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Doney, R. A. and Maller, R. A.

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Manchester Institute for Mathematical Sciences

School of Mathematics

The University of Manchester

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Abstract

We give necessary and sufficient conditions for the almost sure (a.s.) relative stability of the overshoot of a random walk when it exits from a two-sided symmetric region with curved boundaries. The boundaries are of power-law type, $\pm rn^b$, $r > 0$, $n = 1, 2, \cdots$, where $0 \leq b < 1$, $b \neq 1/2$. In these cases, the a.s. stability occurs if and only if the mean step length of the random walk is finite and nonzero, or the step length has a finite variance and mean zero.

1 Introduction

This paper continues the investigation begun in [3] of the asymptotic behaviour of the overshoot of a random walk when it exits from a two-sided symmetric region with curved boundaries of power-law form. In what follows, $S = (S_n, n \geq 0)$, $S_0 = 0$, will denote the random walk with step size $X_n = S_n - S_{n-1}$ (sometimes we write $X(n)$), which is assumed to be non-degenerate with distribution function F, and b will be a constant in the range $[0, 1)$. We define the exit time

$$T_r = \min\{n \geq 1 : |S_n| > rn^b\}, \ r > 0 \tag{1.1}$$

(with $T_r = \infty$ if $|S_n| \leq rn^b$ for all $n \geq 1$), and the overshoot by

$$O_r := |S_{T_r}| - rT_r^b. \tag{1.2}$$

Our aim is to find necessary and sufficient conditions for the overshoot to be almost surely (a.s.) asymptotically small as compared to the boundary, viz, for

$$\frac{O_r}{rT_r^b} = \frac{1}{r}\left(\frac{|S_{T_r}|}{T_r^b} - r\right) \overset{a.s.}{\to} 0 \text{ as } r \to \infty; \tag{1.3}$$

we refer to this as *almost sure relative stability of the overshoot.*

For the case $0 \leq b < 1, b \neq 1/2$, it was shown in [3] and [4] (see also [5] for the case $0 < b < 1/2$ and [9] and [6] for the case $b = 0$) that the *in probability* version of (1.3) can occur in two and only two situations; S has to be relatively stable, in the sense that $S_n/c_n \overset{P}{\to} \pm 1$ for some positive norming sequence c_n, or else, if $b < 1/2$, S must belong to the domain of attraction of the Normal distribution, without centering. (We write $S \in RS$ or $S \in D_0(N)$.) Moreover in case $b = 0$ it was also shown in [6] that a necessary and sufficient condition (NASC) for the a.s. result (1.3) is:

$$\text{either } EX^2 < \infty \text{ and } EX = 0, \text{ or } 0 < |EX| \leq E|X| < \infty. \tag{1.4}$$

In the present paper we generalise this result to the case of curved (power law) boundaries. This is not a straightforward exercise since the techniques required to deal with almost sure results for non-constant boundaries have not previously been worked out, and it was not at all obvious that we could expect such a clearcut equivalence as in (1.4). (Nevertheless, our methods rely on some basic relationships worked out in [3] and [4].) As it turns out, we do get a very easily interpretable answer. The dichotomy in (1.4) essentially extends to our situation, and its simplicity augers well for possible applications of our result in statistics and elsewhere.

As another application, we provide in Proposition 5 below an alternative derivation of a key result in [10] concerning the limsup behaviour of the random walk.

Theorem 1 *If $0 \leq b < 1$ and $b \neq \frac{1}{2}$, the following are equivalent:*

$$(i) \qquad \begin{cases} (1.4), & \text{for } 0 \leq b < 1/2, \\ 0 < |EX| \leq E|X| < \infty, & \text{for } 1/2 < b < 1; \end{cases} \qquad (1.5)$$

$$(ii) \qquad T_r \overset{a.s.}{<} \infty \text{ for all } r > 0 \text{ and } \frac{O_r}{rT_r^b} \overset{a.s.}{\to} 0 \text{ as } r \to \infty; \qquad (1.6)$$

and

$$(iii) \quad T_r \overset{a.s.}{<} \infty \text{ for all } r > 0 \text{ and } \limsup_{r \to \infty} \frac{|S_{T_r}|}{rT_r^b} \overset{a.s.}{<} L \text{ for some } L \in (1, \infty). \quad (1.7)$$

Remark 2 *From Theorem 3.1 in [6] we can read off two more probabilistic conditions which are also equivalent to (1.4) and hence to (1.5)–(1.7) for the values of b specified; with $S_n^* = \max_{1 \leq r \leq n} |S_r|$ and $|X_n^{(1)}| = \max_{1 \leq r \leq n} |X_r|$ they are*

$$\lim_{n \to \infty} \frac{S_n^*}{|X_n|} \overset{a.s.}{=} \infty, \qquad (1.8)$$

and

$$\lim_{n \to \infty} \frac{S_n^*}{|X_n^{(1)}|} \overset{a.s.}{=} \infty. \qquad (1.9)$$

We will also need an analytic condition equivalent to (1.4) which can be found in Lemma 4.2 of [6]; to state it we need some notation. We write X for a generic step in the random walk, and put, for $x > 0$,

$$\begin{aligned} G(x) &= P(|X| > x), & U(x) &= \int_0^x 2yG(y)dy, \\ A(x) &= E((X \wedge x) \vee (-x)), & k(x) &= x^{-2}U(x) + x^{-1}|A(x)|. \end{aligned} \qquad (1.10)$$

Then the condition is

$$I = \int_{[0,\infty)} \frac{x^2 |dG(x)|}{U(x) + x|A(x)|} = \int_{[0,\infty)} \frac{|dG(x)|}{k(x)} < \infty. \qquad (1.11)$$

(We remark that [3], [4] and [12] use a slightly different but closely related function $h(\cdot)$ rather than $k(\cdot)$ in (1.11) and elsewhere; these are equivalent in our context, as pointed out in [6].)

Proof of Theorem 1: The result is known from [6] for $b = 0$ (note that we always have $T_r < \infty$ a.s. for all $r > 0$ in this case, as long as X is not degenerate at 0), so take $0 < b < 1$. First assume (1.5). Then for $0 < b \leq 1/2$, $T_r \overset{a.s.}{<} \infty$ for all $r > 0$ follows from the law of the iterated logarithm or from the strong law of large numbers, according as $EX^2 < \infty$ and $EX = 0$, or $0 < |EX| \leq E|X| < \infty$; while for $1/2 < b < 1$ it follows from the strong law of large numbers, under the assumption that $0 < |EX| \leq E|X| < \infty$. Also (1.5) implies (1.4) which implies (1.8). So, replacing n by T_r, we can argue that

$$\frac{|X_{T_r}|}{|S_{T_r}|} \overset{a.s.}{\to} 0 \text{ as } r \to \infty,$$

and hence, because $|S_{T_r}| \leq rT_r^b + |X_{T_r}|$, we have for large enough r

$$0 < \frac{1}{r}\left(\frac{|S_{T_r}|}{T_r^b} - r\right) \leq \frac{|X_{T_r}|}{rT_r^b} \leq \frac{|X_{T_r}|}{|S_{T_r}| - |X_{T_r}|} \overset{a.s.}{\to} 0,$$

which is (1.6). Clearly this implies (1.7), and our major task is to show that (1.7) implies (1.5), which we will do by establishing (1.11), and then arguing that this gives (1.5) for the values of b specified.

So, let (1.7) hold. First note that, since $|S_{T_{r-1}}| \leq rT_r^b$, a consequence of (1.7) is that

$$\limsup_{r\to\infty} \frac{|X_{T_r}|}{rT_r^b} \leq \limsup_{r\to\infty} \frac{|S_{T_r}|}{rT_r^b} + 1 \overset{a.s.}{<} L + 1 := K, \tag{1.12}$$

where of course $K \in (2, \infty)$. Moreover (1.7) implies the corresponding in probability condition, so by Theorems 2.4 and 2.5 of [4] (see also Theorems 2.1–2.3 of [3]) we know that either $0 < b < 1$, $b \neq 1/2$, and $S \in RS$, or $0 < b < 1/2$ and $S \in D_0(N)$. In the first case it is known that either $A(x)$ is positive for all large enough x, or negative for all large enough x, that $|A(x)|$ is slowly varying at ∞, and that $U(x) = o(x|A(x)|)$ as $x \to \infty$; and thus that

$$k(x) = x^{-2}U(x) + x^{-1}|A(x)| \backsim x^{-1}|A(x)|.$$

In the second case it is known that U is slowly varying at ∞, and that $x|A(x)| = o(U(x))$ as $x \to \infty$; and thus that $k(x) \backsim x^{-2}U(x)$. Furthermore, in both cases,

$$\frac{G(x)}{k(x)} \to 0 \text{ as } x \to \infty. \tag{1.13}$$

So from now on we can assume that $k(\cdot) \in RV(-1)$ or that $0 < b < 1/2$ and $k(\cdot) \in RV(-2)$. (Here $RV(\alpha)$ is the class of positive functions which are regularly varying at ∞ with index α; see [1]). This allows us to rewrite (1.11) in a simpler form: we have

$$J := \int_1^\infty \frac{G(x)dx}{xk(x)} = \int_1^\infty |dG(x)| \int_1^x \frac{dy}{yk(y)},$$

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and since the inner integral is asymptotic to $c/k(x)$ as $x \to \infty$ (here and throughout $c, c_1, c_2, \cdots$ denote generic positive constants whose values can change from one line to the next), we see that $J < \infty$ is equivalent to (1.11). Next note that for $r > 0$ and $x > 2r$

$$P\left(\frac{|X_{T_r}|}{T_r^b} > x\right) = \sum_{i \geq 1} P(T_r = i, |X_i| > xi^b) = \sum_{i \geq 1} G(xi^b) P(T_r \geq i). \quad (1.14)$$

To exploit this, we need some results about $P(T_r \geq i)$ and $P(T_r \leq i)$; these are proved in [3] and [4] by establishing extensions of the classical results for $b = 0$ in [12]. (Γ_r is defined using the $h(\cdot)$ function for $0 < b < 1/2$ in [3] and extended to $0 < b < 1$ in [4], Section 4. Again we can equivalently use the $k(\cdot)$ function as in (1.15) and [11].)

Proposition 3 *If $0 < b < 1$ we can define a function Γ by*

$$\Gamma_r = \inf\{x : xk(rx^b) \geq 1\}, \ r > 0. \quad (1.15)$$

Then $\Gamma_r \to \infty$ as $r \to \infty$,

$$\Gamma_r k(r\Gamma_r^b) = 1, \ r > 0, \quad (1.16)$$

there is a $\delta > 0$ with

$$P(T_r \leq \delta\Gamma_r) \leq 1/2, \quad (1.17)$$

and for any $a > 0$

$$\sum_{m=1}^{\infty} m^{a-1} P(T_r \geq m) \approx \Gamma_r^a. \quad (1.18)$$

[Here and elsewhere $\approx$ means that the ratio of the two sides is in $[c_1, c_2]$, for some $0 < c_1 < c_2 < \infty$, for all sufficiently large r.]

Another useful fact, valid in all cases, is that

$$\frac{x^2}{y^2} \leq \frac{k(y)}{k(x)} \leq 3, \ y \geq x > 0. \quad (1.19)$$

In our particular situation, we have extra information (see [1], pp. 28–29). As the inverse of a function in $RV(1 - b)$ or $RV(1 - 2b)$, we have $\Gamma \in RV(1/1 - b)$ when $k(\cdot) \in RV(-1)$, and $\Gamma \in RV(1/1 - 2b)$ when $k(\cdot) \in RV(-2)$. (Recall in this latter case that $b < 1/2$.)

Using (1.17) in (1.14) gives, for $x > 2r > 0$,

$$P\left(\frac{|X_{T_r}|}{T_r^b} > x\right) \geq \frac{1}{2} \sum_{k \leq \delta\Gamma_r} G(xk^b) \geq \frac{\delta\Gamma_r}{2} G(x\delta^b\Gamma_r^b)$$

$$= \frac{\delta G(\delta^b x\Gamma_r^b)}{2k(r\Gamma_r^b)}, \quad (1.20)$$

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where we have also used (1.16). Now write $g(r) = r\Gamma_r^b$, let g^{-1} denote an asymptotic inverse of g; that is, $g^{-1}(x) = \inf\{r : g(r) > x\}$, for large x, [1], p. 28, and put

$$r(n) = g^{-1}(2^n), \ n \geq 1.$$

Then $g \in RV(1/1-b)$ or $g \in RV((1-b)/(1-2b))$, and we have $g^{-1} \in RV(1-b)$ or $RV((1-2b)/(1-b))$. Also, g^{-1} is nondecreasing. So we can fix n_0 such that $r(n)$ is strictly increasing for $n \geq n_0$ and

$$\sup_{x \geq 2^{n_0}} |\frac{g(g^{-1}(x))}{x} - 1| \leq \frac{1}{2}.$$

We will also write $r(\cdot)$ for a continuous and increasing interpolant of r on $[n_0, \infty)$, so that

$$\frac{1}{2} \leq \frac{r(x)\Gamma_{r(x)}^b}{2^x} \leq \frac{3}{2} \text{ for all } x \geq n_0,$$

and define the event

$$E_n := \left\{ \frac{|X_{T_{r(n)}}|}{T_{r(n)}^b} > 2Kr(n) \right\}.$$

Putting $r = r(n)$ and $x = 2Kr(n)/\delta^b$, $K > \delta^b$, in (1.20) gives, for $n \geq n_0$,

$$
\begin{aligned}
P(E_n) &= P\left(\frac{|X_{T_{r(n)}}|}{T_{r(n)}^b} > 2Kr(n) \right) \geq \frac{\delta G(2Kr(n)\Gamma_{r(n)}^b)}{2k(r(n)\Gamma_{r(n)}^b)} \\
&= \frac{\delta G(2Kg(r(n)))}{2k(g(r(n)))} \geq \frac{c_1 G(4K2^n)}{k(2^n)},
\end{aligned}
\tag{1.21}
$$

where $c_1 > 0$ and we have used (1.19). Hence

$$
\begin{aligned}
\sum_{m \geq n \geq n_0} P(E_n) &\geq c_1 \sum_{m \geq n \geq n_0} \frac{G(4K2^n)}{k(2^n)} \geq c_2 \sum_{m \geq n \geq n_0} \sum_{2^n \leq j < 2^{n+1}} \frac{G(4Kj)}{jk(j)} \\
&= c_2 \sum_{2^{m+1} > j \geq 2^{n_0}} \frac{G(4Kj)}{jk(j)} \geq c_3 \int_{4K2^{n_0}}^{8K2^m} \frac{G(x)dx}{xk(x)}.
\end{aligned}
\tag{1.22}
$$

So if we can show that $\sum_{n \geq n_0} P(E_n)$ is finite, this would give $J < \infty$, thus (1.11), hence (1.4). Since we already know that either $0 < b < 1$, $b \neq 1/2$, and $S \in RS$, or $0 < b < 1/2$ and $S \in D_0(N)$, we deduce (1.5).

We reach this conclusion by contradiction, showing that the assumption

$$\sum_{n \geq n_0} P(E_n) = \infty \tag{1.23}$$

leads to $P(E_n \ i.o.) > 0$, which of course would contradict (1.7). So now we assume (1.23) and aim to apply the generalized Borel-Cantelli lemma in Spitzer ([13], p. 317); to do this we need to establish that

$$\limsup_{n \to \infty} \frac{\sum_{i=1}^n \sum_{j=i+1}^n P(E_i \cap E_j)}{\left(\sum_1^n P(E_i) \right)^2} < \infty. \tag{1.24}$$

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Establishing (1.17) requires some rather intricate though not routine calculations. So as not to interrupt the main points we relegate this to the Appendix, and complete this section with some comments and the application to [10]. ∎

Remark 4 *The argument in the first paragraph of the proof of Theorem 1 in fact will show that* (1.4) *implies a.s. relative stability of the overshoot whenever $b \geq 0$ and $T_r < \infty$ a.s. for all $r > 0$; thus, when $0 < b < 1$ and* (1.4) *holds, or when $b \geq 1$ and $E|X|^{1/b} = \infty$. So we have sufficient conditions for the a.s. relative stability in all cases. As discussed in [3] and [4], for the converses, the cases $b = 1/2$, $b = 1$ clearly have special features and the case $b > 1$ seems to present special difficulties.*

Finally we point our that our result leads to an alternative proof of the following result, due to Kesten and Maller in [10].

Proposition 5 *Assume that $\frac{1}{2} < b < 1, E|X| < \infty, EX = 0, E(X^+)^{\frac{1}{b}} = \infty$, and*

$$X \overset{a.s.}{\geq} -c, \text{ for some } 0 < c < \infty. \tag{1.25}$$

Then

$$\limsup_{n \to \infty} \frac{S_n}{n^b} \overset{a.s.}{=} \infty. \tag{1.26}$$

Proof: Because we assume that $E|X|^{\frac{1}{b}} = \infty$, it follows from the Markinkiewicz-Zygmund strong law of large numbers (see [2], p. 125) that

$$\limsup_{n \to \infty} \frac{|S_n|}{n^b} \overset{a.s.}{=} \infty,$$

so if (1.26) failed, we would have

$$\limsup_{n \to \infty} \frac{S_n}{n^b} \overset{a.s.}{<} \infty \text{ and } \liminf_{n \to \infty} \frac{S_n}{n^b} \overset{a.s.}{=} -\infty.$$

This would then imply that

$$\lim_{r \to \infty} X_{T_r} \overset{a.s.}{<} 0.$$

In view of (1.25) this would imply that $|X_{T_r}| = -X_{T_r} = O(rT_r^b)$, a.s., hence $|S_{T_r}| = O(rT_r^b)$, a.s., as $r \to \infty$, and so (1.7) would hold, giving a.s. relatively stability of the overshoot. But this is equivalent to (1.5), contradicting our assumptions, and proving our claim. ∎

Remark 6 *Actually it is shown in [10] that the Proposition holds without the assumption (1.25). But the result of Proposition 5 as we state it provides a key step in the proof in [10].*

2 Appendix: Proof of (1.24)

In the case $b = 0$ it is possible to find an upper bound for $P(E_n)$ which differs from the lower bound only by a constant multiple; when $b > 0$ things are a little more complicated. We proceed as follows. From (1.14) and (1.18), for r large enough,

$$
\begin{aligned}
P\left(\frac{|X_{T_r}|}{T_r^b} > x\right) &\leq \sum_{i \leq \Gamma_r} G(xi^b) + G(x\Gamma_r^b) \sum_{i > \Gamma_r} P(T_r \geq i) \\
&\leq \sum_{i \leq \Gamma_r} G(xi^b) + c_1 G(x\Gamma_r^b)\Gamma_r \leq (1 + c_1) \sum_{i \leq \Gamma_r} G(xi^b).
\end{aligned}
$$

Put $r = r(n)$ and $x = 2Kr(n)$, $K > 2$, in this to get

$$
P(E_n) \leq c_2 \sum_{i \leq \Gamma_{r(n)}} G(2Kr(n)i^b) := c_2\varepsilon_n, \quad n \geq n_0. \tag{2.1}
$$

Now note that

$$
\begin{aligned}
\sum_{i=n_0}^{n} \varepsilon_i &= \sum_{i=n_0}^{n} \sum_{m \leq \Gamma_{r(i)}} G(2Kr(i)m^b) \leq \sum_{i=n_0}^{n} \int_{z=0}^{\Gamma_{r(i)}} G(2Kr(i)z^b)dz \\
&= \frac{1}{b} \sum_{i=n_0}^{n} (2Kr(i))^{-1/b} \int_{y=0}^{2Kr(i)\Gamma_{r(i)}^b} y^{\frac{1-b}{b}} G(y)dy \\
&\leq c_1 \int_{n_0}^{n} (r(u))^{-1/b} \int_{y=0}^{4K2^u} y^{\frac{1-b}{b}} G(y)dydu \\
&= c_1 \int_{y=0}^{c_2 2^n} y^{\frac{1-b}{b}} G(y)dy \int_{n \geq u \geq n_0, 2^u \geq y/c_2} (r(u))^{-1/b}du \\
&\leq c_3 \int_{y=0}^{c_2 2^n} y^{\frac{1-b}{b}} G(y)dy \int_{n \geq u \geq n_0, 2^u \geq y/c_2} \frac{du}{2^{\frac{u}{b}}k(2^u)} \\
&\leq c_4 \int_{y=0}^{c_2 2^n} y^{\frac{1-b}{b}} G(y)dy \int_{y/c_2}^{\infty} \frac{dv}{v^{\frac{1+b}{b}}k(v)}. \tag{2.2}
\end{aligned}
$$

Noting that the integrand is regularly varying with index strictly less than -1, we see that

$$
\int_{y/c_2}^{\infty} \frac{dv}{v^{\frac{1+b}{b}}k(v)} \backsim \frac{c_5}{y^{\frac{1}{b}}k(y)} \quad \text{as } y \to \infty,
$$

and hence

$$
\sum_{i=n_0}^{n} \varepsilon_i \leq c_6 \int_{y=0}^{c_2 2^n} \frac{y^{\frac{1-b}{b}}G(y)}{y^{\frac{1}{b}}k(y)}dy = c_6 \int_{y=0}^{c_2 2^n} \frac{G(y)}{yk(y)}dy.
$$

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Going back to (2.1) and writing $V(x) = \int_{y=0}^{x} \frac{G(y)}{yk(y)} dy$, we see that

$$\sum_{i=n_0}^{n} P(E_i) \leq c_7 V(c_2 2^n), \ n \geq n_0, \tag{2.3}$$

and because we are assuming that the lefthand side tends to ∞ as $n \to \infty$, we have

$$\int_{y=0}^{\infty} \frac{G(y)}{yk(y)} dy = \infty.$$

Since

$$\frac{xV'(x)}{V(x)} = \frac{G(x)}{k(x)} \to 0,$$

we see that $V \in RV(0)$, and we conclude from (1.22) and (2.3) that

$$\sum_{i=n_0}^{n} P(E_i) \approx V(2^n). \tag{2.4}$$

Now take $j > i \geq n_0$ fixed and write $P(E_i \cap E_j) = P_{i,j} + Q_{i,j}$, where $P_{i,j} = P(E_i \cap E_j, T_{r(i)} < T_{r(j)})$ and $Q_{i,j} = P(E_i \cap E_j, T_{r(i)} = T_{r(j)})$. Note that, without loss of generality, we can take $2K$ as large as we wish; initially we take $2K > 2$, and write

$$\begin{aligned} P_{i,j} &= \sum_{m=1}^{\infty} \sum_{l=m+1}^{\infty} P(E_i \cap E_j, T_{r(i)} = m, T_{r(j)} = l) \\ &= \sum_{m=1}^{\infty} \sum_{l=m+1}^{\infty} P\{A_{m,i} \cap B_{l,j}\}, \ \text{say}, \end{aligned} \tag{2.5}$$

where

$$A_{m,i} = \left\{ \max_{1 \leq v < m} \frac{|S_v|}{v^b} \leq r(i), |X_m| > 2Kr(i)m^b \right\},$$

$$B_{l,j} = \left\{ \max_{1 \leq v < l} \frac{|S_v|}{v^b} \leq r(j), |X_l| > 2Kr(j)l^b \right\}.$$

(2.5) is valid because $\max_{1 \leq v < m} |S_v|/v^b \leq r(i)$ and $|X_m| > 2Kr(i)m^b$ imply $|S_m|/m^b > r(i)$, thus $T_{r(i)} = m$, and similarly with m replaced by l and i replaced by j.

Next we split this sum and write $P_{i,j} = P_{i,j}^{(1)} + P_{i,j}^{(2)}$, where

$$P_{i,j}^{(1)} = \sum_{m=1}^{\infty} \sum_{1 \leq t < m^*} P\{A_{m,i} \cap B_{m+t,j}\},$$

$\Delta > 0$ is a fixed positive constant, and m^* denotes the integer part of Δm. Note that if $B_{m+t,j}$ occurs for a value of t with $1 \leq t < m^*$, then

$$|S_{m+v}| \leq r(j)(m+v)^b \leq r(j)(1+\Delta)^b m^b \ \text{for} \ v = 0, 1, \cdots t-1;$$

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also $|S_m| \leq r(j)m^b$. So if we write $\hat{S}_v = S_{m+v} - S_m$, which is independent of $A_{m,i}$ for $v = 0, 1, \cdots$, then we have

$$\max_{1 \leq v < t} |\hat{S}_v| \leq r(j)(1 + \Delta)^b m^b + r(j)m^b \leq 2Kr(j)m^b,$$

as long as we take K large enough that $(1 + \Delta)^b + 1 \leq K$. Thus we have, for $l > m$,

$$\sum_{1 \leq t \leq m^*} P\left(B_{m+t,j}|A_{m,i}\right)$$

$$\leq \sum_{1 \leq t \leq m^*} P\left(\max_{1 \leq v < t} |\hat{S}_v| \leq Kr(j)m^b, |X_{m+t}| > 2Kr(j)m^b\right)$$

$$= \sum_{1 \leq t \leq m^*} P\left(T^{(0)}(Kr(j)m^b)) = t, |X_t| > 2Kr(j)m^b\right)$$

$$\leq P\left(T^{(0)}(Kr(j)m^b) \leq m^*\right),$$

where we have written $T^{(0)}(r)$ for T_r in the case $b = 0$. For this we have, from [12],

$$P\left(T^{(0)}(Kr(j)m^b) \leq m^*\right) \leq cm^* k\left(Kr(j)m^b\right).$$

Next we show that

$$\sum_{j=i+1}^{\infty} k\left(Kr(j)m^b\right) \leq ck\left(r(i)m^b\right) \quad \text{for } m \geq 1 \text{ and } i \geq n_0. \tag{2.6}$$

We have $r(n+1)/r(n) \to \gamma$, where $\gamma = 2^{1-b}$ (in case $b < 1$) or $2^{(1-2b)/(1-b)}$ (in case $b < 1/2$), so $1 < \gamma < 2$. Thus without loss of generality we can assume that

$$1 < \gamma_1 \leq \frac{r(n+1)}{r(n)} \leq \gamma_2 < 2 \text{ for all } n \geq n_0.$$

So when $j \geq i \geq n_0$ we have $r(j) \geq r(i)\gamma_1^{j-i}$, and hence by (1.19), as $i \to \infty$,

$$\sum_{j=i+1}^{\infty} k\left(Kr(j)m^b\right) \leq 3\sum_{j=i+1}^{\infty} k\left(Kr(i)\gamma_1^{j-i}m^b\right)$$

$$\leq c\int_0^{\infty} k\left(Kr(i)\gamma_1^x m^b\right) dx = c\int_1^{\infty} k\left(Kr(i)ym^b\right)\frac{dy}{y}$$

$$= c\int_{Kr(i)m^b}^{\infty} k(z)\frac{dz}{z} \sim ck\left(Kr(i)m^b\right) \sim ck(r(i)m^b),$$

and (2.6) follows.

Thus, since $P(A_{m,i}) = P(T_{r(i)} \geq m)G(2Kr(i)m^b)$, and $2^{i-1} \leq r(i)\Gamma^b_{r(i)} \leq 3.2^{i-1}$, we have the bound

$$
\begin{aligned}
\sum_{j=i+1}^{n} P_{i,j}^{(1)} &\leq c \sum_{m=1}^{\infty} mP(T_{r(i)} \geq m)G(2Kr(i)m^b)k\left(r(i)m^b\right) \\
&\leq c \sum_{m \leq \Gamma_{r(i)}} mG(2Kr(i)m^b)k\left(r(i)m^b\right) \\
&\quad + 3cG(2Kr(i)\Gamma^b_{r(i)})k(r(i)\Gamma^b_{r(i)}) \sum_{m > \Gamma_{r(i)}} mP(T_{r(i)} \geq m) \\
&= c\{\sigma(1,i) + \sigma(2,i)\}, \text{ say}
\end{aligned}
$$

(where we used (1.19) in the second inequality). Now by (1.18) with $a = 2$ we have

$$
\begin{aligned}
\sigma(2,i) &\leq 2cG(2Kr(i)\Gamma^b_{r(i)})k(r(i)\Gamma^b_{r(i)})\Gamma^2_{r(i)} \\
&\leq 6\sigma(1,i) \text{ (using (1.19)again)}
\end{aligned}
$$

so we need only consider $\sigma(1,i)$. By repeating the calculation leading to (2.2) we see that

$$
\sum_{i=n_0}^{n} \sigma(1,i) \leq c_4 \int_{y=0}^{c_2 2^n} y^{\frac{2-b}{b}} G(y)k(y)dy \int_{y/c_2}^{\infty} \frac{dv}{v^{\frac{2+b}{b}}\{k(v)\}^2}.
$$

Again the integrand is regularly varying with index strictly less than -1, so it follows that

$$
\int_{y/c_2}^{\infty} \frac{dv}{v^{\frac{2+b}{b}}\{k(v)\}^2} \backsim \frac{c_5}{y^{\frac{2}{b}}\{k(y)\}^2} \text{ as } y \to \infty,
$$

and hence

$$
\sum_{i=n_0}^{n} \sigma(1,i) \leq c_6 \int_{y=0}^{c_2 2^n} \frac{y^{\frac{2-b}{b}}G(y)}{y^{\frac{2}{b}}k(y)}dy = c_6 V(c_2 2^n).
$$

From (2.4) we now deduce

$$
\limsup_{n \to \infty} \frac{\sum_{i=1}^{n}\sum_{j=i+1}^{n} P_{i,j}^{(1)}}{\left(\sum_{1}^{n} P(E_i)\right)^2} < \infty. \tag{2.7}
$$

We can also write

$$
P_{i,j}^{(2)} = \sum_{m=1}^{\infty}\sum_{s=1}^{\infty} P\{A_{m,i} \cap B_{m+m^*+s,j}\}
$$

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Again write $\hat{S}_v = S_{m+v} - S_m$ and now take $K \geq (2+\Delta)((1+\Delta)/\Delta)^b$. Then

$$\sum_{s=1}^{\infty} P(B_{m+m^*+s,j}|A_{m,i})$$

$$\leq \sum_{s=1}^{\infty} P\left(\max_{v<s} \frac{|\hat{S}_{m^*+v}|}{(m+m^*+v)^b} \leq 2r(j), \frac{|X_{m+m^*+s}|}{(m+m^*+s)^b} > 2Kr(j)\right)$$

$$\leq \sum_{s=1}^{\infty} P\left(\max_{v<m+s} \frac{|\hat{S}_{m^*+v}|}{(m^*+v)^b} \leq 2\left(\frac{1+\Delta}{\Delta}\right)^b r(j), \frac{|X_{m+m^*+s}|}{(m+m^*+s)^b} > 2Kr(j)\right)$$

$$\leq \sum_{t=m+1}^{\infty} P\left(\max_{v<t} \frac{|\hat{S}_v|}{v^b} \leq Kr(j), \frac{|X_{m+t}|}{(m+t)^b} > 2Kr(j)\right)$$

$$\leq P\left(|X\left(T_{Kr(j)}\right)| > 2Kr(j)T_{Kr(j)}^b\right).$$

It follows that

$$P_{i,j}^{(2)} \leq \sum_{m=1}^{\infty} P\left(|X_{T_{r(i)}}| \geq m\right) G\left(2Kr(i)m^b\right) P\left(|X_{T_{Kr(j)}}| > 2Kr(j)T_{Kr(j)}^b\right)$$

$$= P\left(|X_{T_{r(i)}}| \geq 2Kr(i)T_{Kr(i)}^b\right) P\left(|X_{T_{Kr(j)}}| > 2Kr(j)T_{Kr(j)}^b\right)$$

$$= P(E_i)P(E_j'),$$

where it is easy to check that $\sum_{n_0}^{n} P(E_j') \approx \sum_{n_0}^{n} P(E_j)$. Thus

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n}\sum_{j=i+1}^{n} P_{i,j}^{(2)}}{\left(\sum_1^n P(E_i)\right)^2} < \infty. \tag{2.8}$$

Finally, we write

$$Q_{i,j} = \sum_{m=1}^{\infty} P\left(\max_{v<m} \frac{|S_v|}{v^b} \leq r(i), |X_m| > 2Kr(j)m^b\right)$$

$$= \sum_{m=1}^{\infty} P\left(\max_{v<m} \frac{|S_v|}{v^b} \leq r(i))\right) G(2Kr(j)m^b)$$

$$\leq \sum_{m=1}^{\Gamma_{r(i)}} G(2Kr(j)m^b) + G(2Kr(j)\Gamma_{r(i)}^b) \sum_{m>\Gamma_{r(i)}} P(T_{r(i)} \leq m)$$

$$\leq \sum_{m=1}^{\Gamma_{r(i)}} G(2Kr(j)m^b) + c\Gamma_{r(i)}G(2Kr(j)\Gamma_{r(i)}^b) \quad \text{(using (1.18))}$$

$$\leq (1+c)\sum_{m=1}^{\Gamma_{r(i)}} G(2Kr(j)m^b).$$

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Recall that when $j \geq i \geq n_0$ we have $r(j) \geq r(i)\gamma_1^{j-i}$; thus

$$Q_{i,j} \leq (1+c) \sum_{m=1}^{\Gamma_{r(i)}} G(2K\gamma_1^{j-i}r(i)m^b). \tag{2.9}$$

At this stage we need the following technical fact;

Lemma 7 *Define*

$$W(x) = \int_x^\infty \frac{G(y)}{y}dy. \tag{2.10}$$

Then if $S \in RS$ or $S \in D_0(N)$ we have, for some $c > 0$,

$$\int_1^x \frac{W(y)}{yk(y)}dy \leq cV(x) \text{ for all sufficiently large } x. \tag{2.11}$$

Proof of Lemma 7: First consider the case that $S \in D_0(N)$, when

$$k(x) \sim \frac{U(x)}{x^2}, \text{ where } U(x) = \int_0^x 2yG(y)dy,$$

and we have

$$U \in RV(0) \text{ and } \frac{x^2G(x)}{U(x)} \to 0 \text{ as } x \to \infty.$$

Thus

$$\begin{aligned}
2W(x) &= \int_x^\infty \frac{2yG(y)}{y^2}dy = \int_x^\infty \frac{U'(y)}{y^2}dy \\
&= -\frac{U(x)}{x^2} + 2\int_x^\infty \frac{U(y)}{y^3}dy = o\left(\frac{U(x)}{x^2}\right).
\end{aligned}$$

Now

$$\int_1^x \frac{W(y)}{yk(y)}dy \frown \int_1^x \frac{yW(y)}{U(y)}dy,$$

$$\text{and } \int_1^x \frac{G(y)}{yk(y)}dy \frown \int_1^x \frac{yG(y)}{U(y)}dy, \text{ as } x \to \infty.$$

But

$$\begin{aligned}
\int_1^x \frac{yG(y)}{U(y)}dy &= \int_1^x \frac{G(y)}{y}\frac{y^2}{U(y)}dy = -\int_1^x W'(y)\frac{y^2}{U(y)}dy \\
&= \frac{W(1)}{U(1)} - \frac{x^2W(x)}{U(x)} + \int_1^x \frac{W(y)}{U(y)^2}\left\{2yU(y) - y^2U'(y)\right\}dy \\
&= O(1) + 2\int_1^x \frac{yW(y)}{U(y)^2}\left\{U(y) - y^2G(y)\right\}dy \\
&= O(1) + \{2+o(1)\}\int_1^x \frac{yW(y)}{U(y)}dy,
\end{aligned}$$

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and the result follows.

If $S \in RS$ we have $A(x) > 0$ for all large enough x, or $A(x) < 0$ for all large enough x, and $|A(x)|$ is in $RV(0)$; and we also have $U(x) = o(xA(x))$ as $x \to \infty$. So, taking $A(x) > 0$ for all large enough x, we have

$$
\begin{aligned}
2W(x) &= \int_x^\infty \frac{2yG(y)}{y^2}dy = \int_x^\infty \frac{U'(y)}{y^2}dy \\
&= -\frac{U(x)}{x^2} + 2\int_x^\infty \frac{U(y)}{y^3}dy \\
&= o\left(\frac{A(x)}{x} + 2\int_x^\infty \frac{A(y)}{y^2}dy\right) = o\left(\frac{A(x)}{x}\right).
\end{aligned}
$$

If we now observe that

$$
\begin{aligned}
\int_1^x \frac{G(y)}{A(y)}dy &= \int_1^x \frac{G(y)}{y}\frac{y}{A(y)}dy = -\int_1^x W'(y)\frac{y}{A(y)}dy \\
&= \frac{W(1)}{A(1)} - \frac{xW(x)}{A(x)} + \int_1^x \frac{W(y)}{A(y)}\{1 - \frac{yA'(y)}{A(y)}\}dy \\
&= O(1) + \{1 + o(1)\}\int_1^x \frac{W(y)}{A(y)}dy,
\end{aligned}
$$

because $(yA'(y))/A(y) \to 0$ as $y \to \infty$, then this case is also proven. A similar proof works if $A(x) < 0$ for all large enough x. ∎

Using Lemma 7, we can argue that

$$
\begin{aligned}
&\sum_{i=1}^n \sum_{j=i+1}^n \sum_{m=1}^{\Gamma_{r(i)}} G(2K\gamma_1^{j-i}r(i)m^b) \\
\leq\ & \int_{x=0}^n \int_{y=0}^\infty \int_{w=0}^{r(x)\Gamma_{r(x)}^b} w^{\frac{1-b}{b}}r(x)^{-\frac{1}{b}}G(2K\gamma_1^y w)dwdydx \\
\leq\ & c\int_{x=0}^n \int_{y=0}^\infty \int_{w=0}^{3.2^{x-1}} \frac{w^{\frac{1-b}{b}}G(2K\gamma_1^y w)}{2^{\frac{x}{b}}k(2^x)}dwdydx \\
=\ & c\int_{v=1}^{2^n} \int_{w=0}^{3.v/2} \int_{z=1}^\infty \frac{w^{\frac{1-b}{b}}G(2Kzw)}{zv^{\frac{1+b}{b}}k(v)}dzdwdv \\
=\ & c\int_{v=1}^{2^n} \int_{w=0}^{3.v/2} \frac{w^{\frac{1-b}{b}}W(2Kw)}{v^{\frac{1+b}{b}}k(v)}dwdv \quad (\text{using}(2.10)) \\
\leq\ & c\int_{w=0}^{3.2^{n-1}} w^{\frac{1-b}{b}}W(2Kw)dw \int_{v=2w/3}^\infty \frac{dv}{v^{\frac{1+b}{b}}k(v)} \\
\leq\ & c\int_{w=0}^{3.2^{n-1}} \frac{W(2Kw)}{wk(w)}dw \\
\leq\ & cV(3.2^n) \quad (\text{using}(2.11)) \\
\sim\ & cV(2^n), \quad \text{as } n \to \infty.
\end{aligned}
$$

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Together with (2.7) and (2.8), this concludes the proof of (1.24), and the theorem is proved. ∎

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