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2005

MIMS EPrint: **2005.11**

ISSN 1749-9097

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Algebra 286 (2005) 492–499

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Vertices for irreducible characters of a class of blocks

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Received 13 April 2004

Communicated by Michel Broué

Abstract

We observe that Navarro's definition of a vertex for an irreducible character of a p-solvable group may be extended to irreducible characters in p-blocks with defect groups contained in a normal p-solvable subgroup N, and show that this definition is independent of the choice of N. We show that the fundamental properties of Navarro's vertices generalize, and as a corollary show that the vertices of the irreducible Brauer characters in blocks of the above form are radical and are intersections of pairs of Sylow p-subgroups.

1. Introduction

In [9] Navarro defined the concept of a vertex for an irreducible character of a p-solvable group. To each irreducible character χ of a finite group G is associated a canonical pair (Q, δ), where Q is a p-subgroup of G and $\delta \in \mathrm{Irr}(Q)$, defined uniquely up to G-congugacy. Call (Q, δ) a vertex of χ, and write $\mathrm{Irr}(G \mid Q, \delta)$ for the set of irreducible characters of G with vertex (Q, δ). One property of vertices is that they respect a strong form of the

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[1] The author is supported by a Royal Society University Research Fellowship, and part of this research was carried out whilst visiting the University of Auckland with the support of the Marsden Fund of New Zealand via grant #9144/3368248.

doi:10.1016/j.jalgebra.2005.01.007

Fong–Swan theorem, in that for each p-subgroup Q of G, restriction to p-regular elements defines a canonical bijection between $\mathrm{Irr}(G \mid Q, 1_Q)$ and $\mathrm{IBr}(G, Q)$, the set of irreducible Brauer characters with vertex Q.

Another property of Navarro's vertices (which we show here), is that if an irreducible character of a p-solvable group lies in a block whose defect groups are contained in a normal subgroup, then it shares a vertex with any irreducible character of the normal subgroup that it covers.

In [6] Gagola extended the Fong–Swan theorem to irreducible characters in blocks whose defect groups lie in a normal p-solvable group. Motivated by this and the previous paragraph, we extend the definition of a vertex to such characters and show that vertices defined in this way have key properties of vertices for p-solvable groups demonstrated in [9].

Let G be a finite group. Let $\mathcal{O}$ be a complete local discrete valuation ring with field of fractions K of characteristic zero and residue field $k = \mathcal{O}/J(\mathcal{O})$ of prime characteristic p. Assume that $\mathcal{O}$ contains a primitive $|G|^3$ root of unity, so that we may use the results of [8]. Except where we state otherwise, blocks are defined with respect to $\mathcal{O}$. Write $\mathrm{Irr}(G, B)$ for the set of irreducible characters of G in B and $\mathrm{Irr}(G, B \mid Q, \delta)$ for the subset consisting of those characters with vertex (Q, δ). We keep this notation for the extended definition of vertices as well as Navarro's. As we will see, the two definitions are compatible. If $N \lhd G$ and $\mu \in \mathrm{Irr}(N)$, then write $\mathrm{Irr}(G, B, \mu)$ for the subset of $\mathrm{Irr}(G, B)$ consisting of those characters covering μ. Write $\mathrm{IBr}(G, B)$ for the set of irreducible Brauer characters of G in B. Write $\mathrm{IBr}(G, B \mid Q)$ for the subset consisting of those Brauer characters with vertex Q (in the sense that the associated simple kG-modules have vertex Q).

In Section 2 we consider p-solvable groups and Navarro's vertices with respect to a normal subgroup containing the defect groups of a block. In Section 3 we define vertices for irreducible character in blocks whose defect groups are contained in a normal p-solvable subgroup and observe that the results of Section 2 show that this definition is independent of the choice of normal p-solvable subgroup containing the defect groups. In Section 4 we show that vertices as defined in Section 3 have some key properties identified in [9], and also give some consequences of these.

2. Blocks of p-solvable groups

We first review Navarro's definition of a vertex for an irreducible character of a p-solvable group G. See [9] for details.

For a set of primes π, an irreducible character $\chi \in \mathrm{Irr}(G)$ is π-special if $\chi(1)$ is a π-number and for every subnormal subgroup N of G and every constituent $\theta \in \mathrm{Irr}(N)$ of $\mathrm{Res}^G_N(\chi)$, the determinantal order of θ is a π-number. χ is called π-factorable if it may be written (uniquely) as a product of a π'-special and a π-special irreducible character. We are interested in the case $\pi = \{p\}$.

We may order pairs (H, θ), where $H \leqslant G$ and $\theta \in \mathrm{Irr}(H)$, by $(H_1, \theta_1) \leqslant (H_2, \theta_2)$ if $H_1 \leqslant H_2$ and θ_1 is a constituent of $\mathrm{Res}^{H_2}_{H_1}(\theta_2)$. A normal p-factorable pair (N, θ) under (G, χ) is one where θ is p-factorable, $N \lhd G$ and $(N, \theta) \leqslant (G, \chi)$. Note that if (G, χ) is p-factorable and $(N, \theta) \leqslant (G, \chi)$ with $N \lhd G$, then (N, θ) is also p-factorable.

By [9, 2.3], there is an unique maximal p-factorable pair under (G,χ) up to G-conjugacy. By [9, 2.4], if (N,θ) is a proper maximal normal p-factorable pair under (G,χ), then $I_G(\theta)\neq G$.

A *nucleus* of χ is a canonical (up to G-conjugacy) p-factorable pair $(W,\gamma)\leqslant(G,\chi)$ such that $\chi=\gamma^G$. This is defined inductively on $|G|$ as follows: if χ is p-factorable, then $(W,\gamma)=(G,\chi)$. If not, then let (N,θ) be a maximal normal p-factorable pair under (G,χ). Then $I=I_G(\theta)<G$, and there is $\zeta\in\mathrm{Irr}(I,\theta)$ with $\zeta^G=\chi$. By induction there is a nucleus (W,γ) defined for (I,ζ), and we make (W,γ) a nucleus for (G,χ). Note that $\gamma^G=(\gamma^I)^G=\zeta^G=\chi$.

Given a nucleus (W,γ) of χ, we may write $\gamma=\alpha\beta$ uniquely where α is p'-special and β is p-special. Let $Q\in\mathrm{Syl}_p(W)$. By [9, 2.1] $\delta=\mathrm{Res}^W_Q(\beta)$ is irreducible. (Q,δ) is called a vertex of χ, and is unique up to G-conjugacy.

Theorem 2.1. *Let B be a block of a finite p-solvable group G, let N be a normal subgroup of G containing the defect groups of B and let b be a block of N covered by B. Let $\mu\in\mathrm{Irr}(N,b)$ and $\chi\in\mathrm{Irr}(G,B,\mu)$. Let Q be a p-subgroup of N and $\delta\in\mathrm{Irr}(Q)$. Then χ has vertex (Q,δ) if and only if μ has vertex a G-conjugate of (Q,δ).*

Proof. We use induction on $|G|$. Thus it suffices to consider two cases: G/N is a (nontrivial) p'-group and $[G:N]=p$.

Let (S,σ) be a maximal normal p-factorable pair under (N,μ). Write $S_1=\langle S^g\colon g\in G\rangle\lhd G$ and choose $\sigma_1\in\mathrm{Irr}(S_1)$ so that $(S,\sigma)\leqslant(S_1,\sigma_1)\leqslant(G,\chi)$. Then by [9, 2.2] (S_1,σ_1) is a normal p-factorable pair under (G,χ), and there is a maximal normal p-factorable pair (T,τ) with $(S_1,\sigma_1)\leqslant(T,\tau)\leqslant(G,\chi)$. Write $J=I_N(\sigma)$ and $I=I_G(\tau)$, so there is $\zeta\in\mathrm{Irr}(I,\tau)$ such that $\zeta^G=\chi$. Let (U,α) be a nucleus for μ over (S,σ) and let (W,γ) be a nucleus for χ over (T,τ). Then $S\leqslant T\cap N$. But $T\cap N\lhd N$ and there is p-factorable $\epsilon\in\mathrm{Irr}(T\cap N)$ with $(T\cap N,\epsilon)\leqslant(N,\mu)$, so by maximality there is an N-conjugate of $(T\cap N,\epsilon)$ under (S,σ). Hence $S=T\cap N$ and by [7, 7.3] $I\cap J\lhd I$, $I\cap J\lhd J$, and $[I:I\cap J]$, $[J:I\cap J]$ are divisible only by primes dividing $[G:N]$. Note that $J\leqslant N<G$.

Suppose first that G/N is p'-group. Then $[I:I\cap J]$ and $[J:I\cap J]$ are not divisible by p. If $G=I$, then (G,χ), and so (N,μ), is p-factorable, i.e., each is their own nucleus. Let $Q\in Syl_p(G)$. Then $Q\in Syl_p(N)$ and restrictions of the p-special parts of χ and μ restrict identically to Q, i.e., χ and μ have identical vertices and we are done in this case.

Suppose that $I\neq G$. Let $\beta\in\mathrm{Irr}(I\cap J,\sigma)$ be a constituent of $\mathrm{Res}^I_{I\cap J}(\zeta)$. Choose $\eta\in\mathrm{Irr}(J,\beta)$ such that $(I\cap J,\beta)\leqslant(J,\eta)\leqslant(G,\chi)$. Then $\eta^N\in\mathrm{Irr}(N)$ and is covered by χ, so η^N is conjugate in G to μ. Note that ζ, respectively η, lies in a block of I, respectively J, with defect groups contained in the normal subgroup $I\cap J$. Hence by induction the vertices of ζ and η are both conjugate to a vertex of β. Now η and η^N by definition have the same vertex, as do ζ and $\chi=\zeta^G$. But η^N is G-conjugate to μ, so the vertices of χ and μ are G-conjugate and we are done in this case.

It remains to consider the case $[G:N]=p$. By [5, V.3.5] B is the unique block of G covering b, and so by [1, 15.1] the stabilizer of b in G is N. Hence $I_G(\mu)=N$, and $\chi=\mu^G$. Suppose that $T\nleqslant N$. Then $G=TN$. If χ has vertex (Q,δ), then by [9, A(e)] Q is contained in a conjugate of D, and so $Q\leqslant N$, i.e., T has Sylow p-subgroups contained in N. Hence G has Sylow p-subgroups contained in N, a contradiction. So $T \lhd N$. We may choose a G-conjugate τ^g of τ covered by μ, so (T, τ^g) is a normal p-factorable pair under (N, μ). Hence $T \leqslant S$ by the maximality of (S, σ). But $S \leqslant T$, so $S = T$, and $J = I \cap N$. By [5, V.1.2] ζ lies in a block B_I of I with Brauer correspondent B, so a defect group P of B_I is contained in a G-conjugate of D. Hence $P \leqslant I \cap N = J$, and the result follows by induction. □

3. Defect groups lying in a normal p-solvable subgroup

Definition 3.1. Let G be a finite group and let N be a normal p-solvable subgroup. Let B be a p-block of G with defect groups contained in N and let χ be an irreducible character in B. Let Q be a p-subgroup of N and $\delta \in \mathrm{Irr}(Q)$. We say that χ has vertex (Q, δ) if χ covers an irreducible character of N with vertex (Q, δ) as defined in [9].

It is clear from Navarro's definition of a vertex that if $\theta \in \mathrm{Irr}(N)$ has vertex (Q, δ) and $g \in G$, then θ^g has vertex $(Q, \delta)^g$. Since the vertices of θ are all conjugate in N (see [9, p. 2764]) and all irreducible characters of N covered by χ are G-conjugate, it follows that the vertices of χ form a G-conjugacy class. By Theorem 2.1, the definition is independent of the choice of normal p-solvable subgroup containing the defect groups (this uses the fact that there is a unique minimal normal subgroup containing the defect groups of a block). Indeed, if G itself is p-solvable, then our definition is compatible with Navarro's.

We keep the same notation $\mathrm{Irr}(G, B \mid Q, \delta)$, etc., with regard to the extended definition of a vertex.

As we might hope, vertices have some bearing on the idea of relative projectivity of irreducible characters as discussed in [8].

Proposition 3.2. *Let B be a p-block of a finite group G with defect groups contained in a normal p-solvable subgroup N. Let $Q \leqslant G$ and $\delta \in \mathrm{Irr}(Q)$. If $\chi \in \mathrm{Irr}(G, B \mid Q, \delta)$, then χ is afforded by a Q-projective $\mathcal{O}G$-module.*

Proof. Let χ cover $\theta \in \mathrm{Irr}(N \mid Q, \delta)$. By definition there are $W \leqslant N$ and $\eta \in \mathrm{Irr}(W)$ such that $Q \in Syl_p(W)$ and $\theta = \eta^N$. Every $\mathcal{O}W$-module affording η is Q-projective.

Since every defect group of B is contained in N, it follows that χ is N-projective. By [8, 2.6] and the remarks following it there is an indecomposable $\mathcal{O}G$-module M affording χ such that $\mathrm{Res}^G_N(M)$ is completely reducible. There is a summand M_θ of $\mathrm{Res}^G_N(M)$ affording θ, where M_θ has no proper nonzero $\mathcal{O}$-pure submodule ($M_\theta = V_\theta \cap \mathrm{Res}^G_N(M)$, where V_θ is an irreducible summand of $\mathrm{Res}^G_N(M) \otimes_{\mathcal{O}} K$ affording θ).

We claim that M_θ is Q-projective. Write $L = \mathrm{Res}^N_W(M_\theta)$. Then $L \otimes_{\mathcal{O}} K$ has a summand X_η affording η. Write $L_\eta = X_\eta \cap L$ (so L_η affords η). Let $t_1, \ldots, t_n$ be a transversal of W in N, with $t_1 = 1$. Define $\gamma \in \mathrm{End}_{\mathcal{O}W}(\mathrm{Ind}^N_W(L_\eta))$ by $\gamma(\sum t_i \otimes l_i) = 1 \otimes l_1$. Then $\mathrm{Tr}^N_W(\gamma) = 1_{\mathrm{Ind}^N_W(L_\eta)}$. Now $(\mathrm{Ind}^N_W(L_\eta)) \otimes_{\mathcal{O}} K \cong M_\theta \otimes_{\mathcal{O}} K$, so $M_\theta = \mathrm{Ind}_W(L_\eta) \cap (M_\theta \otimes_{\mathcal{O}} K)$ is γ-invariant. Hence $\gamma|_{M_\theta} \in \mathrm{End}_{\mathcal{O}W}(M_\theta)$ and $\mathrm{Tr}^N_W(\gamma|_{M_\theta}) = 1_{M_\theta}$, i.e., M_θ is Q-projective as claimed.

Since M is indecomposable with $M \mid \operatorname{Ind}_N^G(\operatorname{Res}_N^G(M))$ and M_θ is a direct summand of $\operatorname{Res}_N^G(M)$, we have $M \mid \operatorname{Ind}_N^G(M_\theta)$, and the result follows. □

4. Properties of vertices

In this section we prove the analogue of [9, A] and derive some consequences.

The following useful result is probably known, for which we adapt part of Alperin's module-theoretic proof of the Harris–Knörr theorem (see [1]).

Lemma 4.1. *Let G be a finite group and $N \lhd G$. Let Q be a p-subgroup of N and L a subgroup of G with $QC_G(Q) \leqslant L \leqslant N_G(Q)$. Write $K = N \cap L$. Suppose that we have blocks b_1, b and B of K, N and G, respectively, such that B covers b and $b_1^N = b$. Then there exists a block B_1 of L such that $B_1^G = B$ and B_1 covers b_1.*

Proof. Consider for now blocks defined over the field k. Recall that we may consider blocks of H as indecomposable $k[H \times H]$-modules (see, for example, [2]). If M_1 and M are kH-modules, write $M_1 \mid M$ if M_1 is isomorphic to a (direct) summand of M.

Since B covers b we have $b \mid B_{N\times N}$, and since $b_1^N = b$ we have $b_1 \mid b_{K\times K}$. Hence $b_1 \mid B_{K\times K}$, and $B_{L\times L}$ has an indecomposable summand M such that $M_{K\times K} \cong b_1$. We will show that we may take $B_1 = M$. Now $B_{L\times L} \mid \bigoplus_{t\in L\backslash G/L} kLtL$. By [2, 13.7] when $t \in G - L$ no indecomposable summand of $kLtL$ has vertex containing the diagonal subgroup δQ. Of course every other indecomposable summand is a block of L. However, $Q \lhd K$, so δQ is contained in a vertex of b_1, so δQ is contained in a vertex of M by [4, 19.14]. Hence $M \mid kL$ and M is a block. Write $B_1 = M$. Since $B_1 \mid B_{L\times L}$ and B_1^G is defined, $B_1^G = B$. Since $b_1 \mid (B_1)_{K\times K}$, we have that B_1 covers b_1. □

Recall that a p-subgroup R of G is called *radical* if $R = O_p(N_G(R))$. For a p-subgroup Q of G, write $R_0 = Q$ and $R_n = O_p(N_G(R_{n-1}))$ for $n > 0$. Since G is finite, eventually $R_{t+1} = R_t$ for some t, so that R_t is radical. Call R_t the *radical closure* Q in G. If $\chi \in \operatorname{Irr}(G)$, recall that the *defect* $d(\chi)$ of χ is the integer such that $p^{d(\chi)}\chi(1)_p = |G|_p$.

Theorem 4.2. *Let G be a finite group with normal p-solvable subgroup N. Let B be a p-block of G with defect groups contained in N. Let Q be a p-subgroup of G and $\delta \in \operatorname{Irr}(Q)$.*

(a) *Restriction to p-regular elements defines a canonical bijection* $\operatorname{Irr}(G, B \mid Q, 1_Q) \to \operatorname{IBr}(Q, B \mid Q)$.
(b) *Suppose that (Q, δ) is G-invariant. Then $\chi \in \operatorname{Irr}(G, B)$ has vertex (Q, δ) if and only if $\operatorname{Res}_Q^G(\chi)$ contains δ and $\chi(1)_p = [G : Q]_p\delta(1)$.*

Let $\chi \in \operatorname{Irr}(G, B \mid Q, \delta)$. Then:

(c) $d(\chi) = d(\delta)$;
(d) $\chi(g) = 0$ *whenever the p-section containing g intersects trivially with Q*;
(e) $O_p(N_G(Q, \delta)) = Q$;

(f) *if R is the radical closure of Q in G, then there is a defect group D of B with $R \leqslant D$ and $C_D(Q) \leqslant Z(Q)$;*
(g) *if R is the radical closure of Q in G, then δ^R is irreducible.*

Proof. (a) Let $\mu \in \mathrm{IBr}(N \mid Q)$. By [9] there is a unique $\theta \in \mathrm{Irr}(N \mid Q, 1_Q)$ such that $\mu = \theta_{p'}$ (the restriction of θ to $G_{p'}$). It suffices to assume that μ (and so θ) lies in a p-block of N covered by B. Let $g \in G$. Then θ^g has vertex $(Q, 1_Q)^g = (Q^g, 1_{Q^g})$ and μ^g has vertex Q^g. By [9] there is an unique $\theta' \in \mathrm{Irr}(N \mid Q^g, 1_{Q^g})$ corresponding to μ^g and an unique $\mu' \in \mathrm{IBr}(N \mid Q^g)$ corresponding to θ^g. Hence $g \in I_G(\mu)$ if and only if $g \in I_G(\theta)$. Hence $I_G(\mu) = I_G(\theta)$, and by [6, Lemma 3] restriction to p-regular elements gives a 1–1 correspondence between the set of irreducible constituents of θ^G which lie in B and the set of irreducible constituents of μ^G which lie in B. It suffices now to prove that every such constituent of θ^G has vertex $(Q, 1_Q)$ and every such constituent of μ^G has vertex Q, since we may then apply the above to representatives of each of the G-conjugacy classes of irreducible Brauer characters of N, in turn. Note that if (Q, δ) is a vertex for an irreducible character of N in a block covered by B, then Q is contained in a defect group of B. That the irreducible constituents of θ in B have vertex $(Q, 1_Q)$ then follows from Frobenius reciprocity.

Let e be the primitive idempotent of $Z(kG)$ corresponding to B and let M be a simple kN-module affording μ. By the proof of [6, Lemma 3] $(M^G)e$ is completely reducible, so the constituents of μ^G in B correspond to the summands of $(M^G)e$. Since M has vertex Q, there is a kQ-module S such that $M \mid S^N$. Hence $(M^G)e \mid M^G \mid S^G$, so every summand of $(M^G)e$ is Q-projective. Suppose that some irreducible constituent φ of μ^G in B has vertex R strictly contained in Q. Let $L \mid (M^G)e$ afford φ. There is a kR-module T such that $L \mid T^G$, and

$$M \bigm| \mathrm{Res}^G_N(L) \bigm| \mathrm{Res}^G_N\bigl(T^G\bigr) = \bigoplus_{RaN} \bigl(\mathrm{Res}^{R^a}_{R^a \cap N}(T^a)\bigr)^N = \bigoplus_{RaN} (T^a)^N,$$

since $R \leqslant Q \leqslant D \leqslant N$, where D is a defect group of B. Hence $M \mid (T^a)^N$ for some $a \in G$. Hence M is R^a-projective, a contradiction. So Q is a vertex for every irreducible constituent of μ^G in B as required.

(b) Suppose that $G = N_G(Q, \delta)$. Let $\chi \in \mathrm{Irr}(G)$ cover $\theta \in \mathrm{Irr}(N)$. Note that Q is contained in every defect group of G, so $Q \leqslant N$. Now $\mathrm{Res}^G_Q(\chi)$ contains δ if and only if $\mathrm{Res}^N_Q(\theta)$ does. Since B has defect groups in N, it follows that χ is afforded by an N-projective $\mathcal{O}G$-module, so by [11, 3.9] $\chi(1)_p = [G : N]_p \theta(1)_p$. Hence $\chi(1)_p = [G : Q]_p \delta(1)_p$ if and only if $\theta(1)_p = [N : Q]_p \delta(1)_p$, and the result follows from [9].

(c) Suppose that χ covers $\theta \in \mathrm{Irr}(N \mid Q, \delta)$. By [9] $d(\theta) = d(\delta)$, i.e.,

$$\left(\frac{|N|}{\theta(1)}\right)_p = \frac{|Q|}{\delta(1)}.$$

As in (b), $\chi(1)_p = [G:N]_p\theta(1)_p$. Hence

$$\left(\frac{|G|}{\chi(1)}\right)_p = \frac{|Q|}{\delta(1)},$$

as required.

(d) This is immediate from Proposition 3.2 and Green's theorem (see [4, 19.27]).

(e) Suppose that χ covers $\theta \in \mathrm{Irr}(N \mid Q,\delta)$. Hence by [9] $Q = O_p(N_N(Q,\delta))$. Write $R = O_p(N_G(Q,\delta))$, so $Q \leqslant R$. We claim that R is contained in some defect group of B. By [10, 5.5] applied to the p-block b of N containing θ it follows that there is some block b_1 of $N_N(Q,\delta)$ with $b_1^N = b$. Then by Lemma 4.1 there is a block of $N_G(Q,\delta)$ with Brauer correspondent B. Since R is contained in every defect group of $N_G(Q,\delta)$ it follows that $R \leqslant D$ for some defect group of D of B as claimed. Now $D \leqslant N$, so $R \leqslant N_N(Q,\delta)$. But then $Q \leqslant R \lhd N_N(Q,\delta)$, so $Q = R$.

(f) It suffices to prove that the radical closure R of Q in G is contained in a defect group of B. The result then follows directly from the corresponding result in [9]. Above we proved that there is a p-block, say B_1, of $N_G(Q,\delta)$ with $B_1^G = B$. Note that every p-block of $N_G(R)$ has a Brauer correspondent in G. Hence $B_1^{N_G(R)}$ has Brauer correspondent B, and R is contained in a defect group of B as required.

(g) By (f) the radical closure of Q in G is contained in N. Hence $O_p(N_G(Q)) \leqslant N$, and it follows that $O_p(N_G(Q)) = O_p(N_N(Q))$. Thus by induction the radical closure of Q in G is the radical closure of Q in N, and the result follows by [9]. □

Using the above, we are able to verify a prediction of Robinson:

Corollary 4.3. *Let G be a finite group with normal p-solvable subgroup N. Let B be a p-block of G with defect groups contained in N. If $\chi \in \mathrm{Irr}(G,B)$, then there is a radical p-subgroup R of G, a defect group D of B containing R and $\eta \in \mathrm{Irr}(R)$ such that $d(\eta) = d(\chi)$ and $C_D(R) = Z(R)$.*

Proof. Suppose χ has vertex (Q,δ). Then the result follows from parts (f) and (g) above by taking R to be the radical closure of Q and $\eta = \delta^R$. □

We may also extend some results of Bessenrodt [3]. As mentioned in [3], parts (ii)–(v) of the following only rely on the presence of a simple $kN_G(Q)$-module with vertex Q, rather than on the structure of G.

Corollary 4.4. *Let G be a finite group with normal p-solvable subgroup N. Let B be a p-block of G with defect groups contained in N. If S is a simple kG-module in B with vertex Q, then*

- (i) *$N_G(Q)$ possesses a simple $kN_G(Q)$-module, lying in a Brauer correspondent of B, with vertex Q*;
- (ii) *$N_G(Q)/Q$ possesses a p-block of defect zero*;
- (iii) $Q = O_p(N_G(Q))$;

(iv) *Q is a tame intersection of Sylow p-subgroups of G;*
(v) *If $P \in Syl_p(G)$ contains Q, then there is $x \in N_G(Q)$ such that $Q = P^x \cap P$.*

Proof. (i) By part (a) of Theorem 4.2, there is $\chi \in \mathrm{Irr}(G, B \mid Q, 1_Q)$. By definition, χ covers an irreducible character θ of N with vertex $(Q, 1_Q)$. Note that θ lies in a block b of N covered by B. By [10, 5.5] there is $\mu \in \mathrm{Irr}(N_G(Q) \mid Q, 1_Q)$ lying in a Brauer correspondent, say b_1, of b. By Lemma 4.1 it follows that b_1 is covered by a block B_1 of $N_G(Q)$ with $B_1^G = B$. Since every defect group of B_1 is contained in a conjugate of D, it follows that every defect group of B_1 is contained in the normal p-solvable subgroup $N_N(Q)$ of $N_G(Q)$. Since B_1 covers b_1 it follows that some $\eta \in \mathrm{Irr}(N_G(Q), B_1)$ covers μ, and by definition η has vertex $(Q, 1_Q)$. The result then follows by part (a) of Theorem 4.2.

The remaining parts follow from (i) in the same way as in [3]. □

Acknowledgments

The author thanks the University of Auckland, and in particular Jianbei An, for their hospitality during his visit, when much of this paper was written.

References

[1] J.L. Alperin, The Green correspondence and normal subgroups, J. Algebra 104 (1986) 74–77.
[2] J.L. Alperin, Local Representation Theory, Cambridge Stud. Adv. Math., vol. 11, Cambridge University Press, 1986.
[3] C. Bessenrodt, Vertices of simple modules over p-solvable groups, J. London Math. Soc. (2) 29 (1984) 257–261.
[4] C.W. Curtis, I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, vol. I, Wiley–Interscience, New York, 1981.
[5] W. Feit, The Representation Theory of Finite Groups, North-Holland, Amsterdam, 1982.
[6] S.M. Gagola Jr., A note on lifting Brauer characters, Proc. Amer. Math. Soc. 53 (1975) 295–300.
[7] I.M. Isaacs, Characters of π-separable groups, J. Algebra 86 (1984) 98–128.
[8] B. Külshammer, G.R. Robinson, Characters of relatively projective modules II, J. London Math. Soc. (2) 36 (1987) 59–67.
[9] G. Navarro, Vertices for characters of p-solvable groups, Trans. Amer. Math. Soc. 354 (2002) 2759–2773.
[10] G. Navarro, Defect of characters and local subgroups in p-solvable groups, J. Group Theory 5 (2002) 241–268.
[11] G.R. Robinson, Characters of relatively projective modules, J. London Math. Soc. (2) 36 (1987) 44–58.