$, $s,t\ge 0$,*

$$P(\tau^+_x - \overline{G}_{\tau^+_x-} \in \mathrm{d}t, \overline{G}_{\tau^+_x-} \in \mathrm{d}s, X_{\tau^+_x} - x \in \mathrm{d}u, x - X_{\tau^+_x-} \in \mathrm{d}v, x - \overline{X}_{\tau^+_x-} \in \mathrm{d}y)$$
$$= \mathcal{U}(x-\mathrm{d}y,\mathrm{d}s)\widehat{\mathcal{U}}(\mathrm{d}v-y,\mathrm{d}t)\Pi_X(\mathrm{d}u+v),$$

where Π_X is the Lévy measure of X.

Before going to the proof, let us give some intuition behind the statement of this result by discussing its analogue for random walks. The latter turns out to be relatively simple to establish.

Suppose then that $S=\{S_n : n\ge 0\}$ is a random walk on the probability space $(\Omega,\mathcal{F},\mathbb{P})$. That is $S_0=0$ and $S_n=\sum_{i=1}^n \xi_i$ where $\{\xi_i : i\ge 1\}$ are independent and identically distributed with some law F. Define the random variables

$$\overline{S}_n = \max(0,S_1,\ldots,S_n)$$
$$\overline{\theta}^n = \max\{k\le n : S_k = \overline{S}_n\}$$
$$\sigma_x = \min\{n\ge 1 : S_n > x\}.$$

Let $\{(T'_n,H'_n) : n\ge 0\}$ be the *weak* ascending ladder process and $\{(\widehat{T}_n,\widehat{H}_n) : n\ge 0\}$ be the *strict* descending ladder height process of S. Associated with each of these ladder processes are their Greens functions

$$\mathbb{U}'(\mathrm{d}x, i) := \sum_{n\geq 0} \mathbb{P}(H'_n \in \mathrm{d}x, T'_n = i) \text{ and } \widehat{\mathbb{U}}(\mathrm{d}x, j) = \sum_{n\geq 0} \mathbb{P}(\widehat{H}_n \in \mathrm{d}x, \widehat{T}_n = j)$$

for $x \geq 0$ and $i, j \in \mathbb{Z}_{\geq 0}$. The equivalent quintuple law for random walks takes the following form.

THEOREM 4. *For each $x > 0$, we have on $u > 0,\ v \geq y,\ y \in [0, x],\ i, j, \in \mathbb{Z}_{\geq 0}$*

$$\begin{aligned}\mathbb{P}(\sigma_x - 1 - \overline{\theta}^{\sigma_x - 1} = i, \overline{\theta}^{\sigma_x - 1} = j, S_{\sigma_x} - x \in du, x - S_{\sigma_x - 1} \in dv, x - \overline{S}_{\sigma_x - 1} \in dy) \\ = \mathbb{U}'(x - dy, j)\widehat{\mathbb{U}}(dv - y, i)F(du + v). \qquad (5)\end{aligned}$$

Proof. Note first that by duality

$$\widehat{\mathbb{U}}(dv - y, i) = \mathbb{P}(S_m < 0, 1 \leq m < i, S_i \in y - dv),$$

so that

$$\begin{aligned}&\text{RHS of (5)}\\ &= \mathbb{P}(S_n < x - y, 1 \leq n < j, S_j \in x - dy)\\ &\qquad \times \mathbb{P}(S_m < 0, 1 \leq m < i, S_i \in y - dv)\mathbb{P}(S_1 \in y + du)\\ &= \mathbb{P}(S_n < x - y, 1 \leq n < j, S_j \in x - dy,\\ &\qquad S_{j+m} < x - y, 1 \leq m < i, S_{j+i} \in x - dv, S_{j+i+1} \in x + du)\\ &= \text{LHS of (5)}\end{aligned}$$

□

REMARK 5. *From the analysis above, if we let $\overline{\theta}_n = \min\{k : S_k = \overline{S}_n\}$ then one can reason similarly that for each $x > 0$, we have on $u > 0,\ v \geq y,\ y \in [0, x],\ i, j, \in \mathbb{Z}_{\geq 0}$*

$$\begin{aligned}\mathbb{P}(\sigma_x - 1 - \overline{\theta}_{\sigma_x - 1} = i, \overline{\theta}_{\sigma_x - 1} = j, S_{\sigma_x} - x \in du, x - S_{\sigma_x - 1} \in dv, x - \overline{S}_{\sigma_x - 1} \in dy) \\ = \mathbb{U}(x - dy, j)\widehat{\mathbb{U}}'(dv - y, i)F(du + v).\end{aligned}$$

Here we have the subtle difference that $\mathbb{U}$ and $\widehat{\mathbb{U}}'$ are the Greens functions of the strict ascending and weak descending ladder processes.

Note that hints concerning the quintuple law for the random walk case can already be seen in the discussion concerning the Wiener-Hopf factorization in Borovkov (1976).

We now move to the proof of the quintuple law for Lévy processes.

Proof of Theorem 3. We prove the result in three steps.

Step 1. Let us suppose that m, k, f, g and h are all positive, continuous functions with compact support satisfying $f(0) = g(0) = h(0) = 0$. We prove

$$E(m(\tau_x^+ - \overline{G}_{\tau_x^+-})k(\overline{G}_{\tau_x^+-})f(X_{\tau_x^+} - x)g(x - X_{\tau_x^+-})h(x - \overline{X}_{\tau_x^+-}))$$
$$= \widehat{E}_x\left(\int_0^{\tau_0^-} m(t - \underline{G}_t)k(\underline{G}_t)h(\underline{X}_t)w(X_t)\mathrm{d}t\right) \tag{6}$$

where $w(z) = g(z)\int_{(z,\infty)} \Pi_X(\mathrm{d}u)f(u - z)$.

The proof of this result follows by an application of the compensation formula applied to the point process of jumps with intensity measure $\mathrm{d}t \times \Pi(\mathrm{d}x)$. We have

$$E(m(\tau_x^+ - \overline{G}_{\tau_x^+-})k(\overline{G}_{\tau_x^+-})f(X_{\tau_x^+} - x)g(x - X_{\tau_x^+-})h(x - \overline{X}_{\tau_x^+-}))$$
$$= E\Bigg(\sum_{t<\infty} m(t - \overline{G}_{t-})k(\overline{G}_{t-})g(x - X_{t-})h(x - \overline{X}_{t-})$$
$$\times \mathbf{1}_{(x-\overline{X}_{t-}>0)}f(X_{t-} + \Delta X_t - x)\mathbf{1}_{(\Delta X_t > x - X_{t-})}\Bigg)$$
$$= E\Bigg(\int_0^\infty \mathrm{d}t \cdot m(t - \overline{G}_{t-})k(\overline{G}_{t-})g(x - X_{t-})h(x - \overline{X}_{t-})$$
$$\times \mathbf{1}_{(x-\overline{X}_{t-}>0)}\int_{(x-X_{t-},\infty)} \Pi_X(\mathrm{d}\phi)f(X_{t-} + \phi - x)\Bigg)$$
$$= E\left(\int_0^\infty \mathrm{d}t \cdot m(t - \overline{G}_{t-})k(\overline{G}_{t-})h(x - \overline{X}_{t-})\mathbf{1}_{(x-\overline{X}_{t-}>0)}w(x - X_{t-})\right)$$
$$= \widehat{E}_x\left(\int_0^\infty \mathrm{d}t \cdot \mathbf{1}_{(t<\tau_0^-)}m(t - \underline{G}_t)k(\underline{G}_t)h(\underline{X}_t)w(X_t)\right)$$

which is equal to the right hand side of (6). In the last equality we have rewritten the previous equality in terms of the path of $-X$. Note that the condition $f(0) = g(0) = h(0) = 0$ has been used implicitly to exclude from the calculation the case of first passage by creeping.

Step 2. Next we prove that

$$E_x\left(\int_0^{\tau_0^-} m(t - \underline{G}_t)k(\underline{G}_t)h(\underline{X}_t)w(X_t)\mathrm{d}t\right)$$
$$= \int_{[0,\infty)}\int_{[0,\infty)} \mathcal{U}(\mathrm{d}\phi, \mathrm{d}t)$$
$$\cdot \int_{[0,x]}\int_{[0,\infty)} \widehat{\mathcal{U}}(\mathrm{d}\theta, \mathrm{d}s)m(t)k(s)h(x - \theta)w(x + \phi - \theta). \tag{7}$$

(Note however, that this result will be applied in conjunction with the conclusion of step 1 to the process $-X$).

The statement and proof of (7) is a generalization of Theorem VI.20 in Bertoin (1996). For $q > 0$,

$$
\begin{aligned}
&E_x\left(\int_0^{\tau_0^-} \mathrm{d}t \cdot m(t - \underline{G}_t)k(\underline{G}_t)h(\underline{X}_t)w(X_t)e^{-qt}\right)\\
&= q^{-1}E_x\left(m(\mathbf{e}_q - \underline{G}_{\mathbf{e}_q})k(\underline{G}_{\mathbf{e}_q})h(\underline{X}_{\mathbf{e}_q})w(X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q} + \underline{X}_{\mathbf{e}_q}); \mathbf{e}_q < \tau_0^-\right)\\
&= q^{-1}\int_{[0,x]}\int_{[0,\infty)} P(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}\theta, \underline{G}_{\mathbf{e}_q} \in \mathrm{d}s)k(s)\\
&\qquad \cdot \int_{[0,\infty)}\int_{[0,\infty)} P(X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q} \in \mathrm{d}\phi, \mathbf{e}_q - \underline{G}_{\mathbf{e}_q} \in \mathrm{d}t)m(t)h(x-\theta)w(x+\phi-\theta)\\
&= q^{-1}\int_{[0,x]}\int_{[0,\infty)} P(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}\theta, \underline{G}_{\mathbf{e}_q} \in \mathrm{d}s)k(s)\\
&\qquad \cdot \int_{[0,\infty)}\int_{[0,\infty)} P(\overline{X}_{\mathbf{e}_q} \in \mathrm{d}\phi, \overline{G}_{\mathbf{e}_q} \in \mathrm{d}t)m(t)h(x-\theta)w(x+\phi-\theta) \qquad (8)
\end{aligned}
$$

where the Wiener-Hopf factorization and duality have been used in the second and third equalities respectively. Next note that for a suitable normalization of the local time at the maximum we have

$$q = \kappa(q,0)\widehat{\kappa}(q,0)$$

(cf. equation (3) of Chapter VI in Bertoin (1996)). Further it is also known from the Wiener-Hopf factorization that

$$\frac{1}{\kappa(q,0)}E\left(e^{-\alpha\overline{G}_{\mathbf{e}_q} - \beta\overline{X}_{\mathbf{e}_q}}\right) = \frac{1}{\kappa(\alpha+q,\beta)}$$

(cf. equation (1) Chapter VI of Bertoin (1996)) and hence recalling (1) it follows that

$$\lim_{q\downarrow 0}\frac{1}{\kappa(q,0)}P(\overline{X}_{\mathbf{e}_q} \in \mathrm{d}\phi, \overline{G}_{\mathbf{e}_q} \in \mathrm{d}t) = \mathcal{U}(\mathrm{d}\phi, \mathrm{d}t)$$

in the sense of vague convergence. A similar convergence holds for $P(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}\theta, \underline{G}_{\mathbf{e}_q} \in \mathrm{d}s)/\widehat{\kappa}(q,0)$. Equality (7) thus follows by taking limits in (8).

Step 3. We combine the conclusions of steps 1 and 2 to conclude that

$$
\begin{aligned}
&E(m(\tau_x^+ - \overline{G}_{\tau_x^+-})k(\overline{G}_{\tau_x^+-})f(X_{\tau_x^+} - x)g(x - X_{\tau_x^+-})h(x - \overline{X}_{\tau_x^+-}))\\
&= \int_{u>0, y\in[0,x], 0<y\le v, s\ge 0, t\ge 0} m(t)k(s)f(u)g(v)h(y)\\
&P(\tau_x^+ - \overline{G}_{\tau_x^+-} \in \mathrm{d}t, \overline{G}_{\tau_x^+-} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y)\\
&= \int_{[0,\infty)}\int_{[0,\infty)} \widehat{\mathcal{U}}(\mathrm{d}\phi, \mathrm{d}t)\int_{[0,\infty)}\int_{[0,x]} \mathcal{U}(\mathrm{d}\theta, \mathrm{d}s)m(t)k(s)\\
&\qquad \cdot h(x-\theta)g(x+\phi-\theta)\int_{(x+\phi-\theta,\infty)} \Pi_X(\mathrm{d}\eta)f(\eta - (x+\phi-\theta)).
\end{aligned}
$$

Substituting $y = x - \theta$, then $y + \phi = v$ and finally $\eta = v + u$ in the right hand side above yields

$$
\begin{aligned}
&E(m(\tau_x^+ - \overline{G}_{\tau_x^+ -})k(\overline{G}_{\tau_x^+ -})f(X_{\tau_x^+} - x)g(x - X_{\tau_x^+ -})h(x - \overline{X}_{\tau_x^+ -}))\\
&= \int_{[0,\infty)} \int_{[0,x]} \mathcal{U}(x - \mathrm{d}y, \mathrm{d}s) \int_{[0,\infty)} \int_{[y,\infty)} \widehat{\mathcal{U}}(\mathrm{d}v - y, \mathrm{d}t)\\
&\qquad \cdot \int_{(0,\infty)} \Pi_X(\mathrm{d}u + v) m(t)k(s)f(u)g(v)h(y)
\end{aligned}
$$

and the statement of the theorem follows. □

The missing case of a compound Poisson process, excluded from Theorem 3, can be handled similarly to the random walk case.

As a consequence of the above identity, we obtain the following corollary which relates $\mathbf{\Pi}(\mathrm{d}t, dh)$, the Lévy measure of (L^{-1}, H), to Π_X.

Corollary 6. *For all $t, h > 0$ we have*

$$
\mathbf{\Pi}(\mathrm{d}t, \mathrm{d}h) = \int_{[0,\infty)} \widehat{\mathcal{U}}(\mathrm{d}\theta, \mathrm{d}t)\Pi_X(\mathrm{d}h + \theta).
$$

Proof. The result will follow by first proving the auxiliary result for the ascending ladder process at *its* first passage time $T_x := \inf\{t > 0 : H_t > x\}$. Let $\Delta L^{-1}_{T_x} = L^{-1}_{T_x} - L^{-1}_{T_x -}$, then

$$
\begin{aligned}
&P(\Delta L^{-1}_{T_x} \in \mathrm{d}t, L^{-1}_{T_x -} \in \mathrm{d}s, x - H_{T_x -} \in \mathrm{d}y, H_{T_x} - x \in \mathrm{d}u)\\
&\qquad = \mathcal{U}(x - \mathrm{d}y, \mathrm{d}s)\mathbf{\Pi}(\mathrm{d}t, \mathrm{d}u + y) \qquad (9)
\end{aligned}
$$

for $t > 0$, $s > 0$, $y \in [0, x]$, $u > 0$. The proof follows from a straightforward calculations using the compensation formula along the lines of the proof of Proposition III.2 in Bertoin (1996). We omit the technicalities for the sake of brevity.

To finish the proof of the corollary, note that $\Delta L^{-1}_{T_x} = \tau_x^+ - \overline{G}_{\tau_x^+ -}$, $L^{-1}_{T_x -} = \overline{G}_{\tau_x^+ -}$, $x - H_{T_x -} = x - X_{\tau_x^+ -}$ and $H_{T_x} - x = X_{\tau_x^+} - x$. Hence from the quintuple law we also know that

$$
\begin{aligned}
&P(\Delta L^{-1}_{T_x} \in \mathrm{d}t, L^{-1}_{T_x -} \in \mathrm{d}s, x - H_{T_x -} \in \mathrm{d}y, H_{T_x} - x \in \mathrm{d}u)\\
&= \mathcal{U}(x - \mathrm{d}y, \mathrm{d}s) \int_{[y,\infty)} \widehat{\mathcal{U}}(\mathrm{d}v - y, \mathrm{d}t)\Pi_X(\mathrm{d}u + v)
\end{aligned}
$$

and from here, by comparing with (9), the statement of the theorem follows. □

Note that by integrating out $\mathrm{d}t$ in the conclusion of the above corollary, we recover the recent identity of Vigon (2002) for the Lévy measure of the ascending ladder height process.

We conclude this section with examples of Lévy processes for which new, explicit identities can be obtained. Before doing so we make the remark that there are limited examples of Lévy processes for which the exponents κ and $\widehat{\kappa}$ are known explicitly in terms of elementary or special functions. Further, of these known examples there are no known cases for which the inversion in (1) can be performed to give the bivariate measures $\mathcal{U}$ and $\widehat{\mathcal{U}}$ explicitly. Not surprisingly then our examples do not explore the quintuple law to its full generality.

Example 7 (Strictly stable processes). Suppose that X is a strictly stable process with index $\gamma \in (0,2)$. That is to say, a Lévy process satisfying the scaling property $X_t \stackrel{d}{=} t^{1/\gamma} X_1$ for all $t > 0$. The Lévy measure is given (up to a multiplicative constant) by

$$\Pi_X(dx) = \mathbf{1}_{(x>0)} \frac{c^+}{x^{1+\gamma}} dx + \mathbf{1}_{(x<0)} \frac{c^-}{|x|^{1+\gamma}} dx$$

where c^+ and c^- are two non-negative real numbers.

For such processes it is known that the ladder process H is a stable subordinator with index $\gamma\rho$ where $\rho = P(X_1 \geq 0)$ and hence up to a multiplicative constant $\kappa(0,\beta) = \beta^{\gamma\rho}$ for $\beta \geq 0$. Similarly, up to a multiplicative constant $\widehat{\kappa}(0,\beta) = \beta^{\gamma(1-\rho)}$. For these facts, the reader is again referred to Bertoin (1996).

Inverting (1) when $\alpha = 0$ we find that (up to a multiplicative constant)

$$U(dx) = \frac{x^{\gamma\rho-1}}{\Gamma(\gamma\rho)} dx$$

with a similar identity holding for $\widehat{U}(dx)$ except that ρ is replaced by $1-\rho$. Marginalizing the quintuple law to a triple law we now obtain a new identity for stable processes. Namely,

$$\begin{aligned} &P(X_{\tau_x^+} - x \in du, x - X_{\tau_x^+-} \in dv, x - \overline{X}_{\tau_x^+-} \in dy) \\ &= const. \frac{(x-y)^{\gamma\rho-1}(v-y)^{\gamma(1-\rho)-1}}{(v+u)^{1+\gamma}} dydvdu \end{aligned}$$

for $y \in [0,x]$, $v \geq y$ and $u > 0$, where the normalizing constant makes the right hand side a distribution (note that stable processes do not creep and hence there is no atom on the event $\{X_{\tau_x^+} = x\}$ to take care of).

Example 8 (Spectrally positive processes). In this case, the downward ladder height process is a linear drift with gradient 1 killed at rate $\widehat{q} \geq 0$. For this reason it follows that $\widehat{U}(dx) = e^{-\widehat{q}x} dx$. This gives the triple law

$$\begin{aligned} &P(X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y) \\ &= e^{-\widehat{q}(v-y)} U(x-dy) \Pi_X(du+v) dv \end{aligned}$$

for $y \in [0,x]$, $v \geq y$ and $u > 0$.

The Wiener-Hopf factors for spectrally positive Lévy processes are well understood (cf. Chapter VII, Bertoin (1996)). Indeed it is known that $\widehat{\kappa}(\alpha,\beta) = \Phi(\alpha) + \beta$

where Φ is the right inverse of the Laplace exponent $\psi(\beta) = \log E(e^{-\beta X_1})$ for $\beta \geq 0$. The identification of U via its Laplace transform in (1) thus simplifies to

$$\int_{[0,\infty)} e^{-\beta x} U(dx) = \frac{\beta}{\psi(\beta)}. \tag{10}$$

When in addition X has bounded variation and drifts to minus infinity it is possible to give a more explicit identity for the measure U and hence for the above expression. In this case X is the difference of a subordinator and a positive drift of rate c such that $E(X_1) < 0$. It is known then that $\widehat{q} = 0$ and $q = |E(X_1)|$ (see for example Section 6 of Klüppelberg et al. (2004)). By taking Laplace transforms we see from (10) that

$$U(dx) = \frac{1}{c} \sum_{n \geq 0} \nu^{*n}(dx)$$

where we understand $\nu^{*0}(dx) = \delta_0(dx)$

$$\nu(dx) = \frac{1}{c} \Pi_X(x, \infty) dx.$$

(Note that the assumption $E(X_1) < 0$ ensures that $c^{-1} \int_{(0,\infty)} \Pi_X(y, \infty) dy < 1$).

Our triple law now takes the form

$$\begin{aligned} &P(X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+ -} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+ -} \in \mathrm{d}y) \\ &= \frac{1}{c} \sum_{n \geq 0} \nu^{*n}(x - dy) \Pi_X(du + v) dv \end{aligned}$$

for $y \in [0, x]$, $v \geq y$ and $u > 0$.

Remark 9. *The latter example is relevant to insurance mathematics. One may compare against similar results in the papers of Gerber and Shiu (1997), Dickson and Drekic (2004) and Sun and Yang (2004), which concern the classical Cramér-Lundberg process (which in our setting is a spectrally positive compound Poisson process drifting to minus infinity).*

4. The asymptotic role of undershoots in overshoots In the following two theorems, we consider the asymptotic overshoot and undershoot in space and time at first passage of X, conditional on making first passage, as the barrier tends to infinity. The spatial undershoot is measured, in the first case, backwards from the barrier and, in the second case, upwards from the origin.

Theorem 10. *Under Assumption 1,*

(i) for $t \geq 0$, $y \geq 0$, $v \geq y$ *and* $u > 0$,

$$\begin{aligned} &\lim_{x \uparrow \infty} P(\tau_x^+ - \overline{G}_{\tau_x^+ -} \in \mathrm{d}t, X_{\tau_x^+} - x \in \mathrm{d}u, \\ &\qquad\qquad x - X_{\tau_x^+ -} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+ -} \in \mathrm{d}y | \tau_x^+ < \infty) \\ &= \frac{\alpha}{q} e^{\alpha y} \mathrm{d}y \cdot \widehat{\mathcal{U}}(\mathrm{d}v - y, \mathrm{d}t) \Pi_X(\mathrm{d}u + v). \end{aligned}$$

(ii) For $u > 0$ we have

$$\int_{v\in(0,\infty)} \lim_{x\uparrow\infty} P(X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v | \tau_x^+ < \infty)$$
$$= \frac{\alpha}{q}\int_0^\infty e^{\alpha y}\Pi_H(\mathrm{d}u + y)\mathrm{d}y.$$

Proof. (i) Starting with the main identity given in Theorem 3, marginalizing out $\overline{G}_{\tau_x^+-}$ and recalling the Pollaczek-Khintchine identity

$$P(\tau_x^+ < \infty) = qU(x,\infty)$$

(cf. Proposition 2.5 of Klüppelberg et al. (2004)), note that the required asymptotic is equal to

$$\lim_{x\uparrow\infty} \frac{U(x - \mathrm{d}y)}{qU(x,\infty)}\widehat{\mathcal{U}}(\mathrm{d}v - y, \mathrm{d}t)\Pi_X(\mathrm{d}u + v).$$

Note that $U(\cdot,\infty) \in \mathcal{S}^{(\alpha)}$ by Assumption 1 and so by the associated property of weak convergence, the limit follows.

(ii) Marginalizing from part (i) shows that the required asymptotic is equal to

$$\frac{\alpha}{q}\int_0^\infty \mathrm{d}y e^{\alpha y}\int_y^\infty \widehat{U}(\mathrm{d}v - y)\Pi_X(\mathrm{d}u + v).$$

Invoking Vigon's identity (as a special case of Corollary 6) in the form

$$\Pi_H(du + y) = \int_{[y,\infty)} \widehat{U}(\mathrm{d}v - y)\Pi_X(\mathrm{d}u + v)$$

concludes the proof. □

Theorem 11. *Under Assumption 1,*

(i) for $s,t \geq 0$, $u > 0$, $\theta \geq 0$ and $\phi \leq \theta$,

$$\lim_{x\uparrow\infty} P(\tau_x^+ - \overline{G}_{\tau_x^+-} \in \mathrm{d}t, \overline{G}_{\tau_x^+-} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u,$$
$$X_{\tau_x^+-} \in \mathrm{d}\phi, \overline{X}_{\tau_x^+-} \in \mathrm{d}\theta | \tau_x^+ < \infty)$$
$$= \mathcal{U}(\mathrm{d}\theta, \mathrm{d}s)\widehat{\mathcal{U}}(\theta - \mathrm{d}\phi, \mathrm{d}t)\frac{\alpha(q + \xi(-\alpha))^2\widehat{\xi}(\alpha)}{q}e^{-\alpha(u-\phi)}\mathrm{d}u.$$

(ii) For $u > 0$

$$\int_{\phi\in(0,\infty)} \lim_{x\uparrow\infty} P(X_{\tau_x^+} - x \in \mathrm{d}u, X_{\tau_x^+-} \in \mathrm{d}\phi | \tau_x^+ < \infty)$$
$$= \alpha e^{-\alpha u}\frac{(q + \xi(-\alpha))}{q}\mathrm{d}u.$$

Proof. (i) With a change of variable in the main identity of Theorem 3 we have,

$$
\begin{aligned}
&P(\tau_x^+ - \overline{G}_{\tau_x^+ -} \in \mathrm{d}t, \overline{G}_{\tau_x^+ -} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u,\\
&\qquad\qquad X_{\tau_x^+ -} \in \mathrm{d}\phi, \overline{X}_{\tau_x^+ -} \in \mathrm{d}\theta | \tau_x^+ < \infty)\\
&= \mathcal{U}(\mathrm{d}\theta, \mathrm{d}s)\widehat{\mathcal{U}}(\theta - \mathrm{d}\phi, \mathrm{d}t)\frac{\Pi_X(\mathrm{d}u + x - \phi)}{qU(x,\infty)}.
\end{aligned}
$$

From (4) and the associated weak convergence it follows that

$$
\lim_{x\uparrow\infty} \frac{\Pi_X(\mathrm{d}u + x - \phi)}{qU(x,\infty)} = \frac{(q+\xi(-\alpha))^2\widehat{\xi}(\alpha)\alpha}{q} e^{-\alpha(u-\phi)}\mathrm{d}u
$$

and the result follows.

(ii) The second part follows by again marginalizing the limiting identity in part (i) with the help of the well known fact that

$$
\alpha \int_0^\infty e^{-\alpha x}\widehat{U}(x)dx = \frac{1}{\widehat{\xi}(\alpha)}
$$

(cf. Bertoin (1996) p172). □

We conclude with some additional remarks following from the results above.

Asymptotic independence. Note that in the last theorem we see an intuitively obvious independence appearing between the overshoot and the undershoot.

Decomposing the law of the asymptotic overshoot. The conclusions of Theorems 10 and 11 both reprove and provide an interesting explanation for the identity in Theorem 2. A straightforward calculation on the identity in Theorem 10 (ii) shows that

$$
\begin{aligned}
&\int_{v\in(0,\infty)} \lim_{x\uparrow\infty} P(X_{\tau_x^+} - x > u, x - X_{\tau_x^+ -} \in \mathrm{d}v | \tau_x^+ < \infty)\\
&= \frac{e^{-\alpha u}}{q}\left\{\int_u^\infty (e^{\alpha y} - e^{\alpha u})\Pi_H(\mathrm{d}y)\right\}.
\end{aligned}
$$

Similarly from Theorem 11 (ii) we have

$$
\int_{\phi\in(0,\infty)} \lim_{x\uparrow\infty} P(X_{\tau_x^+} - x > u, X_{\tau_x^+ -} \in \mathrm{d}\phi | \tau_x^+ < \infty) = \frac{e^{-\alpha u}}{q}(q + \xi(-\alpha)).
$$

Adding these two identities together recovers the conclusion of Theorem 2. It also shows that the distribution of the conditional asymptotic overshoot has a contribution coming from an arbitrarily large jump at a finite position and after a finite time, or a finite jump from a finite distance relative to the barrier after an arbitrarily large time. Note also from part (i) of the two theorems in this section that when, asymptotically, the undershoot is close to the barrier, the time of occurrence of the last maximum prior to first passage was historically close to the first passage time. Further, when there is asymptotic first passage due to an arbitrarily large jump, this jump happens early on in the path of the Lévy process.

For further results concerning asymptotic overshoots of Lévy processes (spectrally positive compound Poisson processes) with subexponential tails, see Asmussen and Klüppelberg (1996).

Other identities. There are a number of other identities one can extract from Theorems 10 and 11. For example one can obtain an expression for the joint law of the asymptotic overshoot of X and undershoot of H measured from the barrier or measured from zero. In the latter case, integrating out the overshoot one easily recovers the identity given in Theorem 4.2 (iii) of Klüppelberg et al. (2004). This identity says that

$$\lim_{x\uparrow\infty} P(\overline{X}_{\tau_x^+-} \leq z | \tau_x^+ < \infty) = \frac{(q+\xi(-\alpha))^2}{q}\int_{[0,z]} e^{\alpha\theta} U(\mathrm{d}\theta).$$

The proof is straightforward and left as an exercise.

Asymptotic creeping. From the distribution G given in Theorem 2 one sees that there is an atom at zero of mass $\alpha \mathrm{c}/q$. This atom corresponds to the asymptotic conditional probability of creeping over the barrier as it tends to infinity. This can also be derived directly by noting from Kesten (1969) that when the drift c of H is positive, U is absolutely continuous and

$$P(X_{\tau_x^+} = x) = \mathrm{c}u(x)$$

where $u(x) = \mathrm{d}U(x)/\mathrm{d}x$. Weak convergence of $U(\mathrm{d}x)/U(x,\infty)$ under Assumption 1 now ensures that

$$\lim_{x\uparrow\infty} P(X_{\tau_x^+} = x | \tau_x^+ < \infty) = \lim_{x\uparrow\infty} \frac{\mathrm{c}u(x)}{qU(x,\infty)} = \frac{\mathrm{c}\alpha}{q}.$$

Applications to insurance mathematics. The motivation for the work in Klüppelberg et al. (2004) came from insurance mathematics and in particular the classical ruin problem. The refinements of their results given here also offer direct insight into ruinous behaviour.

Within the current context, one may think of $-X$ as the capital of an insurance firm, the so called risk process. In which case the event of ruin with an initial capital of x units corresponds to the process X starting at the origin and making first passage at x. Understanding the conditional asymptotics as x tends to infinity thus gives information about how ruin occurs when the initial revenue of the insurance firm is extremely large.

The classical risk process is the Cramér-Lundberg model which corresponds to X being a spectrally positive compound Poisson process with negative drift. A more suitable generalization however corresponds to the case that X is a spectrally positive Lévy process. In this case, recalling the Lévy-Itô decomposition, one sees a more realistic features as follows. Large jumps (of magnitude greater than one) correspond to large claims offset by premiums collected at a constant rate corresponding to linear drift. Large jumps occur spaced out by independent exponentially distributed periods of time and thus reasonably correspond to disasters. The compensated small jumps which occur with countable but none the less unbounded frequency correspond to minor claims; their compensation can be understood as the aggregate of premiums called in to offset the high intensity of claims.

The case that X is spectrally positive also has the advantage that many of the identities given above simplify further. Write $\psi(\theta) = \log E(e^{-\theta X_1})$ for the Laplace exponent. Since the descending ladder height process is nothing more than linear drift, we also have $\widehat{U}(\mathrm{d}x) = \mathrm{d}x$, $\widehat{\xi}(\alpha) = \alpha$ and $q + \xi(-\alpha) = -\psi(-\alpha)/\alpha$. From the latter, it is also straightforward to deduce that $q = |E(X_1)| < \infty$; see Klüppelberg et al. (2004) for further details. Our earlier results now tell us for example that

$$\lim_{x\uparrow\infty} P(X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y | \tau_x^+ < \infty)$$
$$= \frac{\alpha}{|E(X_1)|} e^{\alpha y} \mathrm{d}y \cdot \mathrm{d}v \cdot \Pi_X(\mathrm{d}u + v)$$

for $y \geq 0$, $v \geq y$ and $u > 0$ and

$$\lim_{x\uparrow\infty} P(X_{\tau_x^+} - x \in \mathrm{d}u, X_{\tau_x^+-} \in \mathrm{d}\phi, \overline{X}_{\tau_x^+-} \in \mathrm{d}\theta | \tau_x^+ < \infty)$$
$$= \frac{\psi(-\alpha)^2}{|E(X_1)|} e^{-\alpha(u-\phi)} U(\mathrm{d}\theta) \cdot \mathrm{d}\phi \cdot \mathrm{d}u$$

for $\theta \geq 0$, $\phi \leq \theta$, $u > 0$. Note also that the renewal measure U can now be identified directly in terms of ψ, namely

$$\int_0^\infty e^{-\beta x} U(dx) = \frac{\beta}{\psi(\beta)}, \quad \text{for } \beta > 0.$$

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