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A theme running through Gautschi's work is numerical conditioning. His many papers on this topic fall broadly into two categories: those on conditioning of Vandermonde matrices and those on conditioning of polynomials.

1 Conditioning of Vandermonde Matrices

A Vandermonde matrix has the form

$$V_n = V(x_1, x_2, \dots, x_n) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} \in \mathbb{C}^{n\times n},$$

where $x_1, \dots, x_n \in \mathbb{C}$. It is worth noting that in Gautschi's papers the nodes x_i are always indexed from 1, as here, whereas they tend to be indexed from 0 in papers concerned with numerical solution of Vandermonde systems. Vandermonde matrices have long been of interest in linear algebra and numerical analysis because of the explicit formula for the determinant, $\prod_{1\le j<i\le n}(x_i - x_j)$, the fact that the inverse can be obtained from explicit formulae (see Traub [23, Sec. 14] for a short historical survey), and the general ill conditioning of Vandermonde matrices, all of which make them useful in classroom exercises and as test matrices for computational algorithms.

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In a long sequence of papers starting in 1962, Gautschi investigated the conditioning of Vandermonde matrices, obtaining upper and lower bounds as well as results on the optimal placement of the x_i to minimize the condition number. The condition number in question is the matrix condition number with respect to inversion in the ∞-norm: for nonsingular $A \in \mathbb{C}^{n\times n}$, $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$, where $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$. The original motivation for this work came from experiences in computing Gaussian quadrature rules from moments of the weight function, in which possibly ill conditioned Vandermonde matrices arise.

The Vandermonde matrix V_n is nonsingular precisely when the nodes x_i are distinct. When some of the nodes coincide the appropriate form of V_n for practical applications such as Hermite interpolation is as follows: the x_i are ordered so that equal points are contiguous and if x_i is repeated k times then V_n has columns comprising $[1, x, x^2, \ldots, x^{n-1}]$ and its first $k-1$ derivatives, all evaluated at x_i. Gautschi's papers focus on the cases $k = 1$ and $k = 2$.

In [6], Gautschi obtains the upper bound

$$\|V_n^{-1}\|_\infty \le \max_i \prod_{j\ne i} \frac{1+|x_j|}{|x_i - x_j|}, \tag{1}$$

and shows that there is equality when $x_j = |x_j|e^{i\theta}$ for all j with a fixed θ, so in particular when $x_j \ge 0$ for all j. A bound for the confluent case is also given, and a slightly sharper bound was obtained the following year in [7]. The third paper in this "On inverses ..." series appeared in 1978 [13] and gives the lower bound

$$\|V_n^{-1}\|_\infty \ge \max_i \prod_{j\ne i} \frac{\max(1,|x_j|)}{|x_i - x_j|}, \tag{2}$$

which differs from the upper bound in (1) by at most a factor 2^{n-1}. A practical application of these early results is in [8], where they are applied to a Vandermonde system arising in numerical inversion of the Laplace transform.

In [11], Gautschi specializes to the case where the nodes are located symmetrically with respect to the origin. In particular, he shows that while for nodes equispaced on $[0, 1]$,

$$\kappa_\infty(V_n) \sim \frac{1}{\pi} e^{-\frac{\pi}{4}} e^{\frac{n}{4}(\pi+2\log 2)} \approx \frac{1}{\pi} e^{-\frac{\pi}{4}} (3.1)^n,$$

for the Chebyshev points $x_i = \cos(\frac{2i-1}{2n}\pi)$ the rate of growth is much slower:

$$\kappa_\infty(V_n) \sim \frac{3^{3/4}}{4}(1+\sqrt{2})^n.$$

A natural question is how to choose the nodes to minimize the condition number. This is considered in [12], where some characterizations of the optimal nodes are obtained and optimal configurations either symmetric about the origin or nonnegative are computed explicitly for small n.

The 1988 paper [20] returns to lower bounds, showing that for nonnegative nodes, $\kappa_\infty(V_n) > 2^{n-1}$ for $n \geq 2$, while for real nodes symmetric about the origin, $\kappa_\infty(V_n) > 2^{n/2}$ for $n > 2$. Even larger lower bounds for the 2-norm condition number were subsequently obtained by Beckermann [1]:

$$\kappa_2(V_n) \geq \left(\frac{2}{n}\right)^{1/2}(1+\sqrt{2})^{n-2}, \quad \kappa_2(V_n) \geq \frac{1}{2n^{1/2}}\,[(1+\sqrt{2})^{2(n-1)}+(1+\sqrt{2})^{-2(n-1)}],$$

for arbitrary nodes and nonnegative nodes, respectively.

These exponential lower bounds are alarming, but they do not necessarily rule out the use of Vandermonde matrices in practice. One of the reasons is that there exist specialized algorithms for solving Vandermonde-systems whose accuracy is not dependent on the condition number κ, and which in some cases can be proved to be highly accurate. The first such algorithm is an $O(n^2)$ operation algorithm for solving $V_n x = b$ of Björck and Pereyra [3], whose error analysis was given by Higham [21]. There is now a long list of generalizations of this algorithm in various directions, of which we mention just Demmel and Koev [5] and Bella et al. [2]; various other algorithms up to 2002 are described or cited in the chapter "Vandermonde systems" in [22].

Another important observation is that the exponential lower bounds are for real nodes. For complex nodes V_n can be much better conditioned. Indeed V_n is $n^{1/2}$ times a *unitary* matrix when the x_i are the roots of unity. Moreover, it is shown in [4] that when the nodes are $e^{2\pi i c_j}$, with the c_j from the Van der Corput sequence, $\kappa_2(V_n) < (2n)^{1/2}$.

The matrix V_n corresponds to a monomial basis for the space of polynomials of degree up to $n-1$. Other bases can be chosen—in particular, ones built from polynomials that satisfy a three-term recurrence (and in particular, orthogonal polynomials). The latter *Vandermonde-like* matrices can be much better conditioned than Vandermonde matrices, as shown in [16].

Gautschi gives an excellent summary of his work on Vandermonde matrices up to 1990 in [18]. In his most recent contribution on Vandermonde matrices [19], he refines his earlier results and computations.

2 Conditioning of Polynomials

The papers [9], [10], [14], [15], [17] are concerned with several aspects of the conditioning of polynomials: the conditioning of the bases, the conditioning of zeros of the polynomial, and the conditioning of the problem of generating an orthogonal polynomial from moments of the weight function.

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