The principal series of p-adic groups with disconnected centre

Aubert, Anne-Marie and Baum, Paul and Plymen, Roger and Solleveld, Maarten

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THE PRINCIPAL SERIES OF p-ADIC GROUPS WITH DISCONNECTED CENTRE

ANNE-MARIE AUBERT, PAUL BAUM, ROGER PLYMEN,
AND MAARTEN SOLLEVELD

Abstract. Let $\mathcal{G}$ be a split connected reductive group over a local non-archimedean field. We classify all irreducible complex $\mathcal{G}$-representations in the principal series, irrespective of the (dis)connectedness of the centre of $\mathcal{G}$. This leads to a local Langlands correspondence for principal series representations, which satisfies all expected properties. We also prove that the ABPS conjecture about the geometric structure of Bernstein components is valid throughout the principal series of $\mathcal{G}$.

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1. Introduction

In this paper, we construct (granted a mild restriction on the residual characteristic) a local Langlands correspondence throughout the principal series of any connected split reductive p-adic group $\mathcal{G}$. In addition, we prove that the ABPS geometric structure conjecture is valid throughout the principal series of $\mathcal{G}$.

We do not assume that the centre of $\mathcal{G}$ is connected. Previous results on this subject needed the condition that the centre of $\mathcal{G}$ is connected, see Kazhdan–Lusztig [KaLu], Reeder [Ree2], Aubert–Baum–Plymen–Solleveld [ABPS2].

Let F be a local non-Archimedean field and let $\mathcal{G}$ be the group of the F-rational points of an F-split connected reductive algebraic group, and let $\mathcal{T}$ be a maximal torus in $\mathcal{G}$. The principal series consists of all $\mathcal{G}$-representations that are constituents of parabolically induced representations from irreducible characters of $\mathcal{T}$. Let $\mathfrak{B}(\mathcal{G})$ denote the Bernstein spectrum of $\mathcal{G}$, and let $\mathfrak{B}(\mathcal{G},\mathcal{T})$ be the subset of $\mathfrak{B}(\mathcal{G})$ given by all cuspidal pairs $(\mathcal{T},\chi)$, where χ is a character of $\mathcal{T}$.

For each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$ we construct a commutative triangle of bijections

$$\begin{tikzcd} & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\ \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[rr] & & \Psi(G)^{\mathfrak{s}}_{\mathrm{en}} \end{tikzcd} \tag{1}$$

Here $T^{\mathfrak{s}}$ and $W^{\mathfrak{s}}$ are Bernstein's torus and finite group for $\mathfrak{s}$, and $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is the extended quotient of the second kind resulting from the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$. Equivalently, $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is the set of equivalence classes of irreducible representations of the crossed product algebra $\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$:

$$(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \simeq \mathbf{Irr}(\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}).$$

Here $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is the Bernstein component of $\mathbf{Irr}(\mathcal{G})$ attached to $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$, $\Psi(G)^{\mathfrak{s}}_{\mathrm{en}}$ is the set of enhanced Langlands parameters associated to $\mathfrak{s}$, and G is the Langlands dual of $\mathcal{G}$.

In examples, $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is much simpler to directly calculate than either $\mathbf{Irr}(G)^{\mathfrak{s}}$ or $\Psi(G)^{\mathfrak{s}}_{\mathrm{en}}$.

The point $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$ determines a certain complex reductive group $H^{\mathfrak{s}}$ in the Langlands dual group G. If $\mathcal{G}$ has connected centre, then:

- $H^{\mathfrak{s}}$ is connected
- Bernstein's finite group $W^{\mathfrak{s}}$ is the Weyl group of $H^{\mathfrak{s}}$
- Bernstein's torus $T^{\mathfrak{s}}$ is the maximal torus of $H^{\mathfrak{s}}$
- the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ is the standard action of the Weyl group of $H^{\mathfrak{s}}$ on the maximal torus of $H^{\mathfrak{s}}$.

If $\mathcal{G}$ does not have connected centre, then:

- $H^{\mathfrak{s}}$ can be non-connected
- $W^{\mathfrak{s}}$ is the semidirect product $W^{\mathfrak{s}} = \mathcal{W}^{H_0^{\mathfrak{s}}} \rtimes \pi_0(H^{\mathfrak{s}})$ where $H_0^{\mathfrak{s}}$ is the identity component of $H^{\mathfrak{s}}$, and $\mathcal{W}^{H_0^{\mathfrak{s}}}$ is the Weyl group of $H_0^{\mathfrak{s}}$
- $T^{\mathfrak{s}}$ is the maximal torus T of $H_0^{\mathfrak{s}}$

- $W^{\mathfrak{s}} = \mathrm{N}_{H^{\mathfrak{s}}}(T)/T$. The action on T is the evident conjugation action, and $\mathrm{N}_{H^{\mathfrak{s}}}(T)$ is the normalizer in $H^{\mathfrak{s}}$ of T.

See Lemma 3.2 and Eqn. (79).

Semidirect products by $\pi_0(H^{\mathfrak{s}})$ occur frequently in this paper, e.g.

$$\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(H_0^{\mathfrak{s}}) \rtimes \pi_0(H^{\mathfrak{s}})$$

Here $\mathcal{H}^{\mathfrak{s}}$ is a finite type algebra attached by Bernstein to $\mathfrak{s}$ and $\mathcal{H}(H_0^{\mathfrak{s}})$ is the affine Hecke algebra of $H_0^{\mathfrak{s}}$, with parameter q equal to the cardinality of the residue field. Thus, $\mathcal{H}^{\mathfrak{s}}$ is an extended affine Hecke algebra.

Similarly, $\pi_0(H^{\mathfrak{s}})$ acts on Lusztig's asymptotic algebra $\mathcal{J}(H_0^{\mathfrak{s}})$. The crossed product algebra

$$\mathcal{J}(H_0^{\mathfrak{s}}) \rtimes \pi_0(H^{\mathfrak{s}})$$

features crucially in Section 13.

In the above commutative triangle, the right slanted arrow is constructed and proved to be a natural bijection by suitably generalising the Springer correspondence for finite and affine Weyl groups (Sections 4 and 8), and by comparing the involved parameters (Sections 6 and 7).

The left slanted arrow in (1) is defined and proved to be a bijection by applying the representation theory of affine Hecke algebras and, in particular, Lusztig's asymptotic algebra. However, in order to apply this theory, it is necessary to prove the equality of certain 2-cocycles, see §13. The technical issues that are confronted in this paper arise from Clifford theory and are very closely connected to the analysis of these 2-cocycles.

Similar 2-cocycles for connected non-split groups can be non-trivial. Hence, for connected non-split groups, a twisted extended quotient must be used in the statement of the ABPS geometric structure conjecture. We shall develop ABPS for connected non-split reductive p-adic groups elsewhere.

The horizontal arrow in our main result (see the above commutative triangle and Theorem 15.1 and Proposition 16.1) generalises the Kazhdan–Lusztig parametrization of the irreducible representations of affine Hecke algebras with equal parameters (§9), and also generalises the Reeder–Roche parametrization of the irreducible $\mathcal{G}$-representations in the principal series for groups $\mathcal{G}$ with connected centre (cf. §11). We use the new input from $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ to prove that, although the horizontal arrow in (1) is in general not canonical, every element of $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ does canonically determine a Langlands parameter for $\mathcal{G}$. To establish the horizontal arrow as a local Langlands correspondence for these representations, we also show that it satisfies all the desiderata of Borel, see Sections 16 and 17.

The union over all the $\mathfrak{s} \in \mathfrak{B}(\mathcal{G}, \mathcal{T})$ of the extended quotients of the second kind $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is the extended quotient of the second kind $(\mathbf{Irr}(\mathcal{T})/\!/\mathcal{W}^{\mathcal{G}})_2$, with $\mathcal{W}^{\mathcal{G}} = \mathrm{N}_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$, and the triangles (1) for different $\mathfrak{s}$ combine to a bijective commutative diagram

$$\begin{tikzcd}
& (\mathbf{Irr}(\mathcal{T})/\!/\mathcal{W}^{\mathcal{G}})_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G}, \mathcal{T}) \arrow[rr] & & \Psi(G)^{\mathrm{prin}}_{\mathrm{en}}
\end{tikzcd}$$

where $\Psi(G)^{\mathrm{prin}}_{\mathrm{en}}$ denotes the collection of enhanced L-parameters for the principal series of $\mathcal{G}$, and $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$ denotes the collection of irreducible principal series representations of $\mathcal{G}$. All this holds under the restrictions on the residual characteristic stated in Condition 11.1.

The ABPS geometric structure conjecture includes the assertion that the local Langlands correspondence factors through the appropriate extended quotient. This extended quotient is much easier to directly calculate than either the source or the target of the Langlands correspondence.

In fact, the ABPS conjecture has more precision, as explained in §18 of this paper. With this level of precision, we provide a complete proof, in §18 and §19, of the conjecture for the principal series of all connected split reductive p-adic groups (always with the restriction on the residue characteristic). This includes, in §18, a labelling by unipotent classes in $H^{\mathfrak{s}}$ and, in §19, a complete account of the relation between correcting cocharacters and L-packets.

2. Extended quotients

Let Γ be a finite group acting on a topological space X

$$\Gamma \times X \to X.$$

The quotient space X/Γ is obtained by collapsing each orbit to a point. For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x. The *extended quotient of the first kind* is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}$, a subspace of $\Gamma \times X$. The group Γ acts on $\widetilde{X}$:

$$\begin{aligned} \Gamma \times \widetilde{X} &\to \widetilde{X} \\ \alpha(\gamma, x) &= (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}. \end{aligned}$$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X/\!/\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$. The projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ is Γ-equivariant and so passes to quotient spaces

$$\rho\colon X/\!/\Gamma \to X/\Gamma.$$

This map will be referred to as the projection of the extended quotient onto the ordinary quotient.

The inclusion

$$\begin{aligned} X &\hookrightarrow \widetilde{X} \\ x &\mapsto (e, x) \qquad e = \text{identity element of } \Gamma \end{aligned}$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion $X/\Gamma \hookrightarrow X/\!/\Gamma$. This will be referred to as the inclusion of the ordinary quotient in the extended quotient.

With Γ, X, Γ_x as above, let $\mathbf{Irr}(\Gamma_x)$ be the set of (equivalence classes of) irreducible representations of Γ_x. The *extended quotient of the second kind*, denoted $(X//\Gamma)_2$, is constructed by replacing the orbit of x (for the given action of Γ on X) by $\mathbf{Irr}(\Gamma_x)$. This is done as follows :

Set $\widetilde{X}_2 = \{(x,\tau) \,\big|\, x \in X \text{ and } \tau \in \mathbf{Irr}(\Gamma_x)\}$. Then Γ acts on $\widetilde{X}_2$.

$$\Gamma \times \widetilde{X}_2 \to \widetilde{X}_2,$$
$$\gamma(x,\tau) = (\gamma x, \gamma_* \tau),$$

where $\gamma_* \colon \mathbf{Irr}(\Gamma_x) \to \mathbf{Irr}(\Gamma_{\gamma x})$. Now we define

$$(X//\Gamma)_2 := \widetilde{X}_2/\Gamma,$$

i.e. $(X//\Gamma)_2$ is the usual quotient for the action of Γ on $\widetilde{X}_2$. The projection $\widetilde{X}_2 \to X \quad (x,\tau) \mapsto x \quad$ is Γ-equivariant and so passes to quotient spaces to give the projection of $(X//\Gamma)_2$ onto X/Γ.

$$\rho_2 \colon (X//\Gamma)_2 \longrightarrow X/\Gamma$$

Denote by triv_x the trivial one-dimensional representation of Γ_x. The inclusion

$$X \hookrightarrow \widetilde{X}_2$$
$$x \mapsto (x, \mathrm{triv}_x)$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion

$$X/\Gamma \hookrightarrow (X//\Gamma)_2$$

This will be referred to as the inclusion of the ordinary quotient in the extended quotient of the second kind.

Notice that the fibers $\rho^{-1}(\Gamma x)$ and $\rho_2^{-1}(\Gamma x)$ always have the same number of elements. Hence there exist non-canonical bijections $\epsilon \colon X//\Gamma \to (X//\Gamma)_2$ with commutativity in the diagrams

(2)
$$\begin{tikzcd} X//\Gamma \arrow[rr, "\epsilon"] \arrow[dr, "\rho"'] & & (X//\Gamma)_2 \arrow[dl, "\rho_2"] \\ & X/\Gamma & \end{tikzcd} \qquad \begin{tikzcd} X//\Gamma \arrow[rr, "\epsilon"] & & (X//\Gamma)_2 \\ & X/\Gamma \arrow[ul] \arrow[ur] & \end{tikzcd}$$

To construct a bijection ϵ, some choices must be made. We will make use of a family ψ of bijections

$$\psi_x : c(\Gamma_x) \to \mathbf{Irr}(\Gamma_x)$$

such that for all $x \in X$:

- $\psi_x([1]) = \mathrm{triv}_x$;
- $\psi_{\gamma x}([\gamma g \gamma^{-1}]) = \phi_x([g]) \circ \mathrm{Ad}_\gamma^{-1}$ for all $g \in \Gamma_x, \gamma \in \Gamma$.

We shall refer to such a family of bijections as a c-**Irr** system. Clearly ψ induces a map $\widetilde{X} \to \widetilde{X}_2$ which preserves the X-coordinates. By the second property this map is Γ-equivariant, so it descends to a map

$$\epsilon = \epsilon_\psi : X//\Gamma \to (X//\Gamma)_2.$$

Observe that ϵ_ψ makes the diagrams from (2) commute, the first by construction and the second by the first property of ψ_x. The restriction of ϵ_ψ

to the fiber over $\Gamma x \in X/\Gamma$ is ψ_x, and in particular is bijective. Therefore ϵ_ψ is bijective.

Next we will define a twisted version of an extended quotient. Let $\natural$ be a given function which assigns to each $x \in X$ a 2-cocycle $\natural(x) : \Gamma_x \times \Gamma_x \to \mathbb{C}^\times$ where $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$. It is assumed that $\natural(\gamma x)$ and $\gamma_* \natural(x)$ define the same class in $H^2(\Gamma_x, \mathbb{C}^\times)$, where $\gamma_* : \Gamma_x \to \Gamma_{\gamma x}, \alpha \mapsto \gamma \alpha \gamma^{-1}$. Define

$$\widetilde{X}_2^\natural := \{(x, \rho) : x \in X, \rho \in \mathbf{Irr}\, \mathbb{C}[\Gamma_x, \natural(x)]\}.$$

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} : \mathbb{C}[\Gamma_x, \natural(x)] \to \mathbb{C}[\Gamma_{\gamma x}, \natural(\gamma x)]$$

such that:

- $\phi_{\gamma,x}$ is inner if $\gamma x = x$;
- $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma, x \in X$.

We call these maps connecting homomorphisms, because they are reminiscent of a connection on a vector bundle. Then we can define Γ-action on $\widetilde{X}_2^\natural$ by

$$\gamma \cdot (x, \rho) = (\gamma x, \rho \circ \phi_{\gamma,x}^{-1}).$$

We form the *twisted extended quotient*

$$(X//\Gamma)_2^\natural := \widetilde{X}_2^\natural / \Gamma.$$

Notice that this reduces to the extended quotient of the second kind if $\natural(x)$ is trivial for all $x \in X$. We will apply this construction in the following two special cases.

1. Given two finite groups Γ_1, Γ and a group homomorphism $\Gamma \to \mathrm{Aut}(\Gamma_1)$, we can form the semidirect product $\Gamma_1 \rtimes \Gamma$. Let $X = \mathbf{Irr}\, \Gamma_1$. Now Γ acts on $\mathbf{Irr}\, \Gamma_1$ and we get $\natural$ as follows. Given $x \in \mathbf{Irr}\, \Gamma_1$ choose an irreducible representation $\pi_x : \Gamma_1 \to \mathrm{GL}(V)$ whose isomorphism class is x. For each $\gamma \in \Gamma$ consider π_x twisted by γ *i.e.,* consider $\gamma \cdot \pi_x : \gamma_1 \mapsto \pi_x(\gamma^{-1}\gamma_1\gamma)$. Since $\gamma \cdot \pi_x$ is equivalent to $\pi_{\gamma x}$, there exists a nonzero intertwining operator

$$T_{\gamma,x} \in \mathrm{Hom}_{\Gamma_x}(\gamma \cdot \pi_x, \pi_{\gamma x}).$$

By Schur's lemma it is unique up to scalars, but in general there is no preferred choice. For $\gamma, \gamma' \in \Gamma_x$ there exists a unique $c \in \mathbb{C}^\times$ such that

$$T_{\gamma,x} \circ T_{\gamma',x} = c T_{\gamma\gamma',x}.$$

We define the 2-cocycle by $\natural(x)(\gamma, \gamma') = c$. Let $N_{\gamma,x}$ with $\gamma \in \Gamma_x$ be the standard basis of $\mathbb{C}[\Gamma_x, \natural(x)]$. The algebra homomorphism $\phi_{g,x}$ is essentially conjugation by $T_{g,x}$, but the precise definition is

$$\phi_{g,x}(N_{\gamma,x}) = \lambda N_{g\gamma g^{-1},gx} \quad \text{if} \quad T_{g,x} T_{\gamma,x} T_{g,x}^{-1} = \lambda T_{g\gamma g^{-1},gx}, \lambda \in \mathbb{C}^\times. \tag{3}$$

Notice that (3) does not depend on the choice of $T_{g,x}$. This leads to a new formulation of a classical theorem of Clifford.

Lemma 2.1. *There is a bijection*

$$\mathbf{Irr}(\Gamma_1 \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\, \Gamma_1 // \Gamma)_2^\natural.$$

Proof. The proof proceeds by comparing our construction with the classical theory of Clifford; for an exposition of Clifford theory, see [RaRa]. □

The above bijection is in general not canonical, it depends on the choice of the intertwining operators $T_{\gamma,x}$.

Lemma 2.2. *If* Γ_1 *is abelian, then we have a natural bijection*

$$\mathbf{Irr}(\Gamma_1 \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\,\Gamma_1 /\!/ \Gamma)_2.$$

Proof. The irreducible representations of Γ_1 are 1-dimensional, and we have $\gamma \cdot \pi_x = \pi_x$ for $\gamma \in \Gamma_x$. In that case we take each $T_{\gamma,x}$ to be the identity, so that $\natural(x)$ is trivial. Then the projective representations of Γ_x which occur in the construction are all true representations and (3) simplifies to $\phi_{g,x}(T_{\gamma,x}) = T_{g\gamma g^{-1},gx}$. Thus we recover the extended quotient of the second kind in Lemma 2.1. $\square$

2. Given a $\mathbb{C}$-algebra R, a finite group Γ and a group homomorphism $\Gamma \to \mathrm{Aut}(R)$, we can form the crossed product algebra

$$R \rtimes \Gamma := \{\sum_{\gamma\in\Gamma} r_\gamma \gamma \,:\, r_\gamma \in R\},$$

with multiplication given by the distributive law and the relation

$$\gamma r = \gamma(r)\gamma, \quad \text{for } \gamma \in \Gamma \text{ and } r \in R.$$

Now Γ acts on $X := \mathbf{Irr}\,R$. Assuming that all simple R-modules have countable dimension, so that Schur's lemma is valid, we construct $\natural(V)$ and $\phi_{\gamma,V}$ as above for group algebras. Here we have

$$\widetilde{X}_2^\natural = \{(V,\tau) \,:\, V \in \mathbf{Irr}\,R,\ \tau \in \mathbf{Irr}\,\mathbb{C}[\Gamma_V, \natural(V)]\}.$$

Lemma 2.3. *There is a bijection*

$$\mathbf{Irr}(R \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\,R /\!/ \Gamma)_2^\natural.$$

If all simple R-modules are one-dimensional, then it becomes a natural bijection

$$\mathbf{Irr}(R \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\,R /\!/ \Gamma)_2.$$

Proof. The proof proceeds by comparing our construction with the theory of Clifford as stated in [RaRa, Theorem A.6]. The naturality part can be shown in the same way as Lemma 2.2. $\square$

Notation 2.4. For (V,τ) as above, $V \otimes V_\tau$ is a simple $R \rtimes \Gamma_V$-module, in a way which depends on the choice of intertwining operators $T_{\gamma,V}$. The simple $R \rtimes \Gamma$-module associated to (V,τ) by the bijection of Lemma 2.3 is

$$V \rtimes \tau := \mathrm{Ind}_{R\rtimes\Gamma_V}^{R\rtimes V}(V \otimes V_\tau). \tag{4}$$

Similarly, we shall denote by $\tau_1 \rtimes \tau$ the element of $\mathbf{Irr}(\Gamma_1 \rtimes \Gamma)$ which corresponds to (τ_1,τ) by the bijection of Lemma 2.1.

3. Weyl groups of disconnected groups

Let M be a reductive complex algebraic group. Then M may have a finite number of connected components, M^0 is the identity component of M, and $\mathcal{W}^{M^0}$ is the Weyl group of M^0:

$$\mathcal{W}^{M^0} := \mathrm{N}_{M^0}(T)/T$$

where T is a maximal torus of M^0. We will need the analogue of the Weyl group for the possibly disconnected group M.

Lemma 3.1. *Let M, M^0, T be as defined above. Then we have*

$$\mathrm{N}_M(T)/T \cong \mathcal{W}^{M^0} \rtimes \pi_0(M).$$

Proof. The group $\mathcal{W}^{M^0}$ is a normal subgroup of $\mathrm{N}_M(T)/T$. Indeed, let $n \in \mathrm{N}_{M^0}(T)$ and let $n' \in \mathrm{N}_M(T)$, then $n'nn'^{-1}$ belongs to M^0 (since the latter is normal in M) and normalizes T, that is, $n'nn'^{-1} \in \mathrm{N}_{M^0}(T)$. On the other hand, $n'(nT)n'^{-1} = n'nn'^{-1}(n'Tn'^{-1}) = n'nn'^{-1}T$.

Let B be a Borel subgroup of M^0 containing T. Let $w \in \mathrm{N}_M(T)/T$. Then wBw^{-1} is a Borel subgroup of M^0 (since, by definition, the Borel subgroups of an algebraic group are the maximal closed connected solvable subgroups). Moreover, wBw^{-1} contains T. In a connected reductive algebraic group, the intersection of two Borel subgroups always contains a maximal torus and the two Borel subgroups are conjugate by a element of the normalizer of that torus. Hence B and wBw^{-1} are conjugate by an element w_1 of $\mathcal{W}^{M^0}$. It follows that $w_1^{-1}w$ normalises B. Hence

$$w_1^{-1}w \in \mathrm{N}_M(T)/T \cap \mathrm{N}_M(B) = \mathrm{N}_M(T,B)/T,$$

that is,

$$\mathrm{N}_M(T)/T = \mathcal{W}^{M^0} \cdot (\mathrm{N}_M(T,B)/T).$$

Finally, we have

$$\mathcal{W}^{M^0} \cap (\mathrm{N}_M(T,B)/T) = \mathrm{N}_{M^0}(T,B)/T = \{1\},$$

since $\mathrm{N}_{M^0}(B) = B$ and $B \cap \mathrm{N}_{M^0}(T) = T$. This proves that

$$\mathrm{N}_M(T) \cong \mathrm{N}_{M^\circ}(T) \rtimes \mathrm{N}_M(B,T).$$

Now consider the following map:

$$\mathrm{N}_M(T,B)/T \to M/M^0 \qquad mT \mapsto mM^0. \tag{5}$$

It is injective. Indeed, let $m, m' \in \mathrm{N}_M(T,B)$ such that $mM^0 = m'M^0$. Then $m^{-1}m' \in M^0 \cap \mathrm{N}_M(T,B) = \mathrm{N}_{M^0}(T,B) = T$ (as we have seen above). Hence $mT = m'T$.

On the other hand, let m be an element in M. Then $m^{-1}Bm$ is a Borel subgroup of M^0, hence there exists $m_1 \in M^0$ such that $m^{-1}Bm = m_1^{-1}Bm_1$. It follows that $m_1m^{-1} \in \mathrm{N}_M(B)$. Also $m_1m^{-1}Tmm_1^{-1}$ is a torus of M^0 which is contained in $m_1m^{-1}Bmm_1^{-1} = B$. Hence T and $m_1m^{-1}Tmm_1^{-1}$ are conjugate in B: there is $b \in B$ such that $m_1m^{-1}Tmm_1^{-1} = b^{-1}Tb$. Then $n := bm_1m^{-1} \in \mathrm{N}_M(T,B)$. It gives $m = n^{-1}bm_1$. Since $bm_1 \in M^0$, we obtain $mM^0 = n^{-1}M^0$. Hence the map (5) is surjective. $\square$

Let G be a connected complex reductive group and let T be a maximal torus in G. The Weyl group of G is denoted $\mathcal{W}^G$.

Lemma 3.2. *Let A be a subgroup of T and write $M = \mathrm{Z}_G(A)$. Then the isotropy subgroup of A in $\mathcal{W}^G$ is*

$$\mathcal{W}^G_A = \mathrm{N}_M(T)/T \cong \mathcal{W}^{M^0} \rtimes \pi_0(M).$$

In case that the group M is connected, $\mathcal{W}^G_A$ is the Weyl group of M.

Proof. Let $R(G,T)$ denote the root system of G. According to [SpSt, § 4.1], the group $M = \mathrm{Z}_G(A)$ is the reductive subgroup of G generated by T and those root groups U_α for which $\alpha \in R(G,T)$ has trivial restriction to A together with those Weyl group representatives $n_w \in \mathrm{N}_G(T)$ $(w \in \mathcal{W}^G)$ for which $w(t) = t$ for all $t \in A$. This shows that $\mathcal{W}^G_A = \mathrm{N}_M(T)/T$, which by Lemma 3.1 is isomorphic to $\mathcal{W}^{M^0} \rtimes \pi_0(M)$.

Also by [SpSt, § 4.1], the identity component of M is generated by T and those root groups U_α for which α has trivial restriction to A. Hence the Weyl group $\mathcal{W}^{M^\circ}$ is the normal subgroup of $\mathcal{W}^G_A$ generated by those reflections s_α and

$$\mathcal{W}^G_A / \mathcal{W}^{M^\circ} \cong M/M^\circ .$$

In particular, if M is connected then $\mathcal{W}^G_A$ is the Weyl group of M. $\square$

Consequently, for $t \in T$ such that $M = \mathrm{Z}_G(t)$ we have

$$(T/\!/\mathcal{W}^G)_2 = \{(t,\sigma) : t \in T, \sigma \in \mathbf{Irr}(\mathcal{W}^G_t)\}/\mathcal{W}^G, \tag{6}$$

$$\mathbf{Irr}\, \mathcal{W}^G_t = (\mathbf{Irr}\, \mathcal{W}^{M^0} /\!/ \pi_0(M))^\natural_2 . \tag{7}$$

We fix a Borel subgroup B_0 of M° containing T and let $\Delta(B_0,T)$ be the set of roots of (M°,T) that are simple with respect to B_0. We may and will assume that this agrees with the previously chosen simple reflections in $\mathcal{W}^{M^\circ}$. In every root subgroup U_α with $\alpha \in \Delta(B_0,T)$ we pick a nontrivial element u_α. The data $(M^\circ, T, (u_\alpha)_{\alpha \in \Delta(B_0,T)})$ are called a *pinning* of M°. This notion is useful in the following well-known result:

Lemma 3.3. *The short exact sequence*

$$1 \to M^\circ/\mathrm{Z}(M^\circ) \to M/\mathrm{Z}(M^\circ) \to \pi_0(M) \to 1$$

is split. A splitting can be obtained by sending $C \in \pi_0(M)$ to the unique element of $C/\mathrm{Z}(M^\circ) \subset M/\mathrm{Z}(M^\circ)$ that preserves the chosen pinning.

Proof. The connected reductive group M° acts transitively on the set of pairs (B',T') with B' a Borel subgroup containing a maximal torus T'. Since the different simple roots are independent functions on T, M° also acts transitively on the set of pinnings. The stabilizer of a given pinning is $\mathrm{Z}(M^\circ)$, so $M^\circ/\mathrm{Z}(M^\circ)$ acts simply transitively on the set of pinnings for M°. This shows that the given recipe is valid and produces a splitting. $\square$

4. An extended Springer correspondence

Let M° be a connected reductive complex group. We take $x \in M^\circ$ unipotent and we abbreviate

$$A_x := \pi_0(\mathrm{Z}_{M^0}(x)). \tag{8}$$

Let $x \in M^\circ$ be unipotent, $\mathcal{B}^x = \mathcal{B}^x_{M^\circ}$ the variety of Borel subgroups of M° containing x. All the irreducible components of $\mathcal{B}^x$ have the same dimension $d(x)$ over $\mathbb{R}$, see [ChGi, Corollary 3.3.24]. Let $H_{d(x)}(\mathcal{B}^x,\mathbb{C})$ be its top homology, let ρ be an irreducible representation of A_x and write

$$\tau(x,\rho) = \mathrm{Hom}_{A_x}\big(\rho, H_{d(x)}(\mathcal{B}^x,\mathbb{C})\big). \tag{9}$$

We call $\rho \in \mathbf{Irr}(A_x)$ geometric if $\tau(x,\rho) \neq 0$. The Springer correspondence yields a bijection

$$(x,\rho) \mapsto \tau(x,\rho) \tag{10}$$

between the set of M^0-conjugacy classes of pairs (x,ρ) formed by a unipotent element $x \in M^0$ and an irreducible geometric representation ρ of A_x, and the equivalence classes of irreducible representations of the Weyl group $\mathcal{W}^{M^0}$.

Remark 4.1. *The Springer correspondence which we employ here sends the trivial unipotent class to the trivial $\mathcal{W}^{M^\circ}$-representation and the regular unipotent class to the sign representation. It coincides with the correspondence constructed by Lusztig by means of intersection cohomology. The difference with Springer's construction via a reductive group over a field of positive characteristic consists of tensoring with the sign representation of $\mathcal{W}^{M^0}$, see* [Hot].

Choose a set of simple reflections for $\mathcal{W}^{M^\circ}$ and let Γ be a group of automorphisms of the Coxeter diagram of W. Then Γ acts on $\mathcal{W}^{M^\circ}$ by group automorphisms, so we can form the semidirect product $\mathcal{W}^{M^\circ} \rtimes \Gamma$. Furthermore Γ acts on $\mathbf{Irr}(\mathcal{W}^{M^\circ})$, by $\gamma \cdot \tau = \tau \circ \gamma^{-1}$. The stabilizer of $\tau \in \mathbf{Irr}(\mathcal{W}^{M^\circ})$ is denoted Γ_τ. As described in Section 2, Clifford theory for $\mathcal{W}^{M^\circ} \rtimes \Gamma$ produces a 2-cocycle $\natural(\tau) : \Gamma_\tau \times \Gamma_\tau \to \mathbb{C}^\times$.

By Lemma 3.3 The action of $\gamma \in \Gamma$ on the Coxeter diagram of $\mathcal{W}^{M^\circ}$ lifts uniquely to an action of γ on M° which preserves the pinning chosen in Section 3. In this way we construct the semidirect product $M := M^\circ \rtimes \Gamma$. By Lemma 3.2 we may identify $\mathcal{W}^M$ with $\mathcal{W}^{M^\circ} \rtimes \Gamma$. We want to generalize the Springer correspondence to this kind of group. First we need to prove a technical lemma, which in a sense extension of Lemma 3.3.

Lemma 4.2. *Let $\rho \in \mathbf{Irr}(\pi_0(\mathrm{Z}_{M^\circ}(x)))$ and write*

$$\mathrm{Z}_M(x,\rho) = \{m \in \mathrm{Z}_M(x) | \rho \circ \mathrm{Ad}_m^{-1} \cong \rho\}.$$

The following short exact sequence splits:

$$1 \to \pi_0\big(\mathrm{Z}_{M^\circ}(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \Gamma_{[x,\rho]_{M^\circ}} \to 1.$$

Proof. First we ignore ρ. According to the classification of unipotent orbits in complex reductive groups [Car, Theorem 5.9.6] we may assume that x is distinguished unipotent in a Levi subgroup $L \subset M^\circ$ that contains T. Notice that the derived subgroup $\mathcal{D}(L)$ contains only the part of T generated by the coroots of (L,T). The roots of

$$L' := \mathrm{Z}_{M^\circ}(\mathcal{D}(L))(T \cap \mathcal{D}(L)) = \mathrm{Z}_{M^\circ}(\mathcal{D}(L))T.$$

are precisely those that are orthogonal to the coroots of (L,T). We choose Borel subgroups $B_L \subset L$ and $B'_L \subset L'$ such that $x \in B_L$ and $T \subset B_L \cap B'_L$.

Let $[x]_{M^\circ}$ be the M°-conjugacy class of x and $\Gamma_{[x]_{M^\circ}}$ its stabilizer in Γ. Any $\gamma \in \Gamma_{[x]_{M^\circ}}$ must also stabilize the M°-conjugacy class of L, and $T = \gamma(T) \subset \gamma(L)$, so there exists a $w_1 \in \mathcal{W}^{M^\circ}$ with $w_1\gamma(L) = L$. Adjusting w_1 by an element of $W(L,T) \subset \mathcal{W}^{M^\circ}$, we can achieve that moreover $w_1\gamma(B_L) = B_L$. Then $w_1\gamma(L') = L'$, so we can find a unique $w_2 \in W(L',T) \subset \mathcal{W}^{M^\circ}$ with $w_2 w_1 \gamma(B'_L) = B'_L$. Notice that the centralizer of $\Phi(B_L,T) \cup \Phi(B'_L,T)$ in $\mathcal{W}^{M^\circ}$ is trivial, because it is generated by reflections and no root in

$\Phi(M^\circ, T)$ is orthogonal to this set of roots. Therefore the above conditions completely determine $w_2 w_1 \in \mathcal{W}^{M^\circ}$.

The element $w_1 \gamma \in \mathcal{W}^{M^\circ} \rtimes \Gamma$ acts on $\Delta(B_L, T)$ by a diagram automorphism. So upon choosing $u_\alpha \in U_\alpha \setminus \{1\}$ for $\alpha \in \Delta(B_L, T)$, Lemma 3.3 shows that $w_1 \gamma$ can be represented by a unique element

$$\overline{w_1 \gamma} \in \mathrm{Aut}\big(\mathcal{D}(L), T, (u_\alpha)_{\alpha \in \Delta(B_L,T)}\big).$$

The distinguished unipotent class of $x \in L$ is determined by its Bala–Carter diagram. The classification of such diagrams [Car, §5.9] shows that there exists an element $\bar{x}$ in the same class as x, such that $\mathrm{Ad}_{\overline{w_1 \gamma}}(\bar{x}) = \bar{x}$. We may just as well assume that we had $\bar{x}$ instead of x from the start, and that $\overline{w_1 \gamma} \in \mathrm{Z}_M(x)$. Clearly we can find a representative $\overline{w_2}$ for w_2 in $\mathrm{Z}_M(x)$, so we obtain

$$\overline{w_2}\, \overline{w_1 \gamma} \in \mathrm{Z}_M(x) \cap \mathrm{N}_M(T) \quad \text{and} \quad w_2 w_1 \gamma \in \frac{\mathrm{Z}_M(x) \cap \mathrm{N}_M(T)}{\mathrm{Z}(M^\circ)\, T}.$$

Since $w_2 w_1 \in \mathcal{W}^{M^\circ}$ is unique,

$$s : \Gamma_{[x]_{M^\circ}} \to \frac{\mathrm{Z}_M(x) \cap \mathrm{N}_M(T)}{\mathrm{Z}(M^\circ)\, T}, \; \gamma \mapsto w_2 w_1 \gamma \tag{11}$$

is a group homomorphism.

We still have to analyse the effect of $\Gamma_{[x]_{M^\circ}}$ on $\rho \in \mathbf{Irr}(A_x)$. Obviously composing with Ad_m for $m \in \mathrm{Z}_{M^\circ}(x)$ does not change the equivalence class of any representation of $A_x = \pi_0(\mathrm{Z}_{M^\circ}(x))$. Hence $\gamma \in \Gamma_{[x]_{M^\circ}}$ stabilizes ρ if and only if any lift of γ in $\mathrm{Z}_M(x)$ does. This applies in particular to $\overline{w_2}\, \overline{w_1 \gamma}$, and therefore

$$s(\Gamma_{[x,\rho]_{M^\circ}}) \subset \big(\mathrm{Z}_M(x,\rho) \cap \mathrm{N}_M(T)\big) / \big(\mathrm{Z}(M^\circ)\, T\big).$$

Since the torus T is connected, s determines a group homomorphism from $\Gamma_{[x,\rho]_{M^\circ}}$ to $\pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big)$, which is the required splitting. $\square$

A further step towards a Springer correspondence for $\mathcal{W}^M$ is:

Proposition 4.3. *The class of $\natural(\tau)$ in $H^2(\Gamma_\tau, \mathbb{C}^\times)$ is trivial for all $\tau \in \mathbf{Irr}(\mathcal{W}^{M^\circ})$. There is a bijection between*

$$\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})/\!/\Gamma\big)_2 \qquad \textit{and} \qquad \mathbf{Irr}(\mathcal{W}^{M^\circ} \rtimes \Gamma) = \mathbf{Irr}(\mathcal{W}^M).$$

Proof. There are various ways to construct the Springer correspondence for $\mathcal{W}^{M^\circ}$, for the current proof we use the method with Borel–Moore homology. Let Z_{M° be the Steinberg variety of M° and $H_{\mathrm{top}}(\mathrm{Z}_{M^\circ})$ its homology in the top degree

$$2 \dim_{\mathbb{C}} \mathrm{Z}_{M^\circ} = 4 \dim_{\mathbb{C}} \mathcal{B}_{M^\circ} = 4(\dim_{\mathbb{C}} M^\circ - \dim_{\mathbb{C}} B_0),$$

with rational coefficients. We define a natural algebra isomorphism

$$\mathbb{Q}[\mathcal{W}^{M^\circ}] \to H_{\mathrm{top}}(\mathrm{Z}_{M^\circ}) \tag{12}$$

as the composition of [ChGi, Theorem 3.4.1] and a twist by the sign representation of $\mathbb{Q}[\mathcal{W}^{M^\circ}]$. By [ChGi, Section 3.5] the action of $\mathcal{W}^{M^\circ}$ on $H_*(\mathcal{B}^x, \mathbb{C})$ (as defined by Lusztig) corresponds to the convolution product in Borel–Moore homology.

Since M° is normal in M, the groups Γ, M and $M/\mathrm{Z}(M)$ act on the Steinberg variety Z_{M° via conjugation. The induced action of the connected group M° on $H_{\mathrm{top}}(\mathrm{Z}_{M^\circ})$ is trivial, and it easily seen from [ChGi, Section 3.4] that the action of Γ on $H(\mathrm{Z}_{M^\circ})$ makes (12) Γ-equivariant.

The groups Γ, M and $M/\mathrm{Z}(M)$ also act on the pairs (x,ρ) and on the varieties of Borel subgroups, by

$$\begin{aligned} \mathrm{Ad}_m(x,\rho) &= (mxm^{-1}, \rho \circ \mathrm{Ad}_m^{-1}), \\ \mathrm{Ad}_m &: \mathcal{B}^x \to \mathcal{B}^{mxm^{-1}},\ B \mapsto mBm^{-1}. \end{aligned}$$

Given $m \in M$, this provides a linear bijection $H_*(\mathrm{Ad}_m)$:

$$\mathrm{Hom}_{A_x}(\rho, H_*(\mathcal{B}^x,\mathbb{C})) \to \mathrm{Hom}_{A_{mxm^{-1}}}(\rho \circ \mathrm{Ad}_m^{-1}, H_*(\mathcal{B}^{mxm^{-1}},\mathbb{C})).$$

The convolution product in Borel–Moore homology is compatible with these M-actions so, as in [ChGi, Lemma 3.5.2], the following diagram commutes for all $h \in H_{\mathrm{top}}(\mathrm{Z}_{M^\circ})$:

$$\begin{tikzcd} H_*(\mathcal{B}^x,\mathbb{C}) \arrow[r, "h"] \arrow[d, "H_*(\mathrm{Ad}_m)"] & H_*(\mathcal{B}^x,\mathbb{C}) \arrow[d, "H_*(\mathrm{Ad}_m)"] \\ H_*(\mathcal{B}^{mxm^{-1}},\mathbb{C}) \arrow[r, "m\cdot h"] & H_*(\mathcal{B}^{mxm^{-1}},\mathbb{C}). \end{tikzcd} \tag{13}$$

In case $m \in M^\circ\gamma$ and $m \cdot h$ corresponds to $w \in \mathcal{W}^{M^\circ}$, the element $h \in H(\mathrm{Z}_{M^\circ})$ corresponds to $\gamma^{-1}(w)$, so (13) becomes

$$H_*(\mathrm{Ad}_m) \circ \tau(x,\rho)(\gamma^{-1}(w)) = \tau(mxm^{-1}, \rho \circ \mathrm{Ad}_m^{-1})(w) \circ H_*(\mathrm{Ad}_m). \tag{14}$$

Denoting the M°-conjugacy class of (x,ρ) by $[x,\rho]_{M^\circ}$, we can write

$$\begin{aligned} \Gamma_{\tau(x,\rho)} &= \{\gamma \in \Gamma \mid \tau(x,\rho) \circ \gamma^{-1} \cong \tau(x,\rho)\} \\ &= \{\gamma \in \Gamma \mid [\mathrm{Ad}_\gamma(x,\rho)]_{M^\circ} = [x,\rho]_{M^\circ}\} =: \Gamma_{[x,\rho]_{M^\circ}}. \end{aligned} \tag{15}$$

This group fits in an exact sequence

$$1 \to \pi_0\big(\mathrm{Z}_{M^\circ}(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \Gamma_{[x,\rho]_{M^\circ}} \to 1, \tag{16}$$

which by Lemma 4.2 admits a splitting

$$s : \Gamma_{[x,\rho]_{M^\circ}} \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big).$$

By homotopy invariance in Borel–Moore homology $H_*(\mathrm{Ad}_z) = \mathrm{id}_{H_*(\mathcal{B}^x,\mathbb{C})}$ for any $z \in \mathrm{Z}_{M^\circ}(x,\rho)^\circ \mathrm{Z}(M^\circ)$, so $H_*(\mathrm{Ad}_m)$ is well-defined for $m \in \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big)$. In particular we obtain for every $\gamma \in \Gamma_{\tau(x,\rho)} = \Gamma_{[x,\rho]_{M^\circ}}$ a linear bijection

$$H_*(\mathrm{Ad}_{s(\gamma)}) : \mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}_x,\mathbb{C})) \to \mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}_x,\mathbb{C})),$$

which by (14) intertwines the $\mathcal{W}^{M^\circ}$-representations $\tau(x,\rho)$ and $\tau(x,\rho)\circ\gamma^{-1}$. By construction

$$H_*(\mathrm{Ad}_{s(\gamma)}) \circ H_*(\mathrm{Ad}_{s(\gamma')}) = H_*(\mathrm{Ad}_{s(\gamma\gamma')}). \tag{17}$$

This establishes the triviality of the 2-cocycle $\natural(\tau) = \natural(\tau(x,\rho))$.

Consider any $g \in \Gamma \setminus \Gamma_x$. Then $g\tau$ corresponds to

$$\mathrm{Ad}_g(x,\rho) = (gxg^{-1}, \rho \circ \mathrm{Ad}_g^{-1}).$$

For $\gamma \in \Gamma_x$ we define an intertwining operator in

$$\mathrm{End}_{\mathcal{W}^{M^\circ}}\big(\mathrm{Hom}_{A_{gxg^{-1}}}(\rho \circ \mathrm{Ad}_g^{-1}, H_{d(x)}(\mathcal{B}_{gxg^{-1}},\mathbb{C}))\big)$$

associated to $g\gamma g^{-1} \in \Gamma_{gxg^{-1}}$ as

$$H_{d(x)}(\mathrm{Ad}_{gs(\gamma)g^{-1}}) = H_{d(x)}(\mathrm{Ad}_g) H_{d(x)}(\mathrm{Ad}_{s(\gamma)}) H_{d(x)}(g^{-1}). \tag{18}$$

We do the same for any other point in the Γ-orbit of (x, ρ). Then (17) shows that the resulting intertwining operators do not depend on the choices of the elements g.

We follow the same recipe for any other Γ-orbit of Springer parameters (x', ρ'). As connecting homomorphism $\phi_{g,(x',\rho')}$ we take conjugation by $H_{d(x')}(\mathrm{Ad}_g)$. From this construction and Lemma 2.3 we obtain a bijection between $\mathbf{Irr}(\mathcal{W}^{M^\circ} \rtimes \pi_0(M))$ and the extended quotient of the second kind $\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})//\Gamma\big)_2$. $\square$

We note that the bijection from Proposition 4.3 is in general not canonical, because the splitting from Lemma 4.2 is not. But with some additional effort we can extract a natural description of $\mathbf{Irr}(\mathcal{W}^M)$ from Proposition 4.3.

We say that an irreducible representation ρ_1 of $\mathrm{Z}_M(x)$ is geometric if every irreducible $\mathrm{Z}_{M^\circ}(x)$-subrepresentation of ρ_1 is geometric in the previously defined sense. Notice that this condition forces ρ_1 to factor through the component group $\pi_0(\mathrm{Z}_M(x))$.

We note that $\pi_0(\mathrm{Z}_M(x))$ acts naturally on $H_{d(x)}(\mathcal{B}^x)$ and on $\mathbb{C}[\Gamma]$, via the isomorphism

$$\mathrm{Z}_M(x)/\mathrm{Z}_{M^\circ}(x) \cong \Gamma_{[x]_{M^\circ}}. \tag{19}$$

Theorem 4.4. *There is a natural bijection from*

$$\big\{(x, \rho_1) \mid x \in M^\circ \textit{ unipotent}, \rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big) \textit{ geometric}\big\}/M$$

to $\mathbf{Irr}(\mathcal{W}^M)$*, which sends* (x, ρ_1) *to*

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_M(x))}\big(\rho_1, H_{d(x)}(\mathcal{B}^x) \otimes \mathbb{C}[\Gamma]\big).$$

Proof. Let us take another look at the geometric representations of $A_x = \mathrm{Z}_{M^\circ}(x)$. By construction they factor through $\pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ))$. From (11) we get a group isomorphism

$$\pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(M^\circ)) \cong \pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ)) \rtimes s\big(\Gamma_{[x]_{M^\circ}}\big). \tag{20}$$

Suppose that $\rho \in \mathbf{Irr}(A_x)$ is geometric. Then the operators $H_{d(x)}(\mathrm{Ad}_{s(\gamma)})$ intertwine ρ with the $\pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ))$-representation $s(\gamma) \cdot \rho$ and they satisfy the multiplicativity relation (17). Now it follows from Lemma 2.1 that every irreducible geometric representation of $\pi_0(\mathrm{Z}_M(x))$ can be written in a unique way as $\rho \rtimes \sigma$, with $\rho \in \mathbf{Irr}(A_x)$ geometric and

$$\sigma \in \mathbf{Irr} s(\Gamma_{[x,\rho]_{M^\circ}}) = \mathbf{Irr}(\Gamma_{[x,\rho]_{M^\circ}}).$$

This enables us to rewrite $\widetilde{\mathbf{Irr}(\mathcal{W}^{M^\circ}})$ as a union of pairs $(x, \rho_1 = \rho \rtimes \sigma)$, with x in a finite union of chosen Γ-orbits of unipotent elements. Clearly M acts on the larger space

$$\big\{(x, \rho_1) \mid x \in M^\circ \text{ unipotent}, \rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big) \text{ geometric}\big\}$$

by conjugation of the x-parameter and the action induced by $H_*(\mathrm{Ad}_m)$ on the ρ_1-parameter. By (18) and the construction of $s(\gamma)$ in Lemma 4.2, this extends the action of Γ on $\widetilde{\mathbf{Irr}(\mathcal{W}^{M^\circ}})$. That provides the bijection from

$\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})//\Gamma\big)_2$ to set of the M-association classes of pairs (x,ρ_1). Combining this with Proposition 4.3, we obtain a bijection between $\mathbf{Irr}(\mathcal{W}^M)$ and the latter set. If we work out the definitions and use (4), we see that it sends $(x,\rho_1=\rho\rtimes\sigma)$ to

$$\tau(x,\rho)\rtimes\sigma=\mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ}\rtimes\Gamma}\big(\tau(x,\rho)\otimes\sigma\big).$$

Since every irreducible complex representation of a finite group is isomorphic to its contragredient, we can rewrite this as

$$\begin{aligned}&\mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ}\rtimes\Gamma}\big(\mathrm{Hom}_{A_x}(\rho,H_{d(x)}(\mathcal{B}^x))\otimes\sigma^*\big)\cong\\&\mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ}\rtimes\Gamma}\big(\mathrm{Hom}_{\Gamma_{[x,\rho]_{M^\circ}}}\big(\sigma,\mathrm{Hom}_{A_x}(\rho,H_{d(x)}(\mathcal{B}^x))\otimes\mathbb{C}[\Gamma_{[x,\rho]_{M^\circ}}]\big)\big).\end{aligned}$$

In view of Lemma 4.2, the previous line is isomorphic to

$$\begin{aligned}&\mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ}\rtimes\Gamma}\big(\mathrm{Hom}_{\mathrm{Z}_M(x,\rho)}\big(\rho\otimes\sigma,H_{d(x)}(\mathcal{B}^x)\otimes\mathbb{C}[\Gamma_{[x,\rho]_{M^\circ}}]\big)\big)\cong\\&\mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes\Gamma_{[x]_{M^\circ}}}^{\mathcal{W}^{M^\circ}\rtimes\Gamma}\big(\mathrm{Hom}_{\mathrm{Z}_M(x,\rho)}\big(\rho\otimes\sigma,H_{d(x)}(\mathcal{B}^x)\otimes\mathbb{C}[\Gamma_{[x]_{M^\circ}}]\big)\big).\end{aligned}$$

With Frobenius reciprocity and (19) we simplify the above expression to

$$\begin{aligned}&\mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes\Gamma_{[x]_{M^\circ}}}^{\mathcal{W}^{M^\circ}\rtimes\Gamma}\big(\mathrm{Hom}_{\mathrm{Z}_M(x)}\big(\rho\rtimes\sigma,H_{d(x)}(\mathcal{B}^x)\otimes\mathbb{C}[\Gamma_{[x]_{M^\circ}}]\big)\big)\cong\\&\mathrm{Hom}_{\pi_0(\mathrm{Z}_M(x))}\big(\rho\rtimes\sigma,H_{d(x)}(\mathcal{B}^x)\otimes\mathbb{C}[\Gamma]\big).\end{aligned}$$

The last line is natural in $(x,\rho_1=\rho\rtimes\sigma)$ because the $Z_M(x)$-representation $H_{d(x)}(\mathcal{B}^x)$ depends in a natural way on x, as we observed at the start of the proof of Proposition 4.3. $\square$

There is natural partial order on the unipotent classes in M:

$$\mathcal{O}<\mathcal{O}'\quad\text{when}\quad\overline{\mathcal{O}}\subsetneq\overline{\mathcal{O}'}.$$

Let $\mathcal{O}_x\subset M$ be the class containing x. We transfer this to partial order on our extended Springer data by defining

$$(x,\rho_1)<(x',\rho_1')\quad\text{when}\quad\overline{\mathcal{O}_x}\subsetneq\overline{\mathcal{O}_{x'}}.\tag{21}$$

We will use it to formulate a property of the composition series of some $\mathcal{W}^M$-representations that will appear later on.

Lemma 4.5. *Let $x\in M$ be unipotent and let $\rho\rtimes\sigma$ be a geometric irreducible representation of $\pi_0(\mathrm{Z}_M(x))$. There exist multiplicities $m_{x,\rho\rtimes\sigma,x',\rho'\rtimes\sigma'}\in\mathbb{Z}_{\geq 0}$ such that*

$$\begin{aligned}&\mathrm{Ind}_{W\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{W\rtimes\Gamma}\big(\mathrm{Hom}_{A_x}\big(\rho,H_*(\mathcal{B}^x,\mathbb{C})\big)\otimes\sigma\big)\cong\\&\qquad\tau(x,\rho)\rtimes\sigma\oplus\bigoplus_{(x',\rho'\rtimes\sigma')>(x,\rho\rtimes\sigma)}m_{x,\rho\rtimes\sigma,x',\rho'\rtimes\sigma'}\,\tau(x',\rho')\rtimes\sigma'.\end{aligned}$$

Proof. Consider the vector space $\mathrm{Hom}_{A_x}\big(\rho,H_*(\mathcal{B}^x,\mathbb{C})\big)$ with the $\mathcal{W}^{M^\circ}$-action coming from (12). The proof of Proposition 4.3 remains valid for these representations. By [BaMo, Theorem 4.4] (attributed to Borho and MacPherson) there exist multiplicities $m_{x,\rho,x',\rho'} \in \mathbb{Z}_{\geq 0}$ such that

$$\operatorname{Hom}_{A_x}\big(\rho, H_*(\mathcal{B}^x, \mathbb{C})\big) \cong \tau(x,\rho) \oplus \bigoplus_{(x',\rho')>(x,\rho)} m_{x,\rho,x',\rho'}\, \tau(x',\rho'). \tag{22}$$

By (15) and (14) $\Gamma_{[x,\rho]_{M^\circ}}$ also stabilizes the $\tau(x',\rho')$ with $m_{x,\rho,x',\rho'} > 0$, and by Proposition 4.3 the associated 2-cocycles are trivial. It follows that

$$\begin{aligned}\operatorname{Ind}_{W \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{W \rtimes \Gamma} \big(\operatorname{Hom}_{A_x}\big(\rho, H_*(\mathcal{B}^x, \mathbb{C})\big) \otimes \sigma\big) \cong \\ \tau(x,\rho) \rtimes \sigma \oplus \bigoplus_{(x',\rho')>(x,\rho)} m_{x,\rho,x',\rho'} \operatorname{Ind}_{W \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{W \rtimes \Gamma} \big(\tau(x',\rho') \otimes \sigma\big).\end{aligned} \tag{23}$$

Decomposing the right hand side into irreducible representations then gives the statement of the lemma. ☐

5. Langlands parameters for the principal series

Let $\mathbf{W}_F$ denote the Weil group of F, let $\mathbf{I}_F$ be the inertia subgroup of $\mathbf{W}_F$. Let $\mathbf{W}_F^{\mathrm{der}}$ denote the closure of the commutator subgroup of $\mathbf{W}_F$, and write $\mathbf{W}_F^{\mathrm{ab}} = \mathbf{W}_F/\mathbf{W}_F^{\mathrm{der}}$. The group of units in $\mathfrak{o}_F$ will be denoted $\mathfrak{o}_F^\times$.

We recall the Artin reciprocity map $\mathbf{a}_F : \mathbf{W}_F \to F^\times$ which has the following properties (local class field theory):

1. The map $\mathbf{a}_F$ induces a topological isomorphism $\mathbf{W}_F^{\mathrm{ab}} \simeq F^\times$.
2. An element $x \in \mathbf{W}_F$ is a geometric Frobenius if and only if $\mathbf{a}_F(x)$ is a prime element ϖ_F of F.
3. We have $\mathbf{a}_F(\mathbf{I}_F) = \mathfrak{o}_F^\times$.

We now consider the principal series of $\mathcal{G}$. We recall that $\mathcal{G}$ denotes a connected reductive split p-adic group with maximal split torus $\mathcal{T}$, and that G, T denote the Langlands dual groups of $\mathcal{G}$, $\mathcal{T}$. Next, we consider conjugacy classes in G of continuous morphisms

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

which are rational on $\mathrm{SL}_2(\mathbb{C})$ and such that $\Phi(\mathbf{W}_F)$ consists of semisimple elements in G.

The (conjectural) local Langlands correspondence is supposed to be compatible with respect to inclusions of Levi subgroups. Therefore every Langlands parameter Φ for a principal series representation should have $\Phi(\mathbf{W}_F)$ contained in a maximal torus of G. As Φ is only determined up to G-conjugacy, it should suffice to consider Langlands parameters with $\Phi(\mathbf{W}_F) \subset T$.

In particular, for such parameters $\Phi\big|_{\mathbf{W}_F}$ factors through $\mathbf{W}_F^{ab} \cong F^\times$. We view the domain of Φ to be $F^\times \times \mathrm{SL}_2(\mathbb{C})$:

$$\Phi \colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

In this section we will build such a continuous morphism Φ from $\mathfrak{s}$ and data coming from the extended quotient of second kind. In Section 6 we show how such a Langlands parameter Φ can be enhanced with a parameter ρ.

Throughout this article, a Frobenius element Frob_F has been chosen and fixed. This determines a uniformizer ϖ_F via the equation $\mathbf{a}_F(\mathrm{Frob}_F) = \varpi_F$. That in turn gives rise to a group isomorphism $\mathfrak{o}_F^\times \times \mathbb{Z} \to F^\times$, which sends

$1 \in \mathbb{Z}$ to ϖ_F. Let $\mathcal{T}_0$ denote the maximal compact subgroup of $\mathcal{T}$. As the latter is F-split,

$$\mathcal{T} \cong F^\times \otimes_{\mathbb{Z}} X_*(\mathcal{T}) \cong (\mathfrak{o}_F^\times \times \mathbb{Z}) \otimes_{\mathbb{Z}} X_*(\mathcal{T}) = \mathcal{T}_0 \times X_*(\mathcal{T}). \tag{24}$$

Because $\mathcal{W}$ does not act on $F^\times$, these isomorphisms are $\mathcal{W}$-equivariant if we endow the right hand side with the diagonal $\mathcal{W}$-action. Thus (24) determines a $\mathcal{W}$-equivariant isomorphism of character groups

$$\mathbf{Irr}(\mathcal{T}) \cong \mathbf{Irr}(\mathcal{T}_0) \times \mathbf{Irr}(X_*(\mathcal{T})) = \mathbf{Irr}(\mathcal{T}_0) \times X_{\mathrm{unr}}(\mathcal{T}). \tag{25}$$

The way $\mathbf{Irr}(\mathcal{T}_0)$ is embedded depends on the choice of ϖ_F. However, the isomorphisms

$$\mathbf{Irr}(\mathcal{T}_0) \cong \mathrm{Hom}(\mathfrak{o}_F^\times, T), \tag{26}$$

$$X_{\mathrm{unr}}(\mathcal{T}) \cong \mathrm{Hom}(\mathbb{Z}, T) = T. \tag{27}$$

are canonical.

Lemma 5.1. *Let χ be a character of $\mathcal{T}$, and let $[\mathcal{T}, \chi]_{\mathcal{G}}$ be the inertial class of the pair $(\mathcal{T}, \chi)$. Let*

$$\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}. \tag{28}$$

Then $\mathfrak{s}$ determines, and is determined by, the $\mathcal{W}$-orbit of a smooth morphism

$$c^{\mathfrak{s}} \colon \mathfrak{o}_F^\times \to T.$$

Proof. There is a natural isomorphism

$$\begin{aligned} \mathbf{Irr}(\mathcal{T}) &= \mathrm{Hom}(F^\times \otimes_{\mathbb{Z}} X_*(\mathcal{T}), \mathbb{C}^\times) \\ &\cong \mathrm{Hom}(F^\times, \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(\mathcal{T})) = \mathrm{Hom}(F^\times, T). \end{aligned}$$

Let $\hat{\chi} \in \mathrm{Hom}(F^\times, T)$ be the image of χ under these isomorphisms. By (26) the restriction of $\hat{\chi}$ to $\mathfrak{o}_F^\times$ is not disturbed by unramified twists, so we take that as $c^{\mathfrak{s}}$. Conversely, by (25) $c^{\mathfrak{s}}$ determines χ up to unramified twists. Two elements of $\mathbf{Irr}(\mathcal{T})$ are $\mathcal{G}$-conjugate if and only if they are $\mathcal{W}$-conjugate so, in view of (25), the $\mathcal{W}$-orbit of the $c^{\mathfrak{s}}$ contains the same amount of information as $\mathfrak{s}$. $\square$

Let $H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}})$ and let $M = \mathrm{Z}_H(t)$ for some $t \in T$. Recall that a unipotent element $x \in M^0$ is said to be *distinguished* if the connected center $\mathrm{Z}_{M^0}^0$ of M^0 is a maximal torus of $\mathrm{Z}_{M^0}(x)$. Let $x \in M^0$ unipotent. If x is not distinguished, then there is a Levi subgroup L of M^0 containing x and such that $x \in L$ is distinguished.

Let $X \in \mathrm{Lie}\ M^0$ such that $\exp(X) = x$. A cocharacter $h \colon \mathbb{C}^\times \to M^0$ is said to be *associated to x* if

$$\mathrm{Ad}(h(t))X = t^2 X \quad \text{for each } t \in \mathbb{C}^\times,$$

and if the image of h lies in the derived group of some Levi subgroup L for which $x \in L$ is distinguished (see [Jan, Rem. 5.5] or [FoRo, Rem.2.12]).

A cocharacter associated to a unipotent element $x \in M^0$ is not unique. However, any two cocharacters associated to a given $x \in M^0$ are conjugate under elements of $\mathrm{Z}_{M^0}(x)^0$ (see for instance [Jan, Lem. 5.3]).

We work with the Jacobson–Morozov theorem [ChGi, p. 183]. Let $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ be the standard unipotent matrix in $\mathrm{SL}_2(\mathbb{C})$ and let x be a unipotent element in M^0. There exist rational homomorphisms

$$\gamma\colon \mathrm{SL}_2(\mathbb{C}) \to M^0 \quad \text{with} \quad \gamma\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = x, \tag{29}$$

see [ChGi, §3.7.4]. Any two such homomorphisms γ are conjugate by elements of $\mathrm{Z}_{M^\circ}(x)$.

For $\alpha \in \mathbb{C}^\times$ we define the following matrix in $\mathrm{SL}_2(\mathbb{C})$:

$$Y_\alpha = \left(\begin{smallmatrix}\alpha&0\\0&\alpha^{-1}\end{smallmatrix}\right).$$

Then each γ as above determines a cocharacter $h\colon \mathbb{C}^\times \to M^0$ by setting

$$h(\alpha) := \gamma(Y_\alpha) \quad \text{for } \alpha \in \mathbb{C}^\times. \tag{30}$$

Each cocharacter h obtained in this way is associated to x, see [Jan, Rem. 5.5] or [FoRo, Rem.2.12]. Hence each two such cocharacters are conjugate under $\mathrm{Z}_{M^0}(x)^0$.

We set $\Phi(\varpi_F) = t \in T$. Define the Langlands parameter Φ as follows:

$$\Phi\colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G, \qquad (u\varpi_F^n, Y) \mapsto c^{\mathfrak{s}}(u) \cdot t^n \cdot \gamma(Y) \tag{31}$$

for all $u \in \mathfrak{o}_F^\times$, $n \in \mathbb{Z}$, $Y \in \mathrm{SL}_2(\mathbb{C})$.

Note that the definition of Φ uses the appropriate data: the semisimple element $t \in T$, the map $c^{\mathfrak{s}}$, and the homomorphism γ (which depends on the Springer parameter x).

Since x determines γ up to M°-conjugation, $c^{\mathfrak{s}}, x$ and t determine Φ up to conjugation by their common centralizer in G. Notice also that one can recover $c^{\mathfrak{s}}, x$ and t from Φ and that

$$h(\alpha) = \Phi(1, Y_\alpha). \tag{32}$$

6. Varieties of Borel subgroups

We clarify some issues with different varieties of Borel subgroups and different kinds of parameters arising from them. Let G be a connected reductive complex group and let

$$\Phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

be as in (31). We write

$$\begin{aligned} H &= \mathrm{Z}_G(\Phi(\mathbf{I}_F)) = \mathrm{Z}_G(\operatorname{im} c^{\mathfrak{s}}), \\ M &= \mathrm{Z}_G(\Phi(\mathbf{W}_F)) = \mathrm{Z}_H(t). \end{aligned}$$

Although both H and M are in general disconnected, $\Phi(\mathbf{W}_F)$ is always contained in H° because it lies in the maximal torus T of G and H°. Hence $\Phi(\mathbf{I}_F) \subset \mathrm{Z}(H^\circ)$.

By construction t commutes with $\Phi(\mathrm{SL}_2(\mathbb{C})) \subset M$. For any $q^{1/2} \in \mathbb{C}^\times$ the element

$$t_q := t\Phi\big(Y_{q^{1/2}}\big) \tag{33}$$

satisfies the familiar relation $t_q x t_q^{-1} = x^q$. Indeed

$$\begin{aligned} t_q x t_q^{-1} &= t\Phi(Y_{q^{1/2}})\Phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\Phi(Y_{q^{1/2}}^{-1})t^{-1} \\ &= t\Phi\big(Y_{q^{1/2}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)Y_{q^{1/2}}^{-1}\big)t^{-1} \\ &= t\Phi\left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)t^{-1} = x^q. \end{aligned} \tag{34}$$

Recall that B_2 denotes the upper triangular Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$. In the flag variety of M° we have the subvarieties $\mathcal{B}^x_{M^\circ}$ and $\mathcal{B}^{\Phi(B_2)}_{M^\circ}$ of Borel subgroups containing x and $\Phi(B_2)$, respectively. Similarly the flag variety of H° has subvarieties $\mathcal{B}^{t,x}_{H^\circ}$, $\mathcal{B}^{t_q,x}_{H^\circ}$ and

$$\mathcal{B}^{t,\Phi(B_2)}_{H^\circ} = \mathcal{B}^{t_q,\Phi(B_2)}_{H^\circ}.$$

Notice that $\Phi(\mathbf{I}_F)$ lies in every Borel subgroup of H°, because it is contained in $\mathrm{Z}(H^\circ)$. We abbreviate $\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})))$ and similarly for other groups.

Proposition 6.1. (1) *The inclusion maps*

$$\begin{array}{ccccccc} & & \mathrm{Z}_{M^\circ}(\Phi) & \to & \mathrm{Z}_{M^\circ}(\Phi(B_2)) & \to & \mathrm{Z}_{M^\circ}(x), \\ \mathrm{Z}_H(t_q,x) & \leftarrow & \mathrm{Z}_H(\Phi) & \to & \mathrm{Z}_H(t,\Phi(B_2)) & \to & \mathrm{Z}_H(t,x), \end{array}$$

are homotopy equivalences. In particular they induce isomorphisms between the respective component groups.

(2) *The inclusions* $\mathcal{B}^{\Phi(B_2)}_{M^\circ} \to \mathcal{B}^x_{M^\circ}$ *and* $\mathcal{B}^{t_q,x}_{H^\circ} \leftarrow \mathcal{B}^{t,\Phi(B_2)}_{H^\circ} \to \mathcal{B}^{t,x}_{H^\circ}$ *are homotopy equivalences.*

Proof. It suffices to consider the statements for H and t_q, since the others can be proven in the same way.

(1) Our proof uses some elementary observations from [Ree2, §4.3]. There is a Levi decomposition

$$\mathrm{Z}_{H^\circ}(x) = \mathrm{Z}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C})))U_x$$

with $\mathrm{Z}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C}))) = \mathrm{Z}_{H^\circ}(\Phi(B_2))$ reductive and U_x unipotent. Since $t_q \in \mathrm{N}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C})))$ and $\mathrm{Z}_H(x^q) = \mathrm{Z}_H(x)$, conjugation by t_q preserves this decomposition. Therefore

$$\mathrm{Z}_{H^\circ}(t_q,x) = \mathrm{Z}_{H^\circ}(\Phi)\mathrm{Z}_{U_x}(t_q) = \mathrm{Z}_{H^\circ}(t_q,\Phi(B_2))\mathrm{Z}_{U_x}(t_q). \tag{35}$$

We note that

$$\mathrm{Z}_{U_x}(t_q) \cap \mathrm{Z}_{H^\circ}(t_q,\Phi(B_2)) \subset U_x \cap \mathrm{Z}_{H^\circ}(\Phi(B_2)) = 1$$

and that $\mathrm{Z}_{U_x}(t_q) \subset U_x$ is contractible, because it is a unipotent complex group. It follows that

$$\mathrm{Z}_{H^\circ}(\Phi) = \mathrm{Z}_{H^\circ}(t_q,\Phi(B_2)) \to \mathrm{Z}_{H^\circ}(t_q,x) \tag{36}$$

is a homotopy equivalence. If we want to replace H° by H, we find

$$\mathrm{Z}_H(\Phi)/\mathrm{Z}_{H^\circ}(\Phi) = \{hH^\circ \in \pi_0(H) \mid h\Phi h^{-1} \in \mathrm{Ad}(H^\circ)\Phi\},$$

and similarly with $(t_q,\Phi(B_2))$ or (t_q,x) instead of Φ.

Let us have a closer look at the H°-conjugacy classes of these objects. Given any Φ, we obviously know what t_q and x are. Conversely, suppose that t_q and x are given. We apply a refinement of the Jacobson–Morozov theorem due to Kazhdan and Lusztig. According to [KaLu, §2.3] there exist homomorphisms $\Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$ as above, which return t_q and x in the prescribed way. Moreover all such homomorphisms are conjugate under $\mathrm{Z}_{H^\circ}(t_q, x)$, see [KaLu, §2.3.h] or Section 19. So from (t_q, x) we can reconstruct the $\mathrm{Ad}(H^\circ)$-orbit of Φ, and this gives bijections between H°-conjugacy classes of Φ, $(t_q, \Phi(B_2))$ and (t_q, x). Since these bijections clearly are $\pi_0(H)$-equivariant, we deduce

(37)
$$\mathrm{Z}_H(\Phi)/\mathrm{Z}_{H^\circ}(\Phi) = \mathrm{Z}_H(t_q, \Phi(B_2))/\mathrm{Z}_{H^\circ}(t_q, \Phi(B_2)) = \mathrm{Z}_H(t_q, x)/\mathrm{Z}_{H^\circ}(t_q, x).$$

Equations (36) and (37) imply that

$$\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(t_q, \Phi(B_2)) \to \mathrm{Z}_H(t_q, x)$$

is also a homotopy equivalence.

(2) By the aforementioned result [KaLu, §2.3.h]

$$\mathrm{Z}_{H^\circ}(t_q, x) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathcal{B}_{H^\circ}^{t_q, x}. \tag{38}$$

On the other hand, by (35)

(39)
$$\mathrm{Z}_{H^\circ}(t_q, x) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathrm{Z}_{U_x}(t_q)\mathrm{Z}_H(t_q, \Phi(B_2)) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathrm{Z}_{U_x}(t_q) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}.$$

For any $B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}$ and $u \in \mathrm{Z}_{U_x}(t_q)$ it is clear that

$$u \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \iff \Phi(B_2) \subset uBu^{-1} \iff u^{-1}\Phi(B_2)u \subset B.$$

Furthermore, since $\Phi(B_2) \subset B$ is generated by x and $\{\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \mid \alpha \in \mathbb{C}^\times\}$, the right hand side is equivalent to

$$u^{-1}\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) u \in B \quad \forall \alpha \in \mathbb{C}^\times.$$

In Lie algebra terms this can be reformulated as

$$\mathrm{Ad}_{u^{-1}}(d\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)) \in \mathrm{Lie}\, B \quad \forall \alpha \in \mathbb{C}.$$

Because u is unipotent, this happens if and only if

$$\mathrm{Ad}_{u^\lambda}(d\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)) \in \mathrm{Lie}\, B \quad \forall \lambda, \alpha \in \mathbb{C}.$$

By the reverse chain of arguments the last statement is equivalent with

$$u^\lambda \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \quad \forall \lambda \in \mathbb{C}.$$

Thus $\{u \in \mathrm{Z}_{U_x}(t_q) \mid u \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}\}$ is contractible for all $B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}$, and we already knew that $\mathrm{Z}_{U_x}(t_q)$ is contractible. Together with (38) and (39) these imply that $\mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \to \mathcal{B}_{H^\circ}^{t_q, x}$ is a homotopy equivalence. $\square$

For the affine Springer correspondence we will need more precise information on the relation between the varieties for G, for H and for M°.

Proposition 6.2. (1) *The variety $\mathcal{B}_{H^\circ}^{t,x}$ is isomorphic to $[\mathcal{W}^{H^\circ} : \mathcal{W}^{M^\circ}]$ copies of $\mathcal{B}_{M^\circ}^{x}$, and $\mathcal{B}_{H^\circ}^{t,\Phi(B_2)}$ is isomorphic to the same number of copies of $\mathcal{B}_{M^\circ}^{\Phi(B_2)}$.*

(2) *The group $\mathrm{Z}_{H^\circ}(t, x)/\mathrm{Z}_{M^\circ}(x)$ permutes these two sets of copies freely.*

(3) *The variety $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is isomorphic to $[\mathcal{W}^G : \mathcal{W}^{H^\circ}]$ copies of $\mathcal{B}_{H^\circ}^{t,\Phi(B_2)}$. The group $\mathrm{Z}_G(\Phi)/\mathrm{Z}_{H^\circ}(\Phi)$ permutes these copies freely.*

Proof. (1) Let A be a subgroup of T such that $M^\circ = \mathrm{Z}_{H^\circ}(A)^\circ$ and let $\mathcal{B}^A_{H^\circ}$ denote the variety of all Borel subgroups of H° which contain A. With an adaptation of [ChGi, p.471] we will prove that, for any $B \in \mathcal{B}^A_{H^\circ}$, $B \cap M^0$ is a Borel subgroup of M^0.

Since $B \cap M^\circ \subset B$ is solvable, it suffices to show that its Lie algebra is a Borel subalgebra of Lie M°. Write Lie $T = \mathfrak{t}$ and let

$$\text{Lie } H^\circ = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$$

be the triangular decomposition, where Lie $B = \mathfrak{n} \oplus \mathfrak{t}$. Since $A \subset B$, it preserves this decomposition and

$$\begin{aligned} \text{Lie } M^\circ &= (\text{Lie } H)^A = \mathfrak{n}^A \oplus \mathfrak{t} \oplus \mathfrak{n}_-^A, \\ \text{Lie } B \cap M^\circ &= \text{Lie } B^A = \mathfrak{n}^A \oplus \mathfrak{t}. \end{aligned}$$

The latter is indeed a Borel subalgebra of Lie M°. Thus there is a canonical map

$$\mathcal{B}^A_{H^\circ} \to \operatorname{Flag} M^0, \quad B \mapsto B \cap M^0. \tag{40}$$

The group M acts by conjugation on $\mathcal{B}^A_{H^\circ}$ and (40) clearly is M-equivariant. By [ChGi, p. 471] the M°-orbits form a partition

$$\mathcal{B}^A_{H^\circ} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_m. \tag{41}$$

At the same time these orbits are the connected components of $\mathcal{B}^A_{H^\circ}$ and the irreducible components of the projective variety $\mathcal{B}^A_{H^\circ}$. The argument from [ChGi, p. 471] also shows that (40), restricted to any one of these orbits, is a bijection from the M^0-orbit onto Flag M^0.

The number of components m can be determined as in the proof of [Ste1, Corollary 3.12.a]. The collection of Borel subgroups of M° that contain the maximal torus T is in bijection with the Weyl group $\mathcal{W}^{M^\circ}$. Retracting via (40), we find that every component $\mathcal{B}_i$ has precisely $|\mathcal{W}^{M^\circ}|$ elements that contain T. On the other hand, since $A \subset T$, $\mathcal{B}^A_{H^\circ}$ has $|\mathcal{W}^{H^\circ}|$ elements that contain T, so

$$m = [\mathcal{W}^{H^\circ} : \mathcal{W}^{M^\circ}].$$

To obtain our desired isomorphisms of varieties, we let A be the group generated by t and we restrict $\mathcal{B}_i \to$ Flag M° to Borel subgroups that contain t, x (respectively $t, \Phi(B_2)$).

(2) By Proposition 6.1

$$\mathrm{Z}_{H^\circ}(t, x)/\mathrm{Z}_{M^\circ}(x) \cong \mathrm{Z}_{H^\circ}(t, \Phi(B_2))/\mathrm{Z}_{M^\circ}(\Phi(B_2)).$$

Since the former is a subgroup of M/M° and the copies under consideration are in M-equivariant bijection with the components (41), it suffices to show that M/M° permutes these components freely. Pick B, B' in the same component $\mathcal{B}_i$ and assume that $B' = hBh^{-1}$ for some $h \in M$. Since $\mathcal{B}_i$ is M°-equivariantly isomorphic to the flag variety of M° we can find $m \in M^\circ$ such that $B' = m^{-1}Bm$. Then mh normalizes B, so $mh \in B$. As B is connected, this implies $mh \in M^\circ$ and $h \in M^\circ$.

(3) Apply the proofs of parts 1 and 2 with $A = \Phi(\mathbf{I}_F)$, G in the role of H°, H° in the role of M° and $t\Phi(B_2)$ in the role of x. □

7. COMPARISON OF DIFFERENT PARAMETERS

In the following sections we will make use of several different but related kinds of parameters.

Kazhdan–Lusztig–Reeder parameters (KLR parameters)
For a Langlands parameter as in (31), the variety of Borel subgroups $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is nonempty, and the centralizer $\mathrm{Z}_G(\Phi)$ of the image of Φ acts on it. Hence the group of components $\pi_0(\mathrm{Z}_G(\Phi))$ acts on the homology $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. We call an irreducible representation ρ of $\pi_0(\mathrm{Z}_G(\Phi))$ geometric if it appears in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. We define a Kazhdan–Lusztig–Reeder parameter for G to be a such pair (Φ, ρ). The group G acts on these parameters by

$$g \cdot (\Phi, \rho) = (g \Phi g^{-1}, \rho \circ \mathrm{Ad}_g^{-1}) \tag{42}$$

and we denote the corresponding equivalence class by $[\Phi, \rho]_G$.

Affine Springer parameters
As before, suppose that $t \in G$ is semisimple and that $x \in \mathrm{Z}_G(t)$ is unipotent. Then $\mathrm{Z}_G(t, x)$ acts on $\mathcal{B}_G^{t,x}$ and $\pi_0(\mathrm{Z}_G(t, x))$ acts on the homology of this variety. In this setting we say that $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t, x))\big)$ is geometric if it appears in $H_{\mathrm{top}}(\mathcal{B}_G^{t,x}, \mathbb{C})$, where top refers to highest degree in which the homology is nonzero, the real dimension of $\mathcal{B}_G^{t,x}$. We call such triples (t, x, ρ_1) affine Springer parameters for G, because they appear naturally in the representation theory of the affine Weyl group associated to G. The group G acts on such parameters by conjugation, and we denote the conjugacy classes by $[t, x, \rho_1]_G$.

Kazhdan–Lusztig triples
Next we consider a unipotent element $x \in G$ and a semisimple element $t_q \in G$ such that $t_q x t_q^{-1} = x^q$. As above, $\mathrm{Z}_G(t_q, x)$ acts on the variety $\mathcal{B}_G^{t_q,x}$ and we call $\rho_q \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t_q, x))\big)$ geometric if it appears in $H_*\big(\mathcal{B}_G^{t_q,x}, \mathbb{C}\big)$. We refer to triples (t_q, x, ρ_q) of this kind as Kazhdan–Lusztig triples for G. Again they are endowed with an obvious G-action and we denote the equivalence classes by $[t_q, x, \rho_q]_G$.

We note that in all cases the representations of the component groups stem from the action of G on a variety of Borel subgroups. The centre of G acts trivially on such a variety, so in all three above cases an irreducible representation of the appropriate component group can only be if all elements coming from $\mathrm{Z}(G)$ act trivially.

In [KaLu, Ree2] there are some indications that these three kinds of parameters are essentially equivalent. Proposition 6.1 allows us to make this precise in the necessary generality.

Lemma 7.1. *Let $\mathfrak{s}$ be a Bernstein component in the principal series, associate $c^{\mathfrak{s}} \colon \mathfrak{o}_F^\times \to T$ to it as in Lemma 5.1 and write $H = \mathrm{Z}_G(c^{\mathfrak{s}}(\mathfrak{o}_F^\times))$. There are natural bijections between H°-equivalence classes of:*

- *Kazhdan–Lusztig–Reeder parameters for G with $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$ and $\Phi(\varpi_F) \in H^\circ$;*

- *affine Springer parameters for H°;*
- *Kazhdan–Lusztig triples for H°.*

Proof. Since $\mathrm{SL}_2(\mathbb{C})$ is connected and commutes with $\mathfrak{o}_F^\times$, its image under Φ must be contained in the connected component of H. Therefore KLR-parameters with these properties are in canonical bijection with KLR parameters for H° and it suffices to consider the case $H^\circ = G$.

As in (31) and (33), any KLR parameter gives rise to the ingredients t, x and t_q for the other two kinds of parameters. As we discussed after (31), the pair (t, x) is enough to recover the conjugacy class of Φ. A refined version of the Jacobson–Morozov theorem says that the same goes for the pair (t_q, x), see [KaLu, §2.3] or [Ree2, Section 4.2].

To complete $\Phi, (t, x)$ or (t_q, x) to a parameter of the appropriate kind, we must add an irreducible representation ρ, ρ_1 or ρ_q. For the affine Springer parameters it does not matter whether we consider the total homology or only the homology in top degree. Indeed, it follows from Propositions 6.1 and 6.2 and [Sho, bottom of page 296 and Remark 6.5] that any irreducible representation ρ_1 which appears in $H_*\big(\mathcal{B}_G^{t,x}, \mathbb{C}\big)$, already appears in the top homology of this variety.

This and Proposition 6.1 show that there is a natural correspondence between the possible ingredients ρ, ρ_1 and ρ_q. $\square$

8. The affine Springer correspondence

An interesting instance of Section 4 arises when M is the centralizer of a semisimple element t in a connected reductive complex group G. As before we assume that t lies in a maximal torus T of G and we write $\mathcal{W}^G = W(G, T)$. By Lemma 3.2

$$\mathcal{W}^M := \mathrm{N}_M(T)/\mathrm{Z}_M(T) \cong \mathcal{W}^{M^\circ} \rtimes \pi_0(M) \tag{43}$$

is the stabilizer of t in $\mathcal{W}^G$, so the role of Γ is played by the component group $\pi_0(M)$. In contrast to the setup in Section 4, it is possible that some elements of $\pi_0(M) \setminus \{1\}$ fix W pointwise. This poses no problems however, as such elements never act trivially on T. For later use we record the following consequence of (15):

$$\pi_0(M)_{\tau(x,\rho)} \cong \big(\mathrm{Z}_M(x)/\mathrm{Z}_{M^\circ}(x)\big)_\rho. \tag{44}$$

Recall from Section 2 that

$$\begin{aligned} \widetilde{T}_2 &:= \{(t, \sigma) \,:\, t \in T, \sigma \in \mathbf{Irr}(\mathcal{W}^G_t)\}, \\ (T/\!/\mathcal{W}^G)_2 &:= \widetilde{T}_2/\mathcal{W}^G. \end{aligned}$$

We note that the rational characters of the complex torus T span the regular functions on the complex variety T:

$$\mathcal{O}(T) = \mathbb{C}[X^*(T)].$$

From (6), (7), Lemma 2.2 and Proposition 4.3 we infer the following rough form of the extended Springer correspondence for the affine Weyl group $X^*(T) \rtimes \mathcal{W}^G$.

Theorem 8.1. *There are bijections*

$$(T//\mathcal{W}^G)_2 \simeq \mathbf{Irr}\,(X^*(T) \rtimes \mathcal{W}^G) \simeq \{(t, \tau(x,\varrho) \rtimes \psi)\}/\mathcal{W}^G$$

with $t \in T, \tau(x,\varrho) \in \mathbf{Irr}\,\mathcal{W}^{M^0}, \psi \in \mathbf{Irr}(\pi_0(M)_{\tau(x,\varrho)})$.

Now we recall the geometric realization of irreducible representations of $X^*(T) \rtimes \mathcal{W}^G$ by Kato [Kat]. For a unipotent element $x \in M^\circ$ let $\mathcal{B}_G^{t,x}$ be the variety of Borel subgroups of G containing t and x. Fix a Borel subgroup B of G containing T and let $\theta_{G,B} : \mathcal{B}_G^{t,x} \to T$ be the morphism defined by

$$\theta_{G,B}(B') = g^{-1}tg \text{ if } B' = gBg^{-1} \text{ and } t \in gTg^{-1}. \tag{45}$$

The image of $\theta_{G,B}$ is $\mathcal{W}^G t$, the map is constant on the irreducible components of $\mathcal{B}_G^{t,x}$ and it gives rise to an action of $X^*(T)$ on the homology of $\mathcal{B}_G^{t,x}$. Furthermore $\mathbb{Q}[\mathcal{W}^G] \cong H(\mathcal{Z}_G)$ acts on $H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})$ via the convolution product in Borel–Moore homology, as described in (12). Both actions commute with the action of $\mathrm{Z}_G(t,x)$ induced by conjugation of Borel subgroups. By homotopy invariance, the latter action factors through $\pi_0(\mathrm{Z}_G(t,x))$.

Let $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t,x))\big)$. By [Kat, Theorem 4.1] the $X^*(T) \rtimes \mathcal{W}^G$-module

$$\tau(t,x,\rho_1) := \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t,x))}\big(\rho_1, H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \tag{46}$$

is either irreducible or zero. Moreover every irreducible representation of $X^*(T) \rtimes \mathcal{W}^G$ is obtained is in this way, and the data (t, x, ρ_1) are unique up to G-conjugacy. This generalizes the Springer correspondence for finite Weyl groups, which can be recovered by considering the representations on which $X^*(T)$ acts trivially.

Propositions 4.3 and 6.2 shine some new light on this:

Theorem 8.2. (1) *There are bijections between the following sets:*

- $\mathrm{Irr}(X^*(T) \rtimes \mathcal{W}^G) = \mathrm{Irr}(\mathcal{O}(T) \rtimes \mathcal{W}^G)$*;*
- $(T//\mathcal{W}^G)_2 = \big\{(t, \tilde\tau) \mid t \in T, \tilde\tau \in \mathrm{Irr}(\mathcal{W}^M)\big\}/\mathcal{W}^G$*;*
- $\big\{(t, \tau, \sigma) \mid t \in T, \tau \in \mathrm{Irr}(\mathcal{W}^{M^\circ}), \sigma \in \mathrm{Irr}(\pi_0(M)_\tau)\big\}/\mathcal{W}^G$*;*
- $\big\{(t, x, \rho, \sigma) \mid t \in T, x \in M^\circ \text{ unipotent}, \rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{M^\circ}(x))\big)$ $\text{geometric}, \sigma \in \mathrm{Irr}\big(\pi_0(M)_{\tau(x,\rho)}\big)\big\}/G$*;*
- $\big\{(t, x, \rho_1) \mid t \in T, x \in M^\circ \text{ unipotent}, \rho_1 \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t,x))\big)$ $\text{geometric}\big\}/G$*.*

Here a representation of $\pi_0(\mathrm{Z}_{M^\circ}(x))$ *(or* $\pi_0(\mathrm{Z}_G(t,x))$*) is called geometric if it appears in* $H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})$ *(respectively* $H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})$*). Apart from the third and fourth sets, these bijections are natural.*

(2) *The* $X^*(T) \rtimes \mathcal{W}^G$*-representation corresponding to* (t, x, ρ_1) *via these bijections is Kato's module* (46)*.*

We remark that in the fourth and fifth sets it would be more natural to allow t to be any semisimple element of G. In fact that would give the affine Springer parameters from Lemma 7.1. Clearly G acts on the set of such more general parameters (t, x, ρ, σ) or (t, x, ρ_1), which gives equivalence relations $/G$. The two above $/G$ refer to the restrictions of these equivalence relations to parameters with $t \in T$.

Proof. (1) Recall that the isotropy group of t in $\mathcal{W}^G$ is

$$\mathcal{W}^G_t = \mathcal{W}^M = \mathcal{W}^{M^\circ} \rtimes \pi_0(M).$$

Hence the bijection between the first two sets is an instance of Clifford theory, see Lemma 2.3. The second and third sets are in bijection by Proposition 4.3. The Springer correspondence for $\mathcal{W}^{M^\circ}$ provides the bijection with the fourth collection. To establish a bijection with the fifth collection, we first observe that

$$\begin{aligned} \pi_0(\mathrm{Z}_G(t,x)) = \pi_0(\mathrm{Z}_M(x)) &\cong \pi_0\big(\mathrm{Z}_{M^\circ}(x) \rtimes \pi_0(M)_{[x]_{M^\circ}}\big) \\ &= \pi_0(\mathrm{Z}_{M^\circ}(x)) \rtimes \pi_0(M)_{[x]_{M^\circ}}. \end{aligned} \tag{47}$$

Furthermore $\pi_0(M)_{\tau(x,\rho)} = \pi_0(M)_{[x,\rho]_{M^\circ}}$ by (15). From that and Proposition 4.3 it follows that every irreducible representation of (47) is of the form $\rho \rtimes \sigma$ (see Notation 2.4), with ρ and σ as in the fourth set. By Proposition 6.2

$$H_*(\mathcal{B}^{t,x}_G, \mathbb{C}) \cong H_*(\mathcal{B}^x_{M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_G(t,x)/\mathrm{Z}_{M^\circ}(x)] \otimes \mathbb{C}^{[\mathcal{W}^G : \mathcal{W}^G_t]} \tag{48}$$

as $\mathrm{Z}_G(t,x)$-representations. By [Ree2, §3.1]

$$\mathrm{Z}_G(t,x)/\mathrm{Z}_{M^\circ}(x) \cong \pi_0(M)_{[x]_{M^\circ}}$$

is abelian. Hence $\mathrm{Ind}_{\pi_0(M)_{[x,\rho]_{M^\circ}}}^{\pi_0(M)_{[x]_{M^\circ}}}(\sigma)$ appears exactly once in the regular representation of this group and

$$\begin{aligned} \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t,x))}\big(\rho \rtimes \sigma, H_{d(x)}(\mathcal{B}^{t,x}_G, \mathbb{C})\big) \cong \\ \mathrm{Hom}_{\pi_0(\mathrm{Z}_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})\big) \rtimes \sigma \otimes \mathbb{C}^{[\mathcal{W}^G : \mathcal{W}^G_t]}. \end{aligned} \tag{49}$$

In particular we see that ρ is geometric if and only if $\rho \rtimes \sigma$ is geometric, which establishes the final bijection. Now the resulting bijection between the second and fifth sets is natural by Theorem 4.4.

(2) The $X^*(T) \rtimes \mathcal{W}^G$-representation constructed from $(t, x, \rho \rtimes \sigma)$ by means of our bijections is

$$\mathrm{Ind}_{X^*(T) \rtimes \mathcal{W}^G_t}^{X^*(T) \rtimes \mathcal{W}^G}\big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})\big) \rtimes \sigma\big). \tag{50}$$

On the other hand, by [Kat, Proposition 6.2]

$$\begin{aligned} H_*(\mathcal{B}^{t,x}_G, \mathbb{C}) &\cong \mathrm{Ind}_{X^*(T) \rtimes \mathcal{W}^{M^\circ}}^{X^*(T) \rtimes \mathcal{W}^G}\big(H_*(\mathcal{B}^x_{M^\circ}, \mathbb{C})\big) \\ &\cong \mathrm{Ind}_{X^*(T) \rtimes \mathcal{W}^G_t}^{X^*(T) \rtimes \mathcal{W}^G}\big(H_*(\mathcal{B}^x_{M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_G(t,x)/\mathrm{Z}_{M^\circ}(x)]\big) \end{aligned} \tag{51}$$

as $\mathrm{Z}_G(t,x) \times X^*(T) \rtimes \mathcal{W}^G$-representations. Together with the proof of part 1 this shows that $\tau(t, x, \rho \rtimes \sigma)$ is isomorphic to (50). $\square$

We can extract a little more from the above proof. Recall that $\mathcal{O}_x$ denotes the conjugacy class of x in M. Let us agree that the affine Springer parameters with a fixed $t \in T$ are partially ordered by

$$(t, x, \rho_1) < (t, x', \rho_1') \quad \text{when} \quad \overline{\mathcal{O}_x} \subsetneq \overline{\mathcal{O}_{x'}}.$$

Lemma 8.3. *There exist multiplicities* $m_{t,x,\rho_1,x',\rho_1'} \in \mathbb{Z}_{\geq 0}$ *such that*

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t,x))}\big(\rho_1, H_*(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \cong \\ \tau(t,x,\rho_1) \oplus \bigoplus_{(t,x',\rho_1') > (t,x,\rho_1)} m_{t,x,\rho_1,x',\rho_1'}\, \tau(t,x',\rho_1').$$

Proof. It follows from (51), (48) and (49) that

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t,x))}\big(\rho \rtimes \sigma, H_*(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \cong \\ \mathrm{Ind}_{X^*(T)\rtimes \mathcal{W}_t^G}^{X^*(T)\rtimes \mathcal{W}^G} \mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes \pi_0(M)_{[x,\rho]_{M^\circ}}}^{\mathcal{W}_t^G} \big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C})\big) \otimes \sigma\big). \tag{52}$$

The functor $\mathrm{Ind}_{X^*(T)\rtimes \mathcal{W}_t^G}^{X^*(T)\rtimes \mathcal{W}^G}$ provides an equivalence between the categories

- $X^*(T) \rtimes \mathcal{W}_t^G$-representations with $\mathcal{O}(T)^{\mathcal{W}_t^G}$-character t;
- $X^*(T) \rtimes \mathcal{W}^G$-representations with $\mathcal{O}(T)^{\mathcal{W}^G}$-character $\mathcal{W}^G t$.

Therefore we may apply Lemma 4.5 to the right hand side of (52), which produces the required formula. □

Let us have a look at the representations with an affine Springer parameter of the form $(t, x = 1, \rho_1 = \mathrm{triv})$. Equivalently, the fourth parameter in Theorem 8.2 is $(t, x = 1, \rho = \mathrm{triv}, \sigma = \mathrm{triv})$. The $\mathcal{W}^{M^\circ}$-representation with Springer parameter $(x = 1, \rho = \mathrm{triv})$ is the trivial representation, so $(x = 1, \rho = \mathrm{triv}, \sigma = \mathrm{triv})$ corresponds to the trivial representation of $\mathcal{W}_t^G$. With (50) we conclude that the $X^*(T) \rtimes \mathcal{W}^G$-representation with affine Springer parameter $(t, 1, \mathrm{triv})$ is

$$\tau(t, 1, \mathrm{triv}) = \mathrm{Ind}_{X^*(T)\rtimes \mathcal{W}_t^G}^{X^*(T)\rtimes \mathcal{W}^G}\big(\mathrm{triv}_{\mathcal{W}_t^G}\big). \tag{53}$$

Notice that this is the only irreducible $X^*(T) \rtimes \mathcal{W}^G$-representation with an $X^*(T)$-weight t and nonzero $\mathcal{W}^G$-fixed vectors.

9. GEOMETRIC REPRESENTATIONS OF AFFINE HECKE ALGEBRAS

Let G be a connected reductive complex group, B a Borel subgroup and T a maximal torus of G contained in B. Let $\mathcal{H}(G)$ be the affine Hecke algebra with the same based root datum as (G, B, T) and with a parameter $q \in \mathbb{C}^\times$ which is not a root of unity.

As we will have to deal with disconnected reductive groups, we include some additional automorphisms in the picture. In every root subgroup U_α with $\alpha \in \Delta(B, T)$ we pick a nontrivial element u_α. Let Γ be a finite group of automorphisms of $(G, T, (u_\alpha)_{\alpha \in \Delta(B,T)})$. Since G need not be semisimple, it is possible that some elements of Γ fix the entire root system of (G, T). Notice that Γ acts on the Weyl group $\mathcal{W}^G = W(G, T)$ and on $X^*(T)$ because it stabilizes T. Furthermore Γ acts on the standard basis of $\mathcal{H}(G)$ by

$$\gamma(T_w) = T_{\gamma(w)}, \text{ where } \gamma \in \Gamma, w \in X^*(T) \rtimes \mathcal{W}^G.$$

Since Γ stabilizes B, it determines an algebra automorphism of $\mathcal{H}(G)$. We form the crossed product algebra $\mathcal{H}(G) \rtimes \Gamma$ with respect to this Γ-action.

We define a Kazhdan–Lusztig triple for $\mathcal{H}(G) \rtimes \Gamma$ to be a triple (t_q, x, ρ) such that:

- $t_q \in G$ is semisimple, $x \in G$ is unipotent and $t_q x t_q^{-1} = x^q$;
- ρ is an irreducible representation of the component group $\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x))$, such that every irreducible subrepresentation of the restriction of ρ to $\pi_0(\mathrm{Z}_G(t_q, x))$ appears in $H_*(\mathcal{B}^{t_q,x}, \mathbb{C})$.

The group $G \rtimes \Gamma$ acts on such triples by conjugation, and we denote the conjugacy class of a triples by $[t_q, x, \rho]_{G \rtimes \Gamma}$. Now we generalize [KaLu, Theorem 7.12] and [Ree2, Theorem 3.5.4]:

Theorem 9.1. *There exists a natural bijection between* $\mathrm{Irr}(\mathcal{H}(G) \rtimes \Gamma)$ *and* $G \rtimes \Gamma$*-conjugacy classes of Kazhdan–Lusztig triples. The module corresponding to* (t_q, x, ρ) *is the unique irreducible quotient of the* $\mathcal{H}(G) \rtimes \Gamma$*-module*

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q,x))}\big(\rho, H_*(\mathcal{B}^{t_q,x}, \mathbb{C}) \otimes \mathbb{C}[\Gamma]\big).$$

Proof. First we recall the geometric constructions of $\mathcal{H}(G)$-modules by Kazhdan, Lusztig and Reeder, taking advantage of Lemma 4.2 to simplify the presentation somewhat. As in [Ree2, §1.5], let

$$1 \to C \to \tilde{G} \to G \to 1 \tag{54}$$

be a finite central extension such that $\tilde{G}$ is a connected reductive group with simply connected derived group. The kernel C acts naturally on $\mathcal{H}(\tilde{G})$ and

$$\mathcal{H}(\tilde{G})^C \cong \mathcal{H}(G). \tag{55}$$

The action of Γ on the based root datum of (G, B, T) lifts uniquely to an action on the corresponding based root datum for $\tilde{G}$, so the Γ-actions on G and on $\mathcal{H}(G)$ lift naturally to actions on $\tilde{G}$ and $\mathcal{H}(\tilde{G})$. Let $\mathcal{H}_{\mathbf{q}}(\tilde{G})$ be the variation on $\mathcal{H}(\tilde{G})$ with scalars $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$ where $\mathbf{q}$ is a formal variable (instead of scalars $\mathbb{C}$ and $q \in \mathbb{C}^\times$). In [KaLu, Theorem 3.5] an isomorphism

$$\mathcal{H}_{\mathbf{q}}(\tilde{G}) \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \tag{56}$$

is constructed, where the right hand side denotes the $\tilde{G} \times \mathbb{C}^\times$-equivariant K-theory of the Steinberg variety $\mathcal{Z}_{\tilde{G}}$ of $\tilde{G}$. Since $G \rtimes \Gamma$ acts via conjugation on $\tilde{G}$ and on $\mathcal{Z}_{\tilde{G}}$, it also acts on $K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}})$. However, the connected group G acts trivially, so the action factors via Γ. Now the definition of the generators in [KaLu, Theorem 3.5] shows that (56) is Γ-equivariant. In particular it specializes to Γ-equivariant isomorphisms

$$\mathcal{H}(\tilde{G}) \cong \mathcal{H}_{\mathbf{q}}(\tilde{G}) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_q \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_q. \tag{57}$$

Let $(\tilde{t_q}, \tilde{x}) \in (\tilde{G})^2$ be a lift of $(t_q, x) \in G^2$ with $\tilde{x}$ unipotent. The $\tilde{G}$-conjugacy class of $\tilde{t_q}$ defines a central character of $\mathcal{H}(\tilde{G})$ and

$$\mathcal{H}(\tilde{G}) \otimes_{\mathrm{Z}(\mathcal{H}(\tilde{G}))} \mathbb{C}_{\tilde{t_q}} \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{R(\tilde{G} \times \mathbb{C}^\times)} \mathbb{C}_{\tilde{t_q},q}.$$

According to [ChGi, Proposition 8.1.5] there is an isomorphism

$$K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{R(\tilde{G} \times \mathbb{C}^\times)} \mathbb{C}_{\tilde{t_q},q} \cong H_*(\mathcal{Z}_{\tilde{G}}^{\tilde{t_q},q}, \mathbb{C}). \tag{58}$$

Moreover (58) is Γ-equivariant, because all the maps involved in the proof of [ChGi, Proposition 8.1.5] are functorial with respect to isomorphisms of algebraic varieties. To be precise, one should note that throughout [ChGi, Chapter 8] it is assumed that $\tilde{G}$ is simply connected. However, as we already have (57) at our disposal, [ChGi, §8.1] also applies whenever the derived group of $\tilde{G}$ is simply connected.

Any Borel subgroup of $\tilde{G}$ contains C, so $\mathcal{B}^{\tilde{t}_q,\tilde{x}} = \mathcal{B}^{\tilde{t}_q,\tilde{x}}_{\tilde{G}}$ and $\mathcal{B}^{t_q,x} = \mathcal{B}^{t_q,x}_{G}$ are isomorphic algebraic varieties. From [ChGi, p. 414] we see that the convolution product in Borel–Moore homology leads to an action of $H_*(\mathcal{Z}^{\tilde{t}_q,q}_{\tilde{G}},\mathbb{C})$ on $H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C})$. Notice that for $\tilde{h} \in H_*(\mathcal{Z}^{\tilde{t}_q,q}_{\tilde{G}},\mathbb{C})$ and $g \in G \rtimes \Gamma$ we have

$$g \cdot \tilde{h} \in H_*(\mathcal{Z}^{g\tilde{t}_q g^{-1},q}_{\tilde{G}},\mathbb{C}) \cong \mathcal{H}(\tilde{G}) \otimes_{\mathrm{Z}(\mathcal{H}(\tilde{G}))} \mathbb{C}_{g\tilde{t}_q g^{-1}}.$$

An obvious generalization of [ChGi, Lemma 8.1.8] says that all these constructions are compatible with the above actions of $G \rtimes \Gamma$, in the sense that the following diagram commutes:

$$\begin{tikzcd}
H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C}) \arrow[r, "\tilde{h}"] \arrow[d, "H_*(\mathrm{Ad}_g)"] & H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C}) \arrow[d, "H_*(\mathrm{Ad}_g)"] \\
H_*(\mathcal{B}^{g\tilde{t}_q g^{-1},g\tilde{x}g^{-1}},\mathbb{C}) \arrow[r, "g\cdot\tilde{h}"] & H_*(\mathcal{B}^{g\tilde{t}_q g^{-1},g\tilde{x}g^{-1}},\mathbb{C}).
\end{tikzcd} \tag{59}$$

In particular the component group $\pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x}))$ acts on $H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C})$ by $\mathcal{H}(\tilde{G})$-intertwiners. Let $\tilde{\rho}$ be an irreducible representation of this component group, appearing in $H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C})$. In other words, $(\tilde{t}_q,\tilde{x},\tilde{\rho})$ is a Kazhdan–Lusztig triple for $\mathcal{H}(\tilde{G})$. According to [KaLu, Theorem 7.12]

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x}))}(\tilde{\rho}, H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C})) \tag{60}$$

is a $\mathcal{H}(\tilde{G})$-module with a unique irreducible quotient, say $V_{\tilde{t}_q,\tilde{x},\tilde{\rho}}$.

Following [Ree2, §3.3] we define a group $R_{\tilde{t}_q,\tilde{x}}$ by

$$1 \to \pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x})) \to \pi_0(\mathrm{Z}_{\tilde{G}}(t_q,x)) \to R_{\tilde{t}_q,\tilde{x}} \to 1. \tag{61}$$

Obviously $\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x})$ contains $\mathrm{Z}(\tilde{G})$, so the sequence

$$1 \to \pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x})/\mathrm{Z}(\tilde{G})) \to \pi_0(\mathrm{Z}_{\tilde{G}}(t_q,x)/\mathrm{Z}(\tilde{G})) \to R_{\tilde{t}_q,\tilde{x}} \to 1 \tag{62}$$

is also exact. For the middle term we have

$$\mathrm{Z}_{\tilde{G}}(t_q,x)/\mathrm{Z}(\tilde{G}) \cong \mathrm{Z}_G(t_q,x)/\mathrm{Z}(G)$$

Since the derived group of $\tilde{G}$ is simply connected, $\mathrm{Z}_{\tilde{G}}(t_q)^\circ = \mathrm{Z}_{\tilde{G}}(\tilde{t}_q)$. In the second term of (62) we get

$$\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x})/\mathrm{Z}(\tilde{G}) \cong \mathrm{Z}_{\mathrm{Z}_{\tilde{G}}(t_q)^\circ}(\tilde{x})/\mathrm{Z}(\tilde{G}) \cong \mathrm{Z}_{\mathrm{Z}_G(t_q)^\circ}(x)/\mathrm{Z}(G).$$

Let us abbreviate $M = \mathrm{Z}_G(t_q)$. Then (62) can be written as

$$1 \to \pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(G)) \to \pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(G)) \to R_{\tilde{t}_q,\tilde{x}} \to 1.$$

Like in (62) we can derive another short exact sequence

$$1 \to \pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ)) \to \pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(M^\circ)) \to R_{\tilde{t}_q,\tilde{x}} \to 1. \tag{63}$$

It can also be obtained from (61) by the dividing the two appropriate groups by the inverse image of $\mathrm{Z}(M^\circ)$ in $\tilde{G}$. From Lemma 4.2 (with the trivial representation of $\pi_0(\mathrm{Z}_G(t_q)^\circ(x))$ in the role of ρ) we know that (63) splits. By Proposition 6.2 and (41) $\mathrm{Z}(M^\circ)$ acts trivially on $H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}},\mathbb{C})$. Hence all the 2-cocycles of subgroups of $R_{\tilde{t}_q,\tilde{x}}$ appearing associated to (60) are trivial.

Let $\tilde\sigma$ be any irreducible representation of $R_{\tilde t_q,\tilde x,\tilde\rho}$, the stabilizer of the isomorphism class of $\tilde\rho$ in $R_{\tilde t_q,\tilde x}$. Clifford theory for (63) produces $\tilde\rho \rtimes \tilde\sigma \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(M^\circ))\big)$, a representation which lifts to $\pi_0(\mathrm{Z}_G(t_q,x))$. Moreover by [Ree2, Lemma 3.5.1] it appears in $H_*(\mathcal{B}^{t_q,x},\mathbb{C})$, and conversely every irreducible representation with the latter property is of the form $\tilde\rho \rtimes \tilde\sigma$.

With the above in mind, [Ree2, Lemma 3.5.2] says that the $\mathcal{H}(G)$-module

$$\begin{aligned} M(t_q,x,\tilde\rho\rtimes\tilde\sigma) &:= \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t_q,x))}\big(\tilde\rho\rtimes\tilde\sigma, H_*(\mathcal{B}^{t_q,x},\mathbb{C})\big) \\ &= \mathrm{Hom}_{R_{\tilde t_q,\tilde x,\tilde\rho}}\big(\tilde\sigma, \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x))}(\tilde\rho, H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C}))\big) \end{aligned} \tag{64}$$

has a unique irreducible quotient

$$\pi(t_q,x,\tilde\rho\rtimes\tilde\sigma) = \mathrm{Hom}_{R_{\tilde t_q,\tilde x,\tilde\rho}}(\tilde\sigma, V_{\tilde t_q,\tilde x,\tilde\rho}). \tag{65}$$

According [Ree2, Lemma 3.5.3] this sets up a bijection between $\mathrm{Irr}(\mathcal{H}(G))$ and G-conjugacy classes of Kazhdan–Lusztig triples for G.

Remark 9.2. *The module* (64) *is well-defined for any $q \in \mathbb{C}^\times$, although for roots of unity it may have more than one irreducible quotient. For $q = 1$ the algebra $\mathcal{H}(G)$ reduces to $\mathbb{C}[X^*(T) \rtimes \mathcal{W}^G]$ and* [ChGi, Section 8.2] *shows that Kato's module* (46) *is a direct summand of $M(t_1,x,\rho_1)$.*

Next we study what Γ does to all these objects. There is natural action of Γ on Kazhdan–Lusztig triples for G, namely

$$\gamma\cdot(t_q,x,\rho_q) = \big(\gamma t_q\gamma^{-1}, \gamma x\gamma^{-1}, \rho_q\circ\mathrm{Ad}_\gamma^{-1}\big).$$

From (59) and (64) we deduce that the diagram

$$\begin{tikzcd} \pi(t_q,x,\rho_q) \arrow[r, "h"] \arrow[d, "H_*(\mathrm{Ad}_g)"] & \pi(t_q,x,\rho_q) \arrow[d, "H_*(\mathrm{Ad}_g)"] \\ \pi\big(gt_qg^{-1},gxg^{-1},\rho_q\circ\mathrm{Ad}_g^{-1}\big) \arrow[r, "\gamma(h)"] & \pi\big(gt_qg^{-1},gxg^{-1},\rho_q\circ\mathrm{Ad}_g^{-1}\big) \end{tikzcd} \tag{66}$$

commutes for all $g \in G\gamma$ and $h \in \mathcal{H}(G)$. Hence

(67) Reeder's parametrization of $\mathbf{Irr}(\mathcal{H}(G))$ is Γ-equivariant.

Let $\pi \in \mathbf{Irr}(\mathcal{H}(G))$ and choose a Kazhdan–Lusztig triple such that π is equivalent with $\pi(t_q,x,\rho_q)$. Composition with γ^{-1} on π gives rise to a 2-cocycle $\natural(\pi)$ of Γ_π. Clifford theory tells us that every irreducible representation of $\mathcal{H}(G)\rtimes\Gamma$ is of the form $\pi\rtimes\rho_2$ for some $\pi\in\mathbf{Irr}(\mathcal{H}(G))$, unique up to Γ-equivalence, and a unique $\rho_2\in\mathbf{Irr}(\mathbb{C}[\Gamma_\pi,\natural(\pi)])$. By the above the stabilizer of π in Γ equals the stabilizer of the G-conjugacy class $[t_q,x,\rho_q]_G$. Thus we have parametrized $\mathbf{Irr}(\mathcal{H}(G)\rtimes\Gamma)$ in a natural way with $G\rtimes\Gamma$-conjugacy classes of quadruples (t_q,x,ρ_q,ρ_2), where (t_q,x,ρ_q) is a Kazhdan–Lusztig triple for G and $\rho_2\in\mathbf{Irr}\big(\mathbb{C}[\Gamma_{[t_q,x,\rho_q]_G},\natural(\pi(t_q,x,\rho_q))]\big)$.

The short exact sequence

$$1 \to \pi_0(\mathrm{Z}_G(t_q,x)) \to \pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t_q,x)) \to \Gamma_{[t_q,x]_G} \to 1 \tag{68}$$

yields an action of $\Gamma_{[t_q,x]_G}$ on $\mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t_q,x))\big)$. Restricting this to the stabilizer of ρ_q, we obtain another 2-cocycle $\natural(t_q,x,\rho_q)$ of $\Gamma_{[t_q,x,\rho_q]_G}$, which we want to compare to $\natural(\pi(t_q,x,\rho_q))$. Let us decompose

$$H_*(\mathcal{B}^{t_q,x},\mathbb{C}) \cong \bigoplus\nolimits_{\rho_q} \rho_q\otimes M(t_q,x,\rho_q)$$

as $\pi_0((Z_G(t_q, x)) \times \mathcal{H}(G)$-modules. We sum over all $\rho_q \in \mathbf{Irr}\big(\pi_0(Z_G(t_q,x))\big)$ for which the contribution is nonzero, and we know that for such ρ_q the $\mathcal{H}(G)$-module $M(t_q,x,\rho_q)$ has a unique irreducible quotient $\pi(t_q,x,\rho_q)$. Since $\pi_0(Z_{G\rtimes\Gamma}(t_q,x))$ acts (via conjugation of Borel subgroups) on $H_*(\mathcal{B}^{t_q,x},\mathbb{C})$, any splitting of (68) as sets provides a 2-cocycle $\natural$ for the action of $\Gamma_{[t_q,x,\rho_q]_G}$ on $\rho_q \otimes M(t_q,x,\rho_q)$. Unfortunately we cannot apply Lemma 4.2 to find a splitting of (68) as groups, because $Z_G(t_q)$ need not be connected. Nevertheless $\natural$ can be used to describe the actions of $\Gamma_{[t_q,x,\rho_q]_G}$ on both ρ_q and $\pi(t_q,x,\rho_q)$, so

$$\natural(t_q,x,\rho_q) = \natural = \natural(\pi(t_q,x,\rho_q)) \text{ as 2-cocycles of } \Gamma_{[t_q,x,\rho_q]_G}. \tag{69}$$

It follows that every irreducible representation ρ of $\pi_0(Z_{G\rtimes\Gamma}(t_q,x))$ is of the form $\rho_q \rtimes \rho_2$ for ρ_q and ρ_2 as above. Moreover ρ determines ρ_q up to $\Gamma_{[t_q,x]_G}$-equivalence and ρ_2 is unique if ρ_q has been chosen. Finally, if ρ_q appears in $H_{\text{top}}(\mathcal{B}^{t_q,x},\mathbb{C})$ then every irreducible $\pi_0(Z_G(t_q,x))$-subrepresentation of ρ does, because $\pi_0(Z_{G\rtimes\Gamma}(t_q,x))$ acts naturally on $H_*(\mathcal{B}^{t_q,x},\mathbb{C})$. Therefore we may replace the above quadruples (t_q,x,ρ_q,ρ_2) by Kazhdan–Lusztig triples (t_q,x,ρ).

The module associated to (t_q,x,ρ_q,ρ_2) via the above constructions is the unique irreducible quotient of the $\mathcal{H}(G) \rtimes \Gamma$-module

$$\text{Hom}_{\pi_0(Z_G(t_q,x))}\big(\rho_q, H_*(\mathcal{B}^{t_q,x},\mathbb{C})\big) \rtimes \rho_2. \tag{70}$$

The same reasoning as in the proof of Theorem 4.4 shows that (70) is isomorphic to

$$\text{Hom}_{\pi_0(Z_{G\rtimes\Gamma}(t_q,x))}\big(\rho, H_*(\mathcal{B}^{t_q,x},\mathbb{C}) \otimes \mathbb{C}[\Gamma]\big). \tag{71}$$

Since the $\mathcal{H}(G)$-module $H_*(\mathcal{B}^{t_q,x},\mathbb{C})$ depends in a natural way on (t_q,x), so does the unique irreducible quotient of (71). $\square$

10. Spherical representations

Let G,B,T and Γ be as in the previous section. Let $\mathcal{H}(\mathcal{W}^G)$ be the Iwahori–Hecke algebra of the Weyl group $\mathcal{W}^G$, with a parameter $q \in \mathbb{C}^\times$ which is not a root of unity. This is a deformation of the group algebra $\mathbb{C}[\mathcal{W}^G]$ and a subalgebra of the affine Hecke algebra $\mathcal{H}(G)$. The multiplication is defined in terms of the basis $\{T_w \mid w \in \mathcal{W}^G\}$ by

$$\begin{aligned} T_x T_y &= T_{xy}, \quad \text{if } \ell(xy) = \ell(x) + \ell(y), \text{ and} \\ (T_s - q)(T_s + 1) &= 0, \quad \text{if } s \text{ is a simple reflection.} \end{aligned} \tag{72}$$

Recall that $\mathcal{H}(G)$ also has a commutative subalgebra $\mathcal{O}(T)$, such that the multiplication maps

$$\mathcal{O}(T) \otimes \mathcal{H}(\mathcal{W}^G) \longrightarrow \mathcal{H}(G) \longleftarrow \mathcal{H}(\mathcal{W}^G) \otimes \mathcal{O}(T) \tag{73}$$

are bijective.

The trivial representation of $\mathcal{H}(\mathcal{W}^G) \rtimes \Gamma$ is defined as

$$\text{triv}(T_w\gamma) = q^{\ell(w)} \quad w \in \mathcal{W}^G, \gamma \in \Gamma. \tag{74}$$

It is associated to the idempotent

$$p_{\text{triv}}\, p_\Gamma := \sum_{w \in \mathcal{W}^G} T_w P_{\mathcal{W}^G}(q)^{-1} \sum_{\gamma \in \Gamma} \gamma |\Gamma|^{-1} \quad \in \mathcal{H}(\mathcal{W}^G) \rtimes \Gamma,$$

where $P_{\mathcal{W}^G}$ is the Poincaré polynomial

$$P_{\mathcal{W}^G}(q) = \sum\nolimits_{w \in \mathcal{W}^G} q^{\ell(w)}.$$

Notice that $P_{\mathcal{W}^G}(q) \neq 0$ because q is not a root of unity. The trivial representation appears precisely once in the regular representation of $\mathcal{H}(\mathcal{W}^G) \rtimes \Gamma$, just like for finite groups.

An $\mathcal{H}(G) \rtimes \Gamma$-module V is called spherical if it is generated by the subspace $p_{\mathrm{triv}} p_\Gamma V$ [HeOp, (2.5)]. This admits a nice interpretation for the unramified principal series representations. Recall that $\mathcal{H}(G) \cong \mathcal{H}(\mathcal{G}, \mathcal{I})$ for an Iwahori subgroup $\mathcal{I} \subset \mathcal{G}$. Let $\mathcal{K} \subset \mathcal{G}$ be a good maximal compact subgroup containing $\mathcal{I}$. Then p_{triv} corresponds to averaging over $\mathcal{K}$ and $p_{\mathrm{triv}} \mathcal{H}(\mathcal{G}, \mathcal{I}) p_{\mathrm{triv}} \cong \mathcal{H}(\mathcal{G}, \mathcal{K})$, see [HeOp, Section 1]. Hence spherical $\mathcal{H}(\mathcal{G}, \mathcal{I})$-modules correspond to smooth $\mathcal{G}$-representations that are generated by their $\mathcal{K}$-fixed vectors, also known as $\mathcal{K}$-spherical $\mathcal{G}$-representations. By the Satake transform

$$p_{\mathrm{triv}} \mathcal{H}(\mathcal{G}, \mathcal{I}) p_{\mathrm{triv}} \cong \mathcal{H}(\mathcal{G}, \mathcal{K}) \cong \mathcal{O}(T/\mathcal{W}^G), \tag{75}$$

so the irreducible spherical modules of $\mathcal{H}(G) \cong \mathcal{H}(\mathcal{G}, \mathcal{I})$ are parametrized by $T/\mathcal{W}^G$ via their central characters. We want to determine the Kazhdan–Lusztig triples (as in Theorem 9.1) of these representations.

Proposition 10.1. *For every central character* $(\mathcal{W}^G \rtimes \Gamma)t \in T/(\mathcal{W}^G \rtimes \Gamma)$ *there is a unique irreducible spherical* $\mathcal{H}(G) \rtimes \Gamma$*-module, and it is associated to the Kazhdan–Lusztig triple* $(t, x = 1, \rho = \mathrm{triv})$.

Proof. We will first prove the proposition for $\mathcal{H}(G)$, and only then consider Γ.

By the Satake isomorphism (75) there is a unique irreducible spherical $\mathcal{H}(G)$-module for every central character $\mathcal{W}^G t \in T/\mathcal{W}^G$. The equivalence classes of Kazhdan–Lusztig triples of the form $(t, x = 1, \rho = \mathrm{triv})$ are also in canonical bijection with $T/\mathcal{W}^G$. Therefore it suffices to show that $\pi(t, 1, \mathrm{triv})$ is spherical for all $t \in T$.

The principal series of $\mathcal{H}(G)$ consists of the modules $\mathrm{Ind}_{\mathcal{O}(T)}^{\mathcal{H}(G)} \mathbb{C}_t$ for $t \in T$. This module admits a central character, namely $\mathcal{W}^G t$. By (73) every such module is isomorphic to $\mathcal{H}(\mathcal{W}^G)$ as a $\mathcal{H}(\mathcal{W}^G)$-module. In particular it contains the trivial $\mathcal{H}(\mathcal{W}^G)$-representation once and has a unique irreducible spherical subquotient.

As in Section 9, let $\tilde{G}$ be a finite central extension of G with simply connected derived group. Let $\tilde{T}, \tilde{B}$ be the corresponding extensions of T, B. We identify the roots and the Weyl groups of $\tilde{G}$ and G. Let $\tilde{t} \in \tilde{T}$ be a lift of $t \in T$. From the general theory of Weyl groups it is known that there is a unique $t^+ \in \mathcal{W}^G \tilde{T}$ such that $|\alpha(t^+)| \geq 1$ for all $\alpha \in R(\tilde{B}, \tilde{T}) = R(B, T)$. By (59)

$$H_*(\mathcal{B}_{\tilde{G}}^{\tilde{t}}, \mathbb{C}) \cong H_*(\mathcal{B}_{\tilde{G}}^{t^+}, \mathbb{C})$$

as $\mathcal{H}(\tilde{G})$-modules. These $t^+, \tilde{B}$ fulfill [Ree2, Lemma 2.8.1], so by [Ree2, Proposition 2.8.2]

$$M_{\tilde{t}, \tilde{x}=1, \tilde{\rho}=\mathrm{triv}} = H_*(\mathcal{B}_{\tilde{G}}^{t^+}, \mathbb{C}) \cong \mathrm{Ind}_{\mathcal{O}(\tilde{T})}^{\mathcal{H}(\tilde{G})} \mathbb{C}_{t^+}. \tag{76}$$

According to [Ree1, (1.5)], which applies to t^+, the spherical vector p_{triv} generates $M_{\tilde{t},1,\mathrm{triv}}$. Therefore it cannot lie in any proper $\mathcal{H}(\tilde{G})$-submodule of $M_{\tilde{t},1,\mathrm{triv}}$ and represents a nonzero element of $\pi(\tilde{t},1,\mathrm{triv})$. We also note that the central character of $\pi(\tilde{t},1,\mathrm{triv})$ is that of $M_{\tilde{t},1,\mathrm{triv}}$, $\mathcal{W}^G\tilde{t} = \mathcal{W}^G t^+$.

Now we analyse this is an $\mathcal{H}(G)$-module. The group $R_{\tilde{t},1} = R_{\tilde{t},\tilde{x}=1,\tilde{\rho}=\mathrm{triv}}$ from (61) is just the component group $\pi_0(\mathrm{Z}_G(t))$, so by (65)

$$\pi(\tilde{t},1,\mathrm{triv}) \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t))}(\rho,\pi(\tilde{t},1,\mathrm{triv})) = \bigoplus\nolimits_\rho \pi(t,1,\mathrm{triv}).$$

The sum runs over $\mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t))\big)$, all these representations ρ contribute nontrivially by [Ree2, Lemma 3.5.1]. Recall from Lemma 3.2 that $\pi_0(\mathrm{Z}_G(t))$ can be realized as a subgroup of $\mathcal{W}^G$ and from (75) that $p_{\mathrm{triv}} \in \pi(\tilde{t},1,\mathrm{triv})$ can be regarded as a function on $\tilde{\mathcal{G}}$ which is bi-invariant under a good maximal compact subgroup $\tilde{\mathcal{K}}$. This brings us in the setting of [Cas, Proposition 4.1], which says that $\pi_0(\mathrm{Z}_G(t))$ fixes $p_{\mathrm{triv}} \in \pi(\tilde{t},1,\mathrm{triv})$. Hence $\pi(t,1,\mathrm{triv})$ contains p_{triv} and is a spherical $\mathcal{H}(G)$-module. Its central character is the restriction of the central character of $\pi(\tilde{t},1,\mathrm{triv})$, that is, $\mathcal{W}^G t \in T/\mathcal{W}^G$.

Now we include Γ. Suppose that V is a irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-module. By Clifford theory its restriction to $\mathcal{H}(G)$ is a direct sum of irreducible $\mathcal{H}(G)$-modules, each of which contains p_{triv}. Hence V is built from irreducible spherical $\mathcal{H}(G)$-modules. By (67)

$$\gamma \cdot \pi(t,1,\mathrm{triv}) = \pi(\gamma t,1,\mathrm{triv}),$$

so the stabilizer of $\pi(t,1,\mathrm{triv}) \in \mathbf{Irr}(\mathcal{H}(G))$ in Γ equals the stabilizer of $\mathcal{W}^G t \in T/\mathcal{W}^G$ in Γ. Any isomorphism of $\mathcal{H}(G)$-modules

$$\psi_\gamma : \pi(t,1,\mathrm{triv}) \to \pi(\gamma t,1,\mathrm{triv})$$

must restrict to a bijection between the onedimensional subspaces of spherical vectors in both modules. We normalize ψ_γ by $\psi_\gamma(p_{\mathrm{triv}}) = p_{\mathrm{triv}}$. Then $\gamma \mapsto \psi_\gamma$ is multiplicative, so the 2-cocycle of $\Gamma_{\mathcal{W}^G t}$ is trivial. With Theorem 9.1 this means that the irreducible $\mathcal{H}(G) \rtimes \Gamma$-modules whose restriction to $\mathcal{H}(G)$ is spherical are parametrized by equivalence classes of triples $(t,1,\mathrm{triv} \rtimes \sigma)$ with $\sigma \in \mathbf{Irr}(\Gamma_{\mathcal{W}^G t})$. The corresponding module is

$$\pi(t,1,\mathrm{triv} \rtimes \sigma) = \pi(t,1,\mathrm{triv}) \rtimes \sigma = \mathrm{Ind}_{\mathcal{H}(G)\rtimes\Gamma_{\mathcal{W}^G t}}^{\mathcal{H}(G)\rtimes\Gamma}\big(\pi(t,1,\mathrm{triv}) \otimes \sigma\big).$$

Clearly $\pi(t,1,\mathrm{triv} \rtimes \sigma)$ contains the spherical vector $p_{\mathrm{triv}} p_\Gamma$ if and only if σ is the trivial representation. It follows that the irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-modules are parametrized by equivalence classes of triples $\big(t,1,\mathrm{triv}_{\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t))}\big)$, that is, by $T/(\mathcal{W}^G \rtimes \Gamma)$. $\square$

11. From the principal series to affine Hecke algebras

Let χ be a smooth character of the maximal torus $\mathcal{T} \subset \mathcal{G}$. We recall that

$$\begin{aligned} \mathfrak{s} &= [\mathcal{T},\chi]_{\mathcal{G}}, \\ c^{\mathfrak{s}} &= \hat{\chi}\big|_{\mathfrak{o}_F^\times}, \\ H &= \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}}), \\ W^{\mathfrak{s}} &= \mathrm{Z}_{\mathcal{W}^G}(\mathrm{im}\, c^{\mathfrak{s}}). \end{aligned}$$

Let $\{\text{KLR parameters}\}^{\mathfrak s}$ be the collection of Kazhdan–Lusztig–Reeder parameters for G such that $\Phi\big|_{\mathfrak o_F^\times} = c^{\mathfrak s}$. Notice that the condition forces $\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb C)) \subset H$. This collection is not closed under conjugation by elements of G, only $H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak s})$ acts naturally on it.

Recall that $T^{\mathfrak s}$ and $T^{\mathfrak s}/W^{\mathfrak s}$ are Bernstein's torus and Bernstein's centre associated to $\mathfrak s$. Clearly T acts simply transitively on $T^{\mathfrak s}$, but we need a little more. Consider the bijections

$$T^{\mathfrak s} \longrightarrow \{\text{L-parameters } \Phi \text{ for } \mathcal T \text{ with } \Phi|_{\mathfrak o_F^\times} = c^{\mathfrak s}\} \xrightarrow{\mathrm{ev}_{\varpi_F}} T, \tag{77}$$

where the first map is the restriction of the local Langlands correspondence for $\mathcal T$ to $T^{\mathfrak s}$ and the second map sends Φ to $\Phi(\varpi_F)$. The latter is not natural because it depends on our choice of ϖ_F, but since we use the same uniformizer everywhere this is not a problem.

As $T^{\mathfrak s}$ is a maximal torus in H, every semisimple element of H° is conjugate to one in T. By Lemma 3.2 $W^{\mathfrak s} \cong N_H(T)/T$, so we can identify $T^{\mathfrak s}/W^{\mathfrak s}$ with the space $c(H)_{\mathrm{ss}}$ of semisimple conjugacy classes in H that consist of elements if H°.

In general H need not be connected. Recall from Lemma 3.3 that any choice of a pinning of H° determines a splitting of the short exact sequence

$$1 \to H^\circ/\mathrm{Z}(H^\circ) \to H/\mathrm{Z}(H^\circ) \to \pi_0(H) \to 1. \tag{78}$$

Lemma 3.2 shows that

$$W^{\mathfrak s} = \mathcal W^G_{\mathrm{im}\, c^{\mathfrak s}} \cong \mathcal W^{H^\circ} \rtimes \pi_0(H). \tag{79}$$

We fix a Borel subgroup $B \subset G$ containing T, and a pinning of H° with T as maximal torus and $B_H = B \cap H^\circ$ as Borel subgroup. This determines a conjugation action of $\pi_0(H)$ on H°, and hence on objects associated to H°. Like in Section 9, let $\mathcal H(H^\circ)$ be the affine Hecke algebra with the same based root datum as (H°, B), and with parameter q equal to the cardinailty of the residue field of F. By our conventions $\pi_0(H)$ normalizes B, so it acts on $\mathcal H(H^\circ)$ by algebra automorphisms. Following [Roc, Section 8] we define

$$\mathcal H(H) = \mathcal H(H^\circ) \rtimes \pi_0(H). \tag{80}$$

We denote the Hecke algebra of $\mathcal G$ by $\mathcal H(\mathcal G)$. Recall that its consists of all locally constant compactly supported functions $\mathcal G \to \mathbb C$ and is endowed with the convolution product. The category $\mathrm{Rep}(\mathcal G)$ of smooth $\mathcal G$-representations is naturally equivalent to the category of nondegenerate $\mathcal H(\mathcal G)$-modules. Let $\mathrm{Rep}(\mathcal G)^{\mathfrak s}$ be the block of $\mathrm{Rep}(\mathcal G)$ associated to $\mathfrak s$.

The link between these representations and Section 9 is provided by results of Roche. In [Roc, p. 378–379] Roche imposes some conditions on the residual characteristic of the field.

Condition 11.1. *If the root system $R(H,T)$ is irreducible, then the restriction on the residual characteristic p of F is as follows:*

- *for type A_n* $\quad p > n+1$
- *for types B_n, C_n, D_n* $\quad p \neq 2$
- *for type F_4* $\quad p \neq 2,3$
- *for types G_2, E_6* $\quad p \neq 2,3,5$
- *for types E_7, E_8* $\quad p \neq 2,3,5,7.$

If $R(H,T)$ is reducible, one excludes primes attached to each of its irreducible factors.

Since $R(H,T)$ is a subset of $R(G,T) \cong R(\mathcal{G},\mathcal{T})^\vee$, these conditions are fulfilled when they hold for $R(\mathcal{G},\mathcal{T})$.

Theorem 11.2. *Assume that Condition 11.1 holds. There exists an equivalence of categories*

$$\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \longleftrightarrow \mathrm{Mod}(\mathcal{H}(H))$$

such that:

1. *The cuspidal support of an irreducible $\mathcal{G}$-representation corresponds to the central character of the associated $\mathcal{H}(H)$-module via the canonical bijection $T^{\mathfrak{s}}/W^{\mathfrak{s}} \to c(H)_{\mathrm{ss}}$.*
2. *It does not depend on the choice of χ with $[\mathcal{T},\chi]_{\mathcal{G}} = \mathfrak{s}$.*

Proof. First we note that, although Roche [Roc] works with a p-adic field, it follows from [AdRo] that his arguments apply just as well over local fields of positive characteristic. By [Roc, Corollary 7.9] there exists a type (J,τ) for $\mathfrak{s} = [\mathcal{T},\chi]_{\mathcal{G}}$, where τ is a character. Then the τ-spherical Hecke algebra $\mathcal{H}(\mathcal{G},\tau)$ of $\mathcal{H}(\mathcal{G})$ (see [BuKu, §2]) equals $e_\tau \mathcal{H}(\mathcal{G}) e_\tau$, where $e_\tau \in \mathcal{H}(J)$ is the central idempotent corresponding to τ. According to [BuKu, Theorem 4.3] there exists an equivalence of categories

$$\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \to \mathrm{Mod}(\mathcal{H}(\mathcal{G},\tau)) : V \mapsto V^\tau, \tag{81}$$

where $V^\tau = e_\tau V$ is the τ-isotypical subspace of $V|_J$. From the proof of [BuKu, Proposition 3.3] we see that the inverse of (81) is given by

$$\mathrm{Mod}(\mathcal{H}(\mathcal{G},\tau)) \to \mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} : M \mapsto \mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} M. \tag{82}$$

Theorem 8.2 of [Roc] says that there exists a support preserving algebra isomorphism

$$\mathcal{H}(H) \to \mathcal{H}(\mathcal{G},\tau). \tag{83}$$

The combination of (81) and (83) yields the desired equivalence of categories. It satisfies property (1) by [Roc, Theorem 9.4].

In [Roc, §9] it is shown that (J,τ) is a cover of the type $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$, in the sense of [BuKu, §8]. With [Roc, Theorem 9.4] one sees that the above equivalence of categories does not change if one twists χ by an unramified character of $\mathcal{T}$, basically because that does not effect $\chi|_{\mathcal{T}_0}$.

Every other character of $\mathcal{T}$ determining the same inertial equivalence class $\mathfrak{s}$ can be obtained from χ by an unramified twist and conjugation by an element of $W^{\mathfrak{s}}$. Reeder [Ree2, §6] checked that the latter operation does not change Roche's equivalence of categories. We note that in [Ree2] it is assumed that H is connected. Fortunately this does not play a role in [Ree2, §6], because all the underlying results from [Roc] and [Mor] are known irrespective of the connectedness. □

We emphasize that Theorem 11.2 is the only cause of our conditions on the residual characteristic. If one can prove Theorem 11.2 for a particular Bernstein component and a p which is excluded by Condition 11.1, then everything in our paper (except possibly Lemma 12.1) holds for that case.

For example, for unramified characters χ Theorem 11.2 is already classical, proven without any restrictions on p by Borel [Bor1]. As Roche remarks in [Roc, 4.14], all the main results of [Roc] (and hence Theorem 11.2) are valid without restrictions on p when $\mathcal{G} = \mathrm{GL}_n(F)$ or $\mathcal{G} = \mathrm{SL}_n(F)$. For $\mathrm{GL}_n(F)$ this is easily seen, for $\mathrm{SL}_n(F)$ one can use [GoRo].

Theorems 11.2 and 9.1 provide a bijection

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H)) \to \{\text{KLR-parameters}\}^{\mathfrak{s}}/H. \tag{84}$$

Unfortunately this bijection is not entirely canonical in general.

Example 11.3. *Consider the unramified principal series representations of* $\mathrm{SL}_2(F)$*. Then the type is the trivial representation of an Iwahori subgroup* $\mathcal{I} \subset \mathrm{SL}_2(F)$ *and Theorem 11.2 reduces to* [Bor1]*. The functor sends a* $\mathrm{SL}_2(F)$*-representation to its space of* $\mathcal{I}$*-fixed vectors. The Iwahori subgroup is determined by the choice of a maximal compact subgroup and a Borel subgroup of* $\mathrm{SL}_2(F)$*, and these data also determine the isomorphism* $\mathcal{H}(\mathrm{SL}_2(F), \mathrm{triv}_{\mathcal{I}}) \cong \mathcal{H}(H)$*.*

However, there are two conjugacy classes of maximal compact subgroups in $\mathrm{SL}_2(F)$*. If we pick a maximal compact subgroup in the other class and perform the same operations, we obtain an alternative map* (84)*. The difference is not big, for almost all* $\mathrm{SL}_2(F)$*-representations the two maps have the same image. But look at the parabolically induced representation* $\pi = I_{\mathcal{B}}^{\mathrm{SL}_2(F)}(\chi_{-1})$*, where* χ_{-1} *denotes the unique unramified character of* $\mathcal{T}$ *of order 2. It is well-known that* π *is the direct sum of two inequivalent irreducible representations, say* π_+ *and* π_-*. It turns out that the difference between our two candidates for* (84) *is just interchanging* π_+ *and* π_-*.*

We will determine in Section 14 how canonical (84) is precisely.

12. Main result (special case)

In the current section we will study the relations between $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ and $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$, in the case that H is connected. This happens for most $\mathfrak{s}$, a sufficient condition is:

Lemma 12.1. *Suppose that G has simply connected derived group and that the residual characteristic p satisfies Condition 11.1 for $R(G,T)$. Then H is connected.*

Proof. We consider first the case where $\mathfrak{s} = [\mathcal{T}, 1]_{\mathcal{G}}$. Then we have $c^{\mathfrak{s}} = 1, H = G$ and $W^{\mathfrak{s}} = \mathcal{W}$.

We assume now that $c^{\mathfrak{s}} \neq 1$. Then $\operatorname{im} c^{\mathfrak{s}}$ is a finite abelian subgroup of T which has the following structure: the direct product of a finite abelian p-group A_p with a cyclic group B_{q-1} whose order divides $q-1$. This follows from the well-known structure theorem for the group $\mathfrak{o}_F^{\times}$, see [Iwa, §2.2]:

$$\operatorname{im} c^{\mathfrak{s}} = A_p \cdot B_{q-1}.$$

We have

$$H = \mathrm{Z}_{H_A}(B_{q-1}) \quad \text{where} \quad H_A := \mathrm{Z}_G(A_p).$$

Since G has simply connected derived group, A_p is a p-group and p is not a torsion prime for the root system $R(G,T)$, it follows from Steinberg's connectedness theorem [Ste2, 2.16.b] that the group H_A is connected. It was shown in [Roc, p. 397] that $H_A = \mathrm{Z}_G(x)$ for a well-chosen $x \in T$. Then [Ste2, 2.17] says that the derived group of $H_A^\circ = H_A$ is simply connected.

Now B_{q-1} is cyclic. Applying Steinberg's connectedness theorem to the group H_A, we get that H itself is connected. $\square$

Remark 12.2. *Notice that H does not necessarily have a simply connected derived group in setting of Lemma 12.1. For instance, if G is the exceptional group of type* G_2 *and χ is the tensor square of a ramified quadratic character of $F^\times$, then $H = \mathrm{SO}_4(\mathbb{C})$.*

In the remainder of this section we will assume that H is connected, Then Lemma 3.2 shows that $W^{\mathfrak{s}}$ is the Weyl group of H.

Theorem 12.3. *Let $\mathcal{G}$ be a split reductive p-adic group and let $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ be a point in the Bernstein spectrum of the principal series of $\mathcal{G}$. Assume that H is connected and that Condition 11.1 holds. Then there is a commutative diagram of bijections, in which the triangle is canonical:*

$$\begin{tikzcd}
 & & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[r] & \mathbf{Irr}(\mathcal{H}(H)) \arrow[rr] & & \{\text{KLR parameters}\}^{\mathfrak{s}}/H
\end{tikzcd}$$

In the triangle the right slanted map stems from Kato's affine Springer correspondence [Kat]*. The bottom horizontal map is the bijection established by Reeder* [Ree2] *and the left slanted map can be constructed via the asymptotic Hecke algebra of Lusztig.*

Proof. Theorem 11.2 provides the bijection $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H))$.

The right slanted map is the composition of Theorem 8.2.1 (applied to H) and Lemma 7.1 (with the condition $\Phi(\varpi_F) = t$). We can take as the horizontal map the parametrization of irreducible $\mathcal{H}(H)$-modules by Kazhdan, Lusztig and Reeder as described in Section 9. These are both canonical bijections, so there is a unique left slanted map which makes the diagram commute, and it is also canonical. We want to identify it in terms of Hecke algebras.

Fix a KLR parameter (Φ, ρ) and recall from Theorem 8.2.2 that the corresponding $X^*(T) \rtimes \mathcal{W}^H$-representation is

$$\tau(t, x, \rho) = \mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho, H_{d(x)}(\mathcal{B}_H^{t,x}, \mathbb{C})\big). \tag{85}$$

Similarly, by Theorem 9.1 the corresponding $\mathcal{H}(H)$-module is the unique irreducible quotient of the $\mathcal{H}(H)$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t_q,x))}\big(\rho_q, H_*(\mathcal{B}_H^{t_q,x}, \mathbb{C})\big). \tag{86}$$

In view of Proposition 6.1 both spaces are unchanged if we replace t by t_q and ρ by ρ_q, and the vector space (86) is also naturally isomorphic to

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(\Phi))}\big(\rho, H_*(\mathcal{B}_H^{t,\Phi(B_2)}, \mathbb{C})\big). \tag{87}$$

Recall the asymptotic Hecke algebra $\mathcal{J}(H)$ from [Lus2]. We remark that, although in [Lus2] the underlying reductive group H is supposed to be semisimple, this assumption is shown to be unnecessary in [Lus3]. Lusztig constructs canonical bijections

$$\mathbf{Irr}(\mathcal{H}(H)) \longleftrightarrow \mathbf{Irr}(\mathcal{J}(H)) \longleftrightarrow \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) \tag{88}$$

which we will analyse with our terminology. According to [Lus3, Theorem 4.2] $\mathbf{Irr}(\mathcal{J}(H))$ is naturally parametrized by the set of H-conjugacy classes of Kazhdan–Lusztig triples for H. By Lemma 7.1 we can also use KLR parameters, so may call the $\mathcal{J}(H)$-module with parameters (t_q, x, ρ_q) $\tilde\pi(\Phi, \rho)$. Its retraction to $\mathcal{H}(H)$ via

$$\mathcal{H}(H) \xrightarrow{\phi_q} \mathcal{J}(H) \xleftarrow{\phi_1} X^*(T) \rtimes \mathcal{W}^H \tag{89}$$

is described in [Lus3, 2.5]. It is essentially the ρ_q-isotypical part of the $\langle t_q \rangle \times \mathbb{C}^\times$-equivariant K-theory of the variety $\mathcal{B}^{t_q,x}$. With [ChGi, Theorem 6.2.4] this can be translated to the terminology of Section 9, and one can see that it is none other than (86).

Recall that $\mathbf{q}$ is an indeterminate and let $\mathcal{H}_v(H) = \mathcal{H}_{\mathbf{q}}(H) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_v$ be the affine Hecke algebra with the same based root datum as H and with parameter $v \in \mathbb{C}^\times$. Thus

$$\mathcal{H}_q(H) = \mathcal{H}(H) \quad \text{and} \quad \mathcal{H}_1(H) = \mathbb{C}[X^*(T) \rtimes \mathcal{W}^H].$$

Like in (54), let $\tilde H$ be a central finite extension of H whose derived group is simply connected. By (55) and (57)

$$\mathcal{H}_v(H) \cong \big(K^{\tilde H \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde H}) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_v \big)^{\ker(\tilde H \to H)}. \tag{90}$$

The above, in particular (86), describes the retraction $\tilde\pi(\Phi, \rho) \in \mathbf{Irr}(\mathcal{J}(H))$ to $\mathcal{H}_v(H)$ for any $v \in \mathbb{C}^\times$.

In [Lus2, Corollary 3.6] the a-function is used to single out a particular irreducible quotient $\mathcal{H}_v(H)$-module of (86). This applies when $v = 1$ or v is not a root of unity. For $\mathcal{H}_q(H)$ we saw in (64) that there is only one such quotient, which by definition is $\pi(t_q, x, \rho_q)$. This is our description of the left hand side of (88).

For $v = 1$ we need a different argument. By the above and (87) we obtain the $\mathcal{H}_1(H)$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))} \big(\rho, H_*(\mathcal{B}_H^{t,x}, \mathbb{C}) \big) \tag{91}$$

with the action coming from (90), (58) and the convolution product in Borel–Moore homology. Let us compare this with Kato's action [Kat], as described in Section 8. On the subalgebra $\mathbb{C}[\mathcal{W}^H]$ both are defined in terms of Borel–Moore homology, respectively with $K^{\tilde H \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde H})$ and with $H(\mathcal{Z}_{\tilde H})$. It follows from [ChGi, (7.2.12)] that they agree. An element $\lambda \in X^*(T)$ acts via (90) on K-theory as tensoring with a line bundle over $\mathcal{B}_{\tilde H}^{\tilde x}$ canonically associated to λ, see [ChGi, p. 395] or [KaLu, Theorem 3.5]. From the descriptions given in [ChGi, p. 420] and [Kat, §3] we see that on (91) this reduces to the action coming from (45). In other words, we checked that the $\mathcal{H}_1(H)$-module (91) contains Kato's module (46), as the homology in top degree.

We want to see what the right hand bijection in (88) does to $\tilde\pi(\Phi, \rho)$. By construction it produces a certain irreducible quotient of (91), namely the unique one with minimal a-weight. Unfortunately this is not so easy to analyse directly. Therefore we consider the opposite direction, starting with an irreducible $\mathcal{H}_1(H)$-module V with a-weight a_V. According to [Lus2, Corollary 3.6] the $\mathcal{J}(H)$-module

$$\tilde{V} := \mathcal{H}_1(H)^{a_V} \otimes_{\mathcal{H}_1(H)} V,$$

is irreducible and has a-weight a_V. See [Lus2, Lemma 1.9] for the precise definition of $\tilde{V}$.

Now we fix $t \in T$ and we will prove with induction to $\dim \mathcal{O}_x$ that $\widetilde{\tau(t,x,\rho)}$ is none other than $\tilde{\pi}(\Phi,\rho)$. Our main tool is Lemma 8.3, which says that the constituents of (91) are $\tau(t,x,\rho)$ and irreducible representations corresponding to larger affine Springer parameters (with respect to the partial order defined via the unipotent classes $\mathcal{O}_x \subset M$). For $\dim \mathcal{O}_{x_0} = 0$ we see immediately that only the $\mathcal{J}(H)$-module $\tilde{\pi}(t,x_0,\rho_0)$ can contain $\tau(t,x_0,\rho_0)$, so that must be $\widetilde{\tau(t,x_0,\rho_0)}$. For $\dim \mathcal{O}_{x_n} = n$ Lemma 8.3 says that (91) can only contain $\tau(t,x_n,\rho_n)$ if $x \in \overline{\mathcal{O}_{x_n}}$. But when $\dim \mathcal{O}_x < n$

$$\widetilde{\tau(t,x_n,\rho_n)} \not\cong \tilde{\pi}(\Phi,\rho),$$

because the right hand side already is $\widetilde{\tau(t,x,\rho)}$, by the induction hypothesis and the bijectivity of $V \mapsto \tilde{V}$. So the parameter of $\widetilde{\tau(t,x_n,\rho_n)}$ involves an x with $\dim \mathcal{O}_x = n$. Then another look at Lemma 8.3 shows that moreover (x,ρ) must be M-conjugate to (x_n,ρ_n). Hence $\widetilde{\tau(t,x,\rho)}$ is indeed (91).

We showed that the bijections (88) work out as

$$\begin{array}{ccccc} \mathbf{Irr}(\mathcal{H}(H)) & \leftrightarrow & \mathbf{Irr}(\mathcal{J}(H)) & \leftrightarrow & \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) \\ \pi(t_q,x,\rho_q) & \leftrightarrow & \tilde{\pi}(\Phi,\rho) & \leftrightarrow & \tau(t,x,\rho), \end{array} \tag{92}$$

where all the objects in the bottom line are determined by the KLR parameter (Φ,ρ). $\square$

13. Main result (Hecke algebra version)

In this section $q \in \mathbb{C}^\times$ is allowed to be any element of infinite order. We study how Theorem 12.3 can be extended to the algebras and modules from Section 9. So let Γ be a group of automorphisms of G that preserves a chosen pinning, which involves T as maximal torus. With the disconnected group $G \rtimes \Gamma$ we associate three kinds of parameters:

- The extended quotient of the second kind $(T/\!/\mathcal{W}^G \rtimes \Gamma)_2$.
- The space $\mathbf{Irr}(\mathcal{H}_q(G) \rtimes \Gamma)$ of equivalence classes of irreducible representations of the algebra $\mathcal{H}_q(G) \rtimes \Gamma$.
- Equivalence classes of unramified Kazhdan–Lusztig–Reeder parameters. Let $\Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$ be a group homomorphism with $\Phi(\mathbf{I}_F) = 1$ and $\Phi(\mathbf{W}_F) \subset T$. As in Section 6, the component group
$$\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi)) = \pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi(\mathbf{W}_F \times B_2)))$$
acts on $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)},\mathbb{C})$. We take $\rho \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi))\big)$ such that every irreducible $\pi_0(\mathrm{Z}_G(\Phi))$-subrepresentation of ρ appears in $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)},\mathbb{C})$. The set $\{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}}$ of pairs (Φ,ρ) carries an action of $G \rtimes \Gamma$ by conjugation. We consider the collection $\{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}}/G \rtimes \Gamma$ of conjugacy classes $[\Phi,\rho]_{G\rtimes\Gamma}$.

As in the proof of Theorem 12.3, let $\mathcal{J}(G)$ be the asymptotic Hecke algebra of G. The group Γ acts on the extended affine Weyl group $X^*(T) \rtimes \mathcal{W}^G$ in a length-preserving way. Hence every $\gamma \in \Gamma$ naturally determines an automorphism of $\mathcal{J}(G)$, as described in [Lus3, §1]. This enables us to form the crossed product $\mathcal{J}(G) \rtimes \Gamma$.

Theorem 13.1. *There exists a commutative diagram of natural bijections*

$$\begin{tikzcd}
& (T/\!/\mathcal{W}^G \rtimes \Gamma)_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{H}_q(G) \rtimes \Gamma) \arrow[rr] & & \{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}}/G \rtimes \Gamma
\end{tikzcd}$$

It restricts to bijections between the following subsets:

- *the ordinary quotient* $T/(\mathcal{W}^G \rtimes \Gamma) \subset (T/\!/\mathcal{W}^G \rtimes \Gamma)_2$,
- *the collection of spherical representations in* $\mathbf{Irr}(\mathcal{H}_q(G) \rtimes \Gamma)$,
- *equivalence classes of KLR parameters* (Φ, ρ) *for* $G \rtimes \Gamma$ *with* $\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})) = 1$ *and* $\rho = \mathrm{triv}_{\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(\Phi))}$.

Moreover the left slanted map can be constructed via the (irreducible representations of) the algebra $\mathcal{J}(G) \rtimes \Gamma$.

Proof. The corresponding statement for G, proven in Theorem 12.3, is the existence of natural bijections

$$\begin{tikzcd}
\mathbf{Irr}(\mathcal{J}(G)) \arrow[r] \arrow[d] & (T/\!/\mathcal{W}^G)_2 \arrow[dl] \arrow[d] \\
\mathbf{Irr}(\mathcal{H}_q(G)) \arrow[r] & \{\text{KLR parameters for } G\}^{\mathrm{unr}}/G
\end{tikzcd} \tag{93}$$

Although in Section 12 q was a prime power, we notice that among the objects in (93) only the algebra $\mathcal{H}_q(G)$ depends on q. Fortunately the bottom, slanted and left hand vertical maps in (93) are defined equally well for our more general $q \in \mathbb{C}^\times$, as can be seen from the proofs of Theorems 9.1 and 12.3. Thus we may use (93) as our starting point.

Step 1. The bijections in (93) *are* Γ*-equivariant.*
The action of Γ on $(T/\!/\mathcal{W}^G)_2$ can be written as

$$\gamma \cdot [t, \tilde{\tau}]_{\mathcal{W}^G} = [\gamma(t), \tilde{\tau} \circ \mathrm{Ad}_\gamma^{-1}]_{\mathcal{W}^G}. \tag{94}$$

In terms of the multiplication in $G \rtimes \Gamma$, the action on KLR parameters is

$$\gamma \cdot [\Phi, \rho_1]_G = [\gamma \Phi \gamma^{-1}, \rho_1 \circ \mathrm{Ad}_\gamma^{-1}]_G \tag{95}$$

We recall the right hand vertical map in (93) from Theorem 8.2. Write $M = \mathrm{Z}_G(t)$ and $\mathcal{W}_t^G = W(M^\circ, T) \rtimes \pi_0(M)$. Then the $\mathcal{W}_t^G$-representation $\tilde{\tau}$ can be written as $\tau(x, \rho_3) \rtimes \sigma$ for a unipotent element $x \in M^\circ$, a geometric $\rho_3 \in \mathrm{Irr}(\mathrm{Z}_{M^\circ}(x))$ and a $\sigma \in \mathrm{Irr}(\pi_0(M)_{\tau(x,\rho_3)})$. The associated KLR parameter is $[\Phi, \rho_3 \rtimes \sigma]_G$, where $\Phi\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = x$ and Φ maps a Frobenius element of $\mathbf{W}_F$ to t. From (14) we see that $\tau(x, \rho_3) \circ \mathrm{Ad}_\gamma^{-1}$ is equivalent with $\tau(\gamma x \gamma^{-1}, \rho_3 \circ \mathrm{Ad}_\gamma^{-1})$, so

$$\tilde{\tau} \circ \mathrm{Ad}_\gamma^{-1} \text{ is equivalent with } \tau(\gamma x \gamma^{-1}, \rho_3 \circ \mathrm{Ad}_\gamma^{-1}) \rtimes (\sigma \circ \mathrm{Ad}_\gamma^{-1}).$$

Hence (94) is sent to the KLR parameter (95), which means that the right hand vertical map in (93) is indeed Γ-equivariant.

In view of Proposition 6.1 and (95), we already showed in (67) that the lower horizontal map in (93) is Γ-equivariant. By the commutativity of the triangle, so is the slanted map.

As we checked in the proof of Theorem 12.3, the left hand vertical map is retraction along $\phi_q : \mathcal{H}(G) \to \mathcal{J}(G)$ followed by taking the unique irreducible quotient. The algebra homomorphism ϕ_q is Γ-equivariant because Γ respects the entire setup in [Lus3, §1]. Therefore the left hand vertical map is also Γ-equivariant.

Step 2. Suppose that $\tilde{\pi}(\Phi, \rho), [t, \tilde{\tau}]_{\mathcal{W}^G}, \pi$ *and* $[\Phi, \rho_1]_G$ *are three corresponding objects in* (93)*. Then their stabilizers in* Γ *coincide:*

$$\Gamma_{\tilde{\pi}(\Phi,\rho)} = \Gamma_{[t,\tilde{\tau}]_{\mathcal{W}^G}} = \Gamma_\pi = \Gamma_{[\Phi,\rho_1]_G}.$$

This follows immediately from step 1.

Step 3. Clifford theory produces 2-cocycles $\natural\big(\tilde{\pi}(\Phi, \rho)\big)$, $\natural\big([t, \tilde{\tau}]_{\mathcal{W}^G}\big), \natural(\pi)$ *and* $\natural\big([\Phi, \rho_1]_G\big)$ *of* Γ_x*. We can choose the same cocycle for all four of them.*
For $\natural(\pi)$ and $\natural\big([\Phi, \rho_1]_G\big)$ this was already checked in (69), where we use Proposition 6.1 to translate between Φ and (t_q, x).

From (92), and Theorems 8.2 and 9.1 we see that $\tilde{\pi}(\Phi, \rho), [t, \tilde{\tau}]_{\mathcal{W}^G}$ and π come from three rather similar representations. The difference is that $\tilde{\pi}(\Phi, \rho)$ is built from the entire homology of a variety, whereas the other two are quotients thereof. The Γ_π-actions on these three modules are defined in the same way, so the two cocycles can be chosen equal.

We remark that $\natural\big([t, \tilde{\tau}]_{\mathcal{W}^G}\big)$ is trivial by Proposition 4.3, so the other 2-cocycles are also trivial.

Step 4. Upon applying $X \mapsto (X/\!/\Gamma)_2^\natural$ *to the commutative diagram* (93) *we obtain the corresponding diagram for* $G \rtimes \Gamma$*.*
Here $\natural$ denotes the family of 2-cocycles constructed in steps 2 and 3. For $\mathbf{Irr}(\mathcal{J}(G)$, $(T/\!/\mathcal{W}^G)_2$ and $\mathbf{Irr}(\mathcal{H}_q(G))$ we know from Lemmas 2.1 and 2.3 that this procedure yields the correct parameters. That it works for Kazhdan–Lusztig–Reeder parameters was checked in the last part of the proof of Theorem 9.1. By steps 1 and 3 the construction used in (3) yields the same homomorphisms between the twisted group algebras (called $\phi_{\gamma,x}$ in Section 2) in all four settings. Hence the maps from (93) can be lifted in a natural way to the diagram for $G \rtimes \Gamma$.

The ordinary quotient is embedded in $(T/\!/\mathcal{W}^G \rtimes \Gamma)_2$ as the collection of pairs $\big(t, \mathrm{triv}_{(\mathcal{W}^G \rtimes \Gamma)_t}\big)$. By an obvious generalization of (53) these correspond to the affine Springer parameters $(t, x = 1, \rho = \mathrm{triv})$. It is clear from the above construction that they are mapped to KLR parameters (Φ, triv) with $\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})) = 1$ and $\Phi(\varpi_F) = t$. By Proposition 10.1 the latter correspond to the spherical irreducible $\mathcal{H}(G) \rtimes \Gamma$-modules. $\square$

14. Canonicity

We return to the notation from Section 11. We would like to combine Theorems 12.3 and 13.1 to a version that applies to $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ irrespective of the (dis)connectedness of $H = \mathrm{Z}_G(\mathrm{im} c^{\mathfrak{s}})$. We have observed already that everything in Theorem 13.1 is canonical, but we do not know yet how canonical Theorem 11.2 is. Unfortunately a discussion of this issue is avoided in the sources [Roc] and [Ree2].

For this purpose we need some technical results about the extended affine Hecke algebra $\mathcal{H}(H)$. Let us denote the elements of the Bernstein basis of $\mathcal{H}(H)$ by $\theta_\lambda T_w$, where $\lambda \in X^*(T)$ and $w \in \mathcal{W}^H$. The algebra $\mathcal{H}(T)$ is canonically isomorphic to $\mathcal{O}(T) = \mathbb{C}[X^*(T)]$, so it has a basis $\{[\lambda] : \lambda \in X^*(T)\}$. The assignment $[\lambda] \mapsto \theta_\lambda$ determines an algebra injection

$$t_U : \mathcal{H}(T) \cong \mathcal{O}(T) \to \mathcal{H}(H).$$

It is canonical in the sense that it depends only on the based root datum of (H, T), which was fixed by the choice of a Borel subgroup $B_H = B \cap H$. Via t_U we regard $\mathcal{O}(T)$ as a subalgebra of $\mathcal{H}(H)$. It is well-known from [Lus4, §3] that the centre of $\mathcal{H}(H)$ is $\mathcal{O}(T)^{\mathcal{W}^H}$. Let $\mathbb{C}(T)$ be the field of rational functions on T, the quotient field of $\mathcal{O}(T)$. Then $\mathcal{H}(H) \otimes_{\mathrm{Z}(\mathcal{H}(H))} \mathbb{C}(T)^{\mathcal{W}^H}$ carries a natural algebra structure, and as a vector space it is simply

$$\mathcal{H}(H) \otimes_{\mathcal{O}(T)} \mathbb{C}(T) \cong \mathbb{C}(T) \rtimes \mathcal{W}^H.$$

By [Lus4, §6] or [Sol, Proposition 1.5.1] there is an algebra isomorphism

$$\mathcal{H}(H) \otimes_{\mathrm{Z}(\mathcal{H}(H))} \mathbb{C}(T)^{\mathcal{W}^H} \cong \mathbb{C}(T) \rtimes \mathcal{W}^H, \tag{96}$$

which is the identity on $\mathcal{O}(T)$.

Proposition 14.1. *Let ϕ be an automorphism of $\mathcal{H}(H)$ which is the identity on $\mathcal{O}(T)$.*

1. *It induces an automorphism (also denoted by ϕ) of $\mathbb{C}(T) \rtimes \mathcal{W}^H$.*
2. *There exist $z_w \in \mathbb{C}^\times$ and $\lambda_w \in X^*(T)$ such that $\phi(w) = z_w \theta_{\lambda_w} w$ for all $w \in \mathcal{W}^H$.*
3. *For every reflection s_α with $\alpha \in R(H^\circ, T)$ we have $\lambda_{s_\alpha} \in \mathbb{Z}\alpha$.*
4. *$z_w = 1$ for $w \in \mathcal{W}^{H^\circ}$, and $w \mapsto z_w$ is a character of $\pi_0(H) \cong \mathcal{W}^H / \mathcal{W}^{H^\circ}$.*

Proof. (1) is a direct consequence of (96). By assumption ϕ is the identity on the quotient field $\mathbb{C}(T)$ of $\mathcal{O}(T)$. Hence it is of the form

$$\phi : \sum_{w \in \mathcal{W}^H} f_w w \mapsto \sum_{w \in \mathcal{W}^H} f_w \Phi_w w \tag{97}$$

for suitable $\Phi_w \in \mathbb{C}(T)$. Let $\imath_w^\circ \in \mathcal{H}(H) \otimes_{\mathrm{Z}(\mathcal{H}(H))} \mathbb{C}(T)^{\mathcal{W}^H}$ be the image of $w \in \mathcal{W}^H$ under (96). An explicit formula in the case of a simple reflection s_α is given in [Sol, (1.25)]:

$$1 + T_{s_\alpha} = q \frac{\theta_\alpha - q^{-1}}{\theta_\alpha - 1} (1 + \imath_{s_\alpha}^\circ). \tag{98}$$

Since ϕ preserves $\mathcal{H}(H)$, we see from (97) and (98) that $\Phi_{s_\alpha} \in \mathcal{O}(T)^\times = \mathbb{C}^\times X^*(T)$. Say $\Phi_{s_\alpha} = z\theta_\lambda$. Then we calculate in $\mathbb{C}(T) \rtimes \mathcal{W}^H$:

$$1 = s_\alpha^2 = \phi(s_\alpha)^2 = z\theta_\lambda s_\alpha z \theta_\lambda s_\alpha = z^2 \theta_\lambda \theta_{s_\alpha(\lambda)} s_\alpha^2 = z^2 \theta_{\lambda + s_\alpha(\lambda)}.$$

Therefore $z = \pm 1$ and $s_\alpha(\lambda) = -\lambda$, which means that $\lambda \in \mathbb{Q}\alpha \cap X^*(T)$. Now

$$\begin{aligned}\phi(1 + T_{s_\alpha}) &= \phi\Big(\frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 + \imath^\circ_{s_\alpha})\Big)\\ &= \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 + z\theta_\lambda \imath^\circ_{s_\alpha})\\ &= \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 - z\theta_\lambda) + z\theta_\lambda \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 + \imath^\circ_{s_\alpha})\\ &= \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 - z\theta_\lambda) + z\theta_\lambda(1 + T_{s_\alpha}).\end{aligned}$$

This is an element of $\mathcal{H}(H)$ and $q > 1$, so $\theta_\alpha - 1$ divides $1 - z\theta_\lambda$ in $\mathcal{O}(T)$. We deduce that $z = +1$ and $\lambda = \lambda_{s_\alpha} \in \mathbb{Z}\alpha$. In particular

$$\phi(\imath^\circ_{s_\alpha}) = \theta_{\lambda_{s_\alpha}} \imath^\circ_{s_\alpha},$$

which directly implies that for every $w \in \mathcal{W}^{H^\circ}$ there exists a $\lambda_w \in X^*(T)$ with $\phi(\imath^\circ_w) = \theta_{\lambda_w} \imath^\circ_w$.

If $w \in \mathcal{W}^{H^\circ}$ is any reflection, then $w = s_\beta$ for some $\beta \in R(H^\circ, T)$ and w is conjugate to some simple reflection s_α, say by $v \in \mathcal{W}^{H^\circ}$. Then

$$\begin{aligned}\theta_{\lambda_{s_\beta}} s_\beta = \phi(s_\beta) = \phi(v s_\alpha v^{-1}) &= \theta_{\lambda_v} v \theta_{\lambda_{s_\alpha}} s_\alpha v^{-1} \theta_{-\lambda_v}\\ = \theta_{\lambda_v} \theta_{v(\lambda_{s_\alpha})} \theta_{v s_\alpha v^{-1}(-\lambda_v)} v s_\alpha v^{-1} &= \theta_{v(\lambda_{s_\alpha}) + \lambda_v - s_\beta(\lambda_v)} s_\beta\\ &= \theta_{v(\lambda_{s_\alpha}) + \langle \beta^\vee, \lambda_v \rangle \beta} s_\beta.\end{aligned} \tag{99}$$

As $v(\lambda_{s_\alpha}) \in v(\mathbb{Z}\alpha) = \mathbb{Z}\beta$, we see that $\theta_{\lambda_{s_\beta}} \in \mathbb{Z}\beta$. This proves (ii), (iii) and (iv) on $\mathcal{W}^{H^\circ}$.

Recall from Lemma 3.1 that $\mathcal{W}^H \cong \mathcal{W}^{H^\circ} \rtimes \pi_0(H)$, with $\pi_0(H)$ preserving the simple roots. For $w \in \pi_0(H)$ we have $\imath^\circ_w = T_w$ by [Sol, Proposition 1.5.1], so the argument from (97) and (98) shows that $\Phi_w \in \mathcal{O}(T)^\times$. Therefore (ii) holds on $\mathcal{W}^H$. Knowing this, the multiplication rules in $\mathbb{C}(T) \rtimes \mathcal{W}^H$ entail that $w \mapsto z_w$ must be a character of $\mathcal{W}^H$. □

To investigate the effect of automorphisms as in Proposition 14.1 on $\mathcal{H}(H)$-modules, we take a closer look at (89). Let $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$ be the affine Hecke algebra with the same data as $\mathcal{H}(H)$, but with a formal parameter $\mathbf{q}$ and over the ground ring $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$. This algebra also has a Bernstein presentation and a Bernstein basis like $\mathcal{H}(H)$, only over $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$. Any ϕ as in Proposition 14.1 lifts to a $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-linear automorphism of $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$, just use the same formula as in part (ii).

Like in (80) we define

$$\mathcal{J}(H) = \mathcal{J}(H^\circ) \rtimes \pi_0(H).$$

In [Lus1, §2.4] a homomorphism of $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebras

$$\mathcal{H}_{\sqrt{\mathbf{q}}}(H^\circ) \to \mathcal{J}(H^\circ) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]$$

is constructed, which induces (89) by specialization of $\mathbf{q}^{1/2}$ at $q^{1/2}$ or at 1. Because the actions of $\pi_0(H)$ preserve all the data used to construct these algebras, it induces a homomorphism of $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebras

$$\tilde{\phi} : \mathcal{H}_{\sqrt{\mathbf{q}}}(H) \to \mathcal{J}(H) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}].$$

From the $\mathcal{J}(H)$-module $\tilde{\pi}(\Phi, \rho)$ and $\tilde{\phi}$ we obtain the $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module

$$\tilde{\pi}(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]. \tag{100}$$

We call modules of this form, for any KLR parameter (Φ, ρ) with $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$, standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-modules.

Lemma 14.2. *Let ϕ be any automorphism of the $\mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]$-algebra $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$. The induced map ϕ^* on* $\mathrm{Mod}(\mathcal{H}_{\sqrt{\mathbf{q}}}(H))$ *sends standard modules to standard modules.*

Proof. We saw in the proof of Theorem 12.3 that for every generic $v \in \mathbb{C}^\times$ the specialization of (100) at $\mathbf{q^{1/2}}$ at $v^{1/2}$ is an irreducible $\mathcal{H}_v(H)$-module, namely $\pi(t_v, x, \rho_v)$. All these modules have the same underlying vector space

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(\Phi))}\big(\rho, H_*(\mathcal{B}_H^{t, \Phi(B_2)}, \mathbb{C})\big),$$

and the action of $\mathcal{H}_v(H)$ depends algebraically on $v^{\pm 1/2}$. It follows from Section 9 that, for generic v, there is only one way to embed $\pi(t_v, x, \rho_v)$ in a family of irreducible $\mathcal{H}_v(H)$-modules that depends algebraically on $v^{\pm 1/2}$ (varying in this generic set of parameters). Since ϕ is a $\mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]$-algebra automorphism,

$$\phi^*\big(\tilde{\pi}(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]\big) \tag{101}$$

has irreducible specializations at all generic $v \in \mathbb{C}^\times$, and these still depend algebraically on $v^{\pm 1/2}$. So (101) looks like a standard module as long as only generic parameters are considered, say like $\tilde{\pi}(\Phi', \rho') \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]$. But the set of generic parameters in dense in $\mathbb{C}^\times$, so

$$\phi^*\big(\tilde{\pi}(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]\big) = \tilde{\pi}(\Phi', \rho') \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm \mathbf{1/2}}]. \qquad \square$$

Recall the parametrization of irreducible $\mathcal{H}(H)$-modules in Theorem 9.1.

Lemma 14.3. *Let ϕ be an automorphism of $\mathcal{H}(H)$ which is the identity on $\mathcal{O}(T)$. For every Kazhdan–Lusztig triple (t_q, x, ρ_q) there exists a geometric $\rho'_q \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t_q, x))\big)$ such that $\phi^*(\pi(t_q, x, \rho_q)) = \pi(t_q, x, \rho'_q)$.*

Proof. Consider the standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module (100), where (Φ, ρ) is associated to (t_q, x, ρ_q) via Lemma 7.1. Its specialization at $\mathbf{q^{1/2}} = q^{1/2}$ is a $\mathcal{H}_q(H)$-module with $\pi(t_q, x, \rho_q)$ as unique irreducible quotient. On the other hand, the specialization at $\mathbf{q^{1/2}} = 1$ is the $\mathbb{C}[X^*(T) \rtimes \mathcal{W}^H]$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho_1, H_*(\mathcal{B}_H^{t,x}, \mathbb{C})\big).$$

By Theorem 8.2 its component in top homological degree $d(x)$ is

$$\begin{aligned}\tau(t, x, \rho_1) &= \mathrm{ind}_{X^*(T) \rtimes \mathcal{W}^M}^{X^*(T) \rtimes \mathcal{W}^H} \tau^\circ(t, x, \rho_1)) \in \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H),\\ \tau^\circ(t, x, \rho_1) &= \mathrm{Hom}_{\pi_0(\mathrm{Z}_M(x))}\big(\rho_1, H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C}) \otimes \mathbb{C}[\pi_0(M)]\big),\end{aligned}$$

where $M = \mathrm{Z}_H(t)$. Via Proposition 14.1 ϕ determines an automorphism of $\mathcal{H}_1(H) = \mathbb{C}[X^*(T) \rtimes \mathcal{W}^H]$, and we want to know $\phi^*(\tau(t, x, \rho_1))$. Since ϕ is the identity on $\mathcal{O}(T)$, composition with it does not change the parameter $t \in T$.

In $\tau^\circ(t, x, \rho_1) \in \mathbf{Irr}(\mathcal{W}^M)$ the unipotent class of $x \in M^\circ$ is already determined by the action of $\mathcal{W}^{M^\circ}$, see Proposition 4.3 and Theorem 4.4. Recall from [SpSt, §4.1] that M° is generated by the reflections s_α with $\alpha(t) = 1$. By Propostion 14.1.(4) $\phi(s_\alpha) = \theta_{\lambda_{s_\alpha}} s_\alpha$ for some $\lambda_{s_\alpha} \in \mathbb{Z}\alpha$. We calculate

$$\begin{aligned} \tau^\circ(t, x, \rho_1) \circ \phi^{-1}(s_\alpha) &= \tau^\circ(t, x, \rho_1)(s_\alpha \theta_{\lambda_{s_\alpha}}) \\ &= \tau^\circ(t, x, \rho_1)(s_\alpha)\theta_{\lambda_{s_\alpha}}(t) = \tau^\circ(t, x, \rho_1)(s_\alpha). \end{aligned}$$

Thus $\tau^\circ(t, x, \rho_1) \circ \phi^{-1}\big|_{\mathcal{W}^{H^\circ}} = \tau^\circ(t, x, \rho_1)\big|_{\mathcal{W}^{H^\circ}}$ and

$$\tau^\circ(t, x, \rho_1) \circ \phi^{-1} = \tau^\circ(t, x, \rho_1')$$

for a geometric $\rho_1' \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big)$. It follows that

$$\phi^*\big(\tau(t, x, \rho_1)\big) = \tau(t, x, \rho_1'). \tag{102}$$

Lift ϕ to automorphism of $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$, using Proposition 14.1. By Lemma 14.2

$$\phi^*\big(\tilde\pi(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]\big) \tag{103}$$

is again a standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module. But there is only one standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module whose specialization at $\mathbf{q}^{1/2} = 1$ has $\tau(t, x, \rho_1')$ as component in top homological degree, namely the one with parameter (Φ, ρ'). By (103) the module (102) must be isomorphic to

$$\tilde\pi(\Phi, \rho') \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}].$$

In particular its specialization at $\mathbf{q}^{1/2} = q^{1/2}$ is

$$\phi^*\Big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t_q, x))}\big(\rho_q, H_*(\mathcal{B}_H^{t_q, x}, \mathbb{C})\big)\Big) \cong \mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t_q, x))}\big(\rho_q', H_*(\mathcal{B}_H^{t_q, x}, \mathbb{C})\big),$$

which has $\pi(t_q, x, \rho_q')$ as unique irreducible quotient. Consequently
$\phi^*(\pi(t_q, x, \rho_q)) \cong \pi(t_q, x, \rho_q')$. □

We shall apply Lemma 14.3 to Theorem 11.2. The main role in the proof of that Theorem is played by a cover (J, τ) of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$. Let $I_{\mathcal{B}}^{\mathcal{G}}$ denote the normalized parabolic induction functor, starting from the Borel subgroup $\mathcal{B} \subset \mathcal{G}$ corresponding to $B \subset G$. As shown in [Roc, §9], there exists an algebra injection

$$t_B : \mathcal{H}(\mathcal{T}, \chi|_{\mathcal{T}_0}) \to \mathcal{H}(\mathcal{G}, \tau)$$

such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \arrow[r, "\sim"] & \mathrm{Mod}(\mathcal{H}(\mathcal{G}, \tau)) \arrow[r, "\sim"] & \mathrm{Mod}(\mathcal{H}(H)) \\ \mathrm{Rep}(\mathcal{T})^{[\mathcal{T}, \chi]_{\mathcal{T}}} \arrow[u, "I_{\mathcal{B}}^{\mathcal{G}}"] \arrow[r, "\sim"] & \mathrm{Mod}(\mathcal{H}(\mathcal{T}, \chi|_{\mathcal{T}_0})) \arrow[u, "t_{B*}"] \arrow[r, "\sim"] & \mathrm{Mod}(\mathcal{H}(T)) \arrow[u, "t_{U*}"'] \end{tikzcd} \tag{104}$$

Here $t_{U*}(V) = \mathrm{Hom}_{\mathcal{H}T)}(\mathcal{H}(H), V)$, and similarly for t_{B*}.

Lemma 14.4. *Let B' be a Borel subgroup of G such that $B' \cap H^\circ = B \cap H^\circ$. Suppose that (J', τ') is another $\mathfrak{s}$-type which covers $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$, and that there exists an isomorphism $\mathcal{H}(\mathcal{G}, \tau') \cong \mathcal{H}(H)$ that makes the diagram analogous to* (104)*, but with primes, commute. Then the map*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(\mathcal{G}, \tau')) \to \mathbf{Irr}(\mathcal{H}(H)) \to \{\text{KLR } \textit{parameters}\}^{\mathfrak{s}}/H$$

can only differ from its counterpart for (J, τ) in third ingredient ρ of a KLR parameter.

Proof. Let us denote the copy of $\mathcal{H}(H)$ obtained from $\mathcal{H}(\mathcal{G}, \tau')$ by $\mathcal{H}'(H)$, to distinguish it from the earlier $\mathcal{H}(H)$. The assumptions of the lemma entail an equivalence of categories

$$\text{Mod}(\mathcal{H}(H)) \longleftrightarrow \text{Mod}(\mathcal{H}'(H)), \tag{105}$$

which sends any $\mathcal{H}(H)$-module induced from $\mathcal{H}(T)$ to an isomorphic $\mathcal{H}'(H)$-module. In particular the regular representation of $\mathcal{H}(H)$ is mapped to an $\mathcal{H}'(H)$-module isomorphic to the regular representation. Let $\mathcal{M}$ be a Morita $\mathcal{H}'(H) - \mathcal{H}(H)$-bimodule that implements (105). Then $\mathcal{H}(H) \cong \text{End}_{\mathcal{H}'(H)}(\mathcal{M})$ as algebras and

$$\mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{H}(H)} \mathcal{H}(H) \cong \mathcal{H}'(H)$$

as $\mathcal{H}'(H)$-modules. Hence

$$\mathcal{H}(H) \cong \text{End}_{\mathcal{H}'(H)}(\mathcal{H}'(H)) \cong \mathcal{H}'(H), \tag{106}$$

providing an algebra isomorphism that has the same effect as (105). The map $t_U : \mathcal{H}(T) \to \mathcal{H}(H)$ depends only the choice of a positive system in $R(H^\circ, T)$, so it is the same for B' and B. From that and (104) we see that (106) is the identity on $\mathcal{O}(T) = t_U(\mathcal{H}(T))$. By Lemma 14.3 composition with this isomorphism sends an irreducible $\mathcal{H}(H)$-module $\pi(t_q, x, \rho_q)$ to $\pi(t_q, x, \rho'_q)$ for some ρ'_q. This statement is just another way to formulate the lemma. □

Now we can answer the questions raised by (84) and Example 11.3.

Proposition 14.5. *Assume that Condition 11.1 holds. Consider the bijections*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H)) \to \{\text{KLR } \textit{parameters}\}^{\mathfrak{s}}/H$$

from Theorems 11.2 and 9.1. Suppose that $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is mapped to $[t_q, x, \rho_q]_H$. Then the H-conjugacy class of (t_q, x) is uniquely determined by the condition that the equivalence of categories $\text{Rep}(\mathcal{G})^{\mathfrak{s}} \cong \text{Mod}(\mathcal{H}(H))$ comes from an $\mathfrak{s}$-type which is a cover of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$ for some $\chi \in \mathbf{Irr}(\mathcal{T})$ with $[\mathcal{T}, \chi]_{\mathcal{G}} = \mathfrak{s}$.

Proof. Most of the work was done in 14.4 and Theorem 11.2. We only need to show that it does not depend on the choice of a Borel subgroup $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$, or equivalently of a Borel subgroup $T \subset B \subset G$. Any other Borel subgroup of $\mathcal{G}$ containing $\mathcal{T}$ is of the form $\mathcal{B}' = w\mathcal{B}w^{-1}$ for a unique $w \in \mathcal{W}^G$. If we would use $\mathcal{B}'$ instead of $\mathcal{B}$, we would end up studying the induced representation $I_{\mathcal{B}'}^{\mathcal{G}}(\chi)$ with the extended affine Hecke algebra $\mathcal{H}(H, wBw^{-1} \cap H^\circ)$, whose based root datum is that of $(H, wBw^{-1} \cap H^\circ)$. An irreducible constituent π' of $I_{\mathcal{B}'}^{\mathcal{G}}(\chi)$ would then produce a KLR parameter $[t'_q, x', \rho'_q]_H$. In this setting $wBw^{-1} \cap H^\circ$ is conjugate to $B_H = B \cap H^\circ$ by an element $h \in N_{H^\circ}(T)$, unique up to T. Let γ be its image in $\mathcal{W}^{H^\circ} \subset \mathcal{W}^G$. The map

$$\theta_\lambda T_u \mapsto \theta_{\gamma(\lambda)} T_{\gamma u \gamma^{-1}}$$

on the Bernstein bases determines an algebra isomorphism

$$\mathcal{H}(H, wBw^{-1} \cap H^\circ) \to \mathcal{H}(H) = \mathcal{H}(H, B_H).$$

Conjugating the entire situation by h, we obtain a constituent

$$\gamma \cdot \pi' \cong \pi' \quad \text{of} \quad I^{\mathcal{G}}_{\gamma \mathcal{B}' \gamma^{-1}}(\gamma \cdot \chi) \cong I^{\mathcal{G}}_{\mathcal{B}'}(\chi),$$

and an $\mathcal{H}(H)$-module with KLR parameter

$$[h t'_q h^{-1}, h x' h^{-1}, h \cdot \rho'_q]_H = [t'_q, x', \rho'_q]_H.$$

Notice that the Borel subgroup $B'' := hB'h^{-1} = hwBw^{-1}h^{-1}$ satisfies $B'' \cap H^\circ = B \cap H^\circ$. Any type which we used the produce the KLR parameters can also be conjugated by a lift of γ in $N_{\mathcal{G}}(\mathcal{T})$, and that yield a type which covers $(\mathcal{T}_0, \gamma \cdot \chi|_{\mathcal{T}_0})$. By Theorem 11.2.(2) that is just as good as a cover of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$. Thus we have reduced to the situation of Lemma 14.4. $\square$

We remark that the conditions imposed in Proposition 14.5 seem quite reasonable. We need the algebra $\mathcal{H}(H)$ to arrive at the right parameter space, and we use the covering of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$ to get a relation between $\mathcal{H}(T) \to \mathcal{H}(H)$ and the normalized parabolic induction functor $I^{\mathcal{G}}_{\mathcal{B}}$.

15. Main result (general case)

The preparations for our main theorem are now complete.

Theorem 15.1. *Let $\mathcal{G}$ be a split reductive p-adic group and let $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ be a point in the Bernstein spectrum of the principal series of $\mathcal{G}$. Assume that Condition 11.1 holds. Then there is a commutative triangle of bijections*

$$\begin{tikzcd}
& (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[rr] & & \{\text{KLR parameters}\}^{\mathfrak{s}}/H
\end{tikzcd}$$

The slanted maps are generalizations of the slanted maps in Theorem 12.3 and the horizontal map stems from Theorem 9.1. The right slanted map is natural. If $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ corresponds to a KLR parameter (Φ, ρ), then the Langlands parameter Φ is determined canonically by π.

We denote the irreducible $\mathcal{G}$-representation associated to a KLR parameter (Φ, ρ) by $\pi(\Phi, \rho)$.

1. *The infinitesimal central character of $\pi(\Phi, \rho)$ is the H-conjugacy class*
$$\Phi\big(\varpi_F, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\big) \in c(H)_{\mathrm{ss}} \cong T^{\mathfrak{s}}/W^{\mathfrak{s}}.$$
2. *$\pi(\Phi, \rho)$ is tempered if and only if $\Phi(\mathbf{W}_F)$ is bounded, which is the case if and only if $\Phi(\varpi_F)$ lies in a compact subgroup of H.*
3. *$\pi(\Phi, \rho)$ is essentially square-integrable if and only if $\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}))$ is not contained in any proper Levi subgroup of H°.*

Recall that $\mathcal{G}$ only has irreducible square-integrable representations if $\mathrm{Z}(\mathcal{G})$ is compact. A $\mathcal{G}$-representation is called essentially square-integrable if its restriction to the derived group of $\mathcal{G}$ is square-integrable. This is more general than square-integrable modulo centre, because for that notion $\mathrm{Z}(\mathcal{G})$ needs to act by a unitary character.

Proof. By Proposition 14.5 any $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ canonically determines a Langlands parameter Φ. The larger part of the commutative triangle was already discussed in (79), (80) and Theorem 13.1. It remains to show that the set $\{\text{KLR parameters}\}^{\mathfrak{s}}/H$ (as defined on page 21) is naturally in bijection with $\{\text{KLR parameters for } H^\circ \rtimes \pi_0(H)\}^{\text{unr}}/H^\circ \rtimes \pi_0(H)$.

By (78) we are taking conjugacy classes with respect to the group $H/\mathrm{Z}(H^\circ)$ in both cases. It is clear from the definitions that that in both sets the ingredients Φ are determined by the semisimple element $\Phi(\varpi_F) \in H$. This provides the desired bijection between the Φ's in the two collections, so let us focus on the ingredients ρ.

For $(\Phi, \rho) \in \{\text{KLR parameters}\}^{\mathfrak{s}}$ the irreducible representation ρ of the component group $\pi_0(\mathrm{Z}_H(\Phi)) = \pi_0(\mathrm{Z}_G(\Phi))$ must appear in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. By Proposition 6.2.3 this space is isomorphic, as a $\pi_0(\mathrm{Z}_G(\Phi))$-representation, to a number of copies of

$$\mathrm{Ind}_{\pi_0(\mathrm{Z}_{H^\circ}(\Phi))}^{\pi_0(\mathrm{Z}_G(\Phi))} H_*\big(\mathcal{B}_{H^\circ}^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big).$$

Hence the condition on ρ is equivalent to requiring that every irreducible $\pi_0(\mathrm{Z}_{H^\circ}(\Phi))$-subrepresentation of ρ appears in $H_*\big(\mathcal{B}_{H^\circ}^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. That is exactly the condition on ρ in an unramified KLR parameter for $H^\circ \rtimes \pi_0(H)$. This establishes the properties of the commutative diagram.

(1) From the construction in Section 9 we see that the $\mathcal{H}(H^\circ)$-module with Kazhdan–Lusztig triple (t_q, x, ρ_q) has central character

$$t_q = \Phi\big(\varpi_F, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\big) \in c(H^\circ)_{\text{ss}} \cong T/\mathcal{W}^{H^\circ}.$$

It follows that the $\mathcal{H}(H)$-module with parameter (t_q, x, ρ_q) or (Φ, ρ) has central character $t_q \in c(H)_{\text{ss}} \cong T/\mathcal{W}^H$. Via (77) we can also consider it as an element of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. In view of (83) and (82) the corresponding $\mathcal{G}$-representation is

$$\pi(\Phi, \rho) = \mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(H)} \pi(t_q, x, \rho_q). \tag{107}$$

This tensor product defines an equivalence between $\mathrm{Mod}(\mathcal{H}(H))$ and $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}}$, which by definition transforms the central character into the infinitesimal character.

(2) It was checked in [BHK] that a member of $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is tempered if and only if the corresponding $\mathcal{H}(H)$-module is tempered.

For $z \in \mathbb{C}^\times$ we put $V(z) = \log|z|$. According to [KaLu, Theorem 8.2] the $\mathcal{H}(H)$-module with parameter (Φ, ρ) is V-tempered if and only if all the eigenvalues of $t = \Phi(\varpi_F)$ on Lie H (via the adjoint representation) have absolute value 1. That [KaLu] works with simply connected complex groups is inessential to the argument, it also applies to our H. But V-tempered (for this V) means only that the restriction of the $\mathcal{H}(H)$-module to the subalgebra $\mathcal{H}(H^\circ_{\text{der}})$ is tempered, where H°_{der} denotes the derived group of H°. The $\mathcal{H}(H)$-module is tempered if and only if moreover the subalgebra $\mathcal{H}(\mathrm{Z}(H^0))$ acts on it by a unitary character. This is the case if and only if all the eiqenvalues of t (in some realization of H^0 as complex matrices) have absolute value 1. That in turn means that t lies in the maximal compact subgroup of $T^{\mathfrak{s}}$.

Since $\Phi(\mathbf{W}_F)$ is generated by the finite group $\Phi(\mathbf{I}_F)$ and $t = \Phi(\varpi_F)$, the above condition on t is equivalent to boundedness of $\Phi(\mathbf{W}_F)$.
(3) This is similar to property (2), it follows from [KaLu, Theorem 8.3] and [BHK]. $\square$

16. A local Langlands correspondence

As in the introduction, $\mathbf{Irr}(\mathcal{G},\mathcal{T})$ denotes the space of all irreducible $\mathcal{G}$-representations in the principal series. Considering Theorem 15.1 for all Bernstein components in the principal series simultaneously, we will parametrize $\mathbf{Irr}(\mathcal{G},\mathcal{T})$.

Proposition 16.1. *Let $\mathcal{G}$ be a split reductive p-adic group, with restrictions on the residual characteristic as in Condition 11.1. There exists a commutative, bijective triangle*

$$\begin{tikzcd}
& (\mathbf{Irr}\,\mathcal{T}/\!/\mathcal{W}^G)_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G},\mathcal{T}) \arrow[rr] & & \{\text{KLR } \textit{parameters for } G\}/G
\end{tikzcd}$$

The right slanted map is natural, and via the bottom map any $\pi \in \mathbf{Irr}(\mathcal{G},\mathcal{T})$ canonically determines a Langlands parameter Φ for $\mathcal{G}$.

The restriction of this diagram to a single Bernstein component recovers Theorem 15.1. In particular the bottom arrow generalizes the Kazhdan–Lusztig parametrization of the irreducible $\mathcal{G}$-representations in the unramified principal series.

Proof. Let us work out what happens if in Theorem 15.1 we take the union over all Bernstein components $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$.

On the left we obtain (by definition) the space $\mathbf{Irr}(\mathcal{G},\mathcal{T})$. Notice that in Theorem 15.1, instead of $\{\text{KLR parameters}\}^{\mathfrak{s}}/H$ we could just as well take G-conjugacy classes of KLR parameters (Φ,ρ) such that $\Phi\big|_{\mathbf{I}_F}$ is G-conjugate to $c^{\mathfrak{s}}$. The union of those clearly is the space of all G-conjugacy classes of KLR parameters for G. For the space at the top of the diagram, choose a smooth character $\chi_{\mathfrak{s}}$ of $\mathcal{T}$ such that $(\mathcal{T},\chi_{\mathfrak{s}}) \in \mathfrak{s}$. By definition the $T^{\mathfrak{s}}$ in $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ equals

$$T^{\mathfrak{s}} := \{\chi_{\mathfrak{s}} \otimes t \mid t \in T\},$$

where t is considered as an unramified character of $\mathcal{T}$. On the other hand, $\mathbf{Irr}\,\mathcal{T}$ can be obtained by picking representatives $\chi_{\mathfrak{s}}$ for $\mathbf{Irr}(\mathcal{T}_0) = (\mathbf{Irr}\,\mathcal{T})/T$ and taking the union of the corresponding $T^{\mathfrak{s}}$. Two such spaces $T^{\mathfrak{s}}$ give rise to the same Bernstein component for $\mathcal{G}$ if and only if they are conjugate by an element of $N_{\mathcal{G}}(\mathcal{T})$, or equivalently by an element of $\mathcal{W}^G$. Therefore

$$(\mathbf{Irr}\,\mathcal{T}/\!/\mathcal{W}^G)_2 = \Big(\bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})} \mathcal{W}^G \cdot T^{\mathfrak{s}}/\!/\mathcal{W}^G \Big)_2 = \bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})} \big(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}\big)_2 .$$

Hence the union of the spaces in the commutative triangles from Theorem 15.1 is as desired. The right slanted arrows in these triangles combine to a natural bijection

$$(\mathbf{Irr}\,\mathcal{T}/\!/\mathcal{W}^G)_2 \to \{\text{KLR parameters for } G\}/G,$$

because the $\mathcal{W}^G$-action is compatible with the G-action. Suppose that $(\mathcal{T}, \chi'_{\mathfrak{s}})$ is another base point for $\mathfrak{s}$. Up to an unramified twist, we may assume that $\chi'_{\mathfrak{s}} = w\chi_{\mathfrak{s}}$ for some $w \in \mathcal{W}^G$. Then the Hecke algebras $\mathcal{H}(H)$, and $\mathcal{H}(H')$ are isomorphic by a map that reflects conjugation by w and by Theorem 11.2.(2) this is compatible with the bijections between $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}, \mathbf{Irr}(\mathcal{H}(H))$ and $\mathbf{Irr}(\mathcal{H}(H'))$. It follows that the bottom maps in the triangles from Theorem 15.1 paste to a bijection

$$\mathbf{Irr}(\mathcal{G}, \mathcal{T}) \to \{\text{KLR parameters for } G\}/G.$$

Finally, the map

$$(\mathbf{Irr}\, \mathcal{T}//\mathcal{W}^G)_2 \to \mathbf{Irr}(\mathcal{G}, \mathcal{T})$$

can be defined as the composition of the other two bijections in the above triangle. Then it is the combination the left slanted maps from Theorem 15.1 because the triangles over there are commutative. □

The bottom rows of Theorem 15.1 and Proposition 16.1 can be considered as a Langlands correspondence for $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$. In other words, for a Langlands parameter Φ as in (31) we define the principal series part of the L-packet $\Pi_\Phi(\mathcal{G})$ as

$$\{\pi(\Phi, \rho) \mid \rho \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(\Phi))\big) \text{ geometric }\}. \tag{108}$$

It is expected that $\Pi_\Phi(\mathcal{G})$ contains one $\mathcal{G}$-representation for every irreducible representation of $\pi_0(\mathrm{Z}_G(\Phi))$. Therefore we believe that (108) exhausts $\Pi_\Phi(\mathcal{G})$ if and only if every irreducible representation of $\pi_0(\mathrm{Z}_G(\Phi))$ appears in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$.

To support our partial Langlands correspondence, we will show that it satisfies Borel's "desiderata" [Bor2, Section 10]. Let us recall them here:

Condition 16.2 (Borel's desiderata)**.**

(1) *Let χ_Φ be the character of $Z(\mathcal{G})$ canonically associated to Φ. Then any $\pi \in \Pi_\Phi(\mathcal{G})$ has central character χ_Φ.*
(2) *Let c be a one-cocycle of $\mathbf{W}_F$ with values in $Z(G)$ and let χ_c be the associated character of $\mathcal{G}$. Then $\Pi_{c\Phi}(\mathcal{G}) = \{\pi \otimes \chi_c \mid \pi \in \Pi_\Phi(\mathcal{G})\}$.*
(3) *If one element of $\Pi_\Phi(\mathcal{G})$ is essentially square-integrable, then all elements are. This happens if and only if the image of Φ is not contained in any proper Levi subgroup of the Langlands dual group ${}^L\mathcal{G}$.*
(4) *If one element of $\Pi_\Phi(\mathcal{G})$ is tempered, then all elements are. This is equivalent to $\Phi(\mathbf{W}_F)$ being bounded in G.*
(5) *Suppose that $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ is a morphism of connected reductive F-groups with commutative kernel and cokernel. Let ${}^L\eta : {}^L\mathcal{G} \to {}^L\tilde{\mathcal{G}}$ be the dual morphism and let $\pi \in \Pi_\Phi(\mathcal{G})$. Then $\pi \circ \eta$ is a direct sum of some $\tilde{\pi} \in \Pi_{{}^L\eta \circ \Phi}(\tilde{\mathcal{G}})$.*

Parts (3) and (4) of Condition 16.2 have already been proved in parts (3) and (2) of Theorem 15.1.

Lemma 16.3. *Borel's desiderata (1) and (2) hold for our Langlands correspondence for $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$. More precisely, if c and χ_c are as in (2) and (Φ, ρ) is a KLR parameter for $\mathcal{G}$, then $\pi(c\Phi, \rho) \cong \pi(\Phi, \rho) \otimes \chi_c$.*

Proof. (1) The infinitesimal central character of $\pi(\Phi,\rho)$, as described in Theorem 15.1.(1), is its cuspidal support. For representations in the principal series this boils down to a character of $\mathcal{T}$, uniquely determined up to $\mathcal{W}^G$. Since $\mathrm{Z}(\mathcal{G}) \subset \mathcal{T}$, the central character of $\pi(\Phi,\rho)$ is just the restriction of

$$\Phi\big(\varpi_F, \big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\big)\big) \in T/W^{\mathfrak{s}} \cong T^{\mathfrak{s}}/W^{\mathfrak{s}} \tag{109}$$

to $\mathrm{Z}(\mathcal{G})$. Recall that $\Phi\big(\big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\big)\big)$ comes from a homomorphism $\mathrm{SL}_2(\mathbb{C}) \to G$. The image of $\Phi|_{\mathrm{SL}_2(\mathbb{C})}$ is generated by unipotent elements, so it is contained in the derived group G_{der}. That is the complex dual group of $\mathcal{G}/\mathrm{Z}(\mathcal{G})$, so $\Phi\big(\big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\big)\big)$ does not effect the $\mathrm{Z}(\mathcal{G})$-character of $\pi(\Phi,\rho)$ and we may consider $\Phi(\varpi_F)$ instead of (109). In view of our identification $T \cong T^{\mathfrak{s}}$ from (77), this means that the central character of $\pi(\Phi,\rho)$ is just the restriction to $\mathrm{Z}(\mathcal{G})$ of the $\mathcal{T}$-character determined by $\Phi|_{\mathbf{W}_F}$ via the local Langlands correspondence for (split) tori. This agrees with Borel's construction given in [Bor2, §10.1].

(2) Since c takes values in $Z(G) \subset T$, we can multiply any Langlands parameter for $\mathcal{T}$ with c and obtain another Langlands parameter for $\mathcal{T}$. If we transfer this map to $\mathbf{Irr}(\mathcal{T})$ via the local Langlands correspondence we get $\chi \mapsto \chi \otimes \chi_c$, see [Bor2, §10.2]. The composition

$$T \to T^{\mathfrak{s}} \xrightarrow{\otimes \chi_c} T^{\mathfrak{s}\chi_c} \to T, \tag{110}$$

where both outer maps come from (77), is of the form $t \mapsto c_T t$ for a unique $c_T \in T \cap \mathrm{Z}(G)$.

Because the image of c is contained in $\mathrm{Z}(G)$, the Langlands parameter $c\Phi$ has the same centralizer in G as Φ. Hence $(c\Phi,\rho)$ is a well-defined KLR parameter and the groups H and H_c, associated respectively to $\Phi|_{\mathbf{W}_F}$ and to $c\Phi|_{\mathbf{W}_F}$, coincide. Then (110) gives rise to an isomorphism

$$\phi_c : \mathcal{H}(H_c) \to \mathcal{H}(H) \quad \text{with} \quad \phi_c(T_w \theta_\lambda) = \lambda(c_T) T_w \theta_\lambda$$

for $w \in \mathcal{W}^H$ and $\lambda \in X^*(T)$. The induced map on irreducible representations is

$$\begin{aligned} &\phi_c^* : \mathbf{Irr}(\mathcal{H}(H)) \to \mathbf{Irr}(\mathcal{H}(H_c)), \\ &\qquad \pi(t_q, x, \rho_q) \mapsto \pi(t_q, x, \rho_q) \otimes c_T = \pi(t_q c_T, x, \rho_q). \end{aligned} \tag{111}$$

Since χ_c is a character of $\mathrm{Z}(\mathcal{G})$, the $\mathfrak{s}\chi_c$-type (J_c, τ_c) from [Roc] equals to $(J, \tau \otimes \chi_c)$. Therefore the composition of ϕ_c with the two instances of (83) is

$$\mathcal{H}(\mathcal{G}, \tau \otimes \chi_c) \to \mathcal{H}(\mathcal{G}, \tau) : f \mapsto \chi_c f.$$

(Here $\chi_c f$ denotes pointwise multiplication of functions $\mathcal{G} \to \mathbb{C}$, not a convolution product.) It follows that the composition of ϕ_c^* with the two instances of Theorem 11.2 is just

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \xrightarrow{\otimes \chi_c} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}\chi_c}.$$

This and (111) show that $\pi(c\Phi,\rho) \cong \pi(\Phi,\rho) \otimes \chi_c$. □

17. Functoriality

The fifth of Borel's desiderata in Condition 16.2 says that the LLC should be functorial with respect to some specific morphisms of reductive groups. To show that this holds in our setting, we need substantial technical preparations. The first results of this section are valid without any restriction on the residual characteristic.

Let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ be a morphism of connected reductive split F-groups, with commutative kernel and cokernel. Let $\check{\eta} : G \to \tilde{G}$ be the dual homomorphism, as in [Bor2, §1.2].

Lemma 17.1. *Define $\tilde{\mathcal{T}} = \eta^{-1}(\mathcal{T})$.*

1. *$\tilde{\mathcal{T}}$ is a split maximal torus of $\tilde{\mathcal{G}}$ and $\ker(\eta : \tilde{\mathcal{T}} \to \mathcal{T}) = \ker(\eta : \tilde{\mathcal{G}} \to \mathcal{G})$.*
2. *$\ker \eta \subset \mathrm{Z}(\tilde{\mathcal{G}})$ and $\eta^{-1}(\mathrm{Z}(\mathcal{G})) = \mathrm{Z}(\tilde{\mathcal{G}})$.*
3. *The map $X^*(\mathcal{T}) \to X^*(\tilde{\mathcal{T}}) : \alpha \mapsto \alpha \circ \eta$ induces a bijection $R(\mathcal{G},\mathcal{T}) \to R(\tilde{\mathcal{G}},\tilde{\mathcal{T}})$. Similarly $X^*(\tilde{T}) \to X^*(T) : \beta \mapsto \beta \circ \check{\eta}$ induces a bijection $R(\tilde{G},\tilde{T}) \to R(G,T)$.*
4. *$\mathrm{coker}(\eta : \tilde{\mathcal{T}} \to \mathcal{T}) \cong \mathrm{coker}(\eta : \tilde{\mathcal{G}} \to \mathcal{G})$.*

Proof. (1) Decompose the Lie algebras of $\tilde{\mathcal{G}}$ and $\mathcal{G}$ as

$$\begin{aligned} \mathrm{Lie}(\tilde{\mathcal{G}}) &= \mathrm{Lie}(\tilde{\mathcal{G}}_{\mathrm{der}}) \oplus \mathrm{Lie}(\mathrm{Z}(\tilde{\mathcal{G}})), \\ \mathrm{Lie}(\mathcal{G}) &= \mathrm{Lie}(\mathcal{G}_{\mathrm{der}}) \oplus \mathrm{Lie}(\mathrm{Z}(\mathcal{G})). \end{aligned} \tag{112}$$

The assumptions on the kernel and cokernel of η imply that $\mathrm{Lie}(\eta)$ restricts to an isomorphism

$$\mathrm{Lie}(\eta) : \mathrm{Lie}(\tilde{\mathcal{G}}_{\mathrm{der}}) \to \mathrm{Lie}(\mathcal{G}_{\mathrm{der}}) \tag{113}$$

and to a F-linear map $\mathrm{Lie}(\mathrm{Z}(\tilde{\mathcal{G}})) \to \mathrm{Lie}(\mathrm{Z}(\mathcal{G}))$. Hence $\mathrm{Lie}(\tilde{\mathcal{T}})$ is the Lie algebra of $\mathrm{Z}(\tilde{\mathcal{G}})$ times a maximal split torus of $\tilde{\mathcal{G}}_{\mathrm{der}}$. It follows that the unit component $\tilde{\mathcal{T}}^\circ$ of $\tilde{\mathcal{T}}$ is a split maximal torus of $\tilde{\mathcal{G}}$. Now $\ker(\eta)\mathrm{Z}(\tilde{\mathcal{G}})^\circ/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$ is a finite normal subgroup of $\tilde{\mathcal{G}}/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$, so it is central and contained in the maximal torus $\tilde{\mathcal{T}}^\circ/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$ of $\tilde{\mathcal{G}}/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$. Consequently $\ker \eta$ is contained in $\tilde{\mathcal{T}}^\circ$. As $\mathcal{T}$ is connected and $\tilde{\mathcal{T}} = \eta^{-1}(\mathcal{T})$, $\tilde{\mathcal{T}}$ is also connected.
(2) We just saw that $\ker \eta$ is contained in the maximal split torus $\tilde{\mathcal{T}}$ of $\tilde{\mathcal{G}}$. But $\mathcal{T}$ was arbitrary, so $\ker \eta$ lies in every split maximal torus of $\tilde{\mathcal{G}}$. The intersection of all such tori is contained in the centre of $\tilde{\mathcal{G}}$, so $\ker \eta$ as well.

Since $\mathcal{G}$ is connected, $\mathrm{Z}(\mathcal{G})$ is the kernel of the adjoint representation of $\mathcal{G}$. As $\mathrm{Lie}(\mathrm{Z}(\mathcal{G}))$ is a trivial summand of the adjoint representation, $\mathrm{Z}(\mathcal{G})$ is also the kernel of $\mathrm{Ad} : \mathcal{G} \to \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(\mathcal{G}_{\mathrm{der}}))$. In view of (113) there is a commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[r, "\eta"] \arrow[d, "\tilde{\mathrm{Ad}}"] & \mathcal{G} \arrow[d, "\mathrm{Ad}"] \\ \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(\tilde{\mathcal{G}}_{\mathrm{der}})) \arrow[r] & \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(\mathcal{G}_{\mathrm{der}})). \end{tikzcd}$$

Now we see that

$$\eta^{-1}(\mathrm{Z}(\mathcal{G})) = \eta^{-1}(\ker \mathrm{Ad}) = \ker \tilde{\mathrm{Ad}} = \mathrm{Z}(\tilde{\mathcal{G}}).$$

(3) From part (1) and the isomorphism (113) we deduce that η maps any root subgroup of $(\tilde{\mathcal{G}}, \tilde{\mathcal{T}})$ bijectively to a root subgroup of $(\mathcal{G}, \mathcal{T})$. This yields the bijection $R(\mathcal{G}, \mathcal{T}) \to R(\tilde{\mathcal{G}}, \tilde{\mathcal{T}})$, which is given explicitly by $\alpha \mapsto \alpha \circ \eta$. The second claim follows by dualizing the root data.
(4) In view of (113) η restricts to an isomorphism of root subgroups $\tilde{\mathcal{G}}_{\alpha \circ \eta} \to \mathcal{G}_\alpha$, for every $\alpha \in R(\mathcal{G}, \mathcal{T})$. With part (1) we see that the preimage $\tilde{\mathcal{B}} = \eta^{-1}(\mathcal{B})$ of our Borel subgroup $\mathcal{B} \subset \mathcal{G}$ is a Borel subgroup of $\tilde{\mathcal{G}}$. Let $\mathcal{U}$ (resp. $\tilde{\mathcal{U}}$) be the unipotent radical of $\mathcal{B}$ (resp. $\tilde{\mathcal{B}}$). The Bruhat decomposition says that

$$\mathcal{G} = \mathcal{U} N_{\mathcal{G}}(\mathcal{T}) \mathcal{U} \text{ and } \tilde{\mathcal{G}} = \tilde{\mathcal{U}} N_{\tilde{\mathcal{G}}}(\tilde{\mathcal{T}}) \tilde{\mathcal{U}}.$$

As $\eta : \tilde{\mathcal{U}} \to \mathcal{U}$ is an isomorphism and

$$N_{\mathcal{G}}(\mathcal{T})/\mathcal{T} \cong \mathcal{W}^{\mathcal{G}} \cong \mathcal{W}^{\tilde{\mathcal{G}}} \cong N_{\tilde{\mathcal{G}}}(\tilde{\mathcal{T}})/\tilde{\mathcal{T}},$$

the inclusions $\mathcal{T} \to \mathcal{G}$ and $\tilde{\mathcal{T}} \to \tilde{\mathcal{G}}$ give an isomorphism $\mathrm{coker}(\eta : \tilde{\mathcal{T}} \to \mathcal{T}) \to \mathrm{coker}(\eta : \tilde{\mathcal{G}} \to \mathcal{G})$. □

Recall that $\chi \in \mathbf{Irr}(\mathcal{T})$ and $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$. Let $c_{\mathfrak{s}} : \mathfrak{o}_F^\times \to T$ be the restriction of the Langlands parameter of χ.

Lemma 17.2. *Define* $\tilde{\mathfrak{s}} = [\tilde{\mathcal{T}}, \chi \circ \eta]_{\tilde{\mathcal{G}}}$ *and*

$$c_{\tilde{\mathfrak{s}}} = \check{\eta} \circ c_{\mathfrak{s}} : \mathfrak{o}_F^\times \to \tilde{T}.$$

Then $c_{\tilde{\mathfrak{s}}}$ *is the restriction of the Langlands parameter of* $\chi \circ \eta$ *to* $\mathfrak{o}_F^\times$ *and* η^* *sends* $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}}$ *to* $\mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}}$.

Proof. It follows from the construction of the local Langlands correspondence for split tori that the diagram

$$\begin{tikzcd}
\mathbf{Irr}(\mathcal{T}) \arrow[r, "\eta^*"] \arrow[d] & \mathbf{Irr}(\tilde{\mathcal{T}}) \arrow[d] \\
\{\text{L-parameters for } \mathcal{T}\} \arrow[r, "\check{\eta}"] & \{\text{L-parameters for } \tilde{\mathcal{T}}\}
\end{tikzcd}$$

commutes. With Lemma 5.1 this proves the first claim.

By definition $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}}$ is the category of all smooth $\mathcal{G}$-representations π with the property that every irreducible subquotient of π occurs as a subquotient of $I_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)$ for some unramified character $t \in X_{\mathrm{unr}}(\mathcal{T})$. So for the second claim it suffices to show that $\eta^*\big(I_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)\big) \in \mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}}$ for all $t \in X_{\mathrm{unr}}(\mathcal{T})$.

Clearly $\eta(\tilde{\mathcal{T}}_0) \subset \mathcal{T}_0$, so $\eta^*(t) \in \mathbf{Irr}(\tilde{\mathcal{T}})$ is unramified. Hence $\eta^*(\chi \otimes t)$ is an unramified twist of $\eta^*(\chi)$ and $\eta^*(\chi \otimes t) \in [\tilde{\mathcal{T}}, \chi \circ \eta]_{\tilde{\mathcal{T}}}$. By Lemma 17.1.(4) η induces a homeomorphism $\tilde{\mathcal{G}}/\tilde{\mathcal{B}} \to \mathcal{G}/\mathcal{B}$, which implies that

$$\eta^*\big(\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)\big) = \mathrm{Ind}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{G}}}(\eta^*(\chi \otimes t)).$$

It follows from Lemma 17.1.(3) that the difference between parabolic induction and normalized parabolic induction consists of twisting by essentially the same unramified character on both sides, so we get

$$\eta^*\big(I_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)\big) = I_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{G}}}(\eta^*(\chi \otimes t)) \in \mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}}. \qquad \square$$

As before, we define $H = \mathrm{Z}_G(c_{\mathfrak s})$ and $\tilde H = \mathrm{Z}_{\tilde G}(c_{\tilde{\mathfrak s}})$. By Lemma 17.1.(3) there is a bijection

$$\begin{split} R(H,T) = \{\alpha \in R(G,T) \mid \alpha(c_{\mathfrak s}(\mathfrak o_F^\times)) = 1\} \longleftrightarrow \\ \{\tilde\alpha \in R(\tilde G,\tilde T) \mid \tilde\alpha(\eta \circ c_{\mathfrak s}(\mathfrak o_F^\times)) = 1\} = R(\tilde H,\tilde T). \end{split} \tag{114}$$

Hence $\mathcal W^{H^\circ} \cong \mathcal W^{\tilde H^\circ}$. As $\check\eta(H) \subset \tilde H$, we have a canonical inclusion

$$\mathcal W^\eta : \mathcal W^H \to \mathcal W^{\tilde H}. \tag{115}$$

However, in general it is not a bijection.

Example 17.3. *Consider the canonical homomorphism* $\eta : \tilde{\mathcal G} = \mathrm{SL}_3(F) \to \mathrm{PGL}_3(F) = \mathcal G$. *Let* ζ *be a character of order 3 of* $\mathfrak o_F^\times$ *and put* $c_{\mathfrak s} = (\zeta,\zeta^2,1)$. *Since* $c_{\mathfrak s}$ *is trivial on* $Z(\mathrm{GL}_3(F))$, *it defines a Bernstein component for the standard maximal torus* $\tilde{\mathcal T}$ *of* $\mathrm{PGL}_3(F)$. *In this case* $\mathcal W^H = \{1\}$. *Similarly* $\check\eta \circ c_{\mathfrak s}$ *defines a Bernstein component for the standard maximal torus* $\mathcal T$ *of* $\mathrm{SL}_3(F)$. *For* $(a,b,c) \in \mathcal T(\mathfrak o_F)$*:*

$$c_{\mathfrak s}(a,b,c) = \zeta(a)\zeta(b^2) = \zeta(bc^{-1}) = c_{\mathfrak s}(b,c,a),$$

from which it follows easily that $\mathcal W^{\tilde H} \cong \mathbb Z/3\mathbb Z$.

Let $\tilde H' \subset \tilde H$ be the subgroup generated by $\tilde H^\circ$ and $W^{\mathfrak s} = \mathcal W^H$, via (115). Then $\mathcal H(\tilde H)$ contains a subalgebra

$$\mathcal H(\tilde H') \cong \mathcal H(\tilde H^\circ) \rtimes \pi_0(H).$$

The advantage of this algebra over $\mathcal H(\tilde H)$ is that η induces an algebra homomorphism

$$\begin{split} &\phi_\eta : \mathcal H(\tilde H') \to \mathcal H(H), \\ &\phi_\eta(T_w\theta_\mu) = T_w\theta_{\eta(\mu)} \text{ for } w \in W^{\mathfrak s}, \mu \in X_*(\tilde{\mathcal T}) = X^*(\tilde T). \end{split} \tag{116}$$

Recall that Φ is a Langlands parameter for $\mathcal G$ as in (31). As remarked in Section 7, all the geometric representations ρ of $\pi_0\big(\mathrm{Z}_G(\Phi)$, which can be used to enhance Φ to a KLR parameter, factor through $\pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G)\big)$.

Lemma 17.4. $\check\eta$ *induces injective group homomorphisms*

$$\begin{aligned} \pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G)\big) &\to \pi_0\big(\mathrm{Z}_{\tilde G}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G)\big), \\ \pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(H)\big) &\to \pi_0\big(\mathrm{Z}_{\tilde G}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde H')\big). \end{aligned}$$

Proof. Clearly $\check\eta$ restricts to a group homomorphism

$$\mathrm{Z}_H(\Phi) = \mathrm{Z}_G(\Phi) \to \mathrm{Z}_{\tilde G}(\check\eta \circ \Phi) = \mathrm{Z}_{\tilde H}(\eta \circ \Phi).$$

By Lemma 17.1.(4) it induces an injective homomorphism

$$\check\eta : \mathrm{Z}_H(\Phi)/\mathrm{Z}(G) \to \mathrm{Z}_{\tilde H}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G). \tag{117}$$

By (115) Lie(η) maps Lie($Z(H)$) to Lie($Z(\tilde H')$), so by Lemma 17.1.(4) also

$$\check\eta : \mathrm{Z}_H(\Phi)/\mathrm{Z}(H) \to \mathrm{Z}_{\tilde H}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde H') \tag{118}$$

is injective. We will show that (117) and (118) are isogenies.

The properties of η imply that

$$\begin{aligned} (119)\quad \mathrm{Lie}(\check{\eta}) : \mathrm{Lie}(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G)) = \mathrm{Lie}(H)/\mathrm{Z}(\mathrm{Lie}(G)) \to \\ \mathrm{Lie}(\mathrm{Z}_{\tilde H}(\check{\eta} \circ \Phi)/Z(\tilde G)) = \mathrm{Lie}(\mathrm{Z}_{\tilde H}(\check{\eta} \circ \Phi))/\mathrm{ZLie}(\tilde G) \end{aligned}$$

is an isomorphism of reductive Lie algebras. The group $\Phi(\mathrm{SL}_2(\mathbb{C}))$ is contained in H_{der}, so by (119) $\mathrm{Lie}(\check{\eta})$ maps $\mathrm{Lie}\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big)$ bijectively to $\mathrm{Lie}\big(\check{\eta} \circ \Phi(\mathrm{SL}_2(\mathbb{C}))\big)$. Therefore

$$(120)\qquad \mathrm{Lie}\big(\mathrm{Z}_H\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big)/\mathrm{Z}(G)\big) \to \mathrm{Lie}\big(\mathrm{Z}_{\tilde H}\big(\check{\eta} \circ \Phi(\mathrm{SL}_2(\mathbb{C}))\big)/\mathrm{Z}(\tilde G)\big)$$

is another isomorphism of reductive Lie algebras. To reach the Lie algebras of (117), it remains to restrict to elements that commute with $\Phi(\varpi_F)$, respectively $\check{\eta} \circ \Phi(\varpi_F)$. For simplicity we assume that T is a maximal torus of $\mathrm{Z}_H(\Phi)$, this can always be achieved by replacing Φ with a conjugate Langlands parameter. Then we have a bijection

$$\begin{aligned} R\big(\mathrm{Z}_H(\Phi), T\big) = \big\{\alpha \in R\big(\mathrm{Z}_H\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big), T\big) \mid \alpha(\Phi(\varpi_F)) = 1\big\} \longleftrightarrow \\ \big\{\tilde\alpha \in R\big(\mathrm{Z}_{\tilde H}\big(\check{\eta} \circ \Phi(\mathrm{SL}_2(\mathbb{C}))\big), \tilde T\big) \mid \tilde\alpha(\eta \circ \Phi(\varpi_F)) = 1\big\} = R\big(\mathrm{Z}_{\tilde H}(\check{\eta} \circ \Phi), \tilde T\big). \end{aligned}$$

With (120) this this implies that

$$\mathrm{Lie}(\check{\eta}) : \mathrm{Lie}(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G)) \to \mathrm{Lie}\big(\mathrm{Z}_{\tilde H}(\check{\eta} \circ \Phi)/\mathrm{Z}(\tilde G)\big).$$

is an isomorphism, so (117) is indeed an isogeny. The same argument shows that (118) is an isogeny.

As (117) and (118) are also injective, they embed the left hand side in the right hand side as a number of connected components. Hence the induced maps on the component groups are injective. □

Now we reinstate Condition 11.1. Let $(\tilde J, \tilde\tau)$ be Roche's type for $\tilde{\mathfrak{s}}$. The explicit construction in [Roc, §3] shows that $\tilde J = \eta^{-1}(J)$ and $\tilde\tau = \rho \circ \eta$. By [Roc, Theorem 4.15] the support of $\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)$ is $\tilde J(W^{\tilde{\mathfrak{s}}} \rtimes X_*(\tilde{\mathcal{T}}))\tilde J$, where we embed $X_*(\tilde{\mathcal{T}})$ in $\mathcal{T}$ via $\mu \mapsto \mu(\varpi_F)$. By Theorem 11.2

$$(121)\qquad \mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau) \cong \mathcal{H}(\tilde H) = \mathcal{H}(\tilde H^\circ) \rtimes \pi_0(\tilde H).$$

Let $\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)'$ be the subalgebra of $\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)$ isomorphic to $\mathcal{H}(\tilde H')$ and with support $\tilde J(W^{\mathfrak{s}} \rtimes X_*(\tilde{\mathcal{T}}))\tilde J$. We obtain an algebra homomorphism $\mathcal{H}(\eta, \tau) : \mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)' \to \mathcal{H}(\mathcal{G}, \tau)$ with

$$(122)\qquad \mathcal{H}(\eta, \rho)(f)(g) = \begin{cases} f(\eta^{-1}(g)) & g \in J(W^{\mathfrak{s}} \rtimes \eta(X_*(\mathcal{T})))J \\ 0 & g \in G \setminus J(W^{\mathfrak{s}} \rtimes \eta(X_*(\mathcal{T})))J. \end{cases}$$

This is well-defined because any $f \in \mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)'$ takes one common value on the entire preimage $\eta^{-1}(g)$. Taking (82) and Lemma 17.2 into account, we consider the diagram

$$(123)\qquad \begin{tikzcd} \mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \arrow[d, "\eta^*"] & \mathrm{Mod}(\mathcal{H}(\mathcal{G}, \tau)) \arrow[l, "\mathrm{ind}_{\mathcal{H}(\mathcal{G},\tau)}^{\mathcal{H}(\mathcal{G})}"'] \arrow[d, "\mathcal{H}(\eta,\tau)^*"] & \mathrm{Mod}(\mathcal{H}(H)) \arrow[l, "\sim"'] \arrow[d, "\phi_\eta^*"] \\ \mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}} & \mathrm{Mod}(\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)') \arrow[l, "\mathrm{ind}_{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)'}^{\mathcal{H}(\tilde{\mathcal{G}})}"'] & \mathrm{Mod}(\mathcal{H}(\tilde H')) \arrow[l, "\sim"'] \end{tikzcd}$$

The left upper horizontal arrow is invertible with as inverse the map that sends any $\mathcal{G}$-representation to its τ-isotypical component, as in (81).

Lemma 17.5. *The diagram* (123) *commutes up to a natural isomorphism.*

Proof. The right hand square commutes by definition, so consider only the left hand square. Take $V \in \mathrm{Mod}(\mathcal{H}(\mathcal{G},\tau))$. We must compare

$$\mathcal{H}(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'} \mathcal{H}(\eta,\tau)^*(V) \quad \text{with} \quad \eta^*\big(\mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V\big). \tag{124}$$

One problem that we encounter is the lack of a reasonable map $\mathcal{H}(\tilde{\mathcal{G}}) \to \mathcal{H}(\mathcal{G})$. To overcome this we make use of the algebra $\mathcal{H}^\vee(\mathcal{G})$ of essentially left-compact distributions on $\mathcal{G}$, which was introduced in [BeDe]. It naturally contains both $\mathcal{H}(\mathcal{G})$ and a copy of $\mathcal{G}$. From [BeDe, §1.2] it is known that $\mathrm{Mod}(\mathcal{H}^\vee(\mathcal{G}))$ is naturally equivalent with $\mathrm{Rep}(\mathcal{G})$. Moreover $\mathcal{H}(\mathcal{G})$ is a two-sided ideal of $\mathcal{H}^\vee(\mathcal{G})$, so the modules (124) are canonically isomorphic with

$$\mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'} \mathcal{H}(\eta,\tau)^*(V), \text{ respectively } \eta^*\big(\mathcal{H}^\vee(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V\big).$$

An advantage of $\mathcal{H}^\vee(\mathcal{G})$ over $\mathcal{H}(\mathcal{G})$ is that it is functorial in $\mathcal{G}$, see [Moy, Theorem 3.1]. The algebra homomorphism

$$\mathcal{H}^\vee(\eta) : \mathcal{H}^\vee(\tilde{\mathcal{G}}) \to \mathcal{H}^\vee(\mathcal{G}) \quad \text{extends} \quad \mathcal{H}(\eta,\tau) : \mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})' \to \mathcal{H}(\mathcal{G},\tau).$$

This yields a canonical map

$$\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V : \mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'} \mathcal{H}(\eta,\tau)^*(V) \to \mathcal{H}^\vee(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V.$$

By Lemma 17.1.(4) $\mathcal{G} = \eta(\tilde{\mathcal{G}})\mathcal{T}$, and the action of $\mathcal{T}$ on V is already given $\mathcal{H}(\mathcal{G},\tau)$ since $\mathcal{T} \subset JX_*(\mathcal{T})J$. Therefore $\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V$ is surjective.

It is also $\tilde{\mathcal{G}}$-equivariant if we regard its target as $\eta^*\big(\mathcal{H}^\vee(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V\big)$. In particular its kernel is a $\tilde{\mathcal{G}}$-subrepresentation of $\mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'} \mathcal{H}(\eta,\tau)^*(V)$. As $(\tilde{J},\tilde{\tau})$ is a $\tilde{\mathfrak{s}}$-type, $\ker(\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V)$ is of the form $\mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})} N$ for some $\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})$-module

$$N \subset \mathrm{ind}_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'}^{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})} \mathcal{H}(\eta,\tau)^*(V).$$

Let E be a set of representatives for $\mathcal{W}^{\tilde{H}}/\mathcal{W}^\eta(\mathcal{W}^H)$. Then any element of N can be written as $n = \sum_{w \in E} T_w v_w$. We have

$$0 = \mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V(n) = \sum\nolimits_{w \in E} \mathcal{H}^\vee(\eta)(T_w) \otimes_{\mathcal{H}(\mathcal{G},\tau)} v_w.$$

The elements $\mathcal{H}^\vee(\eta)(T_w)$ with $w \in E$ are linearly independent over $\mathcal{H}(\mathcal{G}.\tau)$, because the support of $\mathcal{H}^\vee(\eta)(T_w)$ is $\eta(\tilde{J}w\tilde{J}) = JwJ$. It follows that $v_w = 0$ for all $w \in E$.

Hence $N = 0$ and $\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V$ is injective. This shows that (123) commutes up to the canonical isomorphism between the $\tilde{\mathcal{G}}$-representations (124). $\square$

It is clear that the formula (116) also defines an algebra homomorphism $\phi_\eta : \mathcal{H}_v(\tilde{H}') \to \mathcal{H}_v(H)$ for any $v \in \mathbb{C}^\times$, and that these maps lift to a homomorphism of $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebras

$$\phi_\eta : \mathcal{H}_{\sqrt{\mathbf{q}}}(\tilde{H}') \to \mathcal{H}_{\sqrt{\mathbf{q}}}(H).$$

Denote the category of finite length semisimple modules of an algebra A by $\mathrm{Mod}_{\mathrm{fss}}(A)$.

Lemma 17.6. (1) $\phi_\eta^* : \mathrm{Mod}(\mathcal{H}_v(\tilde{H}')) \to \mathrm{Mod}(\mathcal{H}_v(H))$ *preserves finite length and complete reducibility.*

(2) *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{fss}}(\mathcal{H}_q(H)) \arrow[r, leftrightarrow] \arrow[d, "\phi_\eta^*"] & \mathrm{Mod}_{\mathrm{fss}}(\mathcal{W}^H \ltimes X^*(T)) \arrow[d, "\eta^*"] \\
\mathrm{Mod}_{\mathrm{fss}}(\mathcal{H}_q(\tilde{H}')) \arrow[r, leftrightarrow] & \mathrm{Mod}_{\mathrm{fss}}(\mathcal{W}^H \ltimes X^*(\tilde{T}))
\end{tikzcd}$$

in which the horizontal arrows extend the left slanted map in Theorem 13.1 additively.

Proof. (1) For these considerations the kernel of ϕ_η plays no role, we need only look at the subalgebra $\phi_\eta(\mathcal{H}_v(\tilde{H}'))$ of $\mathcal{H}_v(H)$. It has a basis $\{T_w \theta_\lambda \mid w \in \mathcal{W}^H, \lambda \in \eta(X^*(\tilde{T}))\}$. Since η has commutative cokernel, $\mathcal{W}^H \ltimes \eta(X^*(\tilde{T})) + X^*(T)^{\mathcal{W}^H}$ is of finite index in $\mathcal{W}^H \ltimes X^*(T)$, and it contains $\mathcal{W}^H \ltimes \mathbb{Z}R(H^\circ, T) + X^*(T)^{\mathcal{W}^H}$. The group extension

$$\mathcal{W}^H \ltimes \mathbb{Z}R(H^\circ, T) + X^*(T)^{\mathcal{W}^H} \subset \mathcal{W}^H \ltimes X^*(T) \tag{125}$$

is of the form $X \subset X \rtimes \Gamma$, where $\Gamma \subset \mathcal{W}^H \ltimes X^*(T)$ is the finite group of elements that preserve the fundamental alcove in the Coxeter complex of $\mathcal{W}^H \ltimes \mathbb{Z}R(H^\circ, T) + X^*(T)^{\mathcal{W}^H}$. Hence the inclusion of affine Hecke algebras corresponding to (125) is of the form

$$\mathcal{H}_v(H'') \subset \mathcal{H}_v(H'') \rtimes \Gamma = \mathcal{H}_v(H).$$

It is well-known from Clifford theory [RaRa, Appendix A] that the restriction map $\mathrm{Mod}(\mathcal{H}_v(H'') \rtimes \Gamma) \to \mathrm{Mod}(\mathcal{H}_v(H''))$ preserves finite length and complete reducibility. Since $\phi_\eta(\mathcal{H}_v(\tilde{H}'))$ lies between these two algebras, the same holds for

$$\mathrm{Mod}(\mathcal{H}_v(H)) \to \mathrm{Mod}\big(\phi_\eta(\mathcal{H}_v(\tilde{H}'))\big).$$

(2) Consider the standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module $\tilde{\pi}(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]$, as in (100). Its specialization at a generic $v \in \mathbb{C}^\times$ is irreducible and it equals $\pi(t_v, x, \rho_v)$. Recall from 9 that this is a $\mathcal{H}_v(H)$-submodule of $\mathrm{H}_*(\mathcal{B}_H^{t_v, x}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(H)]$. By Lemma 17.1 H and $\tilde{H}$ have isomorphic varieties of Borel subgroups, and the description of $\mathcal{H}_v(H)$-action entails that

$$\phi_\eta^*\big(\mathrm{H}_*(\mathcal{B}_H^{t_v, x}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(H)]\big) \cong \mathrm{H}_*(\mathcal{B}_{\tilde{H}}^{\check{\eta}(t_v), \check{\eta}(x)}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(H)].$$

By part (1) $\phi_\eta^*(\pi(t_v, x, \rho_v))$ is completely reducible. Hence there is a unique representation $\tilde{\rho} = \oplus_i \tilde{\rho}_i$ of

$$\pi_0\big(Z_{\tilde{H}'}(\check{\eta}(t_v), \check{\eta}(x))\big) = \pi_0\big(Z_{\tilde{H}'}(\check{\eta} \circ \Phi)\big)$$

such that

$$\pi(\check{\eta}(t_v), \check{\eta}(x), \tilde{\rho}) := \oplus_i \pi(\check{\eta}(t_v), \check{\eta}(x), \tilde{\rho}_i).$$

We need to identify $\tilde{\rho}$. Like in the proof of Lemma 14.2, the family of modules $\phi_\eta^*(\pi(t_v, x, \rho_v))$ depends algebraically on v, so $\tilde{\rho}$ does not depend on $v \in \mathbb{C}^\times$ as long as v is generic. As the set of generic parameters is dense in $\mathbb{C}^\times$, we must have

$$\phi_\eta^* \big(\tilde\pi(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}] \big) \cong \tilde\pi(\Phi, \tilde\rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}] = \oplus_i \tilde\pi(\Phi, \tilde\rho_i) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}].$$

In particular this holds for $v = q$ and for $v = 1$. Looking at the unique irreducible quotients (for $v = q$) or at the subrepresentations in top homological degree (for $v = 1$), we find

$$\begin{aligned} \phi_\eta^* (\pi(t_q, x, \rho_q)) &\cong \pi(\check\eta(t_q), \check\eta(x), \tilde\rho) = \oplus_i \pi(\check\eta(t_q), \check\eta(x), \tilde\rho_i), \\ \eta^* (\tau(t_1, x, \rho_1)) &\cong \tau(\check\eta(t_1), \check\eta(x), \tilde\rho) = \oplus_i \tau(\check\eta(t_1), \check\eta(x), \tilde\rho_i). \end{aligned}$$

Thus the diagram in statement commutes for irreducible representations and, being additive, for all semisimple modules of finite length. □

Now we can determine the effect of η^* on irreducible representations. Let (Φ, ρ) be a KLR parameter for $\mathcal{G}$, with (t_q, x, ρ_q) as in Lemma 7.1. Recall that ρ is trivial on the image of $Z(G)$ in $\pi_0(Z_G(\Phi))$.

Proposition 17.7. *Let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}, \check\eta : G \to \tilde G$ and (Φ, ρ) be as above. Identify $\pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G)\big)$ with a subgroup of $\pi_0\big(\mathrm{Z}_{\tilde G}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G)\big)$ via Lemma 17.4. Then*

$$\eta^* \big(\pi(\Phi, \rho) \big) = \pi \big(\check\eta \circ \Phi, \mathrm{ind}_{\pi_0(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde G}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))} \rho \big).$$

Here we use the convention $\pi(\check\eta \circ \Phi, \oplus_i \rho_i) = \oplus_i \pi(\check\eta \circ \Phi, \rho_i)$ for $\rho_i \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_{\tilde G}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G)\big)$.

In particular this proves a precise version of Condition 16.2.(5) for $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$.

Proof. By Lemma 17.5 and (107)

$$\begin{aligned} \eta^*(\pi(\Phi, \rho)) &= \eta^* \big(\mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G}, \tau)} \pi(t_q, x, \rho_q) \big) \\ &\cong \mathcal{H}(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)'} \mathcal{H}(\eta, \tau)^* \pi(t_q, x, \rho_q) \\ &\cong \mathcal{H}(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)} \mathrm{ind}_{\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)'}^{\mathcal{H}(\tilde{\mathcal{G}}, \tilde\tau)} \mathcal{H}(\eta, \tau)^* \pi(t_q, x, \rho_q). \end{aligned} \tag{126}$$

By (121) it suffices to analyse the module $\mathrm{ind}_{\mathcal{H}(\tilde H')}^{\mathcal{H}(\tilde H)} \phi_\eta^* \pi(t_q, x, \rho_q)$. By Lemma 17.6 the module $\phi_\eta^* \pi(t_q, x, \rho_q)$ has finite length and is semisimple, and its parameters can be read off from the $X^*(\tilde T) \rtimes \mathcal{W}^H$-module $\eta^*(\tau(t_1, x, \rho_1))$.

For simplicity we drop the subscripts 1. Recall from (51) that $\tau(t, x, \rho)$ is isomorphic to

$$\mathrm{Ind}_{X^*(T) \rtimes \mathcal{W}_t^H}^{X^*(T) \rtimes \mathcal{W}^H} \big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x)/Z(G))} \big(\rho, H_*(\mathcal{B}_{M^\circ}^x, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_H(t,x)/\mathrm{Z}_{M^\circ}(x)] \big) \big).$$

By Lemma 17.1 $\check\eta$ induces an isomorphism

$$M^\circ/\mathrm{Z}(G) = \mathrm{Z}_H(t)^\circ/\mathrm{Z}(G) \to \mathrm{Z}_{\tilde H'}(\check\eta(t))^\circ/\mathrm{Z}(\tilde G) =: \tilde M^\circ/\mathrm{Z}(\tilde G).$$

As $\check\eta$ also provides an isomorphism between the respective unipotent varieties of M and $\tilde M = \mathrm{Z}_{\tilde H}(\check\eta(t))$,

$$\pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(H)) \cong \pi_0(\mathrm{Z}_{\tilde M^\circ}(\check\eta(x))/\mathrm{Z}(\tilde H')).$$

Steinberg's description of the centralizer of a semisimple element [Ste1, 2.8] and again Lemma 17.1 show that the inclusion $N_{\tilde H'}(\tilde T) \to \tilde H'$ induces a group isomorphism

$$\mathcal{W}^H_{\check\eta(t)} / \mathcal{W}^H_t \xrightarrow{\sim} \mathrm{Z}_{\tilde H'}(\check\eta(t), \check\eta(x)) / \check\eta(\mathrm{Z}_H(t,x)) \mathrm{Z}(\tilde G).$$

It follows that $\tau(t,x,\rho)$ is also isomorphic to

$$\mathrm{Ind}^{X^*(T) \rtimes \mathcal{W}^H}_{X^*(T) \rtimes \mathcal{W}^H_{\check\eta(t)}} \big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))} \big(\rho, \\ H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_{\tilde H'}(\check\eta(t), \check\eta(x)) / \mathrm{Z}_{\tilde M^\circ}(\check\eta(x))] \big) \big).$$

By Lemma 17.4 the composition of this representation with η is

$$\mathrm{Ind}^{X^*(\tilde T) \rtimes \mathcal{W}^H}_{X^*(\tilde T) \rtimes \mathcal{W}^H_{\check\eta(t)}} \Big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))} \big(\rho, \\ H_{d(x)}(\mathcal{B}^{\check\eta(x)}_{\tilde M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_{\tilde H'}(\check\eta(t), \check\eta(x)) / \mathrm{Z}_{\tilde M^\circ}(\check\eta(x))] \big) \Big) \cong \\ \mathrm{Ind}^{X^*(\tilde T) \rtimes \mathcal{W}^H}_{X^*(\tilde T) \rtimes \mathcal{W}^H_{\check\eta(t)}} \Big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}(\tilde G))} \big(\mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))} \rho, \\ H_{d(x)}(\mathcal{B}^{\check\eta(x)}_{\tilde M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_{\tilde H'}(\check\eta(t), \check\eta(x)) / \mathrm{Z}_{\tilde M^\circ}(\check\eta(x))] \big) \Big) \cong \\ \tau \big(\check\eta(t), \check\eta(x), \mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))} \rho \big).$$

Now it follows from Lemma 17.6 and Lemma 7.1 that

$$\phi_\eta^* \pi(t_q, x, \rho_q) \cong \pi \big(\check\eta(t_q), \check\eta(x), \mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))} \rho \big).$$

Next we induce this $\mathcal{H}(\tilde H')$-module to $\mathcal{H}(\tilde H)$:

$$\mathrm{ind}^{\mathcal{H}(\tilde H)}_{\mathcal{H}(\tilde H')} \pi \big(\check\eta(t_q), \check\eta(x), \mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))} \rho \big) = \\ \mathrm{ind}^{\mathcal{H}(\tilde H)}_{\mathcal{H}(\tilde H')} \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta \circ \Phi))} \big(\mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))} \rho, H_*(\mathcal{B}^{\check\eta(x)}_{\tilde H}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(\tilde H')] \big) \cong \\ \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))} \big(\mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))} \rho, H_*(\mathcal{B}^{\check\eta(x)}_{\tilde H}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(\tilde H)] \big) \cong \\ \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))} \big(\mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))} \rho, H_*(\mathcal{B}^{\check\eta(x)}_{\tilde H}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(\tilde H)] \big) \cong \\ \pi \big(\check\eta(t_q), \check\eta(x), \mathrm{Ind}^{\pi_0(\mathrm{Z}_{\tilde H}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))} \rho \big).$$

From this we get to the $\tilde{\mathcal{G}}$-representation

$$\pi \big(\check\eta \circ \Phi, \mathrm{ind}^{\pi_0(\mathrm{Z}_{\tilde G}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G))}_{\pi_0(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G))} \rho \big)$$

via (121) and (126). $\square$

18. THE LABELLING BY UNIPOTENT CLASSES

In this and the next section we will show that the conjectures developed by the authors in [ABP1, ABP2, ABPS1, ABPS2] hold for principal series representation of split groups. These conjectures are expected to hold for every Bernstein component of a quasi-split reductive group. For convenience we recall the version that we will prove.

Let $T^{\mathfrak s}_{\mathrm{cpt}}$ denote the set of tempered representations in $T^{\mathfrak s}$. Then $T^{\mathfrak s}_{\mathrm{cpt}}$ is a compact real torus, and it corresponds to the unique maximal compact subgroup of T under (77). The action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ preserves $T^{\mathfrak{s}}_{\mathrm{cpt}}$, so we can form the compact orbifold $T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}}$.

Conjecture 18.1. *There exists a bijection*

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \tag{127}$$

with the following properties:

(1) The bijection $\mu^{\mathfrak{s}}$ restricts to a bijection

$$\mu^{\mathfrak{s}} \colon T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}_{\mathrm{temp}},$$

where the subscript temp denotes the subset of tempered representations.

(2) The bijection $\mu^{\mathfrak{s}}$ is continuous, where $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ has the Zariski topology and $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ has the Jacobson topology. The composition

$$\pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}} : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

of $\mu^{\mathfrak{s}}$ with the cuspidal support map $\pi^{\mathfrak{s}}$ is a finite morphism of affine algebraic varieties.

(3) There is an algebraic family

$$\theta_z \colon T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^\times$, such that θ_1 is the canonical projection and $\theta_{\sqrt{q}} = \pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}}$.

(4) Correcting cocharacters. For each irreducible component $\mathbf{c}$ of the affine variety $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ there is a cocharacter (i.e. a homomorphism of algebraic groups)

$$h_{\mathbf{c}} \colon \mathbb{C}^\times \longrightarrow T^{\mathfrak{s}}$$

such that

$$\theta_z[w,t] = b(h_{\mathbf{c}}(z)\cdot t)$$

for all $[w,t] \in \mathbf{c}$, where $b \colon T^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$ is the quotient map.

(5) L-packets. This property is conditional on the existence of Langlands parameters for the block $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. In that case, the intersection of an L-packet with that block is well-defined. This property refers to the intersection of such an L-packet with the given Bernstein block.

Let $\{\mathbf{c_1}, \ldots, \mathbf{c_r}\}$ be the irreducible components of the affine variety $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$. There exists a complex reductive group H and, for every irreducible component $\mathbf{c}$ of $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$, a unipotent conjugacy class $\lambda(\mathbf{c})$ in H, such that for every two points $[w,t]$ and $[w',t]$ of $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$:

$$\mu^{\mathfrak{s}}[w,t] \text{ and } \mu^{\mathfrak{s}}[w',t'] \textit{ are in the same L-packet}$$

if and only

- *$\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^\times$;*
- *$\lambda(\mathbf{c}) = \lambda(\mathbf{c}')$, where $[w,t] \in \mathbf{c}$ and $[w',t'] \in \mathbf{c}'$.*

Let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$ and construct $c^{\mathfrak{s}}$ as in Section 5. By Theorem 15.1 we can parametrize $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ with H-conjugacy classes of KLR parameters (Φ,ρ) such that $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$. We note that $\{\text{KLR parameters}\}^{\mathfrak{s}}$ is naturally labelled by the unipotent classes in H:

$$\{\text{KLR parameters}\}^{\mathfrak{s},[x]} := \big\{(\Phi,\rho) \mid \Phi\big(1, \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\big) \text{ is conjugate to } x\big\}. \tag{128}$$

In this way we can associate to any of the parameters in Theorem 15.1 a unique unipotent class in H:

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} = \bigcup\nolimits_{[x]} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s},[x]}, \quad (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 = \bigcup\nolimits_{[x]} (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2^{[x]}. \tag{129}$$

Via the affine Springer correspondence from Section 8 the set of equivalence classes in $\{\text{KLR parameters}\}^{\mathfrak{s}}$ is naturally in bijection with $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$. Recall from Section 2 that

$$\widetilde{T^{\mathfrak{s}}} = \{(w,t) \in W^{\mathfrak{s}} \times T^{\mathfrak{s}} \mid wt = t\}$$

and $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} = \widetilde{T^{\mathfrak{s}}}/W^{\mathfrak{s}}$. In view of Section 2 $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is also in bijection with $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$, albeit not naturally.

Only in special cases a canonical bijection $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is available. For example when $G = \mathrm{GL}_n(\mathbb{C})$, the finite group $\mathcal{W}_t^H$ is a product of symmetric groups: in this case there is a canonical c-**Irr** system, according to the classical theory of Young tableaux.

We have to construct a map (127) with the desired properties, but in general it can already be hard to define any suitable map from $\{\text{KLR parameters}\}^{\mathfrak{s}}$ to $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$, because it is difficult to compare the parameters ρ for different Φ's. It goes better the other way round and with $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ as target. In this way will transfer the labellings (129) to $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$.

From [Roc, Section 8] we know that $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is in bijection with the equivalence classes of irreducible representations of the extended affine Hecke algebra $\mathcal{H}(H)$. To relate it to $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ the parametrization of Kazhdan, Lusztig and Reeder is unsuitable, it is more convenient to use the methods developed in [Opd, Sol].

Let us fix some notations. Choose a Borel subgroup $B \subset H^\circ$ containing T. Let P be a set of roots of (H°, T) which are simple with respect to B and let R_P be the root system that they span. They determine a parabolic subgroup $W_P \subset W^{\mathfrak{s}}$ and a subtorus

$$T^P := \{t \in T \mid \alpha(t) = 1 \; \forall \alpha \in R_P\}^\circ .$$

In [Sol, Theorem 3.3.2] $\mathbf{Irr}(\mathcal{H}(H))$ is mapped, in a natural finite-to-one way, to equivalence classes of triples (P, δ, t). Here P is as above, $t \in T^P$ and δ is a discrete series representation of a parabolic subalgebra $\mathcal{H}_P$ of $\mathcal{H}(H)$. This t is the same as in the affine Springer parameters.

The pair (P, δ) gives rise to a *residual coset* L in the sense of [Opd, Appendix A]. Explicitly, it is the translation of T^P by an element $cc(\delta) \in T$ that represents the central character of δ (a W_P-orbit in a subtorus $T_P \subset T$). The element $cc(\delta)t \in L$ corresponds to t_q. The collection of residual cosets is stable under the action of $W^{\mathfrak{s}}$.

Proposition 18.2. (1) *There is a natural bijection between*

- *H-conjugacy classes of Langlands parameters Φ with $\Phi\big|_{\mathbf{I}_F} = c^{\mathfrak{s}}$;*
- *$W^{\mathfrak{s}}$-conjugacy classes of pairs (t_q, L) with L a residual coset for $\mathcal{H}(H)$ and $t_q \in L$.*

(2) *Let Y^P be the union of the residual cosets of T^P. The stabilizer of Y^P in $W^{\mathfrak{s}}$ is the stabilizer of R_P.*

(3) *Suppose that $w \in W^{\mathfrak{s}}$ fixes $cc(\delta)$. Then w stabilizes R_P.*

Proof. (1) Opdam constructed the maps in both directions for $\mathcal{H}(H^\circ)$. To go from $\mathcal{H}(H^\circ)$ to $\mathcal{H}(H)$ is easy, one just has to divide out the action of $\pi_0(H)$ on both sides.

Let us describe the maps for H° more explicitly. To a residual coset L Opdam [Opd, Proposition B.3] associates a unipotent element $x \in B$ such that $lxl^{-1} = x^q$ for all $l \in L$. Then Φ is a Langlands parameter with data t_q, x.

For the opposite direction we may assume that

$$\Phi\big(\mathbf{W}_F, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\big) \subset T \quad \forall z \in \mathbb{C}^\times$$

and that $x = \Phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) \in B$. Then

$$T^P := \mathrm{Z}_T\big(\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}))\big)^\circ = \mathrm{Z}_T\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big)^\circ$$

is a maximal torus of $\mathrm{Z}_{H^\circ}\big(\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}))\big)$. We take

$$t_q = \Phi\Big(\varpi_F, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\Big) \quad \text{and} \quad L = T^P t_q.$$

This is essentially [Opd, Proposition B.4], but our way to write it down avoids Opdam's assumption that H° is simply connected.
(2) Clear, because any element that stabilizes Y^P also stabilizes T^P.
(3) Since $cc(\delta)$ represents the central character of a discrete series representation of $\mathcal{H}_P$, at least one element (say r) in its W_P-orbit lies in the obtuse negative cone in the subtorus $T_P \subset T$ (see Lemma 2.21 and Section 4.1 of [Opd]). That is, $\log|r|$ can be written as $\sum_{\alpha \in P} c_\alpha \alpha^\vee$ with $c_\alpha < 0$. Some W_P-conjugate w' of $w \in W^{\mathfrak{s}}$ fixes r and hence $\log|r|$. But an element of $W^{\mathfrak{s}}$ can only fix $\log|r|$ if it stabilizes the collection of coroots $\{\alpha^\vee \mid \alpha \in P\}$. It follows that w' and w stabilize R_P. $\square$

In particular the above natural bijection associates to any $W^{\mathfrak{s}}$-conjugacy class of residual cosets L a unique unipotent class $[x]$ in H. Conversely a unipotent class $[x]$ can correspond to more than one $W^{\mathfrak{s}}$-conjugacy class of residual cosets, at most the number of connected components of $\mathrm{Z}_T(x)$ if $x \in B$.

Let $\mathfrak{U}^{\mathfrak{s}} \subset B$ be a set of representatives for the unipotent classes in H. For every $x \in \mathfrak{U}^{\mathfrak{s}}$ we choose an algebraic homomorphism

$$\gamma_x \colon \mathrm{SL}_2(\mathbb{C}) \to H \quad \text{with} \quad \gamma_x \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = x \quad \text{and} \quad \gamma_x \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right) \in T.$$

As noted in (29) all choices for γ_x are conjugate under $\mathrm{Z}_{M^\circ}(x)$. For each $x \in \mathfrak{U}^{\mathfrak{s}}$ we define

$$\{\text{KLR parameters}\}^{\mathfrak{s},x} = \{(\Phi, \rho) \mid \Phi\big|_{\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})} = c^{\mathfrak{s}} \times \gamma_x, \Phi(\varpi_F) \in T\}.$$

We endow this set with the topology such that a subset is open if and only if its image in T under $(\Phi, \rho) \mapsto \Phi(\varpi)$ is open. In this way

$$\bigsqcup\nolimits_{x \in \mathfrak{U}^{\mathfrak{s}}} \{\text{KLR parameters}\}^{\mathfrak{s},x} \tag{130}$$

becomes a nonseparated scheme with maximal separated quotient $\sqcup_{x \in \mathfrak{U}^{\mathfrak{s}}} \mathrm{Z}_T(\mathrm{im}\,\gamma_x)$. Notice that (130) contains representatives for all equivalence classes in $\{\text{KLR parameters}\}^{\mathfrak{s}}$.

Proposition 18.3. *Assume that Condition 11.1 holds. There exists a continuous bijection*

$$\mu^{\mathfrak s} : T^{\mathfrak s}/\!/W^{\mathfrak s} \to \mathbf{Irr}(\mathcal G)^{\mathfrak s}$$

such that:

(1) *The diagram*

$$\begin{tikzcd}
T^{\mathfrak s}/\!/W^{\mathfrak s} \arrow[r, "\mu^{\mathfrak s}"] \arrow[d, "\rho^{\mathfrak s}"] & \mathbf{Irr}(\mathcal G)^{\mathfrak s} \arrow[r] & \{\mathrm{KLR}\ \textit{parameters}\}^{\mathfrak s}/H \arrow[d] \\
T^{\mathfrak s}/W^{\mathfrak s} & & c(H)_{\mathrm{ss}} \arrow[ll]
\end{tikzcd}$$

commutes. Here the unnamed horizontal maps are those from Theorem 15.1 and the right vertical arrow sends (Φ,ρ) *to the* H*-conjugacy class of* $\Phi(\varpi_F)$*.*

(2) *For every unipotent element* $x \in H$ *the preimage*

$$(T^{\mathfrak s}/\!/W^{\mathfrak s})^{[x]} := (\mu^{\mathfrak s})^{-1}\big(\mathbf{Irr}(\mathcal G)^{\mathfrak s,[x]}\big)$$

is a union of connected components of $T^{\mathfrak s}/\!/W^{\mathfrak s}$*.*

(3) *Let* ϵ *be the map that makes the diagram*

$$\begin{tikzcd}
T^{\mathfrak s}/\!/W^{\mathfrak s} \arrow[rr, "\epsilon"] \arrow[dr, "\mu^{\mathfrak s}"] & & (T^{\mathfrak s}/\!/W^{\mathfrak s})_2 \arrow[dl] \\
& \mathbf{Irr}(\mathcal G)^{\mathfrak s} &
\end{tikzcd}$$

commute. Then ϵ *comes from a* c-**Irr** *system.*

(4) $T^{\mathfrak s}/\!/W^{\mathfrak s} \xrightarrow{\mu^{\mathfrak s}} \mathbf{Irr}(\mathcal G)^{\mathfrak s} \to \{\mathrm{KLR}\ \textit{parameters}\}^{\mathfrak s}/H$ *lifts to a map*

$$\widetilde{\mu^{\mathfrak s}} : \widetilde{T^{\mathfrak s}} \to \bigsqcup\nolimits_{x\in\mathfrak U^{\mathfrak s}} \{\textit{KLR parameters}\}^{\mathfrak s,x}$$

such that the restriction of $\widetilde{\mu^{\mathfrak s}}$ *to any connected component of* $\widetilde{T^{\mathfrak s}}$ *is algebraic and an isomorphism onto its image.*

Proof. Proposition 18.2.1 yields a natural finite-to-one map from $\mathbf{Irr}(\mathcal H(H))$ to $W^{\mathfrak s}$-conjugacy classes (t_q, L), namely

$$\pi(\Phi,\rho) \mapsto \Phi \mapsto (t_q, L). \tag{131}$$

In [Sol, Theorem 3.3.2] this map was obtained in a different way, which shows better how the representations depend on the parameters t, t_q, L. That was used in [Sol, Section 5.4] to find a continuous bijection

$$\mu^{\mathfrak s} : T^{\mathfrak s}/\!/W^{\mathfrak s} \to \mathbf{Irr}(\mathcal H(H)) \cong \mathbf{Irr}(\mathcal G)^{\mathfrak s}. \tag{132}$$

The strategy is essentially a step-by-step creation of a c-**Irr** system for $T^{\mathfrak s}/\!/W^{\mathfrak s}$ and $\mathcal H(H)$. One starts with the components of $\widetilde{T}_{\mathfrak s}$ of dimension 0, and proceeds by induction on the dimension. In step d one considers d-dimensional families of representations, and uses that in (132) the fibers over a fixed $t \in T^{\mathfrak s}/W^{\mathfrak s}$ have the same cardinality on both sides.

Only the condition on the unit element and the trivial representation is not considered in [Sol]. Fortunately there is some freedom left, which we can exploit to ensure that $\mu^{\mathfrak s}(1, T^{\mathfrak s})$ is the family of spherical $\mathcal H(H)$-representations, see Section 10. This is possible because every principal series representation of $\mathcal{H}(H)$ has a unique irreducible spherical subquotient, so choosing that for $\mu^{\mathfrak{s}}(1,t)$ does not interfere with the rest of the construction. Via Theorem 15.1 we can transfer this c-**Irr** system to a c-**Irr** system for the two extended quotients of $T^{\mathfrak{s}}$ by $W^{\mathfrak{s}}$, so property (3) holds.

By construction the triple (P,δ,t) associated to the representation $\mu^{\mathfrak{s}}(w,t)$ has the same $t \in T$, modulo $W^{\mathfrak{s}}$. That is, property (1) is fulfilled.

Furthermore $\mu^{\mathfrak{s}}$ sends every connected component of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$ to a family of representations with common parameters (P,δ). Hence these representation are associated to a common residual coset L and to a common unipotent class $[x]$, which verifies property (2).

Let $\mathbf{c}$ be a connected component of $(T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}})^{[x]}$, with $x \in \mathfrak{U}^{\mathfrak{s}}$. The proof of Proposition 18.2.1 shows that $\mathbf{c}$ can be realized in $\mathrm{Z}_T(\mathrm{im}\,\gamma_x)$. In other words, we can find a suitable $w = w(\mathbf{c}) \in W^{\mathfrak{s}}$ with $T^w \subset \mathrm{Z}_T(\mathrm{im}\,\gamma_x)$. Then there is a connected component $T^w_{\mathbf{c}}$ of T^w such that

$$\begin{aligned} &\mathbf{c} := \big(w, T^w_{\mathbf{c}} / \mathrm{Z}(w,\mathbf{c})\big), \\ &\text{where } \mathrm{Z}(w,\mathbf{c}) = \{ g \in \mathrm{Z}_{W^{\mathfrak{s}}}(w) \mid g \cdot T^w_{\mathbf{c}} = T^w_{\mathbf{c}} \}. \end{aligned}$$

In this notation $\widetilde{\mathbf{c}} := (w, T^w_{\mathbf{c}})$ is a connected component of $\widetilde{T^{\mathfrak{s}}}$. We want to define $\widetilde{\mu^{\mathfrak{s}}} : \widetilde{\mathbf{c}} \to \{\text{KLR parameters}\}^{\mathfrak{s},x}$. For every $[w,t] \in \mathbf{c}$, $\mu^{\mathfrak{s}}[w,t]$ determines an equivalence class in $\{\text{KLR parameters}\}^{\mathfrak{s},x}$. Any (Φ,ρ) in this equivalence class satisfies $\Phi(\varpi_F) \in \mathrm{Z}_T(\mathrm{im}\,\gamma_x) \cap W^{\mathfrak{s}} t$. Hence there are only finitely many possibilities for $\Phi(\varpi_F)$, say $t_1,\ldots,t_k$. For every such t_i there is a unique $(\Phi_i,\rho_i) \in \{\text{KLR parameters}\}^{\mathfrak{s},x}$ with $\Phi_i(\varpi_F) = t_i$ and $\pi(\Phi_i,\rho_i) \cong \mu^{\mathfrak{s}}[w,t]$. Every element of $\widetilde{\mathbf{c}}$ lying over $[w,t] \in \mathbf{c}$ is of the form (w,t_i). (Not every t_i is eligible though, for some we would have to modify w.) We put

$$\widetilde{\mu^{\mathfrak{s}}}(w,t_i) := (\Phi_i,\rho_i).$$

Since the ρ's are irrelevant for the topology on $\{\text{KLR parameters}\}^{\mathfrak{s},x}$, $\widetilde{\mu^{\mathfrak{s}}}(\widetilde{\mathbf{c}})$ is homeomorphic to $T^w_{\mathbf{c}}$ and $\widetilde{\mu^{\mathfrak{s}}} : \widetilde{\mathbf{c}} \to \widetilde{\mu^{\mathfrak{s}}}(\widetilde{\mathbf{c}})$ is an isomorphism of affine varieties. This settles the final property (4). ☐

19. CORRECTING COCHARACTERS AND L-PACKETS

In this section we construct the correcting cocharacters on the extended quotient $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$. With part (5) of Conjecture 18.1, these show how to determine when two elements of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$ give rise to $\mathcal{G}$-representations in the same L-packets.

Every KLR parameter (Φ,ρ) naturally determines a cocharacter h_Φ and elements $\theta(\Phi,\rho,z) \in T^{\mathfrak{s}}$ by

$$\begin{aligned} h_\Phi(z) &= \Phi\big(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\big), \\ \theta(\Phi,\rho,z) &= \Phi\big(\varpi_F, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\big) = \Phi(\varpi_F) h_\Phi(z). \end{aligned} \tag{133}$$

Although these formulas obviously do not depend on ρ, it turns out to be convenient to include it in the notation anyway. However, in this way we would end up with infinitely many correcting cocharacters, most of them with range outside T. To reduce to finitely many cocharacters with values in T, we will restrict to KLR parameters associated to $x \in \mathfrak{U}^{\mathfrak{s}}$.

Recall that Proposition 18.3.2 determines a labelling of the connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ by unipotent classes in H. This enables us to define the correcting cocharacters: for a connected component $\mathbf{c}$ of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ with label (represented by) $x \in \mathfrak{U}^{\mathfrak{s}}$ we take the cocharacter

$$h_{\mathbf{c}} = h_x : \mathbb{C}^\times \to T, \quad h_x(z) = \gamma_x \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix} \right). \tag{134}$$

Let $\widetilde{\mathbf{c}}$ be a connected component of $\widetilde{T^{\mathfrak{s}}}$ that projects onto $\mathbf{c}$. We define

$$\begin{aligned} \widetilde{\theta_z} &: \widetilde{\mathbf{c}} \to T^{\mathfrak{s}}, & (w,t) &\mapsto \theta\big(\widetilde{\mu^{\mathfrak{s}}}(w,t), z\big), \\ \theta_z &: \mathbf{c} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}, & [w,t] &\mapsto W^{\mathfrak{s}} \widetilde{\theta_z}(w,t). \end{aligned} \tag{135}$$

For $\widetilde{\mathbf{c}}$ as in the proof of Proposition 18.3, which we can always achieve by adjusting by element of $W^{\mathfrak{s}}$, our construction results in

$$\widetilde{\theta_z}(w,t) = t\, h_{\mathbf{c}}(z).$$

Lemma 19.1. *Let $[w,t], [w',t'] \in T^{\mathfrak{s}}//W^{\mathfrak{s}}$. Then $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet if and only if*

- *$[w,t]$ and $[w',t']$ are labelled by the same unipotent class in H;*
- *$\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^\times$.*

Proof. Suppose that the two $\mathcal{G}$-representations $\mu^{\mathfrak{s}}[w,t] = \pi(\Phi,\rho)$ and $\mu^{\mathfrak{s}}[w',t'] = \pi(\Phi',\rho')$ belong to the same L-packet. By definition this means that Φ and Φ' are G-conjugate. Hence they are labelled by the same unipotent class, say $[x]$ with $x \in \mathfrak{U}^{\mathfrak{s}}$. By choosing suitable representatives we may assume that $\Phi = \Phi'$ and that $\{(\Phi,\rho), (\Phi,\rho')\} \subset \{\text{KLR parameters}\}^{\mathfrak{s},x}$. Then $\theta(\Phi,\rho,z) = \theta(\Phi,\rho',z)$ for all $z \in \mathbb{C}^\times$. Although in general $\theta(\Phi,\rho,z) \neq \widetilde{\theta_z}(w,t)$, they differ only by an element of $W^{\mathfrak{s}}$. Hence $\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^\times$.

Conversely, suppose that $[w,t], [w',t']$ fulfill the two conditions of the lemma. Let $x \in \mathfrak{U}^{\mathfrak{s}}$ be the representative for the unipotent class which labels them. By Proposition 18.2.1 we may assume that $T^w \cup T^{w'} \subset \mathrm{Z}_T(\operatorname{im}\gamma_x)$. Then

$$\widetilde{\theta_z}[w,t] = t\, h_x(z) \quad \text{and} \quad \widetilde{\theta_z}[w',t'] = t'\, h_x(z)$$

are $W^{\mathfrak{s}}$ conjugate for all $z \in \mathbb{C}^\times$. As these points depend continuously on z and $W^{\mathfrak{s}}$ is finite, this implies that there exists a $v \in W^{\mathfrak{s}}$ such that

$$v(t\, h_x(z)) = t'\, h_x(z) \quad \text{for all } z \in \mathbb{C}^\times.$$

For $z = 1$ we obtain $v(t) = t'$, so v fixes $h_x(z)$ for all z. Via the Proposition 18.2.1, $h_x(q^{1/2})$ becomes an element $cc(\delta)$ for a residual coset L_x. By parts (2) and (3) of Proposition 18.2 v stabilizes the collection of residual cosets determined by x, namely the connected components of $\mathrm{Z}_T(\operatorname{im}\gamma_x) h_x(q^{1/2})$.

Let $(t_q, L), (t'_q, L')$ be associated to $\mu^{\mathfrak{s}}[w,t], \mu^{\mathfrak{s}}[w',t']$ by (131). Then $t_q = t h_x(q^{1/2})$ and $t'_q = t' h_x(q^{1/2})$, so the above applies. Hence v sends L to another residual coset determined by x. As $v(L)$ contains t'_q, it must be L'. Thus (t_q, L) and (t'_q, L') are $W^{\mathfrak{s}}$-conjugate, which by Proposition 18.2.1 implies that they correspond to conjugate Langlands parameters Φ and Φ'. So $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet. □

Corollary 19.2. *Properties 1–5 of Conjecture 18.1 hold for $\mu^{\mathfrak{s}}$ as in Proposition 18.3, with the morphism θ_z from* (135) *and the labelling by unipotent classes in H.*

Together with Theorem 15.1 this proves Conjecture 18.1 for all Bernstein components in the principal series of a split reductive p-adic group (under Condition 11.1 on the residual characteristic).

Proof. Property 1 follows from Theorem 15.1.2 and Proposition 18.3.1. The definitions of (134) and (135) establish property 4. The construction of θ_z, in combination with Theorem 15.1.1 and Proposition 18.3.1, shows that properties 2 and 3 are fulfilled. Property 5 is none other than Lemma 19.1. □

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IMJ-PRG, U.M.R. 7586 du C.N.R.S., U.P.M.C., Paris, France
E-mail address: `anne-marie.aubert@imj-prg.fr`

Mathematics Department, Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `baum@math.psu.edu`

School of Mathematics, Southampton University, Southampton SO17 1BJ, England *and* School of Mathematics, Manchester University, Manchester M13 9PL, England
E-mail address: `r.j.plymen@soton.ac.uk` `plymen@manchester.ac.uk`

Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
E-mail address: `m.solleveld@science.ru.nl`