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2014

MIMS EPrint: **2014.41**

ISSN 1749-9097

GEOMETRIC STRUCTURE FOR BERNSTEIN BLOCKS

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Abstract. We consider blocks in the representation theory of reductive p-adic groups. On each such block we conjecture a definite geometric structure, that of an extended quotient. We prove that this geometric structure is present for each block in the representation theory of any inner form of $\mathrm{GL}_n(F)$, and also for each block in the principal series of a connected split reductive p-adic group with connected centre.

Contents

1. Introduction

In [ABP1, ABPS1], we proposed a conjecture that adds a new structure to the space of representations of a reductive p-adic group, and that further studies the interplay between this structure and the local Langlands conjecture. The local Langlands conjecture predicts a relationship between the irreducible representation of a given reductive group $\mathcal{G}$ defined over a local (in the case at hand non-archimedean) field F and certain representations of the Galois group of F as well as closely related groups. Both sides of

Date: August 4, 2014.
2010 *Mathematics Subject Classification.* 20G05, 22E50.
Key words and phrases. reductive p-adic group, representation theory, geometric structure, local Langlands conjecture.
The second author was partially supported by NSF grant DMS-0701184.

this relationship have rich and interesting structure, and their correspondence is at the heart of deep recent developments in different branches of mathematics, in particular of number theory.

The conjecture that we propose refines previous (non-conjectural) work of Bernstein on the structure of the category of smooth admissible representations of $\mathcal{G}(F)$. Bernsteins work decomposes this category into so-called blocks and endows the set of irreducible objects in each block with a finite-to-one map to a certain algebraic variety. We replace this variety with a different object of geometric nature and conjecture that this refines Bernsteins finite-to-one map to a bijection, effectively endowing the set of irreducible objects in each block with the structure of this geometric object.

To support this conjecture, we provide a proof for the case of the general linear group, as well as for the simplest class of irreducible representations of any split connected reductive group with connected centre: the principal series representations. This conjecture and the supporting evidence provide an interesting new insight into the structure of the smooth dual of p-adic groups and in this way also shed some additional light onto the local Langlands correspondence.

Let $\mathcal{G}$ be a connected reductive p-adic group. The smooth dual of $\mathcal{G}$ — denoted $\mathbf{Irr}(\mathcal{G})$ — is the set of equivalence classes of smooth irreducible representations of $\mathcal{G}$. Let $\mathfrak{B}(\mathcal{G})$ denote the Bernstein spectrum of $\mathcal{G}$, let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$, and let $T^{\mathfrak{s}}, W^{\mathfrak{s}}$ denote the complex torus, finite group, attached by Bernstein to $\mathfrak{s}$. For more details at this point, we refer the reader to [Renard]. The Bernstein decomposition provides us, inter alia, with the following data: a canonical disjoint union

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

and, for each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$, a finite-to-one surjective map

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

onto the quotient variety $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. The geometric conjecture amounts to a refinement of these statements. The refinement comprises the assertion that we have a *bijection*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \simeq T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \tag{1}$$

where $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ is the *extended quotient* of the torus $T^{\mathfrak{s}}$ by the finite group $W^{\mathfrak{s}}$. If the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ is free, then the extended quotient is equal to the ordinary quotient $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. If the action is not free, then the extended quotient is a finite disjoint union of quotient varieties, one of which is the ordinary quotient. The bijection (1) is subject to certain constraints, itemised in §4.

In this paper, among other things, we construct an admissible bijective map

$$\mu^{\mathfrak{s}} \colon T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

for each point $\mathfrak{s}$ in the Bernstein spectrum of $\mathrm{GL}_m(D)$ and for each point $\mathfrak{s}$ in the principal series of a split reductive group $\mathcal{G}$ with connected centre, subject to a mild restriction on the residual characteristic p.

It is interesting to compare this with the structure of the space of Langlands parameters for G, in cases where the local Langlands correspondence is known. For $\mathrm{GL}_n(F)$, one can fix a L-parameter ϕ and consider the collection of L-parameters which on the inertia group of F take the same values as ϕ. This set parametrizes precisely one Bernstein component $\mathbf{Irr}(\mathrm{GL}_n(F))^{\mathfrak{s}}$, and it is canonically in bijection with an extended quotient as above. The same holds for inner forms of $\mathrm{GL}_n(F)$. But these cases are really special, because all R-groups are trivial, both on the Galois side and on the representation side.

For other groups one has to enhance the Langlands parameters to see the extended quotients. For example, for $\mathrm{SL}_n(F)$ the space of Langlands parameters enhanced with an irreducible representation of the component R-group is in bijection with $\mathbf{Irr}(\mathrm{SL}_n(F))$ via the local Langlands correspondence. However, a Bernstein component for $\mathrm{SL}_n(F)$ is not necessarily in bijection with the set of L-parameters which have a fixed restriction to the inertia group of F, there may also be a condition on the representations of the component groups.

More precisely, a cuspidal pair (M, σ) for $\mathrm{SL}_n(F)$ corresponds to an elliptic L-parameter ϕ_M for M, plus an irreducible representation ρ_M of the component group of ϕ_M. Then it follows from the LLC for $\mathrm{GL}_n(F)$ that $\mathbf{Irr}(\mathrm{SL}_n(F))^{[M,\sigma]}$ is in bijection with equivalence classes of pairs (ϕ, ρ) where ϕ is an L-parameter for $\mathrm{SL}_n(F)$ which on the inertia group of F agrees with ϕ_M, and ρ is an irreducible representation of the component group of ϕ that extends ρ_M. We refer to [HiSa, ABPS2] for more background. According to our conjecture, the set of these pairs carries the structure of an extended quotient.

For general split groups the situation is much more complicated. Consider the group of type G_2 and the Bernstein component $\mathbf{Irr}(\mathrm{G}_2)^{\mathfrak{s}}$ of irreducible G_2-representations in the unramified principal series. As shown in [ABP2], there is an admissible bijection from $\mathbf{Irr}(\mathrm{G}_2)^{\mathfrak{s}}$ to an extended quotient $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$. The most natural choice for the associated set of Langlands parameters is the space of enhanced *unramified* (i.e. trivial on the inertia subgroup) Langlands parameters. For most unramified L-parameters this works fine, every enhancement with an irreducible representation of the component group gives rise to one representation in this Bernstein component. But for a few unramified L-parameters it is trickier, some enhancements yield representations in the principal series, whereas others point to supercuspidal G_2-representations. In that case, the extended quotient is hidden in the space of enhanced L-parameters: it is not the pre-image of any set under the forgetful map

$$\{\text{enhanced L-parameters}\} \to \{\text{L-parameters}\}.$$

To highlight it one needs intricate conditions on the representations of the component groups. It seems fair to say that in general the structure of extended quotients can not be detected with Langlands parameters alone. In this sense our conjecture reveals some geometric structure that is not present in the local Langlands conjecture.

The new results in this paper appear in §4, namely Theorem 4.10, Theorem 4.11, and Corollary 4.13. We have constructed the simplest and most direct proofs: for example, in §3, on the general linear group, we do *not* use the local Langlands correspondence, relying instead on the Zelevinsky classification.

An earlier, less precise version of our conjecture was formulated in [ABP1]. That version was proven in [Sol] for Bernstein components which are described nicely by affine Hecke algebras. These include the principal series of split groups (with possibly disconnected centre), symplectic and orthogonal groups and also inner forms of GL_n.

Acknowledgements. Thanks to Mark Reeder for drawing our attention to the article of Kato [Kat]. We thank Joseph Bernstein, David Kazhdan, King Fai Lai, George Lusztig, and David Vogan for enlightening comments and discussions.

2. Statement of the conjecture

2.1. **Extended quotient.** Let Γ be a finite group acting on a complex affine variety X as automorphisms of the affine variety

$$\Gamma \times X \to X.$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x. The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}$. $\widetilde{X}$ is an affine variety and is a subvariety of $\Gamma \times X$. The group Γ acts on $\widetilde{X}$:

$$\begin{aligned}\Gamma \times \widetilde{X} &\to \widetilde{X}\\ \alpha(\gamma, x) &= (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}.\end{aligned}$$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X/\!/\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$. The projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to give a morphism of affine varieties

$$\rho \colon X/\!/\Gamma \to X/\Gamma.$$

This map will be referred to as the projection of the extended quotient onto the ordinary quotient. The inclusion

$$\begin{aligned}X &\hookrightarrow \widetilde{X}\\ x &\mapsto (e, x) \qquad e = \text{identity element of } \Gamma\end{aligned}$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion of affine varieties $X/\Gamma \hookrightarrow X/\!/\Gamma$. This will be referred to as the inclusion of the ordinary quotient in the extended quotient. We will denote $X/\!/\Gamma$ with X/Γ removed by $X/\!/\Gamma - X/\Gamma$.

2.2. **Bernstein spectrum.** We recall some well-known parts of Bernstein's work on p-adic groups, which can be found for example in [Renard].

With $\mathcal{G}$ fixed, a cuspidal pair is a pair $(\mathcal{M}, \sigma)$ where $\mathcal{M}$ is a Levi factor of a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ and σ is an irreducible supercuspidal representation of $\mathcal{M}$. Here supercuspidal means that the support of any matrix coefficient of such a representation is compact modulo the centre of the group. Pairs $(\mathcal{M}, \sigma)$ and $(\mathcal{M}, \sigma')$ with σ isomorphic to σ' are considered equal. The group $\mathcal{G}$ acts on the space of cuspidal pairs by conjugation:

$$g \cdot (\mathcal{M}, \sigma) = (g\mathcal{M}g^{-1}, \sigma \circ \mathrm{Ad}_g^{-1}).$$

We denote the space of $\mathcal{G}$-conjugacy classes by $\Omega(\mathcal{G})$. We can inflate σ to an irreducible smooth $\mathcal{P}$-representation. Normalized smooth induction then produces a $\mathcal{G}$-representation $I_{\mathcal{P}}^{\mathcal{G}}(\sigma)$.

For any irreducible smooth $\mathcal{G}$-representation π there is a cuspidal pair $(\mathcal{M}, \sigma)$, unique up to conjugation, such that π is a subquotient of $I_{\mathcal{P}}^{\mathcal{G}}(\sigma)$. (The collection of irreducible subquotients of the latter representation does not depend on the choice of $\mathcal{P}$.) The $\mathcal{G}$-conjugacy class of (M, σ) is called the *cuspidal support* of π. We write the cuspidal support map as

$$\mathbf{Sc} : \mathbf{Irr}(\mathcal{G}) \to \Omega(\mathcal{G}).$$

For any unramified character ν of $\mathcal{M}$, $(\mathcal{M}, \sigma \otimes \nu)$ is again a cuspidal pair. Two cuspidal pairs $(\mathcal{M}, \sigma)$ and $(\mathcal{M}', \sigma')$ are said to be inertially equivalent, written $(\mathcal{M}, \sigma) \sim (\mathcal{M}', \sigma')$, if there exists an unramified character $\nu \colon \mathcal{M} \to \mathbb{C}^\times$ and an element $g \in \mathcal{G}$ such that

$$g \cdot (\mathcal{M}, \nu \otimes \sigma) = (\mathcal{M}', \sigma').$$

The Bernstein spectrum of $\mathcal{G}$, denoted $\mathfrak{B}(\mathcal{G})$, is the set of inertial equivalence classes of cuspidal pairs. It is a countable set, infinite unless $\mathcal{G}$ is a split torus. Let $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathcal{G}} \in \mathfrak{B}(\mathcal{G})$ be the inertial equivalence class of $(\mathcal{M}, \sigma)$ and let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be the subset of $\mathbf{Irr}(\mathcal{G})$ of representations that have cuspidal support in $\mathfrak{s}$. Then $\mathbf{Irr}(\mathcal{G})$ is the disjoint union of the Bernstein components $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$:

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(\mathcal{G})} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

The space $X_{\mathrm{unr}}(\mathcal{M})$ of unramified characters of $\mathcal{M}$ is in a natural way a complex algebraic torus. Put

$$\mathrm{Stab}(\sigma) = \{\nu \in X_{\mathrm{unr}}(\mathcal{M}) \mid \sigma \otimes \nu \cong \sigma\}. \tag{2}$$

This is known to be a finite group, so $X_{\mathrm{unr}}(M)/\mathrm{Stab}(\sigma)$ is again a complex algebraic torus. The map

$$X_{\mathrm{unr}}(\mathcal{M})/\mathrm{Stab}(\sigma) \to \mathbf{Irr}(\mathcal{M})^{[\mathcal{M},\sigma]_{\mathcal{M}}}, \ \nu \mapsto \sigma \otimes \nu \tag{3}$$

is bijective and thus provides $\mathbf{Irr}(\mathcal{M})^{[\mathcal{M},\sigma]_{\mathcal{M}}}$ with the structure of an algebraic torus. The variety structure is canonical, in the sense that it does not depend on the choice of σ in $\mathbf{Irr}(\mathcal{M})^{[\mathcal{M},\sigma]_{\mathcal{M}}}$.

The Weyl group of $(\mathcal{G}, \mathcal{M})$ is defined as

$$W(\mathcal{G}, \mathcal{M}) := N_{\mathcal{G}}(\mathcal{M})/\mathcal{M}.$$

It is a finite group which generalizes the notion of the Weyl group associated to a maximal torus. The Weyl group of $(\mathcal{G}, \mathcal{M})$ acts naturally on $\mathbf{Irr}(\mathcal{M})$, via the conjugation action on $\mathcal{M}$. The subgroup

$$W^{\mathfrak{s}} := \{w \in W(\mathcal{G}, \mathcal{M}) \mid w \text{ stabilizes } [\mathcal{M}, \sigma]_{\mathcal{M}}\}. \tag{4}$$

acts on $\mathbf{Irr}(\mathcal{M})^{[\mathcal{M},\sigma]_{\mathcal{M}}}$. We define

$$T^{\mathfrak{s}} := \mathbf{Irr}(\mathcal{M})^{[\mathcal{M},\sigma]_{\mathcal{M}}} \tag{5}$$

with the structure (3) as algebraic torus and the $W^{\mathfrak{s}}$-action (4). We note that the $W^{\mathfrak{s}}$-action is literally by automorphisms of the algebraic variety $T^{\mathfrak{s}}$, via (3) they need not become group automorphisms. Two elements of $T^{\mathfrak{s}}$ are $\mathcal{G}$-conjugate if and only if they are in the same $W^{\mathfrak{s}}$-orbit.

An inertially equivalent cuspidal pair $(\mathcal{M}', \sigma')$ would yield a torus $T'^{\mathfrak{s}}$ which is isomorphic to $T^{\mathfrak{s}}$ via conjugation in $\mathcal{G}$. Such an isomorphism $T^{\mathfrak{s}} \cong T'^{\mathfrak{s}}$ is unique up to the action of $W^{\mathfrak{s}}$.

The element of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ associated to any $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is called its *infinitesimal central character*, denoted $\pi^{\mathfrak{s}}(\pi)$. Another result of Bernstein is the existence of a unital finite type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$-algebra $\mathcal{H}^{\mathfrak{s}}$, whose irreducible modules are in natural bijection with $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. The construction is such that $\mathcal{H}^{\mathfrak{s}}$ has centre $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$ and that $\pi^{\mathfrak{s}}(\pi)$ is precisely the central character of the corresponding $\mathcal{H}^{\mathfrak{s}}$-module.

Since $\mathbf{Irr}(\mathcal{H}^{\mathfrak{s}})$ is in bijection with the collection of primitive ideals of $\mathcal{H}^{\mathfrak{s}}$, we can endow it with the Jacobson topology. By transferring this topology to $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$, we make the latter into a (nonseparated) algebraic variety. (In fact this topology agrees with the topology on $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ considered as a subspace of $\mathbf{Irr}(\mathcal{G})$, endowed with the Jacobson topology from the Hecke algebra of $\mathcal{G}$.)

SUMMARY: For each Bernstein component $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ there are:

(1) A finite group $W^{\mathfrak{s}}$ acting on a complex torus $T^{\mathfrak{s}}$;
(2) A subset $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ of $\mathbf{Irr}(\mathcal{G})$;
(3) A morphism of algebraic varieties

$$\pi^{\mathfrak{s}} \colon \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

2.3. **Statement of the conjecture.** As above, $\mathcal{G}$ is a quasi-split connected reductive p-adic group or an inner form of $\mathrm{GL}_n(F)$, and $\mathfrak{s}$ is a point in the Bernstein spectrum of $\mathcal{G}$.

We are going to compare and contrast the two maps

$$\rho^{\mathfrak{s}} : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}} \quad \text{and} \quad \pi^{\mathfrak{s}} : \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

Here $\pi^{\mathfrak{s}}$ is the infinitesimal character and $\rho^{\mathfrak{s}}$ is the projection of the extended quotient on the ordinary quotient. In practice $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ and $\rho^{\mathfrak{s}}$ are much easier to calculate than $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ and $\pi^{\mathfrak{s}}$.

$\pi^{\mathfrak{s}}$ and $\rho^{\mathfrak{s}}$ are both surjective finite-to-one maps and morphisms of algebraic varieties. For $x \in T^{\mathfrak{s}}/W^{\mathfrak{s}}$, denote by $\#(x, \rho^{\mathfrak{s}})$, $\#(x, \pi^{\mathfrak{s}})$ the number of points in the pre-image of x using $\rho^{\mathfrak{s}}$, $\pi^{\mathfrak{s}}$. The numbers $\#(x, \pi^{\mathfrak{s}})$ are of interest in describing exactly what happens when $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is constructed by parabolic induction.

Within $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ there are algebraic sub-varieties $R(\rho^{\mathfrak{s}})$, $R(\pi^{\mathfrak{s}})$ defined by

$$R(\rho^{\mathfrak{s}}) := \{x \in T^{\mathfrak{s}}/W^{\mathfrak{s}} \mid \#(x, \rho^{\mathfrak{s}}) > 1\}$$

$$R(\pi^{\mathfrak{s}}) := \{x \in T^{\mathfrak{s}}/W^{\mathfrak{s}} \mid \#(x, \pi^{\mathfrak{s}}) > 1\}$$

It is immediate that

$$R(\rho^{\mathfrak{s}}) = \rho^{\mathfrak{s}}(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}})$$

$R(\pi^{\mathfrak{s}})$ will be referred to as the *sub-variety of non-isotypicality.* In examples, *sub-schemes* are sometimes needed.

In many examples $R(\rho^{\mathfrak{s}}) \neq R(\pi^{\mathfrak{s}})$. Hence in these examples it is impossible to have a bijection

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

for which

$$\pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}} = \rho^{\mathfrak{s}}.$$

A more precise statement of the conjecture is that after a simple algebraic correction ("correcting cocharacters") $\rho^{\mathfrak{s}}$ becomes isomorphic to $\pi^{\mathfrak{s}}$. Thus $\rho^{\mathfrak{s}}$ is an easily calculable map which can be algebraically corrected to give $\pi^{\mathfrak{s}}$. An implication of this is that within the algebraic variety $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ there is a flat family of sub-varieties connecting $R(\rho^{\mathfrak{s}})$ and $R(\pi^{\mathfrak{s}})$.

NOTATION. As above, $\mathcal{G}$ is a quasi-split connected reductive p-adic group or an inner form of $\mathrm{GL}_n(F)$, and $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$. We have the tempered dual $\mathbf{Irr}(\mathcal{G})_{\mathrm{temp}} \subset \mathbf{Irr}(\mathcal{G})$. Let $T^{\mathfrak{s}}_{\mathrm{cpt}}$ denote the set of tempered representations in $T^{\mathfrak{s}}$. Then $T^{\mathfrak{s}}_{\mathrm{cpt}}$ corresponds to the unique maximal compact subgroup of $X_{\mathrm{unr}}(M)/\mathrm{Stab}(\sigma)$ under (3), provided that σ is tempered. Then $T^{\mathfrak{s}}_{\mathrm{cpt}}$ is a compact real torus. The action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ preserves $T^{\mathfrak{s}}_{\mathrm{cpt}}$, so we can form the compact orbifold $T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}}$.

CONJECTURE. There exists a bijection

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

with the following properties (such a bijection will be called *admissible*):

(1) The bijection $\mu^{\mathfrak{s}}$ restricts to a bijection

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cap \mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$$

(2) The bijection $\mu^{\mathfrak{s}}$ is continuous where $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ has the Zariski topology and $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ has the Jacobson topology — and the composition

$$\pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}} : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

is a finite morphism of affine algebraic varieties.

(3) There is an algebraic family

$$\theta_z : T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^{\times}$, such that

$$\theta_1 = \rho^{\mathfrak{s}}, \quad \theta_{\sqrt{q}} = \pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}}, \quad \text{and} \quad \theta_{\sqrt{q}}(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}}) = R(\pi^{\mathfrak{s}}).$$

(4) Correcting cocharacters. For each irreducible component $\mathbf{c}$ of the affine variety $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ there is a cocharacter (i.e. a homomorphism of algebraic groups)

$$h_{\mathbf{c}} : \mathbb{C}^{\times} \longrightarrow T^{\mathfrak{s}}$$

such that
$$\theta_z[w,t] = b(h_{\mathbf{c}}(z)\cdot t)$$
for all $[w,t]\in \mathbf{c}$, where $b\colon T^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$ is the quotient map.

(5) *L*-packets. This property is conditional on the existence of Langlands parameters for the block $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. In that case, the intersection of an L-packet with that block is well-defined. This property *refers to the intersection of such an L-packet with the given block.*

Let $\{\mathbf{c}_1,\dots,\mathbf{c}_r\}$ be the irreducible components of the affine variety $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$. There exists a complex reductive group H and, for every irreducible component $\mathbf{c}$ of $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$, a unipotent conjugacy class $\lambda(\mathbf{c})$ in H, such that: for every two points $[w,t]$ and $[w',t]$ of $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$:
$$\mu^{\mathfrak{s}}[w,t] \text{ and } \mu^{\mathfrak{s}}[w',t'] \text{ are in the same L-packet}$$
if and only

(i) $\theta_z[w,t]=\theta_z[w',t']$ for all $z\in\mathbb{C}^\times$;
(ii) $\lambda(\mathbf{c})=\lambda(\mathbf{c}')$, where $[w,t]\in\mathbf{c}$ and $[w',t']\in\mathbf{c}'$.

Notes on the conjecture. In brief, the conjecture asserts that — once a Bernstein component has been fixed — intersections of L-packets with that Bernstein component consisting of more than one point are "caused" by repetitions among the correcting cocharacters. If, for any one given Bernstein component, the correcting cocharacters h_1, h_2, …, h_r are all distinct, then (according to the conjecture) the intersections of L-packets with that Bernstein component are singletons.

Note on (3). Here q is the order of the residue field of the p-adic field F over which $\mathcal{G}$ is defined and $R(\pi^{\mathfrak{s}})\subset T^{\mathfrak{s}}/W^{\mathfrak{s}}$ is the sub-variety of non-isotypicality. Setting
$$Y_z=\theta_z(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}})$$
a flat family of sub-varieties of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ is obtained with
$$Y_1=R(\rho^{\mathfrak{s}}),\qquad Y_{\sqrt{q}}=R(\pi^{\mathfrak{s}}).$$

Note on (4). Here, as above, points of $\widetilde{T}_{\mathfrak{s}}$ are pairs (w,t) with $w\in W^{\mathfrak{s}}$, $t\in T^{\mathfrak{s}}$ and $wt=t$. $[w,t]$ is the point in $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ obtained by applying the quotient map $\widetilde{T^{\mathfrak{s}}}\to T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ to (w,t).

The equality $\theta_z[w,t]=b(h_c(z)\cdot t)$ is to be interpreted as follows. Let $\mathbf{c}_1,\dots,\mathbf{c}_r$ be as in (5) and let $h_1,\dots,h_r$ be the cocharacters as in (4). Let
$$\nu^{\mathfrak{s}}\colon \widetilde{T^{\mathfrak{s}}}\longrightarrow T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$$
be the quotient map.

Then irreducible components $\mathbf{d}_1,\dots,\mathbf{d}_r$ of the affine variety $\widetilde{T^{\mathfrak{s}}}$ can be chosen with

- $\nu^{\mathfrak{s}}(\mathbf{d}_j)=\mathbf{c}_j$ for $j=1,2,\dots,r$
- For each $z\in\mathbb{C}^\times$ the map $m_z\colon \mathbf{d}_j\to T^{\mathfrak{s}}/W^{\mathfrak{s}}$, which is the composition
$$\begin{array}{ccccc}\mathbf{d}_j & \longrightarrow & T^{\mathfrak{s}} & \longrightarrow & T^{\mathfrak{s}}/W^{\mathfrak{s}}\\ (w,t) & \longmapsto & h_j(z)t & \longmapsto & b(h_j(z)t),\end{array}$$

satisfies

$$\theta_z \circ \nu^{\mathfrak{s}} = m_z.$$

The cocharacter assigned to $T^{\mathfrak{s}}/W^{\mathfrak{s}} \hookrightarrow T^{\mathfrak{s}}//W^{\mathfrak{s}}$ is always the trivial cocharacter mapping $\mathbb{C}^\times$ to the unit element of $T^{\mathfrak{s}}$. So all the non-trivial correcting is taking place on $T^{\mathfrak{s}}//W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}}$.

3. The general linear group and its inner forms

Theorem 3.1. *Let $\mathcal{G}$ denote an inner form of $\mathrm{GL}_n(F)$. Then there exists an admissible bijection*

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longleftrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

Proof. First we consider $\mathrm{GL}_n(F)$ only. We will use the Zelevinsky classification of the smooth dual $\mathbf{Irr}(\mathcal{G})$, see [Zel]. We will denote a balanced segment of length l by

$$\Delta(\sigma : l) := \{\nu^{(1-l)/2}\sigma, \ldots, \nu^{(l-1)/2}\sigma\}, \qquad \sigma \in \mathcal{C}_F, \nu = |\det|_F \tag{6}$$

The unique irreducible submodule of $\Delta = \nu^{(1-l)/2}\sigma \times \cdots \times \nu^{(l-1)/2}\sigma$ will be denoted $< \Delta >$.

The unique irreducible submodule of $< \Delta_1 > \times \cdots \times < \Delta_r >$ will be denoted $< \Delta_1, \ldots, \Delta_r >$.

Let $\mathcal{O}$ denote the set of finite multisets of segments. For each $\mathfrak{a} \in \mathcal{O}, \mathfrak{a} \neq \emptyset$ one can choose an ordering $(\Delta_1, \ldots, \Delta_r)$ of $\mathfrak{a}$, satisfying [Zel, 6.1(a)]. By [Zel, 6.4] the representation $< \Delta_1, \ldots, \Delta_r >$ depends only on $\mathfrak{a}$, and will be denoted $< \mathfrak{a} >$. For the empty multiset $\emptyset \in \mathcal{O}$ define $\pi(\emptyset) =< \emptyset >$ to be the trivial representation of the group $\mathrm{GL}_0 = \{e\}$.

According to the Zelevinsky classification [Zel], there is a bijection

$$\mathcal{O} \to \bigsqcup \mathbf{Irr}(\mathrm{GL}_n(F)), \qquad \mathfrak{a} \mapsto < \mathfrak{a} >$$

A special case. With $n = dk$, the cuspidal pair

$$(M, \omega) := (\mathrm{GL}_d(F)^k, \sigma^{\otimes k})$$

determines a point $\mathfrak{s}$ in the Bernstein spectrum $\mathfrak{B}(\mathrm{GL}_n(F))$, and the Bernstein variety $\mathcal{D}/\mathfrak{S}_k$, where $\mathcal{D} = \Psi(\mathrm{GL}_d(F)^k$ and $\mathfrak{S}_k$ is the symmetric group.

Let $\gamma \in \mathfrak{S}_k$ be made of N disjoint cycles of lengths $l_1, \ldots, l_N$. Consider the multiset

$$\delta(\sigma : \gamma) := \bigsqcup_{j=1}^{N} \Delta(\sigma : l_j) \tag{7}$$

Give each segment an unramified twist: this is a generalisation of Bernstein's method, who restricts himself to segments of length 1 — the method can be traced to Hecke via §2.4 in Tate's thesis.

Let $\psi = (\psi_1, \ldots, \psi_N)$ with each ψ_j an unramified quasicharacter of $\mathrm{GL}_d(F)$, define

$$\delta(\sigma : \gamma : \psi) := \bigsqcup_{j=1}^{N} \Delta(\psi_j\sigma : l_j) \tag{8}$$

and consider the *orbit*

$$\{\delta(\sigma : \gamma : \psi) : \psi \in \Psi(\mathrm{GL}_d(F)^N)\}$$

of $\delta(\sigma : \gamma)$ via the action of the complex torus $\Psi(\mathrm{GL}_d(F)^N)$. If all the lengths l_j are distinct then this orbit is a complex torus of dimension N; if all the lengths are equal then this orbit is the symmetric product $\mathrm{Sym}^N(\mathbb{C}^\times)$. In any event, this orbit is a complex affine algebraic variety, the quotient of a complex torus by a finite group. This variety creates an analytic neighbourhood of $< \delta(\sigma : \gamma) >$.

Extended quotient Write $(\lambda_1, \dots, \lambda_1, \dots, \lambda_s, \dots, \lambda_s)$ for the cycle type of γ, where the λ_is are distinct, and λ_j occurs with multiplicity e_i in the integer partition λ of k. Define

$$\beta(\sigma : \gamma : \psi) := (M, (\psi_1\sigma)^{\otimes\lambda_1} \otimes (\psi_2\sigma)^{\otimes\lambda_1} \otimes \cdots \otimes (\psi_N\sigma)^{\otimes\lambda_s})$$

We have

$$\mathcal{D}^\gamma = \{\beta(\sigma : \gamma : \psi) : \psi \in \Psi(\mathrm{GL}_d(F)^N)\}$$

The centralizer of γ is the direct product of wreath products:

$$\mathrm{Z}(\gamma) = \prod_{i=1}^{s} (\mathbb{Z}/\lambda_i\mathbb{Z} \wr \mathfrak{S}_{e_i})$$

The cyclic groups act trivially on $\mathcal{D}^\gamma$ and so we have

$$\mathcal{D}^\gamma/\mathrm{Z}(\gamma) = \mathcal{D}^\gamma/(\mathfrak{S}_{e_1} \times \cdots \times \mathfrak{S}_{e_s}) = \mathrm{Sym}^{e_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{e_s}\mathbb{C}^\times$$

a variety isomorphic (modulo an affine space) to a complex torus of dimension s. The smooth dual is (locally) a smooth variety. Now we construct the map

$$\mathcal{D}^\gamma/\mathrm{Z}(\gamma) \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}, \qquad \beta(\sigma : \gamma : \psi) \mapsto \delta(\sigma : \gamma : \psi)$$

Since $\mathcal{D}/\!/\mathfrak{S}_k = \bigsqcup D^\gamma/Z(\gamma)$ we obtain the map

$$\mu^{\mathfrak{s}} : \mathcal{D}/\!/\mathfrak{S}_k \simeq \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \tag{9}$$

The general case. In the general case, define

$$\delta := \bigsqcup_{j=1}^{r} \delta(\sigma_j : \gamma_j)$$

where the σ_j remain inequivalent after unramified twist. Let

$$\mathfrak{s} = \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_k.$$

Then we have

$$T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \simeq T^{\mathfrak{s}_1}/\!/W^{\mathfrak{s}_1} \times \cdots \times T^{\mathfrak{s}_k}/\!/W^{\mathfrak{s}_k} \tag{10}$$

$$\simeq \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \tag{11}$$

The cocharacters. These are already present in each segment:

$$\Delta(\sigma : l) := \{q^{(1-l)/2}\sigma, \dots, q^{(l-1)/2}\sigma\}$$

The flat family. This is given by the family of hypersurfaces

$$\prod_{i \neq j} (z_i - tz_j) = 0$$

the point being that $t = q$ gives the variety of non-isotypicality, thanks to the classical result [Zel, Theorem 4.2].

THE TEMPERED DUAL. If we insist that σ has unitary central character, then the tempered case is immediate by restriction to unramified *unitary* twists.

THE INNER FORM $\mathrm{GL}_m(D)$. Here D is a F-division algebra of dimension d^2 over its centre F. The classification of $\mathbf{Irr}(\mathrm{GL}_m(D))$ is via multisets of segments

$$\delta(\sigma:\ell)=\left\{\nu_D^{s(\sigma)(1-\ell)/2}\sigma,\ldots,\nu_D^{s(\sigma)(\ell-1)/2}\sigma\right\}, \tag{12}$$

where we have $\sigma\in\mathcal{C}_D$, $\nu_D=|\mathrm{Nrd}|_F$, and $s(\sigma)$ is the length of the Zelevinsky segment attached to the inverse image $\mathrm{JL}^{-1}(\sigma)$ of σ by the Jacquet-Langlands correspondence, see [Tad] (combined with [Bad] when F has positive characteristic).

Every unramified character ψ_D of $\mathrm{GL}_m(D)$ is of the form $\psi\circ\mathrm{Nrd}$ for some unramified character ψ of $\mathrm{GL}_{md}(F)$. Since $\mathrm{JL}^{-1}(\psi_D\sigma)=\psi\mathrm{JL}^{-1}(\sigma)$, we get that $s(\psi_D\sigma)=s(\sigma)$. Then the proof of (10) and (11) then carries over without change. $\square$

LANGLANDS PARAMETERS. The local Langlands correspondence

$$\mathrm{rec}_F:\mathbf{Irr}(\mathcal{G})\simeq\Phi(G)$$

is not needed in the above proof. The relation with §1.1 is

$$\mathrm{rec}_F<\mathfrak{a}\sqcup\mathfrak{b}>=\mathrm{rec}_F<\mathfrak{a}>\oplus\,\mathrm{rec}_F<\mathfrak{b}>\qquad\forall a,b\in\mathcal{O} \tag{13}$$

$$\mathrm{rec}_F<\Delta(\sigma:l)>=\mathrm{rec}_F(\sigma)\otimes R(l)\qquad\forall\sigma\in\mathcal{C}_F \tag{14}$$

where $R(l)$ is the l-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

We note that the formula

$$\beta(\sigma:\gamma:\psi)\mapsto\bigoplus_{j=1}^{N}\mathrm{rec}_F(\psi_j\sigma)\otimes R(l_j)$$

secures a bijective map

$$\eta^{\mathfrak{s}}:T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}\to\Phi(G)$$

for which

$$\eta^{\mathfrak{s}}=\mathrm{rec}_F\circ\mu^{\mathfrak{s}}$$

4. PRINCIPAL SERIES OF SPLIT REDUCTIVE GROUPS WITH CONNECTED CENTRE

4.1. **The Langlands parameter Φ.** Let $\mathcal{G}$ be a connected reductive p-adic group, split over F, with connected centre, and let $\mathcal{T}$ be a split maximal torus in $\mathcal{G}$. Let G, T denote the Langlands dual groups of $\mathcal{G}$, $\mathcal{T}$. The principal series consists of all $\mathcal{G}$-representations that are obtained with parabolic induction from characters of $\mathcal{T}$. We will suppose that the residual characteristic p of F satisfies the hypothesis in [Roc, p. 379].

We denote the collection of all Bernstein components of $\mathcal{G}$ of the form $\mathfrak{s}=[\mathcal{T},\chi]_{\mathcal{G}}$ by $\mathfrak{B}(\mathcal{G},\mathcal{T})$ and call these the Bernstein components in the principal series. The union

$$\mathbf{Irr}(\mathcal{G},\mathcal{T}):=\bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})}\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

is by definition the set of all irreducible subquotients of principal series representations of $\mathcal{G}$.

Choose a uniformizer $\varpi_F \in F$. There is a bijection $t \mapsto \nu$ between points in T and unramified characters of $\mathcal{T}$, determined by the relation

$$\nu(\lambda(\varpi_F)) = \lambda(t)$$

where $\lambda \in X_*(\mathcal{T}) = X^*(T)$. The space $\mathbf{Irr}(\mathcal{T})^{[\mathcal{T},\chi]_\mathcal{T}}$ is in bijection with T via $t \mapsto \nu \mapsto \chi \otimes \nu$. Hence Bernstein's torus $T^\mathfrak{s}$ is isomorphic to T. However, because the isomorphism is not canonical and the action of the group $W^\mathfrak{s}$ depends on it, we prefer to denote it $T^\mathfrak{s}$.

Let $\mathbf{W}_F$ denote the Weil group of F, let $\mathbf{I}_F$ be the inertia subgroup of $\mathbf{W}_F$. Let $\mathbf{W}_F^{\mathrm{der}}$ denote the closure of the commutator subgroup of $\mathbf{W}_F$, and write $\mathbf{W}_F^{\mathrm{ab}} = \mathbf{W}_F / \mathbf{W}_F^{\mathrm{der}}$. The group of units in $\mathfrak{o}_F$ will be denoted $\mathfrak{o}_F^\times$.

Next, we consider conjugacy classes in G of continuous morphisms

$$\Phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

which are rational on $\mathrm{SL}_2(\mathbb{C})$ and such that $\Phi(\mathbf{W}_F)$ consists of semisimple elements in G.

Let B_2 be the upper triangular Borel subgroup in $\mathrm{SL}_2(\mathbb{C})$. Let $\mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}$ denote the variety of Borel subgroups of G containing $\Phi(\mathbf{W}_F \times B_2)$. The variety $\mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}$ is non-empty if and only if Φ factors through $\mathbf{W}_F^{\mathrm{ab}}$, see [Reed, §4.2]. In that case, we view the domain of Φ to be $F^\times \times \mathrm{SL}_2(\mathbb{C})$:

$$\Phi\colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

In this section we will build such a continuous morphism Φ from $\mathfrak{s}$ and data coming from the extended quotient of second kind. In Section 4.3 we show how such a Langlands parameter Φ can be enhanced with a parameter ρ.

The uniformizer ϖ_F gives rise to a group isomorphism $\mathfrak{o}_F^\times \times \mathbb{Z} \to F^\times$, which sends $1 \in \mathbb{Z}$ to ϖ_F. Let $\mathcal{T}_0$ denote the maximal compact subgroup of $\mathcal{T}$. As the latter is F-split,

$$\mathcal{T} \cong F^\times \otimes_\mathbb{Z} X_*(\mathcal{T}) \cong (\mathfrak{o}_F^\times \times \mathbb{Z}) \otimes_\mathbb{Z} X_*(\mathcal{T}) = \mathcal{T}_0 \times X_*(\mathcal{T}). \tag{15}$$

Because $\mathcal{W}$ does not act on $F^\times$, these isomorphisms are $\mathcal{W}$-equivariant if we endow the right hand side with the diagonal $\mathcal{W}$-action. Thus (15) determines a $\mathcal{W}$-equivariant isomorphism of character groups

$$\mathbf{Irr}(\mathcal{T}) \cong \mathbf{Irr}(\mathcal{T}_0) \times \mathbf{Irr}(X_*(\mathcal{T})) = \mathbf{Irr}(\mathcal{T}_0) \times X_{\mathrm{unr}}(\mathcal{T}). \tag{16}$$

Lemma 4.1. *Let χ be a character of $\mathcal{T}$, and let $[\mathcal{T},\chi]_\mathcal{G}$ be the inertial class of the pair $(\mathcal{T},\chi)$ as in §3. Let*

$$\mathfrak{s} = [\mathcal{T},\chi]_\mathcal{G}. \tag{17}$$

Then $\mathfrak{s}$ determines, and is determined by, the $\mathcal{W}$-orbit of a smooth morphism

$$c^\mathfrak{s}\colon \mathfrak{o}_F^\times \to T.$$

Proof. There is a natural isomorphism

$$\mathbf{Irr}(\mathcal{T}) = \mathrm{Hom}(F^\times \otimes_\mathbb{Z} X_*(\mathcal{T}), \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times, \mathbb{C}^\times \otimes_\mathbb{Z} X^*(\mathcal{T})) = \mathrm{Hom}(F^\times, T).$$

Together with (16) we obtain isomorphisms

$$\mathbf{Irr}(\mathcal{T}_0) \cong \mathrm{Hom}(\mathfrak{o}_F^\times, T),$$
$$X_{\mathrm{unr}}(\mathcal{T}) \cong \mathrm{Hom}(\mathbb{Z}, T) = T.$$

Let $\hat{\chi} \in \mathrm{Hom}(F^\times, T)$ be the image of χ under these isomorphisms. By the above the restriction of $\hat{\chi}$ to $\mathfrak{o}_F^\times$ is not disturbed by unramified twists, so we take that as $c^{\mathfrak{s}}$. Conversely, by (16) $c^{\mathfrak{s}}$ determines χ up to unramified twists. Two elements of $\mathbf{Irr}(\mathcal{T})$ are $\mathcal{G}$-conjugate if and only if they are $\mathcal{W}$-conjugate so, in view of (17), the $\mathcal{W}$-orbit of the $c^{\mathfrak{s}}$ contains the same amount of information as $\mathfrak{s}$. $\square$

We define

$$H^{\mathfrak{s}} := Z_G(\mathrm{im}\ c^{\mathfrak{s}}).$$

The following crucial result is due to Roche [Roc].

Lemma 4.2. *The group $H^{\mathfrak{s}}$ is connected, and the finite group $W^{\mathfrak{s}}$ is the Weyl group of $H^{\mathfrak{s}}$:*

$$W^{\mathfrak{s}} = \mathcal{W}^{H^{\mathfrak{s}}}$$

Proof. See [Roc, p. 394 – 395]. $\square$

4.2. **Comparison of different parameters.** We clarify some issues with different varieties of Borel subgroups and different kinds of parameters arising from them.

We start with the following data: a point $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ and an L-parameter

$$\Phi\colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G$$

for which

$$\Phi|\mathfrak{o}_F^\times = c^{\mathfrak{s}}.$$

This data creates the following items:

$$t := \Phi(\varpi_F, I), \tag{18}$$

$$H := H^{\mathfrak{s}} = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}}), \tag{19}$$

$$M := M^{\mathfrak{s}} = \mathrm{Z}_H(t). \tag{20}$$

We note that $\Phi(\mathfrak{o}_F^\times) \subset \mathrm{Z}(H)$ and that t commutes with $\Phi(\mathrm{SL}_2(\mathbb{C})) \subset M$.

For $\alpha \in \mathbb{C}^\times$ we define the following matrix in $\mathrm{SL}_2(\mathbb{C})$:

$$Y_\alpha = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right).$$

For any $q^{1/2} \in \mathbb{C}^\times$ the element

$$t_q := t\Phi\big(Y_{q^{1/2}}\big) \tag{21}$$

satisfies the familiar relation $t_q x t_q^{-1} = x^q$. Indeed

$$\begin{aligned} t_q x t_q^{-1} &= t\Phi(Y_{q^{1/2}})\Phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\Phi(Y_{q^{1/2}}^{-1})t^{-1} \\ &= t\Phi\big(Y_{q^{1/2}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)Y_{q^{1/2}}^{-1}\big)t^{-1} \\ &= t\Phi\left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)t^{-1} = x^q. \end{aligned} \tag{22}$$

Recall that B_2 denotes the upper triangular Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$. Notice that $\Phi(\mathfrak{o}_F^\times)$ lies in every Borel subgroup of H, because it is contained in $\mathrm{Z}(H)$. We abbreviate $\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(\mathrm{im}\,\Phi)$ and similarly for other groups.

Lemma 4.3. *The inclusion map $\mathrm{Z}_H(\Phi) \to \mathrm{Z}_H(t, x)$ is a homotopy equivalence.*

Proof. Our proof depends on [CG, Prop. 3.7.23]. There is a Levi decomposition

$$\mathrm{Z}_H(x) = \mathrm{Z}_H(\Phi(\mathrm{SL}_2(\mathbb{C})))U_x$$

where $\mathrm{Z}_H(\Phi(\mathrm{SL}_2(\mathbb{C}))$ a maximal reductive subgroup of $\mathrm{Z}_H(x)$ and U_x is the unipotent radical of $\mathrm{Z}_H(x)$. Therefore

$$\mathrm{Z}_H(t,x) = \mathrm{Z}_H(\Phi)\mathrm{Z}_{U_x}(t) \tag{23}$$

We note that $\mathrm{Z}_{U_x}(t) \subset U_x$ is contractible, because it is a unipotent complex group. It follows that

$$\mathrm{Z}_H(\Phi) \to \mathrm{Z}_H(t,x) \tag{24}$$

is a homotopy equivalence. □

If a group A acts on a variety X, let $\mathcal{R}(A,X)$ denote the set of irreducible representations of A appearing in the homology $H_*(X)$.

The variety of Borel subgroups of G which contain $\Phi(\mathbf{W}_F \times B_2)$ will be denoted $\mathcal{B}_G^{\Phi(\mathbf{W}_F\times B_2)}$ and the variety of Borel subgroups of H containing $\{t,x\}$ will be denoted $\mathcal{B}_H^{t,x}$.

Lemma 4.3 allows us to define

$$A := \pi_0(\mathrm{Z}_H(\Phi)) = \pi_0(\mathrm{Z}_H(t,x)).$$

Theorem 4.4. *We have*

$$\mathcal{R}(A, \mathcal{B}^{\Phi(\mathbf{W}_F\times B_2)}) = \mathcal{R}(A, \mathcal{B}_H^{t,x}).$$

Proof. This statement is equivalent to [Reed, Lemma 4.4.1] with a minor adjustment in his proof. To translate into Reeder's paper, write

$$t_q = \tau, Y_q = \tau_u, x = u, t = s.$$

The adjustment consists in the observation that the Borel subgroup B of H contains $\{x, t_q, Y_q\}$ if and only if B contains $\{x, t, Y_q\}$. This is because $t = t_q Y_q^{-1}$. Therefore, in the conclusion of his proof, $\mathcal{B}_H^{\tau,u}$, which is $\mathcal{B}_H^{t_q,x}$, can be replaced by $\mathcal{B}_H^{t,x}$. □

In the following sections we will make use of two different but related kinds of parameters.

4.3. **Enhanced Langlands parameters.** For a Langlands parameter as in (28), the variety of Borel subgroups $\mathcal{B}_G^{\Phi(\mathbf{W}_F\times B_2)}$ is nonempty, and the centralizer $\mathrm{Z}_G(\Phi)$ of the image of Φ acts on it. Hence the group of components $\pi_0(\mathrm{Z}_G(\Phi))$ acts on the homology $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F\times B_2)}, \mathbb{C}\big)$. We call an irreducible representation ρ of $\pi_0(\mathrm{Z}_G(\Phi))$ *geometric* if

$$\rho \in \mathcal{R}\left(\pi_0(\mathrm{Z}_G(\Phi)), \mathcal{B}_G^{\Phi(\mathbf{W}_F\times B_2)}\right).$$

We define an *enhanced Langlands parameter* for $\mathcal{G}$ to be a such pair (Φ,ρ). The group G acts on these parameters by

$$g\cdot(\Phi,\rho) = (g\Phi g^{-1}, \rho\circ \mathrm{Ad}_g^{-1}) \tag{25}$$

and we denote the corresponding equivalence class by $[\Phi,\rho]_G$.

Definition 4.5. *Let $\Psi(G)^{\mathfrak{s}}_{\rm en}$ denote the set of $H^{\mathfrak{s}}$-conjugacy classes of enhanced parameters (Φ,ρ) for $\mathcal{G}$ such that we have $\Phi|\mathfrak{o}^\times = c^{\mathfrak{s}}$.*

For technical reasons it seems necessary to impose some mild restrictions on the residual characteristic of the local non-archimedean field F. We use the conditions in [Reed, §5], which exclude some primes depending on $\mathcal{G}$.

Theorem 4.6. [Reed]
Assume the above mild restrictions on the residual characteristic.

(1) *There is a canonical bijection*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \simeq \Psi(G)^{\mathfrak{s}}_{\rm en}.$$

(2) *It maps $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cap \mathbf{Irr}(\mathcal{G})_{\rm temp}$ onto the set of enhanced Langlands parameters (Φ,ρ) for which $\Phi(F^\times)$ is bounded.*

(3) *If $\sigma \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ corresponds to (Φ,ρ), then the cuspidal support $\pi^{\mathfrak{s}}(\sigma) \in T^{\mathfrak{s}}/W^{\mathfrak{s}}$, considered as a semisimple conjugacy class in $H^{\mathfrak{s}}$, equals $\Phi\big(\varpi_F, Y_{q^{1/2}}\big)$.*

Proof. (1) This is Reeder's classification of the constituents of a given principal series representation, see [Reed, Theorem 1, p.101 − 102].
(2) Reeder's work is based on that of Kazhdan–Lusztig, and it is known from [KL, §8] that the tempered $\mathcal{G}$-representations correspond precisely to the set of bounded enhanced L-parameters in the setting of [KL]. As the constructions in [Reed] preserve temperedness, this characterization remains valid in Reeder's setting.
(3) The element $\Phi\big(\varpi_F, Y_{q^{1/2}}\big) \in H^{\mathfrak{s}}$ is the same as t_q in Subsection 4.2 (up to $H^{\mathfrak{s}}$-conjugacy). In the setting of Kazhdan–Lusztig, it is known from [KL, 5.12 and Theorem 7.12] that property (3) holds. As for (2), this is respected by the constructions of Reeder that lead to (1). ☐

4.4. **Affine Springer parameters.** As before, suppose that $t \in H$ is semisimple and that $x \in \mathrm{Z}_H(t)$ is unipotent. Then $\mathrm{Z}_H(t,x)$ acts on $\mathcal{B}_H^{t,x}$ and $\pi_0(\mathrm{Z}_H(t,x))$ acts on the homology of this variety. In this setting we say that $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_H(t,x))\big)$ is *geometric* if it belongs to $\mathcal{R}\left(\pi_0(\mathrm{Z}_H(t,x)), \mathcal{B}_H^{t,x}\right)$.

For the affine Springer parameters it does not matter whether we consider the total homology or only the homology in top degree. Indeed, it follows from [Shoji, bottom of page 296 and Remark 6.5] that any irreducible representation ρ_1 which appears in $H_*\big(\mathcal{B}_H^{t,x},\mathbb{C}\big)$, already appears in the top homology of this variety. Therefore, we may refine Theorem 4.4 as follows:

Theorem 4.7.

$$\mathcal{R}(A, \mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}) = \mathcal{R}^{\rm top}(A, \mathcal{B}_H^{t,x}),$$

where top *refers to highest degree in which the homology is nonzero, the real dimension of $\mathcal{B}_H^{t,x}$.*

We call such triples (t,x,ρ_1) affine Springer parameters for H, because they appear naturally in the representation theory of the affine Weyl group associated to H. The group H acts on such parameters by conjugation, and we denote the conjugacy classes by $[t,x,\rho_1]_H$.

Definition 4.8. *The set of H-conjugacy classes of affine Springer parameters will be denoted $\Psi(H)_{\rm aff}$.*

For use in Theorem 4.10 we recall the parametrization of irreducible representations of $X^*(T) \rtimes \mathcal{W}^H$ from [Kat]. Kato defines an action of $X^*(T) \rtimes \mathcal{W}^H$ on the top homology $H_{d(x)}(\mathcal{B}_H^{t,x}, \mathbb{C})$, which commutes with the action of $\pi_0(\mathrm{Z}_H(t,x))$ induced by conjugation of Borel subgroups. Let $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_H(t,x))\big)$. By [Kat, Theorem 4.1] the $X^*(T) \rtimes \mathcal{W}^H$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho_1, H_{d(x)}(\mathcal{B}_H^{t,x}, \mathbb{C})\big)$$

is either irreducible or zero. Moreover every irreducible representation of $X^*(T) \rtimes \mathcal{W}^H$ is obtained in this way, and the data (t, x, ρ_1) are unique up to H-conjugacy. So Kato's results provide a natural bijection

$$\Psi(H)_{\mathrm{aff}} \to \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H). \tag{26}$$

This generalizes the Springer correspondence for finite Weyl groups, which can be recovered by considering the representations on which $X^*(T)$ acts trivially.

In [KL, Reed] there are some indications that the above kinds of parameters are essentially equivalent. Theorem (4.9) allows us to make this precise in the necessary generality.

Theorem 4.9. *Let $\mathfrak{s}$ be a Bernstein component in the principal series, associate $c^{\mathfrak{s}} \colon \mathfrak{o}_F^\times \to T$ to it as in Lemma 4.1 and let $H^{\mathfrak{s}}$ be as in (18). There are natural bijections between $H^{\mathfrak{s}}$-equivalence classes of:*

- *enhanced Langlands parameters for $\mathcal{G}$ with $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$;*
- *affine Springer parameters for $H^{\mathfrak{s}}$.*

In other words we have

$$\Psi(G)^{\mathfrak{s}}_{\mathrm{en}} \simeq \Psi(H^{\mathfrak{s}})_{\mathrm{aff}}.$$

Proof. An L-parameter gives rise to the ingredients t, x in an affine Springer parameter in the following way. Consider an L-parameter

$$\Phi \colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G$$

Let $x_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$. Set

$$t := \Phi(\varpi_F, 1), \qquad x := \Phi(1, x_0)$$

Conversely, we work with the Jacobson–Morozov theorem [CG, p. 183]. Let x be a unipotent element in M^0. There exist rational homomorphisms

$$\gamma \colon \mathrm{SL}_2(\mathbb{C}) \to M^0 \quad \text{with} \quad \gamma(x_0) = x, \tag{27}$$

see [CG, §3.7.4]. Any two such homomorphisms γ are conjugate by elements of $\mathrm{Z}_{M^\circ}(x)$. Define the Langlands parameter Φ as follows:

$$\Phi \colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G, \qquad (u\varpi_F^n, Y) \mapsto c^{\mathfrak{s}}(u) \cdot t^n \cdot \gamma(Y) \tag{28}$$

for all $u \in \mathfrak{o}_F^\times$, $n \in \mathbb{Z}$, $Y \in \mathrm{SL}_2(\mathbb{C})$.

Note that the definition of Φ uses the appropriate data: the semisimple element $t \in T$, the map $c^{\mathfrak{s}}$, and the homomorphism γ (which depends on x).

Since x determines γ up to M°-conjugation, $c^{\mathfrak{s}}, x$ and t determine Φ up to conjugation by their common centralizer in G. Notice also that one can recover $c^{\mathfrak{s}}, x$ and t from Φ and that

$$h(\alpha) := \Phi(1, Y_\alpha) \tag{29}$$

defines a *cocharacter* $\mathbb{C}^\times \to T$.

The pair (t, x) is enough to recover the conjugacy class of Φ. A refined version of the Jacobson–Morozov theorem says that the same goes for the pair (t_q, x), see [KL, §2.3] or [Reed, Section 4.2].

To complete Φ, (t, x) or (t_q, x) to a parameter of the appropriate kind, we must add an irreducible representation ρ or ρ_1. Then the result follows from Theorem 4.7. $\square$

4.5. **The labelling by unipotent classes.** Let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G}, \mathcal{T})$ and construct $c^{\mathfrak{s}}$ as in Section 4.1. We note that the set of enhanced Langlands parameters $\Phi(G)^{\mathfrak{s}}_{\mathrm{en}}$ is naturally labelled by the unipotent classes in H:

$$\Phi(G)^{\mathfrak{s},[x]}_{\mathrm{en}} := \big\{ (\Phi, \rho) \in \Phi(G)^{\mathfrak{s}}_{\mathrm{en}} \mid \Phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) \text{ is conjugate to } x \big\}. \tag{30}$$

Via Theorem 4.9 the set $\Phi(G)^{\mathfrak{s}}_{\mathrm{en}}$ is naturally in bijection with $\Psi(H)_{\mathrm{aff}}$. In this way we can associate to any of the parameters in Theorem 4.9 a unique unipotent class in H:

$$\begin{aligned} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} &= \bigcup\nolimits_{[x]} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s},[x]}, \\ \Psi(H)_{\mathrm{aff}} &= \bigcup\nolimits_{[x]} \Psi(H)^{[x]}_{\mathrm{aff}}. \end{aligned} \tag{31}$$

Recall from Section 2.1 that

$$\widetilde{T^{\mathfrak{s}}} = \{(w, t) \in W^{\mathfrak{s}} \times T^{\mathfrak{s}} \mid wt = t\}$$

and $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} = \widetilde{T^{\mathfrak{s}}}/W^{\mathfrak{s}}$. In general it can already be hard to define any suitable map from $\Phi(G)^{\mathfrak{s}}_{\mathrm{en}}$ to $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$, because it is difficult to compare the parameters ρ for different Φ's. It goes better the other way round and with $\Psi(H)_{\mathrm{aff}}$ as target. In this way will transfer the labellings (31) to $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$.

Theorem 4.10. *There exists a continuous bijection $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \to \Psi(H^{\mathfrak{s}})_{\mathrm{aff}}$ such that:*

- *it respects the canonical projections to $T^{\mathfrak{s}}/W^{\mathfrak{s}}$;*
- *for every unipotent class x of $H^{\mathfrak{s}}$, the inverse image of the set of affine Springer parameters with unipotent part x is a union of connected components of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$.*

Proof. First we take another look at (26). By Clifford theory (confer the appendix of [RaRa]) the number of irreducible representations of $X^*(T) \rtimes \mathcal{W}^H$ which have an $X^*(T)$-weight $t \in T$ equals $|\mathbf{Irr}(\mathcal{W}^H_t)|$, where $\mathcal{W}^H_t$ denotes the isotropy group of t in $\mathcal{W}^H$. Hence this is also the number of affine Springer parameters with this particular t. Recall from Subsection 2.1 that also

$$|\mathbf{Irr}(W^{\mathfrak{s}}_t)| = |\{y \in T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \mid \rho^{\mathfrak{s}}(y) = W^{\mathfrak{s}} t\}|.$$

Fix a Borel subgroup B_H of H containing T, and choose a set of representatives $\mathfrak{U}^{\mathfrak{s}} \subset B_H$ for the unipotent classes of H. Every commuting pair (t, x) with $t \in H$ semisimple and $x \in H$ unipotent is conjugate to one in the Borel group B_H, because the union of all Borel groups is H. Conjugating by a suitable element of B_H, one can simultaneously achieve that $t \in T$. Hence every affine Springer parameter is conjugate to one with $t \in T$ and $x \in U^s$.

For $x \in \mathfrak{U}^{\mathfrak{s}}$ we endow $\mathrm{Z}_T(x)$ with a "multiplicity" function m_x, that assigns to every $t \in \mathrm{Z}_T(x)$ the number of $\rho_1 \in \mathbf{Irr}(\mathrm{Z}_H(t, x))$ such that $\tau(t, x, \rho_1)$ is nonzero. Two pairs (t,x) and (t',x) with $t,t' \in T$ can only be conjugate if $t' = w(t)$ for some $w \in W^{\mathfrak s}$. Hence one can count the number of equivalence classes of affine Springer parameters with a particular t by looking at the values of m_x on the $W^{\mathfrak s}$-orbit of t. One needs only the maximum of $m_x(wt)$ over $w \in W^{\mathfrak s}$, and this is achieved whenever $m_x(wt) > 0$.

In view of (26) and the above, we obtain for every $t \in T$:

(32)
$$\sum_{x \in \mathfrak{U}^{\mathfrak s} \mid \exists w_x \in W^{\mathfrak s} : w_x(t) \in \mathrm{Z}_T(x)} m_x(w_x(t)) = |\mathbf{Irr}(W^{\mathfrak s}_t)| = |\{y \in T^{\mathfrak s}/\!/W^{\mathfrak s} \mid \rho^{\mathfrak s}(y) = W^{\mathfrak s} t\}|.$$

It follows that every $x \in \mathfrak{U}^{\mathfrak s}$ and every natural number m,

$$\mathrm{Z}_T(x)_{\geq m} := \{t \in \mathrm{Z}_T(x) : m_x(t) \geq m\}$$

is a union of irreducible components $(T^{\mathfrak s})^w_i$ of the varieties $(T^{\mathfrak s})^w$, for suitable $w \in W^{\mathfrak s}$.

We construct maps from the sets $\mathrm{Z}_T(x)_{\geq m}$ to $T^{\mathfrak s}/\!/W^{\mathfrak s}$ with recursion. Start with an $x \in \mathfrak{U}^{\mathfrak s}$ for which $\mathrm{Z}_T(x)$ has minimal dimension. Then consider the largest m for which $\mathrm{Z}_T(x)_{\geq m}$ is nonempty. Choose irreducible components $(T^{\mathfrak s})^w_i$ as above, which together have the same projection on $T^{\mathfrak s}/W^{\mathfrak s}$ as $\mathrm{Z}_T(x)_{\geq m}$, and match these with $\mathrm{Z}_T(x)_{\geq m}$, in a way which respects the canonical projections on $T^{\mathfrak s}/W^{\mathfrak s}$.

For the next step, remove these components from $T^{\mathfrak s}/\!/W^{\mathfrak s}$ and decrease the multiplicity $m_x(t)$ by 1 for every $t \in \mathrm{Z}_T(x)_{\geq m}$. Repeat the above construction with the new data. This is possible because (32) remains valid for the new data.

Combining all these maps gives a bijection $T^{\mathfrak s}/\!/W^{\mathfrak s} \to \Psi(H)_{\mathrm{aff}}$ which satisfies (1) and (2). It is continuous if we endow $\Psi(H)_{\mathrm{aff}}$ with the following topology: a set $V \subset \Psi(H)_{\mathrm{aff}}$ is open if and only if

$$\{(t,x) \mid (t,x,\rho_1) \in V \text{ for some } \rho_1\}$$

is open in the direct product of T with the set of unipotent classes in H. $\square$

Theorem 4.11. *Let $\mathcal G$ be a split reductive p-adic group with connected centre, with a mild restriction on the residual characteristic p. Then, for each point $\mathfrak s$ in the principal series of $\mathcal G$, we have a continuous bijection*

$$\mu^{\mathfrak s} : T^{\mathfrak s}/\!/W^{\mathfrak s} \to \mathbf{Irr}(\mathcal G)^{\mathfrak s}.$$

It maps $T^{\mathfrak s}_{\mathrm{cpt}}/\!/W^{\mathfrak s}$ onto $\mathbf{Irr}(\mathcal G)^{\mathfrak s} \cap \mathbf{Irr}(\mathcal G)_{\mathrm{temp}}$.

Proof. To get $\mu^{\mathfrak s}$, apply Theorems 4.9, 4.6.(1) and 4.10.

By Theorem 4.10 $T^{\mathfrak s}_{\mathrm{cpt}}/\!/W^{\mathfrak s}$ is first mapped bijectively to the set of parameters in $\Psi(H)_{\mathrm{aff}}$ with t compact. From the proof of Theorem 4.9 we see that the latter set is mapped onto the set of enhanced Langlands parameters (Φ,ρ) with $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak s}$ and $\Phi(\varpi_F)$ compact. These are just the bounded enhanced Langlands parameters, so by Theorem 4.6.(2) they correspond to $\mathbf{Irr}(\mathcal G)^{\mathfrak s} \cap \mathbf{Irr}(\mathcal G)_{\mathrm{temp}}$. $\square$

4.6. **Correcting cocharacters and L-packets.** In this section we construct the correcting cocharacters on the extended quotient $T^{\mathfrak s}/\!/W^{\mathfrak s}$. As conjectured in Section 2.3, these show how to determine when two elements of $T^{\mathfrak s}/\!/W^{\mathfrak s}$ give rise to $\mathcal G$-representations in the same L-packets.

Every enhanced Langlands parameter (Φ, ρ) naturally determines a cocharacter h_Φ and elements $\theta(\Phi, \rho, z) \in T^{\mathfrak{s}}$ by

$$\begin{aligned} h_\Phi(z) &= \Phi\big(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\big), \\ \theta(\Phi, \rho, z) &= \Phi\big(\varpi_F, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\big) = \Phi(\varpi_F) h_\Phi(z). \end{aligned} \tag{33}$$

Although these formulas obviously do not depend on ρ, it turns out to be convenient to include it in the notation anyway. However, in this way we would end up with infinitely many correcting cocharacters, most of them with range outside T. To reduce to finitely many cocharacters with values in T, we will restrict to enhanced Langlands parameters associated to $x \in \mathfrak{U}^{\mathfrak{s}}$, as in the proof of Theorem 4.10.

Recall that (31) and Theorem 4.10 determine a labelling of the connected components of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$ by unipotent classes in H. This enables us to define the correcting cocharacters: for a connected component $\mathbf{c}$ of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$ with label (represented by) $x \in \mathfrak{U}^{\mathfrak{s}}$ we take the cocharacter

$$h_{\mathbf{c}} = h_x : \mathbb{C}^\times \to T, \quad h_x(z) = \gamma_x \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right). \tag{34}$$

Let $\widetilde{\mathbf{c}}$ be a connected component of $\widetilde{T^{\mathfrak{s}}}$ that projects onto $\mathbf{c}$ and appears as $(T^{\mathfrak{s}})_i^w$ in the proof of Theorem 4.10. This can always be achieved by adjusting by element of $W^{\mathfrak{s}}$. We define

$$\begin{aligned} \widetilde{\theta_z} &: \widetilde{\mathbf{c}} \to T^{\mathfrak{s}}, & (w, t) &\mapsto t\, h_{\mathbf{c}}(z), \\ \theta_z &: \mathbf{c} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}, & [w, t] &\mapsto W^{\mathfrak{s}} t\, h_{\mathbf{c}}(z). \end{aligned} \tag{35}$$

Lemma 4.12. *Let $[w, t], [w', t'] \in T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$. Then $\mu^{\mathfrak{s}}[w, t]$ and $\mu^{\mathfrak{s}}[w', t']$ are in the same L-packet if and only if*

- *$[w, t]$ and $[w', t']$ are labelled by the same unipotent class in H;*
- *$\theta_z[w, t] = \theta_z[w', t']$ for all $z \in \mathbb{C}^\times$.*

Proof. Suppose that the two $\mathcal{G}$-representations $\mu^{\mathfrak{s}}[w, t] = \pi(\Phi, \rho)$ and $\mu^{\mathfrak{s}}[w', t'] = \pi(\Phi', \rho')$ belong to the same L-packet. By definition this means that Φ and Φ' are G-conjugate. Hence they are labelled by the same unipotent class, say $[x]$ with $x \in \mathfrak{U}^{\mathfrak{s}}$. By choosing suitable representatives we may assume that $\Phi = \Phi'$ and that $\{(\Phi, \rho), (\Phi, \rho')\} \subset \Phi(G)_{\mathrm{en}}^{\mathfrak{s}, x}$. Then

$$\theta(\Phi, \rho, z) = \theta(\Phi, \rho', z) \text{ for all } z \in \mathbb{C}^\times.$$

Although in general $\theta(\Phi, \rho, z) \neq \widetilde{\theta_z}(w, t)$, they differ only by an element of $W^{\mathfrak{s}}$. Hence $\theta_z[w, t] = \theta_z[w', t']$ for all $z \in \mathbb{C}^\times$.

Conversely, suppose that $[w, t], [w', t']$ fulfill the two conditions of the lemma. Let $x \in \mathfrak{U}^{\mathfrak{s}}$ be the representative for the unipotent class which labels them. From the constructions in Theorem 4.10 we see that there are representatives for $[w, t]$ and $[w', t']$ such that $t(T^w)^\circ$ and $t'(T^{w'})^\circ$ centralize x. Then

$$\widetilde{\theta_z}(w, t) = t\, h_x(z) \quad \text{and} \quad \widetilde{\theta_z}(w', t') = t'\, h_x(z)$$

are $W^{\mathfrak{s}}$ conjugate for all $z \in \mathbb{C}^\times$. As these points depend continuously on z and $W^{\mathfrak{s}}$ is finite, this implies that there exists a $v \in W^{\mathfrak{s}}$ such that

$$v(t\, h_x(z)) = t'\, h_x(z) \quad \text{for all } z \in \mathbb{C}^\times.$$

For $z = 1$ we obtain $v(t) = t'$, so v fixes $h_x(z)$ for all z.

Consider the minimal parabolic root subsystem R_P of $R(G,T)$ that supports h_x. In other words, the unique set of roots P such that h_x lies in a facet of type P in the chamber decomposition of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We write

$$T^P = \{t \in T \mid \alpha(t) = 1 \; \forall \alpha \in P\}^\circ .$$

Then $t(T^w)^\circ$ and $t'(T^{w'})^\circ$ are subsets of T^P and v stabilizes T^P. It follows from [Opd, Proposition B.4] that $h_x(q^{1/2})tT^P$ and $h_x(q^{1/2})t'T^P$ are residual cosets in the sense of Opdam. By the above, these two residual cosets are conjugate via $v \in W^{\mathfrak{s}}$. Now [Opd, Corollary B.5] says that the pairs $(h_x(q^{1/2})t, x)$ and $(h_x(q^{1/2})t', x)$ are H-conjugate. Hence the associated Langlands parameters are conjugate, which means that $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet. □

Corollary 4.13. *Properties 1–5 from Section 2.3 hold for $\mu^{\mathfrak{s}}$ as in Theorem 4.11, with the morphism θ_z from* (35) *and the labelling by unipotent classes in $H^{\mathfrak{s}}$ from* (31) *and Theorem 4.10.*

Together with Theorem 4.11 this proves the conjecture from Section 2.3 for all Bernstein components in the principal series of a split reductive p-adic group with connected centre (with mild restrictions on the residual characteristic).

Proof. Property (1) was already shown in Theorem 4.11. By the definition of θ_z (35), property (4) holds. Property (3) is a consequence of property (4), in combination with Theorems 4.6.(3), 4.11 and 4.10. Property (2) follows from Theorem 4.11 and property (3). Property 5 is none other than Lemma 4.12. □

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