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Stochastic evolution equations driven by compensated Poisson measures: existence, uniqueness and large deviation estimates

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Abstract

Existence and uniqueness results are established for solutions of stochastic evolution equations driven by Poisson point processes. Large deviation estimates are obtained for the case of additive Poisson noise. Illustrating examples are provided.

AMS Subject Classification: Primary 60H15 Secondary 93E20, 35R60.

1 Introduction

Stochastic evolution equations and stochastic partial differential equations driven by Wiener processes have been studied by many people. There exists a great amount of literature on the subject, see, for example the monograph [DZ]. In contrast, there has not been very much study of stochastic partial differential equations driven by jump processes. However, it begun to gain attention recently. In [AWZ] we obtained existence and uniqueness for solutions of stochastic reaction equations driven by Poisson random mesasures. In [F], Malliavin calculus was applied to study the absolute continuity of the law of the solutions of stochastic reaction equations driven by Poisson random mesasures. In [MC], a minimal solution was obtained for the stochastic heat equation driven by non-negative Levy noise with coefficients of polynomial growth. In [ML], a weak solution is established for stochastic heat equation driven by stable noise with coefficients of polynomial growth.

In this paper, we consider the following evolution equation:

$$dY_t = -AY_t dt + b(Y_t)dt + \sigma(Y_t)dB_t + \int_X f(Y_{t-}, x)\tilde{N}(dt, dx), \quad (1.1)$$

$$Y_0 = h \in H \quad (1.2)$$

in the framework of a Gelfand triple :

$$V \subset H \cong H^* \subset V^*. \quad (1.3)$$

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We look for solutions in the space of V instead of the mild solutions in H in contrast to the literature. The stochastic evolution equations of this type driven by Wiener processes were first studied by E.Pardoux in [P]. A large deviation principle for this type of stochastic evolution equations driven by Wiener process was obtained by P.Chow in [C]. The purpose of this paper has two folds. The first one is to establish the existence and uniqueness for solutions of equation (1.1). Our approach is similar to the one in [P]. But, some extra care need to be taken for the jumps. We don't use the Galerkin approximations as in [P]. Instead, we got the solution via successive approximations. Secondly we will study the large deviation principle and exponential integrability of solutions of equation (1.1). We will prove that the solution is exponentially integrable both as a random variable in $D([0,1]\to H)$ and in $L^2([0,1]\to V)$. Thes estimates are of their own interest and also necessary for the study of large deviations. For the large deviation principle, we confine ourself to the case of additive Poisson noise. The situation is quite different from the Gaussian case. If $X_t, t\geq 0$ is a Wiener process, the solution of the equation:

$$dY_t^n = -AY_t^n dt + \frac{1}{n} dX_t$$

is still Gaussian. The large deviations of Y^n follows from the well known large deviations of Gaussian processes. However, if $X_t, t \geq 0$ is a Lévy process, the solution Y^n is no longer a Lévy process. The additive noise case is already hard. Large deviations for Lévy processes on Banach spaces and large deviations for solutions of stochastic differential equations driven by Poisson measures were studied by de Acosta in [A1], [A2].

2 Framework

Let V, H be two separable Hilbert spaces such that V is continuously, densely imbedded in H. Identifying H with its dual we have

$$V \subset H \cong H^* \subset V^*, \tag{2.1}$$

where V^* stands for the topological dual of V. Let A be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$2\langle Au, u\rangle + \lambda |u|_H^2 \geq \alpha ||u||_V^2 \qquad \text{for all } u \in V, \tag{2.2}$$

where $\langle Au, u\rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.
Remark that A is generally not bounded as an operator from H into H. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let $\{B_t, t \geq 0\}$ be a real-valued $\mathcal{F}_t$- Brownian motion. Let $(X, \mathcal{B}(X))$ be a measurable space and $\nu(dx)$ a σ-finite measure

on it. Let $p = (p(t)), t \in D_p$ be a stationary $\mathcal{F}_t$-Poisson point process on X with characteristic measure ν. See [IW] for details on Poisson point processes. Denote by $N(dt, dx)$ the Poisson counting measure associated with p, i.e., $N(t, A) = \sum_{s \in D_p, s \le t} I_A(p(s))$. Let $\tilde{N}(dt, dx) := N(dt, dx) - dt\nu(dx)$ the compensated Poisson mesasure. Let $b(y)$, $\sigma(y)$ be measurable mappings from H into H, and $f(y, x)$ a measurable mapping from $H \times X$ into H. For a separable Hilbert space L, we denote by $M^2([0, T], L)$ the Hilbert space of progressively measurable, square integrable, L-valued processes equipped with the inner product $< a, b >_M = E[\int_0^T < a_t, b_t >_L dt]$. Denote by $M^{\nu,2}([0, T] \times X, L)$ the collection of predictable mappings:

$$f(s, x, \omega) : [0, T] \times X \times \Omega \to L$$

such that $E[\int_0^T \int_X |f(s, x, \omega)|_L^2 ds\nu(dx)] < \infty$. Denote by $D([0, T], H)$ the space of all cadlag paths from $[0, T]$ into H. Consider the stochastic evolution equation:

$$\begin{aligned} dY_t &= -AY_t dt + b(Y_t)dt + \sigma(Y_t)dB_t + \int_X f(Y_{t-}, x)\tilde{N}(dt, dx), && (2.3) \\ Y_0 &= h \in H && (2.4) \end{aligned}$$

We introduce

(H.1) There exists a constant $C < \infty$ such that

$$|b(y)|_H^2 + |\sigma(y)|_H^2 + \int_X |f(y, x)|_H^2 \nu(dx) \le C(1 + |y|_H^2) \qquad (2.5)$$

for all $y \in H$.

(H.2) There exists a constant $C < \infty$ such that

$$\begin{aligned} & |b(y_1) - b(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2 \\ + & \int_X |f(y_1, x) - f(y_2, x)|_H^2 \nu(dx) \\ \le & C|y_1 - y_2|_H^2 \qquad (2.6) \end{aligned}$$

for all $y_1, y_2 \in H$.

3 Existence and uniqueness

Proposition 3.1 *Let $b \in M^2([0, T], H), \sigma \in M^2([0, T], H)$ and $f \in M^{\nu,2}([0, T] \times X, H)$. There exists a unique solution $Y_t, t \ge 0$ to the following equation:*

$$\begin{aligned} Y &\in M^2([0, T], V) \cap D([0, T], H) \\ dY_t &= -AY_t dt + b(t, \omega)dt + \sigma(t, \omega)dB_t \\ & \quad + \int_X f(t, x, \omega)\tilde{N}(dt, dx). && (3.1) \\ Y_0 &= h \in H && (3.2) \end{aligned}$$

Proof. We prove the existence in two steps.
Step 1. Assume $b \in M^2([0,T],V), \sigma \in M^2([0,T],V)$ and $f \in M^{\nu,2}([0,T] \times X, V)$. Put

$$U_t = \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \int_X f(s,x)\tilde{N}(ds,dx)$$

It is easy to see that $U \in M^2([0,T],V)$. Consider the random equation:

$$\begin{aligned} dv_t &= (-Av_t - AU_t)dt \\ v_0 &= h \end{aligned} \tag{3.3}$$

It is known from [L] that there exists a unique solution v to equation (3.3) such that $v \in M^2([0,T],V) \cap C([0,T],H)$. Set $Y_t := v_t + U_t$. Then $Y \in M^2([0,T],V) \cap D([0,T],H)$. Moreover, it solves equation (3.1).

Step 2. General case.

Choose $b_n \in M^2([0,T],V), \sigma_n \in M^2([0,T],V)$ and $f_n \in M^{\nu,2}([0,T] \times X, V)$ such that $b_n \to b$, $\sigma_n \to \sigma$ in $M^2([0,T],H)$ and $f_n \to f$ in $M^{\nu,2}([0,T] \times X, H)$ as $n \to \infty$. Denote by Y_t^n the unique solution to equation (3.1) with b, σ, f replaced by b_n, σ_n, f_n. Such a Y^n exists by step 1. By Ito's formula, we have

$$\begin{aligned}
& |Y_t^n - Y_t^m|_H^2 \\
= & -2\int_0^t < Y_s^n - Y_s^m, A(Y_s^n - Y_s^m) > ds \\
+ & 2\int_0^t < Y_s^n - Y_s^m, b_n(s) - b_m(s) > ds + 2\int_0^t < Y_s^n - Y_s^m, \sigma_n(s) - \sigma_m(s) > dB_s \\
+ & \int_0^t |\sigma_n(s) - \sigma_m(s)|_H^2 ds \\
+ & \int_0^t \int_X \Big(|f_n(s,x) - f_m(s,x)|_H^2 + 2 < Y_{s-}^n - Y_{s-}^m, f_n(s,x) - f_m(s,x) >_H \Big) \tilde{N}(ds,dx) \\
+ & \int_0^t \int_X |f_n(s,x) - f_m(s,x)|_H^2 ds\nu(dx)
\end{aligned} \tag{3.4}$$

In the following, C will denote a generic constant whose values might change from line to line. Put

$$M_t = \int_0^{t+} \int_X \Big(|f_n(s,x) - f_m(s,x)|_H^2 + 2 < Y_{s-}^n - Y_{s-}^m, f_n(s,x) - f_m(s,x) >_H \Big) \tilde{N}(ds,dx)$$

Then,

$$[M,M]_t^{\frac{1}{2}} =$$

$$\Big\{ \sum_{s \in D_p, s \le t} \Big(|f_n(s,p(s)) - f_m(s,p(s))|_H^2 + 2 < Y_{s-}^n - Y_{s-}^m, f_n(s,p(s)) - f_m(s,p(s)) >_H \Big)^2 \Big\}^{\frac{1}{2}}$$

$$\leq C\bigg(\sum_{s\in D_p, s\leq t}|f_n(s,p(s))-f_m(s,p(s))|_H^4\bigg)^{\frac{1}{2}}$$

$$+C\bigg(\sum_{s\in D_p, s\leq t}|Y^n_{s-}-Y^m_{s-}|_H^2|f_n(s,p(s))-f_m(s,p(s))|_H^2\bigg)^{\frac{1}{2}}$$

$$\leq C\sum_{s\in D_p, s\leq t}|f_n(s,p(s))-f_m(s,p(s))|_H^2$$

$$+C\sup_{0\leq s\leq t}(|Y^n_{s-}-Y^m_{s-}|_H)\bigg(\sum_{s\in D_p, s\leq t}|f_n(s,p(s))-f_m(s,p(s))|_H^2\bigg)^{\frac{1}{2}}$$

$$\leq C\sum_{s\in D_p, s\leq t}|f_n(s,p(s))-f_m(s,p(s))|_H^2+\frac{1}{4}\sup_{0\leq s\leq t}(|Y^n_{s-}-Y^m_{s-}|_H^2$$

By Bürkholder's inequality,

$$E[\sup_{0\leq s\leq t}|M_s|]\leq CE([M,M]_t^{\frac{1}{2}})$$

$$\leq CE[\sum_{s\in D_p, s\leq t}|f_n(s,p(s))-f_m(s,p(s))|_H^2]+\frac{1}{4}E[\sup_{0\leq s\leq t}|Y^n_{s-}-Y^m_{s-}|_H^2]$$

$$=CE[\int_0^t\int_X|f_n(s,x)-f_m(s,x)|_H^2ds\nu(dx)]+\frac{1}{4}E[\sup_{0\leq s\leq t}|Y^n_s-Y^m_s|_H^2] \quad (3.5)$$

It follows from (3.4) and (2.2) that

$$E[\sup_{0\leq s\leq t}|Y^n_s-Y^m_s|_H^2]\leq -\alpha E[\int_0^t||Y^n_s-Y^m_s||_V^2ds]$$

$$+(\lambda+C)E[\int_0^t|Y^n_s-Y^m_s|_H^2ds]+CE[\int_0^t|b_n(s)-b_m(s)|_H^2ds]$$

$$+CE[\bigg(\int_0^t<Y^n_s-Y^m_s,\sigma_n(s)-\sigma_m(s)>_H^2ds\bigg)^{\frac{1}{2}}]+CE[\int_0^t|\sigma_n(s)-\sigma_m(s)|_H^2ds]$$

$$+CE([M,M]_t^{\frac{1}{2}})+CE[\int_0^t\int_X|f_n(s,x)-f_m(s,x)|_H^2ds\nu(dx)]$$

Applying (3.5) we have

$$E[\sup_{0\leq s\leq t}|Y^n_s-Y^m_s|_H^2]$$

$$\leq -\alpha E[\int_0^t||Y^n_s-Y^m_s||_V^2ds]+CE[\int_0^t|Y^n_s-Y^m_s|_H^2ds]$$

$$+\frac{1}{2}\sup_{0\leq s\leq t}(|Y^n_{s-}-Y^m_{s-}|_H^2+CE[\int_0^t|b_n(s)-b_m(s)|_H^2ds]$$

$$+CE[\int_0^t |\sigma_n(s)-\sigma_m(s)|_H^2 ds]+CE[\int_0^t\int_X |f_n(s,x)-f_m(s,x)|_H^2 ds\nu(dx)] \quad (3.6)$$

By Gronwall's inequality, this implies that

$$E[\sup_{0\le s\le t}|Y_s^n-Y_s^m|_H^2]\le Ce^{Ct}\{E[\int_0^t |b_n(s)-b_m(s)|_H^2 ds]$$

$$+CE[\int_0^t |\sigma_n(s)-\sigma_m(s)|_H^2 ds]+CE[\int_0^t\int_X |f_n(s,x)-f_m(s,x)|_H^2 ds\nu(dx)]\} \quad (3.7)$$

Therefore,

$$\lim_{n,m\to\infty} E[\sup_{0\le s\le t}|Y_s^n-Y_s^m|_H^2]=0 \quad (3.8)$$

This further implies by (3.6) that $Y^n, n\ge 1$ is also a Cauchy sequence in $M^2([0,T],V)$. Let $Y_t, t\ge 0$ denote an element in $M^2([0,T],V)$ such that

$$\lim_{n\to\infty} E[\sup_{0\le s\le t}|Y_s^n-Y_s|_H^2]=0$$

and

$$\lim_{n\to\infty} E[\int_0^t ||Y_s^n-Y_s||_V^2 ds]=0$$

Letting $n\to\infty$, we see that $Y_t, t\ge 0$ is a solution to equation (3.1).

Uniqueness: If X_t, Y_t are two solutions to equation (3.1), then

$$\begin{cases} \frac{d(X_t-Y_t)}{dt} = & -A(X_t-Y_t) \\ X_0-Y_0 = & 0 \end{cases}$$

By chain rule, we have

$$|X_t-Y_t|_H^2=-2\int_0^t < X_s-Y_s, A(X_s-Y_s) > ds$$

$$\le -\alpha\int_0^t ||X_s-Y_s||_V^2 ds+\lambda\int_0^t |X_s-Y_s|_H^2 ds$$

By Gronwall's inequality, we obtain that $Y_t=X_t$, which completes the proof.

Theorem 3.2 *Assume (H.1)and (H.2). Then there exists a unique H-valued progressively measurable process* (Y_t) *such that*

(i) $Y\in M^2(0,T;V)\cap D(0,T;H)$ *for any* $T>0$.

(ii) $Y_t=h-\int_0^t AY_s ds+\int_0^t b(Y_s)ds+\int_0^t \sigma(Y_s)dB_s+\int_0^{t+}\int_X f(Y_{s-},x)\tilde{N}(ds,dx)$ *a.s..*

(iii) $Y_0=h\in H$.

Proof.

Existence of solution.

Let $Y_t^0 := h, t \geq 0$. For $n \geq 0$, define $Y^{n+1} \in M^2(0,T;V) \cap D(0,T;H)$ to be the unique solution to the following equation:

$$\begin{aligned} dY_t^{n+1} &= -AY_t^{n+1}dt + b(Y_t^n)dt + \sigma(Y_t^n)dB_t \\ &\quad + f(Y_{t-}^n, x)\tilde{N}(dt,dx) \qquad (3.9) \\ Y_0^n &= h \qquad (3.10) \end{aligned}$$

The solution Y^{n+1} of above equation exists according to Proposition 3.1. We are going to show that $\{Y^n, n \geq 1\}$ forms a Cauchy sequence. Using Itô's formula, we find that

$$\begin{aligned} &|Y_t^{n+1} - Y_t^n|_H^2 \\ &= -2\int_0^t \langle Y_s^{n+1} - Y_s^n, A(Y_s^{n+1} - Y_s^n)\rangle ds \\ &+ 2\int_0^t < Y_s^{n+1} - Y_s^n, b(Y_s^n) - b(Y_s^{n-1}) > ds \\ &+ 2\int_0^t < Y_s^{n+1} - Y_s^n, \sigma(Y_s^n) - \sigma(Y_s^{n-1}) > dB_s \\ &+ \int_0^t |\sigma(Y_s^n) - \sigma(Y_s^{n-1})|_H^2 ds \\ &+ \int_0^{t+}\int_X [|f(Y_{s-}^n, x) - f(Y_{s-}^{n-1}, x)|_H^2 + 2 < Y_s^{n+1} - Y_s^n, f(Y_{s-}^n, x) - f(Y_{s-}^{n-1}, x) >]\tilde{N}(ds,dx) \\ &+ \int_0^t \int_X |f(Y_s^n, x) - f(Y_s^{n-1}, x)|_H^2 ds\nu(dx). \qquad (3.11) \end{aligned}$$

By a similar calculation as in Proposition 3.1, it follows from (3.11) that

$$\begin{aligned} &E[\sup_{0\leq s\leq t} |Y_s^{n+1} - Y_s^n|_H^2] \\ &\leq -\alpha E[\int_0^t ||Y_s^{n+1} - Y_s^n||_V^2 ds] + CE[\int_0^t |Y_s^{n+1} - Y_s^n|_H^2 ds] \\ &+\frac{1}{2}E[\sup_{0\leq s\leq t} |Y_s^{n+1} - Y_s^n|_H^2] + CE[\int_0^t |b(Y_s^n) - b(Y_s^{n-1})|_H^2 ds] \\ &+CE[\int_0^t |\sigma(Y_s^n)-\sigma(Y_s^{n-1})|_H^2 ds]+CE[\int_0^t\int_X |f(Y_s^n, x)-f(Y_s^{n-1}, x)|_H^2 ds\nu(dx)] \qquad (3.12) \end{aligned}$$

Using (H.1), this implies that

$$\begin{aligned} E[\sup_{0\leq s\leq t} |Y_s^{n+1} - Y_s^n|_H^2] \leq& CE[\int_0^t |Y_s^{n+1} - Y_s^n|_H^2 ds] \\ &+CE[\int_0^t |Y_s^n - Y_s^{n-1}|_H^2 ds] \qquad (3.13) \end{aligned}$$

Define

$$g_t^n = E[\sup_{0\le s\le t} |Y_s^n - Y_s^{n-1}|_H^2], \quad G_t^n = \int_0^t g_s^n ds$$

We have

$$g_t^{n+1} \le CG_t^{n+1} + CG_t^n \tag{3.14}$$

Multiplying above inequality by e^{-Ct}, we get that

$$\frac{d(G_t^{n+1}e^{-Ct})}{dt} \le Ce^{-Ct}G_t^n \tag{3.15}$$

Therefore,

$$G_t^{n+1} \le Ce^{Ct}\int_0^t e^{-Cs}G_s^n ds \le Ce^{Ct}tG_t^n \tag{3.16}$$

Combining (3.14) and (3.16) we see that for a fixed $T > 0$, and $t \le T$,

$$g_t^{n+1} \le C^2e^{Ct}tG_t^n + CG_t^n \le C_T\int_0^t g_s^n ds \tag{3.17}$$

for some constant C_T. Iterating (3.17), we obtain that

$$E[\sup_{0\le s\le T} |Y_s^{n+1} - Y_s^n|_H^2] \le C\frac{(C_T T)^n}{n!}$$

This implies that there exists $Y \in D([0,T], H)$ such that

$$\lim_{n\to\infty} E[\sup_{0\le s\le T} |Y_s^n - Y_s|_H^2] = 0$$

Using (3.12) we see that Y^n also converges to Y in $M^2(0,T;V)$. Leting $n\to\infty$ in (3.9) it is seen that Y is a solution to equation (ii) in the theorem.

Uniqueness.

Let X, Y be two solutions to (ii) in $M^2(0,T;V)\cap D(0,T;H)$. By Ito's formula,we have

$$\begin{aligned}
&|Y_t - X_t|_H^2\\
&= -2\int_0^t \langle Y_s - X_s, A(Y_s - X_s)\rangle ds\\
&+ 2\int_0^t < Y_s - X_s, b(Y_s) - b(X_s) > ds\\
&+ 2\int_0^t < Y_s - X_s, \sigma(Y_s) - \sigma(X_s) > dB_s\\
&+ \int_0^t |\sigma(Y_s) - \sigma(X_s)|_H^2 ds\\
&+ \int_0^{t+}\int_X [|f(Y_{s-},x) - f(X_{s-},x)|_H^2 + 2 < Y_s - X_s, f(Y_{s-},x) - f(X_{s-},x) >]\tilde{N}(ds,dx)\\
&+ \int_0^t\int_X |f(Y_s,x) - f(X_s,x)|_H^2 ds\nu(dx).
\end{aligned} \tag{3.18}$$

Using (H.2), it follows that

$$
\begin{aligned}
& E[|Y_t - X_t|_H^2] \\
\leq & -\alpha E[\int_0^t ||Y_s - X_s||_V^2 ds] + CE[\int_0^t |Y_s - X_s|_H^2 ds] \\
& + \frac{1}{2} E[\sup_{0\leq s\leq t} |Y_s - X_s|_H^2] + CE[\int_0^t |b(Y_s) - b(X_s)|_H^2 ds] \\
& + CE[\int_0^t |\sigma(Y_s) - \sigma(X_s)|_H^2 ds] + CE[\int_0^t \int_X |f(Y_s, x) - f(X_s, x)|_H^2 ds \nu(dx)] \\
\leq & \, CE[\int_0^t |Y_s - X_s|_H^2 ds]
\end{aligned}
\tag{3.19}
$$

Hence, $X_t = Y_t$.

Next we move to a more general equation which includes terms involving also Poisson measures. Let U be a set in $B(X)$ such that $\nu(X \setminus U) < \infty$. Let $g(y, x)$ be a measurable mapping from $H \times X$ into H. Introduce the following conditions:

(H.3) There exists a constant $C < \infty$ such that

$$|b(y)|_H^2 + |\sigma(y)|_H^2 + \int_U |g(y, x)|_H^2 \nu(dx) \leq C(1 + |y|_H^2) \tag{3.20}$$

for all $y \in H$.

(H.4) There exists a constant $C < \infty$ such that

$$
\begin{aligned}
& |b(y_1) - b(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2 \\
+ & \int_U |g(y_1, x) - g(y_2, x)|_H^2 \nu(dx) \qquad (3.21) \\
\leq & \; C|y_1 - y_2|_H^2 \qquad (3.22)
\end{aligned}
$$

for all $y_1, y_2 \in H$.

Consider the stochastic evolution equation:

$$
\begin{aligned}
Y_t = h - \int_0^t AY_s ds + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) dB_s \\
+ \int_0^{t+} \int_U g(Y_{t-}, x) \tilde{N}(dt, dx) + \int_0^{t+} \int_{X\setminus U} g(Y_{t-}, x) N(dt, dx)
\end{aligned}
\tag{3.23}
$$

Theorem 3.3 *Assume (H.3) and (H.4). Then there exists a unique H-valued progressively measurable process (Y_t) such that*

(i) $Y \in M^2(0, T; V) \cap D(0, T; H)$ for any $T > 0$.

(ii)

$$Y_t = h - \int_0^t AY_s ds + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$$

$$+\int_0^{t+}\int_U g(Y_{t-},x)\tilde{N}(dt,dx) + \int_0^{t+}\int_{X\setminus U} g(Y_{t-},x)N(dt,dx) \qquad (3.24)$$

(iii) $Y_0 = h \in H$.

Proof. Having Theorem 3.1 in hand, this theorem can be proved in the same way as in the finite dimensional case (see [IW]). For completeness we sketch the proof. Let $\tau_1 < \tau_2 < \cdots$ be the enumeration of all elements in $D = \{s \in D_p; \quad p(s) \in X \setminus U\}$. It is clear that τ_n is an $(\mathcal{F}_t)$-stopping time and $\lim_{n\to\infty} \tau_n = \infty$. First we solve the equation on the time interval $[0, \tau_1]$. Consider the equation

$$\begin{aligned} X_t &= h - \int_0^t AX_s ds + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \\ &\quad + \int_0^{t+}\int_U g(X_{t-},x)\tilde{N}(dt,dx) \qquad (3.25) \end{aligned}$$

Following the same proof as Theorem 3.1 , it is seen that there exists a unique solution $X_t, t \geq 0$ to equation (3.25). Set

$$Y_t^1 = X_t, 0 \leq t < \tau_1, = Y_{\tau_1 -} + g(Y_{\tau_1 -}, p(\tau_1)), t = \tau_1$$

Clearly the process $\{Y_t^1\}_{t\in[0,\tau_1]}$ is the unique solution to equation (3.24). Now, set $\hat{B}_t = B_{t+\tau_1} - B_{\tau_1}$, $\hat{p}(s) = p(s + \tau_1)$. We can construct the process Y_t^2 on $[0, \hat{\tau}_1]$ with respect to initial value $Y_0^2 = Y_{\tau_1}^1$, Brownian motion $\hat{B}$ and Poisson point process $\hat{p}$ in the same way as Y_t^1. Note that $\hat{\tau}_1$ defined with respect to $\hat{p}$ coincides with $\tau_2 - \tau_1$. Define

$$Y_t = Y_t^1, t \in [0, \tau_1], = Y_{t-\tau_1}, t \in [\tau_1, \tau_2]$$

It is easy to see that $\{Y_t\}_{t\in[0,\tau_2]}$ is the unique solution to equation (3.24) in the interval $[0, \tau_2]$. Continuing this procedure successively, we get the unique solution Y to equation (3.24).

Example 3.4 Let $H = L^2(\mathbf{R}^d)$, and set

$$V = H_2^1(\mathbf{R}^d) = \{u \in L^2(R^d); \nabla u \in L^2(\mathbf{R}^d \to \mathbf{R}^d)\}$$

Denote by $a(x) = (a_{ij}(x))$ a matrix-valued function on $\mathbf{R}^d$ satisfing the uniform ellipticity condition:

$$\frac{1}{c} I_d \leq a(x) \leq cI_d \qquad \text{for some constant } \ c \in (0, \infty).$$

Let $f(x)$ be a vector field on $\mathbf{R}^d$ with $f \in L^p(\mathbf{R}^d)$ for some $p > d$. Define

$$Au = -div(a(x)\nabla u(x)) + f(x) \cdot \nabla u(x)$$

Then (2.2) is fulfilled for (H, V, A). Thus, for any choice of Brownian motion B and Poisson point process p and any coefficients satisfying (H.3), (H.4) the main results apply.

Example 3.5 Stochastic evolution equations associated with fractional Laplacian:

$$dY_t = \Delta_\alpha Y_t dt + b(Y_t)dt + \sigma(Y_t)dB_t + f(Y_{t-}, x)\tilde{N}(dt, dx), \quad (3.26)$$
$$Y_0 = h \in H, \quad (3.27)$$

where Δ_α denotes the generator of the symmetric α-stable process in R^d, $0 < \alpha \leq 2$. Δ_α is called the fractional Laplacian operator. It is well known that the Dirichlet form associated with Δ_α is given by

$$\mathcal{E}(u, v) = K(d, \alpha) \int\int_{R^d \times R^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dxdy$$

$$D(\mathcal{E}) = \{u \in L^2(R^d) : \quad \int\int_{R^d \times R^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} dxdy < \infty\}$$

where $K(d, \alpha) = \alpha 2^{\alpha-3}\pi^{-\frac{d+2}{2}} sin(\frac{\alpha\pi}{2})\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{\alpha}{2})$. To study equation (3.26), we choose $H = L^2(\mathbf{R}^d)$, and $V = D(\mathcal{E})$ with the inner product $< u, v >= \mathcal{E}(u, v) + (u, v)_{L^2(R^d)}$.
Define

$$Au = -\Delta_\alpha$$

Then (2.2) is fulfilled for (H, V, A). See [FOT] for details about the fractional Laplacian operator. Thus, for any choice of Brownian motion B and Poisson point process p and any coefficients satisfying (H.3), (H.4) the main results apply.

4 Large deviations estimates

(H.5) There exists a measurable function $\bar{f}$ on X satisfying

$$\sup_{y \in H} |f(y, x)|_H \leq \bar{f}(x) \quad (4.28)$$

and

$$\int_X (\bar{f}(x))^2 exp(a\bar{f}(x))\nu(dx) < \infty, \quad \text{for all} \quad a > 0 \quad (4.29)$$

In this section, for simplicity we assume that $b = 0, \sigma = 0$ in equation (1.1). Again denote by Y_t the solution of (1.1).

Lemma 4.1 *For* $g \in C_b^2(H)$, $M_t^g = exp(g(Y_t) - g(y) - \int_0^t h(Y_s)ds)$ *is an* $\mathcal{F}_t$*-local martingale, where*

$$h(y) =< -Ay, g'(y) > + \int_X (exp[g(y+f(y,x))-g(y)]-1- < g'(y), f(y,x) >)\nu(dx)$$

Proof. Applying Itô's formula first to $exp(g(Y_t))$ and then integration by parts to $exp(g(Y_t) - g(y))exp(-\int_0^t h(Y_s)ds)$ proves the lemma.

Proposition 4.2 *Assume (2.2) with* $\lambda = 0$ *and also (H.5). Then for any* $l > 0$,

$$E[exp(l \sup_{0\leq t\leq 1} |Y_t|_H)] < \infty.$$

Proof. It is sufficient to show that for any $l > 0$, there exists a constant C_l such that

$$P(\sup_{0\leq t\leq 1} |Y_t|_H > r) \leq C_l e^{-lr} \tag{4.30}$$

For $\lambda > 0$, set $g(y) = (1 + \lambda|y|_H^2)^{\frac{1}{2}}$. Then

$$g'(y) = \lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}}y$$

$$g''(y) = -\lambda^2(1 + \lambda|y|_H^2)^{-\frac{3}{2}}y \times y + \lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}}I_H$$

where I_H stands for the identity operator. It is easy to see that

$$\sup_y |g''(y)| \leq \lambda, \qquad \sup_y |g'(y)| \leq \lambda^{\frac{1}{2}}.$$

Moreover,

$$< -Ay, g'(y) >= \lambda(1 + \lambda|y|_H^2)^{-\frac{1}{2}} < -Ay, y >\leq 0 \tag{4.31}$$

for $y \in V$. Write $G(y) = e^{g(y)}$. By Taylor expansion, there exists θ between 0 and 1 such that

$$\begin{aligned} & exp[g(y + f(y,x)) - g(y)] - 1- < g'(y), f(y,x) > \\ &= e^{-g(y)}[G(y + f(y,x)) - G(y) - G(y) < g'(y), f(y,x) >] \\ &= \frac{1}{2}e^{-g(y)} < G''(y + \theta f(y,x)), f(y,x) \times f(y,x) > \end{aligned} \tag{4.32}$$

Note that

$$G''(y) = G(y)g'(y) \times g'(y) + G(y)g''(y)$$

It follows that

$$|G''(y)|_{L(H)} \leq \lambda G(y), \qquad \text{for all} \quad y \in H \tag{4.33}$$

By (4.32),

$$|exp[g(y + f(y,x)) - g(y)] - 1- < g'(y), f(y,x) > |$$

$$\leq \lambda exp(g(y+\theta f(y,x)) - g(y))|f(y,x)|_H^2$$
$$= \lambda exp(< g'(y+\theta_1 f(y,x)), \theta f(y,x) >)|f(y,x)|_H^2$$
$$\leq \lambda exp(\lambda^{\frac{1}{2}}|f(y,x)|_H)|f(y,x)|_H^2. \quad (4.34)$$

Applying Lemma 4.1, with the above choice of g, $M_t^g = exp(g(Y_t) - g(y) - \int_0^t h(Y_s)ds)$ is an $\mathcal{F}_t$-local martingale, where

$$h(y) =< -Ay, g'(y) > + \int_X (exp[g(y+f(y,x)) - g(y)] - 1 - < g'(y), f(y,x) >)\nu(dx)$$
$$\leq \int_X \lambda exp(\lambda^{\frac{1}{2}}|f(y,x)|_H)|f(y,x)|_H^2 \nu(dx)$$
$$\leq \int_X \lambda exp(\lambda^{\frac{1}{2}}|\bar{f}(x)|_H)|\bar{f}(x)|_H^2 \nu(dx) = M_\lambda < \infty \quad (4.35)$$

We now show (4.30). We have

$$P(\sup_{0\leq t\leq 1} |Y_t|_H > r) = P(\sup_{0\leq t\leq 1} g(Y_t) \geq (1+\lambda r^2)^{\frac{1}{2}})$$
$$= P(\sup_{0\leq t\leq 1} (g(Y_t) - g(x) - \int_0^t h(Y_s)ds + g(x) + \int_0^t h(Y_s)ds) \geq (1+\lambda r^2)^{\frac{1}{2}})$$
$$\leq P(\sup_{0\leq t\leq 1} (g(Y_t) - g(x) - \int_0^t h(Y_s)ds) + g(x) + M_\lambda \geq (1+\lambda r^2)^{\frac{1}{2}})$$
$$= P(\sup_{0\leq t\leq 1} (g(Y_t) - g(x) - \int_0^t h(Y_s)ds) \geq (1+\lambda r^2)^{\frac{1}{2}} - g(x) - M_\lambda)$$
$$\leq E[\sup_{0\leq t\leq 1} M_t^g] exp(-(1+\lambda r^2)^{\frac{1}{2}} + g(x) + M_\lambda) \quad (4.36)$$

Choosing λ large enough and using the martingale inequality, we obtain (4.30).

Proposition 4.3 *Assume (2.2) with* $\lambda = 0$ *and also (H.5). Then for any* $l > 0$,

$$E[exp(l||Y||_{L^2([0,1]\to V)})] < \infty.$$

Proof. Let $Z_\lambda = \int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}}||Y_s||_V^2 ds$. We first prove that

$$P(Z_\lambda > r) \leq exp(-\alpha\lambda r + M_\lambda + (1+\lambda|x|_H^2)^{\frac{1}{2}}), \quad (4.37)$$

where M_λ is the same constant as in (4.35). For $\lambda > 0$, define $g(y) = (1+\lambda|y|_H^2)^{\frac{1}{2}}$. Using (2.2) we have

$$< -Ay, g'(y) >= \lambda(1+\lambda|y|_H^2)^{-\frac{1}{2}} < -Ay, y >$$
$$\leq -\alpha\lambda(1+\lambda|y|_H^2)^{-\frac{1}{2}}||y||_V^2 \quad (4.38)$$

So the estimate in (4.35) can be strengthened as follows:

$$h(y) \le -\alpha\lambda(1+\lambda|y|_H^2)^{-\frac{1}{2}}||y||_V^2 + M_\lambda \tag{4.39}$$

Let $M_t^g, t \ge 0$ be defined as in the proof of Proposition 4.2. By (4.39), we have

$$\begin{aligned}
P(Z_\lambda > r) &= P(\alpha\lambda \int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}}||Y_s||_V^2 ds > \alpha\lambda r) \\
&\le P(g(Y_1) + \alpha\lambda \int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}}||Y_s||_V^2 ds > \alpha\lambda r) \\
&= P(g(Y_1) - g(x) - \int_0^1 h(Y_s)ds + g(x) + \int_0^1 h(Y_s)ds \\
&\quad + \alpha\lambda \int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}}||Y_s||_V^2 ds > \alpha\lambda r) \\
&\le P(g(Y_1) - g(x) - \int_0^1 h(Y_s)ds) + g(x) + M_\lambda > \alpha\lambda r) \\
&= P(g(Y_1) - g(x) - \int_0^1 h(Y_s)ds) > \alpha\lambda r - g(x) - M_\lambda) \\
&\le E[M_1^g] exp(-\alpha\lambda r + g(x) + M_\lambda) \\
&\le exp(-\alpha\lambda r + g(x) + M_\lambda)
\end{aligned} \tag{4.40}$$

which proves (4.37). It is easy to see from (4.37) that for any $l > 0$, one can choose $\lambda_l > 0$ large enough so that $E[exp(lZ_{\lambda_l})] < \infty$.Now for every $\lambda > 0$,

$$\begin{aligned}
||Y||_{L^2([0,1]\to V)} &= \left(\int_0^1 ||Y_s||_V^2 ds\right)^{\frac{1}{2}} \\
&\le \left(\int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}}||Y_s||_V^2 ds\right)^{\frac{1}{2}} \left(1+\lambda \sup_{0\le s\le 1}|Y_s|_H^2\right)^{\frac{1}{4}} \\
&\le \frac{1}{2}Z_\lambda + \frac{1}{2}\left(1+\lambda \sup_{0\le s\le 1}|Y_s|_H^2\right)^{\frac{1}{2}}
\end{aligned} \tag{4.41}$$

By Hölder inequality, for $l > 0$,

$$\begin{aligned}
& E[exp(l||Y||_{L^2([0,1]\to V)})] \\
&\le E[exp(\frac{1}{2}lZ_\lambda) exp(\frac{1}{2}l\left(1+\lambda \sup_{0\le s\le 1}|Y_s|_H^2\right)^{\frac{1}{2}})] \\
&\le \left(E[exp(lZ_\lambda)]\right)^{\frac{1}{2}} \times \left(E[exp(l\left(1+\lambda \sup_{0\le s\le 1}|Y_s|_H^2\right)^{\frac{1}{2}})]\right)^{\frac{1}{2}}
\end{aligned} \tag{4.42}$$

According to (4.37), we can choose λ such that $E[exp(lZ_\lambda)] < \infty$. On the other hand $E[exp(l\left(1+\lambda \sup_{0\le s\le 1}|Y_s|_H^2\right)^{\frac{1}{2}})] < \infty$ for all $\lambda > 0$ according to Proposition 4.2. So we conclude that $E[exp(l||Y||_{L^2([0,1]\to V)})] < \infty$ proving the Propiosition.

From now on we confine ourself to the special situation where $f(y,x) = f(x)$ is independent of y. Let $f^m(x), m \ge 1$ denote a sequence of Borel measurable functions in $L^2(X \to H, \nu)$. Consider the stochastic evolution equations with additive noise:

$$Y_t^n = x - \int_0^t AY_s^n ds + \frac{1}{n}\int_0^t \int_X f(x)\tilde{N}_n(ds,dx) \tag{4.43}$$

$$Y_t^{n,m} = x - \int_0^t AY_s^{n,m} ds + \frac{1}{n}\int_0^t \int_X f^m(x)\tilde{N}_n(ds,dx) \tag{4.44}$$

where $\tilde{N}_n(ds,dx)$ denotes the compensated Poisson measure with intensity measure $n\nu$.

Lemma 4.4 *If $a_m := \sup_x |f^m(x) - f(x)|_H \to 0$ as $m \to \infty$, then for any $\delta > 0$*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} logP(\sup_{0\le t\le 1} |Y_t^{n,m} - Y_t^n|_H > \delta) = -\infty \tag{4.45}$$

Proof. Set $X_t^{n,m} = \frac{n}{a_m}(Y_t^{n,m} - Y_t)$. Then it is seen that

$$X_t^{n,m} = -\int_0^t AX_s^{n,m} ds + \int_0^t \int_X \frac{1}{a_m}(f^m(x) - f(x))\tilde{N}_n(ds,dx)$$

Let $h(y)$ be the function defined in the proof of Proposition 4.2 with f replaced by $\frac{1}{a_m}(f^m - f)$. Similarly as in the proof of Proposition 4.2 we have

$$h(y) \le c\int_X exp(c\frac{1}{a_m}|f^m(x) - f(x)|_H)(\frac{1}{a_m}|f^m(x) - f(x)|_H)^2 n\nu(dx) \le cn$$

Applying the estimate (4.36) we get that for $r > 0$,

$$P(\sup_{0\le t\le 1} |X_t^{n,m}|_H > r) \le exp(-(1+\lambda r^2)^{\frac{1}{2}} + 1 + cn)$$

This gives that

$$P(\sup_{0\le t\le 1} |Y_t^{n,m} - Y_t^n|_H > \delta) = P(\sup_{0\le t\le 1} |X_t^{n,m}|_H > \frac{n}{a_m}\delta)$$

$$\le exp(-(1+\lambda(\frac{n}{a_m}\delta)^2)^{\frac{1}{2}} + 1 + cn)$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} logP(\sup_{0\le t\le 1} |Y_t^{n,m} - Y_t^n|_H > \delta)$$

$$\leq \limsup_{n\to\infty} \frac{1}{n}[-(1+\lambda(\frac{n}{a_m}\delta)^2)^{\frac{1}{2}}+1+cn]$$

$$\leq -\lambda\frac{\delta}{a_m}+c$$

Taking $m \to \infty$ we get that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} logP(\sup_{0\leq t\leq 1} |Y_t^{n,m} - Y_t^n|_H > \delta) = -\infty.$$

For $g \in D([0,1] \to V)$, define $\phi(g) \in D([0,1] \to H) \cap L^2([0,1] \to V)$ as the solution to the following equation:

$$\phi_t(g) = x - \int_0^t A\phi_s(g)ds + g(t) \tag{4.46}$$

Lemma 4.5 *The mapping ϕ from $D([0,1] \to V)$ into $\phi(g) \in D([0,1] \to H) \cap L^2([0,1] \to V)$ is continuous in the topology of uniform convergence.*

Proof. Let $v_t(g) = \phi_t(g) - g(t)$. it is easy to see that $v(g)$ satisfies the equation:

$$v_t(g) = x - \int_0^t Av_s(g)ds - \int_0^t Ag(s)ds$$

It suffices to show that the mapping

$$v(\cdot) : D([0,1] \to V) \to D([0,1] \to H) \cap L^2([0,1] \to V)$$

is continuous. Taking $\beta < \alpha$, where α is the constant in (2.2), by chain rule and using (2.2),

$$|v_t(g_n) - v_t(g)|_H^2 = -2\int_0^t < A(v_s(g_n) - v_s(g)), v_s(g_n) - v_s(g) > ds$$

$$-2\int_0^t < A(g_n - g)(s), v_s(g_n) - v_s(g) > ds$$

$$\leq -\alpha \int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds + \lambda \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds$$

$$+2\int_0^t ||v_s(g_n) - v_s(g)||_V ||A(g_n - g)(s)||_{V^*} ds$$

$$\leq -\alpha \int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds + \lambda \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds$$

$$+\beta \int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds + C_\beta \int_0^t ||A(g_n - g)(s)||_{V^*}^2 ds$$

This gives that

$$|v_t(g_n) - v_t(g)|_H^2 + (\alpha - \beta)\int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds$$

$$\leq \lambda \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds + C_\beta ||A|| \int_0^t ||g_n - g(s)||_V^2 ds$$

Applying Gronwall's inequality it is easy to deduce that the mapping $v(\cdot)$ is continuous, which completes the proof.

Let $f \in L^2(X \to V, \nu)$. For $l \in V^*$, define $F(l) = \int_X [exp(< f(x), l >) - 1- < f(x), l >]\nu(dx)$. Set, for $z \in V$,

$$F^*(z) = \sup_{l \in V^*}[< z, l > -F(l)] \tag{4.47}$$

Let Y^n be the solution of (4.43) with $f \in L^2(X \to V, \nu)$. Let μ_n denote the law of Y^n on $D([0,1] \to H)$.

Proposition 4.6 *$\{\mu_n, n \geq 1\}$ satisfies a large deviation principle on $D([0,1] \to H)$ with a rate functional I defined as follows: let $k \in D([0,1] \to H)$, if $g(t) = k(t) - x + \int_0^t Ak(s)ds, t \geq 0$ belongs to $D([0,1] \to V)$ and $g' \in L^1([0,1] \to V)$, $I(k) = \int_0^1 F^*(g'(s))ds$; otherwise $I(k) = \infty$.*

Proof. Let ν_n be the law of $\frac{1}{n}\int_0^{\cdot} \int_X f(x)\tilde{N}_n(ds, dx)$ on $D([0,1] \to V)$. It is proved in [A2] that $\{\nu_n, n \geq 1\}$ satisfies a large deviation principle on $D([0,1] \to V)$ with a rate functional I_0 given as follows: if $g \in D([0,1] \to V)$ and $g' \in L^1([0,1] \to V)$, $I_0(g) = \int_0^1 F^*(g'(s))ds$; otherwise $I(g) = \infty$. By lemma 4.5, we know that μ_n is the image measure of ν_n under the continuous map ϕ. The Proposition 4.6 follows now from the contraction principle.

Now assume $f \in L^2(X \to H, \nu)$. Let μ_n be the law of the solution Y^n in equation (4.43). Combing Lemma 4.4, Proposition 4.6 and Theorem 4.2.16 in [DZ] we obtain the following proposition.

Proposition 4.7 *If there exists $f^m \in L^2(X \to V, \nu), m \geq 1$ such that $a_m = \sup_x |f^m(x) - f(x)|_H \to 0$ as $m \to \infty$, then $\{\mu_n, n \geq 1\}$ satisfies a weak large deviation principle on $D([0,1] \to H)$ with rate function given by*

$$I(k) := \sup_{\delta > 0} \liminf_{m \to \infty} \inf_{z \in B_{k,\delta}} I_m(z),$$

where $B_{k,\delta} = \{z; \sup_{0 \leq s \leq 1} |z_s - k_s|_H < \delta\}$, I_m is the rate function defined in Proposition 4.6 with f replaced by f^m.

Let $f \in L^2(X \to H, \nu)$. For $l \in H$, define $F(l) = \int_X [exp(< f(x), l >) - 1- < f(x), l >]\nu(dx)$. Set, for $z \in H$,

$$F^*(z) = \sup_{l \in H}[< z, l > -F(l)] \tag{4.48}$$

Define a functional $I_0(\cdot)$ on $D([0,1] \to H)$ as follows: if $g \in D([0,1] \to H)$ and $g' \in L^1([0,1] \to H)$, $I_0(g) = \int_0^1 F^*(g'(s))ds$; otherwise $I(g) = \infty$.

Lemma 4.8 *Let $a > 0$. Then $\mathcal{G} = \{|g'|; I_0(g) \leq a\}$ is uniformly integrable on the probability space $([0,1], \mathcal{B}, m)$, where m denotes the Lebesgue measure.*

Proof. $\mathcal{G}$ is uniformly integrable if and only if

(i) $\mathcal{G}$ is equi-absolutely continuous, i.e., for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $m(A) < \delta$ implies $\int_A |g'|_H m(ds) < \varepsilon$ for all $g \in \mathcal{G}$.

(ii) $\sup_{g\in\mathcal{G}} \int_0^1 |g'|_H m(ds) < \infty$.

We will modify the proof of Theorem 3.1 in [A2] to get (i) and (ii). Let $a_i, b_i, i = 1, ...n$ be any given numbers such that $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq 1$. For any partition $\tau^i = \{t_0^i = a_i < t_1^i < \cdots < t_{i_m}^i = b_i\}$ of $[a_i, b_i]$ and any $\eta_k^i \in H$ with $|\eta_k^i|_H \leq 1$, define $\beta \in M([0,1], H)$ by

$$\beta = \sum_{i=1}^{n} \sum_{k=0}^{i_m} \eta_k^i (\delta_{t_k^i} - \delta_{t_{k-1}^i}),$$

where $M([0,1], H)$ denotes the space of $H-$ valued vector measures on $([0,1], \mathcal{B})$. Let μ be the law of $\int_0^1 \int_X f(x) \tilde{N}(ds, dx)$ on H. Denote by $\hat{\mu}$ the characteristic functional of μ. Then,

$$\int_0^1 log\hat{\mu}(\beta(s,1])ds = \sum_{i=1}^{n} \sum_{k=0}^{i_m} log\hat{\mu}(\eta_k^i)(t_k^i - t_{k-1}^i)$$

Let $\rho > 0$. By the characterization of I_0 in [A2], for $g \in \mathcal{G}$, we have

$$\rho \int_0^1 < g, d\beta >= \rho \sum_{i=1}^{n} \sum_{k=0}^{i_m} < g(t_k^i) - g(t_{k-1}^i), \eta_k^i >$$

$$\leq \int_0^1 log\hat{\mu}(\rho\beta(s,1])ds + I_0(g)$$

$$\leq \sup_{i,k} |log\hat{\mu}(\rho\eta_k^i)| \sum_{i=1}^{n} \sum_{k=0}^{i_m} (t_k^i - t_{k-1}^i) + I_0(g)$$

$$\leq log(\int_H exp(\rho|x|_H)\mu(dx)) \sum_{i=1}^{n} (b_i - a_i) + a \tag{4.49}$$

Taking superemum in (4.48) over all possible $\eta_k^i \in H$ with $|\eta_k^i|_H \leq 1$ we get

$$\sum_{i=1}^{n} \sum_{k=0}^{i_m} |g(t_k^i) - g(t_{k-1}^i)|_H \leq \rho^{-1} log(\int_H exp(\rho|x|_H)\mu(dx)) \sum_{i=1}^{n} (b_i - a_i) + \rho^{-1} a \tag{4.50}$$

Let $V(g)[a,b]$ denote the total variation of g over the interval $[a,b]$. Taking superemum in (4.49) over all possible partitions we obtain

$$\sum_{i=1}^{n} V(g)[a_i, b_i] = \sum_{i=1}^{n} \int_{a_i}^{b_i} |g'(s)|_H ds = \int_{\cup_{i=1}^n (a_i, b_i)} |g'(s)|_H ds$$

$$\leq \rho^{-1}log(\int_H exp(\rho|x|_H)\mu(dx))\sum_{i=1}^{n}(b_i - a_i) + \rho^{-1}a \tag{4.51}$$

For every $\varepsilon > 0$, choose first ρ_0 large enough such that $\rho_0^{-1}a \leq \frac{\varepsilon}{2}$. Set $\delta = \frac{1}{3}[\rho_0^{-1}log(\int_H exp(\rho|x|_H)\mu(dx))]^{-1}\varepsilon$. If $\cup_{i=1}^n(a_i, b_i) \subset [0,1]$ with $m(\cup_{i=1}^n(a_i, b_i)) < \delta$, by (4.50) we have $\int_{\cup_{i=1}^n(a_i,b_i)} |g'(s)|_H ds < \varepsilon$. for all $g \in \mathcal{G}$. This implies (i). Take particularly $a_1 = 0, b_1 = 1$ in the above proof to see that (ii) also holds.

Let $T_t, t \geq 0$ denote the semigroup generated by $-A$. For $g \in L^1([0,1] \to H)$, define the operator

$$Rg(t) = \int_0^t T_{t-s}g(s)ds, \quad t \geq 0,$$

which is the mild solution of the equation:

$$\phi(t) = -\int_0^t A\phi(s)ds + \int_0^t g(s)ds.$$

Proposition 4.9 *Assume that $T_t, t > 0$ are compact operators. If $\mathcal{G} \subset L^1([0,1] \to H)$ is uniformly integrable, then $\mathcal{S} = R(\mathcal{G})$ is relatively compact in $C([0,1] \to H)$.*

Proof. The proof is a modification of the proof of Proposition 8.4 in [PZ]. According to the infinite dimensional version of the Ascoli-Arzela theorem we need to show

(i) for every $t \in [0,1]$ the set $\{Rg(t); g \in \mathcal{G}\}$ is relatively compact in H;

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \leq s \leq t \leq 1$,$t - s \leq \delta$,

$$|Rg(t) - Rg(s)|_H \leq \varepsilon \quad \text{for all} \quad g \in \mathcal{G} \tag{4.52}$$

To prove (i), fix $t \in (0,1]$ and define for $\varepsilon > 0$ $R^\varepsilon g(t) = \int_0^{t-\varepsilon} T_{t-s}g(s)ds$. Since

$$R^\varepsilon g(t) = T_\varepsilon \int_0^{t-\varepsilon} T_{t-\varepsilon-s}g(s)ds$$

and $T_\varepsilon, \varepsilon > 0$ is compact, $\{R^\varepsilon g(t), g \in \mathcal{G}\}$ is relatively compact in H for every $\varepsilon > 0$. On the other hand,

$$|R^\varepsilon g(t) - Rg(t)|_H \leq M \int_{t-\varepsilon}^t |g(s)|_H ds, \tag{4.53}$$

where $M = sup_{t\in[0,1]}||T_t||$. Since $\mathcal{G}$ is uniformly integrable, (4.52)implies that

$$lim_{\varepsilon\to 0} \sup_{g\in\mathcal{G}} |R^\varepsilon g(t) - Rg(t)|_H = 0$$

which further implies that $\{Rg(t); g \in \mathcal{G}\}$ is also relatively compact. Let us now prove (ii). For $0 \leq t \leq t+u \leq 1$, we have

$$|Rg(t+u) - Rg(t)|_H$$

$$\leq \int_0^t ||T_{t+u-s} - T_{t-s}|||g(s)|_H ds + \int_t^{t+u} ||T_{t+u-s}|||g(s)|_H ds$$
$$:= I_g^u + II_g^u$$

By the uniform integrability of $\mathcal{G}$, it is clear that

$$\lim_{u\to 0} \sup_{g\in\mathcal{G}} II_g^u \leq M \lim_{u\to 0} \sup_{g\in\mathcal{G}} \int_t^{t+u} |g(s)|_H ds = 0$$

Since the semigroup T. is compact, $||T_{t+u-s} - T_{t-s}|| \to 0$ for any $t - s > 0$ as $u \to 0$. By the dominated convergence theorem, we have that

$$\lim_{u\to 0} \int_0^t ||T_{t+u-s} - T_{t-s}||ds = 0 \tag{4.54}$$

Now we prove

$$\lim_{u\to 0} \sup_{g\in\mathcal{G}} I_g^u = 0 \tag{4.55}$$

For given $\varepsilon > 0$, sine $\mathcal{G}$ is uniformly integrable one choose $\rho > 0$ such that $2M \int_{|g|>\rho} |g(s)|_H ds < \frac{\varepsilon}{2}$ for all $g \in \mathcal{G}$. For the fixed $\rho > 0$ above, there exists $\delta > 0$ such that $u \leq \delta$ implies that

$$\rho \int_0^t ||T_{t+u-s} - T_{t-s}||ds \leq \frac{\varepsilon}{2}$$

for all $t \in [0,1]$. Therefore if $u \leq \delta$, for all $g \in \mathcal{G}$, $t \in [0,1]$,

$$I_g^u = \int_{|g|>\rho} ||T_{t+u-s} - T_{t-s}|||g(s)|_H ds + \int_{|g|\leq\rho} ||T_{t+u-s} - T_{t-s}|||g(s)|_H ds$$
$$\leq 2M \int_{|g|>\rho} |g(s)|_H ds + \rho \int_0^t ||T_{t+u-s} - T_{t-s}||ds$$
$$\leq \varepsilon \tag{4.56}$$

This proves (ii), hence the Proposition.

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