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# Convergence rate of extremes of generalized Maxwell distribution *

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**Abstract.** Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution $F$ following the generalized Maxwell distribution. In this paper, we obtain the exact uniform convergence rate of the distribution of the maximum to its extreme value distribution.

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## 1 Introduction and main results

Let $\{X_n,\ n \geq 1\}$ be an independent and identically distributed random sequence with each $X_n$ obeying common distribution function $F(x)$. Let $M_n = \max(X_1, X_2, \ldots, X_n)$ represent the partial maximum. If there exist constants $a_n > 0$ and $b_n \in R$, and a distribution $G(x)$ which is nondegenerate such that

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{1.1}$$

for all continuity points of $G$. Then $G$ must be in one of the following three classes:

$$\text{Class I ( Gumbel ) :}\quad \Lambda(x) = \exp\{-e^{-x}\},\ x \in \mathbb{R};$$

$$\text{Class II ( Fréchet ) :}\quad \Phi_\alpha(x) = \begin{cases} 0, & x < 0, \\ \exp\{-x^{-\alpha}\}, & x \geq 0; \end{cases}$$

$$\text{Class III ( Weibull ) :}\quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x < 0, \\ 1, & x \geq 0; \end{cases}$$

where $\alpha$ is a positive constant. If (1.1) holds, we say that distribution $F$ belongs to one of the domain of attraction of $G$, denoted by $F \in D(G)$.

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One interesting problem in extreme value theory is to study the convergence rate of the distribution of the maximum tending to some above extreme value distribution. For the uniform convergence rate for some important and widely applied distributions under linear normalization, see Hall (1979), Peng et al. (2010), Lin et al. (2011) and Liu and Liu (2013). For the exact uniform convergence rate for the given distribution under nonlinear normalization, see Chen et al. (2012) and Chen and Feng (2014). For the uniform convergence rate under the second-order conditions, see Balkema and de Hann (1990), de Haan and Resnik (1996), Cheng and Jiang (2001) and Peng and Nadarajah (2012). For the Maxwell distribution, Liu and Liu (2013) showed the below result:

$$\frac{c_1}{\log n} < \sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - \Lambda(x)| < \frac{c_2}{\log n}$$

for $n > n_0$, where constants $0 < c_1 < c_2$ and $F$ stands for the ordinary Maxwell distribution function, and the norming constants $a_n$ and $b_n$ are given by

$$\sqrt{\frac{\pi}{2}}\frac{\sigma}{b_n}\exp\left(\frac{b_n^2}{2\sigma^2}\right) = n, \quad a_n = \sigma^2 b_n^{-1}. \tag{1.2}$$

Meanwhile, Liu and Liu (2013) proved that $1/\log n$ is the optimal convergence rate of extremes for the ordinary Maxwell distribution.

We are interested in this article is to consider the uniform rate of convergence of extremes from a sequence of independent and identically distributed random variables with common distribution $F$ which haves the generalized Maxwell distribution ( for short GMD ). The GMD being a generalization of the Maxwell distribution is one of the most widely applied and popular distribution in theoretical analysis and application of statistics and physics. The probability density function of the GMD is given by

$$F'(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)} x^{2k}\exp\left(-\frac{x^{2k}}{2\sigma^2}\right), \quad x \geq 0,$$

where parameter $k,\ \sigma > 0$ and $\Gamma(\cdot)$ denotes the gamma function. If $k = 1$, the GMD is the ordinary Maxwell distribution. Huang and Chen (2014) investigated the tail property of the GMD and the asymptotic distribution of the maximum. In order to derive the uniform rate of convergence of extremes for the GMD, we need to cite some results from Huang and Chen (2014).

In the sequel, let $\{X_n,\ n \geq 1\}$ be an independent and identically distributed random sequence with common distribution $F$ following the GMD, and let $M_n$ denote the partial maximum of $\{X_n,\ n \geq 1\}$. Huang and Chen (2014) proved the bellow result:

$$\lim_{n\to\infty} P(M_n \leq \alpha_n x + \beta_n) = \lim_{n\to\infty} F^n(\alpha_n x + \beta_n) = \Lambda(x),$$

for $k > \frac{1}{2}$ and all $x \in \mathbb{R}$, where

$$\alpha_n = \frac{\sigma^{1/k}}{k(2\log n)^{1-1/(2k)}} \tag{1.3}$$

and

$$\beta_n = \sigma^{1/k}(2\log n)^{1/(2k)} + \frac{\sigma^{1/k}[\log\log n + (1-k^2)\log 2 - 2k\log\Gamma(1+k/2)]}{2k^2(\log n)^{1-1/(2k)}}. \tag{1.4}$$

For the distributional tail representation of the GMD, Huang and Chen (2014) showed that

$$1 - F(x) = c(x)\exp\left(-\int_1^x \frac{g(t)}{f(t)}\,\mathrm{dt}\right)$$

for $k > 1/2$ and $x$ sufficiently large, where

$$c(x) \to \frac{\exp(-1/(2\sigma^2))}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)} \quad \text{as} \quad x \to \infty,$$

$$\text{and} \quad f(t) = \frac{\sigma^2}{k}t^{1-2k}, \quad g(t) = 1 - \frac{\sigma^2}{k}t^{-2k}.$$

Noting that $f'(t) \to 0$ and $g(t) \to 1$ as $t \to \infty$. By proposition 1.1($a$) and Corollary 1.7 of Resinick (1987), we can choose the norming constants $a_n$ and $b_n$ in such a way that $b_n$ is determined by

$$2^{1/k}\sigma^{1/k}\Gamma(1+k/2)b_n^{-1}\exp\left(\frac{b_n^{2k}}{2\sigma^2}\right) = n \tag{1.5}$$

and

$$a_n = f(b_n) = \sigma^2 k^{-1} b_n^{1-2k}. \tag{1.6}$$

Observing that, when $k = 1$, (1.5) and (1.6) reduce to (1.2).

In this paper we prove that the optimal uniform convergence rate of $F^n(a_n x + b_n)$ to its extreme value limit is proportional to $1/\log n$. However, the pointwise convergence rate of $F^n(\alpha_n x + \beta_n)$ converging to its limit is proportional to $(\log\log n)^2/\log n$ which is no better than it even though $\alpha_n/a_n \to 1$ and $(\beta_n - b_n)/a_n \to 0$ as $n \to \infty$. The main results are described as follows:

**Theorem 1.1.** *Let $\{X_n, n \geq 1\}$ denote a sequence of independent and identically distributed random variables with common distribution $F$ following the GMD and parameter $k > 1/2$. For norming constants $b_n$ and $a_n$ defined by (1.5) and (1.6), then there exist absolute constants $0 < c_1(k,\sigma) < c_2(k,\sigma)$ such that*

$$\frac{c_1(k,\sigma)}{\log n} < \sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - \Lambda(x)| < \frac{c_2(k,\sigma)}{\log n}$$

*for $n > n_0$.*

**Theorem 1.2.** *For $\alpha_n$ and $\beta_n$ are given by (1.3) and (1.4), respectively. Then*

$$F^n(\alpha_n x + \beta_n) - \Lambda(x) \sim \frac{(2k-1)e^{-x}\exp\{-e^x\}}{16k^3}\frac{(\log\log n)^2}{\log n},$$

*for large $n$.*

## 2 Auxiliary results

In order to prove main results, we will provide some important properties of the generalized Maxwell distribution. The first one is the distributional tail representation of the GMD, described as follows.

**Lemma 2.1.** *Let $F$ be the distribution function of GMD with $k > 1/2$. For $x > 0$, we have*

$$1 - F(x) = \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}x\left(1 + \frac{\sigma^2}{k}x^{-2k}\right)\exp\left(-\frac{x^{2k}}{2\sigma^2}\right) - r(x) \tag{2.1}$$

$$= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}x\left(1 + \frac{\sigma^2}{k}x^{-2k} - \frac{(2k-1)\sigma^4}{k^2}x^{-4k}\right)\exp\left(-\frac{x^{2k}}{2\sigma^2}\right) + s(x), \tag{2.2}$$

*where*

$$0 < r(x) = \frac{2k-1}{2^{k/2}\sigma^{1/k-2}k\Gamma(1+k/2)}\int_x^{+\infty} t^{-2k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{d}t \qquad (2.3)$$
$$< \frac{2k-1}{2^{k/2}\sigma^{1/k-4}k^2\Gamma(1+k/2)}x^{1-4k}\exp\left(-\frac{x^{2k}}{2\sigma^2}\right)$$

*and*

$$0 < s(x) = \frac{(2k-1)(4k-1)}{2^{k/2}\sigma^{1/k-4}k^2\Gamma(1+k/2)}\int_x^{+\infty} t^{-4k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{d}t \qquad (2.4)$$
$$< \frac{(2k-1)(4k-1)}{2^{k/2}\sigma^{1/k-6}k^3\Gamma(1+k/2)}x^{1-6k}\exp\left(-\frac{x^{2k}}{2\sigma^2}\right).$$

**Proof.** By integration by parts we have

$$1-F(x) = \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}x\left(1+\frac{\sigma^2}{k}x^{-2k}\right)\exp\left(-\frac{x^{2k}}{2\sigma^2}\right) -$$
$$\frac{2k-1}{2^{k/2}\sigma^{1/k-2}k\Gamma(1+k/2)}\int_x^{+\infty} t^{-2k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{d}t$$
$$=: \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}x\left(1+\frac{\sigma^2}{k}x^{-2k}\right)\exp\left(-\frac{x^{2k}}{2\sigma^2}\right) - r(x),$$

so (2.1) follows. Similarly,

$$r(x) = \frac{2k-1}{2^{k/2}\sigma^{1/k-4}k^2\Gamma(1+k/2)}x^{1-4k}\exp(-\frac{x^{2k}}{2\sigma^2}) - s(x). \qquad (2.5)$$

Putting (2.5) into (2.1), we obtain (2.2), where

$$s(x) = \frac{(2k-1)(4k-1)}{2^{k/2}\sigma^{1/k-4}k^2\Gamma(1+k/2)}\int_x^{+\infty} t^{-4k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{d}t$$
$$= \frac{(2k-1)(4k-1)}{2^{k/2}\sigma^{1/k-6}k^3\Gamma(1+k/2)}x^{1-6k}\exp(-\frac{x^{2k}}{2\sigma^2})$$
$$- \frac{(2k-1)(4k-1)(6k-1)}{2^{k/2}\sigma^{1/k-6}k^3\Gamma(1+k/2)}\int_x^{+\infty} t^{-6k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{d}t$$
$$< \frac{(2k-1)(4k-1)}{2^{k/2}\sigma^{1/k-6}k^3\Gamma(1+k/2)}x^{1-6k}\exp(-\frac{x^{2k}}{2\sigma^2}).$$

Hence, the lemma follows. □

For the norming constants $a_n$ and $b_n$ defined by (1.6) and (1.5), respectively, let

$$a_n^* = a_n r_n, \quad b_n^* = b_n + \delta_n a_n, \qquad (2.6)$$

where $r_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$. The following decomposition is needed.

**Lemma 2.2.** *Let $a_n^*$ and $b_n^*$ be defined by (2.6). For fixed $x \in \mathbb{R}$ and sufficiently large $n$,*

$$F^n(a_n^*x+b_n^*) - \Lambda(x) = \Lambda(x)e^{-x}\{((2k-1)x^2-2x-2)a_nb_n^{-1}/2$$
$$+ (r_n-1)x + \delta_n + O((a_nb_n^{-1})^2 + (r_n-1)^2 + \delta_n^2)\}.$$

**Proof.** Note that $b_n \sim \sigma^{1/k}(2\log n)^{1/(2k)}$ by (1.5), which induces that

$$a_n b_n^{-1} \sim (2k\log n)^{-1} \to 0$$

by (1.6). So, by (1.5) and (1.6) we have

$$\begin{aligned}
&\frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}(a_n^* x + b_n^*)\exp\left(-\frac{(a_n^* x + b_n^*)^{2k}}{2\sigma^2}\right) \\
=&n^{-1}(1+a_n b_n^{-1}(r_n x+\delta_n))\exp\left\{-\frac{b_n^{2k}}{2\sigma^2}((1+a_n b_n^{-1}(r_n x+\delta_n))^{2k}-1)\right\} \\
=&n^{-1}(1+a_n b_n^{-1}(r_n x+\delta_n))\exp\bigg\{-x-(r_n-1)x-\delta_n-\frac{1}{2}(2k-1)a_n b_n^{-1}(r_n x+\delta_n)^2 \\
&+O((a_n b_n^{-1})^2(r_n x+\delta_n)^3)\bigg\} \\
=&n^{-1}e^{-x}(1+a_n b_n^{-1}(r_n x+\delta_n))\bigg\{1-(r_n-1)x-\delta_n-\frac{1}{2}(2k-1)a_n b_n^{-1}(r_n x+\delta_n)^2 \\
&+O((a_n b_n^{-1})^2(r_n x+\delta_n)^3)+\frac{1}{2}\left((r_n-1)x+\delta_n+\frac{1}{2}(2k-1)a_n b_n^{-1}(r_n x+\delta_n)^2\right)^2 \\
&+O((a_n b_n^{-1})^3+(r_n-1)^3+\delta_n^3)\bigg\} \\
=&n^{-1}e^{-x}\left\{1-(r_n-1)x-\delta_n-\frac{1}{2}((2k-1)x^2-2x)a_n b_n^{-1}+O((a_n b_n^{-1})^2+(r_n-1)^2+\delta_n^2)\right\}. \quad (2.7)
\end{aligned}$$

Since

$$(a_n^* x + b_n^*)^{-2k} = \frac{k}{\sigma^2}(a_n b_n^{-1} - 2k(a_n b_n^{-1})^2 + O((a_n b_n^{-1})^3)), \tag{2.8}$$

we have

$$(a_n^* x + b_n^*)^{-4k} = \frac{k^2}{\sigma^4}((a_n b_n^{-1})^2 + O((a_n b_n^{-1})^3)). \tag{2.9}$$

Similarly,

$$(a_n^* x + b_n^*)^{1-6k}\exp\left(-\frac{(a_n^* x + b_n^*)^{2k}}{2\sigma^2}\right) = O(n^{-1}(a_n b_n^{-1})^3),$$

so,

$$s(a_n^* x + b_n^*) = O(n^{-1}(a_n b_n^{-1})^3), \tag{2.10}$$

here $s(x)$ is defined by (2.4). Hence, by (2.2) and (2.7)-(2.10), we have

$$\begin{aligned}
1-F(a_n^* x + b_n^*) =&n^{-1}e^{-x}\bigg\{1-(r_n-1)x-\delta_n-\frac{1}{2}((2k-1)x^2-2x-2)a_n b_n^{-1} \\
&+O((a_n b_n^{-1})^2+(r_n-1)^2+\delta_n^2)\bigg\}
\end{aligned}$$

for large enough $n$. Therefore,

$$F^n(a_n^*x+b_n^*)-\Lambda(x)=\Bigg\{1-n^{-1}e^{-x}\Bigg\{1-(r_n-1)x-\delta_n-\frac{1}{2}((2k-1)x^2-2x-2)a_nb_n^{-1}$$

$$+O((a_nb_n^{-1})^2+(r_n-1)^2+\delta_n^2)\Bigg\}\Bigg\}^n-\Lambda(x)$$

$$=\Lambda(x)e^{-x}\Bigg\{\frac{1}{2}((2k-1)x^2-2x-2)a_nb_n^{-1}+(r_n-1)x$$

$$+\delta_n+O((a_nb_n^{-1})^2+(r_n-1)^2+\delta_n^2)\Bigg\},$$

which derives the desired result. □

# 3 The proofs

Firstly, we give the proof of Theorem 1.2 as it is relatively easy.

**Proof of Theorem** 1.2. Firstly, we gain the following asymptotic expansions of $b_n$ defined by (1.5):

$$b_n=\beta_n+o((\log n)^{1/(2k)-1}), \tag{3.1}$$

and

$$b_n=\beta_n-\frac{\sigma^{1/k}}{2k^3}\frac{1}{(2\log n)^{2-1/(2k)}}\left(\frac{2k-1}{4k}t_n^2-t_n\right)+O\left(\frac{(\log\log n)^2}{(\log n)^{3-1/(2k)}}\right), \tag{3.2}$$

here

$$t_n=\log\log n+(1-k^2)\log 2-2k\log\Gamma(1+k/2),$$

and $\beta_n$ is defined by (1.4). By Corollary 1.7 of Resnick (1987) we have

$$P(M_n\le a_nx+b_n)\to\Lambda(x).$$

By arguments similar to those used in Example 2 of Resnick (1987) ( pp.71-72), we can derive (3.1). Now set

$$b_n=\beta_n+\theta_n,$$

where $\theta_n=o((\log n)^{1/(2k)-1})$.

Substituting $b_n=\beta_n+\theta_n$ into

$$\log(2^{k/2}\sigma^{1/k}\Gamma(1+k/2))-\log b_n+\frac{b_n^{2k}}{2\sigma^2}=\log n,$$

observing that

$$\log(1+x)=x-\frac{1}{2}x^2+O(x^3)\quad\text{as}\quad x\to 0$$

and

$$(1+x)^k=1+kx+\frac{k(k-1)}{2}x^2+O(x^3)\quad\text{as}\quad x\to 0,$$

we can derive

$$
\begin{aligned}
&\left(\frac{1}{8\sigma^{2/k}\log^2 n}+\frac{k(2k-1)}{4\sigma^{2/k}\log n}\right)\frac{\theta_n^2}{(2\log n)^{1/k-2}}\\
&+\left(-\frac{1}{2\sigma^{1/k}\log n}+\frac{t_n}{8k^2\sigma^{1/k}\log^2 n}+\frac{k}{\sigma^{1/k}}+\frac{(2k-1)t_n}{4k\sigma^{1/k}\log n}\right)\\
&\times\frac{\theta_n}{(2\log n)^{1/(2k)-1}}\\
&=-\frac{1}{4k^2\log n}\left(\frac{2k-1}{4k}t_n^2-t_n\right)+O\left(\frac{(\log\log n)^3}{\log^2 n}\right),
\end{aligned}
$$

by the above equality we can have

$$\theta_n \sim -\frac{\sigma^{1/k}}{2k^3(2\log n)^{2-1/(2k)}}\left(\frac{2k-1}{4k}t_n^2-t_n\right).$$

Once again, let

$$\theta_n = -\frac{\sigma^{1/k}}{2k^3(2\log n)^{2-1/(2k)}}\left(\frac{2k-1}{4k}t_n^2-t_n\right)+\vartheta_n,$$

here $\vartheta_n = o\left((\log\log n)^2/(\log n)^{2-1/(2k)}\right)$. By similar arguments, we can derive (3.2). Observe that

$$a_n b_n^{-1} \sim \frac{1}{2k\log n}, \quad r_n - 1 = \frac{a_n^*}{a_n} - 1 \sim \frac{2k-1}{4k^2}\frac{\log\log n}{\log n},$$

and

$$\delta_n = \frac{b_n^* - b_n}{a_n} \sim \frac{2k-1}{16k^3}\frac{(\log\log n)^2}{\log n}$$

for large $n$. Therefore, the result follows by Lemma 2.2. $\square$

**Proof of Theorem** 1.1 Setting $r_n = 1, \quad \delta_n = 0$ in (2.6), and noting that $a_n b_n^{-1} \sim 1/(2k\log n)$, by Lemma 2.2 we can prove that there exists an absolute constant $c_1(k,\sigma) > 0$ such that

$$\sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - \Lambda(x)| > \frac{c_1(k,\sigma)}{\log n}$$

for $n > n_0$. In order to obtain the upper bound, we need to prove that

$$\sup_{0\le x<\infty} |F^n(a_n x + b_n) - \Lambda(x)| < \mathcal{D}_1 a_n b_n^{-1}, \tag{3.3}$$

$$\sup_{-c_n\le x<0} |F^n(a_n x + b_n) - \Lambda(x)| < \mathcal{D}_2 a_n b_n^{-1}, \tag{3.4}$$

$$\sup_{-\infty<x<-c_n} |F^n(a_n x + b_n) - \Lambda(x)| < \mathcal{D}_3 a_n b_n^{-1}, \tag{3.5}$$

for $n > n_0$, where $\mathcal{D}_i = \mathcal{D}_i(k,\sigma) > 0, \quad i = 1,2,3$ are absolute constants and $c_n =: \log\log b_n^{2k}$ is positive for $n > n_0$. Observe that, from (3.1),

$$2\sigma^2\log n < b_n^{2k} < 2\sigma^2(1+c_0)^{2k}\log n \tag{3.6}$$

and

$$\sup_{n>n_0}\frac{1}{b_n^{2k}}\log\log b_n^{2k} < \sup_{n>n_0}\frac{\log\log(2\sigma^2(1+c_0)^{2k}\log n)}{2\sigma^2\log n} < \frac{k}{\sigma^2} \tag{3.7}$$

for $n > n_0$, where $c_0$ is a absolute positive constant. So, $b_n - a_n c_n > 0$ for $n > n_0$.

For simplicity, throughout the rest of this paper, let $\mathcal{C}_{ij}$ $(i \in N, j \in N)$ denote absolute positive constants whose value may vary from line by line.

Firstly, suppose that $x \geq -c_n$. Let

$$R_n(x) = -\{n \log F(a_n x + b_n) + n\Psi_n(x)\}, \quad B_n(x) = \exp\{-R_n(x)\},$$

$$A_n(x) = \exp\{-n\Psi_n(x) + e^{-x}\},$$

where $\Psi_n(x) = 1 - F(a_n x + b_n)$. Note that, for $k > \frac{1}{2}$,

$$b_n^{2k} - (b_n - a_n c_n)^{2k} = \int_{b_n - a_n c_n}^{b_n} 2kt^{2k-1}\,\mathrm{d}t < 2k a_n c_n b_n^{2k-1},$$

and, by (2.1), (1.5) and (1.6), we have

$$\begin{aligned}
\Psi_n(x) &\leq \Psi_n(-c_n) \\
&< \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}(b_n - a_n c_n)\left(1 + \frac{\sigma^2}{k}(b_n - a_n c_n)^{-2k}\right)\exp\left(-\frac{(b_n - a_n c_n)^{2k}}{2\sigma^2}\right) \\
&= n^{-1}(1 - a_n b_n^{-1} c_n)\left(1 + \frac{\sigma^2}{k}(b_n - a_n c_n)^{-2k}\right)\exp\left(\frac{b_n^{2k}}{2\sigma^2} - \frac{(b_n - a_n c_n)^{2k}}{2\sigma^2}\right) \\
&< n^{-1}(1 - a_n b_n^{-1} c_n)(1 + a_n b_n^{-1}(1 - a_n b_n^{-1} c_n)^{-2k})\exp\left(\frac{k}{\sigma^2} a_n c_n b_n^{2k-1}\right) \\
&< n^{-1}\left(1 + \frac{\sigma^2}{k} b_n^{-2k}\left(1 - \frac{\sigma^2}{k} b_n^{-2k}\log\log b_n^{2k}\right)^{-2k}\right)\exp(\log\log b_n^{2k}) \\
&< \sup_{n > n_0}\left\{\left(1 + \frac{\sigma^2}{k} b_n^{-2k}\left(1 - \frac{\sigma^2}{k} b_n^{-2k}\log\log b_n^{2k}\right)^{-2k}\right) n^{-1}\log(2\sigma^2(1 + c_0)^{2k}\log n)\right\} \\
&< \mathcal{C}_{11} \\
&< 1.
\end{aligned}$$

So,

$$\inf_{x \geq -c_n}\{1 - \Psi_n(x)\} > 1 - \mathcal{C}_{11} > 0.$$

Noting that the following inequalities

$$\log(1-x) > -x - \frac{x^2}{2(1-x)}, \quad \log(1-x) < -x, \quad \text{as} \quad 0 < x < 1,$$

we have

$$\begin{aligned}
0 < R_n(x) &\leq \frac{n\Psi_n^2(x)}{2(1 - \Psi_n(x))} \\
&< \frac{n\Psi_n^2(-c_n)}{2(1 - \Psi_n(-c_n))} \\
&< \frac{n^{-1}(1 + a_n b_n^{-1}(1 - a_n b_n^{-1} c_n)^{-2k})^2 a_n b_n^{-1}\exp\{2c_n\}}{2(1 - \Psi_n(-c_n)) a_n b_n^{-1}}
\end{aligned}$$

$$< \left(\frac{n_0\mathcal{C}_{11}}{\log(2\sigma^2\log n_0)}\right)^2 \frac{k}{2\sigma^2(1-\mathcal{C}_{11})}\frac{n^{-1}b_n^{2k}\exp\{2c_n\}}{a_n^{-1}b_n}. \tag{3.8}$$

By (3.6), we can have

$$n^{-1}b_n^{2k}\exp\{2c_n\} < n^{-1}b_n^{6k} < n^{-1}(2\sigma^2(1+c_0)^{2k}\log n)^3 < \mathcal{C}_{12}, \quad \text{as} \quad n > n_0. \tag{3.9}$$

Putting (3.9) into (3.8), we have

$$R_n(x) < \left(\frac{n_0\mathcal{C}_{11}}{\log(2\sigma^2\log n_0)}\right)^2 \frac{k\mathcal{C}_{12}}{2\sigma^2(1-\mathcal{C}_{11})}a_n b_n^{-1} = \mathcal{C}_{13}a_n b_n^{-1}.$$

So, by $1-e^{-x} < x, \quad x > 0$, we have

$$|B_n(x)-1| < R_n(x) < \mathcal{C}_{13}a_n b_n^{-1}, \quad \text{for} \quad n > n_0. \tag{3.10}$$

By inequality (3.10) we obtain

$$\begin{aligned} |F^n(a_nx+b_n)-\Lambda(x)| &= |\exp\{-R_n(x)\}A_n(x)\Lambda(x)-\Lambda(x)| \\ &\le \Lambda(x)B_n(x)|A_n(x)-1| + |B_n(x)-1| \\ &< \Lambda(x)|A_n(x)-1| + \mathcal{C}_{13}a_n b_n^{-1}, \end{aligned} \tag{3.11}$$

for $x \ge -c_n$.

Next we prove (3.3). Note that, as $k > 1$,

$$(1+x)^k > 1+kx, \quad \text{for} \quad x > 0,$$

combining with (1.6), we obtain

$$x - \frac{(a_nx+b_n)^{2k}-b_n^{2k}}{2\sigma^2} < 0, \quad \text{for} \quad x > 0. \tag{3.12}$$

By (1.5), (1.6), (3.12) and the definition of $A_n(x)$, we have

$$\begin{aligned} A_n'(x) &= \exp\{-n\Psi_n(x)+e^{-x}\}[-n\Psi_n(x)+e^{-x}]' \\ &= -A_n(x)e^{-x}[1-na_ne^xF'(a_nx+b_n)] \\ &= -A_n(x)e^{-x}\left[1-\frac{k}{\sigma^2}a_nb_n^{-1}(a_nx+b_n)^{2k}\exp\left(\frac{b_n^{2k}}{2\sigma^2}\right)e^x\exp\left(-\frac{(a_nx+b_n)^{2k}}{2\sigma^2}\right)\right] \\ &= -A_n(x)e^{-x}\left[1-(1+a_nb_n^{-1}x)^{2k}\exp\left\{x-\frac{(a_nx+b_n)^{2k}-b_n^{2k}}{2\sigma^2}\right\}\right] \\ &< 0 \end{aligned}$$

for $x > 0$. Since $A_n(x) \to 1$ as $x \to \infty$, we have

$$\begin{aligned} &\sup_{x\ge 0}|A_n(x)-1| \\ =&|A_n(0)-1| \\ =&|\exp\{-\frac{\sigma^2}{k}b_n^{-2k}+nr(b_n)\}-1| \end{aligned}$$

$$
\begin{aligned}
&< \exp\{nr(b_n)\} - 1 \\
&\le nr(b_n)\exp\{nr(b_n)\} \\
&\le (2k-1)a_n b_n^{-1}\exp\left\{\frac{2k-1}{2k\log n_0}\right\} \\
&= \mathcal{C}_{14} a_n b_n^{-1}.
\end{aligned}
$$

The inequalities come from the facts that $e^x - 1 \le xe^x$ for $0 \le x \le 1$,

$$
0 < nr(b_n) = \frac{(2k-1)\sigma^4}{k^2} b_n^{-4k} < (2k-1)a_n b_n^{-1},
$$

and

$$
\exp\{nr(b_n)\} < \exp\{(2k-1)a_n b_n^{-1}\} < \exp\left\{\frac{2k-1}{2k\log n_0}\right\}, \quad \text{for } \ n > n_0.
$$

Combining with (3.11), we have

$$
\sup_{0\le x<\infty} |F^n(a_n x + b_n) - \Lambda(x)| < (\mathcal{C}_{13} + \mathcal{C}_{14}) a_n b_n^{-1}.
$$

Secondly, we consider the case of $-c_n \le x < 0$. By (1.5), (1.6) and Lemma 2.1, we have

$$
\begin{aligned}
&-n\Psi_n(x) + e^{-x} \\
&= -n\left[\frac{(a_n x + b_n)}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\left(1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k}\right)\exp\left(-\frac{(a_n x + b_n)^{2k}}{2\sigma^2}\right) - r(a_x + b_n)\right] \\
&\quad + e^{-x} \\
&= -n\Bigg[\frac{(a_n x + b_n)}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\left(1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k}\right)\exp\left(-\frac{(a_n x + b_n)^{2k}}{2\sigma^2}\right) \\
&\quad - \frac{(2k-1)(a_n x + b_n)^{1-4k}}{2^{k/2}\sigma^{1/k-4}k^2\Gamma(1+k/2)} d_n(a_n x + b_n)\exp\left(-\frac{(a_n x + b_n)^{2k}}{2\sigma^2}\right)\Bigg] + e^{-x} \\
&= -(1 + a_n b_n^{-1}x)\left(1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k}\right)\exp\left(\frac{b_n^{2k} - (a_n x + b_n)^{2k}}{2\sigma^2}\right) \\
&\quad + \frac{(2k-1)\sigma^4}{k^2}(1 + a_n b_n^{-1}x)^{1-4k} b_n^{-4k} d_n(a_n x + b_n)\exp\left(\frac{b_n^{2k} - (a_n x + b_n)^{2k}}{2\sigma^2}\right) + e^{-x} \\
&= (1 + a_n b_n^{-1}x)e^{-x}\Bigg\{ -\left[1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k} - \frac{(2k-1)\sigma^4}{k^2}(1 + a_n b_n^{-1}x)^{1-4k} b_n^{-4k} d_n(a_n x + b_n)\right] \\
&\quad \times \exp\left(-\frac{(a_n x + b_n)^{2k} - b_n^{2k} - 2\sigma^2 x}{2\sigma^2}\right) + (1 + a_n b_n^{-1}x)^{-1}\Bigg\},
\end{aligned}
$$

where $0 < d_n(a_n x + b_n) < 1$ and

$$
\begin{aligned}
D_n(x) = &-\left\{1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k} - \frac{(2k-1)\sigma^4}{k^2}(1 + a_n b_n^{-1}x)^{1-4k} b_n^{-4k} d_n(a_n x + b_n)\right\} \\
&\times \exp\left(-\frac{(a_n x + b_n)^{2k} - b_n^{2k} - 2\sigma^2 x}{2\sigma^2}\right) + (1 + a_n b_n^{-1}x)^{-1}.
\end{aligned}
$$

Noting that

$$(1+x)^k > 1-kx, \quad (1+x)^{-k} < 1-kx \quad \text{for} \quad -1<x<0, \tag{3.13}$$

and

$$a_n x + b_n > 0 \quad \text{for} \quad x > -c_n \quad \text{and} \quad e^{-x} > 1-x \quad \text{for} \quad x>0,$$

we obtain

$$\begin{aligned}
D_n(x) <& -\left(1 - \frac{(a_n x+b_n)^{2k} - b_n^{2k} - 2\sigma^2 x}{2\sigma^2}\right)\left\{1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k}\right.\\
&\left. - \frac{(2k-1)\sigma^4}{k^2}(1+a_n b_n^{-1}x)^{1-4k} b_n^{-4k} d_n(a_n x+b_n)\right\} + (1+a_n b_n^{-1}x)^{-1}\\
&< -\left(1 - \frac{(a_n x+b_n)^{2k} - b_n^{2k} - 2\sigma^2 x}{2\sigma^2}\right)\left\{1 - \frac{(2k-1)\sigma^4}{k^2} b_n^{-4k}(1-(1-4k)a_n b_n^{-1}x)\right\}\\
&+ (1+a_n b_n^{-1}x)^{-1}\\
&< -1 + \frac{(2k-1)\sigma^4}{k^2} b_n^{-4k}(1-(1-4k)a_n b_n^{-1}x) + \frac{(a_n x+b_n)^{2k} - b_n^{2k} - 2\sigma^2 x}{2\sigma^2} + (1+a_n b_n^{-1}x)^{-1}\\
&< (2k-1)(a_n b_n^{-1})^2(1-(1-4k)a_n b_n^{-1}x) - a_n b_n^{-1}x(1+a_n b_n^{-1}x)^{-1}\\
&< (2k-1)a_n b_n^{-1} - a_n b_n^{-1}x(1+a_n b_n^{-1}x)^{-1}.
\end{aligned}$$

Meanwhile, utilizing (3.13), we have

$$\begin{aligned}
D_n(x) &> -1 - \frac{\sigma^2}{k}(a_n x+b_n)^{-2k} + (1+a_n b_n^{-1}x)^{-1}\\
&> -(1+a_n b_n^{-1}x)^{-2k} a_n b_n^{-1}\\
&> -(1-2k a_n b_n^{-1}x) a_n b_n^{-1}.
\end{aligned}$$

Hence,

$$|D_n(x)| < \{2k-1-x(1+a_n b_n^{-1}x)^{-1} + (1-2k a_n b_n^{-1}x)\} a_n b_n^{-1}.$$

Therefore, for $n > n_0$,

$$\begin{aligned}
|-n\Psi_n(x) + e^{-x}| &< (1+a_n b_n^{-1}x)e^{-x}|D_n(x)|\\
&< \{2k(1-a_n b_n^{-1}x) - x(1+a_n b_n^{-1}x)\}(1+a_n b_n^{-1}x)e^{-x} a_n b_n^{-1}\\
&< \{2k(1+(a_n b_n^{-1})^2 c_n^2) + c_n\} e^{c_n} a_n b_n^{-1}\\
&< \mathcal{C}_{21}.
\end{aligned}$$

So,

$$\begin{aligned}
\Lambda(x)|A_n(x)-1| =& \Lambda(x)|\exp\{-n\Psi_n(x)+e^{-x}\} - 1|\\
<& \Lambda(x)|-n\Psi_n(x)+e^{-x}|\{\exp\{-n\Psi_n(x)+e^{-x}\}+1\}\\
<& (e^{\mathcal{C}_{21}}+1)\{2k(1-(a_n b_n^{-1})^2 x^2) - x\} e^{-x}\Lambda(x) a_n b_n^{-1}\\
<& \mathcal{C}_{22} a_n b_n^{-1}.
\end{aligned}$$

Now Combining the above inequalities with (3.11), the proof of (3.4) is complete.

In the end, we consider the case of $-\infty < x < -c_n$. If $a_n x + b_n \leq 0$, we have $F^n(a_n x + b_n) = 0$. Noting that $\Lambda(-x) < 1/x$, as $x > 1$, we obtain

$$\begin{aligned}\sup_{x \leq -b_n/a_n} |F^n(a_n x + b_n) - \Lambda(x)| &= \sup_{x \leq -b_n/a_n} \Lambda(x) \\ &< \Lambda(-\frac{k}{\sigma^2} b_n^{2k}) < a_n b_n^{-1}.\end{aligned}$$

So, we only need to consider the case of $a_n x + b_n > 0$. Applying the monotonicity of $\Lambda(x)$, we have

$$\sup_{-\infty < x < -c_n} \Lambda(x) \leq \Lambda(-c_n) = \frac{k}{\sigma^2} a_n b_n^{-1},$$

moreover,

$$\begin{aligned}\sup_{-\infty < x < -c_n} |F^n(a_n x + b_n) - \Lambda(x)| &< F^n(b_n - a_n c_n) + \Lambda(-c_n) \\ &< \sup_{-c_n \leq x < 0} |F^n(a_n x + b_n) - \Lambda(x)| + 2\Lambda(-c_n) \\ &< (\mathcal{C}_{22} + \mathcal{C}_{13}) a_n b_n^{-1} + \frac{2k}{\sigma^2} a_n b_n^{-1} \\ &= \mathcal{C}_{31} a_n b_n^{-1}.\end{aligned}$$

The proof of (3.5) is complete. This completes the proof of Theorem 1.1. □

# References


[1] Balkema, A.A., de Haan, L., 1990. A convergence rate in extreme-value theory. Journal of Applied Probability 27, 577-585.

[2] Chen, S., Feng, B., 2014. Rates of convergence of extreme for STSD under power normalization. Journal of the Korean Statistical Society, Available on-line at http://dx.doi.org/10.1016/j.jkss.2014.02.001.

[3] Cheng, S., Jiang, C., 2001. The Edgeworth expansion for distributions of extreme values. Science in China Series A: Mathematics 44, 427-437.

[4] Chen, S., Wang, C., Zhang, G., 2012. Rates of convergence of extreme for general error distribution under power normalization. Statistics and Probability Letters 82, 385-395.

[5] de Haan, L., Resnick, S.I., 1996. Second order regular variation and rates of convergence in extreme value theory. The Annals of Probability 24, 97-124.

[6] Hall, P., 1979. On the rate of convergence of normal extremes. Journal of Applied Probability 16, 433-439.

[7] Huang, J., Chen, S., 2014. Tail behavior of the generalized Maxwell distribution. Communications in Statistics-Theory and Methods, (Accepted).

[8] Lin, F., Zhang, X., Peng, Z., Jiang, Y., 2011. On the rate of convergence of STSD extremes. Communications in StatisticsTheory and Methods 40, 1795-1806.

[9] Liu, C., Liu, B., 2013. Convergence rate of extremes from Maxwell sample. Journal of Inequalities and Applications 477, Available on-line at http://www.journalofinequalitiesandapplications.com/content/2013/1/477.

[10] Peng, Z., Nadarajah, S., 2012. Convergence rates for the moments of extremes. Bulletin of the Korean Mathematical Society 49, 495-510.

[11] Peng, Z., Nadarajah, S., Lin, F., 2010. Convergence rate of extremes from general error distribution. Journal of Applied Probability 47, 668-679.

[12] Resnick, S.I., 1987. Extreme value, Regular Variation, and Point Processes. Springer-Verlag, New York.