L-packets and depth for $SL_2(K)$ with $K$ local function field of characteristic 2

Mendes, Sergio and Plymen, Roger

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School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

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Abstract. Let $G = SL_2(K)$ with $K$ a local function field of characteristic 2. We review Artin-Schreier theory for the field $K$, and show that this leads to a parametrization of certain $L$-packets in the smooth dual of $G$. We relate this to a recent geometric conjecture. The $L$-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal $L$-packets of cardinality 4 are parametrised by biquadratic extensions. Each supercuspidal packet of cardinality 4 is accompanied by a singleton packet for $SL_1(D)$. We compute the depths of the irreducible constituents of all these $L$-packets for $SL_2(K)$ and its inner form $SL_1(D)$.

1. Introduction

The special linear group $SL_2$ has been a mainstay of representation theory for at least 45 years, see [GPGS]. In that book, the authors show how the unitary irreducible representations of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Q}_p)$ can be woven together in the context of automorphic forms. This comes about in the following way. The classical notion of a cusp form $f$ in the upper half plane leads first to the concept of a cusp form on the adele group of $GL_2$ over $\mathbb{Q}$, and thence to the idea of an automorphic cuspidal representation $\pi_f$ of the adele group of $GL_2$. We recall that the adele group of $GL_2$ is the restricted product of the local groups $GL_2(\mathbb{Q}_p)$ where $p$ is a place of $\mathbb{Q}$. If $p$ is infinite then $\mathbb{Q}_p$ is the real field $\mathbb{R}$; if $p$ is finite then $\mathbb{Q}_p$ is the $p$-adic field. The unitary representation $\pi_f$ may be expressed as $\otimes \pi_p$ with one local representation for each local group $GL_2(\mathbb{Q}_p)$. It is this way that the unitary representation theory of groups such as $GL_2(\mathbb{Q}_p)$ enters into the modern theory of automorphic forms.

Let $X$ be a smooth projective curve over $\mathbb{F}_q$. Denote by $F$ the field $\mathbb{F}_q(X)$ of rational functions on $X$. For any closed point $x$ of $X$ we denote by $F_x$ the completion of $F$ at $x$ and by $\mathfrak{o}_x$ its ring of integers. If we choose a local coordinate $t_x$ at $x$ (i.e., a rational function on $X$ which vanishes at $x$ to order one), then we obtain isomorphisms $F_x \simeq \mathbb{F}_{q_x}(t_x)$ and $\mathfrak{o}_x \simeq \mathbb{F}_{q_x}[[t_x]]$, where $\mathbb{F}_{q_x}$ is the residue field of $x$; in general, it is a finite extension of $\mathbb{F}_q$ containing $q_x = q^{\deg(x)}$ elements. Thus, we now have a local function field attached to each point of $X$.

With all this in the background, it seems natural to us to study the representation theory of $SL_2(K)$ with $K$ a local function field. The case when $K$ has characteristic 2 has many special features – and we focus on this case in this article. A local function field $K$ of characteristic 2 is of the form $K = \mathbb{F}_q((t))$, the field of Laurent series with coefficients in $\mathbb{F}_q$, with $q = 2^f$. This example is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((t))$. 

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\textbf{Key words and phrases.} Representation theory, $L$-packets, depth.
Artin-Schreier theory is a branch of Galois theory, and more specifically is a positive characteristic analogue of Kummer theory, for Galois extensions of degree equal to the characteristic $p$. Artin and Schreier (1927) introduced Artin-Schreier theory for extensions of prime degree $p$, and Witt (1936) generalized it to extensions of prime power degree $p^n$. If $K$ is a field of characteristic $p$, a prime number, any polynomial of the form

$$X^p - X + \alpha$$

for $\alpha \in K$, is called an Artin-Schreier polynomial. When $\alpha$ does not lie in the subset \(\{y \in K \mid y = x^p - x \text{ for } x \in K\}\), this polynomial is irreducible in $K[X]$, and its splitting field over $K$ is a cyclic extension of $K$ of degree $p$. This follows since for any root $\beta$, the numbers $\beta + i$, for $1 \leq i \leq p$, form all the root – by Fermat’s little theorem – so the splitting field is $K(\beta)$. Conversely, any Galois extension of $K$ of degree $p$ equal to the characteristic of $K$ is the splitting field of an Artin-Schreier polynomial. This can be proved using additive counterparts of the methods involved in Kummer theory, such as Hilbert’s theorem 90 and additive Galois cohomology. These extensions are called Artin-Schreier extensions.

For the moment, let $F$ be a local nonarchimedean field with odd residual characteristic. The $L$-packets for $\text{SL}_2(F)$ are classified in the paper LR by Lansky-Rhaguram. They comprise: the principal series $L$ is a quadratic extension; the unramified supercuspidal character. The $L$-packets for all the root – by Fermat’s little theorem – so the splitting field is $K(\beta)$. Conversely, any Galois extension of $K$ of degree $p$ equal to the characteristic of $K$ is the splitting field of an Artin-Schreier polynomial. This can be proved using additive counterparts of the methods involved in Kummer theory, such as Hilbert’s theorem 90 and additive Galois cohomology. These extensions are called Artin-Schreier extensions.

In this article, we do not consider supercuspidal $L$-packets of cardinality 2. We consider $\text{SL}_2(K)$. Drawing on the accounts in [Da, Th1, Th2], we review Artin-Schreier theory, adapted to the local function field $K$, with special emphasis on the quadratic extensions of $K$.

The $L$-packets in the principal series of $\text{SL}_2(K)$ are parametrized by quadratic extensions, and the supersingular $L$-packets of cardinality 4 are parametrized by bi-quadratic extensions $L/K$. There are countably many such supercuspidal $L$-packets. In this article, we do not consider supersingular $L$-packets of cardinality 2.

The concept of depth can be traced back to the concept of level of a character. Let $\chi$ be a non-trivial character of $K^\times$. The level of $\chi$ is the least integer $n \geq 0$ such that $\chi$ is trivial on the higher unit group $U^{n+1}_K$, see [BH, p.12]. The depth of a Langlands parameter $\phi$ is defined as follows. Let $r$ be a real number, $r \geq 0$, let $\text{Gal}(K_s/K)^r$ be the $r$-th ramification subgroup of the absolute Galois group of $K$. Then the depth of $\phi$ is the smallest number $d(\phi) \geq 0$ such that $\phi$ is trivial on $\text{Gal}(K_s/K)^r$ for all $r > d(\phi)$.

The depth $d(\pi)$ of an irreducible $G$-representation $\pi$ was defined by Moy and Prasad [MoPr1, MoPr2] in terms of filtrations $P_{x,r}(r \in \mathbb{R}_{\geq 0})$ of the parahoric subgroups $P_x \subset G$.

Let $G = \text{SL}_2(K)$. Let $\text{Irr}(G)$ denote the smooth dual of $G$. Thanks to a recent article [ABPS1], we have, for every Langlands parameter $\phi \in \Phi(G)$ with $L$-packet $\Pi_{\phi}(G) \subset \text{Irr}(G)$

$$d(\phi) = d(\pi) \quad \text{for all } \pi \in \Pi_{\phi}(G).$$

The equation (1) is a big help in the computation of the depth $d(\pi)$. To each biquadratic extension $L/K$, there is attached a Langlands parameter $\phi = \phi_{L/K}$, and an $L$-packet $\Pi_{\phi}$ of cardinality 4. The depth of the parameter $\phi_{L/K}$ depends on the extension $L/K$. More precisely, the numbers $d(\phi)$ depend on the breaks in the
upper ramification filtration of the Galois group
\[ \text{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \]

For certain extensions \( L/K \) the allowed depths can be any odd number 1, 3, 5, 7, \ldots. For the other extensions \( L/K \), the allowed depths are 3, 5, 7, 9, \ldots. Accordingly, the depth of each irreducible supercuspidal representation \( \pi \) in the packet \( \Pi_\phi \) is given by the formula
\[ d(\pi) = 2n + 1 \]
where \( n = 0, 1, 2, 3, \ldots \) or 1, 2, 3, 4, \ldots depending on \( L/K \). Let \( D \) be a central division algebra of dimension 4 over \( K \). The parameter \( \phi \) is relevant for the inner form \( \text{SL}_1(D) \), which admits singleton \( L \)-packets, and the depths are given by the formula \[2\].

This contrasts with the case of \( \text{SL}_2(\mathbb{Q}_p) \) with \( p > 2 \). Here there is a unique biquadratic extension \( L/K \), and a unique tamely ramified parameter \( \phi : \text{Gal}(L/K) \to \text{SO}_3(\mathbb{R}) \) of depth zero.

We move on to consider the geometric conjecture in [ABPS]. Let \( \mathfrak{B}(G) \) denote the Bernstein spectrum of \( G \), let \( s \in \mathfrak{B}(G) \), and let \( T^s, W^s \) denote the complex torus, finite group, attached by Bernstein to \( s \). For more details at this point, we refer the reader to [R]. The Bernstein decomposition provides us, inter alia, with the following data: a canonical disjoint union
\[ \text{Irr}(G) = \bigsqcup \text{Irr}(G)^s \]
and, for each \( s \in \mathfrak{B}(G) \), a finite-to-one surjective map
\[ \text{Irr}(G)^s \to T^s/W^s \]
on to the quotient variety \( T^s/W^s \). The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a bijection
\[ \text{Irr}(G)^s \simeq T^s//W^s \]
where \( T^s//W^s \) is the extended quotient of the torus \( T^s \) by the finite group \( W^s \). If the action of \( W^s \) on \( T^s \) is free, then the extended quotient is equal to the ordinary quotient \( T^s/W^s \). If the action is not free, then the extended quotient is a finite disjoint union of quotient varieties, one of which is the ordinary quotient. The bijection \[3\] is subject to certain constraints, itemised in [ABPS].

In the case of \( \text{SL}_2 \), the torus \( T^s \) is of dimension 1, and the finite group \( W^s \) is either 1 or \( \mathbb{Z}/2\mathbb{Z} \). So, in this context, the content of the conjecture is rather modest: but a proof is required, and such a proof is duly given in \S7.

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2. Artin-Schreier theory

Let \( K \) be a local field with positive characteristic \( p \), containing the \( n \)-th roots of unity \( \zeta_n \). The cyclic extensions of \( K \) whose degree \( n \) is coprime with \( p \) are described by Kummer theory. It is well known that any cyclic extension \( L/K \) of degree \( n \), \( (n, p) = 1 \), is generated by a root \( \alpha \) of an irreducible polynomial \( x^n - a \in K[x] \). We fix an algebraic closure \( \overline{K} \) of \( K \) and a separable closure \( K^s \) of \( K \) in \( \overline{K} \). If \( \alpha \in K^s \)
is a root of \(x^n - a\) then \(K(\alpha)/K\) is a cyclic extension of degree \(n\) and is called a Kummer extension of \(K\).

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by \(\text{ch}(K) = p\). It is therefore an analogue of Kummer theory, where the role of the polynomial \(x^n - a\) is played by \(x^n - x - a\). Essentially, every cyclic extension of \(K\) with degree \(p = \text{ch}(K)\) is generated by a root \(\alpha\) of \(x^p - x - a \in K[x]\).

Let \(\varphi\) denote the Artin-Schreier endomorphism of the additive group \(K^s\): 
\[
\varphi : K^s \to K^s, \quad x \mapsto x^p - x.
\]

Given \(a \in K\) denote by \(K(\varphi^{-1}(a))\) the extension \(K(\alpha)\), where \(\varphi(\alpha) = a\) and \(\alpha \in K^s\). We have the following characterization of finite cyclic Artin-Schreier extensions of degree \(p\):

**Theorem 2.1.**

(i) Given \(a \in K\), either \(\varphi(x) - a \in K[x]\) has one root in \(K\) in which case it has all the \(p\) roots are in \(K\), or is irreducible.

(ii) If \(\varphi(x) - a \in K[x]\) is irreducible then \(K(\varphi^{-1}(a))/K\) is a cyclic extension of degree \(p\), with \(\varphi^{-1}(a) \subset K^s\).

(iii) If \(L/K\) be a finite cyclic extension of degree \(p\), then \(L = K(\varphi^{-1}(a))\), for some \(a \in K\).

(See [Th1] p.34 for more details.)

We fix some notation. \(K\) is a local field with characteristic \(p > 1\) with finite residue field \(k\). The field of constants \(k = \mathbb{F}_q\) is a finite extension of \(\mathbb{F}_p\), with degree \([k : \mathbb{F}_p] = f\) and \(q = p^f\).

Let \(\mathfrak{o}\) be the ring of integers in \(K\) and denote by \(\mathfrak{p} \subset \mathfrak{o}\) the (unique) maximal ideal of \(\mathfrak{o}\). This ideal is principal and any generator of \(\mathfrak{p}\) is called a uniformizer. A choice of uniformizer \(\varpi \in \mathfrak{o}\) determines isomorphisms \(K \cong \mathbb{F}_q((\varpi)), \mathfrak{o} \cong \mathbb{F}_q[[\varpi]]\) and \(\mathfrak{p} = \varpi \mathfrak{o} \cong \varpi \mathbb{F}_q[[\varpi]]\).

A normalized valuation on \(K\) will be denoted by \(\nu\), so that \(\nu(\varpi) = 1\) and \(\nu(K) = \mathbb{Z}\). The group of units is denoted by \(\mathfrak{o}^\times\).

### 2.1. The Artin-Schreier symbol

Let \(L/K\) be a finite Galois extension. Let \(N_{L/K}\) be the norm map and denote by \(\text{Gal}(L/K)^{ab}\) the abelianization of \(\text{Gal}(L/K)\). The reciprocity map is a group isomorphism 
\[
K^\times / N_{L/K}L^\times \xrightarrow{\sim} \text{Gal}(L/K)^{ab}.
\]

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism \(K^\times \to K^\times / N_{L/K}L^\times\)
\[
b \in K^\times \mapsto (b, L/K) \in \text{Gal}(L/K)^{ab}.
\]

From the Artin symbol we obtain a pairing 
\[
K \times K^\times \to \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha,
\]
where \(\varphi(\alpha) = a\), \(\alpha \in K^s\) and \(L = K(\alpha)\).

**Definition 2.2.** Given \(a \in K\) and \(b \in K^\times\), the Artin-Schreier symbol is defined by 
\[
[a, b] = (b, L/K)(\alpha) - \alpha.
\]
The Artin-Schreier symbol is a bilinear map satisfying the following properties, see [Ne, p.341]:

(7) \[[a_1 + a_2, b] = [a_1, b] + [a_2, b];\]
(8) \[[a, b_1b_2] = [a, b_1] + [a, b_2];\]
(9) \[[a, b] = 0, \forall a \in K \iff b \in N_{L/K} \L^\times, L = K(\alpha) \text{ and } \wp(\alpha) = a;\]
(10) \[[a, b] = 0, \forall b \in K^\times \iff a \in \wp(K).\]

2.2. The groups $K/\wp(K)$ and $K^\times/K^{\times p}$. In this section we recall some properties of the groups $K/\wp(K)$ and $K^\times/K^{\times p}$ and use them to redefine the pairing (6). Dalawat [Da2, Da] interprets $K/\wp(K)$ and $K^\times/K^{\times p}$ as $\mathbb{F}_p$-spaces. This interpretation will be particularly useful in §4.

Consider the additive group $K$. By [Da, Proposition 11], the $\mathbb{F}_p$-space $K/\wp(K)$ is countably infinite. Hence, $K/\wp(K)$ is infinite as a group.

**Proposition 2.3.** $K/\wp(K)$ is a discrete abelian torsion group.

**Proof.** The ring of integers decomposes as a (direct) sum
\[
o = \mathbb{F}_q + p\]
and we have
\[
\wp(n) = \wp(\mathbb{F}_q) + \wp(p).
\]
The restriction $\wp : p \to p$ is an isomorphism, see [Da, Lemma 8]. Hence,
\[
\wp(n) = \wp(\mathbb{F}_q) + p
\]
and $p \subset \wp(K)$. It follows that $\wp(K)$ is an open subgroup of $K$ and $K/\wp(K)$ is discrete. Since $\wp(K)$ is annihilated by $p$, $K/\wp(K)$ is a torsion group. □

Now we concentrate on the multiplicative group $K^\times$. For any $n > 0$, let $U_n$ be the kernel of the reduction map from $\mathfrak{o}^\times$ to $(\mathfrak{o}/\mathfrak{p}^n)^\times$. In particular, $U_1 = ker(\mathfrak{o}^\times \to k^\times)$. The $U_n$ are $\mathbb{Z}_p$-modules, because they are commutative pro-$p$-groups. By [Da2, Proposition 20], the $\mathbb{Z}_p$-module $U_1$ is not finitely generated. As a consequence, $K^\times/K^{\times p}$ is infinite, see [Da2, Corollary 21]. The next result gives a characterization of the topological group $K^\times/K^{\times p}$.

**Proposition 2.4.** $K^\times/K^{\times p}$ is a profinite abelian $p$-torsion group.

**Proof.** There is a canonical isomorphism $K^\times \cong \mathbb{Z} \times \mathfrak{o}^\times$. The group of units is a direct product $\mathfrak{o}^\times \cong \mathbb{F}_q^\times \times U_1$, with $q = p^f$. By [Iw, p.25], the group $U_1$ is a direct product of countable many copies of the ring of $p$-adic integers
\[
U_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots = \prod_{n} \mathbb{Z}_p.
\]

Give $\mathbb{Z}$ the discrete topology and $\mathbb{Z}_p$ the $p$-adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \mathbb{Z}/(q-1) \mathbb{Z} \times \prod_{n} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

Now, $K^{\times p}$ decomposes as a product of countable many components
\[
K^{\times p} \cong p\mathbb{Z} \times \mathbb{Z}/(q-1) \mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times \ldots
\]
\[ = p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_N p\mathbb{Z}_p. \]

Note that \( p\mathbb{Z}/(q-1)\mathbb{Z} = \mathbb{Z}/(q-1)\mathbb{Z}, \) since \( p \) and \( q-1 \) are coprime. Denote by \( z = \prod_n z_n \) an element of \( \prod_N \mathbb{Z}_p, \) where \( z_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p, \) for every \( n. \)

The map

\[ \varphi : \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_N \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z} \times \prod_N \mathbb{Z}_p/p\mathbb{Z} \]

defined by

\[ (x, y, z) \mapsto (x \mod p, \prod_n pr_0(z_n)) \]

where \( pr_0(z_n) = a_{0,n} \) is the projection, is clearly a group homomorphism.

Now, \( \mathbb{Z}/p\mathbb{Z} \times \prod_N \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z} \) is a topological group for the product topology, where each component \( \mathbb{Z}/p\mathbb{Z} \) has the discrete topology. It is compact, Hausdorff and totally disconnected. Therefore, \( \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z} \) is a profinite group.

Since

\[ ker\varphi = p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_N p\mathbb{Z}_p, \]

it follows that there is an isomorphism of topological groups

\[ K^\times/K^\times p \cong \prod_{N_0} \mathbb{Z}/p\mathbb{Z}, \]

where \( K^\times/K^\times p \) is given the quotient topology. Therefore, \( K^\times/K^\times p \) is profinite. \( \square \)

From propositions 2.3 and 2.4, \( K/\varphi(K) \) is a discrete abelian group and \( K/K^\times p \) is an abelian profinite group, both annihilated by \( p = ch(K). \) Therefore, Pontryagin duality coincides with \( Hom(\cdot, \mathbb{Z}/p\mathbb{Z}) \) on both of these groups, see [Th2]. The pairing \( (6) \) restricts to a pairing

\[ (11) \quad [,.,] : K/\varphi(K) \times K^\times/K^\times p \to \mathbb{Z}/p\mathbb{Z}. \]

which we refer from now on to the Artin-Schreier pairing. It follows from \( (9) \) and \( (10), \) that the pairing is nondegenerate (see also [Th2] Proposition 3.1]). The next result shows that the pairing is perfect.

**Proposition 2.5.** The Artin-Schreier symbol induces isomorphisms of topological groups

\[ K^\times/K^\times p \cong \text{Hom}(K/\varphi(K), \mathbb{Z}/p\mathbb{Z}), bK^\times p \mapsto (a + \varphi(K) \mapsto [a, b]) \]

and

\[ K/\varphi(K) \cong \text{Hom}(K^\times/K^\times p, \mathbb{Z}/p\mathbb{Z}), a + \varphi(K) \mapsto (bK^\times p \mapsto [a, b]) \]

**Proof.** The result follows by taking \( n = 1 \) in Proposition 5.1 of [Th2], and from the fact that Pontryagin duality for the groups \( K/\varphi(K) \) and \( K^\times/K^\times p \) coincide with \( Hom(-, \mathbb{Z}/p\mathbb{Z}) \) duality. Hence, there is an isomorphism of topological groups between each such group and its bidual. \( \square \)
Let $B$ be a subgroup of the additive group of $K$ with finite index such that $\wp(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of exponent $p$ is again a finite abelian Galois extension of exponent $p$. Therefore, the composite

$$K_B = K(\wp^{-1}(B)) = \prod_{a \in B} K(\wp^{-1}(a))$$

is a finite abelian Galois extension of exponent $p$. On the other hand, if $L/K$ is a finite abelian Galois extension of exponent $p$, then $L = K_B$ for some subgroup $\wp(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent $p$, which we denote by $K_p = K(\wp^{-1}(K))$. The extension $K_p/K$ is infinite and Galois. The corresponding Galois group $G_p = Gal(K_p/K)$ is an infinite profinite group and may be identified, under class field theory, with $K^\times/K^\times p$, see [Th2, Proposition 5.1]. The case $ch(K) = 2$ leads to $G_2 \cong K^\times/K^\times 2$ and will play a fundamental role in the sequel.

3. Quadratic characters

From now on we take $K$ to be a local function field with $ch(K) = 2$. Therefore, $K$ is of the form $\mathbb{F}_q((\varpi))$ with $q = 2^f$.

When $K = \mathbb{F}_q((\varpi))$, we have, according to [Wk p.25],

$$U_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots = \prod_{N} \mathbb{Z}_2$$

with countably many copies of $\mathbb{Z}_2$, the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_q((\varpi))$. By proposition 2.4, there is a bijection between the set of quadratic extensions of $\mathbb{F}_q((\varpi))$ and the group

$$\mathbb{F}_q((\varpi))^\times / \mathbb{F}_q((\varpi))^\times 2 \cong \prod_{N} \mathbb{Z} / 2\mathbb{Z} = G_2$$

where $G_2$ is the Galois group of the maximal abelian extension of exponent 2. Since $G_2$ is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension $K(\alpha)/K$, with $\alpha^2 - \alpha = a$, we associate the Artin-Schreier symbol

$$[a, \cdot] : K^\times / K^\times 2 \to \mathbb{Z} / 2\mathbb{Z}.$$ 

Now, let $\varphi$ denote the isomorphism $\mathbb{Z} / 2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$ with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character

$$(12) \quad \chi_a : K^\times \to \mathbb{C}^\times, \chi_a = \varphi([a, \cdot])$$

Proposition 2.5 shows that every quadratic character of $\mathbb{F}_q((\varpi))^\times$ arises in this way.

Example 3.1. The unramified quadratic extension of $K$ is $K(\wp^{-1}(\mathfrak{o}))$, see Da, proposition 12. According to Dalawat, the group $K/\wp(K)$ may be regarded as an $\mathbb{F}_2$-space and the image of $\mathfrak{o}$ under the canonical surjection $K \to K/\wp(K)$ is an $\mathbb{F}_2$-line, i.e., isomorphic to $\mathbb{F}_2$. Since $\wp_\mathfrak{p} : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, the image of $\mathfrak{p}$ in $K/\wp(K)$ is $\{0\}$, see lemma 8 in Da. Now, choose any $a_0 \in \mathfrak{o}\setminus \mathfrak{p}$ such that the image of $a_0$ in $\mathfrak{o}/\mathfrak{p}$ has nonzero trace in $\mathbb{F}_2$, see Da, Proposition 9. The
quadratic character \( \chi_{a_0} = \varphi([a_0, \cdot]) \) associated with \( K(\varphi^{-1}(\mathcal{O})) \) via class field theory is precisely the unramified character \( n \mapsto (-1)^n \) from above. Note that any other choice \( b_0 \in \mathcal{O}\setminus \mathfrak{p} \), with \( a_0 \neq b_0 \), gives the same unique unramified character, since there is only one nontrivial coset \( a_0 + \varphi(K) \) for \( a_0 \in \mathcal{O}\setminus \mathfrak{p} \).

Let \( G \) denote \( SL_2(K) \), let \( B \) be the standard Borel subgroup of \( G \), let \( T \) be the diagonal subgroup of \( G \). Let \( \chi \) be a character of \( T \). Then, \( \chi \) inflates to a character of \( B \). Denote by \( \pi(\chi) \) the (unitarily) induced representation \( \text{Ind}_{B}^{G}(\chi) \). The representation space \( V(\chi) \) of \( \pi(\chi) \) consists of locally constant complex valued functions \( f : G \to \mathbb{C} \) such that, for every \( a \in K^\times \), \( b \in K \) and \( g \in G \), we have

\[
f\left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = |a|\chi(a)f(g)
\]

The action of \( G \) on \( V(\chi) \) is by right translation. The representations \( (\pi(\chi), V(\chi)) \) are called (unitary) principal series of \( G = SL_2(K) \).

Let \( \chi \) be a quadratic character of \( K^\times \). The reducibility of the induced representation \( \text{Ind}_{B}^{G}(\chi) \) is well known in zero characteristic. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic \( p \).

**Theorem 3.2.** [Ca] [Ca2] The representation \( \pi(\chi) = \text{Ind}_{B}^{G}(\chi) \) is reducible if, and only if, \( \chi \) is either \( |.|^\pm \) or a nontrivial quadratic character of \( K^\times \).

For a proof see [Ca], Theorems 1.7, 1.9 and [Ca2] §9.

From now on, \( \chi \) will be a quadratic character. It is a classical result that the unitary principal series for \( GL_2 \) are irreducible. For a representation of \( GL_2 \) parabolically induced by \( 1 \otimes \chi \), Clifford theory tells us that the dimension of the intertwining algebra of its restriction to \( SL_2 \) is 2. This is exactly the induced representation of \( SL_2 \) by \( \chi \):

\[
\text{Ind}_{B}^{GL_2(K)}(1 \otimes \chi)_{SL_2(K)} \cong \text{Ind}_{B}^{SL_2(K)}(\chi)
\]

where \( B \) denotes the standard Borel subgroup of \( GL_2(K) \). This leads to reducibility of the induced representation \( \text{Ind}_{B}^{G}(\chi) \) into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

\[
\pi(\chi) = \text{Ind}_{B}^{G}(\chi) = \pi(\chi)^+ \oplus \pi(\chi)^-
\]

define an \( L \)-packet \( \{\pi(\chi)^+, \pi(\chi)^-\} \) for \( SL_2 \).

4. **Biquadratic extensions of \( \mathbb{F}_q((\varpi)) \)**

Biquadratic extensions \( L/K \) are obtained by adjoining an \( \mathbb{F}_2 \)-line \( D \subset K/\varphi(K) \). Therefore, \( L = K(\varphi^{-1}(D)) = K(\alpha) \) where \( D = \text{span}\{a + \varphi(K)\} \), with \( \alpha^2 - \alpha = a \). In particular, if \( a_0 \in \mathcal{O}\setminus \mathfrak{p} \) such that the image of \( a_0 \) in \( \mathcal{O}/\mathfrak{p} \) has nonzero trace in \( \mathbb{F}_2 \), the \( \mathbb{F}_2 \)-line \( V_0 = \text{span}\{a_0 + \varphi(K)\} \) contains all the cosets \( a_0 + \varphi(K) \) where \( a_0 \) is an integer and so \( K(\varphi^{-1}(a)) = K(\varphi^{-1}(V_0)) = K(a_0) \) where \( a_0^2 - a_0 = a_0 \) gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering \( \mathbb{F}_2 \)-planes \( W = \text{span}\{a + \varphi(K), b + \varphi(K)\} \subset K/\varphi(K) \). Therefore, if \( a + \varphi(K) \) and \( b + \varphi(K) \) are \( \mathbb{F}_2 \)-linearly independent then \( K(\varphi^{-1}(W)) := K(\alpha, \beta) \) is biquadratic, where \( \alpha^2 - \alpha = a \) and \( \beta^2 - \beta = b \), \( \alpha, \beta \in K^\times \). Therefore, \( K(\alpha, \beta)/K \) is biquadratic if \( b - a \notin \varphi(K) \).
A biquadratic extension containing the line $V_0$ is of the form $K(\alpha_0, \beta)/K$. There are countably many quadratic extensions $L_0/K$ containing the unramified quadratic extension. They have ramification index $e(L_0/K) = 2$. And there are countably many biquadratic extensions $L/K$ which do not contain the unramified quadratic extension. They have ramification index $e(L/K) = 4$.

So, there is a plentiful supply of biquadratic extensions $K(\alpha, \beta)/K$.

4.1. Ramification. The space $K/\varphi(K)$ comes with a filtration

\[(14) \quad 0 \subset_1 V_0 \subset_1 f V_1 = V_2 \subset_1 f V_3 = V_4 \subset_1 f \ldots \subset_1 K/\varphi(K)\]

where $V_0$ is the image of $\mathfrak{o}_K$ and $V_i$ ($i > 0$) is the image of $p^{-i}$ under the canonical surjection $K \to K/\varphi(K)$. For $K = \mathbb{F}_q((\wp))$ and $i > 0$, each inclusion $V_{2i} \subset_1 f V_{2i+1}$ is a sub-$\mathbb{F}_2$-space of codimension $f$. The $\mathbb{F}_2$-dimension of $V_n$ is

\[(15) \quad \dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil f,\]

for every $n \in \mathbb{N}$, where $\lceil x \rceil$ is the smallest integer bigger than $x$.

Let $L/K$ denote a Galois extension with Galois group $G$. For each $i \geq -1$ we define the $i^{th}$-ramification subgroup of $G$ (in the lower numbering) to be:

\[G_i = \{\sigma \in G : \sigma(x) - x \in p_L^{i+1}, \forall x \in \mathfrak{o}_L}\].

An integer $t$ is a break for the filtration $\{G_i\}_{i \geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i \geq -1}$ is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i \geq -1}$ and defined by the Hasse-Herbrand function $\psi = \psi_{L/K}$:

\[G^u = G_{\psi(u)}\].

In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(0) = 0$.

Let $G_2 = \text{Gal}(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\varphi^{-1}(K))$. Since $G_2 \cong \mathbb{K}^x/K^x \times K/\varphi(K)$ from (11) coincides with the pairing $G_2 \times K/\varphi(K) \to \mathbb{Z}/2\mathbb{Z}$.

The profinite group $G_2$ comes equipped with a ramification filtration $(G_2^u)_{u \geq -1}$ in the upper numbering, see [Da] p.409. For $u \geq 0$, we have an orthogonal relation for $u \geq 0$:

\[(16) \quad (G_2^u)^\perp = p^{-\lceil u \rceil + 1} = V_{[u]}^{-1}\]

under the pairing $G_2 \times K/\varphi(K) \to \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will compute the upper breaks. By using the Hasse-Herbrand function it is then possible to compute the lower breaks in order to obtain the lower ramification filtration.

According to [Da] Proposition 17, the positive breaks in the filtration $(G^u)_v$ occur precisely at integers prime to $p$. So, for $ch(K) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If $G$ is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^u)_v$ (see [Da], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.
Given a plane \( W \subset \mathbb{K}/\wp(\mathbb{K}) \), the filtration (14) \((V_i)\) on \( \mathbb{K}/\wp(\mathbb{K}) \) induces a filtration \((W_i)\) on \( W \), where \( W_i = W \cap V_i \). There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

**Case 1:** \( W \) contains the line \( V_0 \), i.e. \( L_0 = \mathbb{K}(\wp^{-1}(V_0)) \) contains the unramified quadratic extension \( \mathbb{K}(\wp^{-1}(V_0)) = \mathbb{K}(\alpha_0) \) of \( \mathbb{K} \). The extension has residue degree \( f(L_0/\mathbb{K}) = 2 \) and ramification index \( e(L_0/\mathbb{K}) = 2 \). In this case, there is an integer \( t > 0 \), necessarily odd, such that the filtration \((W_i)\) looks like 

\[ 0 \subset W_0 = W_{t-1} \subset W_t = W. \]

By the orthogonality relation (16), the upper ramification filtration on \( G = \text{Gal}(L_0/\mathbb{K}) \) looks like 

\[ \{1\} = ... = G^{t+1} \subset G^t = ... = G^0 \subset G^{-1} = G. \]

Therefore, the upper ramification breaks occur at \(-1\) and \( t \).

The number of such \( W \) is equal to the number of planes in \( V_t \) containing the line \( V_0 \) but not contained in the subspace \( V_{t-1} \). This number can be computed and equals the number of biquadratic extensions of \( \mathbb{K} \) containing the unramified quadratic extensions and with a pair of upper ramification breaks \((-1, t)\), \( t > 0 \) and odd. Here is an example.

**Example 4.1.** The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks \((-1, t)\), \( t > 0 \) and odd.

\[ 1 + 2 + 2^2 + ... + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1 \]

of such biquadratic extensions.

**Case 2.1:** \( W \) does not contain the line \( V_0 \) and the induced filtration on the plane \( W \) looks like 

\[ 0 = W_{t-1} \subset W_t = W \]

for some integer \( t \), necessarily odd.

The number of such \( W \) is equal to the number of planes in \( V_t \) whose intersection with \( V_{t-1} \) is \( \{0\} \). Note that, there are no such planes when \( f = 1 \). So, for \( K = \mathbb{F}_2((\wp)) \), **case 2.1** does not occur.

Suppose \( f > 1 \). By the orthogonality relation, the upper ramification filtration on \( G = \text{Gal}(L/\mathbb{K}) \) looks like 

\[ \{1\} = ... = G^{t+1} \subset G^t = ... = G^0 \subset G^{-1} = G. \]

Therefore, there is a single upper ramification break occurring at \( t > 0 \) and is necessarily odd.

For \( f = 1 \) there is no such biquadratic extension. For \( f > 1 \), the number of these biquadratic extensions equals the number of planes \( W \) contained in an \( \mathbb{F}_2 \)-space of dimension \( 1 + fi, t = 2i - 1 \), which are transverse to a given codimension-\( f \) \( \mathbb{F}_2 \)-space.

**Case 2.2:** \( W \) does not contain the line \( V_0 \) and the induced filtration on the plane \( W \) looks like 

\[ 0 = W_{t1-1} \subset W_{t1} = W_{t2-1} \subset W_{t2} = W \]
for some integers \( t_1 \) and \( t_2 \), necessarily odd, with \( 0 < t_1 < t_2 \).

The orthogonality relation for this case implies that the upper ramification filtration on \( G = \text{Gal}(L/K) \) looks like
\[
\{1\} = \ldots = G^{t_2+1} \subset G^{t_2} = \ldots = G^{t_1+1} \subset G^{t_1} = \ldots = G
\]
The upper ramification breaks occur at odd integers \( t_1 \) and \( t_2 \).

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks \((t_1, t_2)\).

5. **Langlands parameter**

We have the following canonical homomorphism:
\[
W_K \rightarrow W_K^{ab} \cong K^\times \rightarrow K^\times/K^{\times 2}.
\]

According to §2, we also have
\[
K^\times/K^{\times 2} \cong \prod \mathbb{Z}/2\mathbb{Z}
\]
the product over countably many copies of \( \mathbb{Z}/2\mathbb{Z} \). Using the countable axiom of choice, we choose two copies of \( \mathbb{Z}/2\mathbb{Z} \). This creates a homomorphism
\[
W_K \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

There are countably many such homomorphisms.

Following [We], denote by \( \alpha, \beta, \gamma \) the images in \( \text{PSL}_2(\mathbb{C}) \) of the elements
\[
z_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad z_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z_\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]
in \( \text{SL}_2(\mathbb{C}) \).

Note that \( z_\alpha, z_\beta, z_\gamma \in \text{SU}_2(\mathbb{C}) \) so that
\[
\alpha, \beta, \gamma \in \text{PSU}_2(\mathbb{C}) = \text{SO}_3(\mathbb{R}).
\]

Denote by \( J \) the group generated by \( \alpha, \beta, \gamma \):
\[
J := \{e, \alpha, \beta, \gamma\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]
The group \( J \) is unique up to conjugacy in \( G = \text{PSL}_2(\mathbb{C}) \).

The pre-image of \( J \) in \( \text{SL}_2(\mathbb{C}) \) is the group \( \{\pm 1, \pm z_\alpha, \pm z_\beta, \pm z_\gamma\} \) and is isomorphic to the group \( U_8 \) of unit quaternions \( \{\pm 1, \pm i, \pm j, \pm k\} \).

The centralizer and normalizer of \( J \) are given by
\[
C_G(J) = J, \quad N_G(J) = O
\]
where \( O \cong S_4 \) the symmetric group on 4 letters. The quotient \( O/J \cong \text{GL}_2(\mathbb{Z}/2) \) is the full automorphism group of \( J \).

Each biquadratic extension \( L/K \) determines a Langlands parameter
\[
(17) \quad \phi : \text{Gal}(L/K) \rightarrow \text{SO}_3(\mathbb{R}) \subset \text{SO}_3(\mathbb{C})
\]
Define
\[
(18) \quad S_\phi = C_{\text{PSL}_2(\mathbb{C})}(\text{im } \phi)
\]
Then we have \( S_\phi = J \), since \( C_G(J) = J \), and whose conjugacy class depends only on \( L \), since \( O/J = \text{Aut}(J) \).

Define the new group
\[
S_\phi = C_{\text{SL}_2(\mathbb{C})}(\text{im } \phi)
\]
To align with the notation in [ABPS2], replace $\phi^*$ in [ABPS2] by $\phi$ in the present article. We have the short exact sequence

$$1 \to Z_\phi \to S_\phi \to S_\phi \to 1$$

with $Z_\phi = \mathbb{Z}/2\mathbb{Z}$.

Let $D$ be a central division algebra of dimension 4 over $K$, and let $Nrd$ denote the reduced norm on $D^\times$. Define $SL_1(D) = \{x \in D^\times : Nrd(x) = 1\}$.

Then $SL_1(D)$ is an inner form of $SL_2(K)$. In the local Langlands correspondence [ABPS2] for the inner forms of $SL_2$, the L-parameter $\phi$ is enhanced by elements $\rho \in \text{Irr}(S_\phi)$. Now the group $S_\phi \simeq U_8$ admits four characters $\rho_1, \rho_2, \rho_3, \rho_4$ and one irreducible representation $\rho_0$ of degree 2.

The parameter $\phi$ creates a big packet with five elements, which are allocated to $SL_2(K)$ or $SL_1(D)$ according to central characters. So $\phi$ assigns an L-packet $\Pi_\phi$ to $SL_2(K)$ with 4 elements, and a singleton packet to the inner form $SL_1(D)$. None of these packets contains the Steinberg representation of $SL_2(K)$ and so each $\Pi_\phi$ is a supercuspidal L-packet with 4 elements.

To be explicit: $\phi$ assigns to $SL_2(K)$ the supercuspidal packet

$$\{\pi(\phi, \rho_1), \pi(\phi, \rho_2), \pi(\phi, \rho_3), \pi(\phi, \rho_4)\}$$

and to $SL_1(D)$ the singleton packet

$$\{\pi(\phi, \rho_0)\}$$

and this phenomenon occurs countably many times.

Each supercuspidal packet of four elements is the JL-transfer of the singleton packet, in the following sense: the irreducible supercuspidal representation $\theta$ of $GL_2(K)$ which yields the 4-packet upon restriction to $SL_2(K)$ is the image in the JL-correspondence of the irreducible smooth representation $\psi$ of $GL_1(D)$ which yields two copies of $\pi(\phi, \rho_0)$ upon restriction to $SL_1(D)$:

$$\theta = JL(\psi).$$

Each parameter $\phi : W_K \to PGL_2(\mathbb{C})$ lifts to a Galois representation

$$\phi : W_K \to GL_2(\mathbb{C}).$$

This representation is triply imprimitive, as in [We]. Let $\mathfrak{T}(\phi)$ be the group of characters $\chi$ of $W_K$ such that $\chi \otimes \phi \simeq \phi$. Then $\mathfrak{T}(\phi)$ is non-cyclic of order 4.

6. Depth

Let $L/K$ be a biquadratic extension. We fix an algebraic closure $\overline{K}$ of $K$ such that $L \subset \overline{K}$. From the inclusion $L \subset \overline{K}$, there is a natural surjection

$$\pi_{L/K} : \text{Gal}(\overline{K}/K) \to \text{Gal}(L/K)$$

Let $K^{ur}$ be the maximal unramified extension of $K$ in $\overline{K}$ and let $K^{ab}$ be the maximal abelian extension of $K$ in $\overline{K}$. We have a commutative diagram, where the horizontal maps are the canonical maps and the vertical maps are the natural projections.
Let \( G = \text{Gal}(L/K) \), \( G^r \) be the filtration of the relative inertia subgroup, and so on. Note that 
\[
\text{Gal}(L/K) \subset I_3(L/K) \subset I_2(L/K) \subset I_1(L/K) = \text{Gal}(L/K)
\]
be the filtration of the absolute inertia subgroup, and so on. Note that 
\[
G^r = \text{Gal}(L/K)
\]
be the filtration of the absolute Galois group of \( L/K \) to the subgroup \( J_{L/K} \), i.e., \( J^{(r)} = \iota_3(G^r) \).

Let 
\[
\iota_1(I^{(r)}) \subset I^{(2)} \subset I^{(1)} \subset I^{(0)} \subset \text{Gal}(L/K)
\]
be the filtration of the absolute inertia subgroup \( I^{(0)} = I_{\mathbb{R}/K} \) of \( \text{Gal}(K^s/K) \), \( I^{(1)} \) is the wild inertia subgroup, and so on...

**Lemma 6.1.** We have 
\[
(\forall r) \pi_{L/K} I^{(r)} = J^{(r)}
\]

**Proof.** This follows immediately from the above diagram. Here, we identify \( I^{(r)} \) with \( \iota_1(I^{(r)}) \) and \( J^{(r)} \) with \( \iota_3(J^{(r)}) \).

**Lemma 6.2.** Let \( L/K \) be a biquadratic extension, let \( \phi \) be the Langlands parameter \( I_{\mathbb{R}/K}, \phi = \alpha \circ \pi_{L/K} \) with \( \alpha : \text{Gal}(L/K) \rightarrow SO_3(\mathbb{R}) \). Then we have \( d(\phi) = r - 1 \) where \( r \) is the least integer for which \( J^{(r)} = 1 \).

**Proof.** The depth of a Langlands parameter \( \phi \) is easy to define. For \( r \in \mathbb{R} \geq 0 \) let \( \text{Gal}(F_s/F)^r \) be the \( r \)-th ramification subgroup of the absolute Galois group of \( F \). Then the depth of \( \phi \) is the smallest number \( d(\phi) \geq 0 \) such that \( \phi \) is trivial on \( \text{Gal}(F_s/F)^r \) for all \( r > d(\phi) \).

Note that \( \alpha \) is injective. Therefore 
\[
\phi(I^{(r)}) = 1 \iff (\alpha \circ \pi_{L/K}) I^{(r)} = 1 \iff \alpha(J^{(r)}) = 1 \iff J^{(r)} = 1.
\]

For example, the parameter \( \phi \) has depth zero if it is tamely ramified, i.e. the least integer \( r \) for which \( J^{(r)} = 1 \) is \( r = 1 \). The relative wild inertia group is 1, but the relative inertia group is not 1.

**Case 1:** There are two ramification breaks occurring at \(-1\) and some odd integer \( t > 0 \):
\[
\{1\} = \ldots = J^{(t+1)} \subset J^{(t)} = \ldots J^{(0)} = J_{L/K} \subset \text{Gal}(L/K), \quad d(\varphi) = t
\]
The allowed depths are 1, 3, 5, 7, ...
Case 2.1: One single ramification break occurs at some odd integer \( t > 0 \):

\[
\{1\} = \ldots = \mathfrak{m}^{(t+1)} \subset \mathfrak{m}^{(t)} = \ldots = \mathfrak{m}^{(0)} = \mathfrak{m}_{L/K} = \text{Gal}(L/K); \quad d(\varphi) = t
\]

The allowed depths are 1, 3, 5, 7, \ldots.

Case 2.2: There are two ramification breaks occurring at some odd integers \( t_1 < t_2 \)

\[
\{1\} = \ldots = \mathfrak{m}^{(t_2+1)} \subset \mathfrak{m}^{(t_2)} = \ldots = \mathfrak{m}^{(t_1+1)} \subset \mathfrak{m}^{(t_1)} = \ldots = \mathfrak{m}^{(0)} = \mathfrak{m}_{L/K} = \text{Gal}(L/K); \quad d(\varphi) = t_2
\]

The allowed depths are 3, 5, 7, 9, \ldots.

(In the above, \( \mathfrak{m}^{(0)} = \mathfrak{m}_{L/K} \))

**Theorem 6.3.** Let \( L/K \) be a biquadratic extension, let \( \phi \) be the Langlands parameter \([17]\). For every \( \pi \in \Pi_\phi(\text{SL}_2(K)) \) and \( \pi \in \Pi_\phi(\text{SL}_1(D)) \) there is an equality of depths:

\[
d(\pi) = d(\phi).
\]

The depth of each element in the \( L \)-packet \( \pi_\phi \) is given by the largest break in the ramification of the Galois group \( \text{Gal}(L/K) \). The allowed depths are 1, 3, 5, 7, \ldots except in Case 2.2, when the allowed depths are 3, 5, 7, \ldots.

**Proof.** This follows from Lemma (6.2), the above computations, and Theorem 3.4 in [ABPS1]. □

This contrasts with the case of \( \text{SL}_2(\mathbb{Q}_p) \) with \( p > 2 \). Here there is a unique biquadratic extension \( L/K \), and a unique tamely ramified parameter \( \phi : \text{Gal}(L/K) \to \text{SO}_3(\mathbb{R}) \) of depth zero.

6.1. **Quadratic extensions.** Let \( E/K \) be a quadratic extension. There are two kinds: the unramified one \( E_0 = K(\alpha_0) \) and countably many totally (and wildly) ramified \( E = K(\alpha) \).

**Theorem 6.4.** For the unramified principal series \( L \)-packet \( \{\pi^1_E, \pi^2_E\} \), we have

\[
d(\pi^1_E) = d(\pi^2_E) = -1.
\]

For the ramified principal series \( L \)-packet \( \{\pi^1_E, \pi^2_E\} \), we have

\[
d(\pi^1_E) = d(\pi^2_E) = n
\]

with \( n = 1, 2, 3, 4, \ldots \).

**Proof.** Case 1: \( E_0/K \) unramified. Then, \( f(E_0/K) = 2 \). In this case, we have \( G_0 = \{1\} \), and \( G_0 = G^0 = \mathfrak{m}_{E_0/K} \). There is only one ramification break at \( t = 0 \) and the filtration of \( G = \text{Gal}(E_0/K) \) in the upper numbering is

\[
\{1\} = G^0 \subset G^{-1} = G = \mathbb{Z}/2\mathbb{Z}.
\]

The filtration on the relative inertia \( \mathfrak{m}^{(t)} \) is

\[
\{1\} = \mathfrak{m}_{L_0/K} \subset G = \mathbb{Z}/2\mathbb{Z}
\]

with only one break at \( t = 0 \). Negative depth, as expected.

Case 2: \( E/K \) is totally ramified. Then, \( e(E/K) = 2 \), which is divisible by the residue degree, so the extension is wildly ramified. In this case, there is one break
at some \( t \geq 1 \). This is because of wild ramification, since \( G^1 = \{1\} \) if and only if the extension is tamely ramified. The filtration of \( G \) in the upper numbering is
\[
\{1\} = G^{t+1} \subset G^t = \ldots = G^0 = G = \mathbb{Z}/2\mathbb{Z}
\]
The filtration on the relative inertia \( J^{(r)} \) is
\[
\{1\} = J^{(t+1)} \subset J^{(t)} = \ldots = G = \mathbb{Z}/2\mathbb{Z}
\]
with only one break at \( t \geq 1 \).

\[ \square \]

7. A COMMUTATIVE TRIANGLE

In this section we confirm part of the geometric conjecture in [ABPS] for \( \text{SL}_2(\mathbb{F}_q((\varpi))) \).

We begin by recalling the underlying ideas of the conjecture.

Let \( G \) be the group of \( \mathbb{K} \)-points of a connected reductive group over a nonarchimedean local field \( \mathbb{K} \). The Bernstein decomposition provides us, inner alia, with the following data: a canonical disjoint union

\[
\text{Irr}(G) = \bigsqcup \text{Irr}(G)^s
\]
and, for each \( s \in \mathfrak{B}(G) \), a finite-to-one surjective map

\[
\text{Irr}(G)^s \to T^s/W^s
\]

The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a bijection

\[
\text{Irr}(G)^s \simeq (T^s/W^s)_2
\]
where \((T^s/W^s)_2\) is the extended quotient of the second kind of the torus \( T^s \) by the finite group \( W^s \). This bijection is subject to certain constraints, itemised in [ABPS].

We proceed to define the extended quotient of the second kind. Let \( W \) be a finite group and let \( X \) be a complex affine algebraic variety. Suppose that \( W \) is acting on \( X \) as automorphisms of \( X \). Define

\[
\tilde{X}_2 := \{(x, \tau) : \tau \in \text{Irr}(W_x)\}.
\]

Then \( W \) acts on \( \tilde{X}_2 \):

\[
\alpha(x, \tau) = (\alpha \cdot x, \alpha \ast \tau).
\]

**Definition 7.1.** The extended quotient of the second kind is defined as

\[
(X/W)_2 := \tilde{X}_2/W.
\]

Thus the extended quotient of the second kind is the ordinary quotient for the action of \( W \) on \( \tilde{X}_2 \).

We recall that \((G, T)\) are the complex dual groups of \((\mathfrak{G}, \mathfrak{T})\), so that \( G = \text{PSL}_2(\mathbb{C}) \).

Let \( W_K \) denote the Weil group of \( K \). If \( \phi \) is an \( L \)-parameter

\[
W_K \times \text{SL}_2(\mathbb{C}) \to G
\]
then an enhanced Langlands parameter is a pair \((\phi, \rho)\) where \( \phi \) is a parameter and \( \rho \in \text{Irr}(S_\phi) \).
Theorem 7.2. Let $\mathcal{G} = \text{SL}_2(K)$ with $K = \mathbb{F}_q((\tau))$. Let $s = [T, \chi]_G$ be a point in the Bernstein spectrum for the principal series of $\mathcal{G}$. Let $\text{Irr}(\mathcal{G})^s$ be the corresponding Bernstein component in $\text{Irr}(\mathcal{G})$. Then there is a commutative triangle of natural bijections

$$
\begin{array}{c}
(T^s//W^s)_2 \\
\downarrow \\
\text{Irr}(\mathcal{G})^s \\
\downarrow \\
\mathcal{L}(G)^s
\end{array}
$$

where $\mathcal{L}(G)^s$ denotes the equivalence classes of enhanced parameters attached to $s$.

Proof. We recall that $T^s = \{\psi \chi : \psi \in \Psi(T)\}$ where $\Psi(T)$ is the group of all unramified quasicharacters of $T$. With $\lambda \in T^s$, we define the parameter $\phi(\lambda)$ as follows:

$$
\phi(\lambda) : W_K \times \text{SL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C}), \quad (w\Phi_K^\alpha, Y) \mapsto \left( \lambda(\tau)^\alpha 0 \atop 0 1 \right),
$$

where $A_\alpha$ is the image in $\text{PSL}_2(\mathbb{C})$ of $A \in \text{SL}_2(\mathbb{C})$, $Y \in \text{SL}_2(\mathbb{C})$, $w \in I_K$ the inertia group, and $\Phi_K$ is a geometric Frobenius. Define, as in §3,

$$
\pi(\lambda) := \text{Ind}^G_B(\lambda).
$$

**Case 1.** $\lambda^2 \neq 1$. Send the pair $(\lambda, 1) \in T^s//W^s$ to $\pi(\lambda) \in \text{Irr}(\mathcal{G})^s$ (via the left slanted arrow) and to $\phi(\lambda) \in \mathcal{L}(G)^s$ (via the right slanted arrow).

**Case 2.** Let $\lambda^2 = 1, \lambda \neq 1$. Let $\phi = \phi(\lambda)$. To compute $S_\phi$, let $1, w$ be representatives of the Weyl group $W = W(G)$. Then we have

$$
C_G(\text{im } \phi) = T \sqcup wT
$$

So $\phi$ is a non-discrete parameter, and we have

$$
S_\phi \simeq \mathbb{Z}/2\mathbb{Z}.
$$

We have two enhanced parameters, namely $(\phi, 1)$ and $(\phi, \epsilon)$ where $\epsilon$ is the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$.

Since $\lambda^2 = 1$, there is a point of reducibility. We send

$$
(\lambda, 1) \mapsto \pi(\lambda)^+, \quad (\lambda, \epsilon) \mapsto \pi(\lambda)^-
$$

via the left slanted arrow, and

$$
(\lambda, 1) \mapsto (\phi(\lambda), 1), \quad (\lambda, \epsilon) \mapsto (\phi(\lambda), \epsilon)
$$

via the right slanted arrow. Note that this includes the case when $\lambda$ is the unramified quadratic character of $K^\times$.

**Case 3.** Let $\lambda = 1$. The principal parameter

$$
\phi_0 : W_K \times \text{SL}_2(\mathbb{C}) \to \text{SL}_2(\mathbb{C}) \to \text{PSL}(2, \mathbb{C})
$$

is a discrete parameter for which $S_{\phi_0} = 1$. In the local Langlands correspondence for $\mathcal{G}$, the enhanced parameter $(\phi_0, 1)$ corresponds to the Steinberg representation $\text{St}$ of $\text{SL}_2(K)$. Note also that, when $\phi = \phi(1)$, we have $S_\phi = 1$. We send

$$
(1, 1) \mapsto \pi(1), \quad (1, \epsilon) \mapsto \text{St}
$$

via the left slanted arrow and

$$
(1, 1) \mapsto (\phi(1), 1), \quad (1, \epsilon) \mapsto (\phi_0, 1)
$$
Let \( L/K \) be a quadratic extension of \( K \). Let \( \lambda \) be the quadratic character which is trivial on \( N_{L/K}L^\times \). Then \( \lambda \) factors through \( \text{Gal}(L/K) \simeq K^\times/N_{L/K}L^\times \simeq \mathbb{Z}/2\mathbb{Z} \) and \( \phi(\lambda) \) factors through \( \text{Gal}(L/K) \times \text{SL}_2(\mathbb{C}) \). The parameters \( \phi(\lambda) \) serve as parameters for the \( L \)-packets in the principal series of \( \text{SL}_2(K) \).

It follows from \( \S 3 \) that, when \( K = \mathbb{F}_q((\varpi)) \), there are countably many \( L \)-packets in the principal series of \( \text{SL}_2(K) \).

7.1. The tempered dual. If we insist, in the definition of \( T^\mathfrak{s} \), that the unramified character \( \psi \) shall be unitary, then we obtain a copy \( T^\mathfrak{s} \) of the circle \( T \). We then obtain a compact version of the commutative triangle, in which the tempered dual \( \text{Irr}^\text{temp}(G)^\mathfrak{s} \) determined by \( \mathfrak{s} \) occurs on the left, and the bounded enhanced parameters \( \mathcal{L}^\mathfrak{s}(G)^\mathfrak{s} \) determined by \( \mathfrak{s} \) occur on the right. We now isolate the bijective map

\[
(\mathbb{T}^\mathfrak{s}/W^\mathfrak{s})_2 \to \text{Irr}^\text{temp}(G)^\mathfrak{s}
\]

and restrict ourselves to the case where \( \mathbb{T}^\mathfrak{s} \) contains two ramified quadratic characters. Let \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \), \( W := \mathbb{Z}/2\mathbb{Z} \). We then have \( \mathbb{T}^\mathfrak{s} = \mathbb{T} \), \( W^\mathfrak{s} = W \) and the generator of \( W \) acts on \( \mathbb{T} \) sending \( z \) to \( z^{-1} \).

The left-hand-side and the right-hand-side of the map \((21)\) each has its own natural topology, as we proceed to explain.

The topology on \((\mathbb{T}/W)_2\) comes about as follows. Let \( \text{Prim}(C(\mathbb{T}) \rtimes W) \) denote the primitive ideal space of the noncommutative \( C^* \)-algebra \( C(\mathbb{T}) \rtimes W \). By the classical Mackey theory for semidirect products, we have a canonical bijection

\[
\text{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T}/W)_2.
\]

The primitive ideal space on the left-hand-side of \((22)\) admits the Jacobson topology. So the right-hand side of \((22)\) acquires, by transport of structure, a compact non-Hausdorff topology. The following picture is intended to portray this topology.

The reduced \( C^* \)-algebra of \( G \) is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of \( G \). Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of \( G \), see [Dix 3.1.1, 4.4.1, 18.3.2]. This makes \( \text{Irr}^\text{temp}(G)^\mathfrak{s} \) into a compact space, in the induced topology.

We conjecture that these two topologies make \((21)\) into a homeomorphism. This is a strengthening of the geometric conjecture [ABPS]. In that case, the double-points in the picture arise precisely when the corresponding (parabolically) induced representation has two irreducible constituents. This conjecture is true for \( \text{SL}_2(\mathbb{Q}_p) \) with \( p > 2 \), see [P] Lemma 1. While in conjectural mode, we mention the following point: the standard Borel subgroup of \( \text{SL}_2(K) \) admits countably many ramified quadratic characters and so, following the construction in [ChP], the geometric conjecture predicts that tetrahedra of reducibility will occur countably many times; however, the
$R$-group machinery is not, to our knowledge, available in positive characteristic, so this remains conjectural.

REFERENCES


ISCTE - LISBON University INSTITUTE, AV. DAS FORÇAS ARMADAS, 1649-026, LISBON, PORTUGAL

E-mail address: sergio.mendes@iscte.pt

School of Mathematics, Southampton University, Southampton SO17 1BJ, England
and School of Mathematics, Manchester University, Manchester M13 9PL, England

E-mail address: r.j.plymen@soton.ac.uk plymen@manchester.ac.uk