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Nilpotent blocks of quasisimple groups for the prime two [1]

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Abstract

We investigate the nilpotent blocks of positive defect of the quasisimple groups for the prime 2. We show that every nilpotent block of a quasisimple group has abelian defect groups, and give explicit characterisations in many cases. A conjecture of Puig concerning the recognition of nilpotent blocks is also shown to hold for these groups.

1 Introduction

Let G be a finite group and k an algebraically closed field of characteristic p. A block B of kG with defect group D is said to be nilpotent if for each $Q \leq D$ and each block b_Q of $C_G(Q)$ with Brauer correspondent B we have that $N_G(Q, b_Q)/C_G(Q)$ is a p-group, where $N_G(Q, b_Q)$ is the stabilizer of b_Q under conjugation in $N_G(Q)$. In the case of the principal block B_0, D is a Sylow p-subgroup of G and $N_G(Q, b_Q) = N_G(Q)$ for each $Q \leq D$, so that B_0 is nilpotent if and only if G is p-nilpotent (i.e., G has a normal p-complement).

In this paper and in [6], which deals with the odd primes, we are concerned with two problems relating to nilpotent blocks of quasisimple groups: their occurrence and their recognition.

Note that blocks of defect zero are nilpotent, and the determination of their existence for finite simple groups was completed in [16]. We therefore consider the existence of nilpotent blocks of positive defect of the quasisimple groups.

Explicit characterizations of nilpotent blocks are obtained for classical groups, and these are used to prove:

Theorem 1.1 *Let G be a finite quasisimple group and let B be a nilpotent 2-block of G. Then B has abelian defect groups.*

In a recent paper of Malle and Navarro [23], it has been shown that if B is a p-block of a quasisimple group G which is not a faithful block of the double cover of an alternating group A_n for $n \geq 14$, and is not a quasi-isolated block of an exceptional group of Lie type for p a bad prime, and if every irreducible character of height zero has the same degree, then B has abelian defect groups. In this case it follows that nilpotent blocks have abelian defect groups.

With regards to the recognition of nilpotent groups, we consider a conjecture of Puig, which predicts that a block B of G is nilpotent if and only if $l(b_Q) = 1$ for each p-subgroup Q and each block b_Q of $C_G(Q)$ with Brauer correspondent B (where $l(b_Q)$ is the number of irreducible Brauer characters in b_Q). The necessary condition for nilpotency is well-known. The converse is known for blocks with abelian defect groups (see [25]), and is also known to be a consequence of Alperin's weight conjecture (see [29]).

We prove:

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Theorem 1.2 *Let G be a finite quasisimple group and let B be a 2-block of G. Then B is nilpotent if and only if $l(b_Q) = 1$ for each 2-subgroup Q and each block b_Q of $C_G(Q)$ with $(b_Q)^G = B$.*

For the finite groups of Lie type, we show that 2-blocks either have abelian defect groups (in which case the theorem holds by [25]) or a strong form of non-nilpotency holds which also implies the theorem.

The main part of the paper concerns the representation theory of finite groups of Lie type in non-defining characteristic, and makes use of the examination of subpairs of blocks of classical groups, similar to that given in [13]. The exceptional groups of Lie type are then treated by examination of the centralizer of an element of the centre of a defect group, and the results for the classical groups applied.

2 Notation and general results

Let G be a finite group and p a prime. Let k be an algebraically closed field of characteristic p. Write $\mathrm{Blk}(G)$ for the set of blocks of kG and denote by $B_0(G)$ the principal block of G.

Let B be a p-block of a finite group G. A B-subgroup is a subpair (Q, b_Q), where Q is a p-subgroup of G and b_Q is a block of $QC_G(Q)$ with Brauer correspondent $(b_Q)^G = B$. The B-subgroups with $|Q|$ maximized are called the Sylow B-subgroups, and they are the B-subgroups for which Q is a defect group for B. We will usually write $D(B)$ for a defect group of B when one may be chosen freely.

A useful result, which follows from [1, 4.21], is the following:

Proposition 2.1 *Let B be a block of a finite group G. Suppose a defect group D of B is abelian. Then B is nilpotent if and only if $N_G(D, b_D) = C_G(D)$, where (D, b_D) is a Sylow B-subgroup.*

We have the following lemma by [22, Proposition 6.5]:

Lemma 2.2 *Let N be a normal subgroup of a finite group G such that G/N is a p-group. Suppose that B is a block of G and that $b \in \mathrm{Blk}(N)$ is covered by B. Then B is nilpotent if and only if b is nilpotent.*

Let B be a block of G and $Z \leq Z(O_p(G))$. Let $\overline{B}$ be the unique block of $\overline{G} = G/Z$ dominated by B. If (Q, b_Q) is a B-subgroup with $Z \leq Q$, then $C_{\overline{G}}(\overline{Q})/\overline{C_G(Q)}$ is a p-group, and by [29, Lemma 1] there is a unique $\overline{B}$-subgroup $(\overline{Q}, b_{\overline{Q}})$ corresponding to (Q, b_Q) (where $b_{\overline{Q}}$ is dominated by the unique block of the preimage in G of $C_{\overline{G}}(\overline{Q})$ covering b_Q), and every $\overline{B}$-subgroup may be expressed in this way. It is clear that $l(b_Q) = l(b_{\overline{Q}})$ in each case, so we have:

Lemma 2.3 *Let B be a block of G and $Z \leq O_p(Z(G))$. Let $\overline{B}$ be the unique block of $\overline{G} = G/Z$ dominated by B. If $l(b_Q) = 1$ for each B-subgroup (Q, b_Q), then $l(b_{\overline{Q}}) = 1$ for each $\overline{B}$-subgroup $(\overline{Q}, b_{\overline{Q}})$.*

Lemma 2.4 *Let B be a 2-block of a finite group K, and let $(P, g) \leq (R, b)$ be B-subgroups with $P \leq Z(R)$ and let $Z \leq O_2(Z(C_K(P)))$. Suppose $y \in N_{C_K(P)}(R, b) \setminus C_K(R)$ for some y with $|y| = 3$. Suppose also that either*
(i) $D(g) \cong \mathbb{Z}_{2^c} \wr \mathbb{Z}_2$ for some integer $c \geq 2$ and $R \cong Z(D(g))Q_8$, or
(ii) $D(g)/Z \leq D_{2^{c+1}}$ and $\overline{R} := (RZ)/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Then $\ell(g) \geq 2$.

Proof: Suppose $D(g) \cong \mathbb{Z}_{2^c} \wr \mathbb{Z}_2$. We have $[N_K(R, b){:}C_K(R)R] \geq 3$. Thus case (bB) of [21] cannot occur (see [21, p.533]), and by [21, Propositions (7.G), (14.E), (14.G)], $\ell(g) \geq 2$.

Suppose $D(g)/Z \leq D_{2^{c+1}}$, so that $D(g)/Z$ is $\overline{R}$ or dihedral as $\overline{R} \leq D(g)/Z$. Let $\bar{g}$ and $\bar{b}$ be blocks of $C_G(P)/Z$ and $C_G(R)/Z$ such that $\bar{g}$ and $\bar{b}$ are dominated by g and b, respectively, so that $D(\bar{g}) = D(g)/Z$ and $\ell(g) = \ell(\bar{g})$. By [28], $\ell(\bar{g})$ is the number of 2-weights of $\bar{g}$. If $D(\bar{g}) = \overline{R}$, then $\bar{y} \in N_{K/Z}(D(\bar{g}), \bar{b})$ with $\bar{y} = yZ$ and $\bar{g}$ has three weights $(D(\bar{g}), \psi_i)$, where each ψ_i is an extension of the canonical character $\bar{\theta}$ of $\bar{b}$ to $N_{K/Z}(D(\bar{g}), \bar{b})$. If $D(\bar{g})$ is dihedral, then $N_{K/Z}(\overline{R}, \bar{b})/C_{K/Z}(\overline{R})$ is isomorphic to the symmetric group S_3 and $\bar{g}$ has at least two weights $(\overline{R}, \psi)$, $(D(\bar{g}), \overline{\psi})$, where ψ is the extension of $\bar{\theta}$ to $N_{K/Z}(\overline{R}, \bar{b})$ with $\psi(1) = 2\bar{\theta}(1)$, and $\overline{\psi}$ is the canonical character of the root block $b_{D(\bar{g})}$ of $\bar{g}$. Thus $\ell(g) = \ell(\bar{g}) \geq 2$. □

Recall that for $N \lhd G$, a block B of G is said to dominate the block $\overline{B}$ of G/N if the inflation to G of a simple $k\overline{G}$-module in $\overline{B}$ lies in B.

The next lemma follows from [29, Lemma 2].

Lemma 2.5 *Let Z be a central p-subgroup of a finite group G, $B \in \mathrm{Blk}(G)$ and $\overline{B}$ the block of $\overline{G} := G/Z$ dominated by B. Then B is nilpotent if and only if $\overline{B}$ is nilpotent.*

Let Z be a central p'-subgroup of a finite group G, and write $\overline{H} = HZ/Z$, where $H \leq G$. Let $\overline{B} \in \mathrm{Blk}(\overline{G})$. There is a unique block $B \in \mathrm{Blk}(G)$ dominating $\overline{B}$. By [24, Theorem 5.8.8], $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$ and if D is a defect group of B, then $DZ/Z \cong D$ is a defect group of $\overline{B}$.

If Q is a p-subgroup of G, then $C_{\overline{G}}(\overline{Q}) = C_G(Q)/Z$ (since Z is a central p'-subgroup). Let $(\overline{Q}, b_{\overline{Q}})$ be a $\overline{B}$-subgroup. Then $\overline{Q} = QZ/Z$ for a unique p-subgroup Q of G. Since $C_{\overline{G}}(\overline{Q}) = C_G(Q)/Z$, we may consider the unique subpair (Q, b_Q) with b_Q dominating $b_{\overline{Q}}$, which we call the Brauer pair *dominating* $(\overline{Q}, b_{\overline{Q}})$.

The next lemma, proved in [6], says that (Q, b_Q) must be a B-subgroup, and that dominance of subpairs respects the usual partial order on B-subgroups:

Lemma 2.6 *Let Z be a central p'-subgroup of a finite group G, and let $(\overline{Q}, b_{\overline{Q}})$ and $(\overline{P}, b_{\overline{P}})$ be $\overline{B}$-subgroups, where $\overline{B}$ is the block of $\overline{G}$ dominated by B. Suppose (Q, b_Q) and (P, b_P) are subpairs of G dominating $(\overline{Q}, b_{\overline{Q}})$ and $(\overline{P}, b_{\overline{P}})$, respectively. Then $(\overline{Q}, b_{\overline{Q}}) \leq (\overline{P}, b_{\overline{P}})$ if and only if $(Q, b_Q) \leq (P, b_P)$. In particular, (Q, b_Q) is a B-subgroup.*

As a consequence (see [6]):

Proposition 2.7 *Let G be a finite group, $Z \leq Z(G)$ and $\overline{G} = G/Z$. Suppose $\overline{B} \in \mathrm{Blk}(\overline{G})$ and $B \in \mathrm{Blk}(G)$ dominating $\overline{B}$. Then $\overline{B}$ is nilpotent if and only if B is nilpotent.*

3 The symmetric and alternating groups

Write $\hat{S}_n$ for the double cover of the symmetric group S_n. Then the 2-blocks of $\hat{S}_n$ and of S_n are in one-to-one correspondence under the natural epimorphism, and the block corresponding to a nilpotent block is nilpotent (by [29, Lemma 2]). Hence it suffices to consider S_n (except for the exceptional cases $n = 6, 7$, which we treat separately).

Proposition 3.1 *Let $G = S_n$ with $n \geq 2$. If D is a non-trivial defect group of a nilpotent 2-block, then D is generated by a transposition and $n = 2 + m(m+1)/2$ for some positive integer m. Conversely, if $n = 2 + m(m+1)/2$ for some positive integer m, then S_n possesses precisely one nilpotent 2-block.*

PROOF: Let B be a nilpotent block with non-trivial defect group D. Let $y \in D$ be an involution. Suppose that y is a product of $t > 1$ disjoint transpositions and fixes every other point. Then $C_{S_n}(y) \cong (\mathbb{Z}_2 \wr S_t) \times S_{n-2t}$, and in particular $C_{S_n}(y)$ contains a normal elementary abelian 2-group R generated by t disjoint transpositions. Then $R \leq O_2(C_{S_n}(y))$, and so is contained in every defect group of $C_{S_n}(y)$. Hence by a well-known property of the Brauer correspondence R is contained in a conjugate of D. So D contains an elementary subgroup Q of order 4 such that $C_{S_n}(Q) \cong Q \times S_{n-4}$ and $N_{S_n}(Q) \cong S_4 \times S_{n-4}$. Now every 2-block of $C_{S_n}(Q)$ is $N_{S_n}(Q)$-stable. Since $[N_{S_n}(Q) : C_{S_n}(Q)] = 6$, it follows that B cannot be nilpotent. Hence D is generated by a transposition.

Note that every block with defect groups of order two is nilpotent. The blocks of S_n with defect group D generated by a transposition are in one-to-one correspondence with the blocks of $N_{S_n}(D) \cong D \times S_{n-2}$ with defect group D, and hence in one-to-one correspondence with the blocks of defect zero of S_{n-2}. Since a Young diagram for S_{n-2} is a 2-core precisely when $n - 2 = m(m+1)/2$ for some positive integer m, and in this case there is precisely one such diagram, the result follows by Nakayama's Conjecture. $\square$

Corollary 3.2 *Let $G = A_n$ with $n \geq 3$. Then G possesses no nilpotent 2-block of positive defect.*

PROOF: This follows immediately from Proposition 3.1 and Lemma 2.2, since the defect groups of nilpotent blocks of positive defect of S_n do not lie in A_n. $\square$

Corollary 3.3 *Let G be a quasisimple group with $G/Z(G) \cong A_n$ for some n. Then G has no nilpotent 2-block with non-central defect groups.*

PROOF: The exceptional covers $6.A_6$ and $6.A_7$ follow by observing that every block with non-central defect groups has at least two irreducible Brauer characters (see [14]). The result then follows from Corollary 3.2 and Lemma 2.5. $\square$

Proposition 3.4 *Let B be a 2-block of a quasisimple group G such that $G/Z(G) \cong A_n$ for some n. Then B is nilpotent if and only if $l(b_Q) = 1$ for each B-subgroup (Q, b_Q).*

PROOF: It suffices to show the sufficient condition for nilpotency. Let D be a defect group for B. The exceptional covers $6.A_6$ and $6.A_7$ follow by observing that every block with non-central defect groups has at least two irreducible Brauer characters (see [14]). Hence we may suppose that G is simple or a double cover of some A_n. By Lemmas 2.3 and 2.5 it suffices to consider the case $G \cong A_n$. Let $G \leq E \cong S_n$, and let C be the unique 2-block of E covering B.

By, for example, [17, 6.2.2], $l(C) = p(w)$, where w is the weight of C and $p(w)$ is the number of partitions of w. If $|D| = 2$, then B is nilpotent. Suppose that $|D| \geq 4$, so C has defect at least two. Then $w \geq 2$, and $l(C) = p(w) \geq 2$. Hence we must have $l(B) > 1$, and the result follows. □

4 Sporadic groups and their covers

In this section we determine the nilpotent blocks with non-central defect groups of quasisimple groups G where $G/Z(G)$ is one of the 26 sporadic simple groups. Note that due to Lemma 2.5 it suffices to consider the case $Z(G)$ has odd order.

We show that each 2-block of such a group (where $|Z(G)|$ is odd) with defect greater than one has at least two irreducible Brauer characters. In most cases this may be seen using the library in [14]. As a consequence every nilpotent block of *any* quasisimple group G with $G/Z(G)$ sporadic has abelian defect groups.

Proposition 4.1 *Let G be a quasisimple group such that $G/Z(G)$ is a sporadic simple group. Let B be a 2-block of G and let D be a defect group of B. If $|D| \geq 4$, then $l(B) > 1$.*

PROOF: It suffices to consider the case $|Z(G)|$ odd. If D is cyclic, then the result follows from the theory of blocks with cyclic defect groups. In the following table we list the numbers of irreducible Brauer characters in 2-blocks with non-cyclic defect groups, along with a reference. A '*' will be used to denote a faithful block in a group with non-trivial centre. The result then follows from examination of the table.

Note that since $|Z(G)|$ is odd, if Q is a 2-subgroup of G, then

$$C_{G/Z(G)}(QZ(G))/Z(G)) \cong C_G(Q)/Z(G).$$

We treat each of the cases for which the number of simple kG-modules in each block is not currently given either in [14] or other references of which we are aware.

Suppose $G \cong Ly$. Then G as two blocks of positive defect by [26], the principal block B_0 and a block B_1 of defect seven, and $k(B_0) := |\mathrm{Irr}(B_0)| = 25$. But G has one conjugate class of involutions and one of elements of order 4, and two of order 8, so $k(B_0) = l(B_0) + l(b_1) + l(b_2) + l(b_3) + l(b_3')$, where $b_i = B_0(C_G(z_i))$ with $|z_i| = 2^i$ and $b_3' = B_0(C_G(z_3'))$ such that z_3' is another element of order 8 which is non-conjugate to z_3. Now $C_G(z_1) = \hat{A}_{11}$, $C_G(z_2) = (4 \circ \hat{A}_7).2$, $C_G(z_3) = 8 \circ \hat{A}_5$ and $C_G(z_3') = 8.A_4$, so

$$l(b_1) = l(B_0(A_{11})), \quad l(b_2) = l(B_0(S_7)), \quad l(b_3) = l(B_0(A_5)) \quad \text{and} \quad l(b_3') = 3.$$

The Brauer characters of A_{11} and A_5 are given by [18], and a calculation shows that $l(B_0(A_{11})) = 7$ and $l(B_0(A_5)) = 3$. Since $l(B_0(S_7))$ is the number of partitions of the weight 3, it follows that $l(b_2) = 3$, and $l(B_0) = 25 - 7 - 3 - 3 - 3 = 9$. Since G has 10 blocks of defect 0 and 27 regular classes, it follows that $l(B_1) = 27 - 9 - 10 = 8$.

$G/Z(G)$	$\|D\|$	$\ell(B)$	reference
M_{11}	2^4	3	[14]
M_{12}	$2^6/2^2$	$3/3$	[14]
M_{22}	$2^7/2^7/2^7$	$7/5^*/5^*$	[14]
M_{23}	2^7	9	[14]
M_{24}	2^{10}	13	[14]
J_1	2^3	5	[14]
J_2	$2^7/2^2$	$7/3$	[14]
J_3	$2^7/2^7/2^7$	$10/10^*/10^*$	[14]
J_4	2^{21}	22	[8]
HS	$2^9/2^2$	$6/3$	[14]
McL	$2^7/2^7/2^7$	$8/8^*/8^*$	[14]
Suz	$2^{13}/2^3/2^{13}/2^{13}$	$14/3/14^*/14^*$	[14]
Ly	$2^8/2^7$	$9/8$	[26]
He	$2^{10}/2^3$	$11/3$	[14]
Ru	$2^{14}/2^2$	$6/3$	[14]
$O'N$	$2^9/2^3/2^9/2^9$	$5/3/5^*/5^*$	[14]
Co_3	$2^{10}/2^3$	$10/5$	[14]
Co_2	2^{18}	12	[14]
Co_1	$2^{21}/2^3$	$26/2$	[7]
Fi_{22}	$2^{17}/2^{17}/2^{17}$	$14/11^*/11^*$	[14]
Fi_{23}	$2^{18}/2^3$	$20/2$	[14]
Fi'_{24}	$2^{21}/2^2/2^3/2^{21}/2^{21}/2^3/2^3$	$33/3/3/22^*/22^*/3^*/3^*$	[5]
Th	2^{15}	18	[27]
HN	$2^{14}/2^4$	$17/3$	[14]
$F_2 = B$	$2^{41}/2^3$	$25/2$	
$F_1 = M$	$2^{46}/2^4$	$52/3$	

Table 1: Numbers of irreducible Brauer characters in 2-blocks with non-cyclic, non-central defect groups of covering groups of sporadic simple groups

Suppose $G \cong Th$. Then G has three 2-blocks of defect zero and the principal block (see [27]). By [11] $l(G) = 21$, so $l(B_0(G)) = 18$.

Suppose $G \cong 3.Fi'_{24}$. By [5] there is a unique block of maximal defect and a unique block with D_8 defect groups covering each block of $Z(G)$. Further, there is a non-faithful block with Klein-four defect groups. The non-faithful blocks have 33 or 3 simple kG-modules. Now (see [20]), if $D \cong D_8$ is a defect group, then $(DZ(G)/Z(G))C_{G/Z(G)}(DZ(G)/Z(G)) \cong D_8 \times Sp_6(2)$, so $DC_G(D) \cong Z(G) \times D \times Sp_6(2)$ and the blocks with defect group D have the same number of simple kG-modules, i.e., three. By consideration of character degrees in [11], G as no faithful blocks of defect one. Further, there are 25 simple kG-modules covering each faithful simple $kZ(G)$-module. So the faithful blocks of maximal defect each have 22 simple modules.

Suppose $G \cong F_2$, the baby monster. It is clear from [11] that G has no 2-blocks of defect zero. By [20] G has two blocks of positive defect, the principal block $B_0(G)$ and another B_1 with defect group D_8, where $l(B_1) = 2$. Since $l(G) = 27$, we have $l(B_0(G)) = 25$.

Suppose $G \cong M$, the monster group. By [20] G has two blocks of positive defect: the principal block $B_0(G)$ and one B_1 with semi-dihedral defect groups of order 16. From [11] we see that G has three 2-blocks of defect zero. By [19] we have $l(B_1) = 3$. Since $l(G) = 58$, it follows that $l(B_0(G)) = 52$. $\square$

Corollary 4.2 *Let G be a quasisimple group such that $G/Z(G)$ is a sporadic simple group. Then G has no nilpotent block with defect group D such that $D/O_p(Z(G))$ has of order greater than two. In particular, every nilpotent block of G has abelian defect groups.*

Corollary 4.3 *Let G be a quasisimple group such that $G/Z(G)$ is a sporadic simple group. Then B is nilpotent if and only if $l(b_Q) = 1$ for each B-subgroup (Q, b_Q).*

5 Notation for classical groups

Let V be a linear, unitary, non-degenerate orthogonal or symplectic space over the field $\mathbb{F}_q$, where $q = r^a$ for some odd prime r. We will follow the notation of [4], [9], [12] and [13].

If V is orthogonal, then there is a choice of equivalence classes of quadratic forms. Write $\eta(V)$ for the type of V as defined in [13], so $\eta(V) = \eta = +$ or $-$. Let $\eta(V) = +$ if V is linear and $\eta(V) = -$ if V is unitary. If V is non-degenerate orthogonal or symplectic, then denote by $I(V)$ the group of isometries on V and let $I_0(V) = I(V) \cap \mathrm{SL}(V)$.

If V is symplectic, then $I(V) = I_0(V) = \mathrm{Sp}_{2n}(q)$.

If V is a $(2n+1)$-dimensional orthogonal space, then $I(V) = \langle -1_V \rangle \times I_0(V)$ with $I_0(V) = \mathrm{SO}_{2n+1}(q)$.

If V is a $2n$-dimensional orthogonal space, then $I(V) = O^\eta(V) = O^\eta_{2n}(q)$ and $I_0(V) = \mathrm{SO}^\eta_{2n}(q)$.

If V is a $2n$-dimensional non-degenerate orthogonal or symplectic space, then denote by $J_0(V)$ the conformal isometries of V with square determinant. If V is orthogonal of dimensional at least two, then write $D_0(V)$ for the special Clifford group of V (cf. [13]).

Denote by $\mathrm{GL}^+(V)$ the general linear group $\mathrm{GL}(V)$ and $\mathrm{GL}^-(V)$ the unitary group $U(V)$.

Let $G = \mathrm{GL}^\eta(V)$ or $I(V)$. Write $\mathcal{F}_q = \mathcal{F}_q(G)$ for the set of (monic) polynomials serving as elementary divisors for semisimple elements of G (cf. [4, p.6]). If $G =$

$\mathrm{GL}^-(V)$, then partition $\mathcal{F}_q$ as in [12] by writing:

$$\begin{aligned}\mathcal{F}_1 &= \{\Delta \in \mathcal{F}_q : \Delta \text{ irreducible},\ \Delta \neq X,\ \Delta = \tilde{\Delta}\},\\ \mathcal{F}_2 &= \{\Delta\tilde{\Delta} \in \mathcal{F}_q : \Delta \text{ irreducible},\ \Delta \neq X,\ \Delta \neq \tilde{\Delta}\},\end{aligned}$$

where $\tilde{\Delta}$ is the monic irreducible polynomial whose roots are the q-th power of those of Δ. If $G = I(V)$, then partition $\mathcal{F}_q$ as in [13] by writing:

$$\begin{aligned}\mathcal{F}_0 &= \{X \pm 1\},\\ \mathcal{F}_1 &= \{\Delta \in \mathcal{F}_q - \mathcal{F}_0 : \Delta \text{ irreducible},\ \Gamma \neq X,\ \Delta = \Delta^*\},\\ \mathcal{F}_2 &= \{\Delta\Delta^* \in \mathcal{F}_q - \mathcal{F}_0 : \Delta \text{ irreducible},\ \Delta \neq X,\ \Delta \neq \Delta^*\},\end{aligned}$$

where Δ^* is the monic irreducible polynomial whose roots are the inverses of those of Δ. Let d_Γ be the degree of $\Gamma \in \mathcal{F}_q$, and define the reduced degree δ_Γ as in [4] and [13], so that $\delta_\Gamma = d_\Gamma$ if $G = \mathrm{GL}^\eta(V)$ or $\Gamma \in \mathcal{F}_0$, and $\delta_\Gamma = \frac{1}{2}d_\Gamma$ if $G = I(V)$ and $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$.

For integers c, m, we write $2^c \| m$ when $2^c \mid m$ and $2^{c+1} \nmid m$, and we let $a \geq 2$ be the integer such that $2^{a+1} \| (q^2 - 1)$. Let $\epsilon = \pm 1$, chosen so that $2^a \mid (q - \epsilon)$. For each $\Gamma \in \mathcal{F}_q$, define $\epsilon_\Gamma = 1$ when $G = \mathrm{GL}(V)$, otherwise define

$$\epsilon_\Gamma = \begin{cases} \epsilon & if \quad \Gamma \in \mathcal{F}_0,\\ -1 & if \quad \Gamma \in \mathcal{F}_1,\\ 1 & if \quad \Gamma \in \mathcal{F}_2.\end{cases}$$

Let e_Γ be the multiplicative order of $\epsilon_\Gamma q^{\delta_\Gamma}$ modulo 4. Thus we may write $e_\Gamma \delta_\Gamma = e2^{\alpha_\Gamma}\delta'_\Gamma$ for some α_Γ and δ'_Γ with odd δ'_Γ, where $e = e_{X-1}$.

Given a semisimple element $s \in G$, there is a unique orthogonal decomposition $V = \sum_{\Gamma \in \mathcal{F}_q} V_\Gamma(s)$, with $s = \prod_{\Gamma \in \mathcal{F}_q} s(\Gamma)$, where the $V_\Gamma(s)$ are nondegenerate subspaces of V and $s(\Gamma) \in U(V_\Gamma(s))$, $\mathrm{GL}(V_\Gamma(s))$ or $I(V_\Gamma(s))$ (depending on G) has minimal polynomial Γ. This is called the primary decomposition of s. Write $m_\Gamma(s)$ for the multiplicity of Γ in $s(\Gamma)$. We have $C_G(s) = \prod_{\Gamma \in \mathcal{F}_q} C_\Gamma(s)$, where $C_\Gamma(s) = I(V_\Gamma(s))$ or $\mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma})$ as appropriate.

6 Blocks of linear and unitary groups

Suppose $G = \mathrm{GL}_n^\eta(q) = \mathrm{GL}^\eta(V)$ and r is an odd prime, and let B be a 2-block of G. Then $B = \mathcal{E}_2(G, (s))$ for some semisimple $2'$-element $s \in G$. For convenience we denote $\mathrm{GL}^\eta(V)$ by $G(V)$ and $\mathrm{SL}^\eta(V)$ by $S(V)$.

Theorem 6.1 *Let $G = \mathrm{GL}^\eta(V) = \mathrm{GL}^\eta(n, q)$. Then the following are equivalent.*

(*a*) *B is a nilpotent block of G.*

(*b*) *$m_\Gamma(s) \leq 1$ for all $\Gamma \in \mathcal{F}_q$.*

(*c*) *Let (D, b_D) be a Sylow B-subgroup and θ the canonical character of b_D. Write $C = C_G(D)$. Then $\theta = \pm R_T^C(s)$ with $T = C_C(s)$ a torus of both G and C, and $D = O_2(T)$. Here $R_T^C(s)$ is the Deligne-Lusztig generalized character.*

In particular, if B is nilpotent, then D is abelian.

PROOF: In this proof, to avoid confusion with notation we write $\mathbf{S}(n)$ for the symmetric group on n letters.

Let $s = \prod_\Gamma s(\Gamma)$ be a primary decomposition, so that $V = \bigoplus_\Gamma V_\Gamma$ with $V_\Gamma = V_\Gamma(s)$ the underlying space of $s(\Gamma)$. Thus

$$C_G(s) = \prod_\Gamma C_\Gamma, \quad C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma, q^{\delta_\Gamma}) \tag{6.1}$$

with $m_\Gamma = m_\Gamma(s)$ and $C_\Gamma = C_\Gamma(s)$. We may suppose $D \in \mathrm{Syl}_2(C_G(s))$, so that

$$D = \prod_\Gamma D_\Gamma, \quad D_\Gamma \in \mathrm{Syl}_2(C_\Gamma). \tag{6.2}$$

So D is a direct product of iterated wreath product 2-groups. Write

$$m_\Gamma = n_\Gamma + 2^{c_1+1} + \cdots + 2^{c_t+1} \tag{6.3}$$

where $0 \leq c_1 < \cdots < c_t$ and $n_\Gamma = 0$ or 1 according as m_Γ is even or odd.

Let U_Γ be the underlying space of $\mathrm{GL}^\eta(d_\Gamma, q)$ and $W_\Gamma = U_\Gamma \perp U_\Gamma$ the underlying space of $\mathrm{GL}^\eta(2d_\Gamma, q)$. Let

$$W_\Gamma(i) = W_\Gamma \perp \cdots \perp W_\Gamma \quad (2^{c_i} \text{ terms}) \tag{6.4}$$

for $i > 0$, and $W_\Gamma(0) = U_\Gamma$ or $W_\Gamma(0) = 0$ according as $n_\Gamma = 1$ or 0.

Let $Y_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(2, q^{\delta_\Gamma})$ be a regular subgroup of $G(W_\Gamma) = \mathrm{GL}^\eta(2d_\Gamma, q)$ and $S_\Gamma \in \mathrm{Syl}_2(Y_\Gamma)$. Then

$$V_\Gamma = W_\Gamma(0) \perp W_\Gamma(1) \perp \cdots \perp W_\Gamma(t), \quad D_\Gamma = S_0 \prod_{i \geq 1} (S_\Gamma \wr X_{2^{c_i}}), \tag{6.5}$$

where $S_\Gamma \wr X_{2^{c_i}} \leq (Y_\Gamma)^{2^{c_i}} \leq G(W_\Gamma(i))$ for $i > 1$, and $S_0 = O_2(\mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma}))$ or 1 according as $n_\Gamma = 1$ or 0, and $X_{2^{c_i}} \in \mathrm{Syl}_2(\mathbf{S}(2^{c_i}))$.

For $i \geq 1$, set $E(S_\Gamma \wr X_{2^{c_i}}) = (S_\Gamma)^{2^{c_i}}$ to be the base subgroup of $S_\Gamma \wr X_{2^{c_i}}$ and let

$$E(D) := \prod_\Gamma E(D_\Gamma), \quad E(D_\Gamma) := S_0 \prod_{i \geq 1} E(S_\Gamma \wr X_{2^{c_i}}) = S_0 \times (S_\Gamma)^{(m_\Gamma - n_\Gamma)/2}, \tag{6.6}$$

so that $E(D) \lhd D$ and each component is either cyclic or equal to S_Γ for some Γ. Thus $E(D)$ is abelian if and only if D is abelian if and only if each $m_\Gamma \leq 1$ if and only if $C_G(s)$ is a maximal torus of G.

Let $H_\Gamma \lhd Y_\Gamma$ such that $H_\Gamma \cong \mathrm{SL}(2, q^{\delta_\Gamma})$, so that $S_\Gamma \cap H_\Gamma \in \mathrm{Syl}_2(H_\Gamma)$ and $S_\Gamma \cap H_\Gamma$ is generalized quaternion. By [3, (1H) (a)], $S_\Gamma \cap H_\Gamma$ contains a quaternion subgroup $Q_\Gamma \cong Q_8$ such that

$$N_{H_\Gamma}(Q_\Gamma)/Q_\Gamma = \mathbb{Z}_3 \quad \text{or} \quad \mathbf{S}(3) \tag{6.7}$$

according as $q^{\delta_\Gamma} \equiv \pm 3 \bmod 8$ or $q^{\delta_\Gamma} \equiv \pm 1 \bmod 8$. Note that Q_Γ is unique up to Y_Γ-conjugacy (see [3, (1G)]).

Let $x_\Gamma \in Q_\Gamma$ such that $|x_\Gamma| = 4$ and x_Γ is primary viewed as an element of $G(W_\Gamma)$, so that x_Γ is uniquely determined up to conjugacy in $G(W_\Gamma)$. Let $A(S_\Gamma) := Z(S_\Gamma)\langle x_\Gamma\rangle \leq S_\Gamma$ and $L_0(\Gamma) = C_{G(U_\Gamma)}(S_0)$, so that

$$L(\Gamma) := C_{G(W_\Gamma)}(A(S_\Gamma)) = \mathrm{GL}^\epsilon(\delta'_\Gamma, q^{e2^{\alpha_\Gamma}}) \times \mathrm{GL}^\epsilon(\delta'_\Gamma, q^{e2^{\alpha_\Gamma}}) \text{ or } \mathrm{GL}^\epsilon(d_\Gamma, q^{e_\Gamma}) \tag{6.8}$$

and $L_0(\Gamma) = \mathrm{GL}^\epsilon(\delta'_\Gamma, q^{e2^{\alpha_\Gamma}})$ or $G(U_\Gamma)$ according as $4 \mid (q^{\delta_\Gamma} - \epsilon_\Gamma)$ or $4 \nmid (q^{\delta_\Gamma} - \epsilon_\Gamma)$. Let

$$A(D) := \prod_\Gamma A(D_\Gamma), \quad A(D_\Gamma) := S_0 \times A(S_\Gamma)^{(m_\Gamma - n_\Gamma)/2} \leq E(D_\Gamma).$$

Then $A(D)$ is abelian and $A(D) = D$ if and only if D is abelian. In addition,

$$C_G(A(D)) = \prod_\Gamma (L_0(\Gamma)^{n_\Gamma} \times L(\Gamma)^{(m_\Gamma - n_\Gamma)/2}), \tag{6.9}$$

where $L(\Gamma)$ is given by (6.8) and $L_0(\Gamma)^{n_\Gamma} := 1$ when $n_\Gamma = 0$. Thus $C_G(A(D))$ is regular in G, $s \in C_G(A(D))$ and

$$C_{C_G(A(D))}(s) = \prod_\Gamma (\mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})^{n_\Gamma} \times \mathrm{GL}^{\epsilon_\Gamma}(1, q^{e_\Gamma \delta_\Gamma})^{(m_\Gamma - n_\Gamma)/e_\Gamma}). \tag{6.10}$$

Let $(A(D), b)$ be a B-subgroup, so that $b = \mathcal{E}_2(C_G(A(D)), (s))$ and $D(b)$ is a Sylow 2-subgroup of $C_{C_G(A(D))}(s)$. But $C_{C_G(A(D))}(s)$ is a maximal torus, so $D(b) = O_2(C_{C_G(A(D))}(s))$ is abelian.

Suppose $m_\Delta \geq 2$ for some Δ. By (6.7), there exists $u_\Delta \in N_{H_\Delta}(Q_\Delta)$ with $|u_\Delta| = 3$ and three elements $x_{\Delta,1}, x_{\Delta,2}, x_{\Delta,3}$ of Q_Δ such that $|x_{\Delta,i}| = 4$ and u_Δ acts transitively on the set $\{x_{\Delta,1}, x_{\Delta,2}, x_{\Delta,3}\}$, Q_Δ is generated by any two distinct elements in $\{x_{\Delta,1}, x_{\Delta,2}, x_{\Delta,3}\}$ and $[u_\Delta, Z(S_\Delta)] = 1$.

Let $y_\Delta \in G(V_\Delta)$ such that

$$[\,V_\Delta, y_\Delta\,] = [\,V_\Delta, H_\Delta\,] = W_\Delta, \quad \text{and} \quad (y_\Delta)|_{[\,V_\Delta, y_\Delta\,]} = u_\Delta$$

and set $y_\Gamma = 1$ when $\Gamma \neq \Delta$. Define $A_i(S_\Delta) := Z(S_\Delta)\langle x_{\Delta,i}\rangle \leq Z(S_\Delta)Q_\Delta$, $A_i(D_\Delta) := S_0 \times A(S_\Delta)^{(m_\Delta - n_\Delta)/2 - 1} \times A_i(S_\Delta) \leq E(D_\Delta)$, and

$$A_i(D) := \prod_{\Gamma \neq \Delta} A(D_\Gamma) \times A_i(D_\Delta), \quad y := \prod_\Gamma y_\Gamma. \tag{6.11}$$

Then $y \in C_G(s)$, $|y| = 3$, $\det y = 1$ and y permutes transitively the three abelian subgroups $\{A_1(D), A_2(D), A_3(D)\}$. Note that $C_G(A_i(D))$ is given by (6.9),

$$y = 1_{C_V(y)} \times u_\Delta, \quad [V, y] = W_\Delta, \quad A_i(S_\Delta) \leq Z(S_\Delta)Q_\Delta \leq Z(S_\Delta)H_\Delta \leq Y_\Delta \leq G(W_\Delta).$$

Let $Q(D_\Delta) = \langle A_1(D_\Delta), A_2(D_\Delta)\rangle$ and $Q(D) = \langle A_1(D), A_2(D)\rangle$, so that

$$Q(D) = \prod_{\Gamma \neq \Delta} A(D_\Gamma) \times Q(D_\Delta), \; Q(D_\Delta) = S_0 \times A(S_\Delta)^{(m_\Delta - n_\Delta)/2 - 1} \times Z(S_\Delta)Q_\Delta.$$

Thus $C_{G(V_\Delta)}(Q(D_\Delta)) = L_0(\Delta)^{n_\Delta} \times L(\Delta)^{(m_\Delta - n_\Delta)/2-1} \times C_{G(W_\Delta)}(Z(S_\Delta)Q_\Delta)$,

$$C_{G(W_\Delta)}(Z(S_\Delta)Q_\Delta) = \mathrm{GL}^\epsilon(\delta'_\Delta, q^{e2^{\alpha_\Delta}}) \otimes I_2 := \{x \otimes I_2 : x \in \mathrm{GL}^\epsilon(\delta'_\Delta, q^{e2^{\alpha_\Delta}})\}, \quad (6.12)$$

and

$$C_G(Q(D)) = \prod_{\Gamma \neq \Delta} (L_0(\Gamma)^{n_\Gamma} \times L(\Gamma)^{(m_\Gamma - n_\Gamma)/2}) \times C_{G(V_\Delta)}(Q(D_\Delta)). \quad (6.13)$$

Since $A_1(D) \leq Q(D) \leq D \leq C_G(s)$ and since $C_G(A_1(D)) \cap C_G(s)$ is a torus, it follows that $C_{C_G(Q(D))}(s)$ is a torus.

Let $(Q(D), b_Q)$ be a B-subgroup, so that we may suppose $D(b_Q) \leq C_{C_G(Q(D))}(s)$. Thus $D(b_Q)$ is abelian and $D(b_Q) = O_2(C_{C_G(Q(D))}(s))$.

If we identify $C_{G(W_\Delta)}(Z(S_\Delta)Q_\Delta)$ with $\mathrm{GL}^\epsilon(\delta'_\Delta, q^{e2^{\alpha_\Delta}})$, then by (6.12),

$$\langle Z(S_\Delta)Q_\Delta, u_\Delta\rangle \leq C_{G(W_\Delta)}(C_{G(W_\Delta)}(Z(S_\Delta)Q_\Delta)) \cong \mathrm{GL}_2(q)Z(\mathrm{GL}^\epsilon(\delta'_\Delta, q^{e2^{\alpha_\Delta}})),$$

since $\mathrm{GL}_2(q) \otimes \mathrm{GL}^\eta(d_\Delta, q) \leq G(W_\Delta)$ and $\mathrm{GL}^\epsilon(\delta'_\Delta, q^{e2^{\alpha_\Delta}}) \leq G(U_\Delta) = \mathrm{GL}^\eta(d_\Delta, q)$. Thus

$$\langle Q(D), y\rangle \leq C_G(C_G(Q(D))) \quad (6.14)$$

and in particular, y centralizes $C_G(Q(D))$, and so $b_Q^y = b_Q$. Thus if B is nilpotent, then $m_\Gamma(s) \leq 1$ for all $\Gamma \in \mathcal{F}_q$.

Conversely, if $m_\Gamma(s) \leq 1$ for all $\Gamma \in \mathcal{F}_q$, then $C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$ and so $N_{C_\Gamma}(D_\Gamma) = C_{C_\Gamma}(D_\Gamma) = C_\Gamma$. Thus D is abelian,

$$N_{C_G(s)}(D) = C_{C_G(s)}(D) = C_G(s).$$

The canonical character of b_D is labelled by $(s, 1)$ and is stable in $N_G(D, b_D)$. Let $x \in N_G(D, b_D)$. Then s^x and s are $C_G(D)$-conjugate elements of the abelian group $C_G(D)$, and so $s^x = s$. Hence $x \in C_G(s) \leq C_G(D)$, and we have shown $N_G(D, b_D) = C_G(D)$. By Proposition 2.1, B is nilpotent.

The equivalence of (b) and (c) follows by [9]. $\square$

Recall that for integers c, m, we write $2^c \| m$ when $2^c \mid m$ and $2^{c+1} \nmid m$, and we let $a \geq 2$ be the integer such that $2^{a+1} \| (q^2 - 1)$.

Remark 6.2 (*a*) *Suppose $G = \mathrm{GL}^\eta(2d_\Delta, q) = G(W_\Delta) = G(V)$ with $2 \mid d_\Delta$. Following the notation in the proof above, let $Q'(D) = Q_\Delta \leq S(W_\Delta) \cap D$ and $A'_i(D) = \langle x_{\Delta,i}\rangle \leq Q_\Delta$. Then $Q'(D) \leq S(V)$,*

$$C_G(A'_i(D)) = \mathrm{GL}^\eta(d_\Delta, q) \times \mathrm{GL}^\eta(d_\Delta, q) \quad \text{or} \quad \mathrm{GL}^\epsilon(d_\Delta, q^e) \quad (6.15)$$

according as $4 | (q - \eta)$ or $2 \| (q - \eta)$, and $C_G(Q'(D)) = \mathrm{GL}^\epsilon(d_\Delta, q) \otimes I_2$. In addition, if $(Q'(D), b_{Q'})$ is a B-subgroup, then $D(b_{Q'})$ is abelian, y centralizes $C_G(Q'(D))$ and so $b_{Q'}^y = b_{Q'}$.

(*b*) *Suppose $m_\Delta \geq 2$ and follow the notation in the proof above. Then*

$$V = V_1 \perp V_2, \quad D = D_1 \times D_2, \quad (6.16)$$

where D_1 is abelian and each direct component of D_2 is a wreath product of the form $S_\Gamma \wr X_{2^c}$, $V_1 = [D_1, V]$, $V_2 = [D_2, V]$. Thus

$$Q(D) = D_1 \times Q(D_2), \quad Q(D_2) = Q_0(D_2) \times Q_1(D_2), \quad Q_1(D_2) = Z(S_\Delta)Q_\Delta,$$

where each direct component of $Q_0(D_2)$ has the form $Z(S_\Gamma)\langle x_\Gamma\rangle$. Let $Q = Q(D)$ or $Z(Q(D))$, $Q_2 = Q(D_2)$ or $Z(Q(D_2))$ and $R = Q \cap S(V)$. Then $C_G(R) = C_G(Q)$ except when the rank of $\Omega_1(D_1)$ is 2 and $D_1 \cap S(V_1) = \{\pm 1_{V_1}\}$, in which case,

$$C_G(R) = G(V_1) \times C_{G(V_2)}(Q_2).$$

In particular, if $(R, b_R) \le (Q, g_Q) \le (Q(D), b_Q)$, then $D(b_R) = D(g_Q)$ and $b_R^y = b_R$ as $y \in S(V_2)$ and y centralizes $C_{G(V_2)}(Q(D_2))$.

(*c*) *Following the notation of (b), let $P = Z(Q(D))$, so that $C_G(P)$ is regular and $y \in C_G(P) \cap S(V)$. If $Z_G := O_2(C_G(P))$ and $(P, b_P) \le (Q(D), b_Q)$, then $D(b_P)/Z_G \cong D_{2^{c+1}}$ with $c \ge a$, $Z_G \le Z(C_G(P))$, and yZ_G normalizes $(Q(D)Z_G)/Z_G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof of (*a*) Since $2 \mid d_\Delta$, it follows that $\epsilon_\Delta = 1$ and $D = \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ with $2^{a+1+\alpha} \| (q^{d_\Delta} - 1)$. Since $\mathrm{SL}(2, q^{e2^{\alpha_\Delta}})$ contains subgroups $\mathbb{Z}_{q^{e2^{\alpha_\Delta}} \pm 1}$, it follows that $D \le S(V)$. The rest of (a) is clear.

Proofs of (*b*) *and* (*c*) If $4 | (q^{\delta_\Gamma} - \epsilon_\Gamma)$, then $S_\Gamma = \mathbb{Z}_{2^c} \wr \mathbb{Z}_2$ and $Z(S_\Gamma) = O_2(\mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma}))$, where $c \ge a \ge 2$. If $4 \nmid (q^{\delta_\Gamma} - \epsilon_\Gamma)$, then S_Γ is a semidihedral group $SD_{2^{a+2}}$ and $Z(S_\Gamma) = \{\pm 1_{W_\Gamma}\}$. So $C_{G(V_2)}(R_2) = C_{G(V_2)}(Q_2)$ with $R_2 = Q_2 \cap S(V_2)$ and the rest of (b) is clear.

Let $\eta_\Gamma = 0$ or 2 according as $\Gamma \ne \Delta$ or $\Gamma = \Delta$. Then $P = D_1 \times Q_0(D_2) \times Z(S_\Delta)$,

$$C_G(P) = \prod_\Gamma (L_0(\Gamma)^{n_\Gamma} \times L(\Gamma)^{(m_\Gamma - n_\Gamma)/2 - \eta_\Gamma}) \times \mathrm{GL}^\epsilon(2\delta'_\Delta, q^{e2^{\alpha_\Delta}}).$$

Let $M(H) = C_H([H, H])$ for any $H \le G$. Then $M(S_\Gamma) \cong \mathbb{Z}_{2^c} \times \mathbb{Z}_{2^c}$ or $\mathbb{Z}_{2^{a+1}}$ according as $4 \mid (q^{\delta_\Gamma} - \epsilon_\Gamma)$ or $4 \nmid (q^{\delta_\Gamma} - \epsilon_\Gamma)$, and in the former case, $M(S_\Gamma)$ is the base subgroup of S_Γ. In both cases, $C_{G(W_\Gamma)}(M(S_\Gamma)) = L(\Gamma)$ and $O_2(L(\Gamma)) = M(S_\Gamma)$. Thus

$$O_2(C_G(P)) = \prod_\Gamma (O_2(L_0(\Gamma)^{n_\Gamma}) \times M(S_\Gamma)^{(m_\Gamma - n_\Gamma)/2 - \eta_\Gamma}) \times Z(S_\Delta), \qquad (6.17)$$

$D(b_P) = O_2(C_{C_G(P)}(s)) = \prod_\Gamma (O_2(L_0(\Gamma))^{n_\Gamma} \times M(S_\Gamma)^{(m_\Gamma - n_\Gamma)/2 - \eta_\Gamma}) \times S_\Delta$ and $O_2(C_G(P)) \le Z(C_G(P))$. Thus $D(b_P)/Z_G \cong S_\Delta/Z(S_\Delta) \cong D_{2^{c+1}}$ or $D_{2^{a+1}}$. Since $y \in N_{C_G(P)}(Q(D))$ and $Z_G \le Z(C_G(P))$, it follows that yZ_G normalizes $(Q(D)Z_G)/Z_G$.

7 A condition related to nilpotency

In order to investigate nilpotent blocks of exceptional groups of Lie type it is not sufficient just to find the nilpotent blocks of classical groups. We need in addition a condition stronger than nilpotency which we will see occurs for blocks of classical groups with nonabelian defect groups. In particular it species an element of order three causing the block to be nonnilpotency, and relates blocks with those of covering groups. We have in mind groups of Lie type and extensions by diagonal automorphisms.

Property 7.1 *Let K be a normal subgroup of a finite group H and $p = 2$, and let $B \in \mathrm{Blk}(K)$ and $B_H \in \mathrm{Blk}(H)$ such that B_H covers B.*

(a) *There exist B-subgroups $(P, g) \le (R, b)$ and an element $y \in N_{C_K(P)}(R) \setminus C_K(R)$ such that $P \le Z(R)$, $\ell(g) \ge 2$, $D(b)$ is abelian,*

$$|y| = 3 \quad \text{and} \quad b^y = b.$$

Moreover, if (R, b_H) is a B_H-subgroup with b_H covering b, then $D(b_H)$ is abelian and $b_H^y = b_H$.

(b) *Both $D(B)$ and $D(B_H)$ are abelian.*

Proposition 7.2 *Suppose one of Property 7.1 (a) or (b) holds for a block B. Then B is nilpotent if and only if $l(b_Q) = 1$ for every B-subgroup (Q, b_Q).*

Proof: If (a) holds, then the result is immediate since $l(g) > 1$. If (b) holds, then this is [25, Theorem 3]. □

We prove a lemma which will be useful in establishing the given properties. Let H be a finite group, $K \lhd H$, $Z \le Z(H) \cap K$ and $\overline{K} := K/Z \le \overline{H} := H/Z$. Let $\overline{B} \in \mathrm{Blk}(\overline{K})$ and $B \in \mathrm{Blk}(K)$ dominating $\overline{B}$, and (Q, b_Q) a B-subgroup. Let $\gamma : H \to \overline{H}$ be the natural homomorphism and $\overline{X} = \gamma(X)$ for any $X \subseteq H$,

If Z is a p'-group, then $(\overline{Q}, \bar{b}_Q)$ is defined in Section 2 and it is a $\overline{B}$-subgroup. Suppose Z is a p-group. Then $\gamma^{-1}(C_{\overline{K}}(\overline{Q})) \le N_K(QZ)$ and $\gamma^{-1}(C_{\overline{K}}(\overline{Q}))/C_K(Q)$ is a p-group. Thus $\gamma^{-1}(C_{\overline{K}}(\overline{Q}))$ has a unique block $\hat{b}_Q$ covering b_Q and we denote $\bar{b}_Q$ the block of $C_{\overline{K}}(\overline{Q})$ corresponding to $\hat{b}_Q$, so that by [29, Lemma 1], $(\overline{Q}, \bar{b}_Q)$ is also a $\overline{B}$-subgroup.

In general, since $\overline{K} \cong (K/O_p(Z))/(Z/O_p(Z))$ and $Z/O_p(Z) \le Z(K/O_p(Z))$, it follows that $(\overline{Q}, \bar{b}_Q)$ is defined and is a $\overline{B}$-subgroup.

Lemma 7.3 *Let H be a finite group, $K \lhd H$, $Z \le Z(H) \cap K$. Define $\overline{K} := K/Z$ and $\overline{H} := H/Z$. Let $\overline{B} \in \mathrm{Blk}(\overline{K})$ and $B \in \mathrm{Blk}(K)$ dominating $\overline{B}$. Suppose B-subgroups $(P, g) \le (R, b)$ satisfy Property 7.1 (a). In addition, if $Z = O_2(Z)$, then suppose, moreover that $C_H(P)/Z = C_{\overline{H}}(\overline{P})$ and $C_H(R)R/Z = C_{\overline{H}}(\overline{R})$. Then the $\overline{B}$-subgroups $(\overline{P}, \bar{g}) \le (\overline{R}, \bar{b})$ satisfy Property 7.1 (a).*

Proof: Let $\overline{B}_{\overline{H}} \in \mathrm{Blk}(\overline{H})$ covering $\overline{B}$, and $B_H \in \mathrm{Blk}(H)$ dominating $\overline{B}_{\overline{H}}$ and $\chi \in \mathrm{Irr}(\overline{B}_{\overline{H}})$, so that χ covers some $\psi \in \mathrm{Irr}(\overline{B})$. But $\mathrm{Irr}(\overline{B}_{\overline{H}}) \subseteq \mathrm{Irr}(B_H)$ and $\mathrm{Irr}(\overline{B}) \subseteq \mathrm{Irr}(B)$, so B_H covers B.

Let f be the unique block of Z covered by B. Then each character χ of $\mathrm{Irr}(B)$ covers a character of $\mathrm{Irr}(f)$. Since $\mathrm{Irr}(\overline{B}) \subseteq \mathrm{Irr}(B)$, it follows that f is principal. Since (P, g) is a B-subgroup and $Z \le Z(K)$ and since B covers f, it follows that g covers f, and similarly, b covers f.

Since $C_K(PZ) = C_K(P)$, it follows that we may suppose $O_2(Z) \le P$. Let $y \in N_{C_K(P)}(R) \setminus C_K(R)$ with $y^3 = 1$ and $b^y = b$.

We may suppose Z is either a 2-group or a $2'$-group. Then by [29, Lemma 1 (iii)] and Lemma 2.6, $(\overline{P}, \bar{g}) \le (\overline{R}, \bar{b})$ are $\overline{B}$-subgroups. If Z is a $2'$-group, then $C_K(R)/Z =$

$C_{\overline{K}}(\overline{R})$. If Z is a 2-group, then $C_H(R)/Z = C_{\overline{H}}(\overline{R})$, $\gamma^{-1}(C_{\overline{H}}(\overline{R})) = C_H(R)$ and so $\gamma^{-1}(C_{\overline{K}}(\overline{R})) = C_K(R)$. Thus $C_K(R)/Z = C_{\overline{K}}(\overline{R})$ and $D(b)/Z = D(\bar{b})$. So $D(\bar{b})$ is abelian as $D(b)$ is abelian.

Let $(\overline{R}, \bar{b}_{\overline{H}})$ be a $\overline{B}_{\overline{H}}$-subgroup with $\bar{b}_{\overline{H}}$ covering $\bar{b}$. There exists a B_H-subgroup (R, b_H) such that b_H dominates $\bar{b}_{\overline{H}}$. Then b_H covers b and so b_H has an abelian defect group $D(b_H)$ and hence $D(\bar{b}_{\overline{H}}) = D(b_H)Z/Z$ is abelian.

Since $|y| = 3$ and $y \in N_{C_K(P)}(R) \setminus C_K(R)$, it follows that $|\bar{y}| = 3$ and $\bar{y} \in N_{C_{\overline{K}}(\overline{P})}(\overline{R}) \setminus C_{\overline{K}}(\overline{R})$. Since the canonical character θ_H of b_H is the lift of the canonical character $\bar{\theta}_H$ of $\bar{b}_H$ and since $\theta_H^y = \theta_H$, it follows that $\bar{b}_H^{\bar{y}} = \bar{b}_H$. Similarly, $\bar{b}^{\bar{y}} = \bar{b}$ and since $\ell(\bar{g}) = \ell(g)$, it follows that $\ell(\bar{g}) \geq 2$. $\square$

8 Classical groups

Proposition 8.1 *Let $K := \mathrm{SL}_n^\eta(q) \leq H \leq G := \mathrm{GL}_n^\eta(q) = \mathrm{GL}^\eta(V)$, $Z \leq Z(K)$, $B \in \mathrm{Blk}(K)$, $B_H \in \mathrm{Blk}(H)$ covering B. Let $B_G \in \mathrm{Blk}(G)$ be a weakly regular cover of B_H, and $R := Q(D(B_G)) \cap K$, except when $D(B_G) = \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ for some $\alpha \geq 0$, in which case $R = Q'(D)$ as defined in Remark* 6.2 *(a). Then either Property* 7.1 *(a) holds for $(Z(R), g) \leq (R, b)$ with $C_H(Z(R))/Z = C_{\overline{H}}(Z(R)/Z)$ and $C_H(R)R/Z = C_{\overline{H}}(\overline{R})$, or Property* 7.1 *(b) holds.*

Proof: Suppose $B_G = \mathcal{E}_2(G, (s))$. Since B_H covers B, it follows that $D(B) = D(B_H) \cap K$ for some defect group $D(B_H)$. There exists a defect group $D(B_G)$ such that $D(B_H) = D(B_G) \cap H$, so

$$D(B) = D(B_H) \cap K = D(B_G) \cap K \quad \text{and} \quad D(B_H) = D(B_G) \cap H.$$

We may suppose $D(B_G) \in \mathrm{Syl}_2(C_G(s))$.

Set $R_G = Q(D(B_G))$ or $Q'(D(B_G))$ according as $D(B_G) \ncong \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ for all $\alpha \geq 0$ or $D(B_G) \cong \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ for some $\alpha \geq 0$. Then R_G and $C_G(R_G)$ are given by (6.13) or Remark 6.2 (a) with $Q(D)$ or $Q'(D)$ replaced by R_G.

In the notation of the proof of Theorem 6.1, suppose $m_\Gamma \leq 1$ for all Γ. Then $D(B_G)$ is abelian, and both $D(B)$ and $D(B_H)$ are abelian. So we suppose, moreover $m_\Delta \geq 2$ for some Δ. There exists $y \in C_G(s) \cap K$ such that $y \in N_G(R_G) \setminus C_G(R_G)$, $|y| = 3$, $y|_{V_\Gamma} = 1_{V_\Gamma}$ for all $\Gamma \neq \Delta$. Let $R_H := R_G \cap H$ and $R = R_G \cap K$.

(1). Note that $C_G(R)$ is given in Remark 6.2 (b). Let (R, b) be a B-subgroup, (R, b_H) a B_H-subgroup such that b_H covers b, and (R, b_G) a B_G-subgroup such that b_G coves b_H. By Remark 6.2 (b) and the proof of Theorem 6.1, $D(b_G)$ is abelian, so both $D(b)$ and $D(b_H)$ are abelian as we may suppose $D(b) = D(b_H) \cap C_K(R)$ and $D(b_H) = D(b_G) \cap C_H(R)$.

Let $P = Z(R)$, $(P, g) \leq (R, b)$, $(P, g_G) \leq (R, b_G)$ such that g_G covers g and $y \in N_{C_K(P)}(R, b_G) \setminus C_G(R)$ given by (6.11). By (6.14), y centralizes $C_G(R)$, so $b^y = b$ and hence $y \in N_{C_K(P)}(R, b)$ as $C_K(R) \leq C_G(R)$. Similarly, $y \in N_{C_H(P)}(R, b_H)$.

Suppose $m_\Delta = 2$ and $m_\Gamma = 0$ for all $\Gamma \neq \Delta$. Then $R = O_2(Z(K))Q_8$ or Q_8 according as d_Δ is odd or even. Thus $G = \mathrm{GL}^\eta(2d_\Delta, q)$, $P = Z(R) \leq Z(G)$ and so $C_G(P) = G$. Suppose $x \in G$ such that for any $u \in R$, $x^{-1}ux = zu$ for some

$z \in O_2(Z(G))$, so that $x \in N_G(R)$. Since $\mathrm{Out}(Q_8) = O_2^-(2) = \mathrm{GL}_2(2)$, it follows that if $x \notin C_G(R)R$, then $x^{-1}y_1x = y_1$ and $x^{-1}y_2x = y_1y_2$ for some $y_i \in R$ with $\langle y_1, y_2\rangle \cong Q_8$. Now $zy_2 = x^{-1}y_2x = y_1y_2$ and $y_1 = z \in Z(G)$, which is impossible. It follows that $x \in C_G(R)R$ and so $C_G(R)R/Z = C_{G/Z}(R/Z)$.

Suppose $m_\Delta \geq 3$ or $m_\Delta = 2$ but $m_\Gamma \geq 1$ for some $\Gamma \neq \Delta$. A similar proof to above shows that for any $1 \neq Z \leq Z(K)$, $C_G(P)/Z = C_{G/Z}(P/Z)$ and $C_G(R)R/Z = C_{G/Z}(R/Z)$.

(2). Suppose $G = \mathrm{GL}^\eta(2d_\Delta, q)$ and $D(B_G) = \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ with $\alpha \geq 0$ for some Δ. By Remark 6.2 (a), $D(B) = D(B_G)$, $R = Q_8$, $P \leq Z(K)$ and $g = B$. But $Q(D(B)) = Z(D(B))Q_8$ and $y \in N_K(Q(D), b_Q)$, so by Lemma 2.4, $\ell(B) \geq 2$.

Suppose $D(B_G) \ncong \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$, so that $R_G = Q(D(B_G))$. Let $P_G = Z(R_G)$, $Z_G = O_2(C_G(P_G))$ and $Z_K = Z_G \cap K$. By Remark 6.2 (c), $Z_G \leq Z(C_G(P_G))$ and $D(g_G)/Z_G = D_{2^{c+1}}$ with $c \geq a$. Since $R \leq D(g) = D(g_G) \cap K$, it follows that $\overline{R} := R/Z_K \leq D(g)/Z_K \leq D_{2^{c+1}}$. By Lemma 2.4, $\ell(g) \geq 2$ and Property 7.1 (a) holds. □

Let V be a non-degenerate orthogonal or symplectic space, $G = I_0(V)$ and let G^* be the dual group of G. Then

$$\mathrm{Sp}_{2n}(q)^* = \mathrm{SO}_{2n+1}(q), \quad \mathrm{SO}_{2n+1}(q)^* = \mathrm{Sp}_{2n}(q), \quad \mathrm{SO}_{2n}^\eta(q)^* = \mathrm{SO}_{2n}^\eta(q).$$

If B is a 2-block of $I_0(V)$, then $B = \mathcal{E}_2(I_0(V), (s))$ for a semisimple $2'$-element $s \in I_0(V)^*$. Let (D, b_D) be a Sylow B-subgroup. By [2, (5.1)], V and D have decompositions

$$V = V_0 \perp V_+, \quad D = D_0 \times D_+, \tag{8.1}$$

where $D_0 \leq I_0(V_0)$ is an elementary abelian 2-subgroup and $D_+ \leq I_0(V_+)$. Let $G_0 := I_0(V_0)$, $G_+ := I_0(V_+)$, $C_+ := C_{I_0(V_+)}(D_+)$ and let V^* be the underlying space of $I_0(V)^*$.

Let $z \in D$ be a primitive element. Then z is given by [2, Remark 2.2.9 (2)], so $|z| = 4$ and $[\,V, D_+\,] = [\,V, z\,] = V_+$. Thus

$$z = z_0 \times z_+, \quad L := C_G(z) = L_0 \times L_+, \quad L_0 = G_0, \quad L_+ := \mathrm{GL}^\epsilon(m, q), \tag{8.2}$$

where $z_0 = 1_{V_0}$, $z_+ \leq D_+$ and $\dim V_+ = 2m$. Then L is a regular subgroup of G and we may suppose $s \in L^* \leq G^*$. In particular,

$$V^* = U_0 \perp U_+ \quad \text{and} \quad s = s_0 \times s_+, \tag{8.3}$$

where $U_0 = V_0^*$, $s_0 \in L_0^* = I_0(U_0)$, U_+ is the underlying space of L_+^* and $s_+ \in L_+^* \leq I_0(U_+)$.

Let $C_{I(U_+)}(s_+) = \prod_\Gamma C_\Gamma$ and let U_Γ be the underlying vector space of C_Γ, so that

$$C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s_+), q^{\delta_\Gamma}) \quad \text{or} \quad I(U_\Gamma) \tag{8.4}$$

according as $\Gamma \neq X \pm 1$ or $\Gamma = X \pm 1$.

Proposition 8.2 *Let $K := \Omega_{2n}^\eta(q) := \Omega^\eta(V) \leq H \leq J_0(V)$, $B_K \in \mathrm{Blk}(K)$ and $B_H \in \mathrm{Blk}(H)$ covering B_K. Either Property 7.1 (a) holds for some B_K-subgroups $(Z(R), g) \leq (R, b)$ with $C_H(Z(R))/Z = C_{H/Z}(Z(R)/Z)$ and $C_H(R)R/Z = C_{H/Z}(R/Z)$ for any $Z \leq Z(K)$, or Property 7.1 (b) holds.*

Proof: Let $G := \mathrm{SO}^{\eta}_{2n}(q) := \mathrm{SO}(V)$ and $B \in \mathrm{Blk}(G)$ covering B_K. Then $D(B_K) = D \cap K$ for some defect group $D := D(B)$. Since G is self dual, we have $V = V^*, U_0 = V_0, U_+ = V_+$ in (8.3).

(1). We may suppose $D \in \mathrm{Syl}_2(C_G(s))$ and so $D(B_K) \in \mathrm{Syl}_2(C_K(s))$. In particular, D is abelian if and only if $m_\Gamma(s) \leq 1$ if and only if $C_G(s)$ is a maximal torus with $D = O_2(C_G(s))$, if and only if $D(B_K)$ is abelian.

Suppose $m_\Delta(s) \geq 2$ for some Δ, so that $m_\Delta(s_+) \geq 2$. In the notation given in (8.1), (8.2) and (8.3), we have $D_0 \in \mathrm{Syl}_2(C_{G_0}(s_0))$ and $D_+ \in \mathrm{Syl}_2(C_{G_+}(s_+))$. In general, $z \notin K$ and we modify z as follows. Since $L_+ = \mathrm{GL}^\epsilon(m, q)$ and $D_+ \in \mathrm{Syl}_2(C_{L_+}(s_+))$, it follows that if we view s_+ as an element of L_+, we still have $m_\Gamma(s_+) \geq 2$ for some Γ, and so by Remark 6.2 (b), $D_+ = D_1 \times D_2$ given by (6.16). Thus $z_+ = z_1 \times z_2$ for some $z_1 \in D_1$ and $z_2 \in D_2$. Let $z'_+ := 1_{D_1} \times z_2$ and $z' = z_0 \times z'_+$. Then $z'_+ \in D_+$, $z' \in K$ and $z' \in D_K = D \cap K \in \mathrm{Syl}_2(C_K(s))$.

For simplicity of notation, we write $z = z'$, so $V = V_0 \perp V_+$ with $V_0 = C_V(z)$, $V_+ = [V, z]$, $z = z_0 \times z_+$, $s = s_0 \times s_+$ with $s_0 \in K_0$ and $s_+ \in K_+$, $D = D_0 \times D_+$ with $D_0 \in \mathrm{Syl}_2(C_{G_0}(s_0))$ and $D_+ \in \mathrm{Syl}_2(C_{G_+}(s_+))$, where $K_0 = \Omega(V_0)$, $K_+ = \Omega(V_+)$, $G_0 = \mathrm{SO}(V_0)$ and $G_+ = \mathrm{SO}(V_+)$. In particular, if $\Gamma \neq X \pm 1$, then $m_\Gamma(s_+)$ is even and if $\Gamma = X \pm 1$, then $4 \mid m_\Gamma(s_+)$ (cf. the decomposition of (6.16)), and $m_\Delta(s_+) \geq 2$. In addition, $L = C_G(z) = L_0 \times L_+$, $L_0 = G_0$ and $L_+ = \mathrm{GL}^\epsilon(m, q)$ with m even, and hence $\eta(V_+) = +$ and $4 \mid \dim V_+$.

Let (z, B_z) be a B_K-element and (z, B_L) a B-element such that B_L covers B_z. Then $B_L = B_{L_0} \times B_{L_+}$, where $B_{L_0} \in \mathrm{Blk}(L_0)$ and $B_{L_+} \in \mathrm{Blk}(L_+)$. Thus $B_{L_0} = \mathcal{E}_2(L_0, (s_0))$ and $B_{L_+} = \mathcal{E}_2(L_+, (s_+))$. Let $R_G = D_0 \times Q''(D_+)$, $R = R_G \cap K$, $R_0 := D_0 \cap \Omega(V_0)$ and $R_+ := Q''(D_+) \cap \mathrm{SL}^\epsilon(m, q) \leq \Omega(V_+)$, where $Q''(D_+) = Q'(D(B_{L_+}))$ or $Q(D(B_{L_+}))$ as in Proposition 8.1. So $z \in R$,

$$C_K(R) \leq C_G(R) = C_{L_0}(R_0) \times C_{L_+}(R_+) \leq L.$$

Let (R, b) be a B_z-subgroup and (R, b_G) a B_L-subgroup such that b_G covering b. Then

$$b_G = b_0 \times b_+, \quad b_0 \in \mathrm{Blk}(C_{L_0}(R_0)), \quad b_+ \in \mathrm{Blk}(C_{L_+}(R_+)),$$

and so (R_+, b_+) is a B_{L_+}-subgroup. It follows by Proposition 8.1 that $C_K(Z(R))/Z = C_{K/Z}(Z(R)/Z)$ and $C_K(R)R/Z = C_{K/Z}(R/Z)$ for any $Z \leq Z(K)$. By Proposition 8.1 again, $D(b_+)$ is abelian, and there exists $y \in \mathrm{SL}^\epsilon(m, q) \leq L_+$ such that $y \in N_{L_+}(R_+) \setminus C_{L_+}(R_+)$, $|y| = 3$ and $b_+^y = b_+$. In particular, $b_G^y = b_G$. Note that $C_{C_{L_+}(R_+)}(s_+)$ is a torus of both L_+ and $I_0(V_+)$, $D(b_+) = O_2(C_{C_{L_+}(R_+)}(s_+))$, and $C_{C_{L_+}(R_+)}(s_+) = C_{C_{G_+}(R_+)}(s_+)$.

Since $[C_G(R){:}C_K(R)] \leq 2$ and $|y| = 3$, it follows that $b^y = b$ and $D(b) = D(b_G) \cap C_K(R)$ is abelian. Let $P = Z(R)$, $(P, g) \leq (R, b)$ and $(P, g_G) \leq (R, b_G)$ such that g_G covers g. Then $g_G = g_0 \times g_+$ with $g_0 = b_0$ and $(Z(R_+), g_+) \leq (R_+, b_+)$. Thus $D(g_+) = \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ or $D(g_G)/Z_G \cong D_{2^{c+1}}$, where $Z_G = O_2(C_G(R)) \leq Z(C_G(R))$. If $D(g_+) = \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$, then $D(g_+) \leq \mathrm{SL}^\epsilon(m, q) \leq D(b)$ and $D(g)/Z_K = D(g_+)$ and $Z_K := R_0 \leq Z(C_K(P))$. By Lemma 2.4, $\ell(g) \geq 2$.

(2). Let (R, b_H) be a B_H-subgroup such that b_H covers b. Since $C_H(R)/C_K(R)Z(H)$ is a 2-group, it follows that $b_H^y = b_H$. In order to show that $D(b_H)$ is abelian and

$C_H(Z(R))/Z = C_{H/Z}(Z(R)/Z)$ and $C_H(R)R/Z = C_{H/Z}(R/Z)$ for $Z = Z(K)$ we suppose $H = J_0(V)$. It follows by [13, (1A)] that

$$C_H(R) = \langle C_{L_0}(R_0) \times C_{L_+}(R_+), \tau\rangle, \quad \tau := \tau_0 \times \tau_+,$$

where $\tau_0 \in C_{J_0(V_0)}(R_0)$ and $\tau_+ \in C_{J_0(V_+)}(R_+)$ with $J_0(V) = \langle G, \tau\rangle$. Thus $N_H(R) = \langle N_G(R), \tau\rangle$. If $xZ \in C_{H/Z}(R/Z)$ for some $x \in H$, then $x \in N_H(R)$ and $x^{-1}ux = \pm u$ for any $u \in R$. We may suppose $x = x_0 \times x_+ \in C_{L_0/Z_0}(R_0/Z_0) \times C_{L_+/Z_+}(R_+/Z_+)$ with $Z_0 = Z(K_0)$ and $Z_+ = Z(K_+)$. By Proposition 8.1, $x_+ \in C_{L_+}(R_+)$ and so $x \in C_H(R)$. Similarly, $C_H(Z(R))/Z = C_{H/Z}(Z(R)/Z)$.

Since $C_{C_{J_0(V_0)}(R_0)}(s_0)$ is a maximal torus of $C_{J_0(V_0)}(R_0)$ and $C_{C_{J_0(V_+)}(R_+)}(s_+)$ is a torus of $C_{J_0(V_+)}(R_+)$, it follows that $C_{C_H(R)}(s)$ is a torus of $C_H(R)$. But $D(b_H) \in \mathrm{Syl}_2(C_{C_H(R)}(s))$, so $D(b_H)$ is abelian. This proves Property 7.1 (a).

(3). Suppose $m_\Gamma(s) \leq 1$, so that $C_G(s)$ is a maximal torus of G and $D = O_2(C_G(s))$ is abelian. By [13, (1B)], $C_G(s)$ is a maximal torus of G if and only if $C_{J_0(V)}(s)$ is a maximal torus of $J_0(V)$. To show that $D(B_H)$ is abelian we may suppose $H = J_0(V)$, so that $D(B_H) \in \mathrm{Syl}_2(C_{J_0(V)}(s))$ and hence $D(B_H)$ is abelian. □

Remark 8.3 *In the notation of proof* (1) *above let $A_+(i) = A_i(D_+) \leq R_+$ or $A'_i(D_+) \leq R_+$ according as $R_+ = Q(D_+) \cap \mathrm{SL}^\epsilon(m, q)$ or $Q'(D_+)$, $A(i) := R_0 \times A_+(i)$ for* $1 \leq i \leq 3$, *where $A_i(D_+)$ is defined in* (6.11) *and $A'_i(D_+)$ is given by* (6.15). *Then $A(i) \leq R$, $R = \langle A(1), A(2)\rangle$ and y acts transitively on $\{A(1), A(2), A(3)\}$. If $(A_+(i), b_{A_+(i)}) \leq (R_+, b_+)$, then $b_{A_+(i)} = \mathcal{E}_2(C_{G_+}(A_+(i)), (s_+))$, $C_{C_{L_+}(A_+(i))}(s_+)$ is a maximal torus of both L_+ and G_+, and $D(b_{A_+(i)}) = O_2(C_{C_{L_+}(A_+(i))}(s_+))$. Note also that $C_{G_+}(R_+) = C_{L_+}(R_+) \leq C_{L_+}(A_+(i))$, $\mathrm{SO}^\eta(2, q) = \mathrm{GL}^\eta(1, q)$, and we have that*

$$C_{C_G(A(i))}(s) = \prod_\Gamma (\mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})^{n_\Gamma} \times \mathrm{GL}^{\epsilon_\Gamma}(1, q^{e_\Gamma\delta_\Gamma})^{(m_\Gamma - n_\Gamma)/e_\Gamma}) \tag{8.5}$$

is a maximal torus of G, where $m_\Gamma = m_\Gamma(s)$ and $n_\Gamma = 1$ or 0 according as m_Γ is odd or even.

Proposition 8.4 *Let $K := \Omega_{2n+1}(q) := \Omega(V)$ or $K := \mathrm{Sp}_{2n}(q) = \mathrm{Sp}(V)$, and*

$$K \leq H \leq J_0(V),$$

$B_K \in \mathrm{Blk}(K)$ and $B_H \in \mathrm{Blk}(H)$ covering B_K, where $H = \mathrm{SO}(V)$ when $K = \Omega(V)$. Either Property 7.1 *(a) holds for some B_K-subgroups $(Z(R), g) \leq (R, b)$ with $C_H(Z(R))/Z = C_{H/Z}(Z(R)/Z)$ and $C_H(R)R/Z = C_{H/Z}(R/Z)$ for any $Z \leq Z(K)$, or Property* 7.1 *(b) holds.*

Proof: Let s^* be a dual element of s in $I_0(V)$ given by [3, (4A)]. Replacing G by H and s by $s^* \in I_0(V)$ in the proof (1) of Proposition 8.2 with some obvious modification, we have that Property 7.1 (a) holds for some B_K-subgroup (R, b). A proof similar to the proof (3) of Proposition 8.2 with s replaced by $s^* \in I_0(V)$ shows that Property 7.1 (b) holds. □

Theorem 8.5 *Let $G = I_0(V)$, $B \in \mathrm{Blk}(G)$, and (D, b_D) a Sylow B-subgroup. Follow the notation in* (8.1), (8.2) *and* (8.3). *Then B is nilpotent if and only if $C_{I_0(V^*)}(s)$ is a maximal torus T^* of $I_0(V^*)$ and $D = O_2(T)$, where $T \leq I_0(V)$ is a dual of T^*.*

PROOF: Note that $C_{G^*}(s)$ is regular and $B = \mathcal{E}_2(G, (s))$. By [3, (5A)], we may suppose $D \in \mathrm{Syl}_2(C_G(s^*))$, where s^* is a dual element of s in G defined in [3, (4A)]. If B is nilpotent, then by Propositions 8.2 and 8.4, D is abelian and so $C_G(s^*)$ is a maximal torus of G.

Conversely, suppose $T := C_G(s^*)$ is a maximal torus of G and θ is the canonical character of b_D. Then $D = O_2(T)$, $C_{C_G(D)}(s^*) = T$, and $\theta = \pm R_T^{C_G(D)}(s)$. Thus $N_G(D, \theta) = T$ and $B = \mathcal{E}_2(G, (s))$ is nilpotent. □

Proposition 8.6 *Let $K := \mathrm{Spin}^\eta(V) \lhd H$ such that H/K is abelian, $C_H(K) \leq Z(H)$ and $H/Z(H) \leq J_0(V)/Z(J_0(V))$. Let $B \in \mathrm{Blk}(K)$, $B_H \in \mathrm{Blk}(H)$ covering B, and $Z \leq Z(K)$ such that $K_c := K/Z = \Omega^\eta(V)$, so that $|Z| = \gcd(2, q - \eta)$. Either Property* 7.1 *(a) holds for some B-subgroups $(Z(R), g) \leq (R, b)$ with $C_H(Z(R))/Z = C_{H/Z}(Z(R)/Z)$ and $C_H(R)R/Z = C_{H/Z}(R/Z)$ for any $Z \leq Z(K)$, or Property* 7.1 *(b) holds.*

PROOF: We prove the proposition for $K = \mathrm{Spin}_{2n}^\eta(q) = \mathrm{Spin}^\eta(V)$. A similar proof with some modifications works for $K = \mathrm{Spin}_{2n+1}^\eta(q)$.

Let $D := D(B)$, $G := \mathrm{SO}^\eta(V)$ and $Z_+ \leq Z(D_0(V))$ such that $G = D_0(V)/Z_+$, so that $Z = Z_+ \cap K$ and $Z_+ \cong \mathbb{Z}_{q-1}$. Identify G with G^*.

Note that B dominates a unique block $B_c \in \mathrm{Blk}(K_c)$ and B_c is covered by a unique block $B_G \in \mathrm{Blk}(G)$. Thus $B_G = \mathcal{E}_2(G, (s))$ for some semisimple $2'$-element $s \in G$ and $D_G := D(B_G) \in \mathrm{Syl}_2(C_G(s))$. Since $G/K_c = \mathbb{Z}_2$, it follows that $s \in K_c$ and we may suppose $s \in K$, that is, we identify $s \in G = G^* \leq J_0(V^*)$ with its dual $s^* \in K \leq J_0(V^*)^* = D_0(V)$. Note that since s is a $2'$-element, it follows that $C_{J_0(V^*)}(s)$ is a regular subgroup and so $C_{J_0(V^*)}(s)^* \cong C_{D_0(V)}(s^*)$ for a (unique) semisimple $2'$-element $s^* \in K \leq D_0(V)$, so that it is possible to identify s with s^*.

Since B_G covers B_c, it follows that $D/Z \in \mathrm{Syl}_2(C_{K_c}(s))$ and by [13, (2E) (2)], $D \in \mathrm{Syl}_2(C_K(s))$. Thus D_G is abelian if and only if $m_\Gamma(s) \leq 1$ for $\Gamma \neq X - 1$ and $m_{X-1}(s) \leq 2$. This happens if and only if $C_G(s)$ is a maximal torus of G. By [13, (2E) (2)] again, this happens if and only if $C_K(s)$ is a maximal torus of K. In particular, D_G is abelian if and only if D is abelian.

(1). Suppose $m_\Delta(s) \geq 2$ for some Δ. Let z_c be a primary element of D_G with the modification given by the proof (1) of Proposition 8.2, so that $z_c \in D_G \cap K_c$ and $V = V_0 \times V_+$ with $V_0 = C_V(z_c)$ and $V_+ = [V, z_c]$. Let $z_c = z_0 \times z_+$, $D_G = D_0 \times D_+$, $s = s_0 \times s_+$ and $L = C_G(z_c) = L_0 \times L_+$ be the decompositions corresponding to $V = V_0 \perp V_+$, so that $z_0 = 1_{V_0}$, $L_0 = I_0(V_+)$, $L_+ = C_{G_+}(z_+) = \mathrm{GL}^\epsilon(m, q)$ with $G_+ = I_0(G_+)$. Let $R_G = D_0 \times Q(D_+)$ or $D_0 \times Q'(D_+)$ and $A_G(i) := A_i(D_G) = D_0 \times A_i(D_+)$ or $D_0 \times A_i'(D_+)$ according as $D_+ \not\cong \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ for all $\alpha \geq 0$ or $D_+ = \mathbb{Z}_{2^{a+1+\alpha}} \wr \mathbb{Z}_2$ for some $\alpha \geq 0$, where $A_i(D_+)$ and $A_i'(D_+)$ are given by (6.11) and (6.15), respectively. Then $A_G(i) \leq R_G$, $z_c \in A_G(1)$, $R_G = \langle A_G(1), A_G(2) \rangle$, $C_G(A_G(i))$ is regular, $s \in C_G(A_G(i))$ and $T_G(i) := C_{C_G(A_G(i))}(s)$ is a maximal torus given by (8.5).

Let $A_c(i) = A_G(i) \cap K_c$, $R_c = R_G \cap K_c$, and let $(A_c(i), b_{A_c(i)}) \leq (R_c, b_c)$ be B_c subgroups, and $(A_c(i), b_{A_G(i)}) \leq (R_c, b_G)$ B_G-subgroups such that $b_{A_G(i)}$ covers $b_{A_c(i)}$ and

b_G covers b_c, so that $z_c \in A_c(1)$ and $R_c = \langle A_c(1), A_c(2)\rangle$. Thus $T_G(i) = C_{C_G(A_c(i))}(s)$, $C_G(R_c) \cap C_G(s) = C_G(R_G) \cap C_G(s)$, $b_{A_G(i)} = \mathcal{E}_2(C_G(A_c(i)), (s))$, $D(b_{A_G(i)}) = O_2(T_G(i))$, $D(b_G) = O_2(T_G(1) \cap C_G(R_c))$, $D(b_{A_c(i)}) = O_2(T_c(i))$ and $D(b_c) = O_2(T_c(1) \cap C_{K_c}(R_c))$, where $T_c(i) = T_G(i) \cap K_c$. Let $P_c = Z(R_c)$ and $(P_c, g_c) \leq (R_c, b_c)$. As shown in the proof (1) of Proposition 8.2, $\ell(g_c) \geq 2$ and there exists $y \in N_{C_{K_c}(P_c)}(R_c, b_G) \setminus C_G(R_c)$ such that $|y| = 3$ and $y \in N_{C_{K_c}(P_c)}(R_c, b_c)$.

Let $T(i)$ be maximal torus of K such that $T(i)/Z = T_c(i)$, and let $A(i) = O_2(T(i))$ and $R = \langle A(1), A(2)\rangle$. Then $A(i)/Z = A_c(i)$, $R/Z = R_c$ and $R^y = R$. If $b_A(i)$ is the block of $C_K(A(i))$ dominating $b_{A_c(i)}$, then $b_A(i) = \mathcal{E}_2(C_K(A(i)), (s))$ and so $(A(i), b_A(i))$ is a B-subgroup as $B = \mathcal{E}_2(K, (s))$ (note we identify s with s^*). Similarly, if b is the block of $C_K(R)$ dominating b_c, then $b = \mathcal{E}_2(C_K(R), (s))$ and (R, b) is a B-subgroup. But $b_c^y = b_c$, so by the uniqueness, $b^y = b$. Since $D(b)/Z = D(b_c)$ and $D(b_c) \leq T_c(1)$, it follows that $D(b) \leq T(1)$ and hence $D(b)$ is abelian. Let $P \leq Z(R)$ such that $P/Z = P_c$ and $g = \mathcal{E}_2(C_K(P), (s))$. Then g dominates g_c and so $\ell(g) = \ell(g_c) \geq 2$.

It follows by Proposition 8.2 that $C_H(R)R/Z(K) = C_{H/Z(K)}(R/Z(K))$, and so $C_H(R)R/Z' = C_{H/Z'}(R/Z')$ for any $Z' \leq Z(K)$. Similarly, $C_H(P)/Z' = C_{H/Z'}(P/Z')$.

Let (R, b_H) be a B_H-subgroup such that b_H covering b. Since $H/KZ(H)$ is a 2-group, it follows that b_H is the unique block covering b, so that $b_H^y = b_H$. To show that $D(b_H)$ is abelian we may suppose $H/Z(H)K = J_0(V)/K_cZ(J_0(V)) = \mathrm{Outdiag}(K)$.

By (8.5), $T_G := T_G(1) = \prod_\Gamma((T'_\Gamma)^{n_\Gamma} \times (T_\Gamma)^{m'_\Gamma})$, where $T'_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$, $T_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{e_\Gamma\delta_\Gamma})$ and $m'_\Gamma = (m_\Gamma - n_\Gamma)/e_\Gamma$. Let U_Γ and W_Γ be orthogonal spaces over $\mathbb{F}_q$ such that $\dim U_\Gamma = d_\Gamma, \eta(U_\Gamma) = \epsilon_\Gamma$, and $\dim W_\Gamma = e_\Gamma d_\Gamma$ and $\eta(W_\Gamma) = \epsilon_\Gamma^{e_\Gamma}$. Let $J = J_0(V)$, $B_J = \mathcal{E}_2(J, (s))$ and $T_J := C_J(T_G)$. By [13, (1B)], $T_J = \langle T_G, \tau\rangle$ such that $\tau = \prod_\Gamma(\tau'_\Gamma \times \tau_\Gamma)$ and $[\tau'_\Gamma, T_G] = [\tau_\Gamma, T_G] = 1$, where $\tau'_\Gamma \in J_0(U_\Gamma)$ and $\tau_\Gamma \in J_0(W_\Gamma)$ such that $J_0(U_\Gamma) = \langle I_0(U_\Gamma), \tau'_\Gamma\rangle$ and $J_0(W_\Gamma) = \langle I_0(W_\Gamma), \tau_\Gamma\rangle$. Thus for any $t_c \in \mathrm{Outdiag}(K)$ there exists an element $h \in T_J$ such that h induces the same automorphism t_c on K.

Since $H/Z(H) = J/Z(J)$, it follows that we may suppose $V = U_\Gamma$ or W_Γ, so that $C_G(s) = T'_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$ or $\mathrm{GL}^{\epsilon_\Gamma}(e_\Gamma, q^{\delta_\Gamma})$. If $2\|(q^{\delta_\Gamma} - \epsilon_\Gamma)$, then $e_\Gamma = 2$, $C_G(s) = T'_\Gamma$ or $\mathrm{GL}^{\epsilon_\Gamma}(2, q^{\delta_\Gamma})$. If $4 \mid (q^{\delta_\Gamma} - \epsilon_\Gamma)$, then $e_\Gamma = 1$ and $C_G(s) = T_G = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$. Suppose there exists a primary element $z' \in C_G(s)$ with $|z'| = 4$. Then $z'^2 \in Z(G)$, $C_G(z') = \mathrm{GL}^\epsilon(m, q)$ with $\dim V = 2m$ and so $C_K(z')$ and $C_{\mathrm{Inndiag}(K)}(z')$ are given by [15, Tables 4.5.1 and 4.5.2]. If $x \in C_G(z')$ with $|x|$ odd, then $x \in C_{K_c}(z')$. If $x = t_xZ$ for some $t_x \in K$ with odd order, then $t_x^{z'} = t_x$ and $t_x \in C_K(z')$. Conversely, if $t_x \in C_K(z')$ with odd order, then $x := t_xZ \in C_{K_c}(z')$. Since $\mathbb{F}_q$ has the odd characteristic r, it follows that $O^{r'}(C_G(z')) = O^{r'}(C_K(z'))Z/Z$ and in particular, $O^{r'}(C_K(z')) = \mathrm{SL}^\epsilon(m, q)$. Since $T_G = T'_\Gamma$ or T_Γ which is a maximal torus of $C_G(z')$, it follows that $z' \in T_G$ and $T_G \leq C_G(z')$. But $T_J = \langle T_G, \tau\rangle$, so by [13, (1B)], we may suppose $C_J(z') = C_G(z')T_J = C_G(z') \circ \langle\tau\rangle$. Note that $N_{I(V)}(C_G(z')) = N_{I(V)}(\langle z'\rangle) = \langle C_G(z'), w\rangle$ and $N_J(\langle z'\rangle) \leq C_J(z')N_{I(V)}(\langle z'\rangle)$, where $w \notin C_G(z')$ such that w induces the graph automorphism on $O^{r'}(C_G(z'))$, and $w \in G$ when $\eta(V) = +$ and m is even. So the elements of $C_J(z')$ induce inner-diagonal automorphisms on $L := O^{r'}(C_K(z'))$. In addition, since $[\tau, x] = 1$ for any $x \in C_G(z')$ with odd order, it follows that τ centralizes L.

Suppose $z' \in K_c$, so that $z_c = z'$, Let $z \in K$ such that $zZ = z_c$. Thus $C_K(z) = C_K(z')$, $C_H(z)/Z \leq C_J(z')$ and so elements of $C_H(z)$ induce inner-diagonal automorphisms on L. Now $C_K(z)$ is regular, $s \in C_K(z)$ and so $B_z = \mathcal{E}_2(C_K(z), (s))$ is a block with a defect group $D(B_z) = O_2(C_K(zs)) = O_2(T(1))$. Let B_{H_z} be a block of $C_H(z)$

such that B_{H_z} covers B_z and $b_H^{C_H(z)} = B_{H_z}$, so that $D(b_H) \leq D(B_{H_z})$. Since $C_H(z)$ induces inner-diagonal automorphisms on L, it follows by Proposition 8.1 that $D(B_{H_z})$ is abelian and so is $D(b_H)$.

Suppose $z' \notin K_c$, so that $4\|(q^{\delta_\Gamma} - \epsilon_\Gamma)$, $4\|(q-\epsilon)$ and $C_G(s) = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$ with $O_2(C_G(s)) = \mathbb{Z}_4$. Thus $D(b_c) = O_2(K_c) \leq Z(G) = \mathbb{Z}_2$. Note that $D(b_c)$ is also equal to $O_2(K_c)$ when $C_G(s)$ has no primary element z' of order 4.

If $D(b_c) = 1$, then $D(b) = Z(K) = \mathbb{Z}_2$, $\mathrm{Outdiag}(K_c) = \mathbb{Z}_2$, $|D(b_H)| \leq 4$ and $D(b_H)$ is abelian. Suppose $D(b_c) = O_2(K_c) = \mathbb{Z}_2$. Then $D(b) = O_2(K) = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ according as $G = \mathrm{SO}^+(4k, q)$ or $G = \mathrm{SO}^\epsilon(4k+2, q)$. In the former case, $C_G(s) = \mathrm{GL}(1, q^{2k})$ and $O_2(C_G(s)) = \langle z'' \rangle$ with $|z''| = 2^\ell \geq 8$, and hence $z' = z''^{2^{\ell-2}} \in K_c$, which is impossible.

Thus $G = \mathrm{SO}^\epsilon(2m, q)$ with $m = 2k+1$ and $4\|(q-\epsilon)$. Suppose $t \in D(b_H) \setminus D(b)$ induces an element of order 4 in $\mathrm{Outgiag}(K)$. Then there exists $x_t \in J$ such that x_t induces the same action as t on K_c/Z. Since $J/G \cong \mathbb{Z}_{q-\epsilon}$, it follows by [15, Table 4.5.2] that we may suppose $x_t \in J \setminus G$ such that $x_t^4 \in Z(G)$, $x_t^2 \in GZ(J)$ and $C_K(x_t^2) = C_K(t^2) = (\mathrm{SL}^\epsilon(m, q) \circ \mathbb{Z}_{q-\epsilon}).(\gcd(m, q-\epsilon))$. Thus $C_G(x_t) = C_G(x_t^2) = \mathrm{GL}^\epsilon(m, q)$ and x_t centralizes elements of odd order in $C_K(x_t^2)$. In particular, $[x_t, L] = [t, L] = 1$ with $L = O^{r'}(C_G(t^2)) = \mathrm{SL}^\epsilon(m, q)$. By [15, Table 4.5.1], elements of $C_{\mathrm{Inndiag}(K)}(t^2)$ induce inner-diagonal automorphism on L. Let $B_t = \mathcal{E}_2(C_K(t^2), (s))$ and $B_{H_t} \in \mathrm{Blk}(C_H(t^2))$ such that B_{H_t} covers B_t and $b_H^{C_H(t^2)} = B_{H_t}$, so that $D(B_t) = O_2(C_{C_K(t^2)}(s)) = Z(K)$. By Proposition 8.1, $D(B_{H_t})$ is abelian and so is $D(b_H)$. If $D(b_H)/D(b) = \mathbb{Z}_2$, then as shown above $D(b_H) = \langle Z(K), t^2 \rangle$ is abelian.

(2). Suppose D is abelian, so that D_G is abelian, $C_K(s)$ is a maximal torus of K and by [13, (2E)], $C_{D_0(V)}(s)$ is a maximal torus of $D_0(V)$. To show that $D(B_H)$ is abelian we may suppose $H/KZ(H) = J_0(V)/K_cZ(J_0(V)) = \mathrm{Outdiag}(K)$. So the proof is given in (1) above, and Property 7.1 (*b*) holds. $\Box$

Theorem 8.7 *Let K be a finite quasi-simple group of classical type over a field $\mathbb{F}_q$ and $B \in \mathrm{Blk}(K)$, and let $K \lhd H$ such that H/K is abelian, $C_H(K) \leq Z(H)$, H induces inner-diagonal automorphisms on K and $B_H \in \mathrm{Blk}(H)$ covering B. If q is even, then either $D(B) = D(B_H)$ is cyclic or $\ell(B) \geq 2$. Suppose q is odd. Then either Property 7.1 (a) or (b) holds.*

Proof: We will follow the notation of [15]. In particular, K_u denotes the universal group with the same type as K and $K = K_u/Z$ for some $Z \leq Z(K_u)$. If q is even and $D(B)$ is noncyclic, then $D(B)$ is a Sylow subgroup of K and $\ell(B) = \ell(B_0)$ with principal $B_0 := B_0(K) \in \mathrm{Blk}(K)$. But B_0 dominates the principal block $\overline{B}$ of $K/Z(K) = K_a$ and $\ell(\overline{B}) + 1$ is the number of $2'$-conjugacy classes of K_a, so $\ell(B_0) \geq \ell(\overline{B}) \geq 2$.

Suppose q is odd. If $K = A_n^\eta(q)$, then take $\widehat{K} = K_u = \mathrm{SL}_{n+1}^\eta(q) \leq \widehat{H} \leq \mathrm{GL}_{n+1}^\eta(q)$ such that $H = \widehat{H}/Z$.

If $K = B_n(q) = K_a = \Omega_{2n+1}(q)$, then set $\widehat{K} = \Omega_{2n+1}(q) \leq \widehat{H} \leq \mathrm{SO}_{2n+1}(q)$ such that $H = \widehat{H}/Z$. If $K = B_n(q) = K_u = \mathrm{Spin}_{2n+1}(q) = \mathrm{Spin}(V)$, then take $K = \widehat{K} \lhd \widehat{H} = H$ such that $H/Z(K) \leq \mathrm{SO}(V)$.

If $K = C_n(q)$, then we may take $\widehat{K} = \mathrm{Sp}_{2n}(q) = \mathrm{Sp}(V) \leq \widehat{H} \leq J_0(V)$ such that $H = \widehat{H}/Z$.

Suppose $K = D^\eta_n(q)$ with $(n,\eta) = (2k+1,\pm)$ or $(2k,-)$. If $K = \Omega^\eta_{2n}(q) = \Omega(V)$, then $K = \widehat{K} \lhd \widehat{H} = H \leq J_0(V)$. If $K = P\Omega^\eta_{2n}(q) = P\Omega(V)$, then take $\widehat{K} = \Omega^\epsilon_{2n}(q) \leq \widehat{H} \leq J_0(V)$ such that $H = \widehat{H}/Z$. If $K = \mathrm{Spin}^\eta_{2n}(q) = \mathrm{Spin}(V)$, then take $K = \widehat{K} \lhd \widehat{H} = H$ such that $H/Z(K) \leq J_0(V)$.

Suppose $K = D^+_{2k}(q)$, so that $Z(K_u) = \{1, z, z_s, z_c\}$ and $K_u/\langle z\rangle = \Omega^+_{4k}(q)$. If $K = \Omega^+_{4k}(q) = \Omega(V)$, then take $\widehat{K} = K \leq \widehat{H} \leq J_0(V)$. If $K = P\Omega^+_{4k}(q) = P\Omega(V)$, then take $\widehat{K} = \Omega(V) \leq \widehat{H} \leq J_0(V)$ such that $\widehat{H}/Z = H$ with $Z = \langle z\rangle$. If $K = \mathrm{Spin}^+_{4k}(q)/Z'$ for $Z' = \langle z_s\rangle$ or $\langle z_c\rangle$, then we may take $\widehat{K} = \mathrm{Spin}^+_{4k}(q) = \mathrm{Spin}(V) \leq \widehat{H} \leq D_0(V)$ such that $H = \widehat{H}/Z'$. If $K = \mathrm{Spin}^+_{4k}(q) = \mathrm{Spin}(V)$, then take $\widehat{K} = K$ and $\widehat{H} = H$.

Let $\widehat{B} \in \mathrm{Blk}(\widehat{K})$ dominating B and $\widehat{B}_H \in \mathrm{Blk}(\widehat{H})$ dominating B_H, so that $\widehat{B}_H$ covers $\widehat{B}$. By Propositions 8.1, 8.2, 8.4 and 8.6, one of Properties 7.1 (a) and (b) holds for $\widehat{B}$.

If Property 7.1 (a) holds for $\widehat{B}$, then there exist $\widehat{B}$-subgroups $(\widehat{P}, \hat{g}) \leq (\widehat{R}, \hat{b})$ satisfying Property 7.1 (a), where $\widehat{P} \leq Z(\widehat{R})$. By Lemma 7.3, Property 7.1 (a) holds for some B-subgroups $(P, g) \leq (R, b)$.

If Property 7.1 (b) holds for $\widehat{B}$, then $D(\widehat{B})$ and $D(\widehat{B}_H)$ are both abelian. Since $Z \leq Z(\widehat{H}) \cap \widehat{K}$, it follows that $D(B) = D(\widehat{B})Z/Z$ and $D(B_H) = D(\widehat{B}_H)Z/Z$, and so $D(B)$ and $D(B_H)$ are both abelian. □

9 Exceptional groups

We will follow the notation of [15].

Theorem 9.1 *Let K be a finite quasi-simple group of exceptional type over a field $\mathbb{F}_q$, $B \in \mathrm{Blk}(K)$, and let $K \lhd H$ such that $C_H(K) \leq Z(H)$, H/K is cyclic, and H induces inner-diagonal automorphisms on K. If q is even, then either $D(B) = D(B_H)$ is cyclic or $\ell(B) \geq 2$. If q is odd, then either Property 7.1 (a) or (b) holds.*

Proof: If q is even, then a proof similar to that of Theorem 8.7 shows that either $D(B) = D(B_H)$ is cyclic or $\ell(B) \geq 2$.

Suppose q is odd. Let K_u be the universal group, so that $K = K_u/Z$ for some $Z \leq Z(K_u)$. Since $Z(K_u)$ is cyclic of order 1, 2 or 3, it follows that H centralizes $Z(K_u)$.

Let $D := D(B)$. If $Z(K) \neq \Omega_1(Z(D))$, then take an involution $z \in Z(D) \setminus Z(K)$. If $Z(K) = \Omega_1(Z(D))$, then take $z \in D$ such that $|z| = 4$ and $zZ(K) \in Z(D/Z(K))$. Let (z, B_z) be a B-subsection, and in the case $z \in Z(D)$ we take B_z to have defect group D by [1, 4.15]. Write $C := C_G(z)$.

By [15, Theorem 4.2.2] $C = O^{r'}(C)T$, where $O^{r'}(C)$ is a central product

$$O^{r'}(C) = L_1 \circ L_2 \circ \cdots \circ L_\ell$$

with each $L_i \in \mathcal{L}ie(r)$, and T is an abelian r'-group inducing inner-diagonal automorphisms on each L_i. In general, it may be the case that $z \notin O^{r'}(C)$. We introduce some more notation as follows to allow for this inconvenience: If $Z(C) \leq O^{r'}(C)$, then define $s := \ell$ and $L := O^{r'}(C)$. If $Z(C) \not\leq O^{r'}(C)$, then define $s = \ell + 1$, $L_s = Z(C)$ and

$$L := L_1 \circ L_2 \circ \cdots \circ L_s. \tag{9.1}$$

In all cases $C = LT$, $z \in L$ and $L \lhd C$. Let B_L be a block of L covered by B_z. There are uniquely defined blocks $B_i \in \mathrm{Blk}(L_i)$ such that if $\chi \in \mathrm{Irr}(B_L)$ with $\chi = \chi_1 \circ \cdots \circ \chi_s$ for some $\chi_i \in \mathrm{Irr}(L_i)$, then $\chi_i \in \mathrm{Irr}(B_i)$. We write

$$B_L = B_1 \circ B_2 \circ \cdots \circ B_s.$$

Case 1. Suppose that $H = K := {}^2G_2(3^{2a+1}), G_2(q), {}^3D_4(q), F_4(q)$ or $E_6^{-\epsilon}(q)$ with $q \equiv \epsilon \pmod 3$, and $B \in \mathrm{Blk}(K)$. Then either Property 7.1 (a) or (b) holds.

Since z induces an inner automorphism on K, it follows that each L_i is a classical group (with possibly L_s abelian).

If $\ell = 1$, then L_1 is a classical group given by [15, Table 4.5.1]. Thus Theorem 9.1 follows by Theorem 8.7.

Suppose $\ell \geq 2$, so that by [15, Table 4.5.1], $\ell = 2$, $Z(C) \leq L_1 \circ L_2$, $s = \ell$, $L = L_1 \circ L_2$ and the possible (K, C) are given in Table 2, where $\eta = -$ or $+$. Here $C = (L_1 \circ L_2).(2{:}2)$ means that $C = \langle L_1 \circ L_2, x\rangle$ such that x induces inner-diagonal automorphism of order 2 on each L_i.

K	C	K	C
${}^3D_4(q)$	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q^3)).(2{:}2)$	$G_2(q)$	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).(2{:}2)$
$F_4(q)$	$(\mathrm{SL}_2(q) \circ \mathrm{Sp}_6(q)).(2{:}2)$	$E_6^{\eta}(q)_u$	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_6^{\eta}(q)).(2{:}2)$
$E_7(q)_u$	$(\mathrm{SL}_2(q) \circ \mathrm{Spin}_{12}^{+}(q)).(2{:}2)$	$E_8(q)$	$(\mathrm{SL}_2(q) \circ E_7(q)_u).(2{:}2)$

Table 2: Possible (K, C) with $s \geq 2$

Write $L_1 := \mathrm{SL}_2(q) \leq H_1 := \langle L_1, x_1\rangle \leq G_1 = \mathrm{GL}_2^{\delta}(q)$, $L_2 = \mathrm{SL}_2(q^3), \mathrm{SL}_2(q)$, $\mathrm{Sp}_6(q)$ or $\mathrm{SL}_6^{-\epsilon}(q)$ and $H_2 = \langle L_2, x_2\rangle$, where δ is the sign such that $2\|(q - \delta 1)$ and $x = x_1 \times x_1 \in C \setminus L$ such that x_i induces outer-diagonal automorphism of order 2 on L_i. Then $C \lhd H := H_1 \circ H_2$. If $B_H \in \mathrm{Blk}(H)$ covers B_z, then B_H covers B_L, $B_H = B_{H_1} \circ B_{H_2}$ for some $B_{H_i} \in \mathrm{Blk}(H_i)$ covering B_i, $D(B_H) = D(B_{H_1}) \circ D(B_{H_1})$ and $D(B_z) = D(B_H) \cap C$. Each B_i satisfies Property 7.1 (a) or (b).

If both B_1 and B_2 satisfy (b), then B_z satisfies (b).

Suppose both B_1 and B_2 satisfy Property 7.1 (a), for $(P_1', g_1') \leq (R_1', b_1')$ and $(P_2, g_2) \leq (R_2, b_2)$, respectively. Then R_1' is non-abelian in $L_1 = \mathrm{SL}_2(q)$ and so b_1' is principal. Thus $P_1' = Z(L_1)$ and g_1' is principal. Let $Q_1 \in \mathrm{Syl}_2(H_1)$ containing R_1', so that $Q_1 = SD_{2^{a+2}}$. Writing $S_1 = C_{Q_1}([Q_1, Q_1])$ and $P_1 = S_1 \cap L_1$, we have $S_1 \cong \mathbb{Z}_{2^{a+1}}$, $P_1 \cong \mathbb{Z}_{2^a}$, $C_{G_1}(P_1) \cong \mathbb{Z}_{q^2-1}$ and $C_{L_1}(P_1) \cong \mathbb{Z}_{q+\delta}$. Set $P = P_1 \circ P_2$, $R = P_1 \circ R_2$, $g_1 = b_1 = B_0(C_{L_1}(P_1))$ and $y = y_1 \times y_2$, where $y_1 = 1$ and $y_2 \in N_{C_{L_2}(P_2)}(R_2, b_2) \backslash C_{L_2}(R_2)$ with $|y_2| = 3$.

Suppose one of the B_i satisfies Property 7.1 (a) and the other (b). Say B_1 satisfies Property 7.1 (a) for $(P_1, g_1) \leq (R_1, b_1)$, and B_2 satisfies Property 7.1 (b). Then let

$(P_2, g_2) = (R_2, b_2)$ be a Sylow B_2-subgroup, and define P and R as above, and $y = y_1 \times y_2$ with $y_1 \in N_{C_{L_1}(P_1)}(R_1, b_1) \setminus C_{L_1}(R_1)$ such that $|y_1| = 3$ and $y_2 = 1$.

In either case $(P, g_1 \circ g_2) \leq (R, b_1 \circ b_2)$ are B_L-subgroups. Let $(P, g) \leq (R, b)$ be B_z-subgroups such that g and b cover $g_1 \circ g_2$ and $b_1 \circ b_2$, respectively. Let $(P, g_H) \leq (R, b_H)$ be B_H-subgroups such that g_H and b_H cover g and b, respectively. Then $g_H = g_{H_1} \circ g_{H_2}$ for some $g_{H_i} \in \mathrm{Blk}(C_{H_i}(P_i))$ covering g_i, and $b_H = b_{H_1} \circ b_{H_2}$ for some $b_{H_i} \in \mathrm{Blk}(C_{H_i}(R_i))$ covering b_i. By Propositions 8.1 and 8.4, each $D(b_{H_i})$ is abelian and $b_{H_i}^{y_i} = b_{H_i}$. So $D(b) = (D(b_{H_1}) \circ D(b_{H_2})) \cap C_C(R)$ is abelian and $b_H^y = b_H$. Since b_H covers b and $[C_H(R):C_C(R)] \leq 2$ and since $|y| = 3$, it follows that $b^y = b$.

By consideration of the Lie rank of L_i, we have that $D(g_i) \not\cong \mathbb{Z}_{2^{a+\alpha}} \wr \mathbb{Z}_2$ for any $\alpha \geq 1$. It follows that $D(g)/O_2(C_C(P))$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or a dihedral group. By Lemma 2.4, $\ell(g) \geq 2$ and Property 7.1 (a) holds for $(P, g) \leq (R, b)$.

Case 2. Suppose $3 \mid (q - \epsilon)$, and let $K = K_u := 3.E_6^\epsilon(q) \leq J := 3.E_6^\epsilon(q).3$, $B \in \mathrm{Blk}(K)$ and $B_J \in \mathrm{Blk}(J)$ covering B. Then either Property 7.1 (a) or (b) holds.

Write $m^* := \gcd(m, q - \epsilon)$ and let $z \in Z(D)$ with $|z| = 2$. By [15, Table 4.5.2], $C := C_K(z) = \langle \mathrm{Spin}_{10}^\epsilon(q) \circ (q-\epsilon), t \rangle$ or is given in Table 2, where $t = 4^*{:}1$. By [15, Table 4.5.1], $C_J(z) = \langle \mathrm{Spin}_{10}^\epsilon(q) \circ (q - \epsilon), t_J \rangle$ or $\langle \mathrm{SL}_2(q) \circ \mathrm{SL}_6^\epsilon(q), x_J \rangle$, where $t_J = 4^*{:}3$ and $x_J = 2{:}6$. As in Case 1 $\ell \leq 2$, and if $\ell = 1$, then we are done by Theorem 8.7. If $\ell = 2$, then define L_i, B_i, H_i and B_{H_i} analogously to Case 1. Property 7.1 (a) or (b) holds for each B_i, and if (a) holds for one or both of B_1 and B_2, then a proof similar to that of Case 1 shows that either Property 7.1 (a) holds for some B-subgroups $(P, g) \leq (R, b)$ or Property 7.1 (b) holds for B.

Suppose Property 7.1 (b) holds for B_1 and B_2, and suppose $D_J \cap K = D$ for some $D_J = D(B_J)$, so that $D(B_{H_i})$ is abelian and so is $D(B_z) = D(B_H) \cap C$. Now B_z is major and so $D = D(B_z) = D_J$ is abelian. Thus Property 7.1 (b) holds for B.

Case 3. Let $K = 2.E_7(q) \leq J := 2.E_7(q).2$, $B \in \mathrm{Blk}(K)$ and $B_J \in \mathrm{Blk}(J)$ covering B, where q is odd. Then either Property 7.1 (a) holds for some B-subgroups $(P, g) \leq (R, b)$ with $C_J(P)/Z = C_{J/Z}(P/Z)$ and $C_J(R)R/Z = C_{J/Z}(R/Z)$, or Property 7.1 (b) holds for B, where $Z = Z(K)$.

Again write $m^* := \gcd(m, q - \epsilon)$. Since z induces an inner automorphism on K, it follows by [15, Table 4.5.2] that

$$C_K(z) = \langle \mathrm{SL}_2(q) \circ \mathrm{Spin}_{12}(q), t \rangle \quad \text{with} \quad t = 2{:}2,$$

$\langle (\mathrm{SL}_8^\epsilon(q)/2) \circ 4, x \rangle$ with $x = (8^*/4){:}1$ or $\langle E_6^\epsilon(q)_u \circ (q - \epsilon), w \rangle$ with $w = 3^*{:}1$, where the sign $\epsilon = \pm$ is chosen so that $4 \mid (q - \epsilon)$.

Using Propositions 8.1 and 8.6 or Case 2, and applying a proof similar to that of Case 1, we have that if Property 7.1 (a) holds for some B_i-subgroups $(P_i, g_i) \leq (R_i, b_i)$, then the first part of Property 7.1 (a) holds for some B_z-subgroup $(P, g) \leq (R, b)$.

Let (R, b_J) be a B_J-subgroup such that b_J covers b. By [15, Tables 4.5.1 and 4.5.2],

$$C_J(z) = \langle \mathrm{SL}_2(q) \circ \mathrm{Spin}_{12}(q), t, t_J \rangle, \quad \text{with} \quad t = 2{:}2, \quad t_J = 1{:}2,$$

$\langle (\mathrm{SL}_8^\epsilon(q)/2) \circ 4, x_J \rangle$ with $x_J = (8^*/2){:}1$ or $\langle E_6^\epsilon(q)_u \circ (q - \epsilon).2, w \rangle$ with w given above. Thus $R \leq C_J(z)$, $C_J(z) = LA$ for some abelian A and A induces inner-diagonal automorphism on each L_i. A proof similar to that of Case 1 shows that $D(b_J)$ is abelian. If $y \in N_{C_K(P)}(R, b) \setminus C_K(R)$ such that $|y| = 3$, then $b_J^y = b_J$ since $|C_J(R){:}C_K(R)| \leq 2$ and b_J is the unique cover of b.

Suppose $z \in D$ is such that $|z| = 4$, $z^2 \in Z$ and $zZ \in Z(D/Z)$. Then $4 \mid (q - \epsilon)$ and

$$C_{J/Z}(zZ) = \langle \mathrm{SL}_8^\epsilon(q)/4 \circ 2, \bar{x}_J, \bar{x} \rangle \quad \text{with} \quad \bar{x}_J = (8^*/2){:}1, \quad \bar{x} = \gamma$$

or $\langle 3^*.E_6^\epsilon(q) \circ (q - \epsilon), \bar{w}, \bar{v} \rangle$ with $\bar{w} = 3^*{:}1$ and $\bar{v} = \gamma{:}i$. Here γ and i are graph and inverse automorphisms, respectively. Since $D \leq K$ and $D/Z \leq C_{J/Z}(zZ)$, it follows that $D/Z \leq C_J(z)/Z$, $z \in Z(D)$ and so (z, B_z) is major.

If $L = \mathrm{SL}_8^\epsilon(q)/2$ or $E_6^\epsilon(q)_u$, then by Proposition 8.1 or Case 2, $C_J(P)/Z = C_{J/Z}(P/Z)$ and $C_J(R)R/Z = C_{J/Z}(R/Z)$. Suppose $L = \mathrm{SL}_2(q) \circ \mathrm{Spin}_{12}(q)$. By [15, Table 4.5.1],

$$C_{J/Z}(zZ) = \langle \mathrm{SL}_2(q) \circ (\mathrm{Spin}_{12}(q)/2), t, t_J \rangle$$

where t and t_J are given above. Thus $C_J(z)/Z = C_{J/Z}(zZ)$ and by Propositions 8.1 and 8.6, $C_J(P)/Z = C_{J/Z}(P/Z)$ and $C_J(R)R/Z = C_{J/Z}(R/Z)$.

Suppose $D(B_i)$ and $D(B_{H_i})$ are both abelian. Then $D = D(B_z)$ is abelian. Let B_J cover B and $D_J = D(B_J)$. If there exists $k \in Z(D_J) \setminus D$, then $k \in C_J(D) \setminus K$ and so $D_J = \langle D, k \rangle$ is abelian. If $Z(D_J) \leq D$, then we may choose $z \in Z(D_J)$ and so $D_J \leq C_J(z) = LA$. A proof similar to that of Case 1 with R replaced by D_J shows that D_J is abelian and Property 7.1 (b) holds.

Case 4. Suppose $K := E_8(q)$, so that (z, B_z) is a major subsection of B. Either L_i is classical, or L_i is exceptional and given in Cases 1, 2 or 3. If L_i is classical, then apply Theorem 8.7. If L_i is exceptional, then apply the results in Cases 1, 2 or 3. A proof similar to that of Case 1 shows that either Property 7.1 (a) holds for B or each $D(B_i)$ is abelian. In the latter case, each $D(B_{H_i})$ is abelian, and so $D = D(B_z) = D(B_H) \cap C$ is abelian. □

Lemma 9.2 *Let G be a quasisimple group such that $G/Z(G)$ is alternating or of Lie type, and G is an exceptional cover. Then every 2-block of G with nonabelian defect groups has at least two irreducible Brauer characters.*

Proof: It suffices to consider the cases where $|Z(G)|$ is odd. We must consider the cases $G/Z(G) \cong A_7$, $PSL_2(9)$, $PSU_4(3)$ and $O_7(3)$. In each case we may use [14] to confirm the result. □

10 Proofs of the main theorems

We may finally prove that nilpotent 2-blocks of quasisimple groups have abelian defect groups:

Proof of Theorem 1.1. If $G/Z(G)$ is an alternating group, then the result follows by Corollary 3.3. For $G/Z(G)$ sporadic see Corollary 4.2. If $G/Z(G)$ is a classical group and G is a non-exceptional cover, see Propositions 8.1, 8.2, 8.4 and 8.6. For $G/Z(G)$ an exceptional group of Lie type and G is a non-exceptional cover, see Theorem 9.1. For the exceptional covers, see Lemma 9.2.

We complete the proof of Puig's conjecture for quasisimple groups for the prime 2:

Proof of Theorem 1.2. The necessary condition for nilpotency follows from [10, 1.2]. By Propositions 7.2, 8.1, 8.2, 8.4 and 8.6 the result holds for the classical groups. By Theorem 9.1 it holds for the exceptional groups of Lie type. The result holds for the double covers of the alternating groups by Proposition 3.4, and when $G/Z(G)$ is sporadic by Corollary 4.3. For the exceptional covers of the alternating groups and of the finite simple groups of Lie type, see Lemma 9.2.

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