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MILDNESS AND THE DENSITY OF RATIONAL POINTS ON CERTAIN TRANSCENDENTAL CURVES

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Abstract. We use a result due to Rolin, Speissegger and Wilkie to show that definable sets in certain o-minimal structures admit definable parameterizations by mild maps. We then use this parameterization to prove a result on the density of rational points on curves defined by restricted Pfaffian functions.

1. Introduction

The main result of this note is a generalization of some results of Pila ([9]) to a wider collection of curves. Before stating the result, we need some definitions. A sequence $f_1, \ldots, f_r : U \to \mathbb{R}$ of analytic functions on an open set $U \subseteq \mathbb{R}^n$ is said to be a Pfaffian chain of order $r$ and degree $\alpha$ if there are polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_{n+j}]$ of degree at most $\alpha$ such that

$$df_j = \sum_{i=1}^n P_{i,j}(\bar{x}, f_1(\bar{x}), \ldots, f_j(\bar{x}))dx_i, \text{ for } j = 1, \ldots, r.$$ 

Given such a chain, we say that a function $f : U \to \mathbb{R}$ is Pfaffian of order $r$ and degree $(\alpha, \beta)$ with chain $f_1, \ldots, f_r$, if there is a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ of degree at most $\beta$ such that $f(\bar{x}) = P(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x}))$.

Let $U \subseteq \mathbb{R}^n$ be an open set containing $[0,1]^n$. To every function $f : U \to \mathbb{R}$, we associate a new function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in [0,1]^n, \\ 0 & \text{otherwise}. \end{cases}$$

Recall that $\mathbb{R}_{an}$ is the expansion of the real ordered field by all functions of the form $\hat{f}$, where $f : U \to \mathbb{R}$ is analytic, $[0,1]^n \subseteq U$ and $n \geq 1$. We let $\mathbb{R}_{resPfaff}$ be the reduct of this structure given by the same description, but with the word ‘analytic’ replaced by ‘Pfaffian’.

For $q \in \mathbb{Q}$, the height of $q$ is $H(q) = \max\{|a|, |b|\}$, where $q = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a, b) = 1$. The height of $\bar{q} \in \mathbb{Q}^n$, again written $H(\bar{q})$, is defined as the...
maximum of the heights of the coordinates of $\bar{q}$. For a set $X \subseteq \mathbb{R}^n$ and $H \geq 1$, we let

$$X(\mathbb{Q}, H) = \{\bar{q} \in X \cap \mathbb{Q}^n : H(\bar{q}) \leq H\}.$$ 

A transcendental function $f : \mathbb{R}^n \to \mathbb{R}$ is one that does not satisfy any non-zero polynomial equation $P(y, x_1, \ldots, x_n) = 0$, for $P \in \mathbb{R}[Y, X_1, \ldots, X_n]$.

**Proposition 1.1.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a transcendental analytic function definable in $\mathbb{R}_{\text{resPfaff}}$, and let $X = \text{graph}(f)$. Then there exist $c > 0$ and $\gamma > 0$ such that for $H \geq 3$

$$\#X(\mathbb{Q}, H) \leq c(\log H)^\gamma.$$ 

When $f$ is Pfaffian, and not assumed to be definable in $\mathbb{R}_{\text{resPfaff}}$, this result is due to Pila ([9]). The extra generality here, as far as functions definable in $\mathbb{R}_{\text{resPfaff}}$ are considered, is to include functions implicitly defined by restricted Pfaffian functions.

The proof of the proposition is a modification of Pila’s proof in [8]. To this end, we need a parameterization result which, although a simple consequence of a result due to Rolin, Speissegger and Wilkie ([11]), may be of some independent interest. We need two further definitions, the first of which is due to Pila ([10]). We use the following multi-index notation: for any $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we define the modulus $|\alpha| := \alpha_1 + \ldots + \alpha_k$, the factorial $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_k!$ and the differential operator

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}}.$$

**Definitions 1.2.** Let $A > 0$, $C \geq 0$. A $C^\infty$ function $\phi : (0, 1)^k \to (0, 1)$ is said to be $(A, C)$-mild if

$$|D^\alpha \phi(\bar{x})| \leq \alpha! (A|\alpha|)^{C|\alpha|}$$

for all $\alpha \in \mathbb{N}^k$, all $\bar{x} \in (0, 1)^k$ (where $0^0 = 1$). We say that a map $\Phi : (0, 1)^k \to (0, 1)^n$ is $(A, C)$-mild if each of its coordinate functions is $(A, C)$-mild.

**Definitions 1.3.** Fix an o-minimal structure $\bar{\mathbb{R}}$ expanding the real field, and let $X \subseteq \mathbb{R}^n$ be definable. A parameterization of $X$ is a finite set $S$ of definable maps $\Phi_1, \ldots, \Phi_l : (0, 1)^{\dim X} \to \mathbb{R}^n$ such that $X = \bigcup \text{Im}(\Phi_i)$. A parameterization is said to be $(A, C)$-mild if each of the parameterizing maps is $(A, C)$-mild. We say that $\bar{\mathbb{R}}$ admits $C$-mild parameterization if for every definable set $X \subseteq (0, 1)^n$ there is an $(A, C)$-mild parameterization of $X$, for some $A$.

**Example 1.4.** For a compact box $B \subseteq \mathbb{R}^n$, suppose that $f = (f_1, \ldots, f_m) : B \to \mathbb{R}^m$ extends to an analytic function in a neighborhood of $B$. Then there exist (for example, by [6, 2.2.10]) positive constants $A$ and $K$ such that

$$|D^\alpha f_i(x)| \leq \alpha!KA^{i|\alpha|}$$

for all $x \in B$, $\alpha \in \mathbb{N}^n$, and $i \in \{1, \ldots, m\}$. If $B = [0, 1]^n$ and $f((0, 1)^n) \subseteq (0, 1)^m$, then by making $A$ larger we may take $K = 1$, in which case the graph of $f\mid_{(0, 1)^n}$ has an $(A, 0)$-mild parameterization consisting of one map, namely $\Phi : (0, 1)^n \to (0, 1)^{n+m}$ defined by $\Phi(\bar{x}) = (\bar{x}, f(\bar{x}))$.

**Proposition 1.5.** Any reduct of $\mathbb{R}_{an}$ expanding the real ordered field admits 0-mild parameterization.
We remark on the relationship between the notion of a mild function and that of a Gevrey function. In [4], van den Dries and Speissegger consider \( \mathbb{R}_G \), the expansion of the real ordered field by the class of Gevrey functions \( G \), which is a certain family of real-valued \( C^\infty \) functions on the sets \([0, R] = \prod_{i=1}^n [0, R_i] \), for each \( n \in \mathbb{N} \) and \( R_1, \ldots, R_n > 0 \), which are analytic on \((0, R] = \prod_{i=1}^n (0, R_i] \). For each \( n \)-ary function \( f : [0, R] \to \mathbb{R} \) in \( G \) there exist constants \( A, B > 0 \) and \( \kappa \in (0, 1] \) such that

\[
|D^n f(\overline{x})| \leq \alpha! AB^{\alpha} |\alpha|^{\kappa |\alpha|}
\]

for all \( \overline{x} \in [0, R] \) and \( \alpha \in \mathbb{N}^n \) (see [4, 2.6]). It follows that \( \mathbb{R}_G \) is definably equivalent to an expansion of the real ordered field by a family of functions, each of which is \((B, \kappa)\)-mild for some \( B > 0 \) and \( \kappa \in (0, 1] \). It is therefore natural to ask whether \( \mathbb{R}_G \) admits 1-mild parameterization. To the best of our knowledge, this question is open and does not follow from the methods of this paper. The proof of Proposition 1.5 considers a set \( X \subseteq (0, 1)^n \) definable in some fixed reduct of \( \mathbb{R}_{an} \), and uses [11] to construct a parameterization \( \Phi_1, \ldots, \Phi_l : (0, 1)^{\text{dim } X} \to (0, 1)^n \) of \( X \) such that the definable maps \( \Phi_1, \ldots, \Phi_l \) all extend to (definable) analytic functions on a neighborhood of \([0, 1]^{\text{dim } X} \), from which Proposition 1.5 follows using Example 1.4. In contrast, [4] relies on the model completeness construction in [3], and therefore represents a set \( X \subseteq (0, 1)^n \) definable in \( \mathbb{R}_G \) as a finite union of projections of manifolds which are zero sets of Gevrey functions, but which are not themselves graphs of Gevrey functions. The question of whether such manifolds have 1-mild parameterizations appears to be open.

2. \( \mathcal{C} \)-parameterization

In this section we observe that the results in [11] imply a parameterization result. So, we work in the setting of [11], and fix, for every compact box \( B \subseteq \mathbb{R}^n \) and every \( n \in \mathbb{N} \), an \( \mathbb{R} \)-algebra \( \mathcal{C}_B \) of functions \( f : B \to \mathbb{R} \) such that the following hold.

\((\mathcal{C}_1)\) Each of the projection functions \( \langle x_1, \ldots, x_n \rangle \mapsto x_i \), restricted to \( B \), is in \( \mathcal{C}_B \), and for every function \( f \in \mathcal{C}_B \) the restriction of \( f \) to the interior of \( B \) is \( C^\infty \).

\((\mathcal{C}_2)\) If \( B' \subseteq \mathbb{R}^m \) is a compact box and \( g_1, \ldots, g_n \in \mathcal{C}_{B'} \) are such that \( g(f(B')) \subseteq B \), where \( g = \langle g_1, \ldots, g_n \rangle \), then for every \( f \in \mathcal{C}_B \), the composition \( f \circ g \) is in \( \mathcal{C}_{B'} \).

\((\mathcal{C}_3)\) For every compact box \( B' \subseteq B \) and function \( f \in \mathcal{C}_B \), the restriction of \( f \) to \( B' \) is in \( \mathcal{C}_{B'} \). For every \( f \in \mathcal{C}_B \) there is a compact box \( B' \subseteq \mathbb{R}^n \), the interior of which contains \( B \), and a function \( g \in \mathcal{C}_{B'} \) such that \( g|_B = f \).

\((\mathcal{C}_4)\) For every \( f \in \mathcal{C}_B \) and \( i = 1, \ldots, n \), the partial derivative \( \frac{\partial f}{\partial x_i} \) is in \( \mathcal{C}_B \).

Note that the partial derivatives in \((\mathcal{C}_4)\) exist by \((\mathcal{C}_1)\) and \((\mathcal{C}_3)\). Since we shall not need the precise statements of the remaining assumptions, we only state rough versions of them. The full details can be found in [11].

\((\mathcal{C}_5)\) For each \( n \geq 1 \) and each box \( B \subseteq \mathbb{R}^n \) containing the origin, the collection of germs at the origin of functions in \( \mathcal{C}_B \) forms a quasianalytic class.

\((\mathcal{C}_6)\) This collection of germs is closed under extraction of implicit functions.

\((\mathcal{C}_7)\) This collection of germs is closed under monomial division.
The example which will interest us is as follows. Suppose that \( \hat{\mathbb{R}} \) is a polynomially bounded o-minimal expansion of the real field. For each compact box, let \( C_B \) be the collection of definable functions \( f : B \to \mathbb{R} \) which admit a definable \( C^\infty \) extension to some open set containing \( B \). By well known properties of o-minimal structures ([2], [7]) these algebras satisfy the above requirements. In particular, if \( \hat{\mathbb{R}} \) is a reduct of \( \mathbb{R}_o \), then each function \( f \) in \( C_B \) is the restriction to \( B \) of an analytic function defined in a neighborhood of \( B \), as in Example 1.4.

We now recall some further definitions from [11]. Given a polyradius \( \vec{r} = \langle r_1, \ldots, r_n \rangle \in (0, \infty)^n \) we let \( I_{\vec{r}} = \prod(-r_i, r_i) \) and let \( I_{\vec{r}} \) be the topological closure of \( I_{\vec{r}} \). Write \( C_{n, r} \) for \( C_{\vec{r}} \).

**Definition 2.1.** A set \( A \subseteq \mathbb{R}^n \) is called a *basic \( C \)-set* if there are \( \vec{r} \in (0, \infty)^n \) and \( f, g_1, \ldots, g_k \in C_{n, r} \) such that

\[
A = \{ \bar{x} \in I_{\vec{r}} : f(\bar{x}) = 0, g_1(\bar{x}) > 0, \ldots, g_k(\bar{x}) > 0 \}.
\]

A finite union of basic \( C \)-sets is called a \( C \)-set. A set \( A \subseteq \mathbb{R}^n \) is called \( C \)-semianalytic if for every \( \bar{a} \in \mathbb{R}^n \) there is an \( \vec{r} \in (0, \infty)^n \) such that

\[
(A - \bar{a}) \cap I_{\vec{r}}
\]

is a \( C \)-set. If \( A \) is also a manifold, we call \( A \) a \( C \)-semianalytic manifold.

Given \( m \leq n \) and an injective \( \lambda : \{1, \ldots, m\} \to \{1, \ldots, n\} \), we write \( \pi_\lambda : \mathbb{R}^n \to \mathbb{R}^m \) for the projection \( \bar{x} \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(m)}) \).

**Definition 2.2.** Let \( \vec{r} \in (0, \infty)^n \). A set \( M \subseteq I_{\vec{r}} \) is said to be \( C \)-trivial if one of the following holds:

1. \( M = \{ \bar{x} \in I_{\vec{r}} : x_i \sqcap 0, \ldots, x_n \sqcap 0 \} \), where \( \sqcap_i \in \{<, =, >\} \) for each \( i \);
2. there exist a permutation \( \lambda \) of \( \{1, \ldots, n\} \), a \( C \)-trivial \( N \subseteq I_{\vec{s}} \) and a \( g \in C_{n-1, s} \), where \( \vec{s} = \langle s_{\lambda(1)}, \ldots, s_{\lambda(n-1)} \rangle \), such that \( g(I_{\vec{s}}) \subseteq (-r_{\lambda(n)}, r_{\lambda(n)}) \) and

\[
\pi_\lambda(M) = \text{graph}(g|_{N})
\]

Note that \( C \)-trivial sets are necessarily manifolds; we shall refer to them as \( C \)-trivial manifolds. A \( C \)-semianalytic manifold \( M \subseteq \mathbb{R}^n \) is called *trivial* if there exist \( \bar{a} \in \mathbb{R}^n \) and a \( C \)-trivial manifold \( N \subseteq \mathbb{R}^n \) such that \( M = N + \bar{a} \).

We need two results from [11].

**Fact 2.3.** ([11, 4.7]) Suppose that \( A \subseteq \mathbb{R}^n \) is a bounded \( C \)-semianalytic set and that \( k \leq n \). Then there are trivial \( C \)-semianalytic manifolds \( N_i \subseteq \mathbb{R}^{n_i} \) for some \( n_i \geq n, i = 1, \ldots, J \), such that

\[
\pi_k(A) = \pi_k(N_1) \cup \cdots \cup \pi_k(N_J)
\]

where \( \pi_k|_{N_i} \) is an immersion, for each \( i \). (Here, \( \pi_k \) is projection onto the first \( k \) coordinates.)

Let \( \mathbb{R}_C \) be the expansion of the real ordered field by all functions \( \hat{f} \), for \( f \in C_{n, \vec{r}}, n \in \mathbb{N}, \vec{r} \in (0, \infty)^n \), where \( \hat{f}(\bar{x}) = f(\bar{x}) \) on \( I_{\vec{r}} \) and \( f(\bar{x}) = 0 \) on \( \mathbb{R}^n \setminus I_{\vec{r}} \).

**Fact 2.4.** ([11, 5.2 and 5.4]) The structure \( \mathbb{R}_C \) is o-minimal, model complete and polynomially bounded.
We now use these results to prove a parameterization result. We work in the structure $\mathbb{R}_C$.

**Definition 2.5.** Let $X \subseteq \mathbb{R}^n$ be definable. A $C$-parameterization of $X$ is a finite set $S$ of maps $\Phi_1, \ldots, \Phi_l$ whose coordinate functions are in $C_{[0,1]}[\text{dim } x]$ such that $\{\Phi_i^1|_{(0,1)^{\text{dim } x}} : i = 1, \ldots, l\}$ is a parameterization of $X$.

**Example 2.6.** Let $\vec{r} \in (0, \infty)^n$. Let $M = \{\vec{x} \in I_\vec{r} : x_1 \square_1 0, \ldots, x_n \square_n 0\}$, where $\square_i \in \{\langle, =, \rangle\}$ for each $i$. Let $\lambda_1, \ldots, \lambda_m$ be, in order, the indices for which $\square_i$ is either $<$ or $>$. For each $i$, define the map $\phi_i : (0, 1)^m \to \mathbb{R}$ by

$$
\phi_i(\vec{x}) = \begin{cases} 
-r_j x_j & \text{if } i = \lambda_j \text{ and } \square_i \text{ is } <, \\
r_j x_j & \text{if } i = \lambda_j \text{ and } \square_i \text{ is } >, \\
0 & \text{otherwise.}
\end{cases}
$$

We now see that $M$ has a $C$-parameterization consisting of one map, namely $\Phi : (0, 1)^m \to \mathbb{R}^n$ given by $\Phi(\vec{x}) := (\phi_1(\vec{x}), \ldots, \phi_m(\vec{x}))$.

Now we easily have the following, by induction on $n$.

**Lemma 2.7.** Suppose that $M \subseteq \mathbb{R}^n$ is a $C$-trivial manifold. Then there is a $C$-parameterization $S$ of $M$ with $\#S = 1$.

**Proposition 2.8.** Suppose that $X \subseteq \mathbb{R}^n$ is a bounded definable set. Then $X$ has a $C$-parameterization.

**Proof.** By model completeness, there is an $m \geq 0$ and a quantifier-free definable set $A \subseteq \mathbb{R}^{n+m}$ such that $X = \pi(A)$. Using the fact that $\mathbb{R}_C$ is an expansion of the real field, we may assume that $A$ is bounded and that $A$ is $C$-semianalytic. By Fact 2.3,

$$
X = \pi(N_1) \cup \cdots \cup \pi(N_k)
$$

for some $C$-trivial manifolds $N_1, \ldots, N_k$, where each $\pi|_{N_i}$ is an immersion. Thus $\dim(X) = \max\{\dim(N_1), \ldots, \dim(N_k)\}$. A $C$-parameterization of $X$ can be constructed by composing the functions in the $C$-parameterizations of each of the $N_i$ with the projections $\pi$, and then trivially extending any of these functions to $(0, 1)^{\dim X}$ if their domain is $(0, 1)^{\dim N_i}$, with $\dim N_i < \dim(X)$. \hfill \Box

Note that Proposition 1.5 follows immediately from applying Proposition 2.8 to the given reduct of $\mathbb{R}_{an}$ and then using Example 1.4.

### 3. Curves

We now prove Proposition 1.1. In fact, we prove a result about the number of points in a fixed number field $k \subseteq \mathbb{R}$ of degree $l$. We use the absolute multiplicative height $H$ on $k$, which agrees with the height on $\mathbb{Q}$ given in the introduction (for the definition of $H$, see [1]). For $X \subseteq \mathbb{R}^n$ and $H \geq 1$, we let $X(k, H) = X \cap \{a \in k^n : H(\bar{a}) \leq H\}$. The following is a special case of [10, Corollary 3.3].

**Fact 3.1.** Suppose that $X \subseteq (0, 1)^2$ is definable in $\mathbb{R}_{an}$ with dimension 1 and that $S$ is an $(A, 0)$-mild parameterization of $X$. Then there is an absolute constant $c_0$ such that $X(k, H)$ is contained in a union of at most

$$
\#S \cdot c_0 \cdot A^{2(1 + o(1))}
$$
intersections of $X$ with algebraic curves of degree $[1 \cdot \log H]$. Here the $1 + o(1)$ is taken as $H \to \infty$ with absolute implied constant, and $[\cdot]$ denotes integer part.

Given a function $F : \mathbb{R}^n \to \mathbb{R}$, we let $V(F) = \{ \bar{x} \in \mathbb{R}^n : F(\bar{x}) = 0 \}$.

**Lemma 3.2.** Suppose that $f : (a, b) \to (0, 1)$, with $(a, b) \subseteq (0, 1)$, is a transcendental analytic function definable in $\mathbb{R}_{\text{resPaff}}$. Suppose further that graph($f$) = $\pi(V(F))$, where $F : \mathbb{R}^{2+n} \to \mathbb{R}$ is a Pfaffian function of order $r$ and degree $(\alpha, \beta)$, and $\pi$ is the projection onto the first two coordinates. If $P : \mathbb{R}^2 \to \mathbb{R}$ is a nonzero polynomial of degree $d$ then

\[
\#(\text{graph}(f) \cap V(P)) \leq 2^{(r+1)/2} (n+2)^r (\alpha + 2d')^{n+r+2}
\]

where $d' = \max\{d, \beta\}$.

**Proof.** Let $\tilde{P} : \mathbb{R}^{2+n} \to \mathbb{R}$ be given by $\tilde{P}(x, y, z) = P(x, y)$. Then $\text{graph}(f) \cap V(P) = \pi(V(F) \cap V(\tilde{P}))$. The number of points in $\text{graph}(f) \cap V(P)$ is thus bounded by the number of connected components of $V(F) \cap V(\tilde{P})$ (there are only finitely many points in $\text{graph}(f) \cap V(P)$, as we have assumed that $f$ is transcendental). By Khovanskii’s theorem (as presented in [5, 3.3]) there are at most

\[
2^{(r-1)/2} d' (\alpha + 2d' - 1)^{n+1} ((2(n+2) - 1)(\alpha + d') - 2n - 2)^r
\]

such components, and clearly this is less than the right hand side of (1). \(\square\)

**Proposition 3.3.** Suppose that $f : (a, b) \to (0, 1)$, with $(a, b) \subseteq (0, 1)$, is a transcendental analytic function definable in $\mathbb{R}_{\text{resPaff}}$ and let $X = \text{graph}(f)$. Then there are $c, \gamma > 0$ such that (for $H \geq e$)

\[
\#X(k, H) \leq c(\log H)^\gamma.
\]

**Proof.** By model completeness of $\mathbb{R}_{\text{resPaff}}$ (see [12]), we may suppose that $X = \pi(V(F))$ for some Pfaffian function $F : \mathbb{R}^{2+n} \to \mathbb{R}$ and some $n \geq 0$. Suppose that $F$ is of order $r$ and degree $(\alpha, \beta)$. By Proposition 1.5 we can take an $(A, 0)$-mild parameterization $S$ of $X$, for some $A$. Combining Fact 3.1 with Lemma 3.2 (with $d = \lceil \log H \rceil$), we have

\[
\#X(k, H) \leq \#S \cdot c_0 \cdot A^{2(1+o(1))} 2^{(r+1)/2}(n+2)^r (\alpha + 2\max\{\beta, d\})^{n+r+2} \leq c(\log H)^\gamma
\]

where $\gamma = n + r + 2$. \(\square\)

The collection of points of a number field $k$ of height at most $H$ is preserved under the inversions $x \to \pm x^{\pm 1}$. Therefore, in counting such points on the graph of a transcendental analytic function $f : \mathbb{R} \to \mathbb{R}$, we may instead consider the graphs of a finite collection of transcendental analytic functions, each defined on a subinterval of $(0, 1)$, together with a finite collection of points in $\mathbb{R}^n$. Proposition 1.1 then follows by repeated application of Proposition 3.3.
References


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