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Transformations of Measures Via Their Generalized Densities

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This note is dedicated to the memory of Professor V. P. Belavkin

Abstract. In this note we describe algorithms for obtaining formulae for transformations of measures on infinite dimensional topological vector spaces or manifolds, generated by transformations of the domains of the measures and by transformations of the range.

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INTRODUCTION.

In this note we describe algorithms for obtaining formulae for transformations of measures on infinite dimensional topological vector spaces or manifolds, generated by transformations of the domains of the measures and by transformations of the range. The important classes of manifolds to which our results can be applied are collections of some functions, of the real variable, taking values in a Riemannian manifold K. If K is a submanifold of $\mathbb{R}^n$, for some n, $t > 0, k \in K$ then the subset $C_k([0,t],K)$ of all continuous functions on $[0,t]$, taking values in K and such that $f(0) = k$, can be considered as a submanifold of $C_k([0,t],\mathbb{R}^n)$ having both finite dimension and finite codimension. By this way one can reduce, using the constructions of the so-called surface measure, some properties of measures and even pseudomeasures on $C_k([0,t],K)$ to the properties of the measures and pseudomeasures on $C_k([0,t],\mathbb{R}^n)$. An investigation that does not depend on the embedding $K \subset \mathbb{R}^n$ is also possible[1]; in both cases one uses Feynman-type formulae obtained via the Chernoff theorem[2]. Both the algorithms and the formulae are similar to what is known for the finite-dimensional setting. But, instead of the usual densities of measures, either with respect to the standard Lebesgue measure on finite dimensional spaces or with respect to the measure generated by the Riemannian volume, we use the so-called generalized densities of measures (see for example [2, 4, 5]) that in the finite dimensional case coincide with these standard densities. We do not formulate any general definition of an infinite dimensional manifold. The use of generalized densities is motivated by the famous theorem by A.Weil which claims that on infinite-dimensional locally convex spaces there does not exist an analog of the Lebesgue measure, i.e, of a Borel σ-additive σ-finite locally finite nonzero measure which is invariant with respect to translations. This theorem implies that an infinite-dimensional space cannot be equipped with a canonical measure.

There exist two ways to overcome this problem: either to take, instead of canonical, any measure with sufficiently good properties, e.g., a Gaussian measure, or to use generalized densities.

In what follows, all topological spaces are assumed to be Radon spaces; the σ-algebra of Borel subsets of a topological space E is denoted by $\mathcal{B}(E)$. If E is a locally convex space (LCS), then $\mathcal{B}(E)$ is assumed to coincide with the σ-algebra generated by the algebra $\mathcal{A}(E)$ of E'-cylindrical subsets of E. We say "measure on E" instead of "measure on $\mathcal{B}(E)$".

If E is a finite-dimensional Euclidean space, then the Lebesgue measure on E is equivalent to any Gaussian measure on E which is not concentrated on a proper subspace of E. Such a Gaussian measure is not invariant with respect to translations, but is quasi-invariant in the sense that any shift of the Gaussian measure is a measure which is equivalent to the original one. If E is an infinite-dimensional LCS, then the nondegenerate (i.e., not concentrated on proper closed

[1]It is worth mentioning that it is just the combination of the Chernoff theorem and the construction of the surface measure has led to the solution of some problem coming back to Onsager–Machlup and related to how geometrical characteristics of the manifold K can be represented by some additional potential.

[2]This approach leads also to an alternative method of investigations of diffusion processes in Riemannian manifolds.

subspaces) Gaussian measures on E are quasi-invariant in a weaker sense: for any such Gaussian measure μ there exists a dense vector (even Hilbert) subspace H of E with the following property: the image of the measure with respect to the shift along any element of H is the measure which is equivalent to the original measure. Hence one can try to use any of such Gaussian measure instead of the Lebesgue measure. This idea has been realized in the so called White Noise Analysis by T. Hida.

But if on an infinite dimensional LCS E there exists at least one nondegenerate Gaussian measure, then on E there exist a continuum of nondegenerate Gaussian measures on E which are pairwise singular; nevertheless in the frame of Hida's approach, it is allowed to consider only one of them.

To make the situation more invariant, it is reasonable to consider all measures on the same level. Of course it is necessary to assume that the measures has "good enough" analytical properties, i.e., that they are sufficiently smooth. Just such measures have the generalized densities that have some properties of usual densities. In particular, the transformations of the generalized densities of smooth measures on a LCS E induced by transformations of E are quite similar to the transformations of usual densities (when they exist, i.e., when $\dim E < \infty$). Also in this frame some nonlinear functions of measures can be defined as measures whose generalized densities are the same functions of generalized densities of the original measures. In particular, in this way one can define the square roots of some measures, including the Gaussian measures (a similar approach can be applied to the Feynman pseudomeasures)[3]. Moreover, some similar results can be obtained for vector-valued measures and transformations that transform both the domains and the ranges of the measures. We conjecture that in this way one obtain certain formulae related to the so-called quantum anomalies [7, 8] (it is worth mentioning that in [7, p. 352], it is written that the explanation of the quantum anomalies given in [8] is false).

We expect also that some of the presented ideas will be applicable to questions of the second quantization of complicated classical systems, such as second quantization of the so-called choreographies (see, for example, [9] and the references therein).

In what follows, we consider, in the first instance, the algebraic structure of the theory and often do not formulate the corresponding analytical assumptions.

The paper is organized as follows. In the first section, we present some essentially known results about differentiable measures. Following that, we introduce the notion of generalized density of a measure defined on an infinite-dimensional space or manifold.

The next two sections are related to transformations of measures, both on vector spaces and on submanifolds of vector spaces (whose dimension and codimension are equal to infinity) generated by transformations of the space on which the measures are defined. After that, we discuss transformations of measures induced by transformations of the range of the measure and also multiplications of measures by functions.

We tried to make the paper as independent as possible of previous results. To do so, we formulate, in a form suitable for our aims, some definitions that can be found elsewhere. Moreover, to clarify the situation, we mention some notions that we do not use.

1. DERIVATIVES OF MEASURE VALUED FUNCTIONS AND LOGARITHMIC DERIVATIVES OF MEASURES.

Let Ω be a set, $\mathcal{B}$ a σ-algebra of subsets of Ω and $\mathcal{M}(\Omega)$ the vector space of all (signed, σ-additive) measures on $\mathcal{B}$. Let $\mathcal{M}_\tau(\Omega)$ denote $\mathcal{M}(\Omega)$ equipped with a locally convex (Hausdorff) topology, and let $m : I \to \mathcal{M}(\Omega)$ be a function on an open interval $I \subset \mathbb{R}$. The function m is said to be τ-differentiable at $t_0 \in I$ (see [3]) if there exists in $\mathcal{M}_\tau(\Omega)$ the limit

$$\lim_{\substack{t \to t_0,\\ t \neq t_0}} \frac{m(t) - m(t_0)}{t}.$$

[3]It is worth mentioninig a serious problem related to generalized density: only within some special classes of measures does there exist a one-to-one correspondences between measures and their generalized densities. But in this note we will not discuss this problem.

This limit is called the τ-derivative of $m(t)$ at t_0, and denoted $m'(t_0)$.

Of course, if τ_1 and τ_2 are (different) comparable topologies and $\tau_1 \supset \tau_2$, then τ_1 differentiability at t_0 implies τ_2-differentiability there, and the corresponding derivatives coincide. For this reason we do not use the label τ in the notation $m'(t)$.

Proposition 1. (see [3]) *Let τ be the topology of setwise convergence. If $m(t) \geqslant 0$ for all t in a neighborhood of t_0 then $m'(t_0)$ is absolutely continuous with respect to $m(t_0)$.*

Proof. If $B \in \mathcal{B}(\Omega)$ and $m(t_0)(B) = 0$, then the function $t \mapsto m(t)(B)$ has a local minimum at t_0 and hence $m'(t_0)(B) = 0$.

If the function $m(t)$ is τ-differentiable at t_0 and $m'(t_0) \ll m(t)$, then the Radon-Nikodym density $\rho(t_0)$ of the measure $m(t_0)$ with respect to $m(t)$ is called the *logarithmic derivative* of m at t_0.

Example 1. Let ν be a nonnegative measure on $\mathcal{B}$ and, for every $t \in I$, $f(t,\cdot)$ be a nonnegative ν-integrable function on Ω, and let

$$m(t) = f(t,\cdot)\nu.$$

Suppose the derivatives of the functions $t \mapsto f(t,x)$ exist at t_0; then, under reasonable assumptions, $m'(t_0) = f'(t_0,\cdot)\nu$. Thus

$$m'(t_0) = \frac{f'(t_0,\cdot)}{f(t_0,\cdot)}\, m(t_0).$$

Consequently, in this case,

$$\rho(t_0)(x) = \frac{f'(t_0,x)}{f(t_0,x)} = \frac{\partial}{\partial t}\, \ln f(t_0,x),$$

which explains the term 'logarithmic derivative'.

Proposition 2 (see [3]). *If a measure-valued function $m(\cdot)$ is differentiable on $I = (a,b)$ and for every $t \in I$ the logarithmic derivative $\rho_m(t)$ of $m(t)$ exists, then for any $c, d \in I$ with $c < d$, the measure $m(d)$ is absolutely continuous with respect to $m(c)$ and the Radon–Nikodym derivative is*

$$\frac{dm(d)}{dm(c)} = \exp\left(\int_c^d \rho_m(\tau)d\tau\right).$$

To prove this, it is sufficient to check that the function μ defined by

$$\mu(t) = \exp\left(\int_a^t \rho_m(\tau)d\tau\right) m(a)$$

satisfies the differential equation $\mu'(t) = \rho_m(t)\mu(t)$ with initial condition $\mu(a) = m(a)$.

Remark 1. The proposition 2 can be considered as an abstract version of the Cameron-Martin–Girsanov–Maruyama–Ramer formula.

Let Ω be a separable Banach space and $\mathcal{B}$ the σ-algebra of Borel subsets of Ω. Let $\nu \in \mathcal{M}(\Omega)$, and for $h \in \Omega$ define the function

$$m_\nu^h : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{M}_\tau(\Omega)$$

by, for $A \in \mathcal{B}$,

$$m_\nu^h(t)(A) = \nu(A + th).$$

The measure ν is said to be (τ-)differentiable along h if the function m_ν^h is differentiable at $t = 0$ and if moreover $(m_\nu^h)'(0) \ll m_\nu^h(0)$; that is, if the logarithmic derivative of m_ν^h exists at $t = 0$. This logarithmic derivative of m_ν^h at $t = 0$ is denoted $\beta_\nu(h,\cdot)$ and is called the *logarithmic derivative of ν along h*.

The logarithmic derivative of a measure along a vector was introduced in [1]. In the same paper, it was also proved that if τ is the topology, in the space of measures, of setwise convergence, then the τ-differentiability of m_ν^h at $t = 0$ alone implies that the logarithmic derivative of m_ν^h exists at $t = 0$. It is also known that if τ_n is the topology defined by the norm equal to the total variation and τ_c is any locally convex topology between τ_n and the topology defined by all smooth bounded cylindrical functions, then τ_c-differentiability along h implies τ_n-differentiability.

2. GENERALIZED DENSITIES OF MEASURES

Let D_ν be the collection of elements $h \in \Omega$ along which the measure ν is differentiable. It is known [1] that D_ν is a vector subspace of Ω. Let H be a vector subspace of D_ν and suppose ν is nonnegative.

Definition 1. A generalized H-density of ν is a function $F_\nu^H : H \to \mathbb{R}^+$ whose logarithmic derivative along each $h \in H$ coincides with $\beta_\nu(h, \cdot)$.

We assume that the function $\beta_\nu : H \times H \to \mathbb{R}$ allows to reconstruct ν in a class of measures (this implies that H is dense in Ω). It follows that the generalized density also allows to reconstruct ν. In some cases, it is convenient to assume that H is a Hilbert subspace of Ω; this means that H is equipped with a structure of a Hilbert space, the embedding $H \to \Omega$ being continuous (with the dense image).

Example 1. Let Ω be a Hilbert space and ν_B be a Gaussian measure on Ω with correlation operator B and mean value equal to zero (this means that the Fourier transform $\widetilde{\nu_B}$ of ν is defined by $\widetilde{\nu_B}(x) = e^{-\frac{(Bx,x)}{2}}$), then the generalized density of ν_B is any function on $\sqrt{B}\Omega$ defined by $F_{\nu_B}^H(x) = ce^{-\frac{(B^{-1}x,x)}{2}}$, where $c > 0$ (each such function can be also called a version of the generalized density). This means that if $dim\Omega < \infty$, then the usual density is a version of the generalized density.

If $\nu = \nu_{B_1} + \nu_{B_2}$ and $H = \sqrt{B_1}\Omega \cap \sqrt{B_2}\Omega$, then the generalized density of ν is any function on H defined by $F_\nu^H(x) = c_1 e^{-\frac{(B_1^{-1}x,x)}{2}} + c_2 e^{-\frac{(B_2^{-1}x,x)}{2}}$ (so the generalized densities of ν constitute a two-dimensional vector space).

To define a generalized density of a measure ν on $C_k([0,t], K)$, where K is a Riemannian manifold, $k \in K$, it is necessary to be able to define the derivative of a measure along a vector field. A vector field on a manifold M (which can be a vector space) is a mapping $h : M \to M$. The definition of a differentiability of the measure ν on Ω and of the logarithmic derivative of ν along vector field (both on vector space and on manifold) is completely similar to the definition of the differentiability of the measure ν and the logarithmic derivative of ν along a vector. We denote the logarithmic derivative of ν along a vector field h by $\beta_\nu^h(\cdot)$. A sketch of a proof of the following proposition can be found in [4, 5].

Proposition 3. *Let H be a Hilbert subspace of Ω, h be a vector field on Ω such that $h(\Omega) \subset H$ and let the derivative $h'_H(x) \in L(H)$ of h along H be a trace class operator for any x, the function $x \mapsto trh'(x)$ being ν-integrable. Then*

$$\beta_\nu^h(x) = \beta_\nu(h(x), x) + trh'(x)$$

Remark 2. If h is a Hamiltonian vector field, then

$$\beta_\nu^h(x) = \beta_\nu(h(x), x);$$

this identity is the infinitesimal version of the Liouville theorem about the conservation of the phase volume (which actually does not exist if $dim\Omega = \infty$).

Corollary 1. (*to Proposition* 3) *A function $F : H \to \mathbb{R}$ is a generalized H-density of ν if and only if for any vector field h on Ω such that $h(\Omega) \subset H$ the following identity holds:*

$$\log F'(x)h(x) = \beta_\nu^h - trh'(x)$$

for any $x \in H$.

This corollary motivates a definition of the generalized density of a measure on a manifold.

Let $\mathcal{M}$ be an infinite-dimensional manifold and let $\mathcal{H}$ be a (dense) Hilbert submanifold of $\mathcal{M}$ (the definition of a Hilbert submanifold is similar to the definition of a Hilbert subspace). We say that a measure η on M is $\mathcal{H}$-differentiable if it is differentiable along any vector field h on $\mathcal{M}$ such that $h(\mathcal{M}) \subset \mathcal{H}$ and $trh'(x) = 0$ for any $x \in \mathcal{M}$.

Definition 2. Let η be a measure on $\mathcal{M}$ which is $\mathcal{H}$-differentiable. A generalized $\mathcal{H}$-density of η is a function $F_\eta^{\mathcal{H}} : \mathcal{H} \to \mathbb{R}^+$ having the following property: for any vector field h on $\mathcal{M}$ such that $h(M) \subset \mathcal{H}$ and $trh'(x) = 0$ one has $(\log F_\eta^{\mathcal{H}})'(x)h(x) = \beta_\nu^h$.

Remark 3. An analog of Corollary 1 also holds here.

3. TRANSFORMATIONS OF MEASURES ON VECTOR SPACES GENERATED BY TRANSFORMATIONS OF THE SPACES

We use the notation and assumptions of the preceding section. Let ψ be a mapping of Ω into itself that is twice (Fréchet) differentiable along H. Suppose that the function

$$H \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{F_\nu(\psi(x))}{F_\nu(x)} \det(\psi'(x))$$

is well-defined and can be extended by continuity to the whole space Ω; for this extension, we keep the same notation. Let μ be the image of ν with respect to ψ; that is, $\mu = \psi_* \nu$.

Theorem 1. *The Radon-Nikodym density* $d\mu/d\nu$ *satisfies*

$$\frac{d\mu}{d\nu}(x) = \frac{F_\nu(\psi(x))}{F_\nu(x)} \det(\psi'(x)).$$

One approach to proving this is based on ideas from [4]. In that paper, one uses, roughly speaking, a homotopy between the mapping ψ and the identity map sending x to x.

One can also use some finite-dimensional approximations for μ and ν.

4. TRANSFORMATIONS OF MEASURES ON MANIFOLDS GENERATED BY TRANSFORMATIONS OF MANIFOLDS

The analog of Theorem 1 of the preceding section is valid for measures on manifolds. This means that to obtain explicit formulae for transformations of the measures it is sufficient to find the generalized density of the measure. The aim of this section is to obtain the generalized density of the Wiener measure on the space of continuous functions, taking values in a compact Riemannian manifold S, generated by the Brownian motion in the manifold. We use here the approach described briefly in the introduction.

Let S be a compact Riemannian submanifold of $\mathbb{R}^n$ (any Riemannian manifold can be embedded isometrically into Euclidean space). Let $\Gamma > 0$, $a \in S$ and define

$$\Omega_{\mathbb{R}^n} = C_a([0,\Gamma], \mathbb{R}^n)$$

to be the vector space of all continuous functions g on $[0,\Gamma]$ taking values in $\mathbb{R}^n$ equipped with the Wiener measure W, and such that $g(0) = a$, and for $X \subset \mathbb{R}^n$ define $\Omega_X = C_a([0,\Gamma], X) \subset \Omega_{\mathbb{R}^n}$ to be the subset consisting of those functions taking values in X.

For any $\varepsilon > 0$, let S_ε be the ε-neighborhood of S in $\mathbb{R}^n$. Let W_S^ε be the measure on (Borel subsets of) $\Omega_{\mathbb{R}^n}$ defined by

$$W_S^\varepsilon(A) = \frac{W(\Omega_{S_\varepsilon} \cap A)}{W(\Omega_{S_\varepsilon})}.$$

One can prove (see [6]) that

$$W_S^\varepsilon \longrightarrow f W_S^{\mathbb{R}^n}$$

in the weak topology on $\mathcal{M}(\Omega_{\mathbb{R}^n})$ defined by the duality between $\mathcal{M}(\Omega_{\mathbb{R}^n})$ and the set of all bounded smooth cylindrical functions. Here $W_S^{\mathbb{R}^n}$ is the measure on $\Omega_{\mathbb{R}^n}$ defined by

$$W_S^{\mathbb{R}^n}(A) = W_S(A \cap \Omega_S),$$

where W_S is the Wiener measure on Ω_S, and the function f is defined by $f(\xi) = 0$ if $\xi \notin \Omega_S$ and otherwise

$$f(\xi) = \exp\left(\int_0^\Gamma \left(\tfrac{1}{8}a(\xi(\tau)) - \tfrac{1}{4}R(\xi(\tau))\right) d\tau\right),$$

where $R(x)$ is the scalar curvature and $a(x)$ the mean curvature of S at x.

The measure $fW_S^{\mathbb{R}^n}$ is just the surface measure which we mentioned in the introduction. But the generalized density of the surface measure is the restriction to Ω_S of the generalized density of W; therefore, to get the generalized density of W_S, it is sufficient to take the product of this restriction and the function f^{-1}. Hence the following theorem holds.

Theorem 2. *Let $\mathcal{H}$ be the Sobolev space $W_2^1([0,\Gamma], S)$. Then the generalized $\mathcal{H}$-density of the Wiener measure W_S is defined by*

$$F(\xi) = \frac{1}{f(\xi)} \exp\left(-\frac{1}{2}\int_0^\Gamma (\dot{\xi}(\tau))^2 d\tau\right).$$

5. SOME OTHER TRANSFORMATIONS OF MEASURES

In this section we consider two special transformations of Gaussian measures: powers of Gaussian measures and multiplications of Gaussian measures by Gaussian exponents. Similar results can also be obtained for some other classes of measures.

Definition 3. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function and let ν be a measure whose generalized H-density is F_ν. Then $\psi(\nu)$ is a measure whose generalized H-density is $\psi \circ F_\nu$.

This measure is well-defined only for special classes of functions ψ and measures ν. We will not discuss these classes here.

Example 2. The function F_ν defined on $\mathbb{R}^n$ by $F_\nu(x) = c_B e^{-\frac{(B^{-1}x,x)}{2}}$ is the usual density of the Gaussian measure, but the function $\psi \circ F_\nu$ need not be a density of a Gaussian measure. Even if $\psi(x) = x^a$, $a > 0$, then already the function $\psi \circ F_\nu$ is not the usual density of the Gaussian measure, but is still the generalized $\mathbb{R}^n$-density of the Gaussian measure.

In the infinite-dimensional case, the density does not exist, but for the latter function ψ the function $x \mapsto \psi(e^{-\frac{(B^{-1}x,x)}{2}})$ is the generalized H-density of the (unique) Gaussian measure.

The powers of measures can be needed to define the Schrödinger quantization of infinite-dimensional Hamiltonian systems.

Namely, if $Q \times P$ ia s phase space of a Hamiltonian system, $\mathbb{H} : Q \times P \to \mathbb{R}^1$ is a classical Hamiltonian function, then the pseudodifferential operator $\widehat{\mathbb{H}_\nu}$ in the $L_2(Q, \nu)$, where ν is a Gaussian measure can be calculated as follows:

$$\widehat{\mathbb{H}}(\varphi\sqrt{\nu}) = (\widehat{\mathbb{H}_\nu}\varphi)\sqrt{\nu}$$

where $\widehat{\mathbb{H}}$ is the naturally defined PDO in the space of measures on Q.

The generalized densities can also be used to calculate the product of a Gaussian measure and a Gaussian exponent or, what is the same, to calculate integrals of the Gaussian exponent, on infinite dimensional spaces, with respect to Gaussian measures. Below we will drop the prefix H in the expression "H-densities". To do this calculation, it is convenient to use, for generalized densities, infinite renormalization and take, for the renormalized generalized density of the Gaussian measure with the correlation operator B, the formal expression

$$\frac{1}{(2\pi)^{n/2}\sqrt{\det B}} e^{-\frac{(B^{-1}x,x)}{2}}.$$

Of course this formal expression does not define a function of x because if B is a trace class operator, then $\det B = 0$.

Nevertheless, formal calculations with such "renormalized generalized density" can lead to correct formulae. Here is an example. Let ν be as above the Gaussian measure with correlation operator B and hence with the renormalized generalized density $\frac{1}{\sqrt{\det B}} e^{-\frac{(B^{-1}x,x)}{2}}$. Then the integral

$$\int e^{-\frac{(Ax,x)}{2}} \nu(dx)$$

can be calculated as follows:

$$\frac{1}{\sqrt{\det B}} e^{-\frac{(Ax,x)}{2}} e^{-\frac{(B^{-1}x,x)}{2}} = \frac{\sqrt{\det(B^{-1}+A)}}{\sqrt{\det B}\sqrt{\det(B^{-1}+A)}} e^{-\frac{((B^{-1}+A)x,x)}{2}} = \frac{\sqrt{\det(B^{-1}+A)}}{\sqrt{\det(I+BA)}} e^{-\frac{((B^{-1}+A)x,x)}{2}}.$$

This implies that

$$\int e^{-\frac{(Ax,x)}{2}} \nu(dx) = (\det(I+BA))^{-\frac{1}{2}}.$$

To obtain a proof of this fact, one can apply the machinery from [12].

Remark 3. Some results of the paper can be extended to the so-called pseudomeasures, including the Feynman pseudomeasure [12].

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